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HEEGNER POINTS, STARK-HEEGNER POINTS, AND DIAGONAL CLASSES

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2000 *Mathematics Subject Classification.* — 11R23, 11R34, 11G05, 11G40.

Key words and phrases. — Elliptic curves, modular forms, p-adic L-functions, Heegner points, Stark-Heegner points, generalised Kato classes.

To Bernadette Perrin-Riou on her 65-th birthday

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Abstract. — This volume comprises four interrelated articles whose unifying theme is the study of Heegner and Stark-Heegner points, and their connections with the p-adic logarithm of certain global cohomology classes attached to a pair of weight one theta series of a common (imaginary or real) quadratic field. These global classes are obtained from p-adic deformations of diagonal classes attached to triples of modular forms of weight > 1, and naturally generalise a construction of Kato which one recovers when the two theta series are replaced by Eisenstein series of weight one. Understanding the extent to which such classes obtained via the p-adic interpolation of motivic cohomology classes are themselves motivic is a key motivation for this study. A second is the desire to show that Stark-Heegner points, whose global nature is still poorly understood theoretically, arise from classes in global Galois cohomology.

Résumé. — Ce volume est constitué de 4 articles interdépendants dont le théme unificateur est l'étude des points de Heegner et de Stark-Heegner, et leurs relation avec certaines classes de cohomologie Galoisienne globales associées à une paire de séries theta de poids un du même corps quadratique (imaginaire ou réel). Ces classes proviennent de déformations p-adiques des classes diagonales associés à des triplets de formes modulaires de poids > 1, et généralisent une construction de Kato que l'on récupère quand les deux séries theta sont remplacés par des séries d'Eisenstein de poids un. Une des motivations pour cette étude est de comprendre dans quelle mesure de telles classes, obtenues par interpolation p-adique à partir de familles de classes motiviques, restent elles-mêmes motiviques. Ces résultats permettent aussi de démontrer que les points de Stark-Heegner, dont les propriétés d'algébricité sont encore complétement conjecturales, proviennent tout au moins de classes de cohomologie globales.

PREFACE

by

Massimo Bertolini, Henri Darmon, Victor Rotger,
Marco Adamo Seveso and Rodolfo Venerucci

To Bernadette Perrin-Riou on her 65-th birthday

Over the last three decades, the method of *Euler systems* has been honed into a powerful and versatile technique for relating the arithmetic of a motive to its associated L-function, in the spirit of the conjectures of Deligne, Bloch-Beilinson, Bloch-Kato and Perrin-Riou. Among its most notable successes is the proof of the weak Birch and Swinnerton-Dyer conjecture asserting the equality of the algebraic and analytic rank of an elliptic curve over $\mathbf{Q}$ when the latter invariant is ≤ 1, as well as the finiteness of the associated Shafarevich-Tate group. These statements are particularly striking in the rank one setting, given the dearth of systematic techniques for constructing rational or algebraic points on elliptic curves with direct connections to L-function behaviour.

An important precursor of the Euler System concept is the seminal work of Coates and Wiles [**CW77**] in the mid 1970's, where certain global cohomology classes constructed from norm-compatible collections of elliptic units in $\mathbf{Z}_p$–extensions of an imaginary quadratic field are used to prove the finiteness of Mordell-Weil groups of elliptic curves with complex multiplication, when the L-function of the associated Grossencharakter does not vanish at its center. The stronger method of Euler systems parlays their *tame deformations*, arising from objects defined over tamely ramified abelian extensions of finite, p-power degree, into an efficient approach for establishing the finiteness of Selmer and Shafarevich-Tate groups in addition to Mordell-Weil groups. The genesis of this approach occurs with the work of Francisco Thaine on circular units [**Th88**] in the late 1980's, whose inspiration can be traced back even further to Kummer. The subsequent transposition of Thaine's approach to the setting of elliptic units is the basis for Karl Rubin's remarkable strengthening [**Ru87**] of the approach of Coates-Wiles, with dramatic consequences for the finiteness of Shafarevich-Tate groups of elliptic curves with complex multiplication. Kolyvagin's almost simultaneous but independent breakthrough [**Ko89**] exploits Heegner points and their connection with special values of L-series exhibited earlier by Gross and Zagier [**GZ86**] to prove the equality of analytic and algebraic ranks and the finiteness of the Shafarevich-Tate group for *all* (modular) elliptic curves over $\mathbf{Q}$ of analytic rank ≤ 1.

Shortly afterwards, Kazuya Kato [**Ka04**] pioneered an entirely different Euler system approach in which Heegner points are replaced by Beilinson elements in the second K-groups of modular curves — more accurately, by their p-adic deformations arising from norm-compatible systems in towers of modular curves, echoing the theme of p-adic variation that was already present in the work of Coates and Wiles. Some 20 years later, it was realised that Kato's approach could be profitably adapted to other closely related settings, in which Beilinson elements are replaced by so-called Beilinson-Flach elements [**BDR15**] and diagonal cycles on a triple product of modular curves [**DR14**], whose p-adic deformations — particularly, those that are germane to the study of the Birch and Swinnerton-Dyer conjecture—are referred to as *generalised Kato-classes* in the articles by Darmon–Rotger ([**DR.v1**] and [**DR.v2**]) or as (specialisations of) *balanced diagonal classes* in the contributions by Bertolini–Seveso–Venerucci ([**BSV.v3**] and [**BSV.v4**]) to this collection. These classes are the key to proving the weak Birch and Swinnerton-Dyer conjecture in analytic rank zero for Mordell–Weil groups of elliptic curves over ring class fields of quadratic fields, both imaginary and real [**DR17**] (see also [**BSV20**] for a simpler variant to this method, applied in greater generality). For instance, if H is the Hilbert class field of a quadratic field K, then the implication

$$\text{“}L(E/H,1) \neq 0 \implies E(H) \text{ is finite”} \tag{1}$$

is known unconditionally via these methods. When K is imaginary, the original pathway to such a result, as described in [**BD05**], rests crucially on the existence of compatible families of Heegner points, as well as building on the theory of congruences between modular forms and on the p-adic uniformisation of Shimura curves. The route to the same result when K is real quadratic is entirely different and makes no use of the theory of complex multiplication, for the simple but compelling reason that no such theory is currently available in the setting of real quadratic fields.

Extending the theory of complex multiplication to real quadratic fields represents the simplest open case of *Hilbert's twelfth problem* aiming to adapt the Jugendtraum of Kronecker to ground fields other than the rational numbers or CM fields. A systematic attempt was initiated around 2000 to formulate a theory of “real multiplication”, involving p-adic rather than complex analytic objects. The resulting real quadratic analogues of Heegner points, defined in [**Dar01**] in terms of Coleman's theory of p-adic integration, are referred to as *Stark-Heegner points*. They are expected to give rise to a systematic norm-compatible supply of global points (on suitable elliptic curves over $\mathbf{Q}$) defined over ring class fields of real quadratic fields. Because of their strong analogy with Heegner points, they form the basis for a *purely conjectural* extension of the approach of Kolyvagin described in [**BD05**] for proving (1) when K is real quadratic, which is discussed for instance in [**BDD07**].

The article [**BD09**] introduces a different approach to Stark–Heegner points, by realising them as derivatives of Hida–Rankin p-adic L-functions. This point of view leads to the proof in loc. cit. of the rationality of Stark–Heegner points attached to genus characters of real quadratic fields. It also provides the crucial bridge to connect Stark–Heegner points to generalised Kato classes arising from suitable p-adic families of diagonal cycles. The results of [**BD07**] can likewise be exploited to make a similar comparison with Heegner points. The explicit comparison between Heegner or Stark-Heegner points and generalised Kato classes, with a view to broadening the scope of the conjecture of Perrin-Riou on rational points on elliptic curves [**PR93**], is the main goal of this volume.

Comparisons of this type between different Euler systems and Heegner points have a number of fruitful antecedents, among which it may be worthwhile to mention the following:

1. A pioneering early work by Rubin [**Ru92**] examines the global Selmer class arising from the Euler system of elliptic units and finds that the logarithm of such a class is proportional to the *square* of the logarithm of a global point arising from a Heegner point construction. This comparison of elliptic units and Heegner points has intriguing consequences for the construction of rational points on CM elliptic curves via the special values of the Katz p-adic L-function of an imaginary quadratic field.
2. In an attempt to extend Rubin's theorem to elliptic curves without complex multiplication, Bernadette Perrin-Riou conjectured in [**PR93**] that the p-adic logarithm of the global Selmer class arising from p-adic families of Beilinson elements via Kato's method should likewise be expressed in terms of the square of the logarithm of a Heegner point. This is proved in [**Ve16**] for elliptic curves with multiplicative reduction at p, and in [**BDV**] in the general case. One of the key ingredients in the latter work are the articles [**BDP13**] and [**BDP12**], the latter of which proposes an alternate approach to Rubin's formula based on special values of p-adic Rankin L-series rather than of the Katz p-adic L-function.
3. The systematic study of "p-adic iterated integrals" undertaken in [**DLR15**] leads to a general conjectural formula relating the p-adic logarithms of generalised Kato classes to certain regulators which are linear combinations with algebraic coefficients of products of two logarithms of global points on elliptic curves. This formula is conceptualised in the framework of a p-adic Birch and Swinnerton-Dyer conjecture in [**BSV21**]. The cases where this conjecture is proved unconditionally (thanks to Heegner points) are an important ingredient in the proof of Perrin-Riou's conjecture described in [**BDV**].

The present volume collects four interrelated articles, partially motivated by the goal of systematically studying the p-adic logarithm of the balanced diagonal class attached to a pair of weight one theta series of an imaginary (resp. real) quadratic field, and of relating it to the *product* of logarithms of two Heegner (resp. Stark–Heegner) points. More precisely, the first article [**DR.v1**] gives an overview of the theory of Stark–Heegner points and of Hida–Rankin p-adic L-functions attached to elliptic curves, and explains the general strategy used to relate Stark–Heegner points to generalised Kato classes. The second article [**DR.v2**] studies the problem of the p-adic interpolation of the image of diagonal cycles under the étale Abel–Jacobi map, leading to a 3-variable Λ-adic class in Iwasawa cohomology. It establishes moreover an explicit reciprocity law, connecting this class to a Hida–Garrett–Rankin p-adic L-function attached to a triple of Hida families of cusp forms. The third article [**BSV.v3**] undertakes the construction of a so-called balanced diagonal class in three variables from a different standpoint, by exploiting the invariant theory of the diagonal embedding of GL_2 into its triple product, combined with the Ash–Stevens theory of p-adic distributions. This analytic approach, formulated in the context of Coleman families of modular forms, lends itself to generalisations to higher groups. It allows to establish an explicit reciprocity law in this context, which is at the base of the results of the subsequent article. In turn the constructions of [**DR.v2**] deal more directly with the geometry of diagonal cycles and have been investigated further for example in [**CS21**]. The fourth article [**BSV.v4**] gives detailed proofs of the formulae relating the product of the p-adic logarithms of two Heegner points or Stark–Heegner points to the specialisation at the weight $(2, 1, 1)$ of the balanced diagonal class. We refer to the extensive introductions of the various chapters for further details.

At present, the collection of Heegner points on a modular elliptic curve, arising from the combination of modularity and of the theory of complex multiplication, still represents the "gold standard" for understanding the Birch and Swinnerton-Dyer conjecture, particularly in analytic rank one, where the crucial issue of producing non-trivial algebraic points of infinite order on elliptic curves becomes inescapable. By contrast, generalised Kato classes, as well as their forebearers arising from elliptic units make *a priori* only tenuous contact with these central issues, upon which further progress on the Birch and Swinnterton-Dyer conjecture would seem to be crucially dependent. Obtaining tight connections between generalised Kato classes and global points on elliptic curves, such as those proved in this volume, is worthwhile for at least two reasons. Firstly, it seems important to understand the extent to which Selmer classes constructed via a p-adic limiting process are related to "motivic" extensions attached to genuine global points on elliptic curves (or more general algebraic cycles on higher dimensional varieties). The results of the present monograph combine with those of [**Ru92**], [**Ve16**], [**BDV**], [**DLR15**] and [**BSV21**] to present a coherent picture in the setting of generalised Kato classes arising from diagonal cycles on triple products. Secondly, it lends some theoretical support for the theory of Stark–Heegner points, towards the hope of extending the available constructions of rational points on elliptic curves beyond the theory of Heegner points.

This monograph owes a tremendous debt to the vision of Perrin-Riou, whose conjecture of [**PR93**] is a basic prototype for the results that are proved here. Perrin-Riou's insights into the connection between Euler systems and p-adic L-functions through her fundamental "dual exponential map in p-adic families" also provides a key ingredient for the proofs of our main results. It is therefore a pleasure to dedicate this collection to Bernadette Perrin-Riou on her 65th birthday.

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M. Bertolini, H. Darmon, V. Rotger, M. Seveso and R. Venerucci

STARK-HEEGNER POINTS AND DIAGONAL CLASSES

by

Henri Darmon and Victor Rotger

Abstract. — Stark-Heegner points are conjectural substitutes for Heegner points when the imaginary quadratic field of the theory of complex multiplication is replaced by a real quadratic field K. They are constructed analytically as local points on elliptic curves with multiplicative reduction at a prime p that remains inert in K, but are conjectured to be rational over ring class fields of K and to satisfy a Shimura reciprocity law describing the action of G_K on them. The main conjectures of [**Da01**] predict that any linear combination of Stark-Heegner points weighted by the values of a ring class character ψ of K should belong to the corresponding piece of the Mordell-Weil group over the associated ring class field, and should be non-trivial when $L'(E/K,\psi,1) \neq 0$. Building on the results on families of diagonal classes described in the remaining contributions to this volume, this note explains how such linear combinations arise from global classes in the idoneous pro-p Selmer group, and are non-trivial when the first derivative of a weight-variable p-adic L-function $\mathscr{L}_p(\mathbf{f}/K,\psi)$ does not vanish at the point associated to $(E/K,\psi)$.

To Bernadette Perrin-Riou on her 65-th birthday

Contents

2000 ***Mathematics Subject Classification.*** — 11G18, 14G35.

1. Introduction

Let E be an elliptic curve over $\mathbf{Q}$ of conductor N and let K be a quadratic field of discriminant D relatively prime to N, with associated Dirichlet character χ_K.

When $\chi_K(-N) = -1$, the Birch and Swinnerton-Dyer conjecture predicts a systematic supply of rational points on E defined over abelian extensions of K. More precisely, if H is any ring class field of K attached to an order $\mathcal{O}$ of K of conductor prime to DN, the Hasse-Weil L-function $L(E/H, s)$ factors as a product

$$L(E/H, s) = \prod_{\psi} L(E/K, \psi, s) \tag{1.1}$$

of twisted L-series $L(E/K, \psi, s)$ indexed by the finite order characters

$$\psi : G = \mathrm{Gal}\,(H/K) \longrightarrow L^\times,$$

taking values in some fixed finite extension L of $\mathbf{Q}$. The L-series in the right-hand side of (1.1) all vanish to odd order at $s = 1$, because they arise from self-dual Galois representations and have sign $\chi_K(-N)$ in their functional equations. In particular, $L(E/K, \psi, 1) = 0$ for all ψ. An equivariant refinement of the Birch and Swinnerton-Dyer conjecture predicts that the ψ-eigenspace $E(H)^\psi \subset E(H) \otimes L$ of the Mordell-Weil group for the action of $\mathrm{Gal}\,(H/K)$ has dimension ≥ 1, and hence, that $E(H) \otimes \mathbf{Q}$ contains a copy of the regular representation of G.

When K is imaginary quadratic, this prediction is largely accounted for by the theory of Heegner points on modular or Shimura curves, which for each ψ as above produces an explicit element $P_\psi \in E(H)^\psi$. The Gross-Zagier formula implies that P_ψ is non-zero when $L'(E/K, \psi, 1) \neq 0$. Thus it follows for instance that $E(H) \otimes \mathbf{Q}$ contains a copy of the regular representation of G when $L(E/H, s)$ vanishes to order $[H : K]$ at the center.

When K is real quadratic, the construction of non-trivial algebraic points in $E(H)$ appears to lie beyond the scope of available techniques. Extending the theory of Heegner points to this setting thus represents a tantalizing challenge at the frontier of our current understanding of the Birch and Swinnerton-Dyer conjecture.

Assume from now on that $D > 0$ and there is an odd prime p satisfying

$$N = pM \text{ with } p \nmid M, \qquad \chi_K(p) = -1, \qquad \chi_K(M) = 1. \tag{1.2}$$

A conjectural construction of Heegner-type points, under the further restriction that $\chi_K(\ell) = 1$ for all $\ell | M$, was proposed in [**Da01**], and extended to the more general setting of (1.2) in [**Gr09**], [**DG12**], [**LRV12**], [**KPM18**] and [**Re15**]. It leads to a canonical collection of so-called *Stark-Heegner points*

$$P_{\mathfrak{a}} \in E(H \otimes \mathbf{Q}_p) = \prod_{\wp | p} E(H_\wp),$$

indexed by the ideal classes $\mathfrak{a}$ of $\mathrm{Pic}(\mathcal{O})$, which are regarded here as *semi-local points*, i.e., $[H : K]$-tuples $P_{\mathfrak{a}} = \{P_{\mathfrak{a},\wp}\}_{\wp|p}$ of local points in $E(K_p)$. This construction, and its equivalence with the slightly different approach of the original one, is briefly recalled in §2.

As a formal consequence of the definitions (cf. Lemma 2.1), the semi-local points $P_{\mathfrak{a}}$ satisfy the Shimura reciprocity law

$$P_{\mathfrak{a}}^{\sigma} = P_{\mathrm{rec}(\sigma)\cdot\mathfrak{a}} \qquad \text{for all } \sigma \in G,$$

where G acts on the group $E(H \otimes \mathbf{Q}_p)$ in the natural way and $\mathrm{rec} : G \to \mathrm{Pic}(\mathcal{O})$ is the Artin map of global class field theory.

The construction of the semi-local point $P_{\mathfrak{a}} \in \prod_{\wp|p} E(H_{\wp})$ is purely p-adic analytic, relying on a theory of p-adic integration of 2-forms on the product $\mathcal{H} \times \mathcal{H}_p$, where $\mathcal{H}$ denotes Poincaré's complex upper half plane and $\mathcal{H}_p$ stands for Drinfeld's rigid analytic p-adic avatar of $\mathcal{H}$, the integration being performed, metaphorically speaking, on two-dimensional regions in $\mathcal{H}_p \times \mathcal{H}$ bounded by Shintani-type cycles associated to ideal classes in K. The following statement of the Stark-Heegner conjectures of loc.cit. is equivalent to [**Da01**, Conj. 5.6, 5.9 and 5.15], and the main conjectures in [**Gr09**], [**DG12**], [**LRV12**], [**KPM18**] and [**Re15**] in the general setting of (1.2):

Stark-Heegner Conjecture. *The semi-local points $P_{\mathfrak{a}}$ belong to the natural image of $E(H)$ in $E(H \otimes \mathbf{Q}_p)$, and the ψ-component*

$$P_{\psi} := \sum_{\mathfrak{a} \in \mathrm{Pic}(\mathcal{O})} \psi^{-1}(\mathfrak{a}) P_{\mathfrak{a}} \;\in\; E(H \otimes \mathbf{Q}_p)^{\psi}$$

is non-trivial if and only if $L'(E/K, \psi, 1) \neq 0$.

The Stark-Heegner Conjecture has been proved in many cases where ψ is a *quadratic* ring class character. When $\psi^2 = 1$, the induced representation

$$V_{\psi} := \mathrm{Ind}_K^{\mathbf{Q}} \psi = \chi_1 \oplus \chi_2$$

decomposes as the sum of two one-dimensional Galois representations attached to quadratic Dirichlet characters satisfying

$$\chi_1(p) = -\chi_2(p), \qquad \chi_1(M) = \chi_2(M),$$

and the pair (χ_1, χ_2) can be uniquely ordered in such a way that the L-series $L(E, \chi_1, s)$ and $L(E, \chi_2, s)$ have sign 1 and -1 respectively in their functional equations.

Define the local sign $\alpha := a_p(E)$, which is equal to either 1 or -1 according to whether E has split or non-split multiplicative reduction at p. Let $\mathfrak{p}$ be a prime of H above p, and let $\sigma_{\mathfrak{p}} \in \mathrm{Gal}(H/\mathbf{Q})$ denote the associated Frobenius element. Because p is inert in $K/\mathbf{Q}$, the unique prime of K above p splits completely in H/K and $\sigma_{\mathfrak{p}}$ belongs to a conjugacy class of reflections in the generalised dihedral group $\mathrm{Gal}(H/\mathbf{Q})$. It depends in an essential way on the choice of $\mathfrak{p}$, but, because ψ cuts out an abelian extension of $\mathbf{Q}$, the Stark-Heegner point

$$P_{\psi}^{\alpha} := P_{\psi} + \alpha \cdot \sigma_{\mathfrak{p}} P_{\psi} \tag{1.3}$$

does not depend on this choice. It can in fact be shown that

$$P_{\psi}^{\alpha} = \begin{cases} 2P_{\psi} & \text{if } \chi_2(p) = \alpha; \\ 0 & \text{if } \chi_2(p) = -\alpha. \end{cases}$$

The recent work [**Mo17**] of Mok and [**LMY17**] of Longo, Martin and Yan, building on the methods introduced in [**BD09**, Thm. 1], [**Mo11**], and [**LV14**], asserts:

Stark-Heegner theorem for quadratic characters. *Let* ψ *be a* quadratic *ring class character of conductor prime to* $2DN$*. Then the Stark-Heegner point* P_ψ^α *belongs to* $E(H) \otimes \mathbf{Q}$ *and is non-trivial if and only if*

$$L(E, \chi_1, 1) \neq 0, \quad L'(E, \chi_2, 1) \neq 0, \quad \textit{and} \quad \chi_2(p) = \alpha. \tag{1.4}$$

The principle behind the proof of this result is to compare P_ψ^α to suitable Heegner points arising from Shimura curve parametrisations, exploiting the fortuitous circumstance that the field over which P_ψ is conjecturally defined is a biquadratic extension of $\mathbf{Q}$ and is thus also contained in ring class fields of imaginary quadratic fields (in many different ways).

The present work is concerned with the less well understood *generic* case where $\psi^2 \neq 1$, when the induced representation V_ψ is irreducible. Note that ψ is either totally even or totally odd, i.e., complex conjugation acts as a scalar $\epsilon_\psi \in \{1, -1\}$ on the induced representation V_ψ.

The field which ψ cuts out cannot be embedded in any compositum of ring class fields of imaginary quadratic fields, and the Stark-Heegner Conjecture therefore seems impervious to the theory of Heegner points in this case.

The semi-local point P_ψ^α of (1.3) now depends crucially on the choice of $\mathfrak{p}$, but it is not hard to check that its image under the localisation homomorphism

$$j_\mathfrak{p} : E(H \otimes \mathbf{Q}_p) \longrightarrow E(H_\mathfrak{p}) = E(K_p)$$

at $\mathfrak{p}$ is independent of this choice, up to scaling by $L^\times$ (cf. Lemma 2.4). It is the local point

$$P_{\psi,\mathfrak{p}}^\alpha := j_\mathfrak{p}(P_\psi^\alpha) \in E(H_\mathfrak{p}) \otimes L = E(K_p) \otimes L$$

which will play a key role in Theorems A and B below.

Theorems A and B are conditional on either one of the two non-vanishing hypotheses below, which apply to a pair (E, K) and a choice of archimedean sign $\epsilon \in \{-1, 1\}$. The first hypothesis is the counterpart, in analytic rank one, of the non-vanishing for simultaneous twists of modular L-series arising as the special case of [**DR17**, Def. 6.8] discussed in (168) of loc.cit., where it plays a similar role in the proof of the Birch and Swinnerton–Dyer conjecture for $L(E/K, \psi, s)$ when $L(E/K, \psi, 1) \neq 0$. The main difference is that we are now concerned with quadratic ring class characters for which $L(E/K, \psi, s)$ vanishes to odd rather than to even order at the center.

Analytic non-vanishing hypothesis: *Given* (E, K) *as above, and a choice of a sign* $\epsilon \in \{1, -1\}$*, there exists a quadratic Dirichlet character* χ *of conductor prime to* DN *satisfying*

$$\chi(-1) = -\epsilon, \quad \chi\chi_K(p) = \alpha, \quad L(E, \chi, 1) \neq 0, \quad L'(E, \chi\chi_K, 1) \neq 0.$$

The second non-vanishing hypothesis applies to an arbitrary ring class character ξ of K.

Weak non-vanishing hypothesis for Stark-Heegner points: *Given* (E, K) *as above, and a sign* $\epsilon \in \{1, -1\}$*, there exists a ring class character* ξ *of* K *of conductor prime to* DN *with* $\epsilon_\xi = -\epsilon$ *for which* $P^{\alpha}_{\xi,\mathfrak{p}} \neq 0$.

That the former hypothesis implies the latter follows by applying the Stark-Heegner theorem for quadratic characters to the quadratic ring class character ξ of K attached to the pair $(\chi_1, \chi_2) := (\chi, \chi\chi_K)$ supplied by the analytic non-vanishing hypothesis. The stronger non-vanishing hypothesis is singled out because it has the virtue of tying in with mainstream questions in analytic number theory on which there has been recent progress [**Mu12**]. On the other hand, the weak non-vanishing hypothesis is known to be true in the classical setting of Heegner points, when K is imaginary quadratic. In fact, for a given E and K, *all but finitely many* of the Heegner points $P_{\mathfrak{a}}$ (as $\mathfrak{a}$ ranges over all ideal classes of all possible orders in K) are of infinite order, and P_ξ and P^{α}_{ξ} are therefore non-trivial for infinitely many ring class characters ξ, and for at least one character of any given conductor, with finitely many exceptions. It seems reasonable to expect that Stark-Heegner points should exhibit a similar behaviour, and the experimental evidence bears this out as one can readily verify on a software package like Pari or Magma. In practice, efficient algorithms for calculating Stark-Heegner points make it easy to produce a non-zero $P^{\alpha}_{\xi,\mathfrak{p}}$ for any given (E, K), and indeed, the extensive experiments carried out so far have failed to produce even a single example of a vanishing P^{α}_{ξ} when ξ has order ≥ 3. Thus, while these non-vanishing hypotheses are probably difficult to prove in general, they are expected to hold systematically. Moreover, they can easily be checked in practice for any specific triple (E, K, ϵ) and therefore play a somewhat ancillary role in studying the infinite collection of Stark-Heegner points attached to a fixed E and K.

Let $V_p(E) := \left(\varprojlim E[p^n]\right) \otimes \mathbf{Q}_p$ denote the Galois representation attached to E and let

$$\mathrm{Sel}_p(E/H) := H^1_{\mathrm{f}}(H, V_p(E))$$

be the pro-p Selmer group of E over H. The ψ-component of this Selmer group is an L_p-vector space, where L_p is a field containing both $\mathbf{Q}_p$ and L, by setting

$$\mathrm{Sel}_p(E/H)^{\psi} := \{\kappa \in H^1_{\mathrm{f}}(H, V_p(E)) \otimes_{\mathbf{Q}_p} L_p \text{ s.t. } \sigma\kappa = \psi(\sigma) \cdot \kappa \text{ for all } \sigma \in \mathrm{Gal}\,(H/K)\}.$$

Since E is defined over $\mathbf{Q}$, the group

$$\mathrm{Sel}_p(E/H) \simeq \oplus_{\varrho} H^1_{\mathrm{f}}(\mathbf{Q}, V_p(E) \otimes \varrho)$$

admits a natural decomposition indexed by the set of irreducible representations ϱ of $\mathrm{Gal}\,(H/\mathbf{Q})$. In this note we focus on the isotypic component singled out by ψ, namely

$$\mathrm{Sel}_p(E, \psi) := H^1_{\mathrm{f}}(\mathbf{Q}, V_p(E) \otimes V_\psi) = \mathrm{Sel}_p(E/H)^{\psi} \oplus \mathrm{Sel}_p(E/H)^{\bar{\psi}} \qquad (1.5)$$

where Shapiro's lemma combined with the inflation-restriction sequence gives the above canonical identifications.

It will be convenient to assume from now on that $E[p]$ is irreducible as a $G_{\mathbf{Q}}$-module. This hypothesis could be relaxed at the cost of some simplicity and transparency in some of the arguments.

Theorem A. *Assume that the (analytic or weak) non-vanishing hypothesis holds for (E, K, ϵ). Let ψ be any non-quadratic ring class character of K of conductor prime to DN, for which $\epsilon_\psi = \epsilon$. Then there is a global Selmer class*

$$\kappa_\psi \in \mathrm{Sel}_p(E, \psi)$$

whose natural image in the group $E(H_{\mathfrak{p}}) \otimes L_p$ of local points agrees with $P^{\alpha}_{\psi,\mathfrak{p}}$.

The Selmer class mentioned in the statement above is constructed as a p-adic limit of diagonal classes. In particular, it follows from Theorem A that

$$P^{\alpha}_{\psi,\mathfrak{p}} \neq 0 \quad \Rightarrow \quad \dim_{L_p} Sel(\mathbb{T})_p(E/H)^{\psi} \geq 1. \tag{1.6}$$

As a corollary, we obtain a criterion for the infinitude of $\mathrm{Sel}_p(E/H)^{\psi}$ in terms of the p-adic L-function $\mathscr{L}_p(\mathbf{f}/K, \psi)$ constructed in [BD09, §3], interpolating the square roots of the central critical values $L(f_k/K, \psi, k/2)$, as f_k ranges over the weight $k \geq 2$ classical specializations of the Hida family passing through the weight two eigenform f associated to E. The interpolation property implies that $\mathscr{L}_p(\mathbf{f}/K, \psi)$ vanishes at $k = 2$, and its first derivative $\mathscr{L}_p'(\mathbf{f}/K, \psi)(2)$ is a natural p-adic analogue of the derivative at $s = 1$ of the classical complex L-function $L(f/K, \psi, s)$. The following result can thus be viewed as a p-adic variant of the Birch and Swinnerton-Dyer Conjecture in this setting.

Theorem B. *If $\mathscr{L}_p'(\mathbf{f}/K, \psi)(2) \neq 0$, then $\dim_{L_p} Sel(\mathbb{T})_p(E/H)^{\psi} \geq 1$.*

Theorem B is a direct corollary of (1.6) in light of the main result of [BD09], recalled in Theorem 4.1 below, which asserts that $P^{\alpha}_{\psi,\mathfrak{p}}$ is non-trivial when $\mathscr{L}_p'(\mathbf{f}/K, \psi)(2) \neq 0$.

Remark 1. Assume the p-primary part of (the ψ-isotypic component of) the Tate-Shafarevich group of E/H is finite. Then Theorem A shows that $P^{\alpha}_{\psi,\mathfrak{p}}$ arises from a *global* point in $E(H) \otimes L_p$, as predicted by the Stark-Heegner conjecture. Moreover, Theorem B implies that $\dim_L E(H)^{\psi} \geq 1$ if $\mathscr{L}_p'(\mathbf{f}/K, \psi)(2) \neq 0$.

Remark 2. The irreducibility of V_ψ when ψ is non-quadratic shows that P^{α}_{ψ} is non-trivial if and only if the same is true for P_ψ. The Stark-Heegner Conjecture combined with the injectivity of the map from $E(H) \otimes L$ to $E(H_{\mathfrak{p}}) \otimes L$ suggests that $P^{\alpha}_{\psi,\mathfrak{p}}$ never vanishes when $P_\psi \neq 0$, but the scenario where P^{α}_{ψ} is a non-trivial element of the kernel of $j_{\mathfrak{p}}$ seems hard to rule out unconditionally, without assuming the Stark-Heegner conjecture a priori.

Remark 3. Section 2 is devoted to review the theory of Stark-Heegner points. For notational simplicity, §2 has been written under the stronger Heegner hypothesis

$$\chi_K(p) = -1, \qquad \chi_K(\ell) = 1 \text{ for all } \ell | M$$

of [**Da01**]. This section merely collects together the basic notations and principal results of [**Da01**], [**BD09**], [**Mo17**] and [**LMY17**]. Exact references for the analogous results needed to cover the more general setting of (1.2) are given along the way. The remaining sections §3, 4, 5, 6 and 7, which form the main body of the article, adapt without change to proving Theorems A and B under the general assumption (1.2). In particular, while *quaternionic* modular forms need to be invoked in the general construction of Stark-Heegner points of [**Gr09**], [**DG12**] and [**LRV12**], the arguments in loc. cit. only employ *classical elliptic modular forms* in order to deal with the general setting.

Remark 4. The proof of Theorems A and B summarized in this note invokes several crucial results on families of diagonal classes that are proved in the remaining contributions to this volume. In particular the articles [**BSVa**] and [**BSVb**] supply essential ingredients in the extension of the Perrin-Riou style reciprocity laws in settings where the idoneous p-adic L-function admits an "exceptional zero". In a previous version of this article it was wrongly claimed that one of the key inputs, namely formula (7.7) in the text, follows from one of the main results in Venerucci's paper [**Ve16**]; the authors are grateful to Bertolini, Seveso and Venerucci for pointing out this error and supplying a proof of this important formula in their contributions to this volume.

History and connection with related work. The first two articles in this volume are the culmination of a project which originated in the summer of 2010 during a two month visit by the first author to Barcelona, where, building on the approach of [**BDP13**], the authors began collaborating on what eventually led to the p-adic Gross-Zagier formula of [**DR14**] relating p-adic Abel-Jacobi images of diagonal cycles on a triple product of modular curves to the special values of certain Garrett-Rankin triple product p-adic L-functions. In October of that year, they realized that Kato's powerful idea of varying Galois cohomology classes in (cyclotomic) p-adic families could be adapted to deforming the étale Abel Jacobi images of diagonal cycles, or the étale regulators of Belinson-Flach elements, along Hida families. The resulting *generalised Kato classes* obtained by specialising these families to weight one seemed to promise significant arithmetic applications, notably for the Birch and Swinnerton-Dyer conjecture over ring class fields of real quadratic fields – a setting that held a special appeal because of its connection with the still poorly understood theory of Stark-Heegner points. This led the authors to formulate a program, whose broad outline was already in place by the end of 2010, and whose key steps involved

— In the setting of "analytic rank zero", a proof of the "weak Birch and Swinnerton Dyer conjecture" for elliptic curves over $\mathbf{Q}$ twisted by certain Artin representations ϱ of dimension ≤ 4 arising in the tensor product of a pair of odd two-dimensional Artin representations, i.e., the statement that

$$L(E,\varrho,1) \neq 0 \quad \Rightarrow (E(H) \otimes \varrho)^{G_{\mathbf{Q}}} = 0.$$

This was carried out in [**DR17**] and [**BDR15**] by showing that the generalised Kato classes fail to be crystalline precisely when $L(E,\varrho,1) \neq 0$.

— In the setting of "analytic rank one", when $L(E,\varrho,1) = 0$ it becomes natural to compare the relevant generalised Kato class to algebraic points in the ϱ-isotypic part of $E(H)$, along the lines of conjectures first formulated by Rubin (for CM elliptic curves) and by Perrin-Riou (in the setting of Kato's work). Several precise conjectures were formulated along those lines, notably in [DLR15], guided by extensive numerical experiments conducted with Alan Lauder. In general, the independent existence of such global points is tied with deep and yet unproved instances of the Birch and Swinnerton-Dyer conjecture, but when ϱ is induced from a ring class character of a real quadratic field K and p is a prime of *multiplicative reduction* for E which is inert in K, it becomes natural to compare the resulting generalised Kato class (a global invariant in the Selmer group, albeit with p-adic coefficients) to Stark-Heegner points (which are defined purely p-adic analytically, but are conjecturally motivic, with $\mathbf{Q}$-coefficients).

Starting roughly in 2012, the idea of exploiting p-adic families of diagonal cycles and Beilinson-Flach elements was taken up by several others, motivated by a broader range of applications. While the authors were fleshing out their strategy for writing the two papers appearing in this volume, they thus benefitted from several key advances made during this time, which have simplified and facilitated the work that is described herein, and which it is a pleasure to acknowledge, most importantly:

— The construction of three variable cohomology classes was further developed and perfected, in the setting of Beilinson-Flach elements by Lei, Loeffler and Zerbes [LLZ14] and several significant improvements were subsequently proposed, notably in the article [KLZ17] in which Kings' Λ-adic sheaves play an essential role. These provide what are often more efficient and general approaches to constructing p-adic families of cohomology classes.

— The article [BSVa] by Bertolini, Seveso and Venerucci that appears in this volume constructs a three-variable Λ-adic class of diagonal cohomology classes by a different method, building on the work of Andreatta-Iovita-Stevens, and makes a more systematic study of such classes in settings where there is an exceptional zero, surveying a wider range of scenarios. Although there is some overlap between the two works as far as the general strategy is concerned, both present a different take on these results. Indeed, the approach in this note eschews the methods of Andreatta-Iovita-Stevens in favour of an approach based on the study of a collection of cycles on the cube of the modular curve $X(N)$ of full level structure. These cycles are of interest in their own right, and shed a useful complementary perspective on the construction of the Λ-adic cohomology classes for the triple product. Indeed, their study forms the basis for the ongoing PhD thesis of David Lilienfeldt [Li], and has let to interesting open questions (cf. e.g. those that are explored in [CS20]).

— Families of cohomology classes based on compatible collections of Heegner points are of course a long-standing theme in the subject, and have been taken up anew, for instance in the more recent works of Castella-Hsieh [CS18], Kobayashi [Ko20] and Jetchev-Loeffler-Zerbes [JLZ20].

Acknowledgements. The first author was supported by an NSERC Discovery grant. The second author also acknowledges the financial support by ICREA under the

ICREA Academia programme. This project has received funding from the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation programme (grant agreement No 682152). It is a pleasure to thank M.L. Hsieh and M. Longo for detailed explanations of their respective recent preprints, and M. Bertolini, M. Seveso, and R. Venerucci for their complementary works [**BSVa**], [**BSVb**] appearing in this volume.

2. Stark-Heegner points

This section recalls briefly the construction of Stark-Heegner points originally proposed in [**Da01**] and compares it with the equivalent but slightly different presentation given in the introduction. As explained in Remark 3, we provide the details under the running assumptions of loc. cit., and we refer to the references quoted in the introduction for the analogous story under the more general hypothesis (1.2).

Let $E/\mathbf{Q}$ be an elliptic curve of conductor $N := pM$ with $p \nmid M$. Since E has multiplicative reduction at p, the group $E(\mathbf{Q}_{p^2})$ of local points over the quadratic unramified extension $\mathbf{Q}_{p^2}$ of $\mathbf{Q}_p$ is equipped with Tate's p-adic uniformisation

$$\Phi_{\mathrm{Tate}} : \mathbf{Q}_{p^2}^\times / q^{\mathbf{Z}} \longrightarrow E(\mathbf{Q}_{p^2}).$$

Let f be the weight two newform attached to E via Wiles' modularity theorem, which satisfies the usual invariance properties under Hecke's congruence group $\Gamma_0(N)$, and let

$$\Gamma := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbf{Z}[1/p]), \qquad c \equiv 0 \pmod{M} \right\}$$

denote the associated p-arithmetic group, which acts by Möbius transformations both on the complex upper-half plane $\mathcal{H}$ and on Drinfeld's p-adic analogue $\mathcal{H}_p := \mathbb{P}_1(\mathbf{C}_p) - \mathbb{P}_1(\mathbf{Q}_p)$. The main construction of Sections 1-3 of [**Da01**] attaches to f a non-trivial *indefinite multiplicative integral*

$$\mathcal{H}_p \times \mathbb{P}_1(\mathbf{Q}) \times \mathbb{P}_1(\mathbf{Q}) \longrightarrow \mathbf{C}_p^\times / q^{\mathbf{Z}}, \qquad (\tau, x, y) \mapsto \times\!\!\!\!\!\!\int^{\tau} \int_x^y \omega_f$$

satisfying

$$\times\!\!\!\!\!\!\int^{\gamma\tau} \int_{\gamma x}^{\gamma y} \omega_f = \times\!\!\!\!\!\!\int^{\tau} \int_x^y \omega_f, \qquad \text{for all } \gamma \in \Gamma, \tag{2.1}$$

along with the requirement that

$$\times\!\!\!\!\!\!\int^{\tau} \int_x^y \omega_f = \left(\times\!\!\!\!\!\!\int^{\tau} \int_y^x \omega_f \right)^{-1}, \qquad \times\!\!\!\!\!\!\int^{\tau} \int_x^y \omega_f \times \times\!\!\!\!\!\!\int^{\tau} \int_y^z \omega_f = \times\!\!\!\!\!\!\int^{\tau} \int_x^z \omega_f. \tag{2.2}$$

This function is obtained, roughly speaking, by applying the Schneider-Teitelbaum p-adic Poisson transform to a suitable harmonic cocycle constructed from the modular symbol attached to f. It is important to note that there are in fact *two distinct* such modular symbols, which depend on a choice of a sign $w_\infty = \pm 1$ at ∞ and are referred to as the plus and the minus modular symbols, and therefore two distinct multiplicative integral functions, with different transformation properties under matrices of determinant -1 in $\mathrm{GL}_2(\mathbf{Z}[1/p])$. More precisely, the multiplicative integral associated to w_∞ satisfies the further invariance property

$$\times\!\!\!\!\!\!\int^{-\tau}\int_{-x}^{-y}\omega_f = \left(\times\!\!\!\!\!\!\int^{\tau}\int_{x}^{y}\omega_f\right)^{w_\infty}.$$

See sections 1-3 of loc. cit., and §3.3. in particular, for further details.

Let K be a real quadratic field of discriminant $D>0$, whose associated Dirichlet character χ_K satisfies the *Heegner hypothesis*

$$\chi_K(p)=-1, \qquad \chi_K(\ell)=1 \text{ for all } \ell|M.$$

It follows that D is a quadratic residue modulo M, and we may fix a $\delta\in(\mathbf{Z}/M\mathbf{Z})^\times$ satisfying $\delta^2=D \pmod M$. Let $K_p\simeq\mathbf{Q}_{p^2}$ denote the completion of K at p, and let $\sqrt{D}$ denote a chosen square root of D in K_p.

Fix an order $\mathcal{O}$ of K, of conductor c relatively prime to DN. The narrow Picard group $G_{\mathcal{O}}:=\mathrm{Pic}(\mathcal{O})$ is in bijection with the set of $\mathrm{SL}_2(\mathbf{Z})$-equivalence classes of binary quadratic forms of discriminant Dc^2. A binary quadratic form $F=Ax^2+Bxy+Cy^2$ of this discriminant is said to be a *Heegner form* relative to the pair (M,δ) if M divides A and $B\equiv\delta c \pmod M$. Every class in $G_{\mathcal{O}}$ admits a representative which is a Heegner form, and all such representatives are equivalent under the natural action of the group $\Gamma_0(M)$. In particular, we can write

$$G_{\mathcal{O}}=\Gamma_0(M)\backslash\left\{Ax^2+Bxy+Cy^2 \quad \text{with } (A,B)\equiv(0,\delta c) \pmod M\right\}.$$

For each class $\mathfrak{a}:=Ax^2+Bxy+Cy^2\in G_{\mathcal{O}}$ as above, let

$$\tau_{\mathfrak{a}}:=\frac{-B+c\sqrt{D}}{2A}\in K_p-\mathbf{Q}_p\subset\mathcal{H}_p, \qquad \gamma_{\mathfrak{a}}:=\begin{pmatrix} r-Bs & -2Cs\\ 2As & r+Bs\end{pmatrix},$$

where (r,s) is a primitive solution to the Pell equation $x^2-Dc^2y^2=1$. The matrix $\gamma_{\mathfrak{a}}\in\Gamma$ has $\tau_{\mathfrak{a}}$ as a fixed point for its action on $\mathcal{H}_p$. This fact, combined with properties (2.1) and (2.2), implies that the period

$$J_{\mathfrak{a}}:=\times\!\!\!\!\!\!\int^{\tau_{\mathfrak{a}}}\int_{x}^{\gamma_{\mathfrak{a}}x}\omega_f \ \in K_p^\times/q^{\mathbf{Z}}$$

does not depend on the choice of $x\in\mathbb{P}_1(\mathbf{Q})$ that was made to define it. Property (2.1) also shows that $J_{\mathfrak{a}}$ depends only on $\mathfrak{a}$ and not on the choice of Heegner representative that was made in order to define $\tau_{\mathfrak{a}}$ and $\gamma_{\mathfrak{a}}$. The local point

$$y(\mathfrak{a}):=\Phi_{\mathrm{Tate}}(J_{\mathfrak{a}})\in E(K_p)$$

is called the *Stark-Heegner point* attached to the class $\mathfrak{a}\in G_{\mathcal{O}}$.

Let H denote the narrow ring class field of K attached to $\mathcal{O}$, whose Galois group is canonically identified with $G_{\mathcal{O}}$ via global class field theory. Because p is inert in $K/\mathbf{Q}$ and $\mathrm{Gal}\,(H/K)$ is a generalised dihedral group, this prime splits completely in H/K. The set $\mathcal{P}$ of primes of H that lie above p has cardinality $[H:K]$ and is endowed with a simply transitive action of $\mathrm{Gal}\,(H/K)=G_{\mathcal{O}}$, denoted $(\mathfrak{a},\mathfrak{p})\mapsto\mathfrak{a}*\mathfrak{p}$.

Set $K_p^{\mathcal{P}} := \mathrm{Hom}(\mathcal{P}, E(K_p)) \simeq K_p^{[H:K]}$. There is a canonical identification

$$H \otimes \mathbf{Q}_p = K_p^{\mathcal{P}}, \tag{2.3}$$

sending $x \in H \otimes \mathbf{Q}_p$ to the function $\mathfrak{p} \mapsto x(\mathfrak{p}) := x_{\mathfrak{p}}$, where $x_{\mathfrak{p}}$ denotes the natural image of x in $H_{\mathfrak{p}} = K_p$. The group $\mathrm{Gal}\,(H/K)$ acts compatibly on both sides of (2.3), acting on the latter via the rule

$$\sigma x(\mathfrak{p}) = x(\sigma^{-1} * \mathfrak{p}). \tag{2.4}$$

Our fixed embedding of H into $\bar{\mathbf{Q}}_p$ determines a prime $\mathfrak{p}_0 \in \mathcal{P}$. Conjecture 5.6 of [Da01] asserts that the points $y(\mathfrak{a})$ are the images in $E(K_p)$ of global points $P_{\mathfrak{a}}^{?} \in E(H)$ under this embedding, and Conjecture 5.9 of loc. cit. asserts that these points satisfy the Shimura reciprocity law

$$P_{\mathfrak{ba}}^{?} = \mathrm{rec}(\mathfrak{b})^{-1} P_{\mathfrak{a}}^{?}, \qquad \text{for all } \mathfrak{b} \in \mathrm{Pic}(\mathcal{O}),$$

where $\mathrm{rec} : \mathrm{Pic}(\mathcal{O}) \longrightarrow \mathrm{Gal}\,(H/K)$ denotes the reciprocity map of global class field theory.

It is convenient to reformulate the conjectures of [Da01] as suggested in the introduction, by parlaying the collection $\{y(\mathfrak{a})\}$ of local points in $E(K_p)$ into a collection of semi-local points

$$P_{\mathfrak{a}} \in E(H \otimes \mathbf{Q}_p) = E(K_p)^{\mathcal{P}}$$

indexed by $\mathfrak{a} \in G_{\mathcal{O}}$. This is done by letting $P_{\mathfrak{a}}$ (viewed as an $E(K_p)$-valued function on the set $\mathcal{P}$) be the element of $E(H \otimes \mathbf{Q}_p)$ given by

$$(P_{\mathfrak{a}})(\mathfrak{b} * \mathfrak{p}_0) := y(\mathfrak{ab}),$$

so that, by definition

$$P_{\mathfrak{ba}}(\mathfrak{p}) = P_{\mathfrak{a}}(\mathfrak{b} * \mathfrak{p}). \tag{2.5}$$

This point of view has the pleasant consequence that the Shimura reciprocity law becomes a formal consequence of the definitions:

Lemma 2.1. — *The semi-local Stark-Heegner points $P_{\mathfrak{a}} \in E(H \otimes \mathbf{Q}_p)$ satisfy the Shimura reciprocity law*

$$\mathrm{rec}(\mathfrak{b})^{-1}(P_{\mathfrak{a}}) = P_{\mathfrak{ba}}.$$

Proof. — By (2.4),

$$\mathrm{rec}(\mathfrak{b})^{-1}(P_{\mathfrak{a}})(\mathfrak{p}) = P_{\mathfrak{a}}(\mathrm{rec}(\mathfrak{b}) * \mathfrak{p}) = P_{\mathfrak{a}}(\mathfrak{b} * \mathfrak{p}), \qquad \text{for all } \mathfrak{p} \in \mathcal{P}.$$

But on the other hand, by (2.5)

$$P_{\mathfrak{a}}(\mathfrak{b} * \mathfrak{p}) = P_{\mathfrak{ba}}(\mathfrak{p}).$$

The result follows from the two displayed identities. □

The modular form f is an eigenvector for the Atkin-Lehner involution W_N acting on $X_0(N)$. Let w_N denote its associated eigenvalue. Note that this is the negative of the sign in the functional equation for $L(E,s)$ and hence that $E(\mathbf{Q})$ is expected to have odd (resp. even) rank if $w_N = 1$ (resp. if $w_N = -1$). Recall the prime $\mathfrak{p}_0$ of H attached to the chosen embedding of H into $\bar{\mathbf{Q}}_p$. The frobenius element at $\mathfrak{p}_0$ in $\mathrm{Gal}\,(H/\mathbf{Q})$ is a reflection in this dihedral group, and is denoted by $\sigma_{\mathfrak{p}_0}$.

Proposition 2.2. — *For all $\mathfrak{a} \in G_{\mathcal{O}}$,*

$$\sigma_{\mathfrak{p}_0} P_{\mathfrak{a}} = w_N P_{\mathfrak{a}^{-1}}.$$

Proof. — Proposition 5.10 of [Da01] asserts that

$$\sigma_{\mathfrak{p}_0} y(\mathfrak{a}) = w_N y(\mathfrak{c}\mathfrak{a})$$

for some $\mathfrak{c} \in G_{\mathcal{O}}$. The definition of $\mathfrak{c}$ which occurs in equation (177) of loc.cit. directly implies that

$$\sigma_{\mathfrak{p}_0} y(1) = w_N y(1), \qquad \sigma_{\mathfrak{p}_0} y(\mathfrak{a}) = w_N y(\mathfrak{a}^{-1}),$$

and the result follows from this. □

Lemma 2.1 shows that the collection of Stark-Heegner points $P_{\mathfrak{a}}$ is preserved under the action of $\mathrm{Gal}\,(H/K)$, essentially by fiat. A corollary of the less formal Proposition 2.2 is the following invariance of the Stark-Heegner points under the full action of $\mathrm{Gal}\,(H/\mathbf{Q})$:

Corollary 2.3. — *For all $\sigma \in \mathrm{Gal}\,(H/\mathbf{Q})$ and all $\mathfrak{a} \in G_{\mathcal{O}}$,*

$$\sigma P_{\mathfrak{a}} = w_N^{\delta_\sigma} P_{\mathfrak{b}}, \qquad \text{for some } \mathfrak{b} \in G_{\mathcal{O}},$$

where

$$\delta_\sigma = \begin{cases} 0 & \text{if } \sigma \in \mathrm{Gal}\,(H/K); \\ 1 & \text{if } \sigma \notin \mathrm{Gal}\,(H/K). \end{cases}$$

Proof. — This follows from the fact that $\mathrm{Gal}\,(H/\mathbf{Q})$ is generated by $\mathrm{Gal}\,(H/K)$ together with the reflection $\sigma_{\mathfrak{p}_0}$. □

To each $\mathfrak{p} \in \mathcal{P}$ we have associated an embedding $j_{\mathfrak{p}} : H \longrightarrow K_p$ and a frobenius element $\sigma_{\mathfrak{p}} \in \mathrm{Gal}\,(H/\mathbf{Q})$. If $\mathfrak{p}' = \sigma * \mathfrak{p}$ is another prime in $\mathcal{P}$, then we observe that

$$j_{\mathfrak{p}'} = j_{\mathfrak{p}} \circ \sigma^{-1}, \qquad \sigma_{\mathfrak{p}'} = \sigma \sigma_{\mathfrak{p}} \sigma^{-1}, \qquad j_{\mathfrak{p}'} \circ \sigma_{\mathfrak{p}'} = j_{\mathfrak{p}} \circ \sigma_{\mathfrak{p}} \circ \sigma^{-1}. \tag{2.6}$$

Let $\psi : \mathrm{Gal}\,(H/K) \longrightarrow L^\times$ be a ring class character, let

$$e_\psi := \frac{1}{\# G_{\mathcal{O}}} \sum_{\sigma \in G_{\mathcal{O}}} \psi(\sigma)\sigma^{-1} \in L[G_{\mathcal{O}}]$$

be the associated idempotent in the group ring, and denote by

$$P_\psi := e_\psi P_1 \in E(H \otimes \mathbf{Q}_p) \otimes L$$

the ψ-component of the Stark-Heegner point. Recall from the introduction the sign $\alpha \in \{-1, 1\}$ which is equal to 1 (resp. -1) if E has split (resp. non-split) multiplicative reduction at the prime p. Following the notations of the introduction, write

$$P_\psi^\alpha = (1 + \alpha \sigma_{\mathfrak{p}}) P_\psi.$$

Lemma 2.4. — *The local point $j_{\mathfrak{p}}(P_\psi^\alpha)$ is independent of the choice of prime $\mathfrak{p} \in \mathcal{P}$ that was made to define it, up to multiplication by a scalar in $\psi(G_{\mathcal{O}}) \subset L^\times$.*

Proof. — Let $\mathfrak{p}' = \sigma * \mathfrak{p}$ be any other element of $\mathcal{P}$. Then by (2.6),

$$\begin{aligned} j_{\mathfrak{p}'}(1+\alpha\sigma_{\mathfrak{p}'})P_\psi &= j_{\mathfrak{p}}\circ\sigma^{-1}(1+\alpha\sigma\sigma_{\mathfrak{p}}\sigma^{-1})e_\psi P_1 = j_{\mathfrak{p}}\circ(1+\alpha\sigma_{\mathfrak{p}})\sigma^{-1}e_\psi P_1 \\ &= \psi(\sigma)^{-1} j_{\mathfrak{p}}\circ(1+\alpha\sigma_{\mathfrak{p}})P_\psi. \end{aligned}$$

The result follows. □

Examples. This paragraph describes a few numerical examples illustrating the scope and applicability of the main results of this paper. By way of illustration, suppose that E is an elliptic curve of prime conductor $N = p$, so that $M = 1$. In that special case the Atkin-Lehner sign w_N is related to the local sign α by

$$w_N = -\alpha.$$

The following proposition reveals that the analytic non-vanishing hypothesis fails in the setting of the Stark-Heegner theorem for quadratic characters of [**BD09**] when $\epsilon = -1$:

Proposition 2.5. — *Let ψ be a totally even quadratic ring class character of K of conductor prime to N. Then P_ψ^α is trivial.*

Proof. — Let $(\chi_1, \chi_2) = (\chi, \chi\chi_K)$ be the pair of even quadratic Dirichlet characters associated to ψ, ordered in such a way that $L(E, \chi_1, s)$ and $L(E, \chi_2, s)$ have signs 1 and -1 respectively in their functional equations. Writing $\mathrm{sign}(E, \chi) \in \{-1, 1\}$ for the sign in the functional equation of the twisted L-function $L(E, \chi, s)$, it is well-known that, if the conductor of χ is relatively prime to N,

$$\mathrm{sign}(E,\chi) = \mathrm{sign}(E)\chi(-N) = -w_N\chi(-1)\chi(p) = \alpha\chi(p)\chi(-1).$$

It follows that

$$\alpha\chi_1(p) = 1, \qquad \alpha\chi_2(p) = -1,$$

but equation (1.4) in the Stark-Heegner theorem for quadratic characters implies $P_\psi^\alpha = 0$. □

The systematic vanishing of P_ψ^α for even quadratic ring class characters of K can be traced to the failure of the analytic non-vanishing hypothesis of the introduction, which arises for simple parity reasons. The failure is expected to occur essentially only when E has prime conductor p, i.e., when $M = 1$, and never when M satisfies $\mathrm{ord}_q(M) = 1$ for some prime q. Because of Proposition 2.5, the main theorem of [**BD09**] gives no information about the Stark-Heegner point P_ψ^α attached to even quadratic ring class characters of conductor prime to p, on an elliptic curve of conductor p.

On the other hand, in the setting of Theorem A of the introduction, where ψ has order > 2, this phenomenon does not occur as the non-vanishing of P_ψ^α and $P_\psi^{-\alpha}$ are *equivalent* to each other, in light of the irreducibility of the induced representation V_ψ. The numerical examples below show many instances of non-vanishing P_ψ^α for ring class characters of both even and odd parity.

Example. Let $E : y^2 + y = x^3 - x$ be the elliptic curve of conductor $p = 37$, whose Mordell-Weil group is generated by the point $(0, 0) \in E(\mathbf{Q})$. Let $K = \mathbf{Q}(\sqrt{5})$ be the real quadratic field of smallest discriminant in which p is inert. It is readily checked that $L(E/K,s)$ has a simple zero at $s=1$ and that $E(K)$ also has Mordell-Weil rank one. The curve E has non-split multiplicative reduction at p and hence $\alpha=-1$ in this case. It is readily verified that the pair of odd characters (χ_1,χ_2) attached to the quadratic imaginary fields of discriminant -4 and -20 satisfy the three conditions in (1.4), and hence the analytic non-vanishing hypothesis is satisfied for the triple $(E,K,\epsilon=1)$. In particular, Theorem A holds for E, K, and all *even* ring class characters of K of conductor prime to 37.

Let $\mathcal{O}$ be an order of $\mathcal{O}_K$ with class number 3, and let H be the corresponding cubic extension of K. The prime $\mathfrak{p}$ of H over p and a generator σ of $\mathrm{Gal}\,(H/K)$ can be chosen so that the components

$$P_1 := P_{\mathfrak{p}}, \qquad P_2 := P_{\sigma\mathfrak{p}}, \qquad P_3 := P_{\sigma^2\mathfrak{p}}$$

in $E(H_{\mathfrak{p}})=E(K_p)$ of the Stark-Heegner point in $E(H\otimes\mathbf{Q}_p)$ satisfy

$$\overline{P}_1=P_1, \qquad \overline{P}_2=P_3, \qquad \overline{P}_3=P_2.$$

Letting ψ be the cubic character which sends σ to $\zeta:=(1+\sqrt{-3})/2$, we find that

$$\begin{aligned} j_{\mathfrak{p}}(P_\psi) &= P_1+\zeta P_2+\zeta^2 P_3,\\ \sigma_{\mathfrak{p}}(j_{\mathfrak{p}}(P_\psi)) &= \overline{P}_1+\zeta\overline{P}_2+\zeta^2\overline{P}_3 = P_1+\zeta P_3+\zeta^2 P_2,\\ j_{\mathfrak{p}}(P_\psi^\alpha) &= \sqrt{-3}\times(P_2-P_3)=\sqrt{-3}\times(P_2-\overline{P}_2). \end{aligned}$$

The following table lists the Stark-Heegner points P_1, P_2, and $P_2-\overline{P}_2$ attached to the first few orders $\mathcal{O}\subset\mathcal{O}_K$ of conductor $c=c(\mathcal{O})$ and of class number three, calculated to a 37-adic accuracy of 2 significant digits. (The numerical entries in the table below are thus to be understood as elements of $(\mathbf{Z}/37^2\mathbf{Z})[\sqrt{5}]$.)

$c(\mathcal{O})$	P_1	P_2	$P_2-\overline{P}_2$
18	$(-635,-256)$	$(319+678\sqrt{5},-481230\sqrt{5})$	$(-360,684+27\sqrt{5})$
38	$(-154,447)$	$(-588+1237\sqrt{5},367+386\sqrt{5})$	$(-437,684+87\sqrt{5})$
46	$(223,12\cdot 37)$	$(-112+629\sqrt{5},(-6+34\sqrt{5})\cdot 37)$	∞
47	$(610,-229)$	$(539+71\sqrt{5},10+439\sqrt{5})$	$(-293,684+1132\sqrt{5})$
54	$(533,-561)$	$(679+984\sqrt{5},391+862\sqrt{5})$	$(93,684+673\sqrt{5})$

Since the Mordell-Weil group of $E(K)$ has rank one, the data in this table is enough to conclude that the pro-37-Selmer groups of E over the ring class fields of K attached to the orders of conductors 18, 38, 47 and 54 have rank at least 3. As for the order of conductor 46, a calculation modulo 37^3 reveals that $P_2-\overline{P}_2$ is non-trivial, and hence the pro-37 Selmer group has rank ≥ 3 over the ring class field of that conductor as well. Under the Stark-Heegner conjecture, more is true: the Stark-Heegner points above are 37-adic approximations of global points rather than mere Selmer classes. But recognising them as such (and thereby proving that the Mordell-Weil ranks are ≥ 3) typically requires a calculations to higher accuracy, depending on the eventual height of the Stark-Heegner point as an algebraic point, about which nothing is known of course a priori, and which can behave somewhat erratically. For example, the x-coordinates of the Stark-Heegner points attached to the order of conductor 47 appear to satisfy the cubic polynomial

$$x^3 - 319x^2 + 190x + 420,$$

while those of the Stark-Heegner points for the order of conductor 46 appear to satisfy the cubic polynomial

$$2352347001x^3 - 34772698791x^2 + 138835821427x - 136501565573$$

with much larger coefficients, whose recognition requires a calculation to at least 7 digits of 37-adic accuracy.

The table above produced many examples of non-vanishing P_ψ^α for ψ even, and in particular it verifies the non-vanishing hypothesis for Stark-Heegner points stated in the introduction, for the sign $\epsilon = -1$. This means that Theorem A is also true for *odd* ring class characters of K, even if the premise of (1.6) is *never verified* for odd *quadratic* characters of K.

3. p-adic L-functions associated to Hida families

Let

$$\mathbf{f} = \sum_{n\geq 1} a_n(\mathbf{f}) q^n \in \Lambda_{\mathbf{f}}[[q]]$$

be the Hida family of tame level M and trivial tame character passing through f; cf. [**BD09**] and [**DRb**, §1.3] for more details on the notations chosen for Hida families.

Let $x_0 \in \mathcal{W}_{\mathbf{f}}^\circ$ denote the point of weight 2 such that $\mathbf{f}_{x_0} = f$. Note that $\mathbf{f}_{x_0} \in S_2(N)$ is new at p, while for any $x \in \mathcal{W}_{\mathbf{f}}^\circ$ with $\mathrm{wt}(x) = k > 2$, $\mathbf{f}_x(q) = \mathbf{f}_x^\circ(q) - \beta \mathbf{f}_x^\circ(q^p)$ is the ordinary p-stabilisation of an eigenform $\mathbf{f}_x^\circ$ of level $M = N/p$. We set $\mathbf{f}_{x_0}^\circ = \mathbf{f}_{x_0} = f$.

Let K be a real quadratic field in which p remains inert and all prime factors of M split, and fix throughout a finite order anticyclotomic character ψ of K of conductor c coprime to DN, with values in a finite extension $L_p/\mathbf{Q}_p$. Note that $\psi(p) = 1$ as the prime ideal $p\mathcal{O}_K$ is principal.

Under our running assumptions, the sign of the functional equation satisfied by the Hasse-Weil-Artin L-series $L(E/K, \psi, s) = L(f, \psi, s)$ is

$$\varepsilon(E/K, \psi) = -1,$$

and in particular the order of vanishing of $L(E/K, \psi, s)$ at $s = 1$ is odd. In contrast, at every classical point x of even weight $k > 2$ the sign of the functional equation satisfied by $L(\mathbf{f}_x/K, \psi, s)$ is

$$\varepsilon(\mathbf{f}_x/K, \psi) = +1$$

and one expects generic non-vanishing of the central critical value $L(\mathbf{f}_x/K, \psi, k/2)$.

In [**BD09**, Definition 3.4], a p-adic L-function

$$\mathscr{L}_p(\mathbf{f}/K, \psi) \in \Lambda_{\mathbf{f}}$$

associated to the Hida family $\mathbf{f}$, the ring class character ψ and a choice of collection of periods was defined, by interpolating the algebraic part of (the square-root of) the critical values $L(\mathbf{f}_x/K, \psi, k/2)$ for $x \in \mathcal{W}_{\mathbf{f}}^\circ$ with $\mathrm{wt}(x) = k = k_\circ + 2 \geq 2$. See also

[**LMY17**, §4.1] for a more general treatment, encompassing the setting considered here.

In order to describe this p-adic L-function in more detail, let $\Phi_{\mathbf{f}_x,\mathbf{C}}$ denote the classical modular symbol associated to $\mathbf{f}_x$ with values in the space $P_{k_\circ}(\mathbf{C})$ of homogeneous polynomials of degree $k_\circ$ in two variables with coefficients in $\mathbf{C}$. The space of modular symbols is naturally endowed with an action of $\mathrm{GL}_2(\mathbf{Q})$ and we let $\Phi^+_{\mathbf{f}_x,\mathbf{C}}$ and $\Phi^-_{\mathbf{f}_x,\mathbf{C}}$ denote the plus and minus eigencomponents of $\Phi_{\mathbf{f}_x,\mathbf{C}}$ under the involution at infinity induced by $w_\infty = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$.

As proved in [**KZ84**, §1.1] (with slightly different normalizations as for the powers of the period $2\pi i$ that appear in the formulas, which we have taken into account accordingly), there exists a pair of collections of complex periods

$$\{\Omega^+_{\mathbf{f}_x,\mathbf{C}}\}_{x\in\mathcal{W}^\circ_{\mathbf{f}}}, \quad \{\Omega^-_{\mathbf{f}_x,\mathbf{C}}\}_{x\in\mathcal{W}^\circ_{\mathbf{f}}} \subset \mathbf{C}^\times$$

satisfying the following two conditions:

(i) the modular symbols

$$\Phi^+_{\mathbf{f}_x} := \frac{\Phi^+_{\mathbf{f}_x,\mathbf{C}}}{\Omega^+_{\mathbf{f}_x,\mathbf{C}}}, \quad \Phi^-_{\mathbf{f}_x} := \frac{\Phi^-_{\mathbf{f}_x,\mathbf{C}}}{\Omega^-_{\mathbf{f}_x,\mathbf{C}}} \quad \text{take values in } \mathbf{Q}(\mathbf{f}_x) = \mathbf{Q}(\{a_n(\mathbf{f}_x)\}_{n\geq 1}),$$

(ii) and $\quad \Omega^+_{\mathbf{f}_x,\mathbf{C}} \cdot \Omega^-_{\mathbf{f}_x,\mathbf{C}} = 4\pi^2\langle \mathbf{f}^\circ_x, \mathbf{f}^\circ_x\rangle.$

Note that conditions (i) and (ii) above only characterize $\Omega^\pm_{\mathbf{f}_x,\mathbf{C}}$ up to multiplication by non-zero scalars in the number field $\mathbf{Q}(\mathbf{f}_x)$.

Fix an embedding $\bar{\mathbf{Q}} \hookrightarrow \bar{\mathbf{Q}}_p \subset \mathbf{C}_p$, through which we regard $\Phi^\pm_{\mathbf{f}_x}$ as $\mathbf{C}_p$-valued modular symbols. In [**GS93**], Greenberg and Stevens introduced measure-valued modular symbols $\mu^+_{\mathbf{f}}$ and $\mu^-_{\mathbf{f}}$ interpolating the classical modular symbols $\Phi^+_{\mathbf{f}_x}$ and $\Phi^-_{\mathbf{f}_x}$ as x ranges over the classical specializations of $\mathbf{f}$.

More precisely, they show (cf. [**GS93**, Theorem 5.13] and [**BD07**, Theorem 1.5]) that for every $x \in \mathcal{W}^\circ_{\mathbf{f}}$, there exist p-adic periods

$$\Omega^+_{\mathbf{f}_x,p},\ \Omega^-_{\mathbf{f}_x,p} \in \mathbf{C}_p \tag{3.1}$$

such that the specialisation of $\mu^+_{\mathbf{f}}$ and $\mu^-_{\mathbf{f}}$ at x satisfy

$$x(\mu^+_{\mathbf{f}}) = \Omega^+_{\mathbf{f}_x,p} \cdot \Phi^+_{\mathbf{f}_x}, \qquad x(\mu^-_{\mathbf{f}}) = \Omega^-_{\mathbf{f}_x,p} \cdot \Phi^-_{\mathbf{f}_x}. \tag{3.2}$$

Since no natural choice of periods $\Omega^\pm_{\mathbf{f}_x,\mathbf{C}}$ presents itself, the scalars $\Omega^+_{\mathbf{f}_x,p}$ and $\Omega^-_{\mathbf{f}_x,p}$ are not expected to vary p-adically continuously. However, conditions (i) and (ii) above imply that the *product* $\Omega^+_{\mathbf{f}_x,p} \cdot \Omega^-_{\mathbf{f}_x,p} \in \mathbf{C}_p$ is a more canonical quantity, as it may also be characterized by the formula

$$x(\mu^+_{\mathbf{f}}) \cdot x(\mu^-_{\mathbf{f}}) = \Omega^+_{\mathbf{f}_x,p}\Omega^-_{\mathbf{f}_x,p} \cdot \frac{\Phi^+_{\mathbf{f}_x,\mathbf{C}} \cdot \Phi^-_{\mathbf{f}_x,\mathbf{C}}}{4\pi^2\langle \mathbf{f}^\circ_x, \mathbf{f}^\circ_x\rangle}, \tag{3.3}$$

which is independent of any choices of periods.

This suggests that the map $x \mapsto \Omega^+_{\mathbf{f}_x,p}\Omega^-_{\mathbf{f}_x,p}$ may extend to a p-adic analytic function, possibly after multiplying it by suitable Euler-like factors at p. And indeed, the following theorem is proved in one of the contributing articles of Bertolini, Seveso and Venerucci to this volume, and we refer to [**BSVb**, §3] for the proof.

Theorem 3.1. — *There exists a rigid-analytic function* $\mathscr{L}_p(\mathrm{Sym}^2(\mathbf{f}))$ *on a neighborhood* $U_{\mathbf{f}}$ *of* $\mathcal{W}_{\mathbf{f}}$ *around* x_0 *such that for all classical points* $x \in U_{\mathbf{f}} \cap \mathcal{W}_{\mathbf{f}}^{\circ}$ *of weight* $k \geq 2$*:*

$$\mathscr{L}_p(\mathrm{Sym}^2(\mathbf{f}))(x) = \mathcal{E}_0(\mathbf{f}_x)\mathcal{E}_1(\mathbf{f}_x) \cdot \Omega^+_{\mathbf{f}_x,p}\Omega^-_{\mathbf{f}_x,p}, \tag{3.4}$$

where $\mathcal{E}_0(\mathbf{f}_x)$ *and* $\mathcal{E}_1(\mathbf{f}_x)$ *are as in* [**DR14**, Theorem 1.3]*. Moreover,* $\mathscr{L}_p(\mathrm{Sym}^2(\mathbf{f}))(x_0) \in \mathbf{Q}^\times$*.*

Remark 3.1. — *The motivation for denoting* $\mathscr{L}_p(\mathrm{Sym}^2(\mathbf{f}))$ *the* p*-adic function appearing above relies on the fact that* $\Omega^{\pm}_{\mathbf{f}_x,p}$ *are* p*-adic analogues of the complex periods* $\Omega^{\pm}_{\mathbf{f}_x,\mathbf{C}}$*. As is well-known, the product* $\Omega^+_{\mathbf{f}_x,\mathbf{C}} \cdot \Omega^-_{\mathbf{f}_x,\mathbf{C}} = 4\pi^2\langle \mathbf{f}_x^\circ, \mathbf{f}_x^\circ\rangle$ *is essentially the near-central critical value of the classical* L*-function associated to the symmetric square of* $\mathbf{f}_x^\circ$*. In addition to this, as M. L. Hsieh remarked to us, it might not be difficult to show that* $\mathscr{L}_p(\mathrm{Sym}^2(\mathbf{f}))$ *is a generator of Hida's congruence ideal in the sense of* [**Hs20**, §1.4, p.4]*.*

The result characterizing the p-adic L-function $\mathscr{L}_p(\mathbf{f}/K, \psi)$ alluded to above is [**BD09**, Theorem 3.5], which we recall below. Although [**BD09**, Theorem 3.5] is stated in loc. cit. only for genus characters, the proof has been recently generalized to arbitrary (not necessarily quadratic) ring class characters ψ of conductor c with $(c, DN) = 1$ by Longo, Martin and Yan in [**LMY17**, Theorem 4.2], by employing Gross-Prasad test vectors to extend Popa's formula [**Po06**, Theorem 6.3.1] to this setting.

Let $\mathfrak{f}_c \in K^\times$ denote the explicit constant introduced at the first display of [**LMY17**, §3.2]. It only depends on the conductor c and its square lies in $\mathbf{Q}^\times$.

Theorem 3.2. — *The* p*-adic* L*-function* $L_p(\mathbf{f}/K, \psi)$ *satisfies the following interpolation property: for all* $x \in \mathcal{W}_{\mathbf{f}}^\circ$ *of weight* $\mathrm{wt}(x) = k = k_\circ + 2 \geq 2$*, we have*

$$\mathscr{L}_p(\mathbf{f}/K, \psi)(x) = \mathfrak{f}_{\mathbf{f},\psi}(x) \times L(\mathbf{f}_x^\circ/K, \psi, k/2)^{1/2}$$

where

$$\mathfrak{f}_{\mathbf{f},\psi}(x) = (1 - \alpha_{\mathbf{f}_x}^{-2} p^{k_\circ}) \cdot \frac{\mathfrak{f}_c \cdot (Dc^2)^{\frac{k_\circ+1}{4}} (\frac{k_\circ}{2})!}{(2\pi i)^{k_\circ/2}} \cdot \frac{\Omega^{\epsilon_\psi}_{\mathbf{f}_x,p}}{\Omega^{\epsilon_\psi}_{\mathbf{f}_x,\mathbf{C}}}.$$

4. A p-adic Gross-Zagier formula for Stark-Heegner points

One of the main theorems of [**BD09**] is a formula for the derivative of $\mathscr{L}_p(\mathbf{f}/K, \psi)$ at the point x_0, relating it to the formal group logarithm of a Stark-Heegner point. This formula shall be crucial for relating these points to generalized Kato classes and eventually proving our main results.

Theorem 4.1. — *The p-adic L-function $\mathscr{L}_p(\mathbf{f}/K,\psi)$ vanishes at the point x_0 of weight 2 and*

$$\frac{d}{dx}\mathscr{L}_p(\mathbf{f}/K,\psi)_{|x=x_0} = \frac{1}{2}\log_p(P_\psi^\alpha). \tag{4.1}$$

Proof. — The vanishing of $\mathscr{L}_p(\mathbf{f}/K,\psi)$ at $x = x_0$ is a direct consequence of the assumptions and definitions, because $x = x_0$ lies in the region of interpolation of the p-adic L-function and therefore $\mathscr{L}_p(\mathbf{f}/K,\psi)(x_0)$ is a non-zero multiple of the central critical value $L(f/K,\psi,1)$. This L-value vanishes as remarked in the paragraph right after (1.1).

The formula for the derivative follows verbatim as in the proof of [**BD09**, Theorem 4.1]. See also [**LMY17**, Theorem 5.1] for the statement in the generality required here. Finally, we refer to [**LV14**] for a formulation and proof of this formula in the setting of quaternionic Stark-Heegner points, under the general assumption of (1.2). □

5. Setting the stage

In this section we set the stage for the proofs of Theorems A and B by introducing a particular choice of triplet of eigenforms (f,g,h) of weights $(2,1,1)$. Let $E/\mathbf{Q}$ be an elliptic curve having multiplicative reduction at a prime p and set $\alpha = a_p(E) = \pm 1$. Let

$$\psi : \mathrm{Gal}\,(H/K) \longrightarrow L^\times$$

be an anticyclotomic character of a real quadratic field K satisfying the hypotheses stated in the introduction.

In particular we assume that a prime ideal $\mathfrak{p}$ above p in H has been fixed and either of the *non-vanishing hypotheses* stated in loc. cit. holds; these hypotheses give rise to a character ξ of K having parity opposite to that of ψ that we fix for the remainder of this note, satisfying that the local Stark-Heegner point $P^\alpha_{\xi,\mathfrak{p}}$ is non-zero.

As shown in [**DR17**, Lemma 6.9], there exists a (not necessarily anti-cyclotomic) character ψ_0 of finite order of K and conductor prime to DN_E such that

$$\psi_0/\psi_0' = \xi/\psi. \tag{5.1}$$

Since by hypothesis ξ/ψ is totally odd, it follows that ψ_0 has mixed signature $(+,-)$ with respect to the two real embeddings of K.

Let $\mathfrak{c} \subset \mathcal{O}_K$ denote the conductor of ψ_0 and let χ denote the odd central Dirichlet character of ψ_0. Let χ_K also denote the quadratic Dirichlet character associated to $K/\mathbf{Q}$.

Let $f \in S_2(pM_f)$ denote the modular form associated to E by modularity. Likewise, set

$$M_g = Dc^2 \cdot \mathrm{N}_{K/\mathbf{Q}}(\mathfrak{c}) \quad \text{and} \quad M_h = D \cdot \mathrm{N}_{K/\mathbf{Q}}(\mathfrak{c})$$

and define the eigenforms

$$g = \theta(\psi_0\psi) \in S_1(M_g, \chi\chi_K) \quad \text{and} \quad h = \theta(\psi_0^{-1}) \in S_1(M_h, \chi^{-1}\chi_K)$$

to be the theta series associated to the characters $\psi_0\psi$ and ψ_0^{-1}, respectively.

Recall from the introduction that $E[p]$ is assumed to be irreducible as a $G_{\mathbf{Q}}$-module. This implies that the mod p residual Galois representation attached to f is irreducible, and thus also non-Eisenstein mod p. The same claim holds for g and h because ψ and ξ have opposite signs and p is odd, hence $\xi \not\equiv \psi^{\pm 1} \pmod{p}$.

Note that $p \nmid M_f M_g M_h$. As in previous sections, we let M denote the least common multiple of M_f, M_g and M_h. The Artin representations V_g and V_h associated to g and h are both odd and unramified at the prime p. Since p remains inert in K, the arithmetic frobenius Fr_p acts on V_g and V_h with eigenvalues

$$\{\alpha_g, \beta_g\} = \{\zeta, -\zeta\}, \qquad \{\alpha_h, \beta_h\} = \{\zeta^{-1}, -\zeta^{-1}\},$$

where ζ is a root of unity satisfying $\chi(p) = -\zeta^2$.

In light of (5.1) we have $\psi_0\psi/\psi_0 = \psi$ and $\psi_0\psi/\psi_0' = \xi$, hence the tensor product of V_g and V_h decomposes as

$$V_{gh} = V_g \otimes V_h \simeq \mathrm{Ind}_K^{\mathbf{Q}}(\psi) \oplus \mathrm{Ind}_K^{\mathbf{Q}}(\xi) \quad \text{as } G_{\mathbf{Q}}\text{-modules} \tag{5.2}$$

and

$$V_g = V_g^{\alpha_g} \oplus V_g^{\beta_g}, \quad V_h = V_h^{\alpha_h} \oplus V_h^{\beta_h}, \quad V_{gh} = \bigoplus_{(a,b)} V_{gh}^{ab} \quad \text{as } G_{\mathbf{Q}_p}\text{-modules}$$

where (a,b) ranges through the four pairs $(\alpha_g, \alpha_h), (\alpha_g, \beta_h), (\beta_g, \alpha_h), (\beta_g, \beta_h)$. Here $V_g^{\alpha_g}$, say, is the $G_{\mathbf{Q}_p}$-submodule of V_g on which Fr_p acts with eigenvalue α_g, and similarly for the remaining terms.

Let W_p be an arbitrary self-dual Artin representation with coefficients in L_p and factoring through the Galois group of a finite extension H of $\mathbf{Q}$. Assume W_p is unramified at p. There is a canonical isomorphism

$$\begin{aligned} H^1(\mathbf{Q}, V_p(E) \otimes W_p) &\simeq (H^1(H, V_p(E)) \otimes W_p)^{\mathrm{Gal}\,(H/\mathbf{Q})} \\ &= \mathrm{Hom}_{\mathrm{Gal}\,(H/\mathbf{Q})}(W_p, H^1(H, V_p(E))), \end{aligned} \tag{5.3}$$

where the the second equality follows from the self-duality of W_p. Kummer theory gives rise to a homomorphism

$$\delta : E(H)^{W_p} := \mathrm{Hom}_{\mathrm{Gal}\,(H/\mathbf{Q})}(W_p, E(H) \otimes L_p) \longrightarrow H^1(\mathbf{Q}, V_p(E) \otimes W_p). \tag{5.4}$$

For each rational prime ℓ, the maps (5.3) and (5.4) admit local counterparts

$$\begin{aligned} H^1(\mathbf{Q}_\ell, V_p(E) \otimes W_p) &\simeq \mathrm{Hom}_{\mathrm{Gal}\,(H/\mathbf{Q})}(W_p, \oplus_{\lambda|\ell} H^1(H_\lambda, V_p(E))), \\ \delta_\ell : \left(\oplus_{\lambda|\ell} E(H_\lambda)\right)^{W_p} &\longrightarrow H^1(\mathbf{Q}_\ell, V_p(E) \otimes W_p), \end{aligned}$$

for which the following diagram commutes:

$$\begin{array}{ccc} E(H)^{W_p} & \xrightarrow{\ \delta\ } & H^1(\mathbf{Q}, V_p(E) \otimes W_p) \\ \downarrow{\scriptstyle \mathrm{res}_\ell} & & \downarrow{\scriptstyle \mathrm{res}_\ell} \\ \left(\oplus_{\lambda|\ell} E(H_\lambda)\right)^{W_p} & \xrightarrow{\ \delta_\ell\ } & H^1(\mathbf{Q}_\ell, V_p(E) \otimes W_p). \end{array} \tag{5.5}$$

For primes $\ell \neq p$, it follows from [Ne98, (2.5) and (3.2)] that $H^1(\mathbf{Q}_\ell, V_p(E) \otimes W_p) = 0$. (We warn however that if we were working with integral coefficients, these cohomology groups may contain non-trivial torsion.) For $\ell = p$, the Bloch-Kato submodule $H^1_{\mathrm{f}}(\mathbf{Q}_p, V_p(E)\otimes W_p)$ is the subgroup of $H^1(\mathbf{Q}_p, V_p(E)\otimes W_p)$ formed by classes of *crystalline* extensions of Galois representations of $V_p(E)\otimes W_p$ by $\mathbf{Q}_p$. It may also be identified with the image of the local connecting homomorphism δ_p.

Lemma 5.1. — *There is a natural isomorphism of L_p-vector spaces*

$$H^1_{\mathrm{f}}(\mathbf{Q}_p, V_p(E)\otimes W_p) = H^1_{\mathrm{f}}(\mathbf{Q}_p, V_f^+\otimes W_p^{\mathrm{Fr}_p=\alpha}) \oplus H^1(\mathbf{Q}_p, V_f^+\otimes W_p/W_p^{\mathrm{Fr}_p=\alpha}),$$

where recall $\alpha = a_p(E) = \pm 1$.

Proof. — We firstly observe that $H^1_{\mathrm{f}}(\mathbf{Q}_p, V_p(E)\otimes W_p) = H^1_{\mathrm{g}}(\mathbf{Q}_p, V_p(E)\otimes W_p)$ by e.g. [**Be09**, Prop. 2.0 and Ex. 2.20], because $V_p(E)\otimes W_p$ contains no unramified *submodule*. As shown in [**Fl90**, Lemma , p.125], it follows that

$$H^1_{\mathrm{f}}(\mathbf{Q}_p, V_p(E)\otimes W_p) = \mathrm{Ker}\big(H^1(\mathbf{Q}_p, V_p(E)\otimes W_p) \longrightarrow H^1(I_p, V_p^-(E)\otimes W_p)\big)$$

is the kernel of the composition of the homomorphism in cohomology induced by the natural projection $V_p(E) \longrightarrow V_p^-(E)$ and restriction to the inertia subgroup $I_p \subset G_{\mathbf{Q}_p}$.

The long exact sequence in Galois cohomology arising from the exact sequence

$$0 \to V_p^+(E) \longrightarrow V_p(E) \longrightarrow V_p^-(E) \to 0$$

shows that the kernel of the map $H^1(\mathbf{Q}_p, V_p(E)\otimes W_p) \longrightarrow H^1(\mathbf{Q}_p, V_p^-(E)\otimes W_p)$ is naturally identified with $H^1(\mathbf{Q}_p, V_p^+(E)\otimes W_p)$. We have $H^1(I_p, \mathbf{Q}_p(\psi\varepsilon_{\mathrm{cyc}})) = 0$ for any nontrivial unramified character ψ. Besides, it follows from e.g. [**DRb**, Example 1.4] that $H^1_{\mathrm{f}}(\mathbf{Q}_p, \mathbf{Q}_p(\varepsilon_{\mathrm{cyc}})) = \ker\big(H^1(\mathbf{Q}_p, \mathbf{Q}_p(\varepsilon_{\mathrm{cyc}})) \to H^1(I_p, \mathbf{Q}_p(\varepsilon_{\mathrm{cyc}}))\big)$ is a line in the two-dimensional space $H^1(\mathbf{Q}_p, \mathbf{Q}_p(\varepsilon_{\mathrm{cyc}}))$, which Kummer theory identifies with $\mathbf{Z}_p^\times \hat{\otimes}_{\mathbf{Z}_p}\mathbf{Q}_p$ sitting inside $\mathbf{Q}_p^\times \hat{\otimes}_{\mathbf{Z}_p}\mathbf{Q}_p$.

Note that $V_p^+(E) = L_p(\psi_f\varepsilon_{\mathrm{cyc}})$ and $V_p^-(E) \simeq L_p(\psi_f)$ where ψ_f is the unramified quadratic character of $G_{\mathbf{Q}_p}$ sending Fr_p to α. The lemma follows. □

The Selmer group $Sel(\mathbb{T})_p(E, W_p)$ is defined as

$$Sel(\mathbb{T})_p(E, W_p) := \{\lambda \in H^1(\mathbf{Q}, V_p(E)\otimes W_p) : \mathrm{res}_p(\lambda) \in H^1_{\mathrm{f}}(\mathbf{Q}_p, V_p(E)\otimes W_p)\}.$$

Here res_p stands for the natural map in cohomology induced by restriction from $G_{\mathbf{Q}}$ to $G_{\mathbf{Q}_p}$.

6. Factorisation of p-adic L-series

The goal of this section is proving a factorisation formula of p-adic L-functions which shall be crucial in the proof of our main theorems.

Keep the notations introduced in the previous section and recall in particular the sign $\alpha := a_p(f) \in \{\pm 1\}$ associated to E. Let g_ζ and $h_{\alpha\zeta^{-1}}$ denote the ordinary p-stabilizations of g and h on which the Hecke operator U_p acts with eigenvalue

$$\alpha_g := \zeta \quad \text{and} \quad \alpha_h := \alpha\zeta^{-1}, \tag{6.1}$$

respectively.

Let $\mathbf{f}$, $\mathbf{g}$ and $\mathbf{h}$ be the Hida families of tame levels M_f, M_g, M_h and tame characters 1, $\chi\chi_K$, $\chi^{-1}\chi_K$ passing respectively through f, g_ζ and $h_{\alpha\zeta^{-1}}$. The existence of these families is a theorem of Wiles [**Wi88**], and their uniqueness follows from a recent result of Bellaïche and Dimitrov [**BeDi16**] (note that the main theorem of loc. cit. indeed applies because $\alpha_g \neq \beta_g$, $\alpha_h \neq \beta_h$ and p does not split in K). Let x_0, y_0, z_0 denote the classical points in $\mathcal{W}_{\mathbf{f}}$, $\mathcal{W}_{\mathbf{g}}$ and $\mathcal{W}_{\mathbf{h}}$ respectively such that $\mathbf{f}_{x_0} = f$, $\mathbf{g}_{y_0} = g_\zeta$ and $\mathbf{h}_{z_0} = h_{\alpha\zeta^{-1}}$.

As explained in [**DR14**], [**DR17**] and recalled briefly in [**DRb**, (5.1)] in this volume, associated to a choice

$$\breve{\mathbf{f}} \in S^{\mathrm{ord}}_{\Lambda_{\mathbf{f}}}(M)[\mathbf{f}], \quad \breve{\mathbf{g}} \in S^{\mathrm{ord}}_{\Lambda_{\mathbf{g}}}(M,\chi\chi_K)[\mathbf{g}], \quad \breve{\mathbf{h}} \in S^{\mathrm{ord}}_{\Lambda_{\mathbf{h}}}(M,\chi^{-1}\chi_K)[\mathbf{h}]$$

of Λ-adic test vectors of tame level M there is a three-variable p-adic L-function $\mathscr{L}_p{}^f(\breve{\mathbf{f}},\breve{\mathbf{g}},\breve{\mathbf{h}})$. Among such choices, Hsieh [**Hs20**] pinned down a particular choice of test vectors with optimal interpolation properties (cf. loc. cit. and [**DRb**, Prop. 5.1] for more details), which we fix throughout this section.

Define

$$\mathscr{L}_p{}^f(\breve{\mathbf{f}},\breve{g}_\zeta,\breve{h}_{\alpha\zeta^{-1}}) \in \Lambda_{\mathbf{f}} \tag{6.2}$$

to be the one-variable p-adic L-function arising as the restriction of $\mathscr{L}_p{}^f(\breve{\mathbf{f}},\breve{\mathbf{g}},\breve{\mathbf{h}})$ to the rigid analytic curve $\mathcal{W}_{\mathbf{f}} \times \{y_0, z_0\}$.

In addition, recall the p-adic L-functions described in §3 associated to the twist of E/K by an anticyclotomic character of K, and set $\mathfrak{f}_{\mathcal{O}}(k_\circ) := (Dc^2)^{\frac{1-k}{2}}/\mathfrak{f}_c^2$, where $\mathfrak{f}_c$ is the constant introduced at the first display of [**LMY17**, §3.2]. Note that the rule $k \mapsto \mathfrak{f}_{\mathcal{O}}(k_\circ)$ extends to an Iwasawa function, that we continue to denote $\mathfrak{f}_{\mathcal{O}}$, because p does not divide Dc^2. Recall also the rigid-analytic function $\mathscr{L}_p(\mathrm{Sym}^2(\mathbf{f}))$ in a neighborhood $U_{\mathbf{f}} \subset \mathcal{W}_{\mathbf{f}}$ of x_0 introduced in (3.4).

Theorem 6.1. — *The following factorization of p-adic L-functions holds in $\Lambda_{\mathbf{f}}$:*

$$\mathscr{L}_p(\mathrm{Sym}^2(\mathbf{f})) \times \mathscr{L}_p{}^f(\breve{\mathbf{f}},\breve{g}_\zeta,\breve{h}_{\alpha\zeta^{-1}}) = \mathfrak{f}_{\mathcal{O}} \cdot \mathscr{L}_p(\mathbf{f}/K,\psi) \times \mathscr{L}_p(\mathbf{f}/K,\xi).$$

Proof. — It follows from [**DRb**, Prop. 5.1] that $\mathscr{L}_p{}^f(\breve{\mathbf{f}},\breve{g}_\zeta,\breve{h}_{\alpha\zeta^{-1}})$ satisfies the following interpolation property for all $x \in \mathcal{W}^\circ_{\mathbf{f}}$ of weight $k \geq 2$:

$$\mathscr{L}_p{}^f(\breve{\mathbf{f}},\breve{g}_\zeta,\breve{h}_{\alpha\zeta^{-1}})(x) = (2\pi i)^{-k} \cdot (\frac{k_\circ}{2}!)^2 \cdot \frac{1-\alpha_{\mathbf{f}_x}^{-2}p^{k_\circ}}{1-\beta_{\mathbf{f}_x}^2 p^{1-k}} \cdot \frac{L(\mathbf{f}^\circ_x, g, h, \frac{k}{2})^{1/2}}{\langle \mathbf{f}^\circ_x, \mathbf{f}^\circ_x\rangle}.$$

Besides, it follows from Theorem 3.2 that the product of $\mathscr{L}_p(\mathbf{f}/K,\psi)$ and $\mathscr{L}_p(\mathbf{f}/K,\xi)$ satisfies that for all $x \in \mathcal{W}^\circ_{\mathbf{f}}$ of weight $k \geq 2$:

$$\mathscr{L}_p(\mathbf{f}/K,\psi)\mathscr{L}_p(\mathbf{f}/K,\xi)(x) = \mathfrak{f}_{\mathbf{f},\psi}(x)\cdot\mathfrak{f}_{\mathbf{f},\xi}(x) \times L(\mathbf{f}^\circ_x/K,\psi,k/2)^{1/2}\cdot L(\mathbf{f}^\circ_x/K,\xi,k/2)^{1/2}$$

where

$$\mathfrak{f}_{\mathbf{f},\psi}(x)\cdot\mathfrak{f}_\xi(x) = (1-\alpha_{\mathbf{f}_x}^{-2}p^{k_\circ})^2 \cdot \frac{\mathfrak{f}_c^2\cdot(Dc^2)^{\frac{k_\circ+1}{2}}\cdot(\frac{k_\circ}{2})!^2}{(2\pi i)^{k_\circ}} \cdot \frac{\Omega^+_{\mathbf{f}_x,p}\Omega^-_{\mathbf{f}_x,p}}{\Omega^+_{\mathbf{f}_x,\mathbf{C}}\Omega^-_{\mathbf{f}_x,\mathbf{C}}}.$$

A direct inspection to the Euler factors shows that for all $x \in \mathcal{W}^\circ_{\mathbf{f}}$ of weight $k \geq 2$:

$$L(\mathbf{f}^\circ_x, g, h, k/2) = L(\mathbf{f}^\circ_x/K,\psi,k/2)\cdot L(\mathbf{f}^\circ_x/K,\xi,k/2). \tag{6.3}$$

Recall finally from Theorem 3.1 that the value of $\mathscr{L}_p(\mathrm{Sym}^2(\mathbf{f}))$ at a point $x \in U_{\mathbf{f}} \cap \mathcal{W}_{\mathbf{f}}^\circ$ is

$$\mathscr{L}_p(\mathrm{Sym}^2(\mathbf{f}))(x) = (1-\beta_{\mathbf{f}_x}^2 p^{1-k})(1-\alpha_{\mathbf{f}_x}^{-2} p^{k_\circ})\Omega^+_{\mathbf{f}_x,p}\Omega^-_{\mathbf{f}_x,p}.$$

Combining the above formulae together with the equality

$$\Omega^+_{\mathbf{f}_x,\mathbf{C}} \cdot \Omega^-_{\mathbf{f}_x,\mathbf{C}} = 4\pi^2\langle \mathbf{f}_x^\circ, \mathbf{f}_x^\circ\rangle,$$

described in §3, it follows that the following formula holds for all $x \in \mathcal{W}_{\mathbf{f}}^\circ$ of weight $k \geq 2$:

$$\mathscr{L}_p(\mathrm{Sym}^2(\mathbf{f}))(x) \times \mathscr{L}_p{}^f(\breve{\mathbf{f}}, \breve{g}_\zeta, \breve{h}_{\alpha\zeta^{-1}})(x) = \lambda_{\mathcal{O}}(k_\circ) \cdot \mathscr{L}_p(\mathbf{f}/K,\psi)(x) \times \mathscr{L}_p(\mathbf{f}/K,\xi)(x).$$

Since $\mathcal{W}_{\mathbf{f}}^\circ$ is dense in $\mathcal{W}_{\mathbf{f}}$ for the rigid-analytic topology, the factorization formula claimed in the theorem follows. □

Recall from Theorem 3.2 that $\mathscr{L}_p(\mathbf{f}/K,\psi)$ and $\mathscr{L}_p(\mathbf{f}/K,\xi)$ both vanish at x_0 and

$$\frac{d}{dx}\mathscr{L}_p(\mathbf{f}/K,\psi)_{|x=x_0} = \frac{1}{2}\cdot \log_p(P_\psi^\alpha), \qquad \frac{d}{dx}\mathscr{L}_p(\mathbf{f}/K,\xi)_{|x=x_0} = \frac{1}{2}\cdot \log_p(P_\xi^\alpha). \tag{6.4}$$

By Theorem 3.1, $\mathscr{L}_p(\mathrm{Sym}^2(\mathbf{f}))(x_0) \in \mathbf{Q}^\times$. It thus follows from Theorem 6.1 that the order of vanishing of $\mathscr{L}_p{}^f(\breve{\mathbf{f}}^\vee, \breve{g}_\zeta, \breve{h}_{\alpha\zeta^{-1}})$ at $x = x_0$ is at least two and

$$\frac{d^2}{dx^2}\mathscr{L}_p{}^f(\breve{\mathbf{f}}^\vee, \breve{g}_\zeta, \breve{h}_{\alpha\zeta^{-1}})_{|x=x_0} = C_1 \cdot \log_p(P_\psi^\alpha)\cdot \log_p(P_\xi^\alpha), \tag{6.5}$$

where C_1 is a non-zero simple algebraic constant.

As recalled at the beginning of this article, $P^\alpha_{\xi,\mathfrak{p}}$ is non-zero. We can also suppose that $P^\alpha_{\psi,\mathfrak{p}}$ is non-zero, as otherwise there is nothing to prove. Hence (6.5) shows that the order of vanishing of $\mathscr{L}_p{}^f(\breve{\mathbf{f}}^\vee, \breve{g}_\zeta, \breve{h}_{\alpha\zeta^{-1}})$ at $x = x_0$ is exactly two.

7. Main results

Let us now explain the proofs of the main theorems stated in the introduction by invoking the results proved in previous sections in combination with some of the main statements proved in the remaining contributions to this volume.

Let

$$\kappa(\mathbf{f},\mathbf{g},\mathbf{h}) \in H^1(\mathbf{Q}, \mathbb{V}^\dagger_{\mathbf{fgh}}(M))$$

be the Λ-adic global cohomology class introduced in [DRb, Def. 5.2].

Define $\mathbb{V}^\dagger_{\mathbf{f}gh}(M)$ as the $\Lambda_{\mathbf{f}}[G_{\mathbf{Q}}]$-module obtained by specialising the $\Lambda_{\mathbf{fgh}}[G_{\mathbf{Q}}]$-module $\mathbb{V}^\dagger_{\mathbf{fgh}}(M)$ at (y_0,z_0). Let

$$\kappa(\mathbf{f}, g_\zeta, h_{\alpha\zeta^{-1}}) := \nu_{y_0,z_0}\kappa(\mathbf{f},\mathbf{g},\mathbf{h}) \in H^1(\mathbf{Q}, \mathbb{V}^\dagger_{\mathbf{f}gh}(M)) \tag{7.1}$$

denote the specialisation of $\kappa(\mathbf{f},\mathbf{g},\mathbf{h})$ at (y_0,z_0), and

$$\kappa(f, g_\zeta, h_{\alpha\zeta^{-1}}) \in H^1(\mathbf{Q}, V_{fgh}(M))$$

denote the class obtained by specializing (7.1) further at x_0.

Let us analyze the above class locally. According to the discussion preceding Lemma 5.1, it follows that $\mathrm{res}_\ell(\kappa(f, g_\zeta, h_{\alpha\zeta^{-1}})) = 0$ at every prime $\ell \neq p$.

In order to study it at p, write $\kappa_p(f, g_\zeta, h_{\alpha\zeta^{-1}}) := \mathrm{res}_p\, \kappa(f, g_\zeta, h_{\alpha\zeta^{-1}}) \in H^1(\mathbf{Q}_p, V_f \otimes V_{gh}(M))$.

After setting $V_{gh}^{ab} = V_g^a \otimes V_h^b$, we find that there is a natural decomposition

$$H^1(\mathbf{Q}_p, V_p(E) \otimes V_{gh}) = \bigoplus_{(a,b)} H^1(\mathbf{Q}_p, V_p(E) \otimes V_{gh}^{ab}) \tag{7.2}$$

where (a, b) ranges through the four pairs $(\alpha_g, \alpha_h), (\alpha_g, \beta_h), (\beta_g, \alpha_h), (\beta_g, \beta_h)$. Analogous decompositions hold for the various Galois cohomology groups appearing in this section. Given a class $\kappa \in H^1(\mathbf{Q}_p, V_p(E) \otimes V_{gh}(M))$, we shall denote κ^{ab} for its projection to the corresponding (a, b)-component.

Note that

$$\alpha_g\alpha_h = \beta_g\beta_h = \alpha, \qquad \alpha_g\beta_h = \beta_g\alpha_h = -\alpha. \tag{7.3}$$

Hence, according to Lemma 5.1, $\kappa(f, g_\zeta, h_{\alpha\zeta^{-1}})$ lies in the Bloch-Kato finite submodule of $H^1(\mathbf{Q}, V_{fgh}(M))$ if and only if

(i) $\kappa_p(f, g_\zeta, h_{\alpha\zeta^{-1}})^{\alpha_g\beta_h}$ and $\kappa_p(f, g_\zeta, h_{\alpha\zeta^{-1}})^{\beta_g\alpha_h}$ lie in $H^1(\mathbf{Q}_p, V_p^+(E) \otimes V_{gh}(M))$,

(ii) $\kappa_p(f, g_\zeta, h_{\alpha\zeta^{-1}})^{\alpha_g\alpha_h}$ and $\kappa_p(f, g_\zeta, h_{\alpha\zeta^{-1}})^{\beta_g\beta_h}$ lie in $H^1_{\mathrm{f}}(\mathbf{Q}_p, V_p^+(E) \otimes V_{gh}(M))$.

By [**DRb**, Proposition 1.5.8], the local class $\kappa_p(f, g_\zeta, h_{\alpha\zeta^{-1}})$ is the specialization at (x_0, y_0, z_0) of a Λ-adic cohomology class with values in the Λ-adic representation $\mathbb{V}^+_{\mathbf{fgh}}(M)$, which recall is defined as the span in $\mathbb{V}^\dagger_{\mathbf{fgh}}(M)$ of (suitably twisted) triple tensor products of the form $\mathbb{V}^\pm_{\mathbf{f}} \otimes \mathbb{V}^\pm_{\mathbf{g}} \otimes \mathbb{V}^\pm_{\mathbf{h}}$, with at least two +'s in the exponents.

Since $V_g^{\beta_g} = V_g^+$ and $V_g^{\alpha_g} = V_g^-$, and similarly for V_h, it follows from the very definition of $\mathbb{V}^+_{\mathbf{fgh}}(M)$ that the (α_g, α_h)-component of $\kappa_p(f, g_\zeta, h_{\alpha\zeta^{-1}})$ in $H^1(\mathbf{Q}_p, V_f \otimes V_{gh}^{\alpha_g\alpha_h}(M))$ vanishes –this yields a fortiori claim (ii) for the (α_g, α_h)-component. The same reasoning also yields that the (α_g, β_h) and (β_g, α_h)-components of the projection of $\kappa_p(f, g_\zeta, h_{\alpha\zeta^{-1}})$ to $H^1(\mathbf{Q}_p, V_f^- \otimes V_{gh}(M))$ vanish, and hence (i) holds.

It only remains to analyze the (β_g, β_h)-component $\kappa_p(f, g_\zeta, h_{\alpha\zeta^{-1}})$. For this purpose we define the $\Lambda_{\mathbf{f}}[G_{\mathbf{Q}_p}]$-modules

$$\mathbb{W} := \mathbb{V}_{\mathbf{f},\beta\beta}(M) := \mathbb{V}_{\mathbf{f}}(M)(\underline{\varepsilon}_{\mathbf{f}}^{-1/2}) \otimes V_{gh}^{\beta_g\beta_h}(M),$$

$$\mathbb{W}^- := \mathbb{V}^-_{\mathbf{f},\beta\beta}(M) := \mathbb{V}^-_{\mathbf{f}}(M)(\underline{\varepsilon}_{\mathbf{f}}^{-1/2}) \otimes V_{gh}^{\beta_g\beta_h}(M).$$

It follows from (6.1) that $V_{gh}^{\beta\beta} = L_p(\alpha)$ is the one-dimensional representation afforded by the character of $\mathrm{Gal}\,(K_p/\mathbf{Q}_p)$ sending Fr_p to $\alpha = a_p(E)$. Hence $\mathbb{W}^-$ is the sub-quotient of $\mathbb{V}^\dagger_{\mathbf{f}gh}(M)$ that is isomorphic to several copies of $\Lambda_{\mathbf{f}}(\Psi_{\mathbf{f}}^{gh}\underline{\varepsilon}_{\mathbf{f}}^{-1/2})$, where as in [**DRb**, (1.5.5)], $\Psi_{\mathbf{f}}^{gh}$ denotes the unramified character of $G_{\mathbf{Q}_p}$ satisfying

$$\Psi_{\mathbf{f}}^{gh}(\mathrm{Fr}_p) = \mathbf{a}_p(\mathbf{f}) a_p^{-1}(\mathbf{g}_1) a_p^{-1}(\mathbf{h}_1) = \alpha \cdot \mathbf{a}_p(\mathbf{f}).$$

Let

$$\boldsymbol{\kappa}_p^f(\mathbf{f}, g_\zeta, h_{\alpha\zeta^{-1}}) \in H^1(\mathbf{Q}_p, \mathbb{W}), \quad \boldsymbol{\kappa}_p^f(\mathbf{f}, g_\zeta, h_{\alpha\zeta^{-1}})^- \in H^1(\mathbf{Q}_p, \mathbb{W}^-) \tag{7.4}$$

denote the image of $\kappa_p(\mathbf{f}, g_\zeta, h_{\alpha\zeta^{-1}})$ under the map induced by the projection $\mathbb{V}^+_{\mathbf{f}gh}(M) \to \mathbb{W} = \mathbb{V}_{\mathbf{f},\beta\beta}(M)$, and further to $\mathbb{W}^- = \mathbb{V}^-_{\mathbf{f},\beta\beta}(M)$ respectively.

Equivalently and in consonance with our notations, $\boldsymbol{\kappa}_p^f(\mathbf{f}, g_\zeta, h_{\alpha\zeta^{-1}})^-$ is the specialization at (y_0, z_0) of the local class $\boldsymbol{\kappa}_p^f(\mathbf{f}, \mathbf{g}, \mathbf{h})^-$ introduced in [**DRb**, (1.5.8)] and invoked in [**DRb**, Theorem 1.5.1]. Hence [**DRb**, Theorem 1.5.1] applies and asserts that the following identity holds in $\Lambda_{\mathbf{f}}$ for any triple $(\breve{\mathbf{f}}, \breve{\mathbf{g}}, \breve{\mathbf{h}})$ of Λ-adic test vectors:

$$\langle \mathcal{L}_{\mathbf{f},\mathbf{gh}}(\boldsymbol{\kappa}_p^f(\mathbf{f}, g_\zeta, h_{\alpha\zeta^{-1}})^-), \eta_{\breve{\mathbf{f}}^*} \otimes \omega_{\breve{g}_\zeta^*} \otimes \omega_{\breve{h}^*_{\alpha\zeta^{-1}}} \rangle = \mathscr{L}_p{}^f(\breve{\mathbf{f}}^\vee, \breve{g}_\zeta, \breve{h}_{\alpha\zeta^{-1}}). \tag{7.5}$$

Let now $\kappa_p^f(f, g_\zeta, h_{\alpha\zeta^{-1}})$ and $\kappa_p^f(f, g_\zeta, h_{\alpha\zeta^{-1}})^-$ denote the specializations at x_0 of the classes in (7.4). According to our previous definitions, we have

$$\kappa_p(f, g_\zeta, h_{\alpha\zeta^{-1}})^{\beta_g\beta_h} = \kappa_p^f(f, g_\zeta, h_{\alpha\zeta^{-1}}). \tag{7.6}$$

Since $a_p(f) = \alpha \in \{\pm 1\}$ and $\underline{\varepsilon}_{\mathbf{f}}(x_0) = 1$, it follows from the above description of $\mathbb{W}$ and the character $\Psi_{\mathbf{f}}^{gh}$ that $\mathbb{W}(x_0) \simeq V_p(E_+)(M)$ as $G_{\mathbf{Q}_p}$-modules, where E_+ is the (trivial or quadratic) twist of E given by α. Hence $\kappa_p^f(f, g_\zeta, h_{\alpha\zeta^{-1}}) \in H^1(\mathbf{Q}_p, V_p(E_+)(M))$.

The Bloch-Kato dual exponential and logarithm maps associated to the p-adic representation $V_p(E_+)(M)$ take values in a space $L_p(M)$ consisting of several copies of the base field L_p. Given a choice of test vectors, it gives rise to a projection $L_p(M) \longrightarrow L_p$. We shall denote by a slight abuse of notation

$$\log_{\mathrm{BK}} : H^1_{\mathrm{f}}(\mathbf{Q}_p, V_p(E_+)(M)) \longrightarrow L_p$$

the composition of the Bloch-Kato logarithm with the projection to L_p.

The following fundamental input comes from the main results due to Bertolini, Seveso and Venerucci in this volume, and we refer to [**BSVa**] and [**BSVb**] for the detailed proof; here we just content to point out to precise references in loc. cit. As explained in the introduction, in a previous version of this paper formula (7.7) below was wrongly attributed to [**Ve16**].

Theorem 7.1. — *(Bertolini, Seveso, Venerucci) The local class $\kappa_p^f(f, g_\zeta, h_{\alpha\zeta^{-1}})$ is crystalline and*

$$\frac{d^2}{dx^2}\mathscr{L}_p{}^f(\breve{\mathbf{f}}^\vee, \breve{g}_\zeta, \breve{h}_{\alpha\zeta^{-1}})_{|x=x_0} = C_2 \cdot \log_{\mathrm{BK}}(\kappa_p^f(f, g_\zeta, h_{\alpha\zeta^{-1}})) \tag{7.7}$$

for some nonzero rational number $C_2 \in \mathbf{Q}^\times$.

Indeed, the first claim of the above theorem follows from [**BSVa**, Theorem B]: since $L(f, g, h, 1) = 0$ it follows from the equivalence between (a) and (c) of [**BSVa**, §9.4] that the dual exponential map vanishes on $\kappa_p^f(f, g_\zeta, h_{\alpha\zeta^{-1}})$ –note that the improved class $\kappa_g^*(f, g_\zeta, h_{\alpha\zeta^{-1}})$ of loc. cit. is simply a non-zero multiple of $\kappa(f, g_\zeta, h_{\alpha\zeta^{-1}})$ in our setting, because of (7.3). This amounts to saying that the class is crystalline. Formula (7.7) follows from [**BSVb**, Proposition 2.2] combined with (7.5).

In light of (7.6) and the above discussion, the above theorem implies that $\kappa(f, g_\zeta, h_{\alpha\zeta^{-1}})$ belongs to the Selmer group $H^1_{\mathrm{f}}(\mathbf{Q}, V_{fgh}(M))$, as conditions (i) and (ii) above are fulfilled.

Recall from (5.2) that $V_{gh} = V_\psi \oplus V_\xi$ decomposes as the direct sum of the induced representations of ψ and ξ. Write

$$\kappa_\psi(f, g_\zeta, h_{\alpha\zeta^{-1}}) \in H^1_{\mathrm{f}}(\mathbf{Q}, V_p(E) \otimes V_\psi(M)),\qquad \kappa_\xi(f, g_\zeta, h_{\alpha\zeta^{-1}}) \in H^1_{\mathrm{f}}(\mathbf{Q}, V_p(E) \otimes V_\xi(M)) \tag{7.8}$$

for the projections of the class $\kappa(f, g_\zeta, h_{\alpha\zeta^{-1}})$ to the corresponding quotients. We denote as in the introduction

$$\kappa^\alpha_\psi(f, g_\zeta, h_{\alpha\zeta^{-1}}) = (1 + \alpha\sigma_{\mathfrak{p}})\kappa_\psi(f, g_\zeta, h_{\alpha\zeta^{-1}}) \in H^1_{\mathrm{f}}(H, V_p(E)(M))^{\psi\oplus\bar\psi}$$

the component of $\kappa_\psi(f, g_\zeta, h_{\alpha\zeta^{-1}})$ on which $\sigma_{\mathfrak{p}}$ acts with eigenvalue α, and likewise with ψ replaced by the auxiliary character ξ.

Lemma 7.1. — *We have*

$$\log_{E,\mathfrak{p}} \kappa^\alpha_\psi(f, g_\zeta, h_{\alpha\zeta^{-1}}) = \log_{E,\mathfrak{p}} \kappa^\alpha_\xi(f, g_\zeta, h_{\alpha\zeta^{-1}}).$$

Proof. — We may decompose the local class

$$\kappa_p := \kappa_p(f, g_\zeta, h_{\alpha\zeta^{-1}}) = (\kappa_p^{\alpha_g\alpha_h}, \kappa_p^{\alpha_g\beta_h}, \kappa_p^{\beta_g\alpha_h}, \kappa_p^{\beta_g\beta_h})$$

in $H^1(\mathbf{Q}_p, V_f \otimes V_{gh}^{\alpha_g\alpha_h}(M))$ as the sum of four contributions with respect to the decomposition (7.2) afforded by the eigen-spaces for the action of $\sigma_{\mathfrak{p}}$. In addition to that, κ_p also decomposes as

$$\kappa_p = (\kappa_{\psi,p}, \kappa_{\xi,p}) \in H^1_{\mathrm{f}}(\mathbf{Q}_p, V_p(E) \otimes V_\psi(M)) \oplus H^1_{\mathrm{f}}(\mathbf{Q}_p, V_p(E) \otimes V_\xi(M)),$$

where $\kappa_{\psi,p}$, $\kappa_{\xi,p}$ are the local components at p of the classes in (7.8). An easy exercise in linear algebra shows that

$$\kappa_p^{\alpha_g\alpha_h} = \kappa^\alpha_{\psi,p} - \kappa^\alpha_{\xi,p}, \qquad \kappa_p^{\beta_g\beta_h} = \kappa^\alpha_{\psi,p} + \kappa^\alpha_{\xi,p}. \tag{7.9}$$

Since we already proved that $\kappa_p^{\alpha_g\alpha_h} = 0$, the above display implies that $\kappa^\alpha_{\psi,p} = \kappa^\alpha_{\xi,p}$ are the same element. The lemma follows. □

Let

$$\log_{\beta_g\beta_h} : H^1_{\mathrm{f}}(\mathbf{Q}_p, V_f \otimes V_{gh}(M)) \overset{\mathrm{pr}_{\beta_g\beta_h}}{\longrightarrow} H^1_{\mathrm{f}}(\mathbf{Q}_p, V_f \otimes V_{gh}^{\beta_g\beta_h}(M)) \overset{\log_{\mathrm{BK}}}{\longrightarrow} L_p$$

denote the composition of the natural projection to the (β_g, β_h)-component with the Bloch-Kato logarithm map associated to the p-adic representation $V_f \otimes V_{gh}^{\beta_g\beta_h}(M) \simeq V_{f_+}(M)$ and the choice of test vectors. Note that $H^1_{\mathrm{f}}(\mathbf{Q}_p, V_p(E_+)) = H^1_{\mathrm{f}}(\mathbf{Q}_p, \mathbf{Q}_p(1))$, which as recalled in [**DRb**, Example 1.1.4 (c)] is naturally identified with the completion of $\mathbf{Z}_p^\times$, and the Bloch-Kato logarithm is nothing but the usual p-adic logarithm on $\mathbf{Z}_p^\times$ under this identification. Lemma 7.1 together with the second identity in (7.9) imply that

$$(i) \qquad \log_{E,\mathfrak{p}} \kappa^\alpha_\psi(f, g_\zeta, h_{\alpha\zeta^{-1}}) = \log_{\beta_g\beta_h}(\kappa_p(f, g_\zeta, h_{\alpha\zeta^{-1}})).$$

Thanks to (7.7) we have

$$(ii) \qquad \log_{\beta_g\beta_h}(\kappa_p(f, g_\zeta, h_{\alpha\zeta^{-1}})) = \frac{d^2}{dx^2}\mathscr{L}_p{}^f(\breve{\mathbf{f}}^\vee, \breve{g}_\zeta, \breve{h}_{\alpha\zeta^{-1}})_{|x=x_0} \pmod{L^\times}.$$

Finally, fix $(\breve{\mathbf{f}}, \breve{\mathbf{g}}, \breve{\mathbf{h}})$ to be Hsieh's choice of Λ-adic test vectors satisfying the properties stated in Theorem 6.1. Recall from (6.5) that, with this choice, we have

$$(iii) \qquad \frac{d^2}{dx^2}\mathscr{L}_p{}^f(\breve{\mathbf{f}}^\vee, \breve{g}_\zeta, \breve{h}_{\alpha\zeta^{-1}})_{|x=x_0} = \log_p(P_\psi^\alpha)\cdot\log_p(P_\xi^\alpha) \pmod{L^\times}.$$

Define

$$\kappa_\psi := \log_{E,\mathfrak{p}}(P_\xi^\alpha)^{-1} \times \kappa_\psi^\alpha(f, g_\alpha, h_\alpha).$$

It follows from the combination of (i)-(ii)-(iii) that κ_ψ fulfills the claims stated in Theorem A, and hence the theorem is proved.

Theorem B also follows, because the non-vanishing of the first derivative $\frac{d}{dx}\mathscr{L}_p(\mathbf{f}/K, \psi)_{|x=x_0}$ implies that $P_{\psi,\mathfrak{p}}^\alpha \neq 0$. Theorem A then implies that the class $\kappa_\psi \in H^1_{\mathrm{f}}(H, V_p(E)(M))^{\psi\oplus\bar\psi}$ is non-trivial.

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Henri Darmon and Victor Rotger, H. D.: Montreal, Canada
E-mail : darmon@math.mcgill.ca • V. R.: IMTech, UPC and Centre de Recerca Matemàtiques, C. Jordi Girona 1-3, 08034 Barcelona, Spain • *E-mail :* victor.rotger@upc.edu

p-ADIC FAMILIES OF DIAGONAL CYCLES

by

Henri Darmon and Victor Rotger

Abstract. — This note provides the construction of a three-variable family of cohomology classes arising from diagonal cycles on a triple product of towers of modular curves, and proves a reciprocity law relating it to the three variable triple-product p-adic L-function associated to a triple of Hida families by means of Perrin-Riou's Λ-adic regulator.

To Bernadette Perrin-Riou on her 65-th birthday

Contents

Acknowledgements. The first author was supported by an NSERC Discovery grant, and the second author was supported by Grant MTM2015-63829-P. The second author also acknowledges the financial support by ICREA under the ICREA Academia programme. This project has received funding from the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation programme (grant agreement No 682152).

2000 ***Mathematics Subject Classification.*** — 11G18, 14G35.

Introduction

The main purpose of this article is to supply a construction of a three-variable family of cycles interpolating the generalized diagonal cycles introduced in [**DR14**], and to prove a reciprocity law relating this family to the three variable triple-product p-adic L-function associated to a triple of Hida families by means of Perrin-Riou's Λ-adic regulator.

In order to give a flavor of our construction, let us describe in more detail the organization and contents of this article.

After reviewing some background in the first section, in section 2 we construct for every $r \geq 1$ a completely explicit family of cycles in the cube X_r^3 of the modular curve $X_r = X_1(Mp^r)$ of $\Gamma_1(Mp^r)$-level structure. This family is parametrized by the space of $\mathrm{SL}_2(\mathbf{Z}/p^r\mathbf{Z})$-orbits of the set

$$\Sigma_r := ((\mathbf{Z}/p^r\mathbf{Z} \times \mathbf{Z}/p^r\mathbf{Z})')^3 \subset ((\mathbf{Z}/p^r\mathbf{Z})^2)^3$$

of triples of primitive row vectors of length 2 with entries in $\mathbf{Z}/p^r\mathbf{Z}$, on which $\mathrm{GL}_2(\mathbf{Z}/p^r\mathbf{Z})$ acts diagonally by right multiplication. Any triple in Σ_r gives rise to a twisted diagonal embedding of the modular curve $\mathbb{X}(p^r)$ of $\Gamma_1(M) \cup \Gamma(p^r)$-level structure into the three-fold X_r^3 and the associated cycle is defined as the image of this map: we refer to (2.4) for the precise recipe.

The parameter space $\Sigma_r/\mathrm{SL}_2(\mathbf{Z}/p^r\mathbf{Z})$ is closely related to $((\mathbf{Z}/p^r\mathbf{Z})^\times)^3$ and as shown throughout §2, the associated family of global cohomology classes introduced in Definition 2.9 can be packaged into a global Λ-adic cohomology class parametrized by three copies of weight space.

Along §3 and §4 we study the higher weight and cristalline specialisations of this family and we eventually prove in Theorem 4.1 that they interpolate the classes introduced in [**DR14**] as claimed above.

Finally, in §5 we recall Garrett-Hida's triple product p-adic L-function associated to a triple of Hida families $(\mathbf{f}, \mathbf{g}, \mathbf{h})$ and prove in Theorem 5.1 a reciprocity law expressing the latter as the image of our three-variable cohomology classes (as specified in Definition 5.2) under Perrin-Riou's Λ-adic regulator.

It is instructive to compare the construction of our family to the approach taken in [**DR17**], which associated to a triple $(f, \mathbf{g}, \mathbf{h})$ consisting of a *fixed* newform f and a pair $(\mathbf{g}, \mathbf{h})$ of Hida families a *one-variable* family of cohomology classes instead of the two-variable family that one might have felt entitled to *a priori*. This shortcoming of the earlier approach can be understood by noting that the space of embeddings of $\mathbb{X}(p^r)$ into $X_1(M) \times X_r \times X_r$ as above in which the projection to the first factor is fixed is naturally parametrized by the coset space $M_2(\mathbf{Z}/p^r\mathbf{Z})'/\mathrm{SL}_2(\mathbf{Z}/p^r\mathbf{Z})$, where $M_2(\mathbf{Z}/p^r\mathbf{Z})'$ denotes the set of 2×2 matrices whose rows are not divisible by p. The resulting cycles are therefore parametrized by the coset space $\mathrm{GL}_2(\mathbf{Z}/p^r\mathbf{Z})/\mathrm{SL}_2(\mathbf{Z}/p^r\mathbf{Z}) = (\mathbf{Z}/p^r\mathbf{Z})^\times$, whose inverse limit with r is the one dimensional p-adic space $\mathbf{Z}_p^\times$ rather than a two-dimensional one.

As mentioned already in our previous article in this volume, these cycles are of interest in their own right, and shed a useful complementary perspective on the construction of the Λ-adic cohomology classes for the triple product when compared to

[**BSVa**]. Indeed, their study forms the basis for the ongoing PhD thesis of David Lilienfeldt [**Li**], and has let to interesting open questions as those that are explored by Castella and Hsieh in [**CS20**].

1. Background

1.1. Basic notations. — Fix an algebraic closure $\bar{\mathbf{Q}}$ of $\mathbf{Q}$. All the number fields that arise will be viewed as embedded in this algebraic closure. For each such K, let $G_K := \mathrm{Gal}\,(\bar{\mathbf{Q}}/K)$ denote its absolute Galois group. Fix an odd prime p and an embedding $\bar{\mathbf{Q}} \hookrightarrow \bar{\mathbf{Q}}_p$; let ord_p denote the resulting p-adic valuation on $\bar{\mathbf{Q}}^\times$, normalized in such a way that $\mathrm{ord}_p(p) = 1$.

For a variety V defined over $K \subset \bar{\mathbf{Q}}$, let $\bar{V}$ denote the base change of V to $\bar{\mathbf{Q}}$. If $\mathcal{F}$ is an étale sheaf on V, write $H^i_{\text{ét}}(\bar{V}, \mathcal{F})$ for the ith étale cohomology group of $\bar{V}$ with values in $\mathcal{F}$, equipped with its continuous action by G_K.

Given a prime p, let $\mathbf{Q}(\mu_{p^\infty}) = \cup_{r\geq 1}\mathbf{Q}(\zeta_r)$ be the cyclotomic extension of $\mathbf{Q}$ obtained by adjoining to $\mathbf{Q}$ a primitive p^r-th root of unity ζ_r. Let

$$\varepsilon_{\mathrm{cyc}} : G_{\mathbf{Q}} \longrightarrow \mathrm{Gal}\,(\mathbf{Q}(\mu_{p^\infty})/\mathbf{Q}) \xrightarrow{\simeq} \mathbf{Z}_p^\times$$

denote the p-adic cyclotomic character. It can be factored as $\varepsilon_{\mathrm{cyc}} = \omega\langle\varepsilon_{\mathrm{cyc}}\rangle$, where

$$\omega : G_{\mathbf{Q}} \longrightarrow \mu_{p-1} \qquad \langle\varepsilon_{\mathrm{cyc}}\rangle : G_{\mathbf{Q}} \longrightarrow 1 + p\mathbf{Z}_p$$

are obtained by composing $\varepsilon_{\mathrm{cyc}}$ with the projection onto the first and second factors in the canonical decomposition $\mathbf{Z}_p^\times \simeq \mu_{p-1} \times (1 + p\mathbf{Z}_p)$. If $\mathcal{M}$ is a $\mathbf{Z}_p[G_{\mathbf{Q}}]$-module and j is an integer, write $\mathcal{M}(j) = \mathcal{M} \otimes \varepsilon_{\mathrm{cyc}}^j$ for the j-th Tate twist of $\mathcal{M}$.

Let

$$\mathring{\Lambda}_r := \mathbf{Z}_p[(\mathbf{Z}/p^r\mathbf{Z})^\times], \qquad \mathring{\Lambda} := \mathbf{Z}_p[[\mathbf{Z}_p^\times]] := \varprojlim_r \mathring{\Lambda}_r$$

denote the group ring and completed group ring attached to the profinite group $\mathbf{Z}_p^\times$. The ring $\mathring{\Lambda}$ is equipped with $p-1$ distinct algebra homomorphisms $\omega^i : \mathring{\Lambda} \to \Lambda$ (for $0 \leq i \leq p-2$) to the local ring

$$\Lambda = \mathbf{Z}_p[[1 + p\mathbf{Z}_p]] = \varprojlim \mathbf{Z}_p[1 + p\mathbf{Z}/p^r\mathbf{Z}] \simeq \mathbf{Z}_p[[T]],$$

where ω^i sends a group-like element $a \in \mathbf{Z}_p^\times$ to $\omega^i(a)\langle a\rangle \in \Lambda$. These homomorphisms identify $\mathring{\Lambda}$ with the direct sum

$$\mathring{\Lambda} = \bigoplus_{i=0}^{p-2} \Lambda.$$

The local ring Λ is called the one variable Iwasawa algebra. More generally, for any integer $t \geq 1$, let

$$\mathring{\Lambda}^{\otimes t} := \mathring{\Lambda}\hat{\otimes}_{\mathbf{Z}_p} \overset{t}{\cdots} \hat{\otimes}_{\mathbf{Z}_p}\mathring{\Lambda}, \qquad \Lambda^{\otimes t} = \Lambda\hat{\otimes}_{\mathbf{Z}_p} \overset{t}{\cdots} \hat{\otimes}_{\mathbf{Z}_p}\Lambda \simeq \mathbf{Z}_p[[T_1, \ldots T_t]].$$

The latter ring is called the Iwasawa algebra in t variables, and is isomorphic to the power series ring in t variables over $\mathbf{Z}_p$, while

$$\mathring{\Lambda}^{\otimes t} = \bigoplus_{\alpha} \Lambda^{\otimes t},$$

the sum running over the $(p-1)^t$ distinct $\mathbf{Z}_p^\times$ valued characters of $(\mathbf{Z}/p\mathbf{Z})^{\times t}$.

1.2. Modular forms and Galois representations. — Let

$$\phi = q + \sum_{n\geq 2} a_n(\phi)q^n \in S_k(M,\chi)$$

be a cuspidal modular form of weight $k \geq 1$, level M and character $\chi : (\mathbf{Z}/M\mathbf{Z})^\times \to \mathbf{C}^\times$, and assume that ϕ is an eigenform with respect to all good Hecke operators T_ℓ, $\ell \nmid M$.

Fix an odd prime number p (that in this section may or may not divide M). Let $\mathcal{O}_\phi$ denote the valuation ring of the finite extension of $\mathbf{Q}_p$ generated by the fourier coefficients of ϕ, and let $\mathbb{T}$ denote the Hecke algebra generated over $\mathbf{Z}_p$ by the good Hecke operators T_ℓ with $\ell \nmid M$ and by the diamond operators acting on $S_k(M,\chi)$. The eigenform ϕ gives rise to an algebra homomorphism

$$\xi_\phi : \mathbb{T} \longrightarrow \mathcal{O}_\phi$$

sending T_ℓ to $a_\ell(\phi)$ and the diamond operator $\langle \ell \rangle$ to $\chi(\ell)$.

A fundamental construction of Shimura, Deligne, and Serre-Deligne attaches to ϕ an irreducible Galois representation

$$\varrho_\phi : G_{\mathbf{Q}} \longrightarrow \mathrm{Aut}(V_\phi) \simeq \mathrm{GL}_2(\mathcal{O}_\phi)$$

of rank 2, unramified at all primes $\ell \nmid Mp$, and for which

$$\det(1 - \varrho_\phi(\mathrm{Fr}_\ell)x) = 1 - a_\ell(\phi)x + \chi(\ell)\ell^{k-1}x^2, \tag{1.1}$$

where Fr_ℓ denotes the arithmetic Frobenius element at ℓ. This property characterizes the semi-simplification of ϱ_ϕ up to isomorphism.

When $k := k_\circ + 2 \geq 2$, the representation V_ϕ can be realised in the p-adic étale cohomology of an appropriate Kuga-Sato variety. Since this realisation is important for the construction of generalised Kato classes, we now briefly recall its salient features. Let $Y = Y_1(M)$ and $X = X_1(M)$ denote the open and closed modular curve representing the fine moduli functor of isomorphism classes of pairs (A,P) formed by a (generalised) elliptic curve A together with a torsion point P on A of exact order M. Let

$$\pi : \mathcal{A}_\circ \longrightarrow Y \tag{1.2}$$

denote the universal elliptic curve over Y.

The $k_\circ$-th open Kuga-Sato variety over Y is the $k_\circ$-fold fiber product

$$\mathcal{A}_\circ^{k_\circ} := \mathcal{A}_\circ \times_Y \overset{(k_\circ)}{\cdots} \times_Y \mathcal{A}_\circ \tag{1.3}$$

of $\mathcal{A}_\circ$ over Y. The variety $\mathcal{A}_\circ^{k_\circ}$ admits a smooth compactification $\mathcal{A}^{k_\circ}$ which is fibered over X and is called the $k_\circ$-th Kuga-Sato variety over X; we refer to Conrad's appendix in [**BDP13**] for more details. The geometric points in $\mathcal{A}^{k_\circ}$ that lie above Y are in bijection with isomorphism classes of tuples $[(A,P),P_1,\dots,P_{k_\circ}]$, where (A,P) is associated to a point of Y as in the previous paragraph and $P_1,\dots,P_{k_\circ}$ are points on A.

The representation V_ϕ is realised (up to a suitable Tate twist) in the middle degree étale cohomology $H^{k_\circ+1}_{\text{ét}}(\bar{\mathcal{A}}^{k_\circ}, \mathbf{Z}_p)$. More precisely, let

$$\mathcal{H}_r := R^1\pi_* \mathbf{Z}/p^r\mathbf{Z}(1), \qquad \mathcal{H} := R^1\pi_* \mathbf{Z}_p(1),$$

and for any $k_\circ \geq 0$, define

$$\mathcal{H}_r^{k_\circ} := \mathrm{TSym}^{k_\circ}(\mathcal{H}_r), \qquad \mathcal{H}^{k_\circ} := \mathrm{TSym}^{k_\circ}(\mathcal{H}) \tag{1.4}$$

to be the sheaves of symmetric $k_\circ$-tensors of $\mathcal{H}_r$ and $\mathcal{H}$, respectively. As defined in e.g. [**BDP13**, (2.1.2)], there is an idempotent $\epsilon_{k_\circ}$ in the ring of rational correspondences of $\mathcal{A}^{k_\circ}$ whose induced projector on the étale cohomology groups of this variety satisfy:

$$\epsilon_{k_\circ}\left(H^{k_\circ+1}_{\text{ét}}(\bar{\mathcal{A}}^{k_\circ}, \mathbf{Z}_p(k_\circ))\right) = H^1_{\text{ét}}(\bar{X}, \mathcal{H}^{k_\circ}). \tag{1.5}$$

Define the $\mathcal{O}_\phi$-module

$$V_\phi(M) := H^1_{\text{ét}}(\bar{X}, \mathcal{H}^{k_\circ}(1)) \otimes_{\mathbb{T},\xi_\phi} \mathcal{O}_\phi, \tag{1.6}$$

and write

$$\varpi_\phi : H^1_{\text{ét}}(\bar{X}, \mathcal{H}^{k_\circ}(1)) \longrightarrow V_\phi(M) \tag{1.7}$$

for the canonical projection of $\mathbb{T}[G_{\mathbf{Q}}]$-modules arising from (1.6). Deligne's results and the theory of newforms show that the module $V_\phi(M)$ is the direct sum of several copies of a locally free module V_ϕ of rank 2 over $\mathcal{O}_\phi$ that satisfies (1.1).

Let α_ϕ and β_ϕ the two roots of the p-th Hecke polynomial $T^2 - a_p(\phi)T + \chi(p)p^{k-1}$, ordered in such a way that $\mathrm{ord}_p(\alpha_\phi) \leq \mathrm{ord}_p(\beta_\phi)$. (If α_ϕ and β_ϕ have the same p-adic valuation, simply fix an arbitrary ordering of the two roots.) We set $\chi(p) = 0$ whenever p divides the primitive level of ϕ and thus $\alpha_\phi = a_p(\phi)$ and $\beta_\phi = 0$ in this case. The eigenform ϕ is said to be *ordinary* at p when $\mathrm{ord}_p(\alpha_\phi) = 0$. In that case, there is an exact sequence of $G_{\mathbf{Q}_p}$-modules

$$0 \to V_\phi^+ \longrightarrow V_\phi \longrightarrow V_\phi^- \to 0, \qquad V_\phi^+ \simeq \mathcal{O}_\phi(\varepsilon_{\mathrm{cyc}}^{k-1}\chi\psi_\phi^{-1}), \quad V_\phi^- \simeq \mathcal{O}_\phi(\psi_\phi), \tag{1.8}$$

where ψ_ϕ is the unramified character of $G_{\mathbf{Q}_p}$ sending Fr_p to α_ϕ.

1.3. Hida families and Λ-adic Galois representations. — Fix a prime $p \geq 3$. The formal spectrum

$$\mathcal{W} := \mathrm{Spf}(\Lambda)$$

of the Iwasawa algebra $\Lambda = \mathbf{Z}_p[[1 + p\mathbf{Z}_p]]$ is called the *weight space* attached to Λ. The A-valued points of $\mathcal{W}$ over a p-adic ring A are given by

$$\mathcal{W}(A) = \mathrm{Hom}_{\mathrm{alg}}(\Lambda, A) = \mathrm{Hom}_{\mathrm{grp}}(1 + p\mathbf{Z}_p, A^\times),$$

where the Hom's in this definition denote continuous homomorphisms of p-adic rings and profinite groups respectively. Weight space is equipped with a distinguished collection of *arithmetic points* $\nu_{k_\circ,\varepsilon}$, indexed by integers $k_\circ \geq 0$ and Dirichlet characters $\varepsilon : (1 + p\mathbf{Z}/p^r\mathbf{Z}) \to \mathbf{Q}_p(\zeta_{r-1})^\times$ of p-power conductor. The point $\nu_{k_\circ,\varepsilon} \in \mathcal{W}(\mathbf{Z}_p[\zeta_r])$ is defined by

$$\nu_{k_\circ,\varepsilon}(n) = \varepsilon(n)n^{k_\circ},$$

and the notational shorthand $\nu_{k_\circ} := \nu_{k_\circ,1}$ is adopted throughout. More generally, if $\tilde{\Lambda}$ is any finite flat Λ-algebra, a point $x \in \tilde{\mathcal{W}} := \mathrm{Spf}(\tilde{\Lambda})$ is said to be arithmetic if its restriction to Λ agrees with $\nu_{k_\circ,\varepsilon}$ for some $k_\circ$ and ϵ. The integer $k = k_\circ + 2$ is called the *weight* of x and denoted $\mathrm{wt}(x)$.

Let

$$\underline{\varepsilon}_{\mathrm{cyc}} : G_{\mathbf{Q}} \longrightarrow \Lambda^\times \tag{1.9}$$

denote the Λ-adic cyclotomic character which sends a Galois element σ to the group-like element $[\langle \varepsilon_{\mathrm{cyc}}(\sigma)\rangle]$. This character interpolates the powers of the cyclotomic character, in the sense that

$$\nu_{k_\circ,\varepsilon} \circ \underline{\varepsilon}_{\mathrm{cyc}} = \varepsilon \cdot \langle \varepsilon_{\mathrm{cyc}} \rangle^{k_\circ} = \varepsilon \cdot \varepsilon_{\mathrm{cyc}}^{k_\circ} \cdot \omega^{-k_\circ}. \tag{1.10}$$

Let $M \geq 1$ be an integer not divisible by p.

Definition 1.1. — *A* Hida family of tame level M and tame character $\chi : (\mathbf{Z}/M\mathbf{Z})^\times \to \bar{\mathbf{Q}}_p^\times$ *is a formal q-expansion*

$$\boldsymbol{\phi} = \sum_{n\geq 1} a_n(\boldsymbol{\phi}) q^n \in \Lambda_{\boldsymbol{\phi}}[[q]]$$

with coefficients in a finite flat Λ-algebra $\Lambda_{\boldsymbol{\phi}}$, such that for any arithmetic point $x \in \mathcal{W}_{\boldsymbol{\phi}} := \mathrm{Spf}(\Lambda_{\boldsymbol{\phi}})$ above $\nu_{k_\circ,\epsilon}$, where $k_\circ \geq 0$ and ε is a character of conductor p^r, the series

$$\boldsymbol{\phi}_x := \sum_{n\geq 1} x(a_n(\boldsymbol{\phi}))q^n \in \bar{\mathbf{Q}}_p[[q]]$$

is the q-expansion of a classical p-ordinary eigenform in the space $S_k(Mp^r, \chi\varepsilon\omega^{-k_\circ})$ of cusp forms of weight $k = k_\circ + 2$, level Mp^r and nebentype $\chi\varepsilon\omega^{-k_\circ}$.

By enlarging $\Lambda_{\boldsymbol{\phi}}$ if necessary, we shall assume throughout that $\Lambda_{\boldsymbol{\phi}}$ contains the M-th roots of unity.

Definition 1.2. — *Let $x \in \mathcal{W}_{\boldsymbol{\phi}}$ be an arithmetic point lying above the point $\nu_{k_\circ,\epsilon}$ of weight space. The point x is said to be*

- tame *if the character ϵ is tamely ramified, i.e., factors through $(\mathbf{Z}/p\mathbf{Z})^\times$.*
- crystalline *if $\epsilon\omega^{-k_\circ} = 1$, i.e., if the weight k specialisation of $\boldsymbol{\phi}$ at x has trivial nebentypus character at p.*

We let $\mathcal{W}_{\boldsymbol{\phi}}^\circ$ denote the set of crystalline arithmetic points of $\mathcal{W}_{\boldsymbol{\phi}}$.

Note that a crystalline point is necessarily tame but of course there are tame points that are not crystalline. The justification for this terminology is that the Galois representation V_{ϕ_x} is crystalline at p when x is crystalline.

If x is a crystalline point, then the classical form $\boldsymbol{\phi}_x$ is always old at p if $k > 2$. In that case there exists an eigenform ϕ_x° of level M such that $\boldsymbol{\phi}_x$ is the ordinary p-stabilization of $\boldsymbol{\phi}_x^\circ$. If the weight is $k = 1$ or 2, $\boldsymbol{\phi}_x$ may be either old or new at p; if it is new at p then we set $\boldsymbol{\phi}_x^\circ = \boldsymbol{\phi}_x$ in order to have uniform notations.

We say $\boldsymbol{\phi}$ is residually irreducible if the mod p Galois representation associated to the Deligne representations associated to ϕ_x° for any crystalline classical point is irreducible.

Finally, the Hida family $\boldsymbol{\phi}$ is said to be *primitive* of tame level $M_{\boldsymbol{\phi}} \mid M$ if for all but finitely many arithmetic points $x \in \mathcal{W}_{\boldsymbol{\phi}}$ of weight $k \geq 2$, the modular form $\boldsymbol{\phi}_x$ arises from a newform of level $M_{\boldsymbol{\phi}}$.

The following theorem of Hida and Wiles associates a two-dimensional Galois representation to a Hida family $\boldsymbol{\phi}$ (cf. e.g. [**MT90**, Théorème 7]).

Theorem 1.1. — *Assume $\boldsymbol{\phi}$ is residually irreducible. Then there is a rank two $\Lambda_{\boldsymbol{\phi}}$-module $\mathbb{V}_{\boldsymbol{\phi}}$ equipped with a Galois action*

$$\varrho_{\boldsymbol{\phi}} : G_{\mathbf{Q}} \longrightarrow \mathrm{Aut}_{\Lambda_{\boldsymbol{\phi}}}(\mathbb{V}_{\boldsymbol{\phi}}) \simeq \mathrm{GL}_2(\Lambda_{\boldsymbol{\phi}}), \tag{1.11}$$

such that, for all arithmetic points $x : \Lambda_{\boldsymbol{\phi}} \longrightarrow \bar{\mathbf{Q}}_p$,

$$\mathbb{V}_{\boldsymbol{\phi}} \otimes_{x,\Lambda_{\boldsymbol{\phi}}} \bar{\mathbf{Q}}_p \simeq V_{\boldsymbol{\phi}_x} \otimes \bar{\mathbf{Q}}_p.$$

Let

$$\psi_{\boldsymbol{\phi}} : G_{\mathbf{Q}_p} \longrightarrow \Lambda_{\boldsymbol{\phi}}^{\times}$$

denote the unramified character sending a Frobenius element Fr_p to $\mathbf{a}_p(\boldsymbol{\phi})$. The restriction of $\mathbb{V}_{\boldsymbol{\phi}}$ to $G_{\mathbf{Q}_p}$ admits a filtration

$$0 \to \mathbb{V}_{\boldsymbol{\phi}}^{+} \to \mathbb{V}_{\boldsymbol{\phi}} \to \mathbb{V}_{\boldsymbol{\phi}}^{-} \to 0 \quad \textit{where } \mathbb{V}_{\boldsymbol{\phi}}^{+} \simeq \Lambda_{\boldsymbol{\phi}}(\psi_{\boldsymbol{\phi}}^{-1}\chi\varepsilon_{\mathrm{cyc}}^{-1}\underline{\varepsilon}_{\mathrm{cyc}}) \ \textit{ and } \ \mathbb{V}_{\boldsymbol{\phi}}^{-} \simeq \Lambda_{\boldsymbol{\phi}}(\psi_{\boldsymbol{\phi}}). \tag{1.12}$$

The explicit construction of the Galois representation $\mathbb{V}_{\boldsymbol{\phi}}$ plays an important role in defining the generalised Kato classes, and we now recall its main features.

For all $0 \leq r < s$, let

$$X_r := X_1(Mp^r), \qquad X_{r,s} := X_1(Mp^r) \times_{X_0(Mp^r)} X_0(Mp^s),$$

where the fiber product is taken relative to the natural projection maps. In particular,
- — the curve $X := X_0 := X_1(M)$ represents the functor of elliptic curves A with $\Gamma_1(M)$-level structure, i.e., with a marked point of order M;
- — the curve X_r represents the functor classifying pairs (A, P) consisting of a generalized elliptic curve A with $\Gamma_1(M)$-level structure and a point P of order p^r on A;
- — the curve $X_{0,s} = X_1(M) \times_{X_0(M)} X_0(Mp^s)$ classifies pairs (A, C) consisting of a generalized elliptic curve A with $\Gamma_1(M)$ structure and a cyclic subgroup scheme C of order p^s on A;
- — the curve $X_{r,s}$ classifies pairs (A, P, C) consisting of a generalized elliptic curve A with $\Gamma_1(M)$ structure, a point P of order r on A and and a cyclic subgroup scheme C of order p^s on A containing P.

The curves X_r and $X_{0,r}$ are smooth geometrically connected curves over $\mathbf{Q}$. The natural covering map $X_r \longrightarrow X_{0,r}$ is Galois with Galois group $(\mathbf{Z}/p^r\mathbf{Z})^{\times}$ acting on the left via the diamond operators defined by

$$\langle a \rangle (A, P) = (A, aP). \tag{1.13}$$

Let

$$\varpi_1 : X_{r+1} \longrightarrow X_r \tag{1.14}$$

denote the natural projection from level $r+1$ to level r which corresponds to the map $(A,P)\mapsto(A,pP)$, and to the map $\tau\mapsto\tau$ on upper half planes. Let

$$\varpi_2 : X_{r+1} \longrightarrow X_r$$

denote the other projection, corresponding to the map $(A,P)\mapsto(A/\langle p^rP\rangle, P+\langle p^rP\rangle)$, which on the upper half plane sends τ to $p\tau$. These maps can be factored as

(1.15)
$$\begin{array}{ccc} X_{r+1} & & \\ \downarrow \mu & \searrow \varpi_1 & \\ X_{r,r+1} & \xrightarrow{\pi_1} & X_r, \end{array} \qquad \begin{array}{ccc} X_{r+1} & & \\ \downarrow \mu & \searrow \varpi_2 & \\ X_{r,r+1} & \xrightarrow{\pi_2} & X_r. \end{array}$$

For all $r\geq 1$, the vertical map μ is a cyclic Galois covering of degree p, while the horizontal maps π_1 and π_2 are non-Galois coverings of degree p. When $r=0$, the map μ is a cyclic Galois covering of degree $p-1$ and π_2 are non-Galois coverings of degree $p+1$.

The Λ-adic representation $\mathbb{V}_{\boldsymbol{\phi}}$ shall be realised (up to twists) in quotients of the inverse limit of étale cohomology groups arising from the tower

$$X_\infty^* \quad : \quad \cdots \xrightarrow{\varpi_1} X_{r+1} \xrightarrow{\varpi_1} X_r \xrightarrow{\varpi_1} \cdots \xrightarrow{\varpi_1} X_1 \xrightarrow{\varpi_1} X_0$$

of modular curves. Define the inverse limit

(1.16)
$$H^1_{\text{ét}}(\bar{X}^*_\infty, \mathbf{Z}_p) := \varprojlim_{\varpi_{1*}} H^1_{\text{ét}}(\bar{X}_r, \mathbf{Z}_p)$$

where the transition maps arise from the pushforward induced by the morphism ϖ_1. This inverse limit is a module over the completed group rings $\mathbf{Z}_p[[\mathbf{Z}_p^\times]]$ arising from the action of the diamond operators, and is endowed with a plethora of extra structures that we now describe.

Hecke operators. The transition maps in (1.16) are compatible with the action of the Hecke operators T_n for all n that are not divisible by p. Of crucial importance for us in this article is Atkin's operator U_p^*, which operates on $H^1_{\text{ét}}(\bar{X}_r,\mathbf{Z}_p)$ via the composition

$$U_p^* := \pi_{1*}\pi_2^*$$

arising from the maps in (1.15).

The operator U_p^* is compatible with the transition maps defining $H^1_{\text{ét}}(\bar{X}^*_\infty,\mathbf{Z}_p)$,

Inverse systems of étale sheaves. The cohomology group $H^1_{\text{ét}}(\bar{X}^*_\infty,\mathbf{Z}_p)$ can be identified with the first cohomology group of the base curve X_1 with values in a certain inverse systems of étale sheaves.

For each $r\geq 1$, let

(1.17)
$$\mathcal{L}_r^* := \varpi_{1*}^{r-1}\mathbf{Z}_p$$

be the pushforward of the constant sheaf on X_r via the map

$$\varpi_1^{r-1} : X_r \longrightarrow X_1$$

The stalk of $\mathcal{L}_r^*$ at a geometric point $x=(A,P)$ on X_1 is given by

$$\mathcal{L}^*_{r,x} = \mathbf{Z}_p[A[p^r]\langle P\rangle],$$

where

$$A[p^r]\langle P\rangle := \{Q \in A[p^r] \text{ such that } p^{r-1}Q = P\}.$$

The multiplication by p map on the fibers gives rise to natural homomorphisms of sheaves

$$[p] : \mathcal{L}^*_{r+1} \longrightarrow \mathcal{L}^*_r, \tag{1.18}$$

and Shapiro's lemma gives canonical identifications

$$H^1_{\text{ét}}(\bar{X}_r, \mathbf{Z}_p) = H^1_{\text{ét}}(\bar{X}_1, \mathcal{L}^*_r),$$

for which the following diagram commutes:

$$\begin{array}{ccc} H^1_{\text{ét}}(\bar{X}_{r+1}, \mathbf{Z}_p) & \xrightarrow{\varpi_{1*}} & H^1_{\text{ét}}(\bar{X}_r, \mathbf{Z}_p) \\ \| & & \| \\ H^1_{\text{ét}}(\bar{X}_1, \mathcal{L}^*_{r+1}) & \xrightarrow{[p]} & H^1_{\text{ét}}(\bar{X}_1, \mathcal{L}^*_r). \end{array}$$

Let $\mathcal{L}^*_\infty := \varprojlim_r \mathcal{L}^*_r$ denote the inverse system of étale sheaves relative to the maps $[p]$ arising in (1.18). By passing to the limit, we obtain an identification

$$H^1_{\text{ét}}(\bar{X}^*_\infty, \mathbf{Z}_p) = \varprojlim_{r\geq 1} H^1_{\text{ét}}(\bar{X}_1, \mathcal{L}^*_r) = H^1_{\text{ét}}(\bar{X}_1, \mathcal{L}^*_\infty). \tag{1.19}$$

Weight k specialisation maps. Recall the p-adic étale sheaves $\mathcal{H}^{k_\circ}$ introduced in (1.4), whose cohomology gave rise to the Deligne representations attached to modular forms of weight $k = k_\circ + 2$ via (1.6). The natural $k_\circ$-th power symmetrisation function

$$A[p^r] \longrightarrow \mathcal{H}^{k_\circ}_r, \qquad Q \mapsto Q^{k_\circ},$$

restricted to $A[p^r]\langle P\rangle$ and extended to $\mathcal{L}^*_{r,x}$ by $\mathbf{Z}_p$-linearity, induces morphisms

$$\text{sp}^*_{k,r} : \mathcal{L}^*_r \longrightarrow \mathcal{H}^{k_\circ}_r \tag{1.20}$$

of sheaves over X_1 (which are thus compatible with the action of $G_{\mathbf{Q}}$ on the fibers). These specialisation morphisms are compatible with the transition maps $[p]$ in the sense that the diagram

$$\begin{array}{ccc} \mathcal{L}^*_{r+1} & \xrightarrow{[p]} & \mathcal{L}^*_r \\ \downarrow{\scriptstyle \text{sp}^*_{k,r+1}} & & \downarrow{\scriptstyle \text{sp}^*_{k,r}} \\ \mathcal{H}^{k_\circ}_{r+1} & \longrightarrow & \mathcal{H}^{k_\circ}_r \end{array}$$

commutes, where the bottom horizontal arrow denotes the natural reduction map. The maps $\text{sp}^*_{k,r}$ can thus be pieced together into morphisms

$$\text{sp}^*_k : \mathcal{L}^*_\infty \longrightarrow \mathcal{H}^{k_\circ}. \tag{1.21}$$

The induced morphism

$$\text{sp}^*_k : H^1_{\text{ét}}(\bar{X}^*_\infty, \mathbf{Z}_p) \longrightarrow H^1_{\text{ét}}(\bar{X}_1, \mathcal{H}^{k_\circ}), \tag{1.22}$$

arising from those on $H^1_{\text{ét}}(\bar{X}_1, \mathcal{L}^*_\infty)$ via (1.19) will be denoted by the same symbol by abuse of notation, and is referred to as the *weight* $k = k_\circ + 2$ *specialisation map.* The existence of such maps having finite cokernel reveals that the Λ-adic Galois representation $H^1_{\text{ét}}(\bar{X}^*_\infty, \mathbf{Z}_p)$ is rich enough to capture the Deligne representations attached to modular forms on X_1 of *arbitrary* weight $k \geq 2$.

For each $a \in 1 + p\mathbf{Z}_p$, the diamond operator $\langle a \rangle$ acts trivially on X_1 and as multiplication by $a^{k_\circ}$ on the stalks of the sheaves $\mathcal{H}^{k_\circ}_r$. It follows that the weight k specialisation map sp^*_k factors through the quotient $H^1_{\text{ét}}(\bar{X}^*_\infty, \mathbf{Z}_p) \otimes_{\Lambda, \nu_{k_\circ}} \mathbf{Z}_p$, i.e., one obtains a map

$$\mathrm{sp}^*_k : H^1_{\text{ét}}(\bar{X}^*_\infty, \mathbf{Z}_p) \otimes_{\Lambda, \nu_{k_\circ}} \mathbf{Z}_p \quad \longrightarrow \quad H^1_{\text{ét}}(\bar{X}_1, \mathcal{H}^{k_\circ}).$$

Remark 1.3. — *The inverse limit $\mathcal{L}^*_\infty$ of the sheaves $\underline{\mathcal{L}}^*_r$ on X_1 has been systematically studied by G. Kings in* [**K15**, §2.3-2.4], *and is referred to as a* sheaf of Iwasawa modules. *Jannsen introduced in* [**J88**] *the étale cohomology groups of such inverse systems of sheaves, and proved the existence of a Hoschild-Serre spectral sequence, Gysin excision exact sequences and cycle map in this context.*

Ordinary projections. Let

$$e^* := \lim_{n\to\infty} U_p^{*n!} \tag{1.23}$$

denote Hida's (anti-)ordinary projector. Since U^*_p commutes with the push-forward maps ϖ_{1*}, this idempotent operates on $H^1_{\text{ét}}(\bar{X}^*_\infty, \mathbf{Z}_p)$. While the structure of the Λ-module $H^1_{\text{ét}}(\bar{X}^*_\infty, \mathbf{Z}_p)$ seems rather complicated, a dramatic simplification occurs after passing to the quotient $e^* H^1_{\text{ét}}(\bar{X}^*_\infty, \mathbf{Z}_p)$, as the following classical theorem of Hida shows.

Theorem 1.2. — [**H86**, Corollaries 3.3 and 3.7] *The Galois representation $e^* H^1_{\text{ét}}(\bar{X}^*_\infty, \mathbf{Z}_p(1))$ is a free Λ-module. For each $\nu_{k_\circ} \in \mathcal{W}$ with $k_\circ \geq 0$, the weight $k = k_\circ + 2$ specialisation map induces maps with bounded cokernel (independent of k)*

$$\mathrm{sp}^*_k \quad : \quad e^* H^1_{\text{ét}}(\bar{X}^*_\infty, \mathbf{Z}_p(1)) \otimes_{\nu_{k_\circ}} \mathbf{Z}_p \longrightarrow e^* H^1_{\text{ét}}(\bar{X}_1, \mathcal{H}^{k_\circ}(1)).$$

Galois representations attached to Hida families. The Galois representation $\mathbb{V}_{\boldsymbol{\phi}}$ of Theorem 1.1 associated by Hida and Wiles to a Hida family $\boldsymbol{\phi}$ of tame level M and character χ can be realised as a quotient of the Λ-module $e^* H^1_{\text{ét}}(\bar{X}^*_\infty, \mathbf{Z}_p(1))$. More precisely, let

$$\xi_{\boldsymbol{\phi}} : \mathbb{T}_\Lambda \longrightarrow \Lambda_{\boldsymbol{\phi}}$$

be the Λ-algebra homomorphism from the Λ-adic Hecke algebra $\mathbb{T}_\Lambda$ to the Λ-algebra $\Lambda_{\boldsymbol{\phi}}$ generated by the fourier coefficients of $\boldsymbol{\phi}$ sending T_ℓ to $a_\ell(\boldsymbol{\phi})$.

Then we have, much as in (1.7), a quotient map of Λ-adic Galois representations

$$\varpi^*_{\boldsymbol{\phi}} : e^* H^1_{\text{ét}}(\bar{X}^*_\infty, \mathbf{Z}_p(1)) \longrightarrow e^* H^1_{\text{ét}}(\bar{X}^*_\infty, \mathbf{Z}_p(1)) \otimes_{\mathbb{T}_\Lambda, \xi_{\boldsymbol{\phi}}} \Lambda_\phi =: \mathbb{V}_{\boldsymbol{\phi}}(M), \tag{1.24}$$

for which the following diagram of $\mathbb{T}_\Lambda[G_{\mathbf{Q}}]$-modules is commutative:

$$\begin{array}{ccc} e^* H^1_{\text{ét}}(\bar{X}^*_\infty, \mathbf{Z}_p(1)) & \xrightarrow{\varpi^*_{\boldsymbol{\phi}}} & \mathbb{V}_{\boldsymbol{\phi}}(M) \\ \downarrow \text{sp}^*_k & & \downarrow x \\ e^* H^1_{\text{ét}}(\bar{X}_1, \mathcal{H}^{k_\circ}(1)) & \xrightarrow{\varpi_{\boldsymbol{\phi}_x}} & V_{\boldsymbol{\phi}_x}(Mp), \end{array} \tag{1.25}$$

for all arithmetic points x of $\mathcal{W}_{\boldsymbol{\phi}}$ of weight $k = k_\circ + 2$ and trivial character.

As in (1.7), $\mathbb{V}_{\boldsymbol{\phi}}(M)$ is non-canonically isomorphic to a finite direct sum of copies of a $\Lambda_{\boldsymbol{\phi}}[G_{\mathbf{Q}}]$-module $\mathbb{V}_{\boldsymbol{\phi}}$ of rank 2 over $\Lambda_{\boldsymbol{\phi}}$, satisfying the properties stated in Theorem 1.1.

One can of course work alternatively with the ordinary projection $e := \lim_{n\to\infty} U_p^{n!}$ rather than the anti-ordinary one, in which case one similarly constructs a quotient map of Λ-adic Galois representations

$$\varpi_{\boldsymbol{\phi}} : eH^1_{\text{ét}}(\bar{X}_\infty, \mathbf{Z}_p(1)) := e \varprojlim_{\varpi_{2*}} H^1_{\text{ét}}(\bar{X}_r, \mathbf{Z}_p(1)) \longrightarrow \mathbb{V}_{\boldsymbol{\phi}}(M). \tag{1.26}$$

1.4. Families of Dieudonné modules. — Let $\mathbf{B}_{\text{dR}}$ denote Fontaine's field of de Rham periods, $\mathbf{B}^+_{\text{dR}}$ be its ring of integers and $\log[\zeta_{p^\infty}]$ denote the uniformizer of $\mathbf{B}^+_{\text{dR}}$ associated to a norm-compatible system $\zeta_{p^\infty} = \{\zeta_{p^n}\}_{n\geq 0}$ of p^n-th roots of unity. (cf. e.g.[**BK93**, §1]). For any finite-dimensional de Rham Galois representation V of $G_{\mathbf{Q}_p}$ with coefficients in a finite extension $L_p/\mathbf{Q}_p$, define the de Rham Dieudonné module

$$D(V) = (V \otimes \mathbf{B}_{\text{dR}})^{G_{\mathbf{Q}_p}}.$$

It is an L_p-vector space of the same dimension as V, equipped with a descending exhaustive filtration $\text{Fil}^j D(V) = (V \otimes \log^j[\zeta_{p^\infty}]\mathbf{B}^+_{\text{dR}})^{G_{\mathbf{Q}_p}}$ by L_p-vector subspaces.

Let $\mathbf{B}_{\text{cris}} \subset \mathbf{B}_{\text{dR}}$ denote Fontaine's ring of crystalline p-adic periods. If V is crystalline (which is always the case if it arises as a subquotient of the étale cohomology of an algebraic variety with good reduction at p), then there is a canonical isomorphism

$$D(V) \simeq (V \otimes \mathbf{B}_{\text{cris}})^{G_{\mathbf{Q}_p}},$$

which furnishes $D(V)$ with a linear action of a Frobenius endomorphism Φ.

In [**BK93**] Bloch and Kato introduced a collection of subspaces of the local Galois cohomology group $H^1(\mathbf{Q}_p, V)$, denoted respectively

$$H^1_{\text{e}}(\mathbf{Q}_p, V) \subseteq H^1_{\text{f}}(\mathbf{Q}_p, V) \subseteq H^1_{\text{g}}(\mathbf{Q}_p, V) \subseteq H^1(\mathbf{Q}_p, V),$$

and constructed homomorphisms

$$\log_{\text{BK}} : H^1_{\text{e}}(\mathbf{Q}_p, V) \xrightarrow{\sim} D(V)/\big(\text{Fil}^0 D(V) + D(V)^{\Phi=1}\big) \tag{1.27}$$

and

$$\exp^*_{\text{BK}} : H^1(\mathbf{Q}_p, V)/H^1_{\text{g}}(\mathbf{Q}_p, V) \xrightarrow{\sim} \text{Fil}^0 D(V) \tag{1.28}$$

that are usually referred to as the Bloch-Kato logarithm and dual exponential map.

We illustrate the above Bloch-Kato homomorphisms with a few basic examples that shall be used several times in the remainder of this article.

Example 1.4. — *As shown e.g. in* [**BK93**], [**B09**, §2.2], *for any unramified character ψ of $G_{\mathbf{Q}_p}$ and all $n \in \mathbf{Z}$ we have:*

(a) If $n \geq 2$, or $n = 1$ and $\psi \neq 1$, then $H^1_{\mathrm{e}}(\mathbf{Q}_p, L_p(\psi\varepsilon^n_{\mathrm{cyc}})) = H^1(\mathbf{Q}_p, L_p(\psi\varepsilon^n_{\mathrm{cyc}}))$ is one-dimensional over L_p and the Bloch-Kato logarithm induces an isomorphism

$$\log_{\mathrm{BK}} : H^1(\mathbf{Q}_p, L_p(\psi\varepsilon^n_{\mathrm{cyc}})) \xrightarrow{\sim} D(L_p(\psi\varepsilon^n_{\mathrm{cyc}})).$$

(b) If $n < 0$, or $n = 0$ and $\psi \neq 1$, then $H^1_{\mathrm{g}}(\mathbf{Q}_p, L_p(\psi\varepsilon^n_{\mathrm{cyc}})) = 0$ and $H^1(\mathbf{Q}_p, L_p(\psi\varepsilon^n_{\mathrm{cyc}}))$ is one-dimensional. The dual exponential gives rise to an isomorphism

$$\exp^*_{\mathrm{BK}} : H^1(\mathbf{Q}_p, L_p(\psi\varepsilon^n_{\mathrm{cyc}})) \xrightarrow{\sim} \mathrm{Fil}^0 D(L_p(\psi\varepsilon^n_{\mathrm{cyc}})) = D(L_p(\psi\varepsilon^n_{\mathrm{cyc}})).$$

(c) Assume $\psi = 1$. If $n = 0$, then $H^1(\mathbf{Q}_p, L_p)$ has dimension 2 over L_p, $H^1_{\mathrm{f}}(\mathbf{Q}_p, L_p) = H^1_{\mathrm{g}}(\mathbf{Q}_p, L_p)$ has dimension 1 and $H^1_{\mathrm{e}}(\mathbf{Q}_p, L_p)$ has dimension 0 over L_p. The Bloch-Kato dual exponential map induces an isomorphism

$$\exp^*_{\mathrm{BK}} : H^1(\mathbf{Q}_p, L_p)/H^1_{\mathrm{f}}(\mathbf{Q}_p, L_p) \xrightarrow{\sim} \mathrm{Fil}^0 D(L_p) = D(L_p) = L_p.$$

Class field theory identifies $H^1(\mathbf{Q}_p, L_p)$ with $\mathrm{Hom}_{\mathrm{cont}}(\mathbf{Q}_p^\times, \mathbf{Q}_p) \otimes L_p$, which is spanned by the homomorphisms ord_p and $\log_p$.

If $n = 1$, then $H^1(\mathbf{Q}_p, L_p(1)) = H^1_{\mathrm{g}}(\mathbf{Q}_p, L_p(1))$ is 2-dimensional and $H^1_{\mathrm{f}}(\mathbf{Q}_p, L_p(1)) = H^1_{\mathrm{e}}(\mathbf{Q}_p, L_p(1))$ has dimension 1 over L_p. As proved e.g. in [**B09**, Prop. 2.9], *Kummer theory identifies the spaces $H^1_{\mathrm{f}}(\mathbf{Q}_p, L_p(1)) \subset H^1(\mathbf{Q}_p, L_p(1))$ with $\mathbf{Z}_p^\times \hat{\otimes} L_p$ sitting inside $\mathbf{Q}_p^\times \hat{\otimes} L_p$. Under this identification, the Bloch-Kato logarithm is the usual p-adic logarithm on $\mathbf{Z}_p^\times$.*

Let $\hat{\mathbf{Z}}_p^{\mathrm{nr}}$ denote the ring of integers of the completion of the maximal unramified extension of $\mathbf{Q}_p$. If V is unramified then there is a further canonical isomorphism

$$D(V) \simeq (V \otimes \hat{\mathbf{Z}}_p^{\mathrm{nr}})^{G_{\mathbf{Q}_p}}. \tag{1.29}$$

Let ϕ be an eigenform (with respect to the good Hecke operators) of weight $k = k_\circ + 2 \geq 2$, level M and character χ, with fourier coefficients in a finite extension L_p of $\mathbf{Q}_p$. The comparison theorem [**F97**] of Faltings-Tsuji combined with (1.6) asserts that there is a natural isomorphism

$$D(V_\phi(M)) \simeq H^1_{\mathrm{dR}}(X_1(M), \mathcal{H}^{k_\circ}(1))[\phi]$$

of Dieudonné modules over L_p. Note that $D(V_\phi(M))$ is the direct sum of several copies of the two-dimensional Dieudonné module $D(V_\phi)$.

Assume that $p \nmid M$ and ϕ is ordinary at p. Then $V_\phi(M)$ is crystalline and Φ acts on $D(V_\phi(M))$ as

$$\Phi = \chi(p)p^{k_\circ+1}U_p^{-1}. \tag{1.30}$$

In particular the eigenvalues of Φ on $D(V_\phi(M))$ are $\chi(p)p^{k_\circ+1}\alpha_\phi^{-1} = \beta_\phi$ and $\chi(p)p^{k_\circ+1}\beta_\phi^{-1} = \alpha_\phi$, the two roots of the Hecke polynomial of ϕ at p. For future reference, recall from [DR14, Theorem 1.3] the Euler factors

$$(1.31)\qquad \mathcal{E}_0(\phi) := 1 - \chi^{-1}(p)\beta_\phi^2 p^{1-k} = 1 - \frac{\beta_{\boldsymbol{\phi}_x^\circ}}{\alpha_{\boldsymbol{\phi}_x^\circ}}, \quad \mathcal{E}_1(\phi) := 1 - \chi(p)\alpha_\phi^{-2}p^{k-2}.$$

Let $\phi^* = \phi \otimes \bar\chi \in S_k(M, \bar\chi)$ denote the twist of ϕ by the inverse of its nebentype character. Poincaré duality induces a perfect pairing

$$\langle\,,\rangle : D(V_\phi(M)) \times D(V_{\phi^*}(M)) \longrightarrow D(L_p) = L_p.$$

The exact sequence (1.8) induces in this setting an exact sequence of Dieudonné modules

$$(1.32)\qquad 0 \longrightarrow D(V_\phi^+(M)) \xrightarrow{i} D(V_\phi(M)) \xrightarrow{\pi} D(V_\phi^-(M)) \longrightarrow 0.$$

Since $V_\phi^-(M$ is unramified, we have $D(V_\phi^-(M)) \simeq (V_\phi^-(M) \otimes \hat{\mathbf{Z}}_p^{\mathrm{nr}})^{G_{\mathbf{Q}_p}}$. This submodule may also be characterized as the eigenspace $D(V_\phi^-(M)) = D(V_\phi(M))^{\Phi=\alpha_\phi}$ of eigenvalue α_ϕ for the action of frobenius.

The rule $\breve\phi \mapsto \omega_{\breve\phi}$ that attaches to a modular form its associated differential form gives rise to an isomorphism $S_k(M,\chi)_{L_p}[\phi] \xrightarrow{\sim} \mathrm{Fil}^0(D(V_\phi(M))) \subset D(V_\phi(M))$. Moreover, the map π of (1.32) induces an isomorphism

$$(1.33)\qquad S_k(M,\chi)_{L_p}[\phi] \xrightarrow{\sim} \mathrm{Fil}^0(D(V_\phi(M))) \xrightarrow{\pi} D(V_\phi^-(M)).$$

Any element $\omega \in D(V_{\phi^*}^-(M))$ gives rise to a linear map

$$\omega : D(V_\phi^+(M)) \longrightarrow L_p, \quad \eta \mapsto \langle \eta, \pi^{-1}(\omega)\rangle.$$

Similarly, any $\eta \in D(V_{\phi^*}^+(M))$ may be identified with a linear functional

$$\eta : D(V_\phi^-(M)) \longrightarrow L_p, \quad \omega \mapsto \langle \pi^{-1}(\omega), \eta\rangle,$$

and given $\breve\phi \in S_k(M,\chi)_{L_p}[\phi]$ we set $\eta_{\breve\phi} : D(V_{\phi^*}^-(M)) \to L_p$, $\varphi \mapsto \eta_{\breve\phi}(\varphi) = \frac{\langle\breve\phi,\varphi\rangle}{\langle\breve\phi,\breve\phi^*\rangle}$.

Let now $\tilde\Lambda$ be a finite flat extension of the Iwasawa algebra Λ and let $\mathbb{U}$ denote a free $\tilde\Lambda$-module of finite rank equipped with an *unramified* $\tilde\Lambda$-linear action of $G_{\mathbf{Q}_p}$. Define the Λ-adic Dieudonné module

$$\mathbb{D}(\mathbb{U}) := (\mathbb{U}\hat\otimes\hat{\mathbf{Z}}_p^{\mathrm{nr}})^{G_{\mathbf{Q}_p}}.$$

As shown in e.g. [O03, Lemma 3.3], $\mathbb{D}(\mathbb{U})$ is a free module over $\tilde\Lambda$ of the same rank as $\mathbb{U}$.

Examples of such Λ-adic Dieudonné modules arise naturally in the context of families of modular forms thanks to Theorem 1.1. Indeed, let $\boldsymbol{\phi}$ be a Hida family of tame level M and character χ, and let $\boldsymbol{\phi}^*$ denote the Λ-adic modular form obtained by twisting $\boldsymbol{\phi}$ by $\bar\chi$.

Let $\mathbb{V}_{\boldsymbol{\phi}}$ and $\mathbb{V}_{\boldsymbol{\phi}}(M)$ denote the global Λ-adic Galois representations described in (1.24). It follows from (1.12) that to the restriction of $\mathbb{V}_{\boldsymbol{\phi}}$ to $G_{\mathbf{Q}_p}$ one might associate two natural unramified $\Lambda[G_{\mathbf{Q}_p}]$-modules of rank one, namely

$$\mathbb{V}_{\boldsymbol{\phi}}^- \simeq \Lambda_{\boldsymbol{\phi}}(\psi_{\boldsymbol{\phi}}) \quad \text{and} \quad \mathbb{U}_{\boldsymbol{\phi}}^+ = \mathbb{V}_{\boldsymbol{\phi}}^+(\chi^{-1}\varepsilon_{\mathrm{cyc}}\underline{\varepsilon}_{\mathrm{cyc}}^{-1}).$$

Define similarly the unramified modules $\mathbb{V}_{\boldsymbol{\phi}}^-(M)$ and $\mathbb{U}_{\boldsymbol{\phi}}^+(M)$.

Let

$$S^{\mathrm{ord}}_\Lambda(M,\chi)[\boldsymbol{\phi}] := \left\{ \breve{\boldsymbol{\phi}} \in S^{\mathrm{ord}}_\Lambda(M,\chi) \quad \text{s.t.} \quad \middle| \begin{array}{l} T_\ell \breve{\boldsymbol{\phi}} = a_\ell(\boldsymbol{\phi})\breve{\boldsymbol{\phi}}, \quad \forall \ell \nmid Mp, \\ U_p \breve{\boldsymbol{\phi}} = a_p(\boldsymbol{\phi})\breve{\boldsymbol{\phi}} \end{array} \right\}, \tag{1.34}$$

For any crystalline arithmetic point $x \in \mathcal{W}^\circ_{\boldsymbol{\phi}}$ of weight k, the specialization of a Λ-adic test vector $\breve{\phi} \in S^{\mathrm{ord}}_\Lambda(M,\chi)[\boldsymbol{\phi}]$ at x is a classical eigenform $\breve{\phi}_x \in S_k(Mp,\chi)$ with coefficients in $L_p = x(\Lambda_{\boldsymbol{\phi}}) \otimes \mathbf{Q}_p$ and the same eigenvalues as ϕ_x for the good Hecke operators.

Likewise, define

$$S^{\mathrm{ord}}_\Lambda(M,\bar\chi)^\vee[\boldsymbol{\phi}] = \left\{ \eta : S^{\mathrm{ord}}_\Lambda(M,\bar\chi) \to \Lambda_{\boldsymbol{\phi}} \;\middle|\; \begin{array}{l} \eta \circ T^*_\ell = a_\ell(\boldsymbol{\phi})\eta, \quad \forall \ell \nmid Mp, \\ \eta \circ U^*_p = a_p(\boldsymbol{\phi})\eta \end{array} \right\}$$

Let $\mathcal{Q}_{\boldsymbol{\phi}}$ denote the field of fractions of $\Lambda_{\boldsymbol{\phi}}$. Associated to any test vector $\breve{\boldsymbol{\phi}} \in S^{\mathrm{ord}}_\Lambda(M,\chi)[\boldsymbol{\phi}]$, [**DR14**, Lemma 2.19] describes a $\mathcal{Q}_{\boldsymbol{\phi}}$-linear dual test vector

$$\breve{\boldsymbol{\phi}}^\vee \in S^{\mathrm{ord}}_\Lambda(M,\bar\chi)^\vee[\boldsymbol{\phi}] \,\hat\otimes\, \mathcal{Q}_{\boldsymbol{\phi}} \tag{1.35}$$

such that for any $\boldsymbol{\varphi} \in S^{\mathrm{ord}}_\Lambda(M,\bar\chi)$ and any point $x \in \mathcal{W}^\circ_{\mathbf{f}}$,

$$x(\breve{\boldsymbol{\phi}}^\vee(\boldsymbol{\varphi})) = \frac{\langle \breve{\boldsymbol{\phi}}_x, \boldsymbol{\varphi}_x \rangle}{\langle \breve{\boldsymbol{\phi}}_x, \breve{\boldsymbol{\phi}}^*_x \rangle}$$

where $\langle\,,\rangle$ denotes the pairing induced by Poincaré duality on the modular curve associated to the congruence subgroup $\Gamma_1(M) \cap \Gamma_0(p)$. This way, the specialization of a Λ-adic dual test vector $\breve{\boldsymbol{\phi}}^\vee \in S^{\mathrm{ord}}_\Lambda(M,\bar\chi)^\vee[\boldsymbol{\phi}]$ at x gives rise to a linear functional

$$\eta_{\breve{\boldsymbol{\phi}}_x} : S_k(Mp,\bar\chi)[\phi^*_x] \longrightarrow L_p.$$

A natural $\mathcal{Q}_{\mathbf{f}}$-basis of $S^{\mathrm{ord}}_\Lambda(M,\chi)[\boldsymbol{\phi}] \,\hat\otimes\, \mathcal{Q}_{\boldsymbol{\phi}}$ is given by the Λ-adic modular forms $\boldsymbol{\phi}(q^d)$ as d ranges over the positive divisors of $M/M_{\boldsymbol{\phi}}$ and it is also obvious that $\{\boldsymbol{\phi}(q^d)^\vee : d \mid \frac{M}{M_{\boldsymbol{\phi}}}\}$ provides a $\mathcal{Q}_{\boldsymbol{\phi}}$-basis of $S^{\mathrm{ord}}_\Lambda(M,\bar\chi)^\vee[\boldsymbol{\phi}] \,\hat\otimes\, \mathcal{Q}_{\boldsymbol{\phi}}$.

The following statement shows that the linear maps described above can be made to vary in families.

Proposition 1.5. — *For any Λ-adic test vector $\breve{\boldsymbol{\phi}} \in S^{\mathrm{ord}}_\Lambda(M,\chi)[\boldsymbol{\phi}]$ there exist homomorphisms of $\Lambda_{\boldsymbol{\phi}}$-modules*

$$\omega_{\breve{\boldsymbol{\phi}}} : D(\mathbb{U}^+_{\boldsymbol{\phi}^*}(M)) \longrightarrow \Lambda_{\boldsymbol{\phi}}, \quad \eta_{\breve{\boldsymbol{\phi}}} : D(\mathbb{V}^-_{\boldsymbol{\phi}^*}(M)) \longrightarrow \mathcal{Q}_{\boldsymbol{\phi}},$$

whose specialization at a classical point $x \in \mathcal{W}^\circ_{\boldsymbol{\phi}}$ such that ϕ_x is the ordinary stabilization of an eigenform $\boldsymbol{\phi}^\circ_x$ of level M are, respectively

1. $x \circ \omega_{\breve{\boldsymbol{\phi}}} = \mathcal{E}_0(\boldsymbol{\phi}^\circ_x)\, e\, \varpi^*_1(\omega_{\breve{\boldsymbol{\phi}}^\circ_x})$ *as functionals on* $D(U^+_{\boldsymbol{\phi}^*_x}(Mp))$.
2. $x \circ \eta_{\breve{\boldsymbol{\phi}}} = \frac{1}{\mathcal{E}_1(\boldsymbol{\phi}^\circ_x)} \cdot e\, \varpi^*_1(\eta_{\breve{\boldsymbol{\phi}}^\circ_x})$ *as functionals on* $D(V^-_{\boldsymbol{\phi}^*_x}(Mp))$.

Proof. — This is essentially a reformulation of [**KLZ17**, Propositions 10.1.1 and 10.1.2], which in turn builds on [**O00**]. Namely, the first claim in Prop. 10.1.2 of loc. cit. asserts that $\omega_{\breve{\boldsymbol{\phi}}}$ exists such that at any $x \in \mathcal{W}^\circ_{\boldsymbol{\phi}}$ as above, $x \circ \omega_{\breve{\boldsymbol{\phi}}} = \omega_{\breve{\boldsymbol{\phi}}_x} =$

$\mathrm{Pr}^{\alpha*}(\omega_{\breve{\phi}_x^\circ})$ where $\mathrm{Pr}^{\alpha*}$ is the map defined in [**KLZ17**, 10.1.3] sending $\breve{\phi}_x^\circ$ to its ordinary p-stablilization $\breve{\phi}_x$. Note that $\varpi_1^*(\phi_x^\circ) = \frac{\alpha_{\phi_x^\circ}\breve{\phi}_x}{\alpha_{\phi_x^\circ}-\beta_{\phi_x^\circ}} - \frac{\beta_{\phi_x^\circ}\breve{\phi}'_x}{\alpha_{\phi_x^\circ}-\beta_{\phi_x^\circ}}$, where $\breve{\phi}'_x$ denotes the non-ordinary specialization of $\breve{\phi}_x^\circ$. Since $e\omega_{\breve{\phi}'_x} = 0$ and $\mathcal{E}_0(\boldsymbol{\phi}_x^\circ) = \frac{\alpha_{\phi_x^\circ}-\beta_{\phi_x^\circ}}{\alpha_{\phi_x^\circ}}$ the claim follows.

The second part of [**KLZ17**, Proposition 10.1.2] asserts that there exists a Λ-adic functional $\tilde{\eta}_{\breve{\boldsymbol{\phi}}}$ such that for all x as above:

$$x \circ \tilde{\eta}_{\breve{\boldsymbol{\phi}}} = \frac{\mathrm{Pr}^{\alpha*}\eta_{\breve{\boldsymbol{\phi}}_x^\circ}}{\lambda(\boldsymbol{\phi}_x^\circ)\mathcal{E}_0(\boldsymbol{\phi}_x^\circ)\mathcal{E}_1(\boldsymbol{\phi}_x^\circ)}$$

as L_p-linear functionals on $D(V^-_{\boldsymbol{\phi}_x^*}(Mp))$. Here $\lambda(\boldsymbol{\phi}_x^\circ) \in \bar{\mathbf{Q}}^\times$ denotes the pseudo-eigenvalue of $\boldsymbol{\phi}_x^\circ$, which we recall is the scalar given by

$$W_M(\boldsymbol{\phi}_x^\circ) = \lambda(\boldsymbol{\phi}_x^\circ)\cdot\boldsymbol{\phi}_x^{\circ*}, \tag{1.36}$$

where $W_M : S_k(M,\chi) \to S_k(M,\chi^{-1})$ stands for the Atkin-Lehner operator. Since we are assuming that $\Lambda_{\boldsymbol{\phi}}$ contains the M-th roots of unity (cf. the remark right after Definition 1.1), Prop. 10.1.1 of loc. cit. shows that there exists an element $\lambda(\boldsymbol{\phi}) \in \Lambda_{\boldsymbol{\phi}}$ interpolating the pseudo-eigenvalues of the classical p-stabilized specializations of $\boldsymbol{\phi}$. The claim follows by taking $\eta_{\breve{\boldsymbol{\phi}}} = \lambda(\boldsymbol{\phi})\tilde{\eta}_{\breve{\boldsymbol{\phi}}}$. The same argument as above yields that for all x as above, $x \circ \eta_{\breve{\boldsymbol{\phi}}} = \mathcal{E}_0(\boldsymbol{\phi}_x^\circ)\frac{e\varpi_1^*\eta_{\breve{\boldsymbol{\phi}}_x^\circ}}{\mathcal{E}_0(\boldsymbol{\phi}_x^\circ)\mathcal{E}_1(\boldsymbol{\phi}_x^\circ)}$, which amounts to the statement of the proposition. □

2. Generalised Kato classes

2.1. A compatible collection of cycles. — This section defines a collection of codimension two cycles in $X_1(Mp^r)^3$ indexed by elements of $(\mathbf{Z}/p^r\mathbf{Z})^{\times 3}$ and records some of their properties.

We retain the notations that were in force in Section 1.3 regarding the meanings of the curves $X = X_1(M)$, $X_r = X_1(Mp^r)$ and $X_{r,s}$. In addition, let

$$\mathbb{Y}(p^r) := Y \times_{X(1)} Y(p^r), \quad \mathbb{X}(p^r) := X \times_{X(1)} X(p^r)$$

denote the (affine and projective, respectively) modular curve over $\mathbf{Q}(\zeta_r)$ with full level p^r structure. The curve $\mathbb{Y}(p^r)$ classifies triples (A, P, Q) in which A is an elliptic curve with $\Gamma_1(M)$ level structure and (P, Q) is a basis for $A[p^r]$ satisfying $\langle P, Q\rangle = \zeta_r$, where $\langle\ ,\ \rangle$ denotes the Weil pairing and ζ_r is a fixed primitive p^r-th root of unity. The curve $\mathbb{X}(p^r)$ is geometrically connected but does not descend to a curve over $\mathbf{Q}$, as can be seen by noting that the description of its moduli problem depends on the choice of ζ_r. The covering $\mathbb{X}(p^r)/X$ is Galois with Galois group $\mathrm{SL}_2(\mathbf{Z}/p^r\mathbf{Z})$, acting on the left by the rule

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}(A, P, Q) = (A, aP + bQ, cP + dQ). \tag{2.1}$$

Consider the natural projection map

$$\varpi_1^r \times \varpi_1^r \times \varpi_1^r : X_r^3 \longrightarrow X^3 \tag{2.2}$$

induced on triple products by the map ϖ_1^r of (1.14). Write $\Delta \subset X^3$ for the usual diagonal cycle, namely the image of X under the diagonal embedding $x \mapsto (x,x,x)$. Let Δ_r be the fiber product $\Delta \times_{X^3} X_r^3$ via the natural inclusion and the map of (2.2), which fits into the cartesian diagram

$$\begin{array}{ccc} \Delta_r & \hookrightarrow & X_r^3 \\ \downarrow & & \downarrow \\ \Delta & \hookrightarrow & X^3. \end{array}$$

An element of a $\mathbf{Z}_p$-module Ω is said to be *primitive* if it does not belong to $p\Omega$, and the set of such primitive elements is denoted Ω'. Let

$$\Sigma_r := ((\mathbf{Z}/p^r\mathbf{Z} \times \mathbf{Z}/p^r\mathbf{Z})')^3 \subset ((\mathbf{Z}/p^r\mathbf{Z})^2)^3$$

be the set of triples of primitive row vectors of length 2 with entries in $\mathbf{Z}/p^r\mathbf{Z}$, equipped with the action of $\mathrm{GL}_2(\mathbf{Z}/p^r\mathbf{Z})$ acting diagonally by right multiplication.

Lemma 2.1. — *The geometrically irreducible components of Δ_r are defined over $\mathbf{Q}(\zeta_r)$ and are in canonical bijection with the set of left orbits*

$$\Sigma_r/\mathrm{SL}_2(\mathbf{Z}/p^r\mathbf{Z}).$$

Proof. — Each triple

$$(v_1, v_2, v_3) = ((x_1, y_1), (x_2, y_2), (x_3, y_3)) \in \Sigma_r$$

determines a morphism

$$\varphi_{(v_1,v_2,v_3)} : \mathbb{X}(p^r) \longrightarrow \Delta_r \subset X_r^3$$

of curves over $\mathbf{Q}(\zeta_r)$, defined in terms of the moduli descriptions on $\mathbb{Y}(p^r)$ by

$$(A, P, Q) \quad \mapsto \quad (\ (A, x_1P + y_1Q),\ (A, x_2P + y_2Q),\ (A, x_3P + y_3Q)\ ).$$

It is easy to see that if two elements (v_1, v_2, v_3) and $(v_1', v_2', v_3') \in \Sigma_r$ satisfy

$$(v_1', v_2', v_3') = (v_1, v_2, v_3)\gamma, \qquad \text{with } \gamma \in \mathrm{SL}_2(\mathbf{Z}/p^r\mathbf{Z}),$$

then

$$\varphi_{(v_1',v_2',v_3')} = \varphi_{(v_1,v_2,v_3)} \circ \gamma,$$

where γ is being viewed as an automorphism of $\mathbb{X}(p^r)$ as in (2.1). It follows that the geometrically irreducible cycle

$$\Delta_r(v_1, v_2, v_3) := \varphi_{(v_1,v_2,v_3)*}(\mathbb{X}(p^r))$$

depends only on the $\mathrm{SL}_2(\mathbf{Z}/p^r\mathbf{Z})$-orbit of (v_1, v_2, v_3).

Since $\mathrm{SL}_2(\mathbf{Z}/p^r\mathbf{Z})$ acts transitively on $(\mathbf{Z}/p^r\mathbf{Z} \times \mathbf{Z}/p^r\mathbf{Z})'$, one further checks that the collection of cycles $\Delta_r(v_1, v_2, v_3)$ for $(v_1, v_2, v_3) \in \Sigma_r/\mathrm{SL}_2(\mathbf{Z}/p^r\mathbf{Z})$ do not overlap on Y_r^3 and cover Δ_r. Hence the irreducible components of Δ_r are precisely $\Delta_r(v_1, v_2, v_3)$ for $(v_1, v_2, v_3) \in \Sigma_r/\mathrm{SL}_2(\mathbf{Z}/p^r\mathbf{Z})$. $\square$

The quotient $\Sigma_r/\mathrm{SL}_2(\mathbf{Z}/p^r\mathbf{Z})$ is equipped with a natural determinant map

$$D : \Sigma_r/\mathrm{SL}_2(\mathbf{Z}/p^r\mathbf{Z}) \longrightarrow (\mathbf{Z}/p^r\mathbf{Z})^3$$

defined by

$$D\left((x_1 y_1), (x_2, y_2), (x_3, y_3)\right) := \left(\begin{vmatrix} x_2 & y_2 \\ x_3 & y_3 \end{vmatrix}, \begin{vmatrix} x_3 & y_3 \\ x_1 & y_1 \end{vmatrix}, \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} \right).$$

For each $[d_1, d_2, d_3] \in (\mathbf{Z}/p^r\mathbf{Z})^3$, we can then write

$$\Sigma_r[d_1, d_2, d_3] := \{(v_1, v_2, v_3) \in \Sigma_r \text{ with } D(v_1, v_2, v_3) = (d_1, d_2, d_3)\}.$$

The group $\mathrm{SL}_2(\mathbf{Z}/p^r\mathbf{Z})$ operates *simply transitively* on $\Sigma_r[d_1, d_2, d_3]$ if (and only if)

$$[d_1, d_2, d_3] \in I_r := (\mathbf{Z}/p^r\mathbf{Z})^{\times 3}. \tag{2.3}$$

In particular, if (v_1, v_2, v_3) belongs to $\Sigma_r[d_1, d_2, d_3]$, then the cycle $\Delta_r(v_1, v_2, v_3)$ depends only on $[d_1, d_2, d_3] \in I_r$ and will henceforth be denoted

$$\Delta_r[d_1, d_2, d_3] \in \mathrm{CH}^2(X_r^3). \tag{2.4}$$

A somewhat more intrinsic definition of $\Delta_r[d_1, d_2, d_3]$ as a curve embedded in X_r^3 is that it corresponds to the schematic closure of the locus of points $((A, P_1), (A, P_2), (A, P_3))$ satisfying

$$\langle P_2, P_3\rangle = \zeta_r^{d_1}, \qquad \langle P_3, P_1\rangle = \zeta_r^{d_2}, \qquad \langle P_1, P_2\rangle = \zeta_r^{d_3}. \tag{2.5}$$

This description also makes it apparent that the cycle $\Delta_r[d_1, d_2, d_3]$ is defined over $\mathbf{Q}(\zeta_r)$ but not over $\mathbf{Q}$. Let $\sigma_m \in \mathrm{Gal}\,(\mathbf{Q}(\zeta_r)/\mathbf{Q})$ be the automorphism associated to $m \in (\mathbf{Z}/p^r\mathbf{Z})^\times$, sending ζ_r to ζ_r^m. The threefold X_r^3 is also equipped with an action of the group

$$\tilde{G}_r := ((\mathbf{Z}/p^r\mathbf{Z})^\times)^3 = \{\langle a_1, a_2, a_3\rangle, \quad a_1, a_2, a_3 \in (\mathbf{Z}/p^r\mathbf{Z})^\times\} \tag{2.6}$$

of diamond operators, where the automorphism associated to a triple $(\langle a_1\rangle, \langle a_2\rangle, \langle a_3\rangle)$ has simply been denoted $\langle a_1, a_2, a_3\rangle$.

Lemma 2.2. — *For all diamond operators $\langle a_1, a_2, a_3\rangle \in \tilde{G}_r$ and all $[d_1, d_2, d_3] \in I_r$,*

$$\langle a_1, a_2, a_3\rangle \Delta_r[d_1, d_2, d_3] = \Delta_r[a_2 a_3 \cdot d_1, a_1 a_3 \cdot d_2, a_1 a_2 \cdot d_3]. \tag{2.7}$$

For all $\sigma_m \in \mathrm{Gal}\,(\mathbf{Q}(\zeta_r)/\mathbf{Q})$,

$$\sigma_m \Delta_r[d_1, d_2, d_3] = \Delta_r[m \cdot d_1, m \cdot d_2, m \cdot d_3]. \tag{2.8}$$

Proof. — Equation (2.7) follows directly from the identity

$$D(a_1 v_1, a_2 v_2, a_3 v_3) = [a_2 a_3, a_1 a_3, a_1 a_2] D(v_1, v_2, v_3).$$

The first equality in (2.8) is most readily seen from the equation (2.5) defining the cycle $\Delta_r[d_1, d_2, d_3]$, since applying the automorphism $\sigma_m \in \mathrm{Gal}\,(\mathbf{Q}(\zeta_r)/\mathbf{Q})$ has the effect of replacing ζ_r by ζ_r^m. □

Remark 2.3. — *Assume m is a quadratic residue in $(\mathbf{Z}/p^r\mathbf{Z})^\times$, which is the case, for instance, when σ_m belongs to $\mathrm{Gal}\,(\mathbf{Q}(\zeta_r)/\mathbf{Q}(\zeta_1))$. Then it follows from (2.7) and (2.8) that*

$$\sigma_m \Delta_r[d_1, d_2, d_3] = \langle m, m, m\rangle^{1/2} \Delta_r[d_1, d_2, d_3]. \tag{2.9}$$

Let us now turn to the compatibility properties of the cycles $\Delta_r[d_1, d_2, d_3]$ as the level r varies. Recall the modular curve $X_{r,r+1}$ classifying generalised elliptic curves together with a distinguished cyclic subgroup of order p^{r+1} and a point of order p^r in it. The maps μ, ϖ_1, π_1, ϖ_2 and π_2 of (1.15) induce similar maps on the triple products:

$$(2.10) \qquad \begin{array}{ccc} X_{r+1}^3 & & \\ \downarrow{\scriptstyle \mu^3} & \searrow^{\varpi_1^3} & \\ X_{r,r+1}^3 & \xrightarrow[\pi_1^3]{} & X_r^3, \end{array} \qquad \begin{array}{ccc} X_{r+1}^3 & & \\ \downarrow{\scriptstyle \mu^3} & \searrow^{\varpi_2^3} & \\ X_{r,r+1}^3 & \xrightarrow[\pi_2^3]{} & X_r^3. \end{array}$$

A finite morphism $j : V_1 \longrightarrow V_2$ of varieties induces maps

$$j_* : \mathrm{CH}^j(V_1) \longrightarrow \mathrm{CH}^j(V_2), \qquad j^* : \mathrm{CH}^j(V_2) \longrightarrow \mathrm{CH}^j(V_1)$$

between Chow groups, and j_*j^* agrees with the multiplication by $\deg(j)$ on $\mathrm{CH}^j(V_2)$. If j is a Galois cover with Galois group G,

$$(2.11) \qquad j^*j_*(\Delta) = \sum_{\sigma \in G} \sigma\Delta.$$

By abuse of notation we will denote the associated maps on cycles (rather than just on cycle classes) by the same symbols.

Lemma 2.4. — *For all $r \geq 1$ and all $[d_1', d_2', d_3'] \in I_{r+1}$ whose image in I_r is $[d_1, d_2, d_3]$,*

$$\begin{aligned} (\varpi_1^3)_*\Delta_{r+1}[d_1', d_2', d_3'] &= p^3\Delta_r[d_1, d_2, d_3], \\ (\varpi_2^3)_*\Delta_{r+1}[d_1', d_2', d_3'] &= (U_p)^{\otimes 3}\Delta_r[d_1, d_2, d_3]. \end{aligned}$$

The cycles $\Delta_r[d_1, d_2, d_3]$ also satisfy the distribution relations

$$\sum_{[d_1', d_2', d_3']} \Delta_{r+1}[d_1', d_2', d_3'] = (\varpi_1^3)^*\Delta_r[d_1, d_2, d_3],$$

where the sum is taken over all triples $[d_1', d_2', d_3'] \in I_{r+1}$ which map to $[d_1, d_2, d_3]$ in I_r.

Proof. — A direct verification based on the definitions shows that the morphisms μ^3 and π_1^3 of (2.10) induce morphisms

$$\Delta_{r+1}[d_1'd_2', d_3'] \xrightarrow{\mu^3} \mu^3_*\Delta_{r+1}[d_1', d_2', d_3'] \xrightarrow{\pi_1^3} \Delta_r[d_1, d_2, d_3],$$

of degrees 1 and p^3 respectively. Hence the restriction of ϖ_1^3 to $\Delta_{r+1}[d_1', d_2', d_3']$ induces a map of degree p^3 from $\Delta_{r+1}[d_1', d_2', d_3']$ to $\Delta_r[d_1, d_2, d_3]$, which implies the first assertion. It also follows from this that

$$(2.12) \qquad \mu^3_*\Delta_{r+1}[d_1', d_2', d_3'] = (\pi_1^3)^*\Delta_r[d_1, d_2, d_3].$$

Applying $(\pi_2^3)_*$ to this identity implies that

$$(\varpi_2^3)_*\Delta_{r+1}[d_1', d_2', d_3'] = (U_p)^{\otimes 3}\Delta_r[d_1, d_2, d_3].$$

The second compatibility relation follows. To prove the distribution relation, observe that the sum that occurs in it is taken over the p^3 translates of a fixed $\Delta_{r+1}[d_1', d_2', d_3']$ for the action of the Galois group of X_{r+1}^3 over $X_{r,r+1}^3$, and hence, by (2.11), that

$$\sum_{[d_1',d_2',d_3']} \Delta_{r+1}[d_1', d_2', d_3'] = (\mu^*)^3 \mu_*^3 \Delta_{r+1}[d_1', d_2', d_3'].$$

The result then follows from (2.12). □

2.2. Galois cohomology classes. — The goal of this section is to parlay the cycles $\Delta_r[d_1, d_2, d_3]$ into Galois cohomology classes with values in $H^1_{\text{ét}}(\bar{X}_r, \mathbf{Z}_p)^{\otimes 3}(2)$, essentially by considering their images under the p-adic étale Abel-Jacobi map:

$$\text{AJ}_{\text{ét}} : \text{CH}^2(X_r^3)_0 \longrightarrow H^1(\mathbf{Q}, H^3_{\text{ét}}(\bar{X}_r^3, \mathbf{Z}_p(2))), \tag{2.13}$$

where

$$\text{CH}^2(X_r^3)_0 := \ker\left(\text{CH}^2(X_r^3) \longrightarrow H^4_{\text{ét}}(\bar{X}_r^3, \mathbf{Z}_p(2))\right)$$

denotes the kernel of the étale cycle class map, i.e., the group of null-homologous algebraic cycles defined over $\mathbf{Q}$. There are two issues that need to be dealt with. Firstly, the cycles $\Delta_r[d_1, d_2, d_2]$ need not be null-homologous and have to be suitably modified so that they lie in the domain of the Abel Jacobi map. Secondly, these cycles are defined over $\mathbf{Q}(\zeta_r)$ and not over $\mathbf{Q}$, and it is desirable to descend the field of definition of the associated extension classes.

To deal with the first issue, let q be any prime not dividing Mp, and let T_q denote the Hecke operator attached to this prime. It can be used to construct an algebraic correspondence on X_r^3 by setting

$$\theta_q := (T_q - (q+1))^{\otimes 3}.$$

Lemma 2.5. — *The element θ_q annihilates the target $H^4_{\text{ét}}(\bar{X}_r^3, \mathbf{Z}_p)$ of the étale cycle class map on $\text{CH}^2(X_r^3)$.*

Proof. — The correspondence T_q acts as multiplication by $(q+1)$ on $H^2_{\text{ét}}(\bar{X}_r, \mathbf{Z}_p)$ and θ_q therefore annihilates all the terms in the Künneth decomposition of $H^4_{\text{ét}}(\bar{X}_r, \mathbf{Z}_p)$. □

The *modified diagonal cycles* in $\text{CH}^2(X_r^3)$ are defined by the rule

$$\Delta_r^\circ[d_1, d_2, d_3] := \theta_q \Delta_r[d_1, d_2, d_3]. \tag{2.14}$$

Lemma 2.5 shows that they are null-homologous and defined over $\mathbf{Q}(\zeta_r)$. Define

$$\kappa_r[d_1, d_2, d_3] \;\; := \;\; \text{AJ}_{\text{ét}}(\Delta_r^\circ[d_1, d_2, d_3]) \in H^1(\mathbf{Q}(\zeta_r), H^1_{\text{ét}}(\bar{X}_r, \mathbf{Z}_p)^{\otimes 3}(2)).$$

To deal with the circumstance that the cycles $\Delta_r^\circ[d_1, d_2, d_3]$ are only defined over $\mathbf{Q}(\zeta_r)$, and hence that the associated cohomology classes $\kappa_r[d_1, d_2, d_3]$ need not (and in fact, do not) extend to $G_{\mathbf{Q}}$, it is necessary to replace the $\mathbf{Z}_p[\tilde{G}_r][G_{\mathbf{Q}}]$-module $H^1_{\text{ét}}(\bar{X}_r, \mathbf{Z}_p)^{\otimes 3}(2)$ by an appropriate twist over $\mathbf{Q}(\zeta_r)$. Let G_r denote the Sylow p-subgroup of the group $\tilde{G}_r$ of (2.6), and let $G_\infty := \varprojlim G_r$. Let

$$\Lambda(G_r) := \mathbf{Z}_p[G_r], \qquad \Lambda(G_\infty) = \mathbf{Z}_p[[G_\infty]]$$

be the finite group ring attached to G_r and the associated Iwasawa algebra, respectively.

Let $\Lambda(G_r)(\pm\frac{1}{2})$ denote the Galois module which is isomorphic to $\Lambda(G_r)$ as a $\Lambda(G_r)$-module, and on which the Galois group $G_{\mathbf{Q}(\zeta_1)}$ is made to act via its quotient $\mathrm{Gal}\,(\mathbf{Q}(\zeta_r)/\mathbf{Q}(\zeta_1)) = 1 + p\mathbf{Z}/p^r\mathbf{Z}$, the element σ_m acting as multiplication by the group-like element $\langle m, m, m\rangle^{\mp 1/2}$. Let $\Lambda(G_\infty)(\pm\frac{1}{2})$ denote the projective limit of the $\Lambda(G_r)(\pm\frac{1}{2})$. It follows from the definitions that if

$$\nu_{k_\circ,\ell_\circ,m_\circ} : \Lambda(G_r) \longrightarrow \mathbf{Z}/p^r\mathbf{Z}, \qquad \text{or} \qquad \nu_{k_\circ,\ell_\circ,m_\circ} : \Lambda(G_\infty) \longrightarrow \mathbf{Z}_p$$

is the homomorphism sending $\langle a_1, a_2, a_3\rangle$ to $a_1^{k_\circ} a_2^{\ell_\circ} a_3^{m_\circ}$, then

$$\Lambda(G_r)(\tfrac{1}{2}) \otimes_{\nu_{k_\circ,\ell_\circ,m_\circ}} \mathbf{Z}/p^r\mathbf{Z} = (\mathbf{Z}/p^r\mathbf{Z})(\varepsilon_{\mathrm{cyc}}^{-(k_\circ+\ell_\circ+m_\circ)/2}), \tag{2.15}$$

where the tensor product is taken over $\Lambda(G_r)$, and similarly for G_∞. In particular if $k_\circ + \ell_\circ + m_\circ = 2t$ is an even integer,

$$\Lambda(G_\infty)(\tfrac{1}{2}) \otimes_{\nu_{k_\circ,\ell_\circ,m_\circ}} \mathbf{Z}_p = \mathbf{Z}_p(-t)(\omega^t) \tag{2.16}$$

as $G_{\mathbf{Q}}$-modules. More generally, if Ω is any $\Lambda(G_\infty)$ module, write

$$\Omega(\tfrac{1}{2}) := \Omega \otimes_{\Lambda(G_\infty)} \Lambda(G_\infty)(\tfrac{1}{2}), \qquad \Omega(\tfrac{-1}{2}) := \Omega \otimes_{\Lambda(G_\infty)} \Lambda(G_\infty)(\tfrac{-1}{2}),$$

for the relevant twists of Ω, which are isomorphic to Ω as a $\Lambda(G_\infty)[G_{\mathbf{Q}(\mu_{p^\infty})}]$-module but are endowed with different actions of $G_{\mathbf{Q}}$.

There is a canonical Galois-equivariant $\Lambda(G_r)$-hermitian bilinear, $\Lambda(G_r)$-valued pairing

$$\langle\!\langle\ ,\ \rangle\!\rangle_r : H^1_{\text{ét}}(\bar{X}_r, \mathbf{Z}_p)^{\otimes 3}(2)(\tfrac{1}{2}) \ \times\ H^1_{\text{ét}}(\bar{X}_r, \mathbf{Z}_p)^{\otimes 3}(1)(\tfrac{1}{2}) \ \longrightarrow\ \Lambda(G_r), \tag{2.17}$$

given by the formula

$$\langle\!\langle a, b\rangle\!\rangle_r := \sum_{\sigma=\langle d_1,d_2,d_3\rangle\in G_r} \langle a^\sigma, b\rangle_{X_r} \cdot \langle d_1, d_2, d_3\rangle,$$

where

$$\langle\ ,\ \rangle_{X_r} : H^1_{\text{ét}}(\bar{X}_r, \mathbf{Z}_p)^{\otimes 3}(2) \times H^1_{\text{ét}}(\bar{X}_r, \mathbf{Z}_p)^{\otimes 3}(1) \longrightarrow H^2_{\text{ét}}(\bar{X}_r, \mathbf{Z}_p(1))^{\otimes 3} = \mathbf{Z}_p$$

arises from the Poincaré duality between $H^3_{\text{ét}}(\bar{X}_r^3, \mathbf{Z}_p)(2)$ and $H^3_{\text{ét}}(\bar{X}_r^3, \mathbf{Z}_p)(1)$. This pairing enjoys the following properties:

— For all $\lambda \in \Lambda(G_r)$,

$$\langle\!\langle \lambda a, b\rangle\!\rangle_r = \lambda^* \langle\!\langle a, b\rangle\!\rangle_r, \qquad \langle\!\langle a, \lambda b\rangle\!\rangle_r = \lambda \langle\!\langle a, b\rangle\!\rangle_r,$$

where $\lambda^* \in \Lambda(G_r)$ is obtained from λ by applying the involution on the group ring which sends every group-like element to its inverse. In particular, the pairing of (2.17) can and will also be viewed as a $\Lambda(G_r)$-valued $*$-hermitian pairing

$$\langle\!\langle\ ,\ \rangle\!\rangle_r : H^1_{\text{ét}}(\bar{X}_r, \mathbf{Z}_p)^{\otimes 3}(2) \times H^1_{\text{ét}}(\bar{X}_r, \mathbf{Z}_p)^{\otimes 3}(1) \longrightarrow \Lambda(G_r).$$

— For all $\sigma \in G_{\mathbf{Q}(\zeta_1)}$, we have $\langle\!\langle \sigma a, \sigma b\rangle\!\rangle_r = \langle\!\langle a, b\rangle\!\rangle_r$.

— The U_p and U_p^* operators are adjoint to each other under this pairing, giving rise to a duality (denoted by the same symbol, by an abuse of notation)

$$\langle\langle\ ,\ \rangle\rangle_r : e^* H^1_{\text{ét}}(\bar{X}_r, \mathbf{Z}_p)^{\otimes 3}(2)(\tfrac{1}{2}) \ \times\ e H^1_{\text{ét}}(\bar{X}_r, \mathbf{Z}_p)^{\otimes 3}(1)(\tfrac{1}{2}) \quad\longrightarrow\quad \Lambda(G_r).$$

Define

$$\begin{aligned}
\mathbb{H}^{111}(X_r) &:= \operatorname{Hom}_{\Lambda(G_r)}(H^1_{\text{ét}}(\bar{X}_r, \mathbf{Z}_p)^{\otimes 3}(1)(\tfrac{1}{2}), \Lambda(G_r)) \simeq H^1_{\text{ét}}(\bar{X}_r, \mathbf{Z}_p)^{\otimes 3}(2)(\tfrac{1}{2}),\\
\mathbb{H}^{111}_{\circ}(X_r) &:= \operatorname{Hom}_{\Lambda(G_r)}(e H^1_{\text{ét}}(\bar{X}_r, \mathbf{Z}_p)^{\otimes 3}(1)(\tfrac{1}{2}), \Lambda(G_r)) \simeq e^* H^1_{\text{ét}}(\bar{X}_r, \mathbf{Z}_p)^{\otimes 3}(2)(\tfrac{1}{2}).
\end{aligned}$$

The above identifications of $\mathbf{Z}_p[G_{\mathbf{Q}(\zeta_1)}]$-modules follow from the pairing (2.17).

To descend the field of definition of the classes $\kappa_r[d_1, d_2, d_3]$, we package them together into elements

$$\boldsymbol{\kappa}_r[a,b,c] \quad\in\quad H^1(\mathbf{Q}(\zeta_r), \mathbb{H}^{111}(X_r))$$

indexed by triples

$$[a,b,c] \in I_1 = (\mathbf{Z}/p\mathbf{Z})^{\times 3} = \mu_{p-1}(\mathbf{Z}_p)^3 \subset (\mathbf{Z}_p^{\times})^3. \tag{2.18}$$

The class $\boldsymbol{\kappa}_r[a,b,c]$ is defined by setting, for all $\sigma \in G_{\mathbf{Q}(\zeta_r)}$ and all $\gamma_r \in H^1_{\text{ét}}(\bar{X}_r, \mathbf{Z}_p)^{\otimes 3}(1)$,

$$\boldsymbol{\kappa}_r[a,b,c](\sigma)(\gamma_r) = \langle\langle \kappa_r[a,b,c](\sigma), \gamma_r \rangle\rangle_r, \tag{2.19}$$

where the elements $a, b, c \in (\mathbf{Z}/p\mathbf{Z})^{\times}$ are viewed as elements of $(\mathbf{Z}/p^r\mathbf{Z})^{\times}$ via the Teichmuller lift alluded to in (2.18). Note that there is a natural identification

$$H^1(\mathbf{Q}(\zeta_r), \mathbb{H}^{111}(X_r)) = \operatorname{Ext}^1_{\Lambda(G_r)[G_{\mathbf{Q}(\zeta_r)}]}(H^1_{\text{ét}}(\bar{X}_r, \mathbf{Z}_p)^{\otimes 3}(1), \Lambda(G_r),$$

because $H^1_{\text{ét}}(\bar{X}_r, \mathbf{Z}_p)^{\otimes 3}(1) = H^1_{\text{ét}}(\bar{X}_r, \mathbf{Z}_p)^{\otimes 3}(1)(\frac{1}{2})$ *as* $G_{\mathbf{Q}(\zeta_r)}$*-modules* and the $\Lambda(G_r)$-dual of the latter is $\mathbb{H}^{111}(X_r)$. With these definitions we have

Lemma 2.6. — *The class* $\boldsymbol{\kappa}_r[a,b,c]$ *is the restriction to* $G_{\mathbf{Q}(\zeta_r)}$ *of a class*

$$\boldsymbol{\kappa}_r[a,b,c] \in H^1(\mathbf{Q}(\zeta_1), \mathbb{H}^{111}(X_r)) = \operatorname{Ext}^1_{\Lambda(G_r)[G_{\mathbf{Q}(\zeta_1)}]}(H^1_{\text{ét}}(\bar{X}_r, \mathbf{Z}_p)^{\otimes 3}(1)(\tfrac{1}{2}), \Lambda(G_r)).$$

Furthermore, for all $m \in \mu_{p-1}(\mathbf{Z}_p)$,

$$\sigma_m\ \boldsymbol{\kappa}_r[a,b,c] = \boldsymbol{\kappa}_r[ma,mb,mc].$$

Proof. — We will prove this by giving a more conceptual description of the cohomology class $\boldsymbol{\kappa}_r[a,b,c]$. Let $|\Delta|$ denote the support of an algebraic cycle Δ, and let

$$\Delta_r^{\circ}[[a,b,c]] := |\Delta_1^{\circ}[a,b,c]| \times_{X_1^3} X_r^3 \tag{2.20}$$

denote the inverse image in X_r^3 of $|\Delta_1^{\circ}[a,b,c]|$, which fits into the cartesian diagram

$$\begin{array}{ccc}
\Delta_r^{\circ}[[a,b,c]] & \hookrightarrow & X_r^3\\
\downarrow & & \downarrow{\scriptstyle (\varpi_1^{r-1})^3}\\
|\Delta_1^{\circ}[a,b,c]| & \hookrightarrow & X_1^3.
\end{array}$$

As in the proof of Lemma 2.1, observe that

$$\Delta_r^\circ[[a,b,c]] = \bigsqcup_{[d_1,d_2,d_3]\in I_r^1} |\Delta_r^\circ[ad_1,bd_2,cd_3]|$$

where I_r^1 denotes the p-Sylow subgroup of I_r. Consider now the commutative diagram of $\Lambda(G_r)[G_{\mathbf{Q}}(\zeta_1)]$-modules with exact rows:

(2.21)

$$\begin{array}{ccccc} & & & & \Lambda(G_r)(\frac{-1}{2}) \\ & & & & \downarrow j \\ H^3_{\text{ét}}(\bar{X}_r^3,\mathbf{Z}_p)(2) & \hookrightarrow & H^3_{\text{ét}}(\bar{X}_r^3 - \Delta_r^\circ[[a,b,c]],\mathbf{Z}_p)(2) & \twoheadrightarrow & H^0_{\text{ét}}(\bar{\Delta}_r^\circ[[a,b,c]],\mathbf{Z}_p)_0 \\ \downarrow p & & & & \\ H^1_{\text{ét}}(\bar{X}_r,\mathbf{Z}_p)^{\otimes 3}(2), & & & & \end{array}$$

where

— the map j is the inclusion defined on group-like elements by

$$j\left(\langle d_1,d_2,d_3\rangle\right) = \mathrm{cl}(\Delta_r^\circ[ad_2d_3, bd_1d_3, cd_1d_2]),$$

which is $G_{\mathbf{Q}(\zeta_1)}$-equivariant by Lemma 2.2;

— the middle row arises from the excision exact sequence in étale cohomology (cf. [J88, (3.6)] and [M, p. 108]);

— the subscript of 0 appearing in the rightmost term in the exact sequence denotes the kernel of the cycle class map, i.e.,

$$H^0_{\text{ét}}(\bar{\Delta}_r^\circ[[a,b,c]],\mathbf{Z}_p)_0 := \ker\left(H^0_{\text{ét}}(\bar{\Delta}_r^\circ[[a,b,c]],\mathbf{Z}_p)_0 \longrightarrow H^4_{\text{ét}}(\bar{X}_r^3,\mathbf{Z}_p(2))\right),$$

and the fact that the image of j is contained in $H^0_{\text{ét}}(\bar{\Delta}_r^\circ[[a,b,c]],\mathbf{Z}_p)_0$ follows from Lemma 2.5;

— the projection p is the one arising from the Künneth decomposition.

Taking the pushout and pullback of the extension in (2.21) via the maps p and j yields an exact sequence of $\Lambda(G_r)[G_{\mathbf{Q}}(\zeta_1)]$-modules

$$(2.22)\qquad 0 \longrightarrow H^1_{\text{ét}}(\bar{X}_r,\mathbf{Z}_p)^{\otimes 3}(2) \longrightarrow E_r \longrightarrow \Lambda(G_r)(\tfrac{-1}{2}) \longrightarrow 0.$$

Taking the $\Lambda(G_r)$-dual of this exact sequence, we obtain

$$0 \longrightarrow \Lambda(G_r)(\tfrac{1}{2}) \longrightarrow \check{E}_r \longrightarrow H^1_{\text{ét}}(\bar{X}_r,\mathbf{Z}_p)^{\otimes 3}(1)^* \longrightarrow 0.$$

where M^* means the $\Lambda(G_r)$-module obtained from M by letting act $\Lambda(G_r)$ on it by composing with the involution $\lambda \mapsto \lambda^*$. Twisting this sequence by $(\frac{-1}{2})$ and noting that $M^*(\frac{-1}{2}) \simeq M(\frac{1}{2})^*$ yields an extension

$$(2.23)\qquad 0 \longrightarrow \Lambda(G_r) \longrightarrow E'_r \longrightarrow H^1_{\text{ét}}(\bar{X}_r,\mathbf{Z}_p)^{\otimes 3}(1)(\tfrac{1}{2})^* \longrightarrow 0.$$

Since

$$H^1_{\text{ét}}(\bar{X}_r,\mathbf{Z}_p)^{\otimes 3}(1)(\tfrac{1}{2})^* = \mathrm{Hom}_{\Lambda(G_r)}(H^1_{\text{ét}}(\bar{X}_r,\mathbf{Z}_p)^{\otimes 3}(2)(\tfrac{1}{2}),\Lambda(G_r)),$$

it follows that the cohomology class realizing the extension E'_r is an element of

$$H^1(\mathbf{Q}(\zeta_1),\mathrm{Hom}_{\Lambda(G_r)}(H^1_{\text{ét}}(\bar{X}_r,\mathbf{Z}_p)^{\otimes 3}(1)(\tfrac{1}{2}),\Lambda(G_r))) = H^1(\mathbf{Q}(\zeta_1),\mathbb{H}^{111}(X_r)),$$

because the duality afforded by $\langle\!\langle\ ,\ \rangle\!\rangle_r$ is hermitian (and not Λ-linear). When restricted to $G_{\mathbf{Q}(\zeta_r)}$, this class coincides with $\boldsymbol{\kappa}_r[a,b,c]$, and the first assertion follows.

The second assertion is an immediate consequence of the definitions, using the Galois equivariance properties of the cycles $\Delta_r[d_1,d_2,d_3]$ given in the first assertion of Lemma 2.2. □

Remark 2.7. — *The extension E'_r of (2.23) can also be realised as a subquotient of the étale cohomology group $H^3_c(\bar{X}^3_r–\Delta^\circ_r[[a,b,c]],\mathbf{Z}_p)(1)$ with compact supports, in light of the Poincaré duality*

$$H^3_{\text{ét}}(\bar{X}^3_r–\Delta^\circ_r[[a,b,c]],\mathbf{Z}_p)(2)\ \times\ H^3_c(\bar{X}^3_r–\Delta^\circ_r[[a,b,c]],\mathbf{Z}_p)(1)\longrightarrow\mathbf{Z}_p.$$

2.3. Λ-adic cohomology classes. — Thanks to Lemma 2.6, we now dispose, for each $[a,b,c]\in\mu_{p-1}(\mathbf{Z}_p)^3$, of a system

$$\boldsymbol{\kappa}_r[a,b,c]\in H^1(\mathbf{Q}(\zeta_1),\mathbb{H}^{111}(X_r)) \tag{2.24}$$

of cohomology classes indexed by the integers $r\geq 1$, so that $e^*\boldsymbol{\kappa}_r[a,b,c]\in H^1(\mathbf{Q}(\zeta_1),\mathbb{H}^{111}_\circ(X_r))$. Let

$$p_{r+1,r}:\Lambda(G_{r+1})\longrightarrow\Lambda(G_r)$$

be the projection on finite group rings induced from the natural homomorphism $G_{r+1}\longrightarrow G_r$.

Lemma 2.8. — *Let $\gamma_{r+1}\in H^1_{\text{ét}}(\bar{X}_{r+1},\mathbf{Z}_p)^{\otimes 3}(1)$ and $\gamma_r\in H^1_{\text{ét}}(\bar{X}_r,\mathbf{Z}_p)^{\otimes 3}(1)$ be elements that are compatible under the pushforward by ϖ_1^3, i.e., that satisfy $(\varpi_1^3)_*(\gamma_{r+1})=\gamma_r$. For all $\sigma\in G_{\mathbf{Q}(\zeta_1)}$,*

$$p_{r+1,r}\left(\boldsymbol{\kappa}_{r+1}[a,b,c](\sigma)(\gamma_{r+1})\right)=\boldsymbol{\kappa}_r[a,b,c](\sigma)(\gamma_r).$$

Proof. — This amounts to the statement that

$$p_{r+1,r}(\langle\!\langle\kappa_{r+1}[a,b,c],\gamma_{r+1}\rangle\!\rangle_{r+1})=\langle\!\langle\kappa_r[a,b,c],\gamma_r\rangle\!\rangle_r.$$

But the left-hand side of this equation is equal to

$$\sum_{G_r}\langle(\mu^3)^*(\mu^3)_*\kappa_{r+1}[ad'_2d'_3,bd'_1d'_3,cd'_1d'_2],\gamma_{r+1}\rangle_{X_{r+1}}\cdot\langle d_1,d_2,d_3\rangle,$$

where the sum runs over $\langle d_1, d_2, d_3\rangle \in G_r$ and $\langle d_1', d_2', d_3'\rangle$ denotes an (arbitrary) lift of $\langle d_1, d_2, d_3\rangle$ to G_{r+1}. The third assertion in Lemma 2.4 allows us to rewrite this as

$$\begin{aligned}
&\sum_{G_r} \langle (\varpi_1^3)^* \kappa_r[ad_2d_3, bd_1d_3, cd_1d_2], \gamma_{r+1}\rangle_{X_{r+1}} \cdot \langle d_1, d_2, d_3\rangle \\
&\quad = \sum_{G_r} \langle \kappa_r[ad_2d_3, bd_1d_3, cd_1d_2], (\varpi_1^3)_* \gamma_{r+1}\rangle_{X_r} \cdot \langle d_1, d_2, d_3\rangle \\
&\quad = \sum_{G_r} \langle \kappa_r[ad_2d_3, bd_1d_3, cd_1d_2], \gamma_r\rangle_{X_r} \cdot \langle d_1, d_2, d_3\rangle \\
&\quad = \langle\langle \kappa_r[a,b,c], \gamma_r\rangle\rangle_r,
\end{aligned}$$

and the result follows. □

Define

$$\begin{aligned}
(2.25) \qquad \mathbb{H}^{111}(X_\infty^*) :=\ & \mathrm{Hom}_{\Lambda(G_\infty)}(H^1_{\text{ét}}(\bar{X}_\infty^*, \mathbf{Z}_p)^{\otimes 3}(1)(\tfrac{1}{2}), \Lambda(G_\infty)) \\
=\ & \mathrm{Hom}_{\Lambda(G_\infty)}(H^1_{\text{ét}}(\bar{X}_1, \mathcal{L}_\infty^*)^{\otimes 3}(1)(\tfrac{1}{2}), \Lambda(G_\infty)),
\end{aligned}$$

where the identification follows from (1.19).

Thanks to Lemma 2.8, the classes $\boldsymbol{\kappa}_r[a,b,c]$ can be packaged into a compatible collection. Namely:

Definition 2.9. — *Set*

$$(2.26) \qquad \boldsymbol{\kappa}_\infty[a,b,c] := (\boldsymbol{\kappa}_r[a,b,c])_{r\geq 1} \in H^1(\mathbf{Q}(\zeta_1), \mathbb{H}^{111}(X_\infty^*)).$$

It will also be useful to replace the classes $\boldsymbol{\kappa}_\infty[a,b,c]$ by elements that are essentially indexed by triples

$$(\omega_1, \omega_2, \omega_3) : (\mathbf{Z}/p\mathbf{Z}^\times)^3 \longrightarrow \mathbf{Z}_p^\times$$

of tame characters of $\tilde{G}_r/G_r$. Assume that the product $\omega_1\omega_2\omega_3$ is an *even* character. This assumption is equivalent to requiring that

$$\omega_1\omega_2\omega_3 = \delta^2, \qquad \text{for some } \delta : (\mathbf{Z}/p\mathbf{Z})^\times \longrightarrow \mathbf{Z}_p^\times.$$

Note that for a given $(\omega_1, \omega_2, \omega_3)$, there are in fact two characters δ as above, which differ by the unique quadratic character of conductor p. With the choices of $\omega_1, \omega_2, \omega_3$ and δ in hand, we set

$$(2.27) \quad \boldsymbol{\kappa}_\infty(\omega_1, \omega_2, \omega_3; \delta) := \frac{p^3}{(p-1)^3} \cdot \sum_{[a,b,c]} \delta^{-1}(abc) \cdot \omega_1(a)\omega_2(b)\omega_3(c) \cdot \boldsymbol{\kappa}_\infty[bc, ac, ab],$$

where the sum is taken over the triples $[a,b,c]$ of $(p-1)$st roots of unity in $\mathbf{Z}_p^\times$. The classes $\boldsymbol{\kappa}_\infty(\omega_1, \omega_2, \omega_3; \delta)$ satisfy the following properties.

Lemma 2.10. — *For all* $\sigma_m \in \mathrm{Gal}(\mathbf{Q}(\zeta_\infty)/\mathbf{Q})$,

$$\sigma_m \boldsymbol{\kappa}_\infty(\omega_1, \omega_2, \omega_3; \delta) = \delta(m) \boldsymbol{\kappa}_\infty(\omega_1, \omega_2, \omega_3; \delta).$$

For all diamond operators $\langle a_1, a_2, a_3\rangle \in \mu_{p-1}(\mathbf{Z}_p)^3$

$$\langle a_1, a_2, a_3\rangle \boldsymbol{\kappa}_\infty(\omega_1, \omega_2, \omega_3; \delta) = \omega_{123}(a_1, a_2, a_3) \cdot \boldsymbol{\kappa}_\infty(\omega_1, \omega_2, \omega_3; \delta).$$

Proof. — This follows from a direct calculation based on the definitions, using the compatibilities of Lemma 2.2 satisfied by the cycles $\Delta_r[d_1, d_2, d_3]$. □

The classes $\boldsymbol{\kappa}_\infty[a, b, c]$ and $\boldsymbol{\kappa}_\infty(\omega_1, \omega_2, \omega_3; \delta)$ are called the Λ-*adic cohomology classes* attached to the triple $[a, b, c] \in \mu_{p-1}(\mathbf{Z}_p)^3$ or the quadruple $(\omega_1, \omega_2, \omega_3; \delta)$. As will be explained in the next section, they are three variable families of cohomology classes parametrised by points in the triple product $\mathcal{W} \times \mathcal{W} \times \mathcal{W}$ of weight spaces, and taking values in the three-parameter family of self-dual Tate twists of the Galois representations attached to the different specialisations of a triple of Hida families.

3. Higher weight balanced specialisations

For every integer $k_\circ \geq 0$ define

$$W_1^{k_\circ} := H^1_{\text{ét}}(\bar{X}_1, \mathcal{H}^{k_\circ})$$

and recall from the combination of (1.19), (1.21) and (1.22) the specialisation map

$$\text{sp}^*_{k_\circ} : H^1_{\text{ét}}(\bar{X}^*_\infty, \mathbf{Z}_p) = H^1_{\text{ét}}(\bar{X}_1, \mathcal{L}^*_\infty) \longrightarrow W_1^{k_\circ}. \tag{3.1}$$

Fix throughout this section a triple

$$k = k_\circ + 2, \qquad \ell = \ell_\circ + 2, \qquad m = m_\circ + 2$$

of integers ≥ 2 for which $k_\circ + \ell_\circ + m_\circ = 2t$ is even. Let

$$\mathcal{H}^{k_\circ, \ell_\circ, m_\circ} := \mathcal{H}^{k_\circ} \boxtimes \mathcal{H}^{\ell_\circ} \boxtimes \mathcal{H}^{m_\circ}$$

viewed as a sheaf on X_1^3, and

$$W_1^{k_\circ, \ell_\circ, m_\circ} := W_1^{k_\circ} \otimes W_1^{\ell_\circ} \otimes W_1^{m_\circ}(2 - t).$$

As one readily checks, the p-adic Galois representation $W_1^{k_\circ, \ell_\circ, m_\circ}$ is Kummer self-dual, i.e., there is an isomorphism of $G_{\mathbf{Q}}$-modules

$$\text{Hom}_{G_{\mathbf{Q}}}(W_1^{k_\circ, \ell_\circ, m_\circ}, \mathbf{Z}_p(1)) \simeq W_1^{k_\circ, \ell_\circ, m_\circ}.$$

The specialisation maps give rise, in light of (2.16), to the triple product specialisation map

$$\text{sp}^*_{k_\circ, \ell_\circ, m_\circ} := \text{sp}^*_{k_\circ} \otimes \text{sp}^*_{\ell_\circ} \otimes \text{sp}^*_{m_\circ} : \mathbb{H}^{111}(X^*_\infty) \longrightarrow W_1^{k_\circ, \ell_\circ, m_\circ} \tag{3.2}$$

and to the associated collection of specialised classes

$$\kappa_1(k_\circ, \ell_\circ, m_\circ)[a, b, c] := \text{sp}_{k_\circ, \ell_\circ, m_\circ}(\boldsymbol{\kappa}_\infty[a, b, c]) \in H^1(\mathbf{Q}(\zeta_1), W_1^{k_\circ, \ell_\circ, m_\circ}). \tag{3.3}$$

Note that for $(k_\circ, \ell_\circ, m_\circ) = (0, 0, 0)$, it follows from the definitions (cf. e.g. the proof of Lemma 2.6) that the class $\kappa_1(k_\circ, \ell_\circ, m_\circ)[a, b, c]$ is simply the image under the étale Abel-Jacobi map of the cycle $\Delta_1^\circ[a, b, c]$.

The main goal of this section is to offer a similar geometric description for the above classes also when (k, ℓ, m) is *balanced* and $k_\circ, \ell_\circ, m_\circ > 0$, which we assume henceforth for the remainder of this section.

In order to do this, it shall be useful to dispose of an alternate description of the extension (2.22) in terms of the étale cohomology of the (open) three-fold $X_1^3 - |\Delta_1^\circ[a,b,c]|$ with values in appropriate sheaves.

Lemma 3.1. — *Let $\mathcal{L}_r^{*\boxtimes 3}$ denote the exterior tensor product of $\mathcal{L}_r^*$, over the triple product X_1^3. There is a commutative diagram*

$$\begin{array}{ccccc} H^3_{\text{ét}}(X_r^3, \mathbf{Z}_p)(2) & \longrightarrow & H^3_{\text{ét}}(X_r^3 - \Delta_r^\circ[[a,b,c]], \mathbf{Z}_p)(2) & \longrightarrow & H^0_{\text{ét}}(\Delta_r^\circ[[a,b,c]], \mathbf{Z}_p) \\ \| & & \| & & \| \\ H^3_{\text{ét}}(\bar{X}_1^3, \mathcal{L}_r^{*\boxtimes 3})(2) & \longrightarrow & H^3_{\text{ét}}(\bar{X}_1^3 - |\Delta_1^\circ[a,b,c]|, \mathcal{L}_r^{*\boxtimes 3})(2) & \longrightarrow & H^0_{\text{ét}}(|\Delta_1^\circ[a,b,c]|, \mathcal{L}_r^{*\otimes 3})), \end{array}$$

in which the leftmost maps are injective and the horizontal sequences are exact.

Proof. — Recall from (1.17) that

$$\mathcal{L}_r^{*\boxtimes 3} = (\varpi_1^{r-1} \times \varpi_1^{r-1} \times \varpi_1^{r-1})_* \mathbf{Z}_p,$$

where

$$\varpi_1^{r-1} \times \varpi_1^{r-1} \times \varpi_1^{r-1} : X_r^3 \longrightarrow X_1^3$$

is defined as in (2.2). The vertical isomorphisms then follow from Shapiro's lemma and the definition of $\Delta_r^\circ[[a,b,c]]$ in (2.20). The horizontal sequence arises from the excision exact sequence in étale cohomology of [**J88**, (3.6)] and [**M**, p. 108]. □

Lemma 3.2. — *For all $[a,b,c] \in I_1$,*

$$H^0_{\text{ét}}(\bar{\Delta}_1[a,b,c], \mathcal{H}^{k_\circ, \ell_\circ, m_\circ}) = \mathbf{Z}_p(t).$$

Proof. — The Clebsch-Gordan formula asserts that the space of tri-homogenous polynomials in $6 = 2+2+2$ variables of tridegree $(k_\circ, \ell_\circ, m_\circ)$ has a unique SL_2-invariant element, namely, the polynomial

$$P_{k_\circ, \ell_\circ, m_\circ}(x_1, y_1, x_2, y_2, x_3, y_3) = \begin{vmatrix} x_2 & y_2 \\ x_3 & y_3 \end{vmatrix}^{k_\circ'} \begin{vmatrix} x_3 & y_3 \\ x_1 & y_1 \end{vmatrix}^{\ell_\circ'} \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix}^{m_\circ'},$$

where

$$k_\circ' = \frac{-k_\circ + \ell_\circ + m_\circ}{2}, \qquad \ell_\circ' = \frac{k_\circ - \ell_\circ + m_\circ}{2} \qquad m_\circ' = \frac{k_\circ + \ell_\circ - m_\circ}{2}.$$

Since the triplet of weights is balanced, it follows that $k_\circ', \ell_\circ', m_\circ' \geq 0$. From the Clebsch-Gordan formula it follows that $H^0_{\text{ét}}(\bar{\Delta}_1[a,b,c], \mathcal{H}^{k_\circ, \ell_\circ, m_\circ})$ is spanned by the global section whose stalk at a point $((A,P_1),(A,P_2),(A,P_3)) \in \Delta_1[a,b,c]$ as in (2.5) is given by

$$(X_2 \otimes Y_3 - Y_2 \otimes X_3)^{\otimes k_\circ'} \otimes (X_1 \otimes Y_3 - Y_1 \otimes X_3)^{\otimes \ell_\circ'} \otimes (X_1 \otimes Y_2 - Y_1 \otimes X_2)^{\otimes m_\circ'},$$

where (X_i, Y_i), $i = 1,2,3$, is a basis of the stalk of $\mathcal{H}$ at the point (A, P_i) in X_1. The Galois action is given by the t-th power of the cyclotomic character because the Weil pairing takes values in $\mathbf{Z}_p(1)$ and $k_\circ' + \ell_\circ' + m_\circ' = t$. □

Write $\mathrm{cl}_{k_\circ,\ell_\circ,m_\circ}(\Delta_1[a,b,c]) \in H^0_{\text{ét}}(|\bar{\Delta}^\circ_1[a,b,c]|, \mathcal{H}^{k_\circ,\ell_\circ,m_\circ})$ for the standard generator given by Lemma 3.2. Define

$$\mathrm{AJ}_{k_\circ,\ell_\circ,m_\circ}(\Delta_1[a,b,c]) \in H^1(\mathbf{Q}(\zeta_1), W_1^{k_\circ,\ell_\circ,m_\circ}) \tag{3.4}$$

to be the extension class constructed by pulling back by j and pushing forward by p in the exact sequence of the middle row of the following diagram:

(3.5)

$$\begin{array}{ccccc} & & & & \mathbf{Z}_p(t) \\ & & & & \big\downarrow j \\ H^3_{\text{ét}}(\bar{X}_1^3, \mathcal{H}^{k_\circ,\ell_\circ,m_\circ})(2) & \hookrightarrow & H^3_{\text{ét}}(\bar{X}_1^3 - \bar{\Delta}, \mathcal{H}^{k_\circ,\ell_\circ,m_\circ})(2) & \twoheadrightarrow & H^0_{\text{ét}}(\bar{\Delta}, \mathcal{H}^{k_\circ,\ell_\circ,m_\circ}) \\ \big\downarrow & & & & \\ W_1^{k_\circ,\ell_\circ,m_\circ}(t), & & & & \end{array}$$

where

— $\Delta = \Delta_1[a,b,c]$;

— the map j is the $G_{\mathbf{Q}(\zeta_1)}$-equivariant inclusion defined by $j(1) = \mathrm{cl}_{k_\circ,\ell_\circ,m_\circ}(\Delta)$;

— the surjectivity of the right-most horizontal row follows from the vanishing of the group $H^4_{\text{ét}}(\bar{X}_1^3, \mathcal{H}^{k_\circ,\ell_\circ,m_\circ})$, which in turn is a consequence of the Künneth formula and the vanishing of the terms $H^2_{\text{ét}}(\bar{X}_1, \mathcal{H}^{k_\circ})$ when $k_\circ > 0$ (cf. [**BDP13**, Lemmas 2.1, 2.2]).

In particular the image of j lies in the image of the right-most horizontal row and this holds regardless whether the cycle is null-homologous or not. The reader may compare this construction with (2.21), where the cycle $\Delta_r^\circ[[a,b,c]]$ is null-homologous and this property was crucially exploited.

Theorem 3.1. — *Set* $\mathrm{AJ}_{k_\circ,\ell_\circ,m_\circ}(\Delta^\circ_1[a,b,c]) = \theta_q \mathrm{AJ}_{k_\circ,\ell_\circ,m_\circ}(\Delta_1[a,b,c])$. *Then the identity*

$$\kappa_1(k_\circ,\ell_\circ,m_\circ)[a,b,c] = \mathrm{AJ}_{k_\circ,\ell_\circ,m_\circ}(\Delta^\circ_1[a,b,c])$$

holds in $H^1(\mathbf{Q}(\zeta_1), W_1^{k_\circ,\ell_\circ,m_\circ})$.

Proof. — Set $\Delta := \Delta^\circ_1[a,b,c]$ in order to alleviate notations. Thanks to Lemma 3.1, the diagram in (2.21) used to construct the extension E_r realising the class $\boldsymbol{\kappa}_r[a,b,c]$ is *the same* as the diagram

(3.6)

$$\begin{array}{ccccccc} & & & & & & \Lambda(G_r)(\frac{-1}{2}) \\ & & & & & & \big\downarrow \\ 0 & \longrightarrow & H^3_{\text{ét}}(\bar{X}_1^3, \mathcal{L}_r^{*\boxtimes 3})(2) & \longrightarrow & H^3_{\text{ét}}(\bar{X}_1^3 - |\bar{\Delta}|, \mathcal{L}_r^{*\boxtimes 3})(2) & \longrightarrow & H^0_{\text{ét}}(|\bar{\Delta}|, \mathcal{L}_r^{*\otimes 3}) \\ & & \big\downarrow & & & & \\ & & H^1_{\text{ét}}(\bar{X}_1, \mathcal{L}_r^{*})^{\otimes 3}(2). & & & & \end{array}$$

Let

$$\nu_{k_\circ,\ell_\circ,m_\circ} : \Lambda(G_r) \longrightarrow \mathbf{Z}/p^r\mathbf{Z}$$

be the algebra homomorphism sending the group like element $\langle d_1, d_2, d_3\rangle$ to $d_1^{k_\circ} d_2^{\ell_\circ} d_3^{m_\circ}$, and observe that the moment maps of (1.20) allow us to identify

$$\mathcal{L}_r^{*\boxtimes 3} \otimes_{\nu_{k_\circ,\ell_\circ,m_\circ}} (\mathbf{Z}/p^r\mathbf{Z}) = \mathcal{H}_r^{k_\circ,\ell_\circ,m_\circ}.$$

Tensoring (3.6) over $\Lambda(G_r)$ with $\mathbf{Z}/p^r\mathbf{Z}$ via the map $\nu_{k_\circ,\ell_\circ,m_\circ} : \Lambda(G_r) \longrightarrow \mathbf{Z}/p^r\mathbf{Z}$, yields the specialised diagram which coincides exactly with the mod p^r reduction of (3.5), with $\Delta = \Delta_1^\circ[a,b,c]$. The result follows by passing to the limit with r. □

Corollary 3.3. — *Let*

$$\Delta_1^\circ(\omega_1,\omega_2,\omega_3;\delta) := \frac{p^3}{(p-1)^3} \cdot \sum_{[a,b,c]\in I_1} \delta^{-1}(abc)\omega_1(a)\omega_2(b)\omega_3(c)\Delta_1^\circ[a,b,c]. \tag{3.7}$$

Then

$$\mathrm{sp}^*_{k_\circ,\ell_\circ,m_\circ}(\boldsymbol{\kappa}_\infty(\omega_1,\omega_2,\omega_3;\delta) = \mathrm{AJ}_{k_\circ,\ell_\circ,m_\circ}(\Delta_1^\circ(\omega_1,\omega_2,\omega_3;\delta)).$$

Proof. — This follows directly from the definitions. □

4. Cristalline specialisations

Let $\mathbf{f}$, $\mathbf{g}$, $\mathbf{h}$ be three arbitrary primitive, residually irreducible p-adic Hida families of tame levels M_f, M_g, M_h and tame characters χ_f, χ_g, χ_h, respectively, with associated weight space $\mathcal{W}_{\mathbf{f}}\times\mathcal{W}_{\mathbf{g}}\times\mathcal{W}_{\mathbf{g}}$. Assume $\chi_f\chi_g\chi_h = 1$ and set $M = \mathrm{lcm}(M_f, M_g, M_h)$. Let $(x,y,z) \in \mathcal{W}_{\mathbf{f}}\times\mathcal{W}_{\mathbf{g}}\times\mathcal{W}_{\mathbf{h}}$ be a point lying above a classical triple $(\nu_{k_\circ,\epsilon_1}, \nu_{\ell_\circ,\epsilon_2}, \nu_{m_\circ,\epsilon_3}) \in \mathcal{W}^3$ of weight space. As in Definition 1.2, the point (x,y,z) is said to be *tamely ramified* if the three characters ϵ_1, ϵ_2 and ϵ_3 are tamely ramified, i.e., factor through the quotient $(\mathbf{Z}/p\mathbf{Z})^\times$ of $\mathbf{Z}_p^\times$, and is said to be *crystalline* if $\epsilon_1\omega^{-k_\circ} = \epsilon_2\omega^{-\ell_\circ} = \epsilon_3\omega^{-m_\circ} = 1$.

Fix such a crystalline point (x,y,z) of balanced weight $(k,\ell,m) = (k_\circ+2, \ell_\circ+2, m_\circ+2)$, and let $(\mathbf{f}_x, \mathbf{g}_y, \mathbf{h}_z)$ be the specialisations of $(\mathbf{f},\mathbf{g},\mathbf{h})$ at (x,y,z). The ordinariness hypothesis implies that, for all but finitely many exceptions, these eigenforms are the p-stabilisations of newforms of level dividing M, denoted f, g and h respectively:

$$\mathbf{f}_x(q) = f(q) - \beta_f f(q^p), \qquad \mathbf{g}_y = g(q) - \beta_g g(q^p), \qquad \mathbf{h}_z(q) = h(q) - \beta_h h(q^p).$$

Since the point (x,y,z) is fixed throughout this section, the dependency of (f,g,h) on (x,y,z) has been suppressed from the notations, and we also write $(f_\alpha, g_\alpha, h_\alpha) := (\mathbf{f}_x, \mathbf{g}_y, \mathbf{h}_z)$ for the ordinary p-stabilisations of f, g and h.

Recall the quotient X_{01} of X_1, having $\Gamma_0(p)$-level structure at p, and the projection map $\mu : X_1 \longrightarrow X_{01}$ introduced in (1.15). By an abuse of notation, the symbol $\mathcal{H}^{k_\circ}$ is also used to denote the étale sheaves appearing in (1.4) over any quotient of X_1, such as X_{01}. Let

$$\begin{aligned} W_1 &:= H^1_{\text{ét}}(\bar{X}_1, \mathcal{H}^{k_\circ}) \otimes H^1_{\text{ét}}(\bar{X}_1, \mathcal{H}^{\ell_\circ}) \otimes H^1_{\text{ét}}(\bar{X}_1, \mathcal{H}^{m_\circ})(2-t), \\ W_{01} &:= H^1_{\text{ét}}(\bar{X}_{01}, \mathcal{H}^{k_\circ}) \otimes H^1_{\text{ét}}(\bar{X}_{01}, \mathcal{H}^{\ell_\circ}) \otimes H^1_{\text{ét}}(\bar{X}_{01}, \mathcal{H}^{m_\circ})(2-t), \end{aligned}$$

be the Galois representations arising from the cohomology of X_1 and X_{01} with values in these sheaves. They are endowed with a natural action of the triple tensor product of the Hecke algebras of weight $k_\circ$, $\ell_\circ$, $m_\circ$ and level Mp.

Let $W_1[f_\alpha, g_\alpha, h_\alpha]$ denote the $(f_\alpha, g_\alpha, h_\alpha)$-isotypic component of W_1 on which the Hecke operators act with the same eigenvalues as on $f_\alpha \otimes g_\alpha \otimes h_\alpha$. Let $\pi_{f_\alpha,g_\alpha,h_\alpha}$: $W_1 \to W_1[f_\alpha, g_\alpha, h_\alpha]$ denote the associated projection. Use similar notations for W_{01}.

Recall the family

$$\boldsymbol{\kappa}_\infty(\epsilon_1\omega^{-k_\circ}, \epsilon_2\omega^{-\ell_\circ}, \epsilon_3\omega^{-m_\circ}; 1) = \boldsymbol{\kappa}_\infty(1,1,1;1) \tag{4.1}$$

that was introduced in (2.27). By Lemma 2.10, this class lies in $H^1(\mathbf{Q}, \mathbb{H}^{111}(X^*_\infty))$.

Recall the choice of auxiliary prime q made in the definition of the modified diagonal cycle (2.14). We assume now that q is chosen so that $C_q := (a_q(f) - q - 1)(a_q(g) - q - 1)(a_q(h) - q - 1)$ is a p-adic unit. Note that this is possible because the Galois representations $\varrho_{\mathbf{f}}$, $\varrho_{\mathbf{g}}$ and $\varrho_{\mathbf{h}}$ were assumed to be residually irreducible and hence f, g and h are non-Eisenstein mod p. Let

$$\kappa_1(f_\alpha, g_\alpha, h_\alpha) := \frac{1}{C_q} \cdot \pi_{f_\alpha,g_\alpha,h_\alpha} \operatorname{sp}^*_{x,y,z} \boldsymbol{\kappa}_\infty(1,1,1;1)) \in H^1(\mathbf{Q}, W_1[f_\alpha, g_\alpha, h_\alpha]) \tag{4.2}$$

be the specialisation at the crystalline point (x, y, z) of (4.1), after projecting it to the $(f_\alpha, g_\alpha, h_\alpha)$-isotypic component of W_1 via $\pi_{f_\alpha,g_\alpha,h_\alpha}$. We normalize the class by multiplying it by the above constant in order to remove the dependency on the choice of q.

The main goal of this section is to relate this class to the generalised Gross-Schoen diagonal cycles that were studied in [**DR14**], arising from cycles in Kuga-Sato varieties which are fibered over X^3 and have *good reduction* at p.

The fact that (x, y, z) is a crystalline point implies that the diamond operators in $\operatorname{Gal}(X_1/X_{01})$ act trivially on the $(f_\alpha, g_\alpha, h_\alpha)$-eigencomponents, and hence the Hecke-equivariant projection $\mu^3_* : W_1 \longrightarrow W_{01}$ induces an isomorphism

$$\mu^3_* : W_1[f_\alpha, g_\alpha, h_\alpha] \longrightarrow W_{01}[f_\alpha, g_\alpha, h_\alpha].$$

The first aim is to give a geometric description of the class

$$\kappa_{01}(f_\alpha, g_\alpha, h_\alpha) := \mu^3_* \kappa_1(f_\alpha, g_\alpha, h_\alpha)$$

in terms of appropriate algebraic cycles. To this end, recall the cycles $\Delta_1[a, b, c] \in \mathrm{CH}^2(X_1^3)$ introduced in (2.4), and let $p^* := \pm p$ be such that $\mathbf{Q}(\sqrt{p^*})$ is the quadratic subfield of $\mathbf{Q}(\zeta_1)$.

Lemma 4.1. — *The cycle $\mu^3_*\Delta_1[a, b, c]$ depends only on the quadratic residue symbol $(\frac{abc}{p})$ attached to $abc \in (\mathbf{Z}/p\mathbf{Z})^\times$. The cycles*

$$\begin{aligned}
\Delta^+_{01} &:= \mu^3_*\Delta_1[a, b, c] \quad \text{for any } a, b, c \text{ with } \left(\frac{abc}{p}\right) = 1, \\
\Delta^-_{01} &:= \mu^3_*\Delta_1[a, b, c] \quad \text{for any } a, b, c \text{ with } \left(\frac{abc}{p}\right) = -1,
\end{aligned}$$

belong to $\mathrm{CH}^2(X^3_{01}/\mathbf{Q}(\sqrt{p^*}))$ and are interchanged by the non-trivial automorphism of $\mathbf{Q}(\sqrt{p^*})$.

Proof. — Arguing as in Lemma 2.2 shows that for all $(d_1, d_2, d_3) \in I_1 = (\mathbf{Z}/p\mathbf{Z})^{\times 3}$,

$$\langle d_1, d_2, d_3\rangle \Delta_1[a, b, c] = \Delta_1[d_2d_3a, d_1d_3b, d_1d_2c].$$

The orbit of the triple $[a, b, c]$ under the action of I_1 is precisely the set of triples $[a', b', c']$ for which $(\frac{a'b'c'}{p}) = (\frac{abc}{p})$. Since X_{01} is the quotient of X_1 by the group I_1, it follows that $\mu_*^3\Delta_1[a, b, c]$ depends only on this quadratic residue symbol, and hence that the classes Δ_{01}^+ and Δ_{01}^- in the statement of Lemma 4.1 are well-defined. Furthermore, Lemma 2.6 implies that, for all $m \in (\mathbf{Z}/p\mathbf{Z})^\times$, the Galois automorphism σ_m fixes Δ_{01}^+ and Δ_{01}^- if m is a square modulo p, and interchanges these two cycle classes otherwise. It follows that they are invariant under the Galois group $\mathrm{Gal}\,(\mathbf{Q}(\zeta_1)/\mathbf{Q}(\sqrt{p^*}))$ and hence descend to a pair of conjugate cycles $\Delta_{01}^{\pm}$ defined over $\mathbf{Q}(\sqrt{p^*})$, as claimed. □

It follows from this lemma that the algebraic cycle

$$\Delta_{01} := \Delta_{01}^+ + \Delta_{01}^- \in \mathrm{CH}^2(X_{01}^3/\mathbf{Q}). \tag{4.3}$$

is defined over $\mathbf{Q}$. To describe it concretely, note that a triple (C_1, C_2, C_3) of distinct cyclic subgroups of order p in an elliptic curve A admits a somewhat subtle discrete invariant in $(\mu_p^{\otimes 3} - \{1\})$ modulo the action of $(\mathbf{Z}/p\mathbf{Z})^{\times 2}$, denoted $o(C_1, C_2, C_3)$ and called the *orientation* of (C_1, C_2, C_3). This orientation is defined by choosing generators P_1, P_2, P_3 of C_1, C_2 and C_3 respectively and setting

$$o(C_1, C_2, C_3) := \langle P_2, P_3\rangle \otimes \langle P_3, P_1\rangle \otimes \langle P_1, P_2\rangle \in \mu_p^{\otimes 3} - \{1\}.$$

It is easy to check that the value of $o(C_1, C_2, C_3)$ in $\mu_p^{\otimes 3} - \{1\}$ only depends on the choices of generators P_1, P_2 and P_3, up to multiplication by a *non-zero square* in $(\mathbf{Z}/p\mathbf{Z})^\times$. In view of (2.5), we then have

$$\Delta_{01} = \{((A, C_1), (A, C_2), (A, C_3)) \quad \text{with} \quad C_1 \neq C_2 \neq C_3\}, \tag{4.4}$$

and

$$\begin{aligned}
\Delta_{01}^+ &= \left\{((A, C_1), (A, C_2), (A, C_3)) \quad \text{with } o(C_1, C_2, C_3) = a\zeta_1^{\otimes 3}, \quad a \in (\mathbf{Z}/p\mathbf{Z})^{\times 2}\right\}, \\
\Delta_{01}^- &= \left\{((A, C_1), (A, C_2), (A, C_3)) \quad \text{with } o(C_1, C_2, C_3) = a\zeta_1^{\otimes 3}, \quad a \notin (\mathbf{Z}/p\mathbf{Z})^{\times 2}\right\}.
\end{aligned}$$

Recall the natural projections

$$\pi_1, \pi_2 : X_{01} \longrightarrow X, \qquad \varpi_1, \varpi_2 : X_1 \longrightarrow X$$

to the curve $X = X_0(M)$ of prime to p level, and set

$$W_0 \;:=\; H^1_{\text{ét}}(\bar{X}_0, \mathcal{H}^{k_\circ}) \otimes H^1_{\text{ét}}(\bar{X}_0, \mathcal{H}^{\ell_\circ}) \otimes H^1_{\text{ét}}(\bar{X}_0, \mathcal{H}^{m_\circ})(2 - t),$$

The Galois representation W_0 is endowed with a natural action of the triple tensor product of the Hecke algebras of weight $k_\circ$, $\ell_\circ$, $m_\circ$ and level M. Let $W_0[f, g, h]$ denote the (f, g, h)- isotypic component of W_0, on which the Hecke operators act with the same eigenvalues as on $f \otimes g \otimes h$. Note that the U_p^* operator does not act naturally on W_0 and hence one cannot speak of the $(f_\alpha, g_\alpha, h_\alpha)$-eigenspace of this Hecke module. One can, however, denote by $W_1[f, g, h]$ and $W_{01}[f, g, h]$ the (f, g, h)-isotypic component of these Galois representations, in which the action of the U_p^* operators on the three factors are not taken into account. Thus, $W_{01}[f_\alpha, g_\alpha, h_\alpha]$ is the image of $W_{01}[f,g,h]$ under the ordinary projection, and likewise for W_1. In other words, denoting by $\pi_{f,g,h}$ the projection to the (f,g,h)-isotypic component on any of these modules, one has

$$\pi_{f_\alpha,g_\alpha,h_\alpha} = e^*\pi_{f,g,h}$$

whenever the left-hand projection is defined.

The projection maps

$$(\pi_1,\pi_1,\pi_1): X_{01}^3 \longrightarrow X^3, \quad (\varpi_1,\varpi_1,\varpi_1): X_1^3 \longrightarrow X^3$$

induces push-forward maps

$$\begin{aligned}(\pi_1,\pi_1,\pi_1)_* &: W_{01}[f_\alpha,g_\alpha,h_\alpha] \longrightarrow W_0[f,g,h],\\ (\varpi_1,\varpi_1,\varpi_1)_* &: W_1[f_\alpha,g_\alpha,h_\alpha] \longrightarrow W_0[f,g,h]\end{aligned}$$

on cohomology, as well as maps on the associated Galois cohomology groups.

The goal is now to relate the class

$$(\varpi_1,\varpi_1,\varpi_1)_*(\kappa_1(f_\alpha,g_\alpha,h_\alpha)) = (\pi_1,\pi_1,\pi_1)_*(\kappa_{01}(f_\alpha,g_\alpha,h_\alpha)) \tag{4.5}$$

to those arising from the diagonal cycles on the curve $X_0 = X$, whose level is prime to p.

To do this, it is key to understand how the maps π_{1*} and $(\pi_1,\pi_1,\pi_1)_*$ interact with the Hecke operators, especially with the ordinary and anti-ordinary projectors e and e^*, which do not act naturally on the target of π_{1*}. Consider the map

$$(\pi_1,\pi_2): W_{01}^{k_\circ} := H^1_{\text{ét}}(\bar X_{01},\mathcal{H}^{k_\circ}) \longrightarrow W_0^{k_\circ} := H^1_{\text{ét}}(\bar X_0,\mathcal{H}^{k_\circ}).$$

It is compatible in the obvious way with the good Hecke operators arising from primes $\ell \nmid Mp$, and therefore induces a map

$$(\pi_1,\pi_2): W_{01}^{k_\circ}[f] \longrightarrow W_0^{k_\circ}[f] \oplus W_0^{k_\circ}[f] \tag{4.6}$$

on the f-isotypic components for this Hecke action. As before, note that $W_{01}^{k_\circ}[f]$ is a priori larger than $W_{01}^{k_\circ}[f_\alpha]$, which is its ordinary quotient.

Let $\xi_f := \chi_f(p)p^{k-1}$ be the determinant of the frobenius at p acting on the two-dimensional Galois representation attached to f, and likewise for g and h.

Lemma 4.2. — *For the map (π_1,π_2) as in (4.6),*

$$\begin{aligned}\begin{pmatrix}\pi_1\\ \pi_2\end{pmatrix} \circ U_p &= \begin{pmatrix} a_p(f) & -1\\ \xi_f & 0\end{pmatrix}\begin{pmatrix}\pi_1\\ \pi_2\end{pmatrix},\\ \begin{pmatrix}\pi_1\\ \pi_2\end{pmatrix} \circ U_p^* &= \begin{pmatrix} 0 & p\\ -\xi_f p^{-1} & a_p(f)\end{pmatrix}\begin{pmatrix}\pi_1\\ \pi_2\end{pmatrix}.\end{aligned}$$

Proof. — The definitions π_1 and π_2 imply that, viewing U_p and U_p^* (resp. T_p) as correspondences on a Kuga-Sato variety fibered over X_{01} (resp. over X_0), we have

$$\begin{aligned}\pi_1 U_p &= T_p\pi_1 - \pi_2, & \pi_1 U_p^* &= p\pi_2\\ \pi_2 U_p &= p[p]\pi_1 & \pi_2 U_p^* &= -[p]\pi_1 + T_p\pi_2,\end{aligned}$$

where $[p]$ is the correspondence induced by the multiplication by p on the fibers and on the prime-to-p part of the level structure. The result follows by passing to the f-isotypic parts, using the fact that $[p]$ induces multiplication by $\xi_f p^{-1}$ on this isotypic part. □

For the next calculations, it shall be notationally convenient to introduce the notations

$$\delta_f = \alpha_f - \beta_f, \qquad \delta_g = \alpha_g - \beta_g, \qquad \delta_h = \alpha_h - \beta_h, \qquad \delta_{fgh} = \delta_f \delta_g \delta_h.$$

Lemma 4.3. — *For (π_1, π_2) as in Lemma 4.2,*

$$\pi_1 \circ e = \frac{\alpha_f \pi_1 - \pi_2}{\delta_f}, \qquad \pi_2 \circ e = \frac{\xi_f \pi_1 - \beta_f \pi_2}{\delta_f} = \beta_f \cdot (\pi_1 \circ e),$$
$$\pi_1 \circ e^* = \frac{-\beta_f \pi_1 + p\pi_2}{\delta_f}, \qquad \pi_2 \circ e^* = \frac{-\xi_f p^{-1}\pi_1 + \alpha_f \pi_2}{\delta_f} = p\alpha_f^{-1} \cdot (\pi_1 \circ e^*).$$

Proof. — The matrix identities

$$\begin{pmatrix} a_p(f) & -1 \\ \xi_f & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ \beta_f & \alpha_f \end{pmatrix} \begin{pmatrix} \alpha_f & 0 \\ 0 & \beta_f \end{pmatrix} \begin{pmatrix} 1 & 1 \\ \beta_f & \alpha_f \end{pmatrix}^{-1},$$
$$\begin{pmatrix} 0 & p \\ -\xi_f p^{-1} & a_p(f) \end{pmatrix} = \begin{pmatrix} \beta_f & \alpha_f \\ \xi_f p^{-1} & \xi_f p^{-1} \end{pmatrix} \begin{pmatrix} \alpha_f & 0 \\ 0 & \beta_f \end{pmatrix} \begin{pmatrix} \beta_f & \alpha_f \\ \xi_f p^{-1} & \xi_f(p) p^{-1} \end{pmatrix}^{-1},$$

show that

$$\lim \begin{pmatrix} a_p(f) & -1 \\ \xi_f & 0 \end{pmatrix}^{n!} = \begin{pmatrix} 1 & 1 \\ \beta_f & \alpha_f \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ \beta_f & \alpha_f \end{pmatrix}^{-1}$$
$$= \delta_f^{-1} \begin{pmatrix} \alpha_f & -1 \\ \xi_f & -\beta_f \end{pmatrix},$$
$$\lim \begin{pmatrix} 0 & p \\ -\xi_f p^{-1} & a_p(f) \end{pmatrix}^{n!} = \delta_f^{-1} \begin{pmatrix} -\beta_f & p \\ -\xi_f p^{-1} & \alpha_f \end{pmatrix},$$

and the result now follows from Lemma 4.2. □

Lemma 4.4. — *Let $\kappa \in H^1(\mathbf{Q}, W_{01}[f,g,h])$ be any cohomology class with values in the (f,g,h)-isotypic subspace of W_{01}, and let $e, e^* : H^1(\mathbf{Q}, W_{01}[fgh]) \longrightarrow H^1(\mathbf{Q}, W_{01}[f_\alpha, g_\alpha, h_\alpha])$ denote the ordinary and anti-ordinary projections. Then*

$$\begin{aligned}(\pi_1,\pi_1,\pi_1)_*(e\kappa) = \delta_{fgh}^{-1} \times \Big\{ & \alpha_f\alpha_g\alpha_h(\pi_1,\pi_1,\pi_1)_* \\ & -\alpha_g\alpha_h(\pi_2,\pi_1,\pi_1)_* - \alpha_f\alpha_h(\pi_1,\pi_2,\pi_1)_* - \alpha_f\alpha_g(\pi_1,\pi_1,\pi_2)_* \\ & +\alpha_f(\pi_1,\pi_2,\pi_2)_* + \alpha_g(\pi_2,\pi_1,\pi_2)_* + \alpha_h \cdot (\pi_2,\pi_2,\pi_1)_* \\ & -(\pi_2,\pi_2,\pi_2)_* \Big\}(\kappa).\end{aligned}$$

$$\begin{aligned}(\pi_1,\pi_1,\pi_1)_*(e^*\kappa) \;=\; & \delta_{fgh}^{-1}\times\Big\{-\beta_f\beta_g\beta_h(\pi_1,\pi_1,\pi_1)_* \\ & +p\beta_g\beta_h(\pi_2,\pi_1,\pi_1)_*+p\beta_f\beta_h(\pi_1,\pi_2,\pi_1)_*+p\beta_f\beta_g(\pi_1,\pi_1,\pi_2)_* \\ & -p^2\beta_f(\pi_1,\pi_2,\pi_2)_*-p^2\beta_g(\pi_2,\pi_1,\pi_2)_*-p^2\beta_h(\pi_2,\pi_2,\pi_1)_* \\ & +p^3(\pi_2,\pi_2,\pi_2)_*\Big\}(\kappa),\end{aligned}$$

where we recall that $\delta_{fgh} := ((\alpha_f-\beta_f)(\alpha_g-\beta_g)(\alpha_h-\beta_h))$.

Proof. — This follows directly from Lemma 4.3. □

Recall the notations

$$k_\circ := k-2, \qquad \ell_\circ = \ell-2, \qquad m_\circ := m-2, \qquad r := (k_\circ+\ell_\circ+m_\circ)/2.$$

Let $\mathcal{A}$ denote the Kuga-Sato variety over X as introduced in 1.2. In [**DR14**, Definitions 3.1,3.2 and 3.3], a generalized diagonal cycle

$$\Delta^{k_\circ,\ell_\circ,m_\circ} = \Delta_0^{k_\circ,\ell_\circ,m_\circ} \in \mathrm{CH}^{r+2}(\mathcal{A}^{k_\circ}\times\mathcal{A}^{\ell_\circ}\times\mathcal{A}^{m_\circ},\mathbf{Q})$$

is associated to the triple $(k_\circ,\ell_\circ,m_\circ)$.

When $k_\circ,\ell_\circ,m_\circ>0$, $\Delta^{k_\circ,\ell_\circ,m_\circ}$ is obtained by choosing subsets A, B and C of the set $\{1,\dots,r\}$ which satisfy:

$$\#A = k_\circ, \qquad \#B = \ell_\circ, \qquad \#C = m_\circ, \qquad A\cap B\cap C = \emptyset,$$

$$\#(B\cap C) = r-k_\circ, \qquad \#(A\cap C) = r-\ell_\circ, \qquad \#(A\cap B) = r-m_\circ.$$

The cycle $\Delta^{k_\circ,\ell_\circ,m_\circ}$ is defined as the image of the embedding $\mathcal{A}^r$ into $\mathcal{A}^{k_\circ}\times\mathcal{A}^{\ell_\circ}\times\mathcal{A}^{m_\circ}$ given by sending $(E,(P_1,\dots,P_r))$ to $((E,P_A),(E,P_B),(E,P_C))$, where for instance P_A is a shorthand for the $k_\circ$-tuple of points P_j with $j\in A$.

Let also $\Delta_{01}^{k_\circ,\ell_\circ,m_\circ}\in\mathrm{CH}^{r+2}(\mathcal{A}^{k_\circ}\times\mathcal{A}^{\ell_\circ}\times\mathcal{A}^{m_\circ})$ denote the generalised diagonal cycle in the product of the three Kuga-Sato varieties of weights (k,ℓ,m) fibered over X_{01}, defined in a similar way as in (4.4) and along the same lines as recalled above.

More precisely, $\Delta_{01}^{k_\circ,\ell_\circ,m_\circ}$ is defined as the schematic closure in $\mathcal{A}^{k_\circ}\times\mathcal{A}^{\ell_\circ}\times\mathcal{A}^{m_\circ}$ of the set of tuples $((E,C_1,P_A),(E,C_2,P_B),(E,C_3,P_C))$ where P_A,P_B,P_C are as above, and C_1,C_2,C_3 is a triple of pairwise distinct subgroups of order p in the elliptic curve E.

Since the triple $(k_\circ,\ell_\circ,m_\circ)$ is fixed throughout this section, in order to alleviate notations in the statements below we shall simply denote $\Delta^\sharp$ and $\Delta_{01}^\sharp$ for $\Delta^{k_\circ,\ell_\circ,m_\circ}$ and $\Delta_{01}^{k_\circ,\ell_\circ,m_\circ}$ respectively.

Lemma 4.5. — *The following identities hold in* $\mathrm{CH}^{r+2}(\mathcal{A}^{k_\circ} \times \mathcal{A}^{\ell_\circ} \times \mathcal{A}^{m_\circ})$*:*

$$\begin{aligned}
(\pi_1,\pi_1,\pi_1)_*(\Delta^\sharp_{01}) &= (p+1)p(p-1)(\Delta^\sharp),\\
(\pi_2,\pi_1,\pi_1)_*(\Delta^\sharp_{01}) &= p(p-1)\times(T_p,1,1)(\Delta^\sharp),\\
(\pi_1,\pi_2,\pi_1)_*(\Delta^\sharp_{01}) &= p(p-1)\times(1,T_p,1)(\Delta^\sharp),\\
(\pi_1,\pi_1,\pi_2)_*(\Delta^\sharp_{01}) &= p(p-1)\times(1,1,T_p)(\Delta^\sharp),\\
(\pi_1,\pi_2,\pi_2)_*(\Delta^\sharp_{01}) &= (p-1)\times((1,T_p,T_p)(\Delta^\sharp)-p^{r-k_\circ}D_1)\\
(\pi_2,\pi_1,\pi_2)_*(\Delta^\sharp_{01}) &= (p-1)\times((T_p,1,T_p)(\Delta^\sharp)-p^{r-\ell_\circ}D_2)\\
(\pi_2,\pi_2,\pi_1)_*(\Delta^\sharp_{01}) &= (p-1)\times((T_p,T_p,1)(\Delta^\sharp)-p^{r-m_\circ}D_3)\\
(\pi_2,\pi_2,\pi_2)_*(\Delta^\sharp_{01}) &= (T_p,T_p,T_p)(\Delta^\sharp)-p^{r-k_\circ}E_1-p^{r-\ell_\circ}E_2-p^{r-m_\circ}E_3\\
&\quad -p^r(p+1)\Delta^\sharp,
\end{aligned}$$

where the cycles D_1, D_2 and D_3 satisfy

$$([p],1,1)_*(D_1)=p^{k_\circ}(T_p,1,1)_*(\Delta^\sharp),\qquad (1,[p],1)_*(D_2)=p^{\ell_\circ}(1,T_p,1)_*(\Delta^\sharp),$$
$$(1,1,[p])_*(D_3)=p^{m_\circ}(1,1,T_p)(\Delta^\sharp),$$

the cycles E_1, E_2 and E_3 satisfy

$$([p],1,1)_*(E_1)=p^{k_\circ}(T_{p^2},1,1)(\Delta^\sharp),\qquad (1,[p],1)_*(E_2)=p^{\ell_\circ}(1,T_{p^2},1)(\Delta^\sharp),$$
$$(1,1,[p])_*(E_3)=p^{m_\circ}(1,1,T_{p^2})(\Delta^\sharp),$$

and $T_{p^2} := T_p^2 - (p+1)[p]$.

Proof. — The first four identities are straightforward: the map $\pi_1\times\pi_1\times\pi_1$ induces a finite map from $\Delta^\sharp_{01}$ to $\Delta^\sharp$ of degree $(p+1)p(p-1)$, which is the number of possible choices of an ordered triple of distinct subgroups of order p on an elliptic curve, and likewise $\pi_2\times\pi_1\times\pi_1$ induces a map of degree $p(p-1)$ from $\Delta^\sharp_{01}$ to $(T_p,1,1)\Delta^\sharp$. The remaining identities follow from combinatorial reasonings based on the explicit description of the cycles $\Delta^\sharp_{01}$ and $\Delta^\sharp$. For the 5th identity, observe that $(\pi_1,\pi_2,\pi_2)_*$ induces a degree $(p-1)$ map from $\Delta^\sharp_{01}$ to the variety X parametrising triples $((E,P_A),(E',P'_B),(E'',P''_C))$ for which there are distinct cyclic p-isogenies $\varphi' : E\longrightarrow E'$ and $\varphi' : E\longrightarrow E''$, the system of points $P'_B\subset E'$ and $P''_C\subset E''$ indexed by the sets B and C satisfy

$$\varphi'(P_{A\cap B}) = P'_{A\cap B},\quad \varphi''(P_{A\cap C}) = P''_{A\cap C},$$

and for which there exists points $Q_{B\cap C}\subset E$ indexed by $B\cap C$ satisfying

$$\varphi'(Q_{B\cap C}) = P'_{B\cap C},\qquad \varphi''(Q_{B\cap C}) = P''_{B\cap C}.$$

On the other hand, $(1,T_p,T_p)$ parametrises triples of the same type, in which E' and E'' need not be distinct. It follows that

$$(1,T_p,T_p)(\Delta^\sharp) = X + p^{r-k_\circ}D_1, \tag{4.7}$$

where the closed points of D_1 correspond to triples $((E,P_A),(E',P'_B),(E',P'_C))$ for which there is a cyclic p-isogeny $\varphi' : E \longrightarrow E'$ satisfying

$$\varphi'(P_{A\cap B}) = P'_{A\cap B}, \qquad \varphi'(P_{A\cap C}) = P'_{A\cap C}.$$

The coefficient of $p^{r-k_\circ}$ in (4.7) arises because each closed point of D_1 comes from $p^{\#(B\cap C)}$ distinct closed points on $(1,T_p,T_p)(\Delta^\sharp)$, obtained by translating the points $P_j \in P_{B\cap C}$ with $j \in B\cap C$ by arbitrary elements of $\ker(\varphi)$. The fifth identity now follows after noting that the map $([p],1,1])$ induces a map of degree $p^{k_\circ}$ from D_1 to $(T_p,1,1)_*\Delta^\sharp$. The 6th and 7th identity can be treated with an identical reasoning by interchanging the three factors in $W^{k_\circ}\times W^{\ell_\circ}\times W^{m_\circ}$. As for the last identity, the map (π_2,π_2,π_2) induces a map of degree 1 to the variety Y consisting of triples (E',E'',E''') of elliptic curves which are p-isogenous to a common elliptic curve E and distinct. But it is not hard to see that

$$(T_p,T_p,T_p)(\Delta^\sharp) = Y + p^{r-k_\circ}E_1 + p^{r-\ell_\circ}E_2 + p^{r-m_\circ}E_3 + p^r(p+1)\Delta^\sharp$$

where the additional terms on the right hand side account for triples (E',E'',E''') where $E'\neq E''=E'''$, where $E''\neq E'=E'''$, where $E'''\neq E'=E''$, and where $E'=E''=E'''$ respectively. □

Assume for the remainder of the section that $k_\circ,\ell_\circ,m_\circ>0$. Recall the projectors $\epsilon_{k_\circ}$ of (1.5). It was shown in [**DR14**, §3.1] that $(\epsilon_{k_\circ},\epsilon_{\ell_\circ},\epsilon_{m_\circ})\Delta^{k_\circ,\ell_\circ,m_\circ}$ is a null-homologous cycle and we may define

$$\kappa(f,g,h) := \pi_{f,g,h}\,\mathrm{AJ}_{\text{ét}}((\epsilon_{k_\circ},\epsilon_{\ell_\circ},\epsilon_{m_\circ})\Delta^{k_\circ,\ell_\circ,m_\circ}) \in H^1(\mathbf{Q},W_0[f,g,h]) \tag{4.8}$$

as the image of this cycle under the p-adic étale Abel-Jacobi map, followed by the natural projection from $H^{2c-1}_{\text{ét}}(\bar{\mathcal{A}}^{k_\circ}\times\bar{\mathcal{A}}^{\ell_\circ}\times\bar{\mathcal{A}}^{m_\circ},\mathbf{Q}_p(c))$ to $W_0^{k_\circ,\ell_\circ,m_\circ}$ induced by the Künneth decomposition and the projection $\pi_{f,g,h}$.

It follows from [**DR14**, (66)], (1.5) and the vanishing of the terms $H^i_{\text{ét}}(\bar{X}_1,\mathcal{H}^{k_\circ})$ for $i\neq 1$ when $k_\circ>0$, that the class $\kappa(f,g,h)$ is realized by the (f,g,h)-isotypic component of the same extension class as in (3.5), after replacing X_1 by the curve $X=X_0$ and $\Delta=\Delta^{0,0,0}$ is taken to be the usual diagonal cycle in X^3. In the notations of (3.4), this amounts to

$$\kappa(f,g,h) = \pi_{f,g,h}\mathrm{AJ}_{k_\circ,\ell_\circ,m_\circ}(\Delta). \tag{4.9}$$

Similar statements holds over the curve X_{01}. Namely, we also have the following:

Proposition 4.6. — *The cycle $(\epsilon_{k_\circ},\epsilon_{\ell_\circ},\epsilon_{m_\circ})\Delta_{01}^{k_\circ,\ell_\circ,m_\circ}$ is null-homologous and the following equality of classes holds in $H^1(\mathbf{Q},W_{01}[f_\alpha,g_\alpha,h_\alpha])$:*

$$\kappa_{01}(f_\alpha,g_\alpha,h_\alpha) = p^3\cdot\pi_{f_\alpha,g_\alpha,h_\alpha}\,\mathrm{AJ}_{\text{ét}}((\epsilon_{k_\circ},\epsilon_{\ell_\circ},\epsilon_{m_\circ})\Delta_{01}^{k_\circ,\ell_\circ,m_\circ}). \tag{4.10}$$

Proof. — Corollary 3.3 together with (4.2) imply that

$$\kappa_1(f_\alpha,g_\alpha,h_\alpha) = \frac{1}{C_q}\cdot\pi_{f_\alpha,g_\alpha,h_\alpha}\,\mathrm{AJ}_{k_\circ,\ell_\circ,m_\circ}(\Delta_1^\circ(1,1,1;\delta)),$$

in which $\delta = 1$ is the trivial character of $(\mathbf{Z}/p\mathbf{Z})^\times$. Since μ^3 induces a finite map of degree $(p-1)^3$ from the support of $\Delta_1(1,1,1;\delta)$ to Δ_{01}, it follows from the convention adopted in (3.7) that

$$\kappa_{01}(f_\alpha, g_\alpha, h_\alpha) := \mu_*^3\,\kappa_1(f_\alpha, g_\alpha, h_\alpha) = \frac{p^3}{C_q} \cdot \pi_{f_\alpha,g_\alpha,h_\alpha}\,\mathrm{AJ}_{k_\circ,\ell_\circ,m_\circ}(\Delta_{01}^\circ),$$

where $\mathrm{AJ}_{k_\circ,\ell_\circ,m_\circ}(\Delta_{01}^\circ)$ is defined to be the class realized by the same extension class as in (3.5), after replacing X_1 by the curve X_{01} and replacing Δ by the cycle Δ_{01}° arising from (4.4). Since $\Delta_{01}^{k_\circ,\ell_\circ,m_\circ}$ is fibered over Δ_{01}, the same argument as in (4.9) then shows that

$$\mathrm{AJ}_{k_\circ,\ell_\circ,m_\circ}(\Delta_{01}) = \mathrm{AJ}_{\text{ét}}((\epsilon_{k_\circ}, \epsilon_{\ell_\circ}, \epsilon_{m_\circ})\Delta_{01}^{k_\circ,\ell_\circ,m_\circ}).$$

Since $\pi_{f_\alpha,g_\alpha,h_\alpha}(\Delta_{01}) = \frac{1}{C_q}\pi_{f_\alpha,g_\alpha,h_\alpha}(\Delta_{01}^\circ)$, the proposition follows. □

Theorem 4.1. — *With notations as before, letting $c = r+2$, we have*

$$(\varpi_1, \varpi_1, \varpi_1)_*\ \kappa_1(f_\alpha, g_\alpha, h_\alpha) = \frac{\mathcal{E}^{\mathrm{bal}}(f_\alpha, g_\alpha, h_\alpha)}{\mathcal{E}(f_\alpha)\mathcal{E}(g_\alpha)\mathcal{E}(h_\alpha)} \times \kappa(f, g, h),$$

where

$$\mathcal{E}^{\mathrm{bal}}(f_\alpha, g_\alpha, h_\alpha) = (1-\alpha_f\beta_g\beta_h p^{-c})(1-\beta_f\alpha_g\beta_h p^{-c})(1-\beta_f\beta_g\alpha_h p^{-c})(1-\beta_f\beta_g\beta_h p^{-c}),$$

and

$$\mathcal{E}(f_\alpha) = 1-\chi_f^{-1}(p)\beta_f^2 p^{1-k}, \quad \mathcal{E}(g_\alpha) = 1-\chi_g^{-1}(p)\beta_g^2 p^{1-\ell}, \quad \mathcal{E}(h_\alpha) = 1-\chi_h^{-1}(p)\beta_h^2 p^{1-m}.$$

Proof. — In view of (4.5), (4.8) and (4.10), it suffices to prove the claim for the cycles $\Delta^{k_\circ,\ell_\circ,m_\circ}$ and $(\pi_1,\pi_1,\pi_1)_*\ e^*\Delta_{01}^{k_\circ,\ell_\circ,m_\circ}$. Since $k_\circ, \ell_\circ, m_\circ$ are fixed throughout the discussion, we again denote $\Delta^\sharp = \Delta^{k_\circ,\ell_\circ,m_\circ}$ and $\Delta_{01}^\sharp = \Delta_{01}^{k_\circ,\ell_\circ,m_\circ}$ to lighten notations.

When combined with Lemma 4.4, Lemma 4.5 equips us with a completely explicit formula for comparing $(\pi_1,\pi_1,\pi_1)_*e^*(\Delta_{01}^\sharp)$ with the generalised diagonal cycle $\Delta^\sharp$. Namely, since the correspondences $([p],1,1)$, $(1,[p],1)$ and $(1,1,[p])$ induce multiplication by $p^{k_\circ}$, $p^{\ell_\circ}$ and $p^{m_\circ}$ respectively on the (f,g,h)-isotypic parts, while $(T_p,1,1)$, $(1,T_p,1)$, and $(1,1,T_p)$ induce multiplication by $a_p(f)$, $a_p(g)$, and $a_p(h)$ respectively, it follows that, with notations as in the proof of Lemma 4.5,

$$\begin{aligned}
\pi_{f,g,h}(D_1) &= a_p(f)\pi_{f,g,h}(\Delta^\sharp),\\
\pi_{f,g,h}(D_2) &= a_p(g)\pi_{f,g,h}(\Delta^\sharp),\\
\pi_{f,g,h}(D_3) &= a_p(h)\pi_{f,g,h}(\Delta^\sharp),
\end{aligned}$$

and that

$$\begin{aligned}
\pi_{f,g,h}(E_1) &= (a_p^2(f) - (p+1)p^{k_\circ})\pi_{f,g,h}(\Delta^\sharp),\\
\pi_{f,g,h}(E_2) &= (a_p^2(g) - (p+1)p^{\ell_\circ})\pi_{f,g,h}(\Delta^\sharp),\\
\pi_{f,g,h}(E_3) &= (a_p^2(h) - (p+1)p^{m_\circ})\pi_{f,g,h}(\Delta^\sharp).
\end{aligned}$$

By projecting the various formulae for $(\pi_1,\pi_1,\pi_1)_*(\Delta_{01}^\sharp)$ that are given in Lemma 4.5 to the (f,g,h)-isotypic component and substituting them into Lemma 4.4, one obtains a expression for $e_{f,g,h}(\pi_1,\pi_1,\pi_1)_*e^*(\Delta_{01}^\sharp)$ as a multiple of $\pi_{f,g,h}(\Delta^\sharp)$ by an explicit factor, which is a rational function in α_f, α_g and α_h. This explicit factor is somewhat tedious to calculate by hand, but the identity asserted in Theorem 4.1 is readily checked with the help of a symbolic algebra package. □

5. Triple product p-adic L-functions and the reciprocity law

Let $(\mathbf{f},\mathbf{g},\mathbf{h})$ be a triple of p-adic Hida families of tame levels M_f, M_g, M_h and tame characters χ_f, χ_g, χ_h as in the previous section. Let also $(\mathbf{f}^*,\mathbf{g}^*,\mathbf{h}^*) = (\mathbf{f}\otimes\bar\chi_f, \mathbf{g}\otimes\bar\chi_g, \mathbf{h}\otimes\bar\chi_h)$ denote the conjugate triple. As before, we assume $\chi_f\chi_g\chi_h = 1$ and set $M = \mathrm{lcm}(M_f, M_g, M_h)$.

Let $\Lambda_{\mathbf{f}}$, $\Lambda_{\mathbf{g}}$ and $\Lambda_{\mathbf{h}}$ be the finite flat extensions of Λ generated by the coefficients of the Hida families $\mathbf{f}$, $\mathbf{g}$ and $\mathbf{h}$, and set $\Lambda_{\mathbf{fgh}} = \Lambda_{\mathbf{f}}\hat\otimes_{\mathbf{Z}_p}\Lambda_{\mathbf{g}}\hat\otimes_{\mathbf{Z}_p}\Lambda_{\mathbf{h}}$. Let also $\mathcal{Q}_{\mathbf{f}}$ denote the fraction field of $\Lambda_{\mathbf{f}}$ and define

$$\mathcal{Q}_{\mathbf{f},\mathbf{gh}} := \mathcal{Q}_{\mathbf{f}}\hat\otimes\Lambda_{\mathbf{g}}\hat\otimes\Lambda_{\mathbf{h}}.$$

Let $\mathcal{W}^\circ_{\mathbf{fgh}} := \mathcal{W}^\circ_{\mathbf{f}}\times\mathcal{W}^\circ_{\mathbf{g}}\times\mathcal{W}^\circ_{\mathbf{h}} \subset \mathcal{W}_{\mathbf{fgh}} = \mathrm{Spf}(\Lambda_{\mathbf{fgh}})$ denote the set of triples of *crystalline* classical points, at which the three Hida families specialize to modular forms with trivial nebentype at p (and may be either old or new at p). This set admits a natural partition, namely

$$\mathcal{W}^\circ_{\mathbf{fgh}} = \mathcal{W}^f_{\mathbf{fgh}} \sqcup \mathcal{W}^g_{\mathbf{fgh}} \sqcup \mathcal{W}^h_{\mathbf{fgh}} \sqcup \mathcal{W}^{\mathrm{bal}}_{\mathbf{fgh}}$$

where

— $\mathcal{W}^f_{\mathbf{fgh}}$ denotes the set of points $(x,y,z)\in\mathcal{W}^\circ_{\mathbf{fgh}}$ of weights (k,ℓ,m) such that $k\geq \ell+m$.

— $\mathcal{W}^g_{\mathbf{fgh}}$ and $\mathcal{W}^h_{\mathbf{fgh}}$ are defined similarly, replacing the role of $\mathbf{f}$ with $\mathbf{g}$ (resp. $\mathbf{h}$).

— $\mathcal{W}^{\mathrm{bal}}_{\mathbf{fgh}}$ is the set of *balanced* triples, consisting of points (x,y,z) of weights (k,ℓ,m) such that each of the weights is strictly smaller than the sum of the other two.

Each of the four subsets appearing in the above partition is dense in $\mathcal{W}_{\mathbf{fgh}}$ for the rigid-analytic topology.

Recall from (1.34) the spaces of Λ-adic test vectors $S^{\mathrm{ord}}_\Lambda(M,\chi_f)[\mathbf{f}]$. For any choice of a triple

$$(\breve{\mathbf{f}},\breve{\mathbf{g}},\breve{\mathbf{h}}) \in S^{\mathrm{ord}}_\Lambda(M,\chi_f)[\mathbf{f}]\times S^{\mathrm{ord}}_\Lambda(M,\chi_g)[\mathbf{g}]\times S^{\mathrm{ord}}_\Lambda(M,\chi_h)[\mathbf{h}]$$

of Λ-adic test vectors of tame level M, in [**DR14**, Lemma 2.19 and Definition 4.4] we constructed a p-adic L-function $\mathscr{L}_p{}^f(\breve{\mathbf{f}},\breve{\mathbf{g}},\breve{\mathbf{h}})$ in $\mathcal{Q}_{\mathbf{f}}\hat\otimes\Lambda_{\mathbf{g}}\hat\otimes\Lambda_{\mathbf{h}}$, giving rise to a meromorphic rigid-analytic function

$$\mathscr{L}_p{}^f(\breve{\mathbf{f}},\breve{\mathbf{g}},\breve{\mathbf{h}}) : \mathcal{W}_{\mathbf{fgh}} \longrightarrow \mathbf{C}_p. \tag{5.1}$$

As shown in [**DR14**, §4], this p-adic L-function is characterized by an interpolation property relating its values at classical points $(x,y,z)\in\mathcal{W}^f_{\mathbf{fgh}}$ to the central critical value of Garrett's triple-product complex L-function $L(\mathbf{f}_x,\mathbf{g}_y,\mathbf{h}_z,s)$ associated to the triple of classical eigenforms $(\mathbf{f}_x,\mathbf{g}_y,\mathbf{h}_z)$. The fudge factors appearing in the interpolation property depend heavily on the choice of test vectors: cf. [**DR14**, §4] and

[**DLR15**, §2] for more details. In a recent preprint, Hsieh [**H17**] has found an explicit choice of test vectors, which yields a very optimal interpolation formula which shall be useful for our purposes. We describe it below:

Proposition 5.1. — *for every $(x,y,z)\in\mathcal{W}^f_{\mathbf{fgh}}$ of weights (k,ℓ,m) we have*

$$\mathscr{L}_p{}^f(\breve{\mathbf{f}},\breve{\mathbf{g}},\breve{\mathbf{h}})^2(x,y,z)=\frac{\mathfrak{a}(k,\ell,m)}{\langle\mathbf{f}_x^\circ,\mathbf{f}_x^\circ\rangle^2}\cdot\mathfrak{e}^2(x,y,z)\cdot\prod_{v\mid N\infty}C_v\times L(\mathbf{f}_x^\circ,\mathbf{g}_y^\circ,\mathbf{h}_z^\circ,c) \tag{5.2}$$

where

i) $c=\frac{k+\ell+m-2}{2}$,

ii) $\mathfrak{a}(k,\ell,m)=(2\pi i)^{-2k}\cdot(\frac{k+\ell+m-4}{2})!\cdot(\frac{k+\ell-m-2}{2})!\cdot(\frac{k-\ell+m-2}{2})!\cdot(\frac{k-\ell-m}{2})!$,

iii) $\mathfrak{e}(x,y,z)=\mathcal{E}(x,y,z)/\mathcal{E}_0(x)\mathcal{E}_1(x)$ *with*

$$\begin{aligned}\mathcal{E}_0(x)&:=1-\chi_f^{-1}(p)\beta_{\mathbf{f}_x}^2p^{1-k},\\ \mathcal{E}_1(x)&:=1-\chi_f(p)\alpha_{\mathbf{f}_x}^{-2}p^{k_\circ},\\ \mathcal{E}(x,y,z)&:=\left(1-\chi_f(p)\alpha_{\mathbf{f}_x}^{-1}\alpha_{\mathbf{g}_y}\alpha_{\mathbf{h}_z}p^{\frac{k-\ell-m}{2}}\right)\times\left(1-\chi_f(p)\alpha_{\mathbf{f}_x}^{-1}\alpha_{\mathbf{g}_y}\beta_{\mathbf{h}_z}p^{\frac{k-\ell-m}{2}}\right)\\ &\quad\times\left(1-\chi_f(p)\alpha_{\mathbf{f}_x}^{-1}\beta_{\mathbf{g}_y}\alpha_{\mathbf{h}_z}p^{\frac{k-\ell-m}{2}}\right)\times\left(1-\chi_f(p)\alpha_{\mathbf{f}_x}^{-1}\beta_{\mathbf{g}_y}\beta_{\mathbf{h}_z}p^{\frac{k-\ell-m}{2}}\right).\end{aligned}$$

iv) The local constant $C_v\in\mathbf{Q}(\mathbf{f}_x,\mathbf{g}_y,\mathbf{h}_z)$ depends only on the admissible representations of $\mathrm{GL}_2(\mathbf{Q}_v)$ associated to $(\mathbf{f}_x,\mathbf{g}_y,\mathbf{h}_z)$ and on the local components at v of the test vectors.

Moreover, there exists a distinguished choice of test vectors $(\breve{\mathbf{f}},\breve{\mathbf{g}},\breve{\mathbf{h}})$ (as specified by Hsieh in [**H17**, §3]*) for which $\mathscr{L}_p{}^f(\breve{\mathbf{f}},\breve{\mathbf{g}},\breve{\mathbf{h}})$ lies in $\Lambda_{\mathbf{fgh}}$ and the local constants may be taken to be $C_v=1$ at all $v\mid N\infty$.*

Proof. — This follows from [**H17**, Theorem A], after spelling out explicitly the definitions involved in Hsieh's formulation.

Let us remark that throughout the whole article [**DR14**], it was implicitly assumed that $\mathbf{f}_x$, $\mathbf{g}_\ell$ and $\mathbf{h}_m$ are all old at p, and note that the definition we have given here of the terms $\mathcal{E}_0(x)$, $\mathcal{E}_1(x)$ and $\mathcal{E}(x,y,z)$ is exactly the same as in [**DR14**] in such cases, because $\beta_{\mathbf{f}_x}=\chi_f(p)\alpha_{\mathbf{f}_x}^{-1}p^{k-1}$ when $\mathbf{f}_x$ is old at p.

In contrast with loc. cit., in the above proposition we also allow any of the eigenforms $\mathbf{f}_x$, $\mathbf{g}_\ell$ and $\mathbf{h}_m$ to be new at p (which can only occur when the weight is 2); in such case, recall the usual convention adopted in §1.2 to set $\beta_\phi=0$ when p divides the primitive level of an eigenform ϕ. With these notations, the current formulation of $\mathcal{E}(x,y,z)$, $\mathcal{E}_0(x)$ and $\mathcal{E}_1(x)$ is the correct one, as one can readily verify by rewriting the proof of [**DR14**, Lemma 4.10]. □

5.1. Perrin-Riou's regulator. — Recall the Λ-adic cyclotomic character $\underline{\varepsilon}_{\mathrm{cyc}}$ and the unramified characters $\Psi_{\mathbf{f}}$, $\Psi_{\mathbf{g}}$, $\Psi_{\mathbf{h}}$ of $G_{\mathbf{Q}_p}$ introduced in Theorem 1.1. As a piece of notation, let $\underline{\varepsilon}_{\mathbf{f}}:G_{\mathbf{Q}_p}\longrightarrow\Lambda_{\mathbf{f}}^\times$ denote the composition of $\underline{\varepsilon}_{\mathrm{cyc}}$ and the natural inclusion $\Lambda^\times\subset\Lambda_{\mathbf{f}}^\times$, and likewise for $\underline{\varepsilon}_{\mathbf{g}}$ and $\underline{\varepsilon}_{\mathbf{h}}$. Expressions like $\Psi_{\mathbf{f}}\Psi_{\mathbf{g}}\Psi_{\mathbf{h}}$ or $\underline{\varepsilon}_{\mathbf{f}}\underline{\varepsilon}_{\mathbf{g}}\underline{\varepsilon}_{\mathbf{h}}$

are a short-hand notation for the $\Lambda_{\mathbf{fgh}}^{\times}$-valued character of $G_{\mathbf{Q}_p}$ given by the tensor product of the three characters.

Let $\mathbb{V}_{\mathbf{f}}$, $\mathbb{V}_{\mathbf{g}}$ and $\mathbb{V}_{\mathbf{h}}$ be the Galois representations associated to $\mathbf{f}$, $\mathbf{g}$ and $\mathbf{h}$ in Theorem 1.1.

The purpose of this section is describing in precise terms the close connection between the diagonal cycles constructed above and the three-variable triple-product p-adic L-function. In order to do that, let us introduce the $\Lambda_{\mathbf{fgh}}$-modules

$$\mathbb{V}_{\mathbf{fgh}}^{\dagger} := \mathbb{V}_{\mathbf{f}} \otimes \mathbb{V}_{\mathbf{g}} \otimes \mathbb{V}_{\mathbf{h}}(-1)(\tfrac{1}{2}) = \mathbb{V}_{\mathbf{f}} \otimes \mathbb{V}_{\mathbf{g}} \otimes \mathbb{V}_{\mathbf{h}}(\varepsilon_{\mathrm{cyc}}^{-1} \underline{\varepsilon}_{\mathbf{f}}^{-1/2} \underline{\varepsilon}_{\mathbf{g}}^{-1/2} \underline{\varepsilon}_{\mathbf{h}}^{-1/2}). \tag{5.3}$$

and

$$\mathbb{V}_{\mathbf{fgh}}^{\dagger}(M) := \mathbb{V}_{\mathbf{f}}(M) \otimes \mathbb{V}_{\mathbf{g}}(M) \otimes \mathbb{V}_{\mathbf{h}}(M)(-1)(\tfrac{1}{2}). \tag{5.4}$$

The pairing defined in (2.17) yields an identification $\mathbb{H}^{111}(X_\infty^*) = H^1_{\mathrm{\acute{e}t}}(\bar{X}_\infty, \mathbf{Z}_p)^{\otimes 3}(2)(\frac{1}{2})$. As explained in (1.26), $\mathbb{V}_{\mathbf{fgh}}^{\dagger}(M)$ is isomorphic to the direct sum of several copies of $\mathbb{V}_{\mathbf{fgh}}^{\dagger}$ and there are canonical projections $\varpi_{\mathbf{f}}$, $\varpi_{\mathbf{g}}$, $\varpi_{\mathbf{h}}$ which assemble into a $G_{\mathbf{Q}}$-equivariant map

$$\varpi_{\mathbf{f},\mathbf{g},\mathbf{h}} : \mathbb{H}^{111}(X_\infty^*) = H^1_{\mathrm{\acute{e}t}}(\bar{X}_\infty, \mathbf{Z}_p)^{\otimes 3}(2)(\tfrac{1}{2}) \longrightarrow \mathbb{V}_{\mathbf{fgh}}^{\dagger}(M).$$

Recall the three-variable Λ-adic global cohomology class

$$\boldsymbol{\kappa}_\infty(\epsilon_1 \omega^{-k_\circ}, \epsilon_2 \omega^{-\ell_\circ}, \epsilon_3 \omega^{-m_\circ}; 1) = \boldsymbol{\kappa}_\infty(1,1,1;1) \in H^1(\mathbf{Q}, \mathbb{H}^{111}(X_\infty^*))$$

introduced in (4.1).

Set $C_q(\mathbf{f},\mathbf{g},\mathbf{h}) := (a_q(\mathbf{f}) - q - 1)(a_q(\mathbf{g}) - q - 1)(a_q(\mathbf{h}) - q - 1)$. Note that $C_q(\mathbf{f},\mathbf{g},\mathbf{h})$ is a unit in $\Lambda_{\mathbf{fgh}}$, because its classical specializations are p-adic units (cf. (4.2)).

Definition 5.2. — *Define*

$$\boldsymbol{\kappa}(\mathbf{f},\mathbf{g},\mathbf{h}) := \frac{1}{C_q(\mathbf{f},\mathbf{g},\mathbf{h})} \cdot \varpi_{\mathbf{f},\mathbf{g},\mathbf{h}*}(\boldsymbol{\kappa}_\infty(\epsilon_1 \omega^{-k_\circ}, \epsilon_2 \omega^{-\ell_\circ}, \epsilon_3 \omega^{-m_\circ}; 1)) \in H^1(\mathbf{Q}, \mathbb{V}_{\mathbf{fgh}}^{\dagger}(M))$$

to be the projection of the above class to the $(\mathbf{f},\mathbf{g},\mathbf{h})$*-isotypical component.*

In the above definition, we normalize $\boldsymbol{\kappa}(\mathbf{f},\mathbf{g},\mathbf{h})$ by the constant $C_q(\mathbf{f},\mathbf{g},\mathbf{h})$ so that the classical specializations of $\boldsymbol{\kappa}(\mathbf{f},\mathbf{g},\mathbf{h})$ at classical points coincide with the classes $\kappa_1(f_\alpha, g_\alpha, h_\alpha)$ introduced in (4.2).

Let

$$\mathrm{res}_p : H^1(\mathbf{Q}, \mathbb{V}_{\mathbf{fgh}}^{\dagger}(M)) \to H^1(\mathbf{Q}_p, \mathbb{V}_{\mathbf{fgh}}^{\dagger}(M))$$

denote the restriction map to the local cohomology at p and set

$$\boldsymbol{\kappa}_p(\mathbf{f},\mathbf{g},\mathbf{h}) := \mathrm{res}_p(\boldsymbol{\kappa}(\mathbf{f},\mathbf{g},\mathbf{h})) \in H^1(\mathbf{Q}_p, \mathbb{V}_{\mathbf{fgh}}^{\dagger}(M)).$$

The main result of this section asserts that the p-adic L-function $\mathscr{L}_p{}^f(\breve{\mathbf{f}}, \breve{\mathbf{g}}, \breve{\mathbf{h}})$ introduced in §5 can be recast as the image of the Λ-adic class $\boldsymbol{\kappa}_p(\mathbf{f},\mathbf{g},\mathbf{h})$ under a suitable three-variable Perrin-Riou regulator map whose formulation relies on a choice of families of periods which depends on the test vectors $\breve{\mathbf{f}}, \breve{\mathbf{g}}, \breve{\mathbf{h}}$.

The recipe we are about to describe depends solely only on the projection of $\boldsymbol{\kappa}_p(\mathbf{f},\mathbf{g},\mathbf{h})$ to a suitable sub-quotient of $\mathbb{V}^{\dagger}_{\mathbf{fgh}}$ which is free of rank one over $\Lambda_{\mathbf{fgh}}$, and whose definition requires the following lemma.

Lemma 5.3. — *The Galois representation $\mathbb{V}^{\dagger}_{\mathbf{fgh}}$ is endowed with a four-step filtration*

$$0 \subset \mathbb{V}^{++}_{\mathbf{fgh}} \subset \mathbb{V}^{+}_{\mathbf{fgh}} \subset \mathbb{V}^{-}_{\mathbf{fgh}} \subset \mathbb{V}^{\dagger}_{\mathbf{fgh}}$$

by $G_{\mathbf{Q}_p}$-stable $\Lambda_{\mathbf{fgh}}$-submodules of ranks 0, 1, 4, 7 *and* 8 *respectively.*

The group $G_{\mathbf{Q}_p}$ acts on the successive quotients for this filtration (which are free over $\Lambda_{\mathbf{fgh}}$ of ranks 1, 3, 3 *and* 1 *respectively) as a direct sum of one dimensional characters,*

$$\mathbb{V}^{++}_{\mathbf{fgh}} = \eta^{\mathbf{fgh}}, \qquad \frac{\mathbb{V}^{+}_{\mathbf{fgh}}}{\mathbb{V}^{++}_{\mathbf{fgh}}} = \eta^{\mathbf{gh}}_{\mathbf{f}} \oplus \eta^{\mathbf{fh}}_{\mathbf{g}} \oplus \eta^{\mathbf{fg}}_{\mathbf{h}}, \qquad \frac{\mathbb{V}^{-}_{\mathbf{fgh}}}{\mathbb{V}^{+}_{\mathbf{fgh}}} = \eta_{\mathbf{gh}}^{\mathbf{f}} \oplus \eta_{\mathbf{fh}}^{\mathbf{g}} \oplus \eta_{\mathbf{fg}}^{\mathbf{h}}, \qquad \frac{\mathbb{V}^{\dagger}_{\mathbf{fgh}}}{\mathbb{V}^{-}_{\mathbf{fgh}}} = \eta_{\mathbf{fgh}},$$

where

$$\eta^{\mathbf{fgh}} = (\Psi_{\mathbf{f}}\Psi_{\mathbf{g}}\Psi_{\mathbf{h}} \times \varepsilon_{\mathrm{cyc}}^{2}(\underline{\varepsilon}_{\mathbf{f}}\underline{\varepsilon}_{\mathbf{g}}\underline{\varepsilon}_{\mathbf{h}})^{1/2}, \qquad \eta_{\mathbf{fgh}} = \Psi_{\mathbf{f}}\Psi_{\mathbf{g}}\Psi_{\mathbf{h}} \times \varepsilon_{\mathrm{cyc}}^{-1}(\underline{\varepsilon}_{\mathbf{f}}\underline{\varepsilon}_{\mathbf{g}}\underline{\varepsilon}_{\mathbf{h}})^{-1/2},$$

$$\eta^{\mathbf{gh}}_{\mathbf{f}} = \chi_f^{-1}\Psi_{\mathbf{f}}\Psi_{\mathbf{g}}^{-1}\Psi_{\mathbf{h}}^{-1} \times \varepsilon_{\mathrm{cyc}}(\underline{\varepsilon}_{\mathbf{f}}^{-1}\underline{\varepsilon}_{\mathbf{g}}\underline{\varepsilon}_{\mathbf{h}})^{1/2}, \quad \eta_{\mathbf{gh}}^{\mathbf{f}} = \chi_f\Psi_{\mathbf{f}}^{-1}\Psi_{\mathbf{g}}\Psi_{\mathbf{h}} \times (\underline{\varepsilon}_{\mathbf{f}}\underline{\varepsilon}_{\mathbf{g}}^{-1}\underline{\varepsilon}_{\mathbf{h}}^{-1})^{1/2},$$

$$\eta^{\mathbf{fh}}_{\mathbf{g}} = \chi_g^{-1}\Psi_{\mathbf{g}}\Psi_{\mathbf{f}}^{-1}\Psi_{\mathbf{h}}^{-1} \times \varepsilon_{\mathrm{cyc}}(\underline{\varepsilon}_{\mathbf{f}}\underline{\varepsilon}_{\mathbf{g}}^{-1}\underline{\varepsilon}_{\mathbf{h}})^{1/2}, \quad \eta_{\mathbf{fh}}^{\mathbf{g}} = \chi_g\Psi_{\mathbf{f}}\Psi_{\mathbf{h}}\Psi_{\mathbf{g}}^{-1} \times (\underline{\varepsilon}_{\mathbf{f}}^{-1}\underline{\varepsilon}_{\mathbf{g}}\underline{\varepsilon}_{\mathbf{h}}^{-1})^{1/2},$$

$$\eta^{\mathbf{fg}}_{\mathbf{h}} = \chi_h^{-1}\Psi_{\mathbf{h}}\Psi_{\mathbf{f}}^{-1}\Psi_{\mathbf{g}}^{-1} \times \varepsilon_{\mathrm{cyc}}(\underline{\varepsilon}_{\mathbf{f}}\underline{\varepsilon}_{\mathbf{g}}\underline{\varepsilon}_{\mathbf{h}}^{-1})^{1/2}, \quad \eta_{\mathbf{fg}}^{\mathbf{h}} = \chi_h\Psi_{\mathbf{f}}\Psi_{\mathbf{g}}\Psi_{\mathbf{h}}^{-1} \times (\underline{\varepsilon}_{\mathbf{f}}^{-1}\underline{\varepsilon}_{\mathbf{g}}^{-1}\underline{\varepsilon}_{\mathbf{h}})^{1/2}.$$

Proof. — Let $\boldsymbol{\phi}$ be a Hida family of tame character χ as in §1.3. Let $\psi_{\boldsymbol{\phi}}$ denote the unramified character of $G_{\mathbf{Q}_p}$ sending a Frobenius element Fr_p to $\mathbf{a}_p(\boldsymbol{\phi})$ and recall from (1.12) that the restriction of $\mathbb{V}_{\boldsymbol{\phi}}$ to $G_{\mathbf{Q}_p}$ admits a filtration

$$0 \to \mathbb{V}^{+}_{\boldsymbol{\phi}} \to \mathbb{V}_{\boldsymbol{\phi}} \to \mathbb{V}^{-}_{\boldsymbol{\phi}} \to 0$$

with

$$\mathbb{V}^{+}_{\boldsymbol{\phi}} \simeq \Lambda_{\boldsymbol{\phi}}(\psi_{\boldsymbol{\phi}}^{-1}\chi\varepsilon_{\mathrm{cyc}}^{-1}\underline{\varepsilon}_{\mathrm{cyc}}), \qquad \mathbb{V}^{-}_{\boldsymbol{\phi}} \simeq \Lambda_{\boldsymbol{\phi}}(\psi_{\boldsymbol{\phi}}).$$

Set

$$\begin{aligned}
\mathbb{V}^{++}_{\mathbf{fgh}} &= \mathbb{V}^{+}_{\mathbf{f}} \otimes \mathbb{V}^{+}_{\mathbf{g}} \otimes \mathbb{V}^{+}_{\mathbf{h}}(\varepsilon_{\mathrm{cyc}}^{-1}\underline{\varepsilon}_{\mathbf{f}}^{-1/2}\underline{\varepsilon}_{\mathbf{g}}^{-1/2}\underline{\varepsilon}_{\mathbf{h}}^{-1/2}),\\
\mathbb{V}^{+}_{\mathbf{fgh}} &= \big(\mathbb{V}_{\mathbf{f}} \otimes \mathbb{V}^{+}_{\mathbf{g}} \otimes \mathbb{V}^{+}_{\mathbf{h}} + \mathbb{V}^{+}_{\mathbf{f}} \otimes \mathbb{V}_{\mathbf{g}} \otimes \mathbb{V}^{+}_{\mathbf{h}} + \mathbb{V}^{+}_{\mathbf{f}} \otimes \mathbb{V}^{+}_{\mathbf{g}} \otimes \mathbb{V}_{\mathbf{h}}\big)(\varepsilon_{\mathrm{cyc}}^{-1}\underline{\varepsilon}_{\mathbf{f}}^{-1/2}\underline{\varepsilon}_{\mathbf{g}}^{-1/2}\underline{\varepsilon}_{\mathbf{h}}^{-1/2})\\
\mathbb{V}^{-}_{\mathbf{fgh}} &= \big(\mathbb{V}_{\mathbf{f}} \otimes \mathbb{V}_{\mathbf{g}} \otimes \mathbb{V}^{+}_{\mathbf{h}} + \mathbb{V}_{\mathbf{f}} \otimes \mathbb{V}^{+}_{\mathbf{g}} \otimes \mathbb{V}_{\mathbf{h}} + \mathbb{V}^{+}_{\mathbf{f}} \otimes \mathbb{V}_{\mathbf{g}} \otimes \mathbb{V}_{\mathbf{h}}\big)(\varepsilon_{\mathrm{cyc}}^{-1}\underline{\varepsilon}_{\mathbf{f}}^{-1/2}\underline{\varepsilon}_{\mathbf{g}}^{-1/2}\underline{\varepsilon}_{\mathbf{h}}^{-1/2}).
\end{aligned}$$

It follows from the definitions that these three representations are $\Lambda_{\mathbf{fgh}}[G_{\mathbf{Q}_p}]$-submodules of $\mathbb{V}^{\dagger}_{\mathbf{fgh}}$ of ranks 1, 4, 7 as claimed. Moreover, since $\chi_f\chi_g\chi_h = 1$, the rest of the lemma follows from (1.12). □

A one-dimensional character $\eta : G_{\mathbf{Q}_p} \longrightarrow \mathbf{C}_p^{\times}$ is said to be *of Hodge-Tate weight* $-j$ if it is equal to a finite order character times the j-th power of the cyclotomic character. The following is an immediate corollary of Lemma 5.3.

Corollary 5.4. — *Let $(x,y,z)\in\mathcal{W}^{\circ}_{\mathbf{fgh}}$ be a triple of classical points of weights (k,ℓ,m). The Galois representation $V^{\dagger}_{\mathbf{f}_x,\mathbf{g}_y,\mathbf{h}_z}$ is endowed with a four-step $G_{\mathbf{Q}_p}$-stable filtration*

$$0\subset V^{++}_{\mathbf{f}_x,\mathbf{g}_y,\mathbf{h}_z}\subset V^{+}_{\mathbf{f}_x,\mathbf{g}_y,\mathbf{h}_z}\subset V^{-}_{\mathbf{f}_x,\mathbf{g}_y,\mathbf{h}_z}\subset V^{\dagger}_{\mathbf{f}_x,\mathbf{g}_y,\mathbf{h}_z},$$

and the Hodge-Tate weights of its successive quotients are:

Subquotient	Hodge-Tate weights
$V^{++}_{\mathbf{f}_x,\mathbf{g}_y,\mathbf{h}_z}$	$\frac{-k-\ell-m}{2}+1$
$V^{+}_{\mathbf{f}_x,\mathbf{g}_y,\mathbf{h}_z}/V^{++}_{\mathbf{f}_x,\mathbf{g}_y,\mathbf{h}_z}$	$\frac{k-\ell-m}{2},\ \frac{-k+\ell-m}{2},\ \frac{-k-\ell+m}{2}$
$V^{-}_{\mathbf{f}_x,\mathbf{g}_y,\mathbf{h}_z}/V^{+}_{\mathbf{f}_x,\mathbf{g}_y,\mathbf{h}_z}$	$\frac{-k+\ell+m}{2}-1,\ \frac{k-\ell+m}{2}-1,\ \frac{k+\ell-m}{2}-1$
$V_{\mathbf{f}_x,\mathbf{g}_y,\mathbf{h}_z}/V^{-}_{\mathbf{f}_x,\mathbf{g}_y,\mathbf{h}_z}$	$\frac{k+\ell+m}{2}-2$

Corollary 5.5. — *The Hodge-Tate weights of $V^{+}_{\mathbf{f}_x,\mathbf{g}_y,\mathbf{h}_z}$ are all strictly negative if and only if (k,ℓ,m) is balanced.*

Let $\mathbb{V}^{\mathbf{gh}}_{\mathbf{f}}$ and $\mathbb{V}^{\mathbf{gh}}_{\mathbf{f}}(M)$ be the subquotient of $\mathbb{V}^{\dagger}_{\mathbf{fgh}}$ (resp. of $\mathbb{V}^{\dagger}_{\mathbf{fgh}}(M)$) on which $G_{\mathbf{Q}_p}$ acts via (several copies of) the character

$$\eta^{\mathbf{gh}}_{\mathbf{f}}:=\Psi^{\mathbf{gh}}_{\mathbf{f}}\times\Theta^{\mathbf{gh}}_{\mathbf{f}} \tag{5.5}$$

where

— $\Psi^{\mathbf{gh}}_{\mathbf{f}}$ is the unramified character of $G_{\mathbf{Q}_p}$ sending Fr_p to $\chi_f^{-1}(p)\mathbf{a}_p(\mathbf{f})\mathbf{a}_p(\mathbf{g})^{-1}\mathbf{a}_p(\mathbf{h})^{-1}$, and

— $\Theta^{\mathbf{gh}}_{\mathbf{f}}$ is the $\Lambda_{\mathbf{fgh}}$-adic cyclotomic character whose specialization at a point of weight (k,ℓ,m) is $\varepsilon^t_{\mathrm{cyc}}$ with $t:=(-k+\ell+m)/2$.

The classical specializations of $\mathbb{V}^{\mathbf{gh}}_{\mathbf{f}}$ are

$$V^{\mathbf{g}_y\mathbf{h}_z}_{\mathbf{f}_x}:=V^{-}_{\mathbf{f}_x}\otimes V^{+}_{\mathbf{g}_y}\otimes V^{+}_{\mathbf{h}_z}\Big(\frac{-k-\ell-m+4}{2}\Big)\simeq L_p\big(\chi_f^{-1}\psi_{\mathbf{f}_x}\psi^{-1}_{\mathbf{g}_y}\psi^{-1}_{\mathbf{h}_z}\big)(t), \tag{5.6}$$

where the coefficient field is $L_p=\mathbf{Q}_p(\mathbf{f}_x,\mathbf{g}_y,\mathbf{h}_z)$. Note that $t>0$ when $(x,y,z)\in\mathcal{W}^{\mathrm{bal}}_{\mathbf{fgh}}$, while $t\le 0$ when $(x,y,z)\in\mathcal{W}^{f}_{\mathbf{fgh}}$.

Recall now from §1.4 the Dieudonné module $D(V^{\mathbf{g}_y\mathbf{h}_z}_{\mathbf{f}_x}(Mp))$ associated to (5.6). As it follows from loc. cit., every triple

$$(\eta_1,\omega_2,\omega_3)\in D(V^{+}_{\mathbf{f}^*_x}(Mp))\times D(V^{-}_{\mathbf{g}^*_y}(Mp))\times D(V^{-}_{\mathbf{h}^*_z}(Mp))$$

gives rise to a linear functional $\eta_1\otimes\omega_2\otimes\omega_3: D(V^{\mathbf{g}_y\mathbf{h}_z}_{\mathbf{f}_x}(Mp))\longrightarrow L_p$.

In order to deal with the p-adic variation of these Dieudonné modules, write $\mathbb{V}^{\mathbf{gh}}_{\mathbf{f}}(M)$ as

$$\mathbb{V}^{\mathbf{gh}}_{\mathbf{f}}(M)=\mathbb{U}(\Theta^{\mathbf{gh}}_{\mathbf{f}})$$

where $\mathbb{U}$ is the unramified $\Lambda_{\mathbf{fgh}}$-adic representation of $G_{\mathbf{Q}_p}$ given by (several copies of) the character $\Psi^{\mathbf{gh}}_{\mathbf{f}}$.

As in §1.4, define the Λ-adic Dieudonné module

$$\mathbb{D}(\mathbb{U}):=(\mathbb{U}\hat{\otimes}\hat{\mathbf{Z}}^{\mathrm{nr}}_p)^{G_{\mathbf{Q}_p}}.$$

In view of (1.29), for every $(x,y,z)\in\mathcal{W}^{\circ}_{\mathbf{fgh}}$ there is a natural specialisation map

$$\nu_{x,y,z}:\mathbb{D}(\mathbb{U})\longrightarrow D(U^{\mathbf{g}_y\mathbf{h}_z}_{\mathbf{f}_x})$$

where $U^{\mathbf{g}_y\mathbf{h}_z}_{\mathbf{f}_x}:=\mathbb{U}\otimes_{\Lambda_{\mathbf{fgh}}}\mathbf{Q}_p(\mathbf{f}_x,\mathbf{g}_y,\mathbf{h}_z)\simeq V^{\mathbf{g}_y\mathbf{h}_z}_{\mathbf{f}_x}(Mp)(-t)$.

Proposition 5.6. — *For any triple of test vectors*

$$(\breve{\mathbf{f}},\breve{\mathbf{g}},\breve{\mathbf{h}})\in S^{\mathrm{ord}}_{\Lambda}(M,\chi_f)[\mathbf{f}]\times S^{\mathrm{ord}}_{\Lambda}(M,\chi_g)[\mathbf{g}]\times S^{\mathrm{ord}}_{\Lambda}(M,\chi_h)[\mathbf{h}],$$

there exists a homomorphism of $\Lambda_{\mathbf{fgh}}$-modules

$$\langle\quad,\eta_{\breve{\mathbf{f}}^*}\otimes\omega_{\breve{\mathbf{g}}^*}\otimes\omega_{\breve{\mathbf{h}}^*}\rangle:\mathbb{D}(\mathbb{U})\longrightarrow\mathcal{Q}_{\mathbf{f},\mathbf{gh}}$$

such that for all $\boldsymbol{\lambda}\in\mathbb{D}(\mathbb{U})$ and all $(x,y,z)\in\mathcal{W}^{\circ}_{\mathbf{fgh}}$ such that $\mathbf{f}_x$ is the ordinary stabilization of an eigenform $\mathbf{f}^{\circ}_x$ of level M:

$$\nu_{x,y,z}\big(\langle\boldsymbol{\lambda},\eta_{\breve{\mathbf{f}}^*}\otimes\omega_{\breve{\mathbf{g}}^*}\otimes\omega_{\breve{\mathbf{h}}^*}\rangle\big)=\frac{1}{\mathcal{E}_0(\mathbf{f}^{\circ}_x)\mathcal{E}_1(\mathbf{f}^{\circ}_x)}\times\langle\nu_{x,y,z}(\boldsymbol{\lambda}),\eta_{\breve{\mathbf{f}}^*_x}\otimes\omega_{\breve{\mathbf{g}}^*_y}\otimes\omega_{\breve{\mathbf{h}}^*_z}\rangle.$$

Recall from (1.31) *that*

$$\mathcal{E}_0(\mathbf{f}^{\circ}_x)=1-\chi^{-1}(p)\beta^2_{\mathbf{f}^{\circ}_x}p^{1-k},\quad\mathcal{E}_1(\mathbf{f}^{\circ}_x)=1-\chi(p)\alpha^{-2}_{\mathbf{f}^{\circ}_x}p^{k-2}.$$

Proof. — Since $\mathbb{U}$ is isomorphic to the unramified twist of $\mathbb{V}^-_{\mathbf{f}}\otimes\mathbb{V}^+_{\mathbf{g}}\otimes\mathbb{V}^+_{\mathbf{h}}$, this follows from Proposition 1.5 because $\mathcal{E}_0(\mathbf{f}^{\circ}_x)=\mathcal{E}_0(\mathbf{f}^{\circ *}_x)$ and $\mathcal{E}_1(\mathbf{f}^{\circ}_x)=\mathcal{E}_1(\mathbf{f}^{\circ *}_x)$. □

It follows from Example 1.4 (a) and (b) that the Bloch-Kato logarithm and dual exponential maps yield isomorphisms

$$\log_{\mathrm{BK}}:H^1(\mathbf{Q}_p,V^{\mathbf{g}_y\mathbf{h}_z}_{\mathbf{f}_x})\xrightarrow{\sim}D(V^{\mathbf{g}_y\mathbf{h}_z}_{\mathbf{f}_x}),\qquad\text{if }t>0,$$

$$\exp^*_{\mathrm{BK}}:H^1(\mathbf{Q}_p,V^{\mathbf{g}_y\mathbf{h}_z}_{\mathbf{f}_x})\xrightarrow{\sim}D(V^{\mathbf{g}_y\mathbf{h}_z}_{\mathbf{f}_x}),\qquad\text{if }t\leq 0.$$

Define

$$\mathcal{E}^{\mathrm{PR}}(x,y,z)=\frac{1-p^{\frac{k-\ell-m}{2}}\alpha^{-1}_{\mathbf{f}_x}\alpha_{\mathbf{g}_y}\alpha_{\mathbf{h}_z}}{1-p^{\frac{\ell+m-k-2}{2}}\alpha_{\mathbf{f}_x}\alpha^{-1}_{\mathbf{g}_y}\alpha^{-1}_{\mathbf{h}_z}}=\frac{1-p^{-c}\beta_{\mathbf{f}_x}\alpha_{\mathbf{g}_y}\alpha_{\mathbf{h}_z}}{1-p^{-c}\alpha_{\mathbf{f}_x}\beta_{\mathbf{g}_y}\beta_{\mathbf{h}_z}}.\tag{5.7}$$

The following is a three-variable version of Perrin-Riou's regulator map constructed in [PR95] and [LZ14].

Proposition 5.7. — *There is a homomorphism*

$$\mathcal{L}_{\mathbf{f},\mathbf{gh}}:H^1(\mathbf{Q}_p,\mathbb{V}^{\mathbf{gh}}_{\mathbf{f}}(M))\longrightarrow\mathbb{D}(\mathbb{U})$$

such that for all $\boldsymbol{\kappa}_p\in H^1(\mathbf{Q}_p,\mathbb{V}^{\mathbf{gh}}_{\mathbf{f}}(M))$ the image $\mathcal{L}_{\mathbf{f},\mathbf{gh}}(\boldsymbol{\kappa}_p)$ satisfies the following interpolation properties:

(i) For all balanced points $(x,y,z)\in\mathcal{W}^{\mathrm{bal}}_{\mathbf{fgh}}$,

$$\nu_{x,y,z}\big(\mathcal{L}_{\mathbf{f},\mathbf{gh}}(\boldsymbol{\kappa}_p)\big)=\frac{(-1)^t}{t!}\cdot\mathcal{E}^{\mathrm{PR}}(x,y,z)\cdot\log_{\mathrm{BK}}(\nu_{x,y,z}(\boldsymbol{\kappa}_p)),$$

(ii) For all points $(x,y,z)\in\mathcal{W}^{f}_{\mathbf{fgh}}$,

$$\nu_{x,y,z}\big(\mathcal{L}_{\mathbf{f},\mathbf{gh}}(\boldsymbol{\kappa}_p)\big)=(-1)^t\cdot(1-t)!\cdot\mathcal{E}^{\mathrm{PR}}(x,y,z)\cdot\exp^*_{\mathrm{BK}}(\nu_{x,y,z}(\boldsymbol{\kappa}_p)).$$

Proof. — This follows by standard methods as in [**KLZ17**, Theorem 8.2.8], [**LZ14**, Appendix B], [**DR17**, §5.1]. □

Proposition 5.8. — *The class $\boldsymbol{\kappa}_p(\mathbf{f},\mathbf{g},\mathbf{h})$ belongs to the image of $H^1(\mathbf{Q}_p,\mathbb{V}^+_{\mathbf{fgh}}(M))$ in*
$H^1(\mathbf{Q}_p,\mathbb{V}^\dagger_{\mathbf{fgh}}(M))$ under the map induced from the inclusion $\mathbb{V}^+_{\mathbf{fgh}}(M)\hookrightarrow\mathbb{V}^\dagger_{\mathbf{fgh}}(M)$.

Proof. — Let $(x,y,z)\in\mathcal{W}^\circ_{\mathbf{fgh}}$ be a triple of classical points of weights (k,ℓ,m). By the results proved in §4, the cohomology class $\kappa_p(\mathbf{f}_x,\mathbf{g}_y,\mathbf{h}_z)$ is proportional to the image under the p-adic étale Abel-Jacobi map of the cycles appearing in (4.8), that were introduced in [**DR14**, §3]. The purity conjecture for the monodromy filtration is known to hold for the variety $\mathcal{A}^{k_\circ}\times\mathcal{A}^{\ell_\circ}\times\mathcal{A}^{m_\circ}$ by the work of Saito (cf. [**S97**], [**N98**, (3.2)]). By Theorem 3.1 of loc.cit., it follows that the extension $\kappa_p(\mathbf{f}_x,\mathbf{g}_y,\mathbf{h}_z)$ is crystalline. Hence $\kappa_p(\mathbf{f}_x,\mathbf{g}_y,\mathbf{h}_z)$ belongs to $H^1_{\mathrm{f}}(\mathbf{Q}_p,V^\dagger_{\mathbf{f}_x,\mathbf{g}_y,\mathbf{h}_z}(Mp))\subset H^1(\mathbf{Q}_p,V^\dagger_{\mathbf{f}_x,\mathbf{g}_y,\mathbf{h}_z}(Mp))$.

Since (k,ℓ,m) is balanced, Corollary 5.5 implies that $V^+_{\mathbf{f}_x,\mathbf{g}_y,\mathbf{h}_z}$ is the subrepresentation of $V^\dagger_{\mathbf{f}_x,\mathbf{g}_y,\mathbf{h}_z}$ on which the Hodge-Tate weights are all *strictly negative*. As is well-known (cf. [**F90**, Lemma 2, p. 125], [**LZ19**, §3.3] for similar results), the finite Bloch-Kato local Selmer group of an ordinary representation can be recast à la Greenberg [**G89**] as

$$H^1_{\mathrm{f}}(\mathbf{Q}_p,V^\dagger_{\mathbf{f}_x,\mathbf{g}_y,\mathbf{h}_z})=\ker\left(H^1(\mathbf{Q}_p,V^\dagger_{\mathbf{f}_x,\mathbf{g}_y,\mathbf{h}_z})\longrightarrow H^1(I_p,V^\dagger_{\mathbf{f}_x,\mathbf{g}_y,\mathbf{h}_z}/V^+_{\mathbf{f}_x,\mathbf{g}_y,\mathbf{h}_z})\right),$$

where I_p denotes the inertia group at p.

Since the set of balanced classical points is dense in $\mathcal{W}_{\mathbf{fgh}}$ for the rigid-analytic topology, it follows that the Λ-adic class $\kappa_p(\mathbf{f},\mathbf{g},\mathbf{h})$ belongs to the kernel of the natural map

$$H^1(\mathbf{Q}_p,\mathbb{V}^\dagger_{\mathbf{fgh}}(M))\longrightarrow H^1(I_p,\mathbb{V}^\dagger_{\mathbf{fgh}}(M)/\mathbb{V}^+_{\mathbf{fgh}}(M)).$$

Since the kernel of the restriction map

$$H^1(\mathbf{Q}_p,\mathbb{V}^\dagger_{\mathbf{fgh}}(M)/\mathbb{V}^+_{\mathbf{fgh}}(M))\longrightarrow H^1(I_p,\mathbb{V}^\dagger_{\mathbf{fgh}}(M)/\mathbb{V}^+_{\mathbf{fgh}}(M))$$

is trivial by Lemma 5.3, the result follows. □

Thanks to Lemma 5.3 and Proposition 5.8, we are entitled to define

$$\boldsymbol{\kappa}_p^f(\mathbf{f},\mathbf{g},\mathbf{h})^- \quad\in\quad H^1(\mathbf{Q}_p,\mathbb{V}^{\mathbf{gh}}_{\mathbf{f}}(M)) \tag{5.8}$$

as the projection of the local class $\boldsymbol{\kappa}_p(\mathbf{f},\mathbf{g},\mathbf{h})$ to $\mathbb{V}^{\mathbf{gh}}_{\mathbf{f}}(M)$.

Theorem 5.1. — *For any triple of Λ-adic test vectors $(\breve{\mathbf{f}},\breve{\mathbf{g}},\breve{\mathbf{h}})$, the following equality holds in the ring $\mathcal{Q}_{\mathbf{f},\mathbf{gh}}$:*

$$\langle\mathcal{L}_{\mathbf{f},\mathbf{gh}}(\,\boldsymbol{\kappa}_p^f(\mathbf{f},\mathbf{g},\mathbf{h})^-\,),\,\eta_{\breve{\mathbf{f}}^*}\otimes\omega_{\breve{\mathbf{g}}^*}\otimes\omega_{\breve{\mathbf{h}}^*}\,\rangle\;=\;\mathscr{L}_p{}^f(\breve{\mathbf{f}},\breve{\mathbf{g}},\breve{\mathbf{h}}).$$

Proof. — It is enough to prove this equality for a subset of classical points that is dense for the rigid-analytic topology, and we shall do so for all balanced triple of crystalline classical points $(x,y,z)\in\mathcal{W}^{\mathrm{bal}}_{\mathbf{fgh}}$ such that $\mathbf{f}_x$, $\mathbf{g}_\ell$ and $\mathbf{h}_m$ are respectively the ordinary stabilization of an eigenform $f:=\mathbf{f}^\circ_x$, $g:=\mathbf{g}^\circ_y$ and $h:=\mathbf{h}^\circ_z$ of level M.

Set $\boldsymbol{\kappa}_p^- := \boldsymbol{\kappa}_p^f(\mathbf{f},\mathbf{g},\mathbf{h})^-$ and $\mathcal{L} = \langle \mathcal{L}_{\mathbf{f},\mathbf{gh}}(\boldsymbol{\kappa}_p^-), \eta_{\breve{\mathbf{f}}^*} \otimes \omega_{\breve{\mathbf{g}}^*} \otimes \omega_{\breve{\mathbf{h}}^*}\rangle$ for notational simplicity. Proposition 5.6 asserts that the following identity holds in L_p:

$$\nu_{x,y,z}(\mathcal{L}) \quad = \quad \langle \nu_{x,y,z}(\mathcal{L}_{\mathbf{f},\mathbf{gh}}(\boldsymbol{\kappa}_p^-)), \eta_{\breve{\mathbf{f}}_x^*} \otimes \omega_{\breve{\mathbf{g}}_y^*} \otimes \omega_{\breve{\mathbf{h}}_z^*}\rangle.$$

Recall also from Proposition 1.5 that

$$\eta_{\breve{\mathbf{f}}_x^*} = \frac{1}{\mathcal{E}_1(f)} e\varpi_1^*(\eta_{\breve{f}^*}), \quad \omega_{\breve{\mathbf{g}}_y^*} = \mathcal{E}_0(g) e\varpi_1^*(\omega_{\breve{g}^*}), \quad \omega_{\breve{\mathbf{h}}_z^*} = \mathcal{E}_0(h) e\varpi_1^*(\omega_{\breve{h}^*})$$

and

$$\nu_{x,y,z}(\mathcal{L}_{\mathbf{f},\mathbf{gh}}(\boldsymbol{\kappa}_p^-)) = \frac{(-1)^t}{t!} \cdot \mathcal{E}^{\mathrm{PR}}(x,y,z) \log_{\mathrm{BK}}(\nu_{x,y,z}(\boldsymbol{\kappa}_p^-))$$

by Proposition 5.7.

Recall the class $\kappa(f,g,h) = \kappa(\mathbf{f}_x^\circ, \mathbf{g}_y^\circ, \mathbf{h}_z^\circ)$ introduced in (4.8) arising from the generalized diagonal cycles of [**DR14**]. As in (5.8), we may define $\kappa_p^f(f,g,h)^- \in H^1(\mathbf{Q}_p, \mathbb{V}_f^{gh}(M))$ as the projection to $V_f^{gh}(M)$ of the restriction at p of the global class $\kappa(f,g,h)$.

It follows from Theorem 4.1 that

$$(\varpi_1,\varpi_1,\varpi_1)_* \nu_{x,y,z}(\boldsymbol{\kappa}_p^-) \quad = \quad \frac{\mathcal{E}^{\mathrm{bal}}(x,y,z)}{(1-\beta_f/\alpha_f)(1-\beta_g/\alpha_g)(1-\beta_h/\alpha_h)} \times \kappa_p^f(f,g,h)^-$$

where

$$\mathcal{E}^{\mathrm{bal}}(x,y,z) = (1-\alpha_f\beta_g\beta_h p^{-c})(1-\beta_f\alpha_g\beta_h p^{-c})(1-\beta_f\beta_g\alpha_h p^{-c})(1-\beta_f\beta_g\beta_h p^{-c}).$$

The combination of the above identities shows that the value of $\mathcal{L}$ at the balanced triple (x,y,z) is

$$\nu_{x,y,z}(\mathcal{L}) = \frac{(-1)^t \cdot \mathcal{E}^{\mathrm{bal}}(x,y,z)\mathcal{E}^{\mathrm{PR}}(x,y,z)}{t! \cdot \mathcal{E}_0(f)\mathcal{E}_1(f)} \times \langle \log_{\mathrm{BK}}(\kappa_p^f(f,g,h)), \eta_{\breve{f}^*} \otimes \omega_{\breve{g}^*} \otimes \omega_{\breve{h}^*}\rangle$$

Besides, since the syntomic Abel-Jacobi map appearing in [**DR14**] is the composition of the étale Abel-Jacobi map and the Bloch-Kato logarithm, the main theorem of loc. cit. asserts in the present notations that

$$\nu_{x,y,z}\big(\mathscr{L}_p{}^f(\breve{\mathbf{f}},\breve{\mathbf{g}},\breve{\mathbf{h}})\big) \quad = \quad \frac{(-1)^t}{t!} \frac{\mathcal{E}^{\mathrm{f}}(x,y,z)}{\mathcal{E}_0(f)\mathcal{E}_1(f)} \langle \log_{\mathrm{BK}}(\kappa_p^f(f,g,h)^-), \eta_{\breve{f}^*} \otimes \omega_{\breve{g}_y^*} \otimes \omega_{\breve{h}^*}\rangle$$

where

$$\mathcal{E}^{\mathrm{f}}(x,y,z) = \left(1-\beta_f\alpha_g\alpha_h p^{-c}\right)\left(1-\beta_f\alpha_g\beta_h p^{-c}\right)\left(1-\beta_f\beta_g\alpha_h p^{-c}\right)\left(1-\beta_f\beta_g\beta_h p^{-c}\right).$$

Since

$$\mathcal{E}^{\mathrm{f}}(x,y,z) = \mathcal{E}^{\mathrm{bal}}(x,y,z) \times \mathcal{E}^{\mathrm{PR}}(x,y,z)$$

and the sign and factorial terms also cancel, we have

$$\nu_{x,y,z}(\mathcal{L}) = \nu_{x,y,z}\big(\mathscr{L}_p{}^f(\breve{\mathbf{f}},\breve{\mathbf{g}},\breve{\mathbf{h}})\big),$$

as we wanted to show. The theorem follows. □

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Henri Darmon and Victor Rotger, H. D.: Montreal, Canada
E-mail : darmon@math.mcgill.ca • V. R.: IMTech, UPC and Centre de Recerca Matemàtiques, C. Jordi Girona 1-3, 08034 Barcelona, Spain • *E-mail :* victor.rotger@upc.edu

RECIPROCITY LAWS
FOR BALANCED DIAGONAL CLASSES

by

Massimo Bertolini, Marco Adamo Seveso, and Rodolfo Venerucci

Abstract. — This article constructs a 3-variable *balanced* diagonal class $\kappa(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})$ in the cohomology of the Galois representation associated to a self-dual triple $(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})$ of p-adic Hida families. Its first main result (Theorem A of Section 1.1) establishes an explicit reciprocity law relating $\kappa(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})$ to the *unbalanced* Garrett–Rankin p-adic L-function attached to $(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})$. The class $\kappa(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})$ arises from the p-adic interpolation of diagonal classes in the Bloch–Kato Selmer groups of the specialisations of $(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})$ at balanced triples of classical weights. As a consequence, the value of $\kappa(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})$ at a specialisation (f, g, h) of $(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})$ at an unbalanced triple of classical weights is a p-adic limit of crystalline classes. Our second main result (Theorem B of Section 1.2) shows that the obstruction to the crystallinity of an appropriate derivative of $\kappa(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})$ at (f, g, h) is encoded in the central critical value of the complex L-function of $f \otimes g \otimes h$.

To Bernadette Perrin-Riou on her 65*th birthday*

Contents

2000 ***Mathematics Subject Classification.*** — 11F67 (11G40 11G35).

1. Description and statement of results

The reciprocity laws alluded to in the title of this work concern the diagonal class arising in the cohomology of the big Galois representation attached to a self-dual triple of Hida p-adic families of cusp forms. Our construction of this class builds on the push-forward of a canonical generator of an invariant space of locally analytic functions along the diagonal morphism of a modular curve into the corresponding triple-product threefold. It constitutes a crucial step towards the proof of the main results of this paper and of those of our other contribution [**BSV20a**] to the present volume.

The specialisations of the diagonal class at triples of classical weights in the so-called *balanced* region, in which each weight is strictly smaller than the sum of the other two, give rise to cohomology classes admitting a similar description in terms of invariant theory which are closely related to diagonal cycles in Chow groups of Kuga–Sato varieties. As a consequence, the diagonal class belongs to a big Selmer group, called the balanced Selmer group, which interpolates in the geometric region of balanced weights the Bloch–Kato Selmer groups of the triple tensor product representations of the corresponding modular forms.

The first main result of this paper – Theorem A of Section 1.1 – pertains to the specialisation of the diagonal class to the three *unbalanced* regions where one weight is at least equal to the sum of the other two. The explicit reciprocity laws proved therein identify the image of the diagonal class by a branch of the Perrin-Riou big logarithm corresponding to the choice of unbalanced region as the 3-variable p-adic L-function interpolating the central critical values of the Garrett–Rankin complex L-functions attached to the triples of weights in that region.

Our second main result – Theorem B of Section 1.2 – proves that the specialisation of the diagonal class at an unbalanced point is crystalline at p if and only if the corresponding central critical value is zero. This criterion follows directly from the reciprocity law of Theorem A combined with Jacquet's conjecture proved by Harris–Kudla when the p-adic L-function for the corresponding unbalanced region does *not* have an exceptional zero in the sense of Mazur–Tate–Teitelbaum. The *exceptional cases* can only occur at unbalanced triples in which the modular form of dominant weight is multiplicative at p. These subtler cases require the proof of an exceptional zero formula for the 3-variable p-adic L-function, combined with an analysis of the derivatives of the Perrin-Riou logarithm at the unbalanced point and the costruction of an improved class.

Applications to the arithmetic of elliptic curves obtained from instances of the exceptional case constitute the object of the main results of our other contribution [**BSV20a**] to this volume, and represent one motivating feature of the present work. The Hida families considered in this setting respectively interpolate the weight-two modular form attached to an elliptic curve A over the rational numbers and two weight-one theta series associated to the same quadratic field K and subject to natural arithmetic conditions. In this setting, we establish a factorisation of the triple product p-adic L-function along the line $(k, 1, 1)$ as a product of two Hida–Rankin p-adic L-functions attached to A/K, which implies a relation between the fourth derivative at weights $(2,1,1)$ of the former p-adic L-function and the product of the second derivatives at $k=2$ of the latter. This translates into a formula for the Bloch–Kato logarithm of the specialisation of the diagonal class at $(2,1,1)$ as a product of formal group logarithms of Heegner points or Stark–Heegner points, depending respectively on whether K is imaginary quadratic or real quadratic. This result provides a bridge between the diagonal class arising from the geometry of higher dimensional varieties and the theory of rational points on elliptic curves, lending also some support to the conjecture on the rationality of Stark–Heegner points.

1.1. The three-variable reciprocity law. — Fix a prime $p \geqslant 5$, algebraic closures $\bar{\mathbf{Q}}$ and $\bar{\mathbf{Q}}_p$ of $\mathbf{Q}$ and $\mathbf{Q}_p$ respectively, and embeddings $\bar{\mathbf{Q}} \hookrightarrow \bar{\mathbf{Q}}_p$ and $\bar{\mathbf{Q}} \hookrightarrow \mathbf{C}$. Let L be a finite extension of $\mathbf{Q}_p$ and let

$$\boldsymbol{f}^\sharp = \sum_{n\geqslant 1} a_n(\boldsymbol{k}) \cdot q^n \in \mathcal{O}(U_{\boldsymbol{f}})[\![q]\!],$$
$$\boldsymbol{g}^\sharp = \sum_{n\geqslant 1} b_n(\boldsymbol{l}) \cdot q^n \in \mathcal{O}(U_{\boldsymbol{g}})[\![q]\!]$$
$$\text{and} \qquad \boldsymbol{h}^\sharp = \sum_{n\geqslant 1} c_n(\boldsymbol{m}) \cdot q^n \in \mathcal{O}(U_{\boldsymbol{h}})[\![q]\!]$$

be primitive, L-rational Hida p-adic families of modular forms of tame conductors $N_{\boldsymbol{f}}, N_{\boldsymbol{g}}$ and $N_{\boldsymbol{h}}$, centres k_o, l_o and m_o and tame characters $\chi_{\boldsymbol{f}}, \chi_{\boldsymbol{g}}$ and $\chi_{\boldsymbol{h}}$ respectively (cf. Section 5). Here $N_{\boldsymbol{f}}$ is a positive integer coprime to p, $U_{\boldsymbol{f}}$ is an L-rational open disc centred at $k_o \in \mathbf{Z}_{\geqslant 1}$ in the p-adic weight space $\mathcal{W}$, and $\mathcal{O}(U_{\boldsymbol{f}})$ is the ring of analytic functions on $U_{\boldsymbol{f}}$. For each k in $U_{\boldsymbol{f}}^{\mathrm{cl}} = \left\{k \in U_{\boldsymbol{f}} \cap \mathbf{Z}_{\geqslant 2} \mid k \equiv k_o \bmod 2(p-1)\right\}$ the weight-k specialisation $\boldsymbol{f}_k^\sharp = \sum_{n\geq 1} a_n(k) \cdot q^n \in L[\![q]\!] \cap S_k(N_{\boldsymbol{f}}p, \chi_{\boldsymbol{f}})$ is a p-stabilised newform of weight k, level $\Gamma_1(N_{\boldsymbol{f}}) \cap \Gamma_0(p)$ and character $\chi_{\boldsymbol{f}}$. In particular the p-th Fourier coefficient $a_p(\boldsymbol{k})$ is a unit in the ring $\Lambda_{\boldsymbol{f}}$ of functions $\alpha \in \mathcal{O}(U_{\boldsymbol{f}})$ satisfying $|\alpha(x)|_p \leqslant 1$ for all $x \in U_{\boldsymbol{f}}$. If $k > 2$ then $\boldsymbol{f}_k^\sharp$ is the ordinary p-stabilisation of a newform $f_k^\sharp$ in $S_k(N_{\boldsymbol{f}}, \chi_{\boldsymbol{f}})$. If $k=2$ then either $\boldsymbol{f}_2^\sharp = f_2^\sharp$ is new or it is the p-stabilisation of a newform $f_2^\sharp$ of level $N_{\boldsymbol{f}}$. A similar discussion applies to $\boldsymbol{g}^\sharp$ and $\boldsymbol{h}^\sharp$.

Let $(\boldsymbol{\xi}^\sharp, u_o)$ denote one of pairs $(\boldsymbol{f}^\sharp, k_o), (\boldsymbol{g}^\sharp, l_o)$ and $(\boldsymbol{h}^\sharp, m_o)$. If $u_o = 1$, then the weight-one specialisation $\boldsymbol{\xi}_1^\sharp$ of $\boldsymbol{\xi}^\sharp$ is a cuspidal-overconvergent (but not necessarily classical) ordinary modular form. Throughout the paper we make the following

Assumption 1.1. — *If $u_o = 1$, then $\boldsymbol{\xi}_1^\sharp$ is a p-stabilisation of a classical, cuspidal and p-regular newform of level $\Gamma_1(N_{\boldsymbol{\xi}})$, without real multiplication by a quadratic field in which p splits.*

A weight-one eigenform has *real multiplication* if it is equal to the theta series $\vartheta_\chi = \sum_{\mathfrak{a}} \chi(\mathfrak{a}) \cdot q^{\mathbf{N}\mathfrak{a}}$ associated with a ray class character χ of a real quadratic field K, where $\mathfrak{a}$ runs over the non-zero ideals of $\mathcal{O}_K$ and $\mathbf{N}\mathfrak{a} = |\mathcal{O}_K/\mathfrak{a}|$. Moreover, a normalised weight-one eigenform $\xi = \sum_{n\geqslant 0} a_n(\xi) \cdot q^n$ of level $\Gamma_1(N_{\boldsymbol{\xi}})$ and character χ_ξ is said to be *p-regular* if its p-th Hecke polynomial $X^2 - a_p(\xi) \cdot X + \chi_\xi(p)$ is separable. We refer to Remarks 1.4 and to Section 5 below for explanations on the relevance of Assumption 1.1 for the main results of this paper.

Let N be the least common multiple of $N_{\boldsymbol{f}}, N_{\boldsymbol{g}}$ and $N_{\boldsymbol{h}}$. A *level-N test vector* for $(\boldsymbol{f}^\sharp, \boldsymbol{g}^\sharp, \boldsymbol{h}^\sharp)$ is a triple $(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})$ of Hida families of common tame level N, having $(\boldsymbol{f}^\sharp, \boldsymbol{g}^\sharp, \boldsymbol{h}^\sharp)$ as associated triple of primitive families (cf. Section 5). For each k in $U_{\boldsymbol{f}}^{\mathrm{cl}}$ the weight-k specialisation $\boldsymbol{f}_k$ of $\boldsymbol{f}$ is an ordinary cusp form of weight k, level $\Gamma_1(N) \cap \Gamma_0(p)$ and character $\chi_{\boldsymbol{f}}$, which is an eigenvector for U_p and T_ℓ for all primes $\ell \nmid Np$, with the same eigenvalues as $\boldsymbol{f}_k^\sharp$. Similarly for $\boldsymbol{g}$ and $\boldsymbol{h}$. Fix a level-N test vector $(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})$ for $(\boldsymbol{f}^\sharp, \boldsymbol{g}^\sharp, \boldsymbol{h}^\sharp)$.

We make throughout this paper the following crucial *self-duality assumption.*

Assumption 1.2. — $\chi_{\boldsymbol{f}} \cdot \chi_{\boldsymbol{g}} \cdot \chi_{\boldsymbol{h}} = 1.$

Set $\Sigma = \tilde{U}_{\boldsymbol{f}}^{\mathrm{cl}} \times \tilde{U}_{\boldsymbol{g}}^{\mathrm{cl}} \times \tilde{U}_{\boldsymbol{h}}^{\mathrm{cl}}$, where $\tilde{U}_{\boldsymbol{f}}^{\mathrm{cl}} = U_{\boldsymbol{f}}^{\mathrm{cl}} \cup \{k_o\}$ (so that $\tilde{U}_{\boldsymbol{f}}^{\mathrm{cl}} = U_{\boldsymbol{f}}^{\mathrm{cl}}$ if $k_o \geqslant 2$), and $\tilde{U}_{\boldsymbol{g}}^{\mathrm{cl}}$ and $\tilde{U}_{\boldsymbol{h}}^{\mathrm{cl}}$ are defined similarly. Assumption 1.2 implies that $k+l+m$ is an even integer for all $w = (k, l, m)$ in $U_{\boldsymbol{f}}^{\mathrm{cl}} \times U_{\boldsymbol{g}}^{\mathrm{cl}} \times U_{\boldsymbol{h}}^{\mathrm{cl}}$, hence $c_w = (k+l+m-2)/2$ is a positive integer. Let $\Sigma_{\boldsymbol{f}}$ be the set of w in Σ such that $k \geqslant l+m$, define similarly $\Sigma_{\boldsymbol{g}}$ and $\Sigma_{\boldsymbol{h}}$ and denote by Σ_{bal} the complement in Σ of the union of $\Sigma_{\boldsymbol{f}}$, $\Sigma_{\boldsymbol{g}}$ and $\Sigma_{\boldsymbol{h}}$. One calls Σ_{bal} the *balanced region.*

Denote by $\boldsymbol{\xi}$ one of the symbols $\boldsymbol{f}, \boldsymbol{g}$ and $\boldsymbol{h}$ and correspondingly by ξ one of f, g and h. Let $\mathscr{O}_{\boldsymbol{\xi}} = \Lambda_{\boldsymbol{\xi}}[1/p]$ be the space of bounded analytic functions on $U_{\boldsymbol{\xi}}$ and set $\mathscr{O}_{\boldsymbol{fgh}} = \mathscr{O}_{\boldsymbol{f}} \hat{\otimes}_L \mathscr{O}_{\boldsymbol{g}} \hat{\otimes}_L \mathscr{O}_{\boldsymbol{h}}$. Associated with $(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})$ one has:

- *Garrett–Rankin square root p-adic L-functions* $\mathscr{L}_p^\xi(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})$ in $\mathscr{O}_{\boldsymbol{fgh}}$, interpolating the square roots of the central critical values $L(f_k^\sharp \otimes g_l^\sharp \otimes h_m^\sharp, c_w)$ of the complex Garrett–Rankin L-functions $L(f_k^\sharp \otimes g_l^\sharp \otimes h_m^\sharp, s)$ for classical triples $w = (k, l, m)$ in the region $\Sigma_{\boldsymbol{\xi}}$ (cf. Remark 1.8(1) and see Section 6 for details).
- An $\mathscr{O}_{\boldsymbol{fgh}}$-adic representation $V(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})$ of $G_{\mathbf{Q}} = \mathrm{Gal}(\bar{\mathbf{Q}}/\mathbf{Q})$, satisfying the following interpolation property (cf. Section 7.2). For each classical triple $w = (k, l, m)$ in Σ let $V(f_k^\sharp, g_l^\sharp, h_m^\sharp)$ be the central critical twist (i.e. the c_w-th Tate twist) of the tensor product of the Deligne representations of $f_k^\sharp, g_l^\sharp$ and $h_m^\sharp$. Then the base change $V(\boldsymbol{f}_k, \boldsymbol{g}_l, \boldsymbol{h}_m)$ of $V(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})$ under evaluation at (k, l, m) on $\mathscr{O}_{\boldsymbol{fgh}}$ is isomorphic to $\bigoplus_{i=1}^a V(f_k^\sharp, g_l^\sharp, h_m^\sharp)$, for some integer $a \geqslant 1$ which is independent of $(k, l, m) \in \Sigma$ (cf. Section 7.2).
- A *balanced Selmer group* $H^1_{\mathrm{bal}}(\mathbf{Q}, V(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})) \subset H^1(\mathbf{Q}, V(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h}))$, which interpolates the Bloch–Kato Selmer groups $\mathrm{Sel}(\mathbf{Q}, V(\boldsymbol{f}_k, \boldsymbol{g}_l, \boldsymbol{h}_m))$ for all *balanced* triples $(k, l, m) \in \Sigma_{\mathrm{bal}}$ (cf. Section 7.2).
- Perrin-Riou big logarithms

$$\mathscr{L}_{\boldsymbol{\xi}} = \mathscr{L}og_{\boldsymbol{\xi}}(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h}) : H^1_{\mathrm{bal}}(\mathbf{Q}_p, V(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})) \longrightarrow \mathscr{O}_{\boldsymbol{fgh}},$$

  satisfying the following interpolation properties. Say that $\boldsymbol{\xi} = \boldsymbol{f}$ to fix ideas. Then for all balanced triples $w = (k, l, m)$ in a subset of Σ_{bal} which is dense in $U_{\boldsymbol{f}} \times U_{\boldsymbol{g}} \times U_{\boldsymbol{h}}$, and for all local balanced classes $\mathscr{Z}$ in $H^1_{\mathrm{bal}}(\mathbf{Q}_p, V(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h}))$

$$\mathscr{L}_{\boldsymbol{f}}\big(\mathrm{res}_p(\mathscr{Z})\big)(w) = \mathscr{E}_{\boldsymbol{f}}(\boldsymbol{f}_k, \boldsymbol{g}_l, \boldsymbol{h}_m) \cdot \log_p(\mathscr{Z}_w)\big(\eta^\alpha_{\boldsymbol{f}_k} \otimes \omega_{\boldsymbol{g}_l} \otimes \omega_{\boldsymbol{h}_m}\big).$$

  Here $\mathscr{E}_{\boldsymbol{f}}(\boldsymbol{f}_k, \boldsymbol{g}_l, \boldsymbol{h}_m)$ is an explicit non-zero algebraic number, the class $\mathscr{Z}_w$ in $H^1_{\mathrm{fin}}(\mathbf{Q}_p, V(\boldsymbol{f}_k, \boldsymbol{g}_l, \boldsymbol{h}_m))$ is the specialisation of $\mathscr{Z}$ at w, $\log_p$ is the Bloch–Kato logarithm and $\eta^{\alpha}_{\boldsymbol{f}_k} \otimes \omega_{\boldsymbol{g}_l} \otimes \omega_{\boldsymbol{h}_m}$ is the differential considered in Section 7.3, to which we refer for details.

According to a conjectural picture envisioned by Perrin-Riou the L-functions $\mathscr{L}^{\xi}_p(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})$ should arise from a global balanced class via the logarithms $\mathscr{L}_{\boldsymbol{\xi}}$. Our first main result confirms this expectation.

Theorem A. — *There is a canonical class $\kappa(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})$ in $H^1_{\mathrm{bal}}(\mathbf{Q}, V(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h}))$ such that, for $\boldsymbol{\xi} = \boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h}$, one has*

$$\mathscr{L}_{\boldsymbol{\xi}}\big(\mathrm{res}_p\big(\kappa(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})\big)\big) = \mathscr{L}^{\xi}_p(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h}).$$

Remarks 1.3. —

1. The equality displayed in Theorem A determines the class $\kappa(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})$ only up to addition by an element in a suitable (conjecturally trivial) restricted Selmer group. Nonetheless Section 8.1 gives a geometric construction of a *canonical* three-variable balanced class $\kappa(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})$ satisfying the conclusions of Theorem A.

2. Theorem 8.1 and Proposition 8.3 express the specialisation of $\kappa(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})$ at a *balanced* triple $(k, l, m) \in \Sigma_{\mathrm{bal}}$ as an explicit multiple of a suitable Selmer *diagonal class* $\kappa(\boldsymbol{f}_k, \boldsymbol{g}_l, \boldsymbol{h}_m) \in \mathrm{Sel}(\mathbf{Q}, V(\boldsymbol{f}_k, \boldsymbol{g}_l, \boldsymbol{h}_m))$ associated in Section 3 with $(\boldsymbol{f}_k, \boldsymbol{g}_l, \boldsymbol{h}_m)$ (cf. Proposition 3.2). The latter is in turn related to the values of $\mathscr{L}^{\xi}_p(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})$ at (k, l, m) by an explicit reciprocity law (cf. Proposition 3.6). Theorem A then follows from analytic continuation.

3. Both the square-root p-adic L-function $\mathscr{L}^{\xi}_p(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})$ and the big logarithm $\mathscr{L}_{\boldsymbol{\xi}} = \mathscr{L}og_{\boldsymbol{\xi}}(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})$ genuinely depend on the choice of the level-N test vector $(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})$ for $(\boldsymbol{f}^{\sharp}, \boldsymbol{g}^{\sharp}, \boldsymbol{h}^{\sharp})$. On the other hand the big Galois representation $V(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h}) = V_N(\boldsymbol{f}^{\sharp}, \boldsymbol{g}^{\sharp}, \boldsymbol{h}^{\sharp})$ and the balanced class

$$\kappa(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h}) = \kappa_N(\boldsymbol{f}^{\sharp}, \boldsymbol{g}^{\sharp}, \boldsymbol{h}^{\sharp})$$

depend on the test vector $(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})$ only through its level N and the systems of eigenvalues defined by $(\boldsymbol{f}^{\sharp}, \boldsymbol{g}^{\sharp}, \boldsymbol{h}^{\sharp})$ (cf. Sections 5 and 8.1).

4. The construction of $\kappa(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})$ given in Section 8.1 applies more generally to a triple $(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})$ of (not necessarily ordinary) Coleman families. The p-adic L-function $\mathscr{L}^{\xi}_p(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})$ has recently been constructed in [**AI21b**], and it is natural to wonder if one can generalise Theorem A to this setting.

Remark 1.4. — Let $(\boldsymbol{\xi}^{\sharp}, u_o)$ denote one of pairs $(\boldsymbol{f}^{\sharp}, k_o)$, $(\boldsymbol{g}^{\sharp}, l_o)$ and $(\boldsymbol{h}^{\sharp}, m_o)$. When $u_o = 1$, Assumption 1.1 guarantees that the big Galois representation $V(\boldsymbol{\xi})$ and its maximal $G_{\mathbf{Q}_p}$-unramified quotient $V(\boldsymbol{\xi})^-$ are free over $\mathscr{O}_{\boldsymbol{\xi}}$ (cf. Section 5 below for more details). It is likely that Theorem A can be proved without this assumption, at the cost of extending scalars to the fraction field of $\mathscr{O}_{\boldsymbol{fgh}}$ in the definition of $\kappa(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})$ and in the statement of the explicit reciprocity law. On the other hand, the freeness of $V(\boldsymbol{\xi})$ and $V(\boldsymbol{\xi})^-$ are crucial in the proofs of Theorem B below and of the main result of our contribution [**BSV20a**].

Remark 1.5. — By using different methods, extending those of [**DR16**], the contribution of Darmon and Rotger [**DR20**] to this volume gives an alternate construction of the 3-variable diagonal class.

Remark 1.6. — The class $\kappa(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})$ is constructed by interpolating diagonal classes in the Bloch-Kato Selmer groups $\mathrm{Sel}(\mathbf{Q}, V(\boldsymbol{f}_k, \boldsymbol{g}_l, \boldsymbol{h}_m))$ for all triples $(k,l,m) \in \Sigma_{\mathrm{bal}}$. By using systems of étale sheaves attached to spaces of locally analytic functions and the big Abel–Jacobi map defined in equation (156), this geometric problem is reduced to the simpler one of constructing a canonical invariant in a space of locally analytic functions. This invariant element plays a central role in the construction, carried out in [**GS20**] (cf. also [**Hsi20**]), of a *balanced* triple-product p-adic L-function interpolating the square-roots of the central critical values $L(f_k^\sharp \otimes g_l^\sharp \otimes h_m^\sharp, c_w)$ for triples $w = (k,l,m)$ in the balanced region Σ_{bal}. We remark that a similar method can be applied in other settings, for example for the interpolation of generalised Heegner cycles. In this case, the relevant invariant function was instrumental for the definition in [**BD07**] of an anticyclotomic two-variable p-adic L-function. The resulting big Heegner class gives rise via an explicit reciprocity law to the p-adic L-functions considered in [**BDP13, AI21a**]. See also [**JLZ20**] for a related construction in the Heegner case.

1.2. Specialisations at unbalanced points. — Let $w_o = (k,l,m)$ be a classical triple in the unbalanced region $\Sigma_{\boldsymbol{f}}$. The following assumption will be in force in this section (cf. Remarks 1.8).

Assumption 1.7. — *The local sign $\varepsilon_\ell(f_k^\sharp, g_l^\sharp, h_m^\sharp)$ is equal to $+1$ for each rational prime ℓ.*

Theorem B stated below relates the specialisation of the big diagonal class $\kappa(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})$ at w_o to the central value of the *complex* Garrett–Rankin L-function $L(f_k^\sharp \otimes g_k^\sharp \otimes h_k^\sharp, s)$. This relation is particularly intriguing and subtle when $\mathscr{L}_p^f(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})$ has an *exceptional zero* at w_o in the sense of Mazur–Tate–Teitelbaum.

Let $\mathcal{H}_{\boldsymbol{g}} = \mathcal{H}_{\boldsymbol{g}}(w_o)$ be the *g-improving plane* in $U_{\boldsymbol{f}} \times U_{\boldsymbol{g}} \times U_{\boldsymbol{h}}$ defined by the equation

$$\boldsymbol{k} - \boldsymbol{l} + \boldsymbol{m} = k - l + m.$$

Let $\mathscr{O}_{\boldsymbol{gh}} = \mathscr{O}_{\boldsymbol{g}} \hat{\otimes}_L \mathscr{O}_{\boldsymbol{h}}$ and (shrinking $U_{\boldsymbol{g}}$ and $U_{\boldsymbol{h}}$ if necessary) let $\nu_{\boldsymbol{g}} : \mathscr{O}_{\boldsymbol{fgh}} \longrightarrow \mathscr{O}_{\boldsymbol{gh}}$ be the map sending $F(\boldsymbol{k},\boldsymbol{l},\boldsymbol{m})$ to its restriction $F(\boldsymbol{l} - \boldsymbol{m} + k + m - l, \boldsymbol{l}, \boldsymbol{m})$ to $\mathcal{H}_{\boldsymbol{g}}$. Set $V(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})|_{\mathcal{H}_{\boldsymbol{g}}} = V(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h}) \otimes_{\nu_{\boldsymbol{g}}} \mathscr{O}_{\boldsymbol{gh}}$ and denote by

$$\kappa(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})|_{\mathcal{H}_{\boldsymbol{g}}} \in H^1(\mathbf{Q}, V(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})|_{\mathcal{H}_{\boldsymbol{g}}})$$

the image of $\kappa(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})$ under the morphism induced in cohomology by $\nu_{\boldsymbol{g}}$. Define the analytic g-Euler factor

$$\mathcal{E}_g(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h}) = 1 - \frac{\bar{\chi}_{\boldsymbol{g}}(p) \cdot b_p(\boldsymbol{l})}{c_p(\boldsymbol{m}) \cdot a_p(\boldsymbol{l} - \boldsymbol{m} + k + m - l)} \cdot p^{(k-l+m-2)/2} \in \mathscr{O}_{\boldsymbol{gh}}. \tag{1}$$

Section 9.3 proves the factorisation

$$\kappa(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})|_{\mathcal{H}_{\boldsymbol{g}}} = \mathcal{E}_g(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h}) \cdot \kappa_g^*(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h}) \tag{2}$$

for a canonical *g-improved balanced diagonal class*

$$\kappa_g^*(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h}) \in H^1_{\mathrm{bal}}(\mathbf{Q}_p, V(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})|_{\mathcal{H}_{\boldsymbol{g}}}).$$

This is not interesting nor surprising if $\mathcal{E}_g(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})$ does not vanish at w_o. On the other hand, if $\mathcal{E}_g(\boldsymbol{f}_k,\boldsymbol{g}_l,\boldsymbol{h}_m)=0$ this implies that the specialisation of $\kappa(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})$ at w_o vanishes independently of whether the complex L-function $L(f_k^\sharp\otimes g_l^\sharp\otimes h_m^\sharp,s)$ vanishes at the central point $s=c_{w_o}$. This phenomenon is the first source of exceptional zeros in the present setting. Since we are limiting our discussion to Hida families, the vanishing of $\mathcal{E}_g(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})$ at w_o is equivalent to the following conditions:

$$w_o=(2,1,1),\quad p\|c(\boldsymbol{f}_2),\quad p\nmid c(\boldsymbol{g}_1)\cdot c(\boldsymbol{h}_1)\ \text{ and }\ \chi_{\boldsymbol{h}}(p)\cdot a_p(2)\cdot b_p(1)=c_p(1), \tag{3}$$

where $c(\boldsymbol{f}_2), c(\boldsymbol{g}_1)$ and $c(\boldsymbol{h}_1)$ denote the conductors of $\boldsymbol{f}_2,\boldsymbol{g}_1$ and $\boldsymbol{h}_1$ respectively. In particular $\boldsymbol{g}_1$ and $\boldsymbol{h}_1$ are classical weight-one eigenforms.

The second source of exceptional zeros for $\mathscr{L}_p^f(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})$ at w_o is of a different (non geometric) nature (cf. Section 9.2). It is related to the vanishing at w_o of the analytic *f-unbalanced* Euler factor

$$\mathcal{E}_f^*(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})=1-\frac{b_p(\boldsymbol{l})\cdot c_p(\boldsymbol{m})}{\bar{\chi}_{\boldsymbol{f}}(p)\cdot a_p(\boldsymbol{l}+\boldsymbol{m}+k-l-m)}p^{(k-l-m)/2}\in\mathscr{O}_{\boldsymbol{gh}}, \tag{4}$$

which on the *f-improving plane* in $U_{\boldsymbol{f}}\times U_{\boldsymbol{g}}\times U_{\boldsymbol{h}}$ defined by the equation

$$\boldsymbol{k}-\boldsymbol{l}-\boldsymbol{m}=k-l-m$$

interpolates a different Euler factor of $\mathscr{L}_p^f(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})$. In the present ordinary scenario, this vanishing is equivalent to the following conditions:

$$w_o=(2,1,1),\quad p\|c(\boldsymbol{f}_2),\quad p\nmid c(\boldsymbol{g}_1)\cdot c(\boldsymbol{h}_1)\ \text{ and }\ \chi_{\boldsymbol{f}}(p)\cdot b_p(1)\cdot c_p(1)=a_p(2). \tag{5}$$

We say that the unbalanced triple w_o in $\Sigma_{\boldsymbol{f}}$ is *exceptional* if the conditions displayed in Equation (3) or those displayed in Equation (5) are satisfied.

Remarks 1.8. —

1. Assumption 1.7 is in place to guarantee that for weights in the unbalanced region the Garrett–Rankin complex L-functions involved in the definition of the triple-product p-adic L-function have sign of the functional equation equal to $+1$, and that the corresponding central values can be described in terms of trilinear forms arising on $\mathrm{GL}_{2,\mathbf{Q}}$ (cf. [**HK91**]). On the other hand, Theorem A holds regardless of this assumption and does not exclude the possibility of vanishing of the diagonal class for sign reasons.

2. The exceptional zero condition (3) is symmetric in $\boldsymbol{g}$ and $\boldsymbol{h}$. Precisely, define $\mathcal{H}_{\boldsymbol{h}}, V(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})|_{\mathcal{H}_h}, \kappa(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})|_{\mathcal{H}_{\boldsymbol{g}}}$ and $\mathcal{E}_h(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})$ by switching in the above definitions the roles of $\boldsymbol{g}$ of $\boldsymbol{h}$. Then

$$\kappa(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})|_{\mathcal{H}_{\boldsymbol{h}}}=\mathcal{E}_h(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})\cdot\kappa_h^*(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})$$

for a unique canonical *h-improved diagonal class* $\kappa_h^*(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})$ in the global Galois cohomology of $V(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})|_{\mathcal{H}_{\boldsymbol{g}}}$.

3. The restriction of the class $\kappa(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})$ to the plane $\mathcal{H}_{\boldsymbol{f}}$ also factors as the product of $\mathcal{E}_f(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})$ and a canonical class $\kappa_f^*(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})$ in the Galois cohomology of $V(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})|_{\mathcal{H}_{\boldsymbol{f}}}$. This factorisation is uninteresting in the present setting, as the Euler factor $\mathcal{E}_f(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})$ does not vanish at any classical point of the region $\Sigma_{\boldsymbol{f}}$.

4. Under Assumption 1.1, the exceptional zero conditions (3) and (5) are mutually exclusive. Indeed, if one of them holds, then the other is satisfied precisely if the form $g_1^\sharp$ (or equivalently $h_1^\sharp$) is p-irregular.

Define the *diagonal class*

$$\kappa^*(\boldsymbol{f}_k, \boldsymbol{g}_l, \boldsymbol{h}_m) \in H^1(\mathbf{Q}, V(\boldsymbol{f}_k, \boldsymbol{g}_l, \boldsymbol{h}_m))$$

by the following recipe. If the conditions stated in Equation (3) are *not* satisfied, then

$$\kappa^*(\boldsymbol{f}_k, \boldsymbol{g}_l, \boldsymbol{h}_m) = \kappa(\boldsymbol{f}_k, \boldsymbol{g}_l, \boldsymbol{h}_m)$$

is the specialisation of $\kappa(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})$ at the classical triple $w_o = (k, l, m)$. If Equation (3) is satisfied, one defines

$$\kappa^*(\boldsymbol{f}_2, \boldsymbol{g}_1, \boldsymbol{h}_1) = \kappa_g^*(\boldsymbol{f}_2, \boldsymbol{g}_1, \boldsymbol{h}_1),$$

where the global class $\kappa_g^*(\boldsymbol{f}_2, \boldsymbol{g}_1, \boldsymbol{h}_1)$ is the specialisation of the g-improved diagonal class $\kappa_g^*(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})$ at $w_o = (2, 1, 1)$. (Note that $\kappa_h^*(\boldsymbol{f}_2, \boldsymbol{g}_1, \boldsymbol{h}_1) = -\kappa_g^*(\boldsymbol{f}_2, \boldsymbol{g}_1, \boldsymbol{h}_1)$.)

Theorem B. — *The diagonal class $\kappa^*(\boldsymbol{f}_k, \boldsymbol{g}_l, \boldsymbol{h}_m)$ is crystalline at p if and only if the complex L-function $L(f_k^\sharp \otimes g_l^\sharp \otimes h_m^\sharp, s)$ vanishes at $s = \frac{k+l+m-2}{2}$.*

Acknowledgements. The authors are grateful to F. Andreatta for helpful conversations about his work with A. Iovita and G. Stevens on overconvergent Eichler–Shimura isomorphisms. They also thank the referees for their detailed comments and corrections, which resulted in a significant improvement of our contributions to this volume.

2. Cohomology of modular curves

In a first reading of this paper it will be sufficient to get acquainted with the main definitions and notations of this section. The precise description of the various Hecke operators will be necessary for crucial computations in the arguments of later sections (see in particular Section 8). The exposition follows [**Kat04**, Section 2].

2.1. Modular curves. — Let $M \geqslant 1$ and $N \geqslant 1$ be positive integers such that $M + N \geqslant 5$. Denote by

$$Y(M, N) \longrightarrow \mathrm{Spec}(\mathbf{Z}[1/MN])$$

the scheme which represents the functor

$$S \longmapsto \big\{\text{isomorphism classes of } S\text{-triples } (E, P, Q)\big\},$$

where S is a $\mathbf{Z}[1/MN]$-scheme, E is an elliptic curve over S, and P and Q are sections of E over S such that $M \cdot P = 0$, $N \cdot Q = 0$ and the map $\mathbf{Z}/M\mathbf{Z} \times \mathbf{Z}/N\mathbf{Z} \to E$ which on (a, b) takes the value $a \cdot P + b \cdot Q$ is injective. More generally, for each rational prime $\ell \geqslant 1$, we consider as in [**Kat04**] the schemes

$$Y(M(\ell), N) \longrightarrow \mathbf{Z}[1/\ell MN] \quad \text{and} \quad Y(M, N(\ell)) \longrightarrow \mathbf{Z}[1/\ell MN].$$

The $\mathbf{Z}[1/\ell MN]$-scheme $Y(M(\ell), N)$ classifies 4-tuples (E, P, Q, C), where (E, P, Q) is as above and C is a cyclic subgroup of E of order ℓM which contains P and is complementary to Q (viz. the map $\mathbf{Z}/N\mathbf{Z} \times C \to E$ which sends (a, x) to $a \cdot Q + x$ is injective). Similarly $Y(M, N(\ell))$ classifies 4-tuples (E, P, Q, C) where C is a cyclic subgroup of order ℓN which contains Q and is complementary to P. Denote by

$$\begin{aligned} E(M,N) &\longrightarrow Y(M,N), \\ E(M(\ell),N) &\longrightarrow Y(M(\ell),N) \\ \text{and} \quad E(M,N(\ell)) &\longrightarrow Y(M,N(\ell)) \end{aligned}$$

the universal elliptic curves over $Y(M, N)$, $Y(M(\ell), N)$ and $Y(M, N(\ell))$ respectively.

Let $\mathbf{H} = \{z \in \mathbf{C} \mid \Im(z) > 0\}$ be the Poincaré upper half-plane and set

$$\Gamma(M,N) = \left\{\gamma \text{ in } \mathrm{SL}_2(\mathbf{Z}) \text{ such that } \gamma \equiv \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right) \bmod \left(\begin{smallmatrix} M & M \\ N & N \end{smallmatrix}\right)\right\}.$$

Then

$$Y(M,N)(\mathbf{C}) \cong (\mathbf{Z}/M\mathbf{Z})^* \times \Gamma(M,N)\backslash\mathbf{H}, \tag{6}$$

where the class of (a, z) in $(\mathbf{Z}/M\mathbf{Z})^* \times \mathbf{H}$ corresponds to the isomorphism class of the triple $(\mathbf{C}/\mathbf{Z} \oplus \mathbf{Z}z, az/M, 1/N)$. The Riemann surfaces $Y(M(\ell), N)(\mathbf{C})$ and $Y(M, N(\ell))(\mathbf{C})$ admit a similar complex uniformisation by $(\mathbf{Z}/M\mathbf{Z})^* \times \mathbf{H}$.

There is an isomorphism of $\mathbf{Z}[1/\ell MN]$-schemes

$$\varphi_\ell : Y(M,N(\ell)) \cong Y(M(\ell),N)$$

which on the 4-tuple $(E, P, Q, C)_{/S}$ in $Y(M, N(\ell))$ (for some $\mathbf{Z}[1/\ell MN]$-scheme S) takes the value

$$\varphi_\ell(E,P,Q,C) = \big(E/NC, P+NC, \ell^{-1}(Q)\cap C+NC, \big(\ell^{-1}(\mathbf{Z}\cdot P)+NC\big)/NC\big),$$

where $\ell^{-1}(\cdot)$ is the inverse image of $\cdot$ under multiplication by ℓ on E. On complex points (cf. Equation (6)) this is induced by the map $(\mathbf{Z}/M\mathbf{Z})^* \times \mathbf{H} \longrightarrow (\mathbf{Z}/M\mathbf{Z})^* \times \mathbf{H}$ which sends (a, z) to $(a, \ell \cdot z)$. If

$$\varphi_\ell^*(E(M(\ell),N)) \longrightarrow Y(M,N(\ell))$$

denotes the base change of $E(M(\ell), N) \to Y(M(\ell), N)$ under φ_ℓ, there is a natural degree-ℓ isogeny

$$\lambda_\ell : E(M,N(\ell)) \longrightarrow \varphi_\ell^*(E(M(\ell),N)).$$

When $M = 1$ one denotes by

$$Y_1(N) = Y(1,N) \quad \text{and} \quad Y_1(N,\ell) = Y(1,N(\ell)) \tag{7}$$

the affine modular curves over $\mathbf{Z}[1/N]$ and $\mathbf{Z}[1/N\ell]$ corresponding to the subgroups $\Gamma_1(N)$ and $\Gamma_1(N, \ell) = \Gamma_1(N) \cap \Gamma_0(\ell^{\mathrm{ord}_l(N)+1})$ of $\mathrm{SL}_2(\mathbf{Z})$ respectively. Similarly one writes

$$E_1(N) = E(1,N) \quad \text{and} \quad E_1(N,\ell) = E(1,N(\ell))$$

for the universal elliptic curves over $Y_1(N)$ and $Y_1(N, \ell)$ respectively.

2.2. Degeneracy maps. — Let M and N be as in the previous section, and let ℓ be a rational prime. Let

$$Y(M, N\ell) \xrightarrow{\mu_\ell} Y(M, N(\ell)) \xrightarrow{\nu_\ell} Y(M, N)$$
$$\text{and} \quad Y(M\ell, N) \xrightarrow{\check{\mu}_\ell} Y(M(\ell), N) \xrightarrow{\check{\nu}_\ell} Y(M, N)$$

be the natural degeneracy maps (e.g. $\mu_\ell(E, P, Q) = (E, P, \ell \cdot Q, \mathbf{Z} \cdot Q)$ and $\nu_\ell(E, P, Q, C) = (E, P, Q)$), and define

$$\mathrm{pr}_1 : Y(M, N\ell) \longrightarrow Y(M, N) \quad \text{and} \quad \mathrm{pr}_\ell : Y(M, N\ell) \longrightarrow Y(M, N)$$

by the formulae

$$\mathrm{pr}_1(E, P, Q) = (E, P, \ell \cdot Q) \quad \text{and} \quad \mathrm{pr}_\ell(E, P, Q) = (E/N\mathbf{Z} \cdot Q, P + N\mathbf{Z} \cdot Q, Q + N\mathbf{Z} \cdot Q).$$

Under the isomorphism (6) the map pr_1 (resp., pr_ℓ) is induced by the identity (resp., multiplication by ℓ) on the complex upper half-plane $\mathbf{H}$. Unwinding the definitions one easily checks the identities

$$\mathrm{pr}_1 = \nu_\ell \circ \mu_\ell \quad \text{and} \quad \mathrm{pr}_\ell = \check{\nu}_\ell \circ \varphi_\ell \circ \mu_\ell. \tag{8}$$

The degeneracy maps $\mu_\ell, \check{\mu}_\ell, \nu_\ell, \check{\nu}_\ell, \mathrm{pr}_1$ and pr_ℓ are finite étale morphisms of $\mathbf{Z}[1/MN\ell]$-schemes.

2.3. Relative Tate modules and Hecke operators. — Let N, M and ℓ be as in the previous section and let S be a $\mathbf{Z}[1/MN\ell p]$-scheme. For every $\mathbf{Z}[1/MN\ell p]$-scheme X write $X_S = X \times_{\mathbf{Z}[1/MN\ell p]} S$ and denote by $A = A_X$ either the locally constant sheaf $\mathbf{Z}/p^m\mathbf{Z}(j)$ or the locally constant p-adic sheaf (cf. [FK88, Definition 12.6]) $\mathbf{Z}_p(j)$ on $X_{\mathrm{\acute{e}t}}$, for fixed $m \geqslant 1$ and $j \in \mathbf{Z}$. Moreover fix an integer $r \geqslant 0$.

The previous sections yield the following commutative diagram, in which the smaller squares are cartesian.

(9)

$$\begin{array}{ccccccccc}
E(M,N)_S & \longleftarrow & E(M,N(\ell))_S & \xrightarrow{\lambda_\ell} & \varphi_\ell^*(E(M(\ell),N)_S) & \longrightarrow & E(M(\ell),N)_S & \longrightarrow & E(M,N)_S \\
\downarrow{\scriptstyle v_{M,N}} & & \downarrow{\scriptstyle v_{M,N(\ell)}} & & \downarrow & & \downarrow{\scriptstyle v_{M(\ell),N}} & & \downarrow{\scriptstyle v_{M,N}} \\
Y(M,N)_S & \xleftarrow{\nu_\ell} & Y(M,N(\ell))_S & = & Y(M,N(\ell))_S & \xrightarrow{\varphi_\ell} & Y(M(\ell),N)_S & \xrightarrow{\check{\nu}_\ell} & Y(M,N)_S
\end{array}$$

Here $v_{M,N}, v_{M(\ell),N}$ and $v_{M,N(\ell)}$ are the structural maps, one writes again ν_ℓ and $\check{\nu}_\ell$ (resp., λ_ℓ) for the base changes to S of the corresponding degeneracy maps (resp., isogeny), and the unlabelled maps are the natural projections.

If $Y(\cdot)_S$ denotes one of $Y(M,N)_S, Y(M(\ell),N)_S$ and $Y(M,N(\ell))_S$, set

$$\mathscr{T}_\cdot(A) = R^1 v_{\cdot *}\mathbf{Z}_p(1) \otimes_{\mathbf{Z}_p} A \quad \text{and} \quad \mathscr{T}_\cdot^*(A) = \mathrm{Hom}_A(\mathscr{T}_\cdot(A), A). \tag{10}$$

Here $R^q v_{\cdot *}$ is the q-th right derivative of $v_{\cdot *} : E(\cdot)_{\mathrm{\acute{e}t}} \longrightarrow Y(\cdot)_{\mathrm{\acute{e}t}}$ and one calls

$$\mathscr{T}_\cdot \overset{\mathrm{def}}{=} \mathscr{T}_\cdot(\mathbf{Z}_p)$$

the *relative Tate module* of the universal elliptic curve $E(\cdot) \longrightarrow Y(\cdot)$. The perfect cup-product pairing

$$\mathscr{T}_\cdot \otimes_{\mathbf{Z}_p} \mathscr{T}_\cdot \longrightarrow R^2 v_{\cdot *}\mathbf{Z}_p(2)$$

and the relative trace $R^2 v_{\cdot *}\mathbf{Z}_p \cong \mathbf{Z}_p(-1)$ give the perfect relative Weil pairing

$$\langle\cdot,\cdot\rangle_{E(\cdot)_{p^\infty}} : \mathscr{T}_{\cdot} \otimes_{\mathbf{Z}_p} \mathscr{T}_{\cdot} \longrightarrow \mathbf{Z}_p(1), \tag{11}$$

under which one identifies $\mathscr{T}_{\cdot}(-1)$ with $\mathscr{T}_{\cdot}^* = \mathrm{Hom}_{\mathbf{Z}_p}(\mathscr{T}_{\cdot}, \mathbf{Z}_p)$. It is a consequence of the smooth base change theorem (cf. Corollary 4.2, Chapter IV of [Mil80]) that $\mathscr{T}_{\cdot}(A)$ and $\mathscr{T}_{\cdot}^*(A)$ are locally constant p-adic sheaves on $Y_1(N)_S$, of formation compatible with base changes along morphisms of $\mathbf{Z}[1/NM\ell p]$-schemes $S' \to S$. (This justifies the choice to suppress the dependence on S from the notations.) Define

$$\mathscr{L}_{\cdot,r}(A) = \mathrm{Tsym}^r_A \mathscr{T}_{\cdot}(A) \quad \text{and} \quad \mathscr{S}_{\cdot,r}(A) = \mathrm{Symm}^r_A \mathscr{T}_{\cdot}^*(A),$$

where for any finite free module M over a profinite $\mathbf{Z}_p$-algebra R one denotes by $\mathrm{Tsym}^r_R M$ the R-submodule of symmetric tensors in $M^{\otimes r}$ and by $\mathrm{Symm}^r_R M$ the maximal symmetric quotient of $M^{\otimes r}$.

Notation. — When $Y(\cdot)_S = Y(1,N)_S$ is the modular curve $Y_1(N)_S$ associated with the congruence subgroup $\Gamma_1(N)$, and the level N is clear from the context, we use the simplified notations

$$\mathscr{L}_r(A) = \mathscr{L}_{1,N,r}(A), \quad \mathscr{L}_r = \mathscr{L}_r(\mathbf{Z}_p), \quad \mathscr{S}_r(A) = \mathscr{S}_{1,N,r}(A) \text{ and } \mathscr{S}_r = \mathscr{S}_r(\mathbf{Z}_p). \tag{12}$$

If there is no risk of confusion, we use the same simplified notations to denote the étale sheaves $\mathscr{L}_{1,N(\ell),r}(A)$ and $\mathscr{S}_{1,N(\ell),r}(A)$ on the modular curve $Y(1,N(\ell))_S = Y_1(N,\ell)_S$ of level $\Gamma_1(N) \cap \Gamma_0(\ell^{\mathrm{ord}_l(N)+1})$ (cf. Equation (7)).

Throughout the rest of this section let $\mathscr{F}^r_{\cdot}$ denote either $\mathscr{L}_{\cdot,r}(A)$ or $\mathscr{S}_{\cdot,r}(A)$. According to the proper base change theorem [Mil80, Chapter VI, Corollary 2.3] and the diagram (9), associated with the finite étale morphisms ν_ℓ and $\check{\nu}_\ell$ one has natural isomorphisms

$$\nu_\ell^*\big(\mathscr{F}^r_{M,N}\big) \cong \mathscr{F}^r_{M,N(\ell)} \quad \text{and} \quad \check{\nu}_\ell^*\big(\mathscr{F}^r_{M,N}\big) \cong \mathscr{F}^r_{M(\ell),N}, \tag{13}$$

which induce pullbacks

(14)

$$H^i_{\text{ét}}(Y(M,N(\ell))_S, \mathscr{F}^r_{M,N(\ell)}) \xleftarrow{\nu_\ell^*} H^i_{\text{ét}}(Y(M,N)_S, \mathscr{F}^r_{M,N}) \xrightarrow{\check{\nu}_\ell^*} H^i_{\text{ét}}(Y(M(\ell),N)_S, \mathscr{F}^r_{M(\ell),N})$$

and traces (cf. [Mil80, Lemma 1.12, pag. 168])

(15)

$$H^i_{\text{ét}}(Y(M,N(\ell))_S, \mathscr{F}^r_{M,N(\ell)}) \xrightarrow{\nu_{\ell *}} H^i_{\text{ét}}(Y(M,N)_S, \mathscr{F}^r_{M,N}) \xleftarrow{\check{\nu}_{\ell *}} H^i_{\text{ét}}(Y(M(\ell),N)_S, \mathscr{F}^r_{M(\ell),N})$$

Similarly the (finite étale) isogeny λ_ℓ induces morphisms

$$\lambda_{\ell *} : \mathscr{F}^r_{M,N(\ell)} \longrightarrow \varphi_\ell^*\big(\mathscr{F}^r_{M(\ell),N}\big) \quad \text{and} \quad \lambda_\ell^* : \varphi_\ell^*\big(\mathscr{F}^r_{M(\ell),N}\big) \longrightarrow \mathscr{F}^r_{M,N(\ell)}. \tag{16}$$

More precisely, associated with the ℓ-isogeny λ_ℓ there is a trace $\lambda_{\ell *} \circ \lambda_\ell^* \longrightarrow \mathrm{id}$. As $v \circ \lambda_\ell = v_{M,N(\ell)}$, where $v : \varphi_\ell^*(E(M(\ell),N)_S) \to Y(M(\ell),N)_S$ is the first projection, it induces a map $v_{M,N(\ell)*} \circ \lambda_\ell^* \longrightarrow v_*$. Applying R^1 and using the natural isomorphisms

$\varphi_\ell^*(R^1 v_{M(\ell),N*}\mathbf{Z}_p(1)) \cong R^1 v_* \mathbf{Z}_p(1)$ and $\lambda_\ell^* \mathbf{Z}_p(1) \cong \mathbf{Z}_p(1)$, this in turn induces a morphism $R^1 v_{M,N(\ell)*}\mathbf{Z}_p(1) \longrightarrow \varphi_\ell^*\big(R^1 v_{M(\ell),N*}\mathbf{Z}_p(1)\big)$, and finally the push-forwards $\lambda_{\ell*}$ which appear in Equation (16). The pullbacks are defined similarly, after replacing the trace $\lambda_{\ell*}\circ\lambda_\ell^* \longrightarrow \mathrm{id}$ with the adjunction morphism $\mathrm{id} = \lambda_{\ell*}\circ\lambda_\ell^*$. Together with φ_ℓ the previous morphisms give a pushforward

$$\Phi_{\ell*} = \varphi_{\ell*}\circ\lambda_{\ell*} : H^i_{\text{ét}}(Y(M,N(\ell))_S, \mathscr{F}^r_{M,N(\ell)}) \longrightarrow H^i_{\text{ét}}(Y(M(\ell),N)_S, \mathscr{F}^r_{M(\ell),N}) \tag{17}$$

and a pullback

$$\Phi_\ell^* = \lambda_\ell^*\circ\varphi_\ell^* : H^i_{\text{ét}}(Y(M(\ell),N)_S, \mathscr{F}^r_{M(\ell),N}) \longrightarrow H^i_{\text{ét}}(Y(M,N(\ell))_S, \mathscr{F}^r_{M,N(\ell)}).$$

Define the *dual ℓ-th Hecke operator*

$$T'_\ell = \nu_{\ell*}\circ\Phi_\ell^*\circ\check{\nu}_\ell^* : H^i_{\text{ét}}(Y(M,N)_S, \mathscr{F}^r_{M,N}) \longrightarrow H^i_{\text{ét}}(Y(M,N)_S, \mathscr{F}^r_{M,N}).$$

We also consider the *ℓ-th Hecke operator*

$$T_\ell = \check{\nu}_{\ell*}\circ\Phi_{\ell*}\circ\nu_\ell^* : H^i_{\text{ét}}(Y(M,N)_S, \mathscr{F}^r_{M,N}) \longrightarrow H^i_{\text{ét}}(Y(M,N)_S, \mathscr{F}^r_{M,N}).$$

As customary, if the prime ℓ divides MN, we also denote by U_ℓ and U'_ℓ the Hecke operators T_ℓ and T'_ℓ respectively.

For each profinite $\mathbf{Z}_p$-algebra R and each finite free R-module M, the evaluation map induces a perfect pairing

$$\mathrm{Tsym}^r_R M \otimes_R \mathrm{Symm}^r_R M^* \longrightarrow R,$$

where $M^* = \mathrm{Hom}_R(M, \mathbf{Z}_p)$. This defines a perfect pairing $\mathscr{L}_r \otimes_{\mathbf{Z}_p} \mathscr{S}_r \longrightarrow \mathbf{Z}_p$, hence a cup-product

$$\langle\cdot,\cdot\rangle_N : H^1_{\text{ét}}(Y_1(N)_{\bar{\mathbf{Q}}}, \mathscr{L}_r(1)) \otimes_{\mathbf{Z}_p} H^1_{\text{ét,c}}(Y_1(N)_{\bar{\mathbf{Q}}}, \mathscr{S}_r) \longrightarrow H^2_{\text{ét,c}}(Y_1(N)_{\bar{\mathbf{Q}}}, \mathbf{Z}_p(1)) \cong \mathbf{Z}_p, \tag{18}$$

which by Poincaré duality is perfect after inverting p. The Hecke operators $T_\ell^{\cdot}$ induce endomorphisms on the compactly supported cohomology $H^1_{\text{ét,c}}(Y_1(N)_{\bar{\mathbf{Q}}}, \mathscr{S}_r)$, and by construction T_ℓ and T'_ℓ (resp., T'_ℓ and T_ℓ) are adjoint to each other under $\langle\cdot,\cdot\rangle_N$. In addition, the Eichler–Shimura isomorphism (cf. Chapter 8 of [Shi71])

$$H^1_{\text{ét}}(Y_1(N)_{\bar{\mathbf{Q}}}, \mathscr{L}_r) \otimes_{\mathbf{Z}_p} \mathbf{C} \cong M_{r+2}(N, \mathbf{C}) \oplus \overline{S_{r+2}(N, \mathbf{C})} \tag{19}$$

(depending on a fixed embedding $\mathbf{Z}_p \hookrightarrow \mathbf{C}$) commutes with the action of the Hecke operators $T_\ell^{\cdot}$ on both sides.

After replacing the left hand square in the diagram (9) with the cartesian square

$$\begin{array}{ccc} E(M,N\ell)_S & \longrightarrow & E(M,N(\ell))_S \\ \downarrow{\scriptstyle v_{M,N\ell}} & & \downarrow{\scriptstyle v_{M,N(\ell)}} \\ Y(M,N\ell)_S & \xrightarrow{\mu_\ell} & Y(M,N(\ell))_S \end{array}$$

one defines as in Equations (14) and (15) the maps μ_ℓ^* and $\mu_{\ell*}$. For $\cdot = 1, \ell$ one can also define as above morphisms

$$H^i_{\text{ét}}(Y(M,N\ell)_S, \mathscr{F}^r_{M,N\ell}) \xrightarrow{\mathrm{pr}_{\cdot*}} H^i_{\text{ét}}(Y(M,N)_S, \mathscr{F}^r_{M,N}) \xrightarrow{\mathrm{pr}_\cdot^*} H^i_{\text{ét}}(Y(M,N\ell)_S, \mathscr{F}^r_{M,N\ell}), \tag{20}$$

which according to Equation (8) satisfy the identities

$$\mathrm{pr}_{1*} = \nu_{\ell*} \circ \mu_{\ell*}, \quad \mathrm{pr}_1^* = \mu_\ell^* \circ \nu_\ell^*, \quad \mathrm{pr}_{\ell*} = \check{\nu}_{\ell*} \circ \Phi_{\ell*} \circ \mu_{\ell*} \text{ and } \mathrm{pr}_\ell^* = \mu_\ell^* \circ \Phi_\ell^* \circ \check{\nu}_\ell^*. \tag{21}$$

As a consequence, if $\deg(\mu_\ell)$ denotes the degree of the finite morphism μ_ℓ, one has the relations

$$\deg(\mu_\ell) \cdot T_\ell = \mathrm{pr}_{\ell*} \circ \mathrm{pr}_1^* \quad \text{and} \quad \deg(\mu_\ell) \cdot T_\ell' = \mathrm{pr}_{1*} \circ \mathrm{pr}_\ell^*. \tag{22}$$

2.3.1. *Diamond and Atkin–Lehner operators.* — We recall here the geometric definition of the diamond and Atkin–Lehner operators on the cohomology groups $H^i_{\text{ét}}(Y(\cdot)_S, \mathscr{F}^r_\cdot)$ (where $\mathscr{F}^r_\cdot$ are the sheaves introduced in the previous section). For simplicity we limit the discussion to the modular curves $Y_1(\cdot)$ of level $\Gamma_1(\cdot)$, and denote by $\mathscr{F}_r$ the étale sheaf $\mathscr{F}^r_{1,\cdot}$ on $Y_1(\cdot)_S$.

For every unit d in $(\mathbf{Z}/N\mathbf{Z})^*$ the *diamond operator* $\langle d \rangle : Y_1(N)_S \to Y_1(N)_S$ is the automorphism of $Y_1(N)_S$ defined on the moduli problem by sending (E, P) to $(E, d \cdot P)$. Denote by $P_1(N)$ the universal point of order N of $E_1(N)_S$. The pair $(E_1(N)_S, d \cdot P_1(N))$ is an elliptic curve with $\Gamma_1(N)$-level structure over $Y_1(N)_S$, hence there exists a unique isomorphism $\langle d \rangle : E_1(N)_S \cong E_1(N)_S$ which makes the following diagram cartesian:

$$\begin{array}{ccc} E_1(N)_S & \xrightarrow{\langle d \rangle} & E_1(N)_S \\ {\scriptstyle v_N}\downarrow & & \downarrow{\scriptstyle v_N} \\ Y_1(N)_S & \xrightarrow{\langle d \rangle} & Y_1(N)_S. \end{array}$$

This induces automorphisms $\langle d \rangle = \langle d \rangle^*$ and $\langle d \rangle' = \langle d \rangle_*$ of $H^i_{\text{ét}}(Y_1(N)_S, \mathscr{F}_r)$ which are inverse to each other.

Assume in the rest of this Section 2.3.1 that p does not divide N and that S is a scheme over $\mathbf{Z}[1/N, \mu_p]$. Set $\zeta_p = e^{2\pi i/p}$. For every elliptic curve E denote by E_p the kernel of multiplication by p and by $\langle \cdot, \cdot \rangle_{E_p} : E_p \times E_p \to \mu_p$ the Weil pairing. Since $p \nmid N$ the curve $Y_1(Np)$ classifies triples (E, P, Q), where E is an elliptic curve and P (resp., Q) is a point of exact order N (resp., p). (More precisely a pair (E, P_{Np}), where E is an elliptic curve over and P_{Np} is a section of exact order Np, corresponds in the above identification to the triple $(E, p \cdot P_{Np}, N \cdot P_{Np})$.) The *Atkin–Lehner operator* $w_p = w_{\zeta_p} : Y_1(Np)_S \cong Y_1(Np)_S$ is the automorphism of $Y_1(Np)_S$ defined by

$$w_p(E, P, Q) = (E/\mathbf{Z} \cdot Q, P + \mathbf{Z} \cdot Q, Q' + \mathbf{Z} \cdot Q),$$

where $Q' \in E_p$ is characterized by $\langle Q, Q' \rangle_{E_p} = \zeta_p$. There is a natural commutative diagram

$$\begin{array}{ccccc} E_1(Np)_S & \xrightarrow{\check{w}_p} & w_p^*(E_1(Np))_S & \longrightarrow & E_1(Np)_S \\ {\scriptstyle v_{Np}}\downarrow & & \downarrow & & \downarrow{\scriptstyle v_{Np}} \\ Y_1(Np)_S & = & Y_1(Np)_S & \xrightarrow{w_p} & Y_1(Np)_S, \end{array}$$

in which the right-hand square is cartesian and $\check{w}_p$ is a degree-p isogeny. As in Equations (13)–(17), associated with the previous diagram one has a *Atkin–Lehner operator*

$$w_p : H^i_{\text{ét}}(Y_1(Np)_S, \mathscr{F}_r) \xrightarrow{w_p^*} H^i_{\text{ét}}(Y_1(Np)_S, w_p^*(\mathscr{F}_r)) \xrightarrow{\check{w}_p^*} H^i_{\text{ét}}(Y_1(Np)_S, \mathscr{F}_r)$$

and a *dual Atkin–Lehner operator*

$$w_p' : H^i_{\text{ét}}(Y_1(Np)_S, \mathscr{F}_r) \xrightarrow{\check{w}_{p*}} H^i_{\text{ét}}(Y_1(Np)_S, w_p^*(\mathscr{F}_r)) \xrightarrow{w_{p*}} H^i_{\text{ét}}(Y_1(Np)_S, \mathscr{F}_r).$$

More generally, let Q be a divisor of Np such that Q and Np/Q are coprime. After replacing the pair (p, N) with $(Q, Np/Q)$ in the previous construction, one defines the *Atkin–Lehner operators* $w_Q^\cdot$ on $H^1_{\text{ét}}(Y_1(Np)_S, \mathscr{F}_r)$.

2.4. Deligne representations. — Let

$$f = \sum_{n\geqslant 1} a_n(f) q^n \in S_k(N, \chi_f)$$

be a normalised cusp form of weight $k \geqslant 2$, level $\Gamma_1(N)$ and character χ_f. Set $N_o = N/p^{\mathrm{ord}_p(N)}$ and assume that f is an eigenvector for the Hecke operator T_ℓ for every prime $\ell \nmid N_o$. (In particular f is an eigenvector for U_p if p divides N.)

Let $L/\mathbf{Q}_p$ be a finite extension containing the Fourier coefficients of f. Define

$$H^1_{\text{ét}}(Y_1(N)_{\bar{\mathbf{Q}}}, \mathscr{L}_{k-2}(1))_L \longrightarrow\!\!\!\!\!\!\rightarrow V(f) \tag{23}$$

to be the maximal L-quotient on which T_ℓ' and $\langle d\rangle' = \langle d\rangle_*$ act as multiplication by $a_\ell(f)$ and $\chi_f(d)$ respectively, for all $\ell \nmid N_o$ and $\langle d\rangle \in (\mathbf{Z}/N\mathbf{Z})^*$. If f is new of conductor N then $V(f)$ is the *dual* of the Deligne representation of f: for every prime $\ell \nmid Np$ an arithmetic Frobenius $\mathrm{Frob}_\ell \in G_{\mathbf{Q}}$ at ℓ acts on it with characteristic polynomial

$$\det\left(1 - \mathrm{Frob}_\ell | V(f) \cdot X\right) = 1 - a_\ell(f) \cdot X + \chi_f(\ell) \cdot \ell^{k-1} \cdot X^2.$$

In general $V(f) \cong \bigoplus_{i=1}^a V(f^{\mathrm{prim}})$ is (non-canonically) isomorphic to the direct sum of a finite number of copies of $V(f^{\mathrm{prim}})$, where f^{prim} is the primitive form (of conductor a divisor of N) associated with f. Dually let

$$V^*(f) \hookrightarrow H^1_{\text{ét,c}}(Y_1(N)_{\bar{\mathbf{Q}}}, \mathscr{S}_{k-2})_L$$

be the maximal L-submodule on which the Hecke operators T_ℓ and $\langle d\rangle = \langle d\rangle^*$ act as multiplication by $a_p(f)$ and $\chi_f(d)$ respectively, for every prime $\ell \nmid N_o$ and unit d modulo N. (Since f is cuspidal, one can replace the compactly supported cohomology $H^1_{\text{ét,c}}$ with the full cohomology $H^1_{\text{ét}}$ in the definition of $V^*(f)$.) If f is new of level N then $V^*(f)$ is the Deligne $G_{\mathbf{Q}}$-representation of f. In general $V^*(f) \cong \bigoplus_{i=1}^a V^*(f^{\mathrm{prim}})$ for a positive integer a.

Because (by construction) T_ℓ' and $\langle d\rangle^*$ are respectively the adjoints of T_ℓ and $\langle d\rangle_*$ under the morphism $\langle\cdot,\cdot\rangle_N$ defined in Equation (18), the latter induces a pairing

$$\langle\cdot,\cdot\rangle_f : V(f) \otimes_L V^*(f) \longrightarrow L, \tag{24}$$

which is perfect by Poincaré duality [Mil80, Chapter VI].

2.5. Comparison with de Rham cohomology. — Let A be a subring of $\mathbf{C}_p$. Write $v : E \to Y$ for one of the universal morphisms $v_{M,N}$ et cetera that as been previously introduced. Denote by

$$\mathscr{S}_{\mathrm{dR}} = \mathscr{S}_{\mathrm{dR}}(v) = \mathbf{R}^1 v_*\big(\mathscr{O}_E \longrightarrow \Omega^1_{E/Y}\big)$$

the relative de Rham cohomology of E/Y and for every $r \geqslant 0$ set

$$\mathscr{S}_{\mathrm{dR},r} = \mathrm{Symm}^r_{\mathscr{O}_Y} \mathscr{S}_{\mathrm{dR}}.$$

Let $\underline{\omega} = v_*\Omega^1_{E/Y}$ be the invertible sheaf of relative differentials on E/Y. The vector bundle $\mathscr{S}_{\mathrm{dR}}$ is equipped with the *Hodge filtration*

$$0 \longrightarrow \underline{\omega} \longrightarrow \mathscr{S}_{\mathrm{dR}} \longrightarrow \underline{\omega}^{-1} \longrightarrow 0$$

and with an integrable Gauß–Manin connection $\nabla : \mathscr{S}_{\mathrm{dR}} \to \mathscr{S}_{\mathrm{dR}} \otimes_{\mathscr{O}_Y} \Omega^1_{Y/K}$. For all $r \geqslant 0$ these give rise to the Hodge filtration

$$\underline{\omega}^r \hookrightarrow \cdots \hookrightarrow \underline{\omega} \otimes \mathscr{S}_{\mathrm{dR},r-1} \hookrightarrow \mathscr{S}_{\mathrm{dR},r} \tag{25}$$

and to an integrable connection on $\mathscr{S}_{\mathrm{dR},r}$, denoted again by ∇.

Set $\mathscr{L}_{\mathrm{dR}} = \mathrm{Hom}_{\mathscr{O}_Y}(\mathscr{S}_{\mathrm{dR}}, \mathscr{O}_Y)$ and $\mathscr{L}_{\mathrm{dR},r} = \mathrm{Tsym}^r_{\mathscr{O}_Y}\mathscr{L}_{\mathrm{dR}}$, equipped with the induced Hodge filtration and integrable connection (denoted again by ∇). If $\mathscr{F} = \mathscr{S}, \mathscr{L}$ define the de Rham cohomology groups

$$H^j_{\mathrm{dR}}(Y, \mathscr{F}_{\mathrm{dR},r}) = \mathbf{H}^j\big(Y, \mathscr{F}_{\mathrm{dR},r} \xrightarrow{\nabla} \mathscr{F}_{\mathrm{dR},r} \otimes_{\mathscr{O}_Y} \Omega^1_{Y/K}\big)$$

(where the complex $\mathscr{F}_{\mathrm{dR},r} \xrightarrow{\nabla} \mathscr{F}_{\mathrm{dR},r} \otimes_{\mathscr{O}_Y} \Omega^1_{Y/K}$ is concentrated in degrees zero and one). As in Section 2.3 one defines on $H^j_{\mathrm{dR}}(Y, \mathscr{F}_{\mathrm{dR},r})$ Hecke operators T_ℓ and T'_ℓ, for every prime ℓ (when $Y = Y(M,N)$), and diamond operators $\langle d \rangle$, for every unit d of $\mathbf{Z}/N\mathbf{Z}$ (when $Y = Y_1(N)$).

Taking $A = \mathbf{Q}_p$ the comparison theorem of Faltings–Tsuji [**Fal88, Tsu99**] (and the Leray spectral sequence for v_N, cf. the proof of [**BDP13**, Lemma 2.2]) gives a natural, Hecke equivariant isomorphism of filtered $\mathbf{Q}_p$-vector spaces

$$D_{\mathrm{dR}}\big(H^1_{\text{ét}}(Y_1(N)_{\bar{\mathbf{Q}}_p}, \mathscr{F}_r)_{\mathbf{Q}_p}\big) \cong H^1_{\mathrm{dR}}(Y_1(N)_{\mathbf{Q}_p}, \mathscr{F}_{\mathrm{dR},r}), \tag{26}$$

where $D_{\mathrm{dR}}(\cdot) = H^0(\mathbf{Q}_p, \cdot \otimes_{\mathbf{Q}_p} B_{\mathrm{dR}})$ with B_{dR} Fontaine's field of p-adic periods, and the filtration on the de Rham cohomology arises from the Hodge filtration on $\mathscr{F}_{\mathrm{dR}}$ (cf. Equation (25)). Denote by $M_{r+2}(N, \mathbf{Z})$ the $\mathbf{Z}$-module of modular forms of weight $r+2$, level $\Gamma_1(N)$ and integral Fourier coefficients, and set $M_{r+2}(N, R) = M_{r+2}(N, \mathbf{Z}) \otimes_{\mathbf{Z}} R$ for every ring R. It then follows that canonically

$$\mathrm{Fil}^i D_{\mathrm{dR}}\big(H^1_{\text{ét}}(Y_1(N)_{\bar{\mathbf{Q}}_p}, \mathscr{S}_r)_{\mathbf{Q}_p}\big) \otimes_{\mathbf{Q}} \mathbf{Q}(\mu_N) \cong M_{r+2}(N, \mathbf{Q}_p) \otimes_{\mathbf{Q}} \mathbf{Q}(\mu_N) \tag{27}$$

for every $1 \leqslant i \leqslant k-1$ (cf. [**BDP13**, Lemma 2.2]). Under the isomorphisms (26) and (27) the space $\mathrm{Fil}^1 H^1_{\mathrm{dR}}(Y_1(N)_{\mathbf{Q}}, \mathscr{S}_{\mathrm{dR},r})$ corresponds to the image of $M_{r+2}(N, \mathbf{Q})$ under the Atkin–Lehner operator w_N.

Let f and $L/\mathbf{Q}_p$ be as in the previous section and assume that L contains $\mathbf{Q}(\mu_N)$. Define

$$V^*_{\mathrm{dR}}(f) \hookrightarrow H^1_{\mathrm{dR}}(Y_1(N)_{\mathbf{Q}_p}, \mathscr{S}_{\mathrm{dR},k-2})_L$$

to be the maximal submodule on which T_ℓ and $\langle d\rangle_*$ act respectively as $a_\ell(f)$ and $\chi_f(d)$ for every prime $\ell \nmid N_o$ and every $d \in (\mathbf{Z}/N\mathbf{Z})^*$, and dually (cf. Section 2.4)

$$H^1_{\mathrm{dR}}(Y_1(N)_{\mathbf{Q}_p}, \mathscr{L}_{k-2}(1))_L \longrightarrow\!\!\!\!\!\rightarrow V_{\mathrm{dR}}(f).$$

(Here $\mathscr{L}_{\mathrm{dR},r}(j) = \mathscr{L}_{\mathrm{dR},r}$ as flat sheaves and $\mathrm{Fil}^i\mathscr{L}_{\mathrm{dR},r}(j) = \mathrm{Fil}^{i+j}\mathscr{L}_{\mathrm{dR},r}$.) The comparison isomorphism (26) gives

$$D_{\mathrm{dR}}(V(f)) \cong V_{\mathrm{dR}}(f) \quad \text{and} \quad D_{\mathrm{dR}}(V^*(f)) \cong V^*_{\mathrm{dR}}(f), \tag{28}$$

and Equation (27) implies that they restrict to canonical isomorphisms

$$\mathrm{Fil}^0 V_{\mathrm{dR}}(f) \cong S_k(N,L)_{f^*} \quad \text{and} \quad \mathrm{Fil}^1 V^*_{\mathrm{dR}}(f) \cong S_k(N,L)_f. \tag{29}$$

Here $f^* = \sum_{n\geqslant 1} \bar{a}_n(f)\cdot q^n \in S_k(N,\bar{\chi}_f)$ is the dual of f and $S_k(N,L)_\cdot$ denotes the L-module of cusp forms in $S_k(N,L)$ which are eigenvectors for the Hecke operators T_ℓ and $\langle d\rangle$, with the same eigenvalues as $\cdot$, for all primes $\ell \nmid N_o$ and units d in $\mathbf{Z}/N\mathbf{Z}$. One denotes by

$$\omega_f \in \mathrm{Fil}^1 V^*_{\mathrm{dR}}(f) \tag{30}$$

the element corresponding to f under the second isomorphism in Equation (29).

The pairing (24) and the isomorphisms (28) induce a perfect duality

$$\langle\cdot,\cdot\rangle_f : V_{\mathrm{dR}}(f) \otimes_L V^*_{\mathrm{dR}}(f) \longrightarrow D_{\mathrm{dR}}(L) = L, \tag{31}$$

which together with the isomorphisms (29) gives rise to perfect pairings

$$\begin{aligned} &\langle\cdot,\cdot\rangle_f : S_k(N,L)_{f^*} \otimes_L V^*_{\mathrm{dR}}(f)/\mathrm{Fil}^1 \longrightarrow L \\ \text{and} \quad &\langle\cdot,\cdot\rangle_f : V_{\mathrm{dR}}(f)/\mathrm{Fil}^0 \otimes_L S_k(N,L)_f \longrightarrow L, \end{aligned} \tag{32}$$

under which we often identify $V^*_{\mathrm{dR}}(f)/\mathrm{Fil}^1$ with the L-linear dual of $S_k(N,L)_{f^*}$.

Denote by

$$f^w = w_N(f) = N^{k-1}\cdot(Nz)^{-k}\cdot f(-1/Nz) \tag{33}$$

the image of f under the Atkin–Lehner isomorphism

$$w_N : S_k(N,\chi_f) \cong S_k(N,\bar{\chi}_f)$$

and define

$$\eta_f \in V^*_{\mathrm{dR}}(f)/\mathrm{Fil}^1 \tag{34}$$

to be the element which represents the linear functional

$$J_f = \frac{(f^w,\cdot)_N}{(f^w,f^w)_N} : S_k(N,L)_{f^*} \longrightarrow L. \tag{35}$$

Here $(\mu,\nu)_N = \iint_{Y_1(N)_{\mathbf{C}}} \bar{\mu}(z)\nu(z)y^k\frac{dxdy}{y^2}$ (with $z = x+iy$) is the Petersson scalar product on $S_k(N,\mathbf{C})$. The *a priori* $\mathbf{C}$-valued functional J_f indeed takes values in L (cf. [**Hid85**, Proposition 4.5]).

Assume that $\mathrm{ord}_p(N) \leqslant 1$, that p does not divide the conductor of χ_f, and that $a_p(f)$ is a unit in $\mathscr{O}$. Then the $G_{\mathbf{Q}_p}$-representations $V^\cdot(f)$ are semistable, viz. $D_{\mathrm{dR}}(V^\cdot(f)) = D_{\mathrm{st}}(V^\cdot(f))$. It follows that $D_{\mathrm{dR}}(V^\cdot(f))$, hence $V^\cdot_{\mathrm{dR}}(f)$ by Equation

(28), are equipped with an L-linear Frobenius endomorphism φ. Enlarging L if necessary, let $\alpha_f \in \mathscr{O}^*$ be the unit root of the Hecke polynomial

$$h_{f,p} = X^2 - a_p(f) \cdot X + \chi_f(p)p^{k-1} = (X - \alpha_f) \cdot (X - \beta_f)$$

of f. As proved in [Sai97] the characteristic polynomial of the Frobenius endomorphism φ acting on $V^*_{\mathrm{dR}}(f)$ is a power of $h_{f,p}$, and

$$V^*_{\mathrm{dR}}(f) = \mathrm{Fil}^1 V^*_{\mathrm{dR}}(f) \oplus V^*_{\mathrm{dR}}(f)^{\varphi=\alpha_f}. \tag{36}$$

As a consequence η_f lifts uniquely to a differential

$$\eta_f^\alpha \in V^*_{\mathrm{dR}}(f)^{\varphi=\alpha_f}. \tag{37}$$

3. Diagonal classes

Notation. In this section $Y_1(N) = Y_1(N)_{\mathbf{Q}}$ denotes the modular curve of level $\Gamma_1(N) = \Gamma(1,N)$ over $\mathbf{Q}$ and $\mathscr{T} = \mathscr{T}_{1,N}$ denotes the relative Tate module of the universal elliptic curve $E_1(N) = E_1(N)_{\mathbf{Q}}$ (cf. Equation (10)).

Fix a geometric point $\eta = \eta_N : \mathrm{Spec}(\bar{\mathbf{Q}}) \to Y_1(N)$ and denote by $\mathcal{G}_N = \pi_1^{\text{ét}}(Y_1(N), \eta)$ the fundamental group of $Y_1(N)$ with base point η. Then the stalk $\mathscr{T}_\eta$ of $\mathscr{T}$ at η is a free $\mathbf{Z}_p$-module of rank two, equipped with a continuous action of $\mathcal{G}_N$. Choose an isomorphism of $\mathbf{Z}_p$-modules $\xi : \mathscr{T}_\eta \cong \mathbf{Z}_p \oplus \mathbf{Z}_p$ satisfying (cf. Equation (11))

$$\langle x, y \rangle_{E_{p^\infty}} = \xi(x) \wedge \xi(y) \tag{38}$$

for every $x, y \in \mathscr{T}_\eta$ (where one identifies $\bigwedge^2 \mathbf{Z}_p^2$ and $\mathbf{Z}_p$ via $(1,0) \wedge (0,1) = 1$) and denote by

$$\varrho_N : \mathcal{G}_N \longrightarrow \mathrm{Aut}_{\mathbf{Z}_p}(\mathscr{T}_\eta) \cong \mathrm{GL}_2(\mathbf{Z}_p)$$

the corresponding continuous group morphism. According to Proposition A I.8 of [FK88] the map which sends $\mathscr{F}$ to its stalk $\mathscr{F}_\eta$ gives an equivalence between the category of locally constant p-adic sheaves on $Y_1(N)_{\text{ét}}$ and that of p-adic representations of $\mathcal{G}_N$. Then restriction via ϱ_N allows to associate with every continuous representation of $\mathrm{GL}_2(\mathbf{Z}_p)$ into a free finite $\mathbf{Z}_p$-module M a smooth sheaf $M^{\text{ét}}$ on $Y_1(N)$ satisfying $M^{\text{ét}}_\eta = M$.

Let $S_i(A)$ be the set of two-variable homogeneous polynomials of degree i in $A[x_1, x_2]$, equipped with the action of $\mathrm{GL}_2(\mathbf{Z}_p)$ defined for every $g \in \mathrm{GL}_2(\mathbf{Z}_p)$ and $P(x_1, x_2) \in S_i(A)$ by

$$gP(x_1, x_2) = P\big((x_1, x_2) \cdot g\big),$$

and let $L_i(A)$ be the A-linear dual of $S_i(A)$, with $\mathrm{GL}_2(\mathbf{Z}_p)$-action defined by $g\mu(P(x_1, x_2)) = \mu(g^{-1}P(x_1, x_2))$ for every $g \in \mathrm{GL}_2(\mathbf{Z}_p)$, $\mu \in L_i(A)$ and $P(x_1, x_2) \in S_i(A)$. Then (as sheaves on $Y_1(N)_{\mathbf{Q}}$) one has (cf. Equation (12))

$$\mathscr{L}_i(A) = L_i(A)^{\text{ét}} \quad \text{and} \quad \mathscr{S}_i(A) = S_i(A)^{\text{ét}}. \tag{39}$$

In particular $\mathscr{T}_\eta$ is isomorphic to $L_1(\mathbf{Z}_p)$, hence $\mathbf{Z}_p(1)_\eta \cong \bigwedge^2 \mathscr{T}_\eta \cong \det^{-1}$, where $\det^j : \mathrm{GL}_2(\mathbf{Z}_p) \to \mathbf{Z}_p^*$ is defined by $\det^j(\cdot) = \det(\cdot)^j$ for $j \in \mathbf{Z}$. As a consequence, for every $j \in \mathbf{Z}$ and every p-adic representation M of $\mathrm{GL}_2(\mathbf{Z}_p)$:

$$H^0(\mathrm{GL}_2(\mathbf{Z}_p), M \otimes \det^{-j}) \hookrightarrow H^0(\mathcal{G}_N, M \otimes \det^{-j}) \cong H^0_{\text{ét}}(Y_1(N), M^{\text{ét}}(j)). \tag{40}$$

Let $\boldsymbol{r} = (r_1, r_2, r_3) \in \mathbf{N}^3$ be a triple of nonnegative integers satisfying the following assumption.

Assumption 3.1. — 1. $r_1 + r_2 + r_3 = 2 \cdot r$ *with* $r \in \mathbf{N}$.
2. *For every permutation* $\{i, j, k\}$ *of* $\{1, 2, 3\}$ *one has* $r_i + r_j \geqslant r_k$.

Let $S_{\boldsymbol{r}}$ denote the $\mathrm{GL}_2(\mathbf{Z}_p)$-representation $S_{r_1}(\mathbf{Z}_p) \otimes_{\mathbf{Z}_p} S_{r_2}(\mathbf{Z}_p) \otimes_{\mathbf{Z}_p} S_{r_3}(\mathbf{Z}_p)$, which we identify with the module of six-variable polynomials in $\mathbf{Z}_p[\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}]$ which are homogeneous of degree r_1, r_2 and r_3 in the variables $\boldsymbol{x} = (x_1, x_2)$, $\boldsymbol{y} = (y_1, y_2)$ and $\boldsymbol{z} = (z_1, z_2)$ respectively. Following the Clebsch–Gordan decomposition of classical invariant theory, define (cf. Assumption 3.1)

$$\mathrm{Det}_N^{\boldsymbol{r}} = \det \begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix}^{r-r_3} \cdot \det \begin{pmatrix} x_1 & x_2 \\ z_1 & z_2 \end{pmatrix}^{r-r_2} \cdot \det \begin{pmatrix} y_1 & y_2 \\ z_1 & z_2 \end{pmatrix}^{r-r_1}, \tag{41}$$

which is a $\mathrm{GL}_2(\mathbf{Z}_p)$-invariant of $S_{\boldsymbol{r}} \otimes \det^{-r}$:

$$\mathrm{Det}_N^{\boldsymbol{r}} \in H^0(\mathrm{GL}_2(\mathbf{Z}_p), S_{\boldsymbol{r}} \otimes \det^{-r}).$$

After setting $\mathscr{S}_{\boldsymbol{r}} = \mathscr{S}_{r_1}(\mathbf{Z}_p) \otimes_{\mathbf{Z}_p} \mathscr{S}_{r_2}(\mathbf{Z}_p) \otimes_{\mathbf{Z}_p} \mathscr{S}_{r_3}(\mathbf{Z}_p)$, denote by

$$\mathtt{Det}_N^{\boldsymbol{r}} \in H^0_{\text{ét}}(Y_1(N), \mathscr{S}_{\boldsymbol{r}}(r)) \tag{42}$$

the class corresponding to $\mathrm{Det}_N^{\boldsymbol{r}}$ under the natural injection (40). Let

$$p_j : Y_1(N)^3 \to Y_1(N)$$

be the natural projections, let

$$\mathscr{S}_{[\boldsymbol{r}]} = p_1^* \mathscr{S}_{r_1}(\mathbf{Z}_p) \otimes_{\mathbf{Z}_p} p_2^* \mathscr{S}_{r_2}(\mathbf{Z}_p) \otimes_{\mathbf{Z}_p} p_3^* \mathscr{S}_{r_3}(\mathbf{Z}_p)$$

and set

$$\mathtt{W}_{N,\boldsymbol{r}} = H^3_{\text{ét}}(Y_1(N)^3_{\bar{\mathbf{Q}}}, \mathscr{S}_{[\boldsymbol{r}]})(r+2).$$

Since $Y_1(N)_{\bar{\mathbf{Q}}}$ is a smooth affine curve over $\bar{\mathbf{Q}}$ one has

$$H^4_{\text{ét}}(Y_1(N)^3_{\bar{\mathbf{Q}}}, \mathscr{S}_{[\boldsymbol{r}]}(r+2)) = 0,$$

hence the Hochschild–Serre spectral sequence

$$H^p(\mathbf{Q}, H^q_{\text{ét}}(Y^3_{\bar{\mathbf{Q}}}, \mathscr{S}_{[\boldsymbol{r}]}(r+2))) \Longrightarrow H^{p+q}_{\text{ét}}(Y_1(N)^3, \mathscr{S}_{[\boldsymbol{r}]}(r+2))$$

defines a morphism

$$\mathtt{HS} : H^4_{\text{ét}}(Y_1(N)^3, \mathscr{S}_{[\boldsymbol{r}]}(r+2)) \longrightarrow H^1(\mathbf{Q}, \mathtt{W}_{N,\boldsymbol{r}}).$$

Let $d : Y_1(N) \longrightarrow Y_1(N)^3$ be the diagonal embedding. As

$$E_1^{2r}(N) = E_1^{\boldsymbol{r}}(N) \times_{Y_1(N)^3} Y_1(N)$$

is isomorphic to the base change of $u_N^{\boldsymbol{r}} : E_1^{\boldsymbol{r}}(N) \to Y_1(N)^3$ under d, there is a natural isomorphism $d^* \mathscr{S}_{[\boldsymbol{r}]} \cong \mathscr{S}_{\boldsymbol{r}}$ of smooth sheaves on $Y_1(N)_{\text{ét}}$. The codimension-2 closed embedding d then gives a pushforward map

$$d_* : H^0_{\text{ét}}(Y_1(N), \mathscr{S}_{\boldsymbol{r}}(r)) \longrightarrow H^4_{\text{ét}}(Y_1(N)^3, \mathscr{S}_{[\boldsymbol{r}]}(r+2)),$$

and one defines the *diagonal class* of level N and weights $\boldsymbol{r}+\mathbf{2}$:

$$\tilde{\kappa}_{N,\boldsymbol{r}} = \mathtt{HS}\circ d_*(\mathtt{Det}_N^{\boldsymbol{r}}) \in H^1(\mathbf{Q}, \mathtt{W}_{N,\boldsymbol{r}}) \tag{43}$$

as the image of $\mathtt{Det}_N^{\boldsymbol{r}}$ under the composition of d_* with $\mathtt{HS}$. Let $W_{N,\boldsymbol{r}} = \mathtt{W}_{N,\boldsymbol{r}} \otimes_{\mathbf{Z}_p} \mathbf{Q}_p$ and let $H^1_{\mathrm{geo}}(\mathbf{Q}, W_{N,\boldsymbol{r}})$ be the geometric Bloch–Kato Selmer group of $W_{N,\boldsymbol{r}}$ over $\mathbf{Q}$, viz. the module of classes in $H^1(\mathbf{Q}, W_{N,\boldsymbol{r}})$ which are unramified at every prime different from p, and whose restrictions at p belong to the geometric subspace

$$H^1_{\mathrm{geo}}(\mathbf{Q}_p, W_{N,\boldsymbol{r}}) = \ker\big(H^1(\mathbf{Q}_p, W_{N,\boldsymbol{r}}) \longrightarrow H^1(\mathbf{Q}_p, W_{N,\boldsymbol{r}} \otimes_{\mathbf{Q}_p} B_{\mathrm{dR}})\big)$$

(cf. [**BK90**, Section 3]). The results of [**NN16**] (cf. the proof of Theorem 5.9) yield the following crucial proposition.

Proposition 3.2. — *The class $\tilde{\kappa}_{N,\boldsymbol{r}}$ belongs to $H^1_{\mathrm{geo}}(\mathbf{Q}, W_{N,\boldsymbol{r}})$.*

The bilinear form $\det^* : L_i(\mathbf{Z}_p) \otimes_{\mathbf{Z}_p} L_i(\mathbf{Z}_p) \to \mathbf{Z}_p \otimes \det^{-i}$ defined by

$$\det{}^*(\mu\otimes\nu) = \mu\otimes\nu\big((x_1y_2 - x_2y_1)^i\big)$$

for all $\mu, \nu \in L_i(\mathbf{Z}_p)$ becomes perfect after extending scalars to $\mathbf{Q}_p$, hence induces an isomorphism of $\mathrm{GL}_2(\mathbf{Z}_p)$-modules

$$\mathtt{s}_i : S_i(\mathbf{Q}_p) = \mathrm{Hom}_{\mathbf{Q}_p}(L_i(\mathbf{Q}_p), \mathbf{Q}_p) \cong L_i(\mathbf{Q}_p) \otimes_{\mathbf{Z}_p} \det{}^i.$$

Under the equivalence $\cdot^{\text{ét}}$ this corresponds by Equation (39) to an isomorphism of sheaves

$$\mathtt{s}_i : \mathscr{S}_i(\mathbf{Q}_p) \cong \mathscr{L}_i(\mathbf{Q}_p) \otimes_{\mathbf{Z}_p} \mathbf{Z}_p(-i). \tag{44}$$

Define the sheaves $\mathscr{L}_{\boldsymbol{r}}$ on $Y_1(N)$ and $\mathscr{L}_{[\boldsymbol{r}]}$ on $Y_1(N)^3$ as above, and set

$$\mathtt{V}_{N,\boldsymbol{r}} = H^3_{\text{ét}}(Y_1(N)^3_{\bar{\mathbf{Q}}}, \mathscr{L}_{[\boldsymbol{r}]})(2-r) \quad \text{and} \quad V_{N,\boldsymbol{r}} = \mathtt{V}_{N,\boldsymbol{r}} \otimes_{\mathbf{Z}_p} \mathbf{Q}_p. \tag{45}$$

The tensor product of the $\mathtt{s}_{r_j}$ gives an isomorphism $\mathtt{s}_{\boldsymbol{r}} : W_{N,\boldsymbol{r}} \cong V_{N,\boldsymbol{r}}$. Set

$$\kappa_{N,\boldsymbol{r}} = \mathtt{s}_{\boldsymbol{r}*}(\tilde{\kappa}_{N,\boldsymbol{r}}) \in H^1_{\mathrm{geo}}(\mathbf{Q}, V_{N,\boldsymbol{r}}). \tag{46}$$

Remarks 3.3. — 1. We strived to define diagonal classes with values in the representations $V_{N,\boldsymbol{r}}$, as the corresponding cohomology groups are those which are extensively studied in the literature (cf. Sections 4 and 5).

2. For every $0 \leqslant j \leqslant i$ denote by $[x_1, x_2]_j$ the projection of $x_1^{\otimes j} \otimes x_2^{\otimes i-j}$ in $S_i(\mathbf{Q}_p)$. Then $[x_1, x_2]_j$ is a $\mathbf{Q}_p$-basis of $S_i(\mathbf{Q}_p)$ and one writes $[x_1, x_2]_j^*$ for the dual basis of $L_i(\mathbf{Q}_p)$. A direct computation shows that $\mathtt{s}_i : S_i(\mathbf{Q}_p) \cong L_i(\mathbf{Q}_p)$ is given by the formula

$$(-1)^j \cdot \binom{i}{j} \cdot \mathtt{s}_i([x_1, x_2]_j) = [x_1, x_2]_j^*.$$

Set $k = r_1 + 2$, $l = r_2 + 2$ and $m = r_3 + 2$, and consider three cuspidal normalised modular forms

$$f = \sum_{n \geqslant 1} a_n(f) \cdot q^n \in S_k(N, \chi_f),$$
$$g = \sum_{n \geqslant 1} a_n(g) \cdot q^n \in S_l(N, \chi_g),$$
$$h = \sum_{n \geqslant 1} a_n(h) \cdot q^n \in S_m(N, \chi_h)$$

of level $\Gamma_1(N)$, weights k, l and m and characters χ_f, χ_g and χ_h. Assume in the rest of this section the following

Assumption 3.4. — 1. *The triple (f, g, h) is* self-dual*, that is $\chi_f \cdot \chi_g \cdot \chi_h = 1$.*
2. *The forms f, g and h are eigenvectors for the Hecke operators T_ℓ, for every $\ell \nmid N$.*
3. *If p divides N then f, g and h are eigenvectors for the Hecke operator U_p.*

Note that Assumption 3.4.1 implies Assumption 3.1.1, id est that $k + l + m$ is an even integer. Moreover, Assumption 3.1.2 states that the triple (k, l, m) is balanced (with the terminology introduced in Section 1.1). Set

$$V(f, g, h) = V(f) \otimes_L V(g) \otimes_L V(h)\big((4 - k - l - m)/2\big). \tag{47}$$

The Künneth decomposition and projection to the (f, g, h)-isotypic component give a morphism of $G_{\mathbf{Q}}$-modules

$$\mathrm{pr}_{fgh} : \mathtt{V}_{N,\boldsymbol{r}} \otimes_{\mathbf{Q}_p} L \longrightarrow\!\!\!\!\rightarrow V(f, g, h) \tag{48}$$

and one defines the diagonal class associated to the triple (f, g, h) by

$$\kappa(f, g, h) = \mathrm{pr}_{fgh}(\kappa_{N,\boldsymbol{r}}) \in H^1_{\mathrm{geo}}(\mathbf{Q}, V(f, g, h)).$$

3.1. The explicit reciprocity law (*cf.* [BSV20b]). — Let $\boldsymbol{r}$ and (f, g, h) be as in the previous section. In particular $\boldsymbol{r}$ and (f, g, h) satisfy Assumption 3.1 and Assumption 3.4 respectively. In addition, assume in this section that $\mathrm{ord}_p(N) \leqslant 1$, that the conductors of χ_f, χ_g and χ_h are all coprime to p, and that the forms f, g and h are *p-ordinary* (viz. their p-th Fourier coefficients are p-adic units).

Lemma 3.5. — *For $\bullet$ in $\{\mathrm{geo}, \mathrm{fin}, \mathrm{exp}\}$, the Bloch–Kato local conditions*

$$H^1_\bullet(\mathbf{Q}_p, V(f, g, h)) \hookrightarrow H^1(\mathbf{Q}_p, V(f, g, h))$$

(cf. [**BK90**, Section 3]*) are all equal.*

Proof. — Set $w = (k, l, m)$. For $\xi = f, g, h$, denote by $\xi^\sharp$ the newform of conductor $N_\xi | N$ and weight $u = k, l, m$ associated to ξ, and set

$$V = V(f^\sharp) \otimes_L V(g^\sharp) \otimes_L V(h^\sharp)\big((4 - k - l - m)/2\big).$$

Since $V(\xi)$ is isomorphic to the direct sum of a finite number of copies of $V(\xi^\sharp)$ (cf. Section 2.4), it is sufficient to prove the statement after replacing $V(f, g, h)$ with V. Moreover, since V is isomorphic to its Kummer dual $V^* = \mathrm{Hom}_L(V, L(1))$, it is sufficient to prove that $H^1_{\mathrm{exp}}(\mathbf{Q}_p, V)$ equals $H^1_{\mathrm{fin}}(\mathbf{Q}_p, V)$ (cf. Proposition 3.8 of [**BK90**]). According to [**BK90**, Corollary 3.8.4], the quotient $H^1_{\mathrm{fin}}(\mathbf{Q}_p, V)/H^1_{\mathrm{exp}}(\mathbf{Q}_p, V)$ is isomorphic to $D/(\varphi - 1)D$, where D is the crystalline module $D_{\mathrm{cris}}(V) = H^0(\mathbf{Q}_p, V \otimes_{\mathbf{Q}_p} B_{\mathrm{cris}})$ associated with the restriction of V to $G_{\mathbf{Q}_p}$, and φ is the crystalline Frobenius acting on it. We are then reduced to prove the claim

(49) $$D^{\varphi=1} = 0.$$

The assumptions $\mathrm{ord}_p(N) \leqslant 1$ and $p \nmid \mathrm{cond}(\chi_\xi)$ guarantee that $V(\xi^\sharp)|_{G_{\mathbf{Q}_p}}$ is semi-stable, hence so is $V|_{G_{\mathbf{Q}_p}}$. Denote by $D_{\mathrm{st}}(\xi^\sharp) = H^0(\mathbf{Q}_p, V(\xi^\sharp) \otimes_{\mathbf{Q}_p} B_{\mathrm{st}})$ and $D_{\mathrm{st}} = H^0(\mathbf{Q}_p, V \otimes_{\mathbf{Q}_p} B_{\mathrm{st}})$ the semi-stable Fontaine modules of $V(\xi^\sharp)|_{G_{\mathbf{Q}_p}}$ and $V|_{G_{\mathbf{Q}_p}}$ respectively. One has

$$D_{\mathrm{st}}(\xi^\sharp) = L \cdot \mathbf{a}_\xi \oplus L \cdot \mathbf{b}_\xi,$$

where $\mathbf{a}_\xi$ and $\mathbf{b}_\xi$ are φ-eigenvectors with eigenvalues $a_p(\xi^\sharp)^{-1}$ and $p^{1-u}\chi_\xi(p)^{-1}a_p(\xi^\sharp)$ respectively (cf. Section 2.5). Moreover the monodromy operator N_ξ on $D_{\mathrm{st}}(\xi^\sharp)$ is zero if $p \nmid N_\xi$, and satisfies $\mathrm{N}_\xi(\mathbf{a}_\xi) = \mathbf{b}_\xi$ and $\mathrm{N}_\xi(\mathbf{b}_\xi) = 0$ if $p \| N_\xi$. Consider the set $\mathcal{B}_w = \{\mathbf{a}_w^\cdot, \mathbf{b}_w^\cdot : \cdot = \emptyset, f, g, h\}$ of elements of

$$D_{\mathrm{st}} \cong D_{\mathrm{st}}(f^\sharp) \otimes_L D_{\mathrm{st}}(g^\sharp) \otimes_L D_{\mathrm{st}}(h^\sharp) \otimes_{\mathbf{Q}_p} D_{\mathrm{cris}}(\mathbf{Q}_p((4-k-l-m)/2))$$

defined by

$$\mathbf{a}_w = \mathbf{a}_f \otimes \mathbf{a}_g \otimes \mathbf{a}_h \otimes t^{(4-k-l-m)/2}, \qquad \mathbf{a}_w^f = \mathbf{b}_f \otimes \mathbf{a}_g \otimes \mathbf{a}_h \otimes t^{(4-k-l-m)/2},$$
$$\mathbf{b}_w^f = \mathbf{a}_f \otimes \mathbf{b}_g \otimes \mathbf{b}_h \otimes t^{(4-k-l-m)/2}, \qquad \mathbf{b}_w = \mathbf{b}_f \otimes \mathbf{b}_g \otimes \mathbf{b}_h \otimes t^{(4-k-l-m)/2}$$

et cetera, where t is the canonical generator of $D_{\mathrm{cris}}(\mathbf{Q}_p(1))$. Then $\mathcal{B}_w$ is an L-basis of φ-eigenvectors of D_{st} with respective eigenvalues $\mathcal{E}_w = \{\alpha_w^\cdot, \beta_w^\cdot : \cdot = \emptyset, f, g, h\}$, where

$$\alpha_w = \frac{p^{c(w)-1}}{a_p(f^\sharp)a_p(g^\sharp)a_p(h^\sharp)}, \qquad \alpha_w^f = \frac{p^{c(w)-k} \cdot a_p(f^\sharp)}{\chi_f(p)a_p(g^\sharp)a_p(h^\sharp)},$$

α_w^g and α_w^h are defined similarly, and $\beta_w^\cdot$ is defined by the equality

$$p \cdot \alpha_w^\cdot \cdot \beta_w^\cdot = 1.$$

Since the forms f, g and h are ordinary and w is balanced, one has

$$\mathrm{ord}_p(\beta_w^\cdot) < 0 \leqslant \mathrm{ord}_p(\alpha_w^\xi) < \mathrm{ord}_p(\alpha_w)$$

for $\cdot = \emptyset, f, g, h$ and $\xi = f, g, h$. In particular the L-module $D_{\mathrm{st}}^{\varphi=1}$ (hence $D^{\varphi=1}$) is contained in the space generated by the eigenvectors $\mathbf{a}_w^\xi$ for $\xi = f, g, h$.

Define $\varepsilon_\xi \in \{0, 1\}$ to be 1 (resp., 0) if p divides (resp., does not divide) the conductor N_ξ of $\xi = f, g, h$, and set $\varepsilon_w = \varepsilon_f + \varepsilon_g + \varepsilon_h$. According to Theorems 4.5.17 (namely the Ramanujan–Petersson conjecture) and 4.6.17 of [**Miy06**] one has

$$|\alpha_w^\xi|_\infty = p^{(\varepsilon_w - 2\cdot\varepsilon_\xi - 1)/2}$$

for $\xi = f, g, h$, where $|\cdot|_\infty$ denotes the complex absolute value. As a consequence $D_{\mathrm{st}}^{\varphi=1}$ vanishes if $\varepsilon_w = 0$ or $\varepsilon_w = 2$. If $\varepsilon_w = 1$, say $\varepsilon_f = 1$, then $D_{\mathrm{st}}^{\varphi=1}$ is contained in $L \cdot \mathbf{a}_w^g \oplus L \cdot \mathbf{a}_w^h$. On the other hand, the monodromy operator N on D_{st} satisfies

$$\mathrm{N}(\mathbf{a}_w^g) = \mathbf{b}_w^h \quad \text{and} \quad \mathrm{N}(\mathbf{a}_w^h) = \mathbf{b}_w^g,$$

hence $D_{\rm st}^{\varphi=1,\mathrm{N}=0}$ vanishes in this case. Finally, if $\varepsilon_w = 3$, then

$$\mathrm{N}(\mathbf{a}_w^{\xi}) = \mathbf{b}_w^{\xi'} + \mathbf{b}_w^{\xi''}$$

for each permutation (ξ, ξ', ξ'') of (f, g, h), hence $D^{\varphi=1} = D_{\rm st}^{\varphi=1,\mathrm{N}=0} = 0$ also in this case, thus proving the claim (49). □

It follows from the previous Lemma 3.5 that, upon setting

$$V_{\rm dR}(f,g,h) = V_{\rm dR}(f) \otimes_L V_{\rm dR}(g) \otimes_L V_{\rm dR}(h)\big((4-k-l-m)/2\big), \tag{50}$$

the Bloch–Kato exponential and the isomorphism (28) give an isomorphism

$$\exp_p : V_{\rm dR}(f,g,h)/\mathrm{Fil}^0 \cong H^1_{\rm geo}(\mathbf{Q}_p, V(f,g,h)).$$

Similarly for the dual representations define

$$V^*_{\rm dR}(f,g,h) = V^*_{\rm dR}(f) \otimes_L V^*_{\rm dR}(g) \otimes_L V^*_{\rm dR}(h)\big((k+l+m-2)/2\big). \tag{51}$$

Then the perfect dualities (31) (for f, g and h) yield a natural isomorphism

$$V_{\rm dR}(f,g,h)/\mathrm{Fil}^0 \cong \mathrm{Fil}^0 V^*_{\rm dR}(f,g,h)^\vee,$$

where $\cdot^\vee = \mathrm{Hom}_L(\cdot, L)$. Its composition with $\exp_p^{-1}$ defines an isomorphism

$$\log_p : H^1_{\rm geo}(\mathbf{Q}_p, V(f,g,h)) \cong \mathrm{Fil}^0 V^*_{\rm dR}(f,g,h)^\vee. \tag{52}$$

For every global Selmer class κ in $H^1_{\rm geo}(\mathbf{Q}, V(f,g,h))$ one simply writes $\log_p(\kappa)$ as a shorthand for $\log_p(\mathrm{res}_p(\kappa))$.

Denote by $\omega_g \in \mathrm{Fil}^{l-1} V^*_{\rm dR}(g)$ and $\omega_h \in \mathrm{Fil}^{m-1} V^*_{\rm dR}(h)$ the differentials corresponding to g and h respectively under the isomorphism (29), and recall the class $\eta_f^\alpha \in V^*_{\rm dR}(f)^{\varphi=\alpha_f}$ defined in Equation (37). Since $\mathrm{Fil}^0 V^*_{\rm dR}(f)$ equals $V^*_{\rm dR}(f)$ and $l+m-2 \geqslant (k+l+m-2)/2$ by Assumption 3.1(2) one has

$$\eta_f^\alpha \otimes \omega_g \otimes \omega_h \in \mathrm{Fil}^0 V^*_{\rm dR}(f,g,h). \tag{53}$$

Assume in the rest of this section that p does not divide N. For every s in $\mathbf{Z}$ denote by

$$\mathbf{M}_s(N, L) \subset \mathbf{Z}_p[\![q]\!] \otimes_{\mathbf{Z}_p} L$$

the space of p-adic modular forms of weight s and level $\Gamma_1(N)$ defined over L. Let

$$\mathbf{S}_s(N, L) \subset q \cdot \mathscr{O}[\![q]\!] \otimes_{\mathbf{Z}_p} \mathbf{Q}_p$$

be the subspace of cuspidal p-adic modular forms. $\mathbf{M}_s(N, L)$ contains naturally the space $M_s(\Gamma_1(N,p), L)$ of classical modular forms of level $\Gamma_1(N,p) = \Gamma_1(N) \cap \Gamma_0(p)$ and q-expansion in $L[\![q]\!]$. It is equipped with the Hecke operators $U = U_p$ and $V = V_p$, which are described on q-expansions by

$$U\Big(\sum_{n\geqslant 0} a_n \cdot q^n\Big) = \sum_{n \geqslant 0} a_{np} \cdot q^n \quad \text{and} \quad V\Big(\sum_{n\geqslant 0} a_n \cdot q^n\Big) = \sum_{n\geqslant 0} a_n \cdot q^{pn}$$

respectively. Serre's derivative operator $d = q \cdot \frac{d}{dq}$ on $L[\![q]\!]$ restricts to a morphism

$$d : \mathbf{M}_s(N, L) \to \mathbf{M}_{s+2}(N, L).$$

For every $s \geqslant 2$ Hida defined in [**Hid85**] an *ordinary projector*

$$e_{\mathrm{ord}} : \mathbf{M}_s(N, L) \longrightarrow M_s^{\mathrm{ord}}(\Gamma_1(N,p), L)$$

onto the space $M_s^{\mathrm{ord}}(\Gamma_1(N,p), L)$ of classical ordinary modular forms of level $\Gamma_1(N,p)$, which is a section of the natural inclusion $M_s^{\mathrm{ord}}(\Gamma_1(N,p), L) \hookrightarrow \mathbf{M}_s(N, L)$. Given $\xi \in S_l(\Gamma_1(N,p), L)$ and $\psi \in S_m(\Gamma_1(N,p), L)$ set

$$\Xi_k^{\mathrm{ord}}(\xi, \psi) = e_{\mathrm{ord}}\big(d^{(k-l-m)/2}\xi^{[p]} \times \psi\big) \in S_k^{\mathrm{ord}}(\Gamma_1(N,p), L),$$

where $\xi^{[p]}$ and $d^{(k-l-m)/2}\xi^{[p]}$ are defined as follows. Note first that $t = (k-l-m)/2$ is a negative integer by Assumption 3.1. The *p-depletion* $\xi^{[p]} \in \mathbf{S}_l(N,p)$ is defined by $\xi^{[p]} = (1 - VU)\xi$. If ξ has q-expansion $\sum_{n \geqslant 1} a_n(\xi) \cdot q^n$ then

$$\xi^{[p]} = \sum_{(n,p)=1} a_n(\xi) \cdot q^n,$$

hence the limit of p-adic modular forms

$$d^t \xi^{[p]} = \lim_{n \to \infty} d^{t+(p-1)p^n} \xi$$

defines a p-adic modular form of weight $l+2t$ such that $d^{-t}(d^t\xi^{[p]}) = \xi^{[p]}$, and $d^t\xi^{[p]} \times \psi$ belongs to $\mathbf{S}_k(N, L)$.

Let $\xi \in S_k(N, \chi_\xi, L)$ be a eigenvector for the Hecke operators T_ℓ, for all primes $\ell \nmid N$. Assume that ξ is p-ordinary, viz. $T_p(\xi) = a_p(\xi) \cdot \xi$ for a unit $a_p(\xi)$ in $\mathscr{O}^*$. Let α_ξ and β_ξ be the roots of the p-th Hecke polynomial $X^2 - a_p(\xi) \cdot X + \chi_\xi(p)p^{k-1}$ of ξ. Enlarging L if necessary, assume that α_ξ and β_ξ belong to L, and order them in such a way that $\alpha_f \in \mathscr{O}^*$ is a p-adic unit and $\beta_f \in p^{k-1} \cdot \mathscr{O}^*$. Then the *(ordinary) p-stabilisation* of ξ:

$$\xi_\alpha(q) = \xi(q) - \beta_\xi \cdot \xi(q^p) \in S_k^{\mathrm{ord}}(\Gamma_1(N,p), \chi_\xi) \tag{54}$$

is a normalised eigenvector for the Hecke operator T_ℓ, with the same eigenvalue as ξ, for every prime $\ell \nmid Np$, and is an eigenvector for U_p with eigenvalue α_ξ. Taking ξ to be one of f, g, h and $f^w = w_N(f)$ gives rise to the p-stabilised forms $f_\alpha, g_\alpha, h_\alpha$ and $f_\alpha^w = (f^w)_\alpha$ in $S_k(\Gamma_1(N,p), L)$. Define (cf. Sections 2.5 and 6)

$$\mathscr{L}_p^f(f_\alpha, g_\alpha, h_\alpha) = \frac{(f_\alpha^w, \Xi_k^{\mathrm{ord}}(g,h))_{Np}}{(f_\alpha^w, f_\alpha^w)_{Np}} \in L. \tag{55}$$

In [**BSV20b**] we proved the following *explicit reciprocity law*. Its proof uses the ideas and techniques introduced in [**BDP13, DR14, BDR15, KLZ20**]. In particular it relies on Besser's generalisation of Coleman's p-adic integration and the work of Bannai–Kings, Nekovář and Nizioł [**Nek04, Niz97, Niz01, Bes00, BK90**], which forces the assumption $p \nmid N$ in the statement.

Proposition 3.6 **([BSV20b]).** — *Assume that p does not divide N, and that the eigenforms f, g and h are p-ordinary. Then*

$$\log_p\big(\kappa(f,g,h)\big)(\eta_f^\alpha \otimes \omega_g \otimes \omega_h) = E(f,g,h) \cdot \mathscr{L}_p^f(f_\alpha, g_\alpha, h_\alpha),$$

where

$$E(f,g,h)=\frac{(-1)^{r-r_1}(r-r_1)!\left(1-\frac{\beta_f}{\alpha_f}\right)\left(1-\frac{\beta_f}{p\alpha_f}\right)}{\left(1-\frac{\beta_f\alpha_g\alpha_h}{p^{r+2}}\right)\left(1-\frac{\beta_f\alpha_g\beta_h}{p^{r+2}}\right)\left(1-\frac{\beta_f\beta_g\alpha_h}{p^{r+2}}\right)\left(1-\frac{\beta_f\beta_g\beta_h}{p^{r+2}}\right)}.$$

3.2. Comparison with Gross–Kudla–Schoen diagonal cycles. — This section elucidates the relation between the diagonal classes introduced above and the Gross–Kudla–Schoen diagonal cycles. It will not be used in the sequel of this paper.

Let the notations and assumptions be as in the previous section. In this section only we also assume $r_j \geqslant 1$ for $j=1,2,3$. As in [DR14, Section 3.1] fix three subsets $A=\{a_1,\dots,a_{r_1}\}$, $B=\{b_1,\dots,b_{r_2}\}$ and $C=\{c_1,\dots,c_{r_3}\}$ of $\{1,\dots,r\}$ of cardinalities r_1, r_2 and r_3 respectively, such that $A\cap B\cap C=\emptyset$. This is possible by Assumption 3.1. For $1\leqslant j\leqslant r$, let $p_j : E_1^r(N)=E_1(N)\times_{Y_1(N)}\cdots\times_{Y_1(N)}E_1(N)\longrightarrow E_1(N)$ be the projection from the r-fold fibered product of $E_1(N)$ over $Y_1(N)$ onto its j-th component. Define

$$\iota_{N,\boldsymbol{r}}=(p_A,p_B,p_C): E_1^r(N)\longrightarrow E_1^{\boldsymbol{r}}(N)\overset{\mathrm{def}}{=}E_1^{r_1}(N)\times_{\mathbf{Q}}E_1^{r_2}(N)\times_{\mathbf{Q}}E_1^{r_3}(N), \tag{56}$$

where $p_A=p_{a_1}\times\cdots\times p_{a_{r_1}} : E_1^r(N)\to E_1^{r_1}(N)$ and p_B and p_C are defined similarly. Then $\iota_{N,\boldsymbol{r}}=\iota_{N,(A,B,C)}$ is a closed immersion of relative dimension $\dim E_1^{\boldsymbol{r}}(N)-\dim E_1^r(N)=r+2$, and one defines the *generalised Gross–Kudla–Schoen diagonal cycle* of level N and weights $\boldsymbol{r}+\mathbf{2}$ (cf. Section 3 of [DR14]) as

$$\Delta_{N,\boldsymbol{r}}=\iota_{N,\boldsymbol{r}*}(E_1^r(N))\in \mathrm{CH}^{r+2}(E_1^{\boldsymbol{r}}(N)), \tag{57}$$

where $\mathrm{CH}^j(\cdot)$ is the Chow group of codimension-j cycles in $\cdot$ modulo rational equivalence.

For $i\in\mathbf{N}$ denote by $\mathfrak{S}_i=\mu_2^i\rtimes\Sigma_i$ the semi-direct product of $\mu_2^i=\{\pm1\}^i$ with the symmetric group Σ_i on i letters. The permutation action of Σ_i on $E_1^i(N)$ and the action of μ_2 on $E_1(N)$ induce an action of $\mathfrak{S}_i$ on $E_1^i(N)$. Define the character $\psi_i : \mathfrak{S}_i\to\{\pm1\}$ by $\psi_i(s_1,\dots,s_i,\sigma)=\mathrm{sgn}(\sigma)\cdot s_1\cdots s_i$, and set $\varepsilon_i=\frac{1}{2^i\cdot i!}\sum_{g\in\mathfrak{S}_i}\psi_i(g)\cdot g$. Then ε_i gives an idempotent in the ring $\mathrm{Corr}(E_1^i(N))_{\mathbf{Q}}$ of correspondences on $E_1^i(N)$ with rational coefficients. Set $\varepsilon_{\boldsymbol{r}}=\varepsilon_{r_1}\otimes\varepsilon_{r_2}\otimes\varepsilon_{r_3}\in\mathrm{Corr}(E_1^{\boldsymbol{r}}(N))_{\mathbf{Q}}$. The Lieberman trick (cf. the proof of Lemme 5.3 of [Del71]) shows that $\varepsilon_{\boldsymbol{r}}$ kills the cohomology group $H^j_{\text{ét}}(E_1^{\boldsymbol{r}}(N)_{\bar{\mathbf{Q}}},\mathbf{Q}_p)$ for every $j\neq 2r+3$, hence the image

$$cl_{\text{ét}}\big(\varepsilon_{\boldsymbol{r}}\cdot\Delta_{N,\boldsymbol{r}}\big)\in H^{2r+4}_{\text{ét}}(E_1^{\boldsymbol{r}}(N),\mathbf{Q}_p(r+2))$$

of $\varepsilon_{\boldsymbol{r}}\cdot\Delta_{N,\boldsymbol{r}}$ under the cycle class map

$$cl_{\text{ét}} : \mathrm{CH}^{r+2}(E_1^{\boldsymbol{r}}(N))_{\mathbf{Q}}\to H^{2r+4}_{\text{ét}}(E_1^{\boldsymbol{r}}(N),\mathbf{Q}_p(r+2))$$

belongs to

$$\begin{aligned}&\mathrm{Fil}^0 H^{2r+4}_{\text{ét}}(E_1^{\boldsymbol{r}}(N),\mathbf{Q}_p(r+2))\\ &\quad=\ker\left(H^{2r+4}_{\text{ét}}(E_1^{\boldsymbol{r}}(N),\mathbf{Q}_p(r+2))\xrightarrow{\pi^*}H^{2r+3}_{\text{ét}}(E_1^{\boldsymbol{r}}(N)_{\bar{\mathbf{Q}}},\mathbf{Q}_p(r+2))\right),\end{aligned}$$

where $\pi : E_1^{\boldsymbol{r}}(N)_{\bar{\mathbf{Q}}} \to E_1^{\boldsymbol{r}}(N)$ is the projection. As a consequence one can consider the Abel–Jacobi image

$$\mathrm{AJ}_p^{\text{ét}}\big(\varepsilon_{\boldsymbol{r}} \cdot \Delta_{N,\boldsymbol{r}}\big) = \mathtt{HS} \circ cl_{\text{ét}}\big(\varepsilon_{\boldsymbol{r}} \cdot \Delta_{N,\boldsymbol{r}}\big) \in H^1(\mathbf{Q}, \varepsilon_{\boldsymbol{r}} \cdot H^{2r+3}_{\text{ét}}(E_1^{\boldsymbol{r}}(N)_{\bar{\mathbf{Q}}}, \mathbf{Q}_p(r+2)))$$

of $\varepsilon_{\boldsymbol{r}} \cdot \Delta_{N,\boldsymbol{r}}$ under the composition of the cycle class map $cl_{\text{ét}}$ with the morphism

$$\mathtt{HS} : \mathrm{Fil}^0 H^{2r+4}_{\text{ét}}(E_1^{\boldsymbol{r}}(N), \mathbf{Q}_p(r+2)) \longrightarrow H^1(\mathbf{Q}, H^{2r+3}_{\text{ét}}(E_1^{\boldsymbol{r}}(N)_{\bar{\mathbf{Q}}}, \mathbf{Q}_p(r+2))) \tag{58}$$

arising from the Hochschild–Serre spectral sequence. According to the Lieberman trick the Leray spectral sequence associated with the structural map $E_1^{\boldsymbol{r}}(N) \to Y_1(N)^3$ induces a natural isomorphism

$$\mathtt{L}_{\boldsymbol{r}} : \varepsilon_{\boldsymbol{r}} \cdot H^{2r+3}_{\text{ét}}(E_1^{\boldsymbol{r}}(N)_{\bar{\mathbf{Q}}}, \mathbf{Q}_p(r+2)) \cong H^1_{\text{ét}}(Y_1(N)^3_{\bar{\mathbf{Q}}}, \mathscr{S}_{[\boldsymbol{r}]}) \otimes_{\mathbf{Z}_p} \mathbf{Q}_p(r+2) = W_{N,\boldsymbol{r}}. \tag{59}$$

Denote by

$$\mathtt{L}_{\boldsymbol{r}*} : H^1(\mathbf{Q}, \varepsilon_{\boldsymbol{r}} \cdot H^{2r+3}_{\text{ét}}(E_1^{\boldsymbol{r}}(N)_{\bar{\mathbf{Q}}}, \mathbf{Q}_p(r+2))) \cong H^1(\mathbf{Q}, W_{N,\boldsymbol{r}})$$

the isomorphism induced in Galois cohomology by $\mathtt{L}_{\boldsymbol{r}}$.

Proposition 3.7. — *The image of* $\mathrm{AJ}_p^{ét}(\varepsilon_{\boldsymbol{r}} \cdot \Delta_{N,\boldsymbol{r}})$ *under the isomorphism* $\mathtt{L}_{\boldsymbol{r}*}$ *is equal (up to sign) to* $\tilde{\kappa}_{N,\boldsymbol{r}}$.

Proof. — To ease notation set $E^{\cdot} = E_1^{\cdot}(N)$, $Y = Y_1(N)$, $\iota_{\boldsymbol{r}} = \iota_{N,\boldsymbol{r}}$, and denote by $u^{\boldsymbol{r}} = u_N^{\boldsymbol{r}}$ the structural morphism

$$u_N^{r_1} \times_{\mathbf{Q}} u_N^{r_2} \times_{\mathbf{Q}} u_N^{r_3} : E_1^{\boldsymbol{r}}(N) \to Y_1(N)^3.$$

Let $\iota_r : E^r \to E^{2r}$ be the proper morphism defined by

$$\iota_r(P_1, \ldots, P_r) = (\{P_{a_j}\}, \{P_{b_j}\}, \{P_{c_j}\}),$$

so that $\iota_{\boldsymbol{r}}$ is the composition of ι_r with the natural map $d_{\boldsymbol{r}} : E^{2r} \to E^{\boldsymbol{r}}$.

Define

$$\mathscr{R}^{2r} = R^{2r} u^{2r}_* \mathbf{Z}_p, \quad \mathscr{R}^r = R^{2r} u^r_* \mathbf{Z}_p \quad \text{and} \quad \mathscr{R}^{[\boldsymbol{r}]} = R^{2r} u^{\boldsymbol{r}}_* \mathbf{Z}_p.$$

Then ι_r induces relative pull-back and pushforward maps

$$\vartheta_r^* : \mathscr{R}^{2r}(r) \longrightarrow \mathbf{Z}_p \quad \text{and} \quad \vartheta_{r*} : \mathbf{Z}_p \longrightarrow \mathscr{R}^{2r}(r)$$

which are adjoint to each other under the perfect relative Poincaré duality

$$\mathscr{R}^{2r}(r) \otimes_{\mathbf{Z}_p} \mathscr{R}^{2r}(r) \longrightarrow R^{4r} u^{2r}_* \mathbf{Z}_p(2r) \cong \mathbf{Z}_p$$

induced by the cup-product pairing. (They induce on the stalks at a geometric point $y : \mathrm{Spec}(\bar{\mathbf{Q}}) \to Y$ the pull-back $H^{2r}_{\text{ét}}(E^{2r}_y, \mathbf{Z}_p(r)) \to H^{2r}_{\text{ét}}(E^r_y, \mathbf{Z}_p(r)) \cong \mathbf{Z}_p$ and push-forward $\mathbf{Z}_p = H^0_{\text{ét}}(E^r_y, \mathbf{Z}_p) \to H^{2r}_{\text{ét}}(E^{2r}_y, \mathbf{Z}_p(r))$ associated with $\iota_r \times_y \bar{\mathbf{Q}}$ respectively.) The Leray spectral sequences associated with the morphisms u^{2r} and $u^{\boldsymbol{r}}$ identify the $\mathbf{Q}_p$-linear extensions of $H^0_{\text{ét}}(Y, \mathscr{R}^{2r}(r))$ and $H^4_{\text{ét}}(Y^3, \mathscr{R}^{[\boldsymbol{r}]}(r+2))$ with direct summands of $H^{2r}_{\text{ét}}(E^{2r}, \mathbf{Q}_p(r))$ and $H^{2r+4}_{\text{ét}}(E^{\boldsymbol{r}}, \mathbf{Q}_p(r+2))$ respectively. (This is again a consequence of the Lieberman trick, cf. [Del71].) By the functoriality of the Leray spectral sequence, under these identifications ϑ_{r*} and d_* are compatible with the absolute push-forward maps attached to ι_r and $d_{\boldsymbol{r}}$, viz. the following diagram is commutative:

(60)

$$\begin{array}{ccccc} \mathbf{Q}_p & \xrightarrow{\vartheta_{r*}} & H^0_{\text{ét}}(Y, \mathscr{R}^{2r}(r))_{\mathbf{Q}_p} & \xrightarrow{d_*} & H^4_{\text{ét}}(Y^3, \mathscr{R}^{[\boldsymbol{r}]}(r+2))_{\mathbf{Q}_p} \\ \| & & \downarrow{\scriptstyle \text{Leray}} & & \downarrow{\scriptstyle \text{Leray}} \\ H^0_{\text{ét}}(E^r, \mathbf{Q}_p) & \xrightarrow{\iota_{r*}} & H^{2r}_{\text{ét}}(E^{2r}, \mathbf{Q}_p(r)) & \xrightarrow{d_{\boldsymbol{r}*}} & H^{2r+4}_{\text{ét}}(E^{\boldsymbol{r}}, \mathbf{Q}_p(r+2)). \end{array}$$

On the other hand the compatibility of the cycle class

$$cl_{\text{ét}} : \mathrm{CH}^{r+2}(E^{\boldsymbol{r}})_{\mathbf{Q}} \to H^{2r+4}_{\text{ét}}(E^{\boldsymbol{r}}, \mathbf{Q}_p(r+2))$$

with proper push-forwards and the definition of the diagonal cycle $\Delta_{\boldsymbol{r}} = \Delta_{N,\boldsymbol{r}}$ yield the identities

$$cl_{\text{ét}}(\Delta_{\boldsymbol{r}}) = cl_{\text{ét}} \circ \iota_{\boldsymbol{r}*}(E^r) = \iota_{\boldsymbol{r}*}(1) = d_{\boldsymbol{r}*} \circ \iota_{r*}(1).$$

In addition, using again the functoriality of the Leray spectral sequences, one has the commutative diagram

$$\begin{array}{ccccc} H^4_{\text{ét}}(Y^3, \mathscr{R}^{[\boldsymbol{r}]}(r+2))_{\mathbf{Q}_p} & \xrightarrow{\mathtt{p}_{[\boldsymbol{r}]}} & H^4_{\text{ét}}(Y^3, \mathscr{S}_{[\boldsymbol{r}]}(r+2))_{\mathbf{Q}_p} & \xrightarrow{\mathtt{HS}} & H^1(\mathbf{Q}, W_{\boldsymbol{r}}) \\ \downarrow{\scriptstyle \text{Leray}} & & & & \downarrow{\scriptstyle \text{Leray}} \\ \mathrm{Fil}^0 H^{2r+4}_{\text{ét}}(E^{\boldsymbol{r}}, \mathbf{Q}_p(r+2)) & & \xrightarrow{\varepsilon_{\boldsymbol{r}*} \circ \mathtt{HS}} & & H^1(\mathbf{Q}, \varepsilon_{\boldsymbol{r}} \cdot H^{2r+3}_{\text{ét}}(E^{\boldsymbol{r}}_{\bar{\mathbf{Q}}}, \mathbf{Q}_p(r+2))), \end{array}$$

where $\mathtt{p}_{[\boldsymbol{r}]} : \mathscr{R}^{[\boldsymbol{r}]} \twoheadrightarrow \mathscr{S}_{[\boldsymbol{r}]}$ is the natural projection and $W_{\boldsymbol{r}} = W_{N,\boldsymbol{r}}$. Since $\varepsilon_{\boldsymbol{r}}$ acts as the identity on $\mathscr{S}_{[\boldsymbol{r}]}$, the previous three equations prove that (cf. Equation (59))

$$\mathtt{L}_{\boldsymbol{r}*}\big(\mathrm{AJ}^{\text{ét}}_p(\varepsilon_{\boldsymbol{r}} \cdot \Delta_{\boldsymbol{r}})\big) = \mathtt{HS} \circ \mathtt{p}_{[\boldsymbol{r}]} \circ d_* \circ \vartheta_{r*}(1).$$

After setting $\mathtt{Det}^{\boldsymbol{r}} = \mathtt{Det}^{\boldsymbol{r}}_N$, to conclude the proof of the proposition it is then sufficient to show that

$$\mathtt{Det}^{\boldsymbol{r}} = \mathtt{p}_{\boldsymbol{r}} \circ \vartheta_{r*}(1) \in H^0_{\text{ét}}(Y, \mathscr{S}_{\boldsymbol{r}}(r)), \tag{61}$$

where $\mathtt{p}_{\boldsymbol{r}} : \mathscr{R}^{2r}(r) \twoheadrightarrow \mathscr{S}_{\boldsymbol{r}}(r)$ is the natural projection. Let $S = S_1(\mathbf{Z}_p)$ be the standard representation of $\mathrm{GL}_2(\mathbf{Z}_p)$. Recall the geometric point $\eta : \mathrm{Spec}(\bar{\mathbf{Q}}) \to Y$ and the isomorphism $\xi : \mathscr{T}_\eta \cong S \otimes \det^{-1}$ fixed above (cf. Equations (39) and (44)). The $\mathrm{GL}_2(\mathbf{Z}_p)$-representation $\mathscr{R}^{2r}(r)_\eta$ contains $S^{\otimes 2r} \otimes \det^{-r}$ as a direct summand, and $\mathtt{p}_{\boldsymbol{r}} : \mathscr{R}^{2r}(r)_\eta \to \mathscr{S}_{\boldsymbol{r}}(r)_\eta = S_{\boldsymbol{r}} \otimes \det^{-r}$ is the composition of $\mathrm{pr} : \mathscr{R}^{2r}(r)_\eta \twoheadrightarrow S^{\otimes 2r} \otimes \det^{-r}$ and the natural projection $\mathrm{pr}_{\boldsymbol{r}} : S^{\otimes 2r} \otimes \det^{-r} \twoheadrightarrow S_{\boldsymbol{r}} \otimes \det^{-r}$. Let $\vartheta^o_{r*} : \mathbf{Z}_p \to \mathscr{R}^{2r}(r)$ be the relative push-forward associated (as above) with the morphism $E^r \to E^{2r}$ which sends the point $(P_1, \dots, P_r)$ to $(P_1, P_1, \dots, P_r, P_r)$. Then

$$\vartheta_{r*} = \sigma_{\boldsymbol{r}} \circ \vartheta^o_{r*}, \tag{62}$$

where $\sigma_{\boldsymbol{r}} = \sigma_{A,B,C}$ is any fixed permutation of $\{1, \dots, 2r\}$ satisfying

$$\sigma_{\boldsymbol{r}}(P_1, P_1, \dots, P_r, P_r) = (P_{a_1}, \dots, P_{a_{r_1}}, P_{b_1}, \dots, P_{b_{r_2}}, P_{c_1}, \dots, P_{c_{r_3}})$$

for every point $(P_1, \dots, P_r)$ of E^r. The image of 1 under the composition

$$\mathrm{pr} \circ \vartheta^o_{r*} : \mathbf{Z}_p = H^0_{\text{ét}}(E^r_\eta, \mathbf{Z}_p) \longrightarrow H^{2r}_{\text{ét}}(E^{2r}_\eta, \mathbf{Z}_p(r)) = \mathscr{R}^{2r}(r)_\eta \twoheadrightarrow S^{\otimes 2r} \otimes \det^{-r}$$

(where one writes again ϑ^o_{r*} for the morphism induced by ϑ^o_{r*} on the stalks at η) is equal to

$$F_r = \big(x \otimes y - y \otimes x\big)^{\otimes r},$$

where x and y give a $\mathbf{Z}_p$-basis of $S \subset \mathbf{Z}_p[x,y]$. It then follows by the definition of $\mathtt{Det}^{\boldsymbol{r}}$ (see Equation (42)) and Equation (62) that in order to prove the claim (61) is it sufficient to prove (setting $\mathrm{Det}^{\boldsymbol{r}} = \mathrm{Det}^{\boldsymbol{r}}_N$)

$$\mathrm{Det}^{\boldsymbol{r}} = \mathrm{pr}_{\boldsymbol{r}} \circ \sigma_{\boldsymbol{r}}(F_r). \tag{63}$$

The previous formula is easily verified if $r \leqslant 2$ or $\boldsymbol{r} = (2,2,2)$ (hence $r = 3$). Assume now $r \geqslant 3$ and $\boldsymbol{r} \neq (2,2,2)$. Then at least one of $|A \cap B|$, $|A \cap C|$ and $|B \cap C|$ is greater or equal than 2. Without loss of generality one can then assume $r_2 = \min\{r_1, r_2, r_3\}$ and that the sets A and C are of the form

$$A = \{1, r, a_3, \dots, a_{r_1}\} \quad \text{ and } \quad C = \{c_1, \dots, c_{r_3-2}, 1, r\}.$$

Let $\boldsymbol{s} = (r_1 - 2, r_2, r_3 - 2)$ and $s = r - 2$. Then $\boldsymbol{s}$ satisfies Assumption 3.1 and one can chose as above a permutation $\sigma_{\boldsymbol{s}} = \sigma_{A_o,B,C_o}$ of $\{1, \dots, 2 \cdot (r-1)\}$ relative to $A_o = \{a_3, \dots, a_{r_1-1}\}$, B and $C_o = \{c_1, \dots, c_{r_3-2}\}$. Extend $\sigma_{\boldsymbol{s}}$ to a permutation (denoted by the same symbol) of $\{1, \dots, 2r\}$ by $\sigma_{\boldsymbol{s}}(i) = i$ for $i = 1, 2, 2r-1, 2r$ Without loss of generality one can then assume that $\sigma_{\boldsymbol{r}} = \sigma_{A,B,C}$ is the composition of $\sigma_{\boldsymbol{s}}$ with the permutation $\sigma_{\boldsymbol{r}|\boldsymbol{s}}$ of $\{1, \dots, 2r\}$ defined by $\sigma_{\boldsymbol{r}|\boldsymbol{s}}(2) = 2r - 1$ and $\sigma_{\boldsymbol{r}|\boldsymbol{s}}(i) = i$ for $i \neq 2, 2r-1$, hence by induction on r one has

$$\mathrm{pr}_{\boldsymbol{r}} \circ \sigma_{\boldsymbol{r}}(F_r) = \mathrm{pr}_{\boldsymbol{r}} \circ \sigma_{\boldsymbol{r}|\boldsymbol{s}}\big(F_1 \otimes \sigma_{\boldsymbol{s}}(F_s) \otimes F_1\big) = \det \begin{pmatrix} x_1 & x_2 \\ z_1 & z_2 \end{pmatrix}^2 \cdot \mathrm{Det}^{\boldsymbol{s}}.$$

Since $r - r_2 = s - s_2 + 2$ and $r - r_j = s - s_j$ for $j \neq 2$, this proves Equation (63), and with it the proposition. □

4. *Big* étale sheaves and Galois representations

Sections 4.1 and 4.2 collect the technical background entering the construction of the three-variable diagonal class of Theorem A. In particular they present a slight extension of the *overconvergent cohomology* theory developed by Ash–Stevens and Andreatta–Iovita–Stevens in [**AS08, AIS15**].

Notation. In this section N is a positive integer coprime with p. Set $\Gamma = \Gamma_1(N,p)$, let Y denote the affine modular curve $Y_1(N,p)$ of level Γ defined over $\mathbf{Z}[1/Np]$ and let $u : E \to Y$ be the universal elliptic curve $E_1(N,p)$. Denote by C_p the universal order-p cyclic subgroup $C_1(N,p)$ of $E_1(N,p)$.

4.1. Locally analytic functions and distributions. — Let L be a finite extension of $\mathbf{Q}_p$ with ring of integers $\mathscr{O}$ and maximal ideal $\mathfrak{m} = \pi \cdot \mathscr{O}$. Let $\mathcal{W}$ be the *weight space* over $\mathbf{Q}_p$, viz. the rigid analytic space over $\mathbf{Q}_p$ which parametrises the continuous characters of $\mathbf{Z}_p^*$. It is isomorphic to $p-1$ copies of the open unit disc, indexed by the powers ω^j of the Teichmüller character $\omega : \mathbf{F}_p^* \to \mathbf{Z}_p^*$. We identify $\mathbf{Z} \times \mathbf{Z}/(p-1)\mathbf{Z}$ with a subset of $\mathcal{W}(\mathbf{Q}_p)$ by sending the pair (n,a) to the character

$(n,a) : \mathbf{Z}_p^* \to \mathbf{Z}_p^*$ defined by $(n,a)(u \cdot \omega) = u^n \cdot \omega^a$ for every $u \in 1 + p\mathbf{Z}_p$ and $\omega \in \mathbf{F}_p^*$. Given $\kappa \in \mathcal{W}$ and $z \in \mathbf{Z}_p^*$ we often write z^κ for $\kappa(z)$.

Let $U \subset \mathcal{W}$ be a connected wide open disc defined over L. Write $U \cap \mathbf{Z}$ for the set of characters in $U(\mathbf{Q}_p)$ of the form (n, i_U) for some $n \in \mathbf{Z}$ with $n\,(\mathrm{mod}\ p-1) = i_U$, where $i_U \in \mathbf{Z}/(p-1)\mathbf{Z}$ satisfies $\kappa|_{\mathbf{F}_p^*} = \omega^{i_U}$ for every $\kappa \in U$. Denote by $\mathcal{O}(U)$ the ring of rigid analytic functions on U, and by $\Lambda_U \subset \mathcal{O}(U)$ the set of $a \in \mathcal{O}(U)$ such that $\mathrm{ord}_p(a(x)) \geqslant 0$ for every $x \in U$. The $\mathscr{O}$-algebra Λ_U is isomorphic to the power series ring $\mathscr{O}[\![T]\!]$. In particular it is a regular local ring, complete with respect to the topology defined by its maximal ideal $\mathfrak{m}_U \cong (\pi, T)$. Let

$$\kappa_U : \mathbf{Z}_p^* \longrightarrow \Lambda_U^*$$

be the character sending $z \in \mathbf{Z}_p^*$ to the analytic function $\kappa_U(z) \in \Lambda_U^*$ which on $t \in U$ takes the value

$$\kappa_U(z)(t) = z^{t-2}.$$

In what follows let (B, κ) denote either the pair (Λ_U, κ_U) or $(\mathscr{O}, r)$ for some $r \in \mathcal{W}(L)$, and write $\mathfrak{m}_B$ for the maximal ideal of B. For every nonnegative integer $m \geqslant 0$ let $LA_m(\mathbf{Z}_p, B)$ be the space of functions $\gamma : \mathbf{Z}_p \to B$ converging on balls of width m, viz. for every $[a] \in \mathbf{Z}/p^m\mathbf{Z}$ one has $\gamma(a + p^m z) = \sum_{n \geqslant 0} c_n(\gamma) \cdot z^n$ for a sequence $c_n(\gamma)$ in B which converges to zero in the $\mathfrak{m}_B$-adic topology. We always assume that U is contained in a connected affinoid domain in $\mathcal{W}$ and that the function sending z to $\kappa_U(1 + pz)$ belongs to $LA_m(\mathbf{Z}_p, \Lambda_U)$. The latter condition is guaranteed by taking $m = m(U)$ big enough.

Define $\mathsf{T} = \mathbf{Z}_p^* \times \mathbf{Z}_p$ and $\mathsf{T}' = p\mathbf{Z}_p \times \mathbf{Z}_p^*$. Right multiplication on $\mathbf{Z}_p^2$ by the semi-group

$$\Sigma_0(p) = \begin{pmatrix} \mathbf{Z}_p^* & \mathbf{Z}_p \\ p\mathbf{Z}_p & \mathbf{Z}_p \end{pmatrix} \subset \mathrm{Mat}_{2\times 2}(\mathbf{Z}_p) \quad \Big(\text{resp.,} \quad \Sigma_0'(p) = \begin{pmatrix} \mathbf{Z}_p & \mathbf{Z}_p \\ p\mathbf{Z}_p & \mathbf{Z}_p^* \end{pmatrix} \subset \mathrm{Mat}_{2\times 2}(\mathbf{Z}_p)\Big)$$

preserves the subset T (resp., T'). In particular both T and T' are preserved by scalar multiplication by $\mathbf{Z}_p^*$ and right multiplication by the Iwahori subgroup

$$\Gamma_0(p\mathbf{Z}_p) = \Sigma_0(p) \cap \Sigma_0'(p)$$

of $\mathrm{GL}_2(\mathbf{Z}_p)$. Define

$$\mathcal{A}_{\kappa,m} = \Big\{ f : \mathsf{T} \longrightarrow B \ \Big|\ f(1,z) \in LA_m(\mathbf{Z}_p, B) \text{ and}$$

$$f(a \cdot t) = \kappa(a) \cdot f(t) \text{ for every } a \in \mathbf{Z}_p^*,\ t \in \mathsf{T} \Big\}, \tag{64}$$

and similarly define $\mathcal{A}'_{\kappa,m}$ as the space of functions $f : \mathsf{T}' \to B$ such that $f(pz, 1)$ belongs to $LA_m(\mathbf{Z}_p, B)$, and $f(a \cdot t) = \kappa(a) \cdot f(t)$ for all $a \in \mathbf{Z}_p^*$ and $t \in \mathsf{T}'$. Set

$$A^{\cdot}_{\kappa,m} = \mathcal{A}^{\cdot}_{\kappa,m} \otimes_{\mathscr{O}} L, \quad \mathcal{D}^{\cdot}_{\kappa,m} = \mathrm{Hom}_B(\mathcal{A}^{\cdot}_{\kappa,m}, B) \quad \text{and} \quad D^{\cdot}_{\kappa,m} = \mathcal{D}^{\cdot}_{\kappa,m} \otimes_{\mathscr{O}} L,$$

where the superscript $\cdot$ denotes either $\emptyset$ or $\prime$. We equip $\mathcal{A}^{\cdot}_{\kappa,m}$ with the $\mathfrak{m}_B$-adic topology and $\mathcal{D}^{\cdot}_{\kappa,m}$ with the weak-$*$ topology, viz. the weakest topology which makes the evaluation-at-f morphism continuous for every f in $\mathcal{A}^{\cdot}_{\kappa,m}$. The B-module $\mathcal{A}^{\cdot}_{\kappa,m}$ is preserved by the left action of $\Sigma^{\cdot}_0(p)$ on functions $f : \mathsf{T}^{\cdot} \to B$ given by $\gamma \cdot f(t) = f(t \cdot \gamma)$, for every $\gamma \in \Sigma^{\cdot}_0(p)$ and $t \in \mathsf{T}^{\cdot}$. This equips $\mathcal{A}^{\cdot}_{\kappa,m}$ with the structure of a

$B[\Sigma_0^{\cdot}(p)]$-module, and induce on $\mathcal{D}^{\cdot}_{\kappa,m}$ the structure of a right $B[\Sigma_0^{\cdot}(p)]$-module. If $(B,\kappa)=(\Lambda_U,\kappa_U)$ we write $\mathcal{A}^{\cdot}_{U,m}$ and $\mathcal{D}^{\cdot}_{U,m}$ as shorthands for $\mathcal{A}^{\cdot}_{\kappa_U,m}$ and $\mathcal{D}^{\cdot}_{\kappa_U,m}$.

Remark 4.1. — For any function $f:\mathsf{T}\to B$ define $f_o:\mathbf{Z}_p\to B$ by $f_o(z)=f(1,z)$. The map which to f associates f_o gives an isomorphism of B-modules between $\mathcal{A}_{\kappa,m}$ and $LA_m(\mathbf{Z}_p,B)$. This intertwines the action of $\Sigma_0(p)$ on $\mathcal{A}_{\kappa,m}$ with the one on $LA_m(\mathbf{Z}_p,B)$ given by

$$\sigma\cdot f_o(z)=(a+cz)^{\kappa}\cdot f_o\left(\frac{b+dz}{a+cz}\right),\quad \text{where}\quad \sigma=\begin{pmatrix} a & b\\ c & d\end{pmatrix}.$$

The B-module $LA_m(\mathbf{Z}_p,B)$ is isomorphic to the product $\prod_{a=0}^{p^m-1}B[\![T]\!]^o$, where $B[\![T]\!]^o$ is the set of power series $\sum_{n\geqslant 0}b_n\cdot T^n$ in $B[\![T]\!]$ with $\lim_{n\to\infty}b_n=0$ in the $\mathfrak{m}_B$-adic topology. Under this isomorphism, for every $0\leqslant a\leqslant p^m-1$ and every $n\geqslant 0$, the power T^n in the a-th factor of $LA_m(\mathbf{Z}_p,B)$ corresponds to an element $f_{a,n}\in\mathcal{A}_{\kappa,m}$. Every $f\in\mathcal{A}_{\kappa,m}$ can be written uniquely as $f=\sum_{0\leqslant a\leqslant p^m-1,n\geqslant 0}b_{a,n}(f)\cdot f_{a,n}$ with $\lim_{n\to\infty}b_{a,n}(f)=0$ for every $0\leqslant a\leqslant p^m-1$. A similar discussion applies to $\mathcal{A}'_{\kappa,m}$.

4.1.1. *Hecke operators.* — Set $\Xi_0^{\cdot}(p)=\Sigma_0^{\cdot}(p)\cap\mathrm{GL}_2(\mathbf{Q}_p)$, and recall that Γ denotes the congruence subgroup $\Gamma_1(N)\cap\Gamma_0(p)$ of $\mathrm{SL}_2(\mathbf{Z})$. Let M be a *right* $\Xi_0^{\cdot}(p)$-module (e.g. $M=\mathcal{D}^{\cdot}_{\kappa,m}$). Given $\sigma\in\Xi_0^{\cdot}(p)$ one defines a Hecke operator

$$T_\sigma: H^j(\Gamma,M)\to H^j(\Gamma,M)$$

as follows (cf. [**AS86a**, Section 1.1]). Write $\Gamma\sigma\Gamma=\coprod_{i=1}^{n_\sigma}\Gamma\sigma_i$ with $\sigma_i\in\Xi_0^{\cdot}(p)$, and define $t_i:\Gamma\longrightarrow\Gamma$ by $\sigma_i\cdot\gamma=t_i(\gamma)\cdot\sigma_{i(\gamma)}$ (for some $1\leqslant i(\gamma)\leqslant n_\sigma$). If $\boldsymbol{\xi}\in H^j(\Gamma,M)$ is represented by the homogeneous j-cochain $\xi:\Gamma^{j+1}\longrightarrow M$ then $T_\sigma(\boldsymbol{\xi})=cl(\xi_\sigma)$, where $\xi_\sigma:\Gamma^{j+1}\longrightarrow M$ is defined by

$$\xi_\sigma(\gamma_0,\dots,\gamma_j)=\sum_{i=1}^{n_\sigma}\xi(t_i(\gamma_0),\dots,t_i(\gamma_j))\cdot\sigma_i.$$

For every prime ℓ denote by σ_ℓ (resp., σ'_ℓ) the diagonal matrix with diagonal $(1,\ell)$ (resp., $(\ell,1)$). If σ_ℓ (resp., σ'_ℓ) belongs to $\Xi_0^{\cdot}(p)$ set $T_\ell=T_{\sigma_\ell}$ (resp., $T'_\ell=T_{\sigma'_\ell}$). As usual one also writes $U^{\cdot}_\ell$ for $T^{\cdot}_\ell$ if ℓ divides Np. The previous discussion then equips $H^i(\Gamma,\mathcal{D}_{\kappa,m})$ (resp., $H^1(\Gamma,\mathcal{D}'_{\kappa,m})$) with the action of the p-th Hecke operator U_p (resp., p-th dual Hecke operator U'_p), as well as with the action of the Hecke operators T_ℓ and T'_ℓ for every prime $\ell\neq p$.

Let N be a *left* $\Xi_0^{\cdot}(p)$-module (e.g. $N=\mathcal{A}^{\cdot}_{\kappa,m}$) and let N^{op} denote the abelian group N equipped with the structure of right $\Xi_0^{\cdot}(p)^{-1}$-module by $n\cdot\tau=\tau^{-1}\cdot n$ for every $n\in N$ and $\tau\in\Xi_0^{\cdot}(p)^{-1}$. After identifying $H^i(\Gamma,N)$ and $H^i(\Gamma,N^{\mathrm{op}})$ define for every $\sigma\in\Xi_0^{\cdot}(p)$ the Hecke operator T_σ on $H^i(\Gamma,N)$ to be the Hecke operator $T_{\sigma^{-1}}$ on $H^i(\Gamma,N^{\mathrm{op}})$ defined in previous paragraph. This equips $H^i(\Gamma,\mathcal{A}_{\kappa,m})$ (resp., $H^i(\Gamma,\mathcal{A}'_{\kappa,m})$) with the action of the p-th Hecke operator $U_p=T_{\sigma_p}$ (resp., p-th dual Hecke operator $U'_p=T_{\sigma'_p}$), as well as with the action of the Hecke operators $T_\ell=T_{\sigma_\ell}$ and $T'_\ell=T_{\sigma'_\ell}$ for every prime ℓ different from p.

4.1.2. *Atkin–Lehner operators.* — Let Q be a positive divisor of Np, such that Q and Np/Q are coprime. Consider any matrix

$$\mathtt{w}_Q = \begin{pmatrix} Qa & b \\ Np & Qd \end{pmatrix} \in M_2(\mathbf{Z})$$

such that $\det(\mathtt{w}_Q) = Q$ and $d \equiv 1 \pmod{Np/Q}$. Such a matrix satisfies

$$\Gamma = \mathtt{w}_Q \cdot \Gamma \cdot \mathtt{w}_Q^{-1}. \tag{65}$$

If p divides Q, then right multiplication by $\mathtt{w}_Q$ on $\mathbf{Z}_p^2$ maps T onto T', hence induces a topological morphism of B-modules $w_Q : \mathcal{A}'_{\kappa,m} \longrightarrow \mathcal{A}_{\kappa,m}$. Together with conjugation by the inverse of $\mathtt{w}_Q$ on Γ (cf. Equation (65)), it yields a morphism of pairs $w_Q : (\Gamma, \mathcal{A}'_{\kappa,m}) \longrightarrow (\Gamma, \mathcal{A}_{\kappa,m})$, which in turn induces a morphism

$$w_Q : H^1(\Gamma, \mathcal{A}'_{\kappa,m}) \longrightarrow H^1(\Gamma, \mathcal{A}_{\kappa,m}). \tag{66}$$

A direct computation proves that, for each x in $H^1(\Gamma, \mathcal{A}'_{\kappa,m})$, one has

$$U_p \circ w_p(x) = w_p \circ U'_p \circ \langle p \rangle_N (x) \quad \text{and} \quad U_p \circ w_{Np}(x) = w_{Np} \circ U'_p(x),$$

where $\langle p \rangle_N = T_{\alpha_p}$ is the Hecke operator on $H^1(\Gamma, \mathcal{A}'_{\kappa,m})$ associated with any matrix α_p in $\mathrm{SL}_2(\mathbf{Z})$ of the form $\alpha_p = \left(\begin{smallmatrix} a & b \\ Npc & d \end{smallmatrix}\right)$ with $d \equiv 1 \pmod p$ and $d \equiv p \pmod N$. The dual of $w_Q : \mathcal{A}'_{\kappa,m} \longrightarrow \mathcal{A}_{\kappa,m}$ yields a map $w_Q : \mathcal{D}_{\kappa,m} \longrightarrow \mathcal{D}'_{\kappa,m}$, which together with conjugation by $\mathtt{w}_Q$ on Γ induces as above a morphism

$$w_Q : H^1(\Gamma, \mathcal{D}_{\kappa,m}) \longrightarrow H^1(\Gamma, \mathcal{D}'_{\kappa,m}). \tag{67}$$

For each y in $H^1(\Gamma, \mathcal{D}_{\kappa,m})$ one has

$$w_p \circ U_p(y) = U'_p \circ w_p \circ \langle p \rangle_N (y) \quad \text{and} \quad w_{Np} \circ U_p(y) = U'_p \circ w_{Np}(y). \tag{68}$$

If p does not divide Q, then $\mathtt{w}_Q$ belongs to $\Gamma_0(p\mathbf{Z}_p)$, and for $\cdot = \emptyset, \prime$ one defines

$$w_Q : H^1(\Gamma, \mathcal{D}^{\cdot}_{\kappa,m}) \longrightarrow H^1(\Gamma, \mathcal{D}^{\cdot}_{\kappa,m}) \ \text{ and } \ w_Q : H^1(\Gamma, \mathcal{A}^{\cdot}_{\kappa,m}) \longrightarrow H^1(\Gamma, \mathcal{A}^{\cdot}_{\kappa,m}) \tag{69}$$

to be the Hecke operators $T_{\mathtt{w}_Q}$ introduced in Section 4.1.1.

4.1.3. *Specialisations and comparison.* — Let $k = r + 2 \in U$ and let $\pi_k \in \Lambda_U$ be a uniformiser at $k - 2$ (hence π and π_k generate $\mathfrak{m}_U$). There are short exact sequences of $\Sigma_0^{\cdot}(p)$-modules (cf. [**AIS15**, Proposition 3.11])

$$\begin{gathered} 0 \longrightarrow \mathcal{A}^{\cdot}_{U,m} \xrightarrow{\pi_k} \mathcal{A}^{\cdot}_{U,m} \xrightarrow{\rho_k} \mathcal{A}^{\cdot}_{r,m} \longrightarrow 0; \\ 0 \longrightarrow \mathcal{D}^{\cdot}_{U,m} \xrightarrow{\pi_k} \mathcal{D}^{\cdot}_{U,m} \xrightarrow{\rho_k} \mathcal{D}^{\cdot}_{r,m} \longrightarrow 0. \end{gathered} \tag{70}$$

The morphisms ρ_k are defined by the formulae

$$\rho_k(f)(x,y) = f(x,y)(k) \quad \text{and} \quad \rho_k(\mu)(\gamma) = \mu(\gamma_U)(k)$$

for every $f \in \mathcal{A}^{\cdot}_{U,m}$, $(x,y) \in \mathsf{T}^{\cdot}$, $\mu \in \mathcal{D}^{\cdot}_{U,m}$, and $\gamma \in \mathcal{A}^{\cdot}_{r,m}$, where $\gamma_U(x,y) = \kappa_U(x) \cdot \gamma(1, y/x)$ if $\mathsf{T}^{\cdot} = \mathsf{T}$ and $\gamma_U(x,y) = \kappa_U(y) \cdot \gamma(x/y, 1)$ if $\mathsf{T}^{\cdot} = \mathsf{T}'$.

Let $r \in U \cap \mathbf{Z}_{\geqslant 0}$ be a nonnegative integer. Viewing two-variable polynomials as analytic functions on $\mathsf{T}^{\cdot}$ gives a natural map of $\Sigma_0^{\cdot}(p)$-modules $S_r(\mathscr{O}) \longrightarrow \mathcal{A}^{\cdot}_{r,m}$, and dually a morphism of $\Sigma_0^{\cdot}(p)$-modules $\mathcal{D}^{\cdot}_{r,m} \longrightarrow L_r(\mathscr{O})$. Together with the comparison isomorphisms between étale and Betti cohomology:

$$H^1_{\text{ét}}(Y_{\bar{\mathbf{Q}}}, \mathscr{S}_r(\mathscr{O})) \cong H^1(\Gamma, S_r(\mathscr{O})) \quad \text{and} \quad H^1_{\text{ét}}(Y_{\bar{\mathbf{Q}}}, \mathscr{L}_r(\mathscr{O})) \cong H^1(\Gamma, L_r(\mathscr{O})) \tag{71}$$

they induce *comparison* morphisms

$$H^1_{\text{ét}}(Y_{\bar{\mathbf{Q}}}, \mathscr{S}_r(\mathscr{O})) \longrightarrow H^1(\Gamma, \mathcal{A}^{\cdot}_{r,m}) \quad \text{and} \quad H^1(\Gamma, \mathcal{D}^{\cdot}_{r,m}) \longrightarrow H^1_{\text{ét}}(Y_{\bar{\mathbf{Q}}}, \mathscr{L}_r(\mathscr{O})). \tag{72}$$

The second isomorphism in Equation (71) is Hecke equivariant, hence so is the second morphism in Equation (72). On the other hand the first isomorphism in Equation (71) (resp., morphism in Equation (72)) intertwines the actions of the Hecke operators $U_p, T_\ell, U'_p, T'_\ell$ on the left hand side with those of Hecke operators $U'_p, T'_\ell, U_p, T_\ell$ respectively on the right hand side (whenever the latter are defined).

4.1.4. *Slope decompositions.* — Let $\mathscr{B}$ be a $\mathbf{Q}_p$-Banach algebra, let N be a module over $\mathscr{B}$, let $\boldsymbol{u}$ be a $\mathscr{B}$-linear endomorphism of N, and let $h \in \mathbf{Q}_{\geqslant 0}$. Following [**AS08**] one says that N admits a *slope $\leqslant h$ decomposition with respect to $\boldsymbol{u}$* if there exists a (necessarily unique) direct sum decomposition

$$N = N^{\leqslant h} \oplus N^{>h}$$

into $\mathscr{B}[\boldsymbol{u}]$-modules such that the conditions 1–3 below are satisfied. One says that a polynomial $P(t)$ in $\mathscr{B}[t]$ has *slope* $\leqslant h$ if every edge of its Newton polygon has slope $\leqslant h$. Let $\mathscr{B}[t]^{\leqslant h}$ be the set of polynomials in $\mathscr{B}[t]$ of slope $\leqslant h$ and whose leading coefficient is a multiplicative unit. For every $P(t) \in \mathscr{B}[t]$ write $P^*(t) = t^{\deg(P)} \cdot P(1/t)$.

1. $N^{\leqslant h}$ is finitely generated over $\mathscr{B}$.
2. There exists $P(t) \in \mathscr{B}[t]^{\leqslant h}$ such that $P^*(\boldsymbol{u})$ kills $N^{\leqslant h}$.
3. For every $P(t) \in \mathscr{B}[t]^{\leqslant h}$ the endomorphism $P^*(\boldsymbol{u})$ of $N^{>h}$ is an isomorphism.

Let m and U be as in Section 4.1, let $k = r + 2 \in U(L)$, and let $h \in \mathbf{Q}_{\geqslant 0}$. Set

$$\mathcal{T}_r = \big\{(L, A_{r,m}, U_p), (L, A'_{r,m}, U'_p), (L, D_{r,m}, U_p), (L, D'_{r,m}, U'_p)\big\}$$

and

$$\mathcal{T}_U = \big\{(\mathscr{O}_U, A_{U,m}, U_p), (\mathscr{O}_U, A'_{U,m}, U'_p), (\mathscr{O}_U, D_{U,m}, U_p), (\mathscr{O}_U, D'_{U,m}, U'_p)\big\},$$

where $\mathscr{O}_U$ is a shorthand for $\Lambda_U[1/p]$. Recall that Λ_U is isomorphic to the power series ring $\mathscr{O}[\![T]\!]$, equipped with the topology defined by the maximal ideal $\mathfrak{m}_U \cong (\pi, T)$, hence $\mathscr{O}_U$ is isomorphic to the L-module $L[\![T]\!]^o$ of power series in $L[\![T]\!]$ with bounded Gauß norm. If s is a real number satisfying $0 < s < 1$, define $|\cdot|_s : L[\![T]\!]^o \longrightarrow \mathbf{R}_{\geqslant 0}$ by $|\sum_{n\geqslant 0} a_n \cdot T^n|_s = \sup_{n\geqslant 0} s^n \cdot |a_n|_p$. Then $|\cdot|_s$ is an L-Banach algebra norm on $L[\![T]\!]^o$, which is independent of s and induces the (π, T)-adic topology on $\mathscr{O}[\![T]\!]$. This corresponds to an L-Banach algebra norm on $\mathscr{O}_U$, which restricts to the $\mathfrak{m}_U$-adic topology on the $\mathscr{O}$-submodule Λ_U. The discussion on slope $\leqslant h$ decompositions then applies to each triple $(\mathscr{B}, M, \boldsymbol{u})$ in $\mathcal{T}_r \cup \mathcal{T}_U$. The following proposition is a consequence of the work of Coleman and Ash–Stevens [**Col97, AS08**] (see also [**AIS15**]).

Proposition 4.2. — *Let $(\mathscr{B}, M, \boldsymbol{u})$ be a triple in $\mathcal{T}_r \cup \mathcal{T}_U$. If $r \in U \cap \mathbf{Z}_{\geqslant 0}$, one also allows $(\mathscr{B}, M, \boldsymbol{u})$ to denote either $(L, S_r(L), U^{\cdot}_p)$ or $(L, L_r(L), U^{\cdot}_p)$, with $U^{\cdot}_p = U_p, U'_p$.*

1. *Up to shrinking U if necessary, the $\mathscr{B}$-module $H^1(\Gamma, M)$ admits a slope $\leqslant h$ decomposition with respect to $\boldsymbol{u}$. Moreover, for $\cdot = \emptyset, \prime$, the specialisation maps ρ_k defined in Equation (70) induce Hecke equivariant isomorphisms*

$$\rho_k : H^1(\Gamma, A^{\cdot}_{U,m})^{\leqslant h} \otimes_{\Lambda_U} \Lambda_U/\pi_k \cong H^1(\Gamma, A^{\cdot}_{r,m})^{\leqslant h}$$
$$\text{and} \quad \rho_k : H^1(\Gamma, D^{\cdot}_{U,m})^{\leqslant h} \otimes_{\Lambda_U} \Lambda_U/\pi_k \cong H^1(\Gamma, D^{\cdot}_{r,m})^{\leqslant h}.$$

2. *Assume that $r = (n, a) \in \mathbf{Z}_{\geqslant 0} \times \mathbf{Z}/(p-1)\mathbf{Z}$ with $n \equiv a \pmod{p-1}$ and $h < n+1$. Then (for $\cdot = \emptyset, \prime$) the natural maps $S_r(L) \longrightarrow A^{\cdot}_{r,m}$ and $D^{\cdot}_{r,m} \longrightarrow L_r(L)$ induce Hecke equivariant isomorphisms*

$$H^1(\Gamma, S_r(L))^{\leqslant^{\cdot} h} \cong H^1(\Gamma, A^{\cdot}_{r,m})^{\leqslant h} \quad \text{and} \quad H^1(\Gamma, D^{\cdot}_{r,m})^{\leqslant h} \cong H^1(\Gamma, L_r(L))^{\leqslant^{\cdot} h},$$

where the superscript $\leqslant^{\cdot} h$ in $H^1(\Gamma, -)^{\leqslant^{\cdot} h}$ refers to the slope decomposition with respect to the endomorphism $U^{\cdot}_p$.

Let r be a nonnegative integer and let $h \in \mathbf{Q}_{\geqslant 0}$ such that $h < r+1$. As the étale cohomology groups $H^1_{\text{ét}}(Y_{\bar{\mathbf{Q}}}, \mathscr{S}_r)_L$ and $H^1_{\text{ét}}(Y_{\bar{\mathbf{Q}}}, \mathscr{L}_r)_L$ are finite-dimensional over L, they admit slope $\leqslant^{\cdot} h$ decompositions with respect to $U^{\cdot}_p$. Part 2 of Proposition 4.2 then implies that the comparison maps defined in Equation (72) induce natural isomorphisms of L-modules (cf. the last lines of the previous section)

$$H^1_{\text{ét}}(Y_{\bar{\mathbf{Q}}}, \mathscr{S}_r)_L^{\leqslant' h} \cong H^1(\Gamma, A_{r,m})^{\leqslant h} \quad \text{and} \quad H^1(\Gamma, D_{r,m})^{\leqslant h} \cong H^1_{\text{ét}}(Y_{\bar{\mathbf{Q}}}, \mathscr{L}_r)_L^{\leqslant h}. \tag{73}$$

One obtains similar isomorphisms after replacing $A_{r,m}$ and $D_{r,m}$ with $A'_{r,m}$ and $D'_{r,m}$ respectively.

4.2. Étale sheaves. — Let $\mathscr{T} = \mathscr{T}_{1(p),N}$ be the relative Tate module $R^1u_*\mathbf{Z}_p(1)$ of E over Y (cf. Equation (10)). Fix a geometric point $\eta : \mathrm{Spec}(\bar{\mathbf{Q}}) \longrightarrow Y$ and denote by $\mathcal{G} = \mathcal{G}_{N,p}$ the fundamental group $\pi_1^{\text{ét}}(Y, \eta)$. Fix in addition an isomorphism $\xi : \mathscr{T}_\eta \cong \mathbf{Z}_p \oplus \mathbf{Z}_p$ of $\mathbf{Z}_p$-modules such that, for every $x, y \in \mathscr{T}_\eta$, one has

$$\langle x, y \rangle_{E_{p^\infty}} = \xi(x) \wedge \xi(y) \quad \text{and} \quad \bar{\xi}(C_{p,\eta}) = \mathbf{F}_p \cdot (1, 0), \tag{74}$$

where $\langle \cdot, \cdot \rangle_{E_{p^\infty}}$ is the Weil pairing, $\bigwedge^2 \mathbf{Z}_p^2 = \mathbf{Z}_p$ via $(1,0) \wedge (0,1) = 1$, and $\bar{\xi} : E_{p,\eta} \cong \mathbf{F}_p \oplus \mathbf{F}_p$ is the reduction of ξ modulo p. The action of $\mathcal{G}$ on $\mathscr{T}_\eta$ and the isomorphism ξ give a continuous morphism $\varrho : \mathcal{G} \to \mathrm{GL}_2(\mathbf{Z}_p)$. Since the subgroup $C_{p,\eta}$ of $E_{p,\eta}$ is preserved by the action of $\mathcal{G}$, the second condition in Equation (74) implies that ϱ factors through a continuous morphism $\varrho : \mathcal{G} \longrightarrow \Gamma_0(p\mathbf{Z}_p)$. Let $\mathbf{S}_f(Y_{\text{ét}})$ be the category of locally constant constructible sheaves on $Y_{\text{ét}}$ with finite stalk of p-power order at η, and for every topological group G denote by $\mathbf{M}_f(G)$ the category of finite sets of p-power order, equipped with a continuous action of G. Taking the stalk at η defines an equivalence of categories $\cdot_\eta : \mathbf{S}_f(Y_{\text{ét}}) \cong \mathbf{M}_f(\mathcal{G})$, whose inverse $\cdot^{\text{ét}} : \mathbf{M}_f(\mathcal{G}) \cong \mathbf{S}_f(Y_{\text{ét}})$ restricts via ϱ to a functor $\cdot^{\text{ét}} : \mathbf{M}_f(\Gamma_0(p\mathbf{Z}_p)) \longrightarrow \mathbf{S}_f(Y_{\text{ét}})$. (Here both $\mathcal{G}$ and $\Gamma_0(p\mathbf{Z}_p)$ have the profinite topology.) Define $\mathbf{M}_{\text{cts}}(G)$ to be the category of G-modules M which are filtered unions $M = \bigcup_{i \in I} M_i$ with $M_i \in \mathbf{M}_f(G)$ for every $i \in I$, and $\mathbf{M}(G) \subset \mathbf{M}_{\text{cts}}(G)^{\mathbf{N}}$ to be the category of inverse systems of objects of $\mathbf{M}_{\text{cts}}(G)$. Define similarly $\mathbf{S}_{\text{cts}}(Y_{\text{ét}})$ and $\mathbf{S}(Y_{\text{ét}}) \subset \mathbf{S}_{\text{cts}}(Y_{\text{ét}})^{\mathbf{N}}$. If G denotes one of $\mathcal{G}$ and $\Gamma_0(p\mathbf{Z}_p)$, the functor $\cdot^{\text{ét}}$ extends to $\cdot^{\text{ét}} : \mathbf{M}(G) \longrightarrow \mathbf{S}(Y_{\text{ét}})$. Let $(M_i)_{i \in \mathbf{N}}$

be an inverse system of G-modules and let $M = \varprojlim M_i$. If the inverse system $(M_i)_i$ defining M is clear from the context, we say that M *belongs* to $\mathbf{M}(G)$ to mean that $(M_i)_i$ does. If this is the case we write $M^{\text{ét}}$ for $(M_i)_i^{\text{ét}}$.

More generally for every scheme S one defines the category $\mathbf{S}(S_{\text{ét}})$ as above. For every $\mathscr{F} = (\mathscr{F}_i)_{i\in\mathbf{N}} \in \mathbf{S}(S_{\text{ét}})$ set

$$H^j_{\text{ét}}(S,\mathscr{F}) = R^j\big(\varprojlim_i \Gamma(S,\cdot)\big)(\mathscr{F}_i)_i \quad \text{and} \quad \mathtt{H}^j_{\text{ét}}(S,\mathscr{F}) = \varprojlim_i H^j_{\text{ét}}(S,\mathscr{F}_i),$$

so that ($H^j_{\text{ét}}(S,\mathscr{F})$ is the continuous étale cohomology in the sense of [**Jan88**] and) there are short exact sequences

$$0 \longrightarrow R^1\varprojlim_i H^{j-1}_{\text{ét}}(S,\mathscr{F}_i) \longrightarrow H^j_{\text{ét}}(S,\mathscr{F}) \longrightarrow \mathtt{H}^j_{\text{ét}}(S,\mathscr{F}) \longrightarrow 0. \tag{75}$$

One similarly defines compactly supported cohomology groups $H^j_{\text{ét,c}}(S,\mathscr{F})$ and $\mathtt{H}^j_{\text{ét,c}}(S,\mathscr{F})$ (cf. [**Jan88**]).

Let (B,κ) be as in Section 4.1. The modules $\mathcal{A}^{\cdot}_{\kappa,m}$ and $\mathcal{D}^{\cdot}_{\kappa,m}$ belong to $\mathbf{M}(\Gamma_0(p\mathbf{Z}_p))$:

$$\mathcal{D}^{\cdot}_{\kappa,m} = \varprojlim_j \mathcal{D}^{\cdot}_{\kappa,m}/\mathrm{Fil}^j\mathcal{D}^{\cdot}_{\kappa,m},$$
$$\mathcal{A}^{\cdot}_{\kappa,m} = \varprojlim_j \mathcal{A}^{\cdot}_{\kappa,m}/\mathfrak{m}_B^j\mathcal{A}^{\cdot}_{\kappa,m}$$
$$\text{and} \quad \mathcal{A}^{\cdot}_{\kappa,m}/\mathfrak{m}_B^i\cdot\mathcal{A}^{\cdot}_{\kappa,m} = \bigcup_{j\geqslant i}\mathrm{Fil}_{i,j}\mathcal{A}^{\cdot}_{\kappa,m},$$

where $(\mathrm{Fil}^j\mathcal{D}^{\cdot}_{\kappa,m})_{j\geqslant 0}$ is a decreasing filtration by $B[\Sigma^{\cdot}_0(p)]$-submodules on $\mathcal{D}^{\cdot}_{\kappa,m}$, such that $\mathcal{D}^{\cdot}_{\kappa,m}/\mathrm{Fil}^j$ is finite for every j, and where $(\mathrm{Fil}_{i,j}\mathcal{A}^{\cdot}_{\kappa,m})_{j\geqslant i}$ is an increasing filtration on $\mathcal{A}^{\cdot}_{\kappa,m}/\mathfrak{m}_B^i\cdot\mathcal{A}^{\cdot}_{\kappa,m}$ by $B[\Sigma^{\cdot}_0(p)]$-submodules of finite cardinality. Precisely one defines

$$\mathrm{Fil}^j\mathcal{D}^{\cdot}_{\kappa,m} = \Big\{\mu\in\mathcal{D}^{\cdot}_{\kappa,m} \;\Big|\; \mu(f_{a,n})\in\mathfrak{m}_B^{j-n} \text{ for every } 0\leqslant a\leqslant p^m-1 \text{ and } n\leqslant j\Big\}$$

(cf. [**AIS15**, Definition 3.9 and Proposition 3.10]) and

$$\mathrm{Fil}_{i,j}\mathcal{A}^{\cdot}_{\kappa,m} = \bigoplus_{0\leqslant a\leqslant p^m-1,\, n\leqslant j} B\cdot\big(f_{a,n}+\mathfrak{m}_B^i\big) \subset \mathcal{A}^{\cdot}_{\kappa,m}/\mathfrak{m}_B^i\cdot\mathcal{A}^{\cdot}_{\kappa,m},$$

where $(f_{a,n})_{0\leqslant a\leqslant p^m-1, n\geqslant 0}$ is the orthonormal basis of $\mathcal{A}^{\cdot}_{\kappa,m}$ defined in Remark 4.1. Denote by

$$\boldsymbol{\mathcal{A}}^{\cdot}_{\kappa,m} = \mathcal{A}^{\cdot\,\text{ét}}_{\kappa,m} \quad \text{and} \quad \boldsymbol{\mathcal{D}}^{\cdot}_{\kappa,m} = \mathcal{D}^{\cdot\,\text{ét}}_{\kappa,m}$$

the images of $\mathcal{A}^{\cdot}_{\kappa,m}$ and $\mathcal{D}^{\cdot}_{\kappa,m}$ respectively under $\cdot^{\text{ét}} : \mathbf{M}(\Gamma_0(p\mathbf{Z}_p)) \to \mathbf{S}(Y_{\text{ét}})$. For every $j\geqslant 0$ set

$$\mathcal{A}^{\cdot}_{\kappa,m,j} = \mathcal{A}^{\cdot}_{\kappa,m}/\mathfrak{m}_B^j\cdot\mathcal{A}^{\cdot}_{\kappa,m},$$
$$\mathcal{D}^{\cdot}_{\kappa,m,j} = \mathcal{D}^{\cdot}_{\kappa,m}/\mathrm{Fil}^j,$$
$$\boldsymbol{\mathcal{A}}^{\cdot}_{\kappa,m,j} = \mathcal{A}^{\cdot\,\text{ét}}_{\kappa,m,j}$$
$$\text{and} \quad \boldsymbol{\mathcal{D}}^{\cdot}_{\kappa,m,j} = \mathcal{D}^{\cdot\,\text{ét}}_{\kappa,m,j},$$

so that $\boldsymbol{\mathcal{A}}^{\cdot}_{\kappa,m}$ is a shortened notation for the inverse system $(\boldsymbol{\mathcal{A}}^{\cdot}_{\kappa,m,j})_{j\in\mathbf{N}}$ and similarly $\boldsymbol{\mathcal{D}}^{\cdot}_{\kappa,m} = (\boldsymbol{\mathcal{D}}^{\cdot}_{\kappa,m,j})_{j\in\mathbf{N}}$. If S is a $\mathbf{Z}[1/Np]$-scheme one can define for every prime $\ell\nmid Np$

(resp., prime $\ell|Np$, unit $d \in (\mathbf{Z}/N\mathbf{Z})^*$) Hecke operators $T_\ell^{\cdot}$ (resp., $U_\ell^{\cdot}$, $\langle d \rangle$) acting on $H^i_{\text{ét}}(Y_S, \boldsymbol{\mathcal{A}}^{\cdot}_{\kappa,m,j})$ and $H^i_{\text{ét}}(Y_S, \boldsymbol{\mathcal{D}}^{\cdot}_{\kappa,m,j})$ (cf. Section 2.3 or [AIS15, Section 5]). We list below some of the basic properties satisfied by $\boldsymbol{\mathcal{A}}^{\cdot}_{\kappa,m}$ and $\boldsymbol{\mathcal{D}}^{\cdot}_{\kappa,m}$. Let S be a $\mathbf{Z}[1/Np]$-scheme and let $\chi : \mathbf{Z}_p^* \to B^*$ be a continuous character. Let $B/\mathfrak{m}^i_B(\chi) \in \mathbf{M}_f(\Gamma_0(p\mathbf{Z}_p))$ be a copy of $B/\mathfrak{m}^i_B$ equipped with the action of $\Gamma_0(p\mathbf{Z}_p)$ defined by $\gamma \cdot b = \chi(\det(\gamma)) \cdot b$, and let $B(\chi) = \lim_{\leftarrow i} B/\mathfrak{m}^i_B(\chi)$. If $\mathcal{C}^{\cdot}_{\kappa,m,\cdot}$ denotes either $\mathcal{A}^{\cdot}_{\kappa,m,\cdot}$ or $\mathcal{D}^{\cdot}_{\kappa,m,\cdot}$ define $\mathcal{C}^{\cdot}_{\kappa,m,\cdot}(\chi) = \mathcal{C}^{\cdot}_{\kappa,m,\cdot} \otimes_B B(\chi)$ and $\boldsymbol{\mathcal{C}}^{\cdot}_{\kappa,m}(\chi) = \mathcal{C}^{\cdot}_{\kappa,m}(\chi)^{\text{ét}} = \boldsymbol{\mathcal{C}}^{\cdot}_{\kappa,m} \otimes (B/\mathfrak{m}^i_B(\chi))^{\text{ét}}_{i \in \mathbf{N}}$. As usual, if $(B,\kappa) = (\Lambda_U, \kappa_U)$, one sets $\boldsymbol{\mathcal{C}}^{\cdot}_{U,m,\cdot} = \boldsymbol{\mathcal{C}}^{\cdot}_{\kappa_U,m,\cdot}$.

- For each $k = r+2 \in U(L)$, each $j \in \mathbf{N}$ and $\cdot = \emptyset, \prime$, the specialisation maps (70) induce morphisms
$$\rho_k : \boldsymbol{\mathcal{A}}^{\cdot}_{U,m,j}(\chi) \to \boldsymbol{\mathcal{A}}^{\cdot}_{r,m,j}(\chi) \quad \text{and} \quad \rho_k : \boldsymbol{\mathcal{D}}^{\cdot}_{U,m,j}(\chi) \to \boldsymbol{\mathcal{D}}^{\cdot}_{r,m,j}(\chi),$$
which in turn induce in cohomology *specialisation maps*
$$\begin{aligned} &\rho_k : H^1_{\text{ét}}(Y_S, \boldsymbol{\mathcal{A}}^{\cdot}_{U,m}(\chi)) \longrightarrow H^1_{\text{ét}}(Y_S, \boldsymbol{\mathcal{A}}^{\cdot}_{r,m}(\chi)) \\ \text{and} \quad &\rho_k : H^1_{\text{ét}}(Y_S, \boldsymbol{\mathcal{D}}^{\cdot}_{U,m}(\chi)) \longrightarrow H^1_{\text{ét}}(Y_S, \boldsymbol{\mathcal{D}}^{\cdot}_{r,m}(\chi)). \end{aligned} \tag{76}$$
- There are natural isomorphisms $H^1_{\text{ét}}(Y_{\bar{\mathbf{Q}}}, \boldsymbol{\mathcal{D}}^{\cdot}_{\kappa,m,j}) \cong H^1(\Gamma, \mathcal{D}^{\cdot}_{\kappa,m,j})$, which induce isomorphisms (cf. Theorem 3.15 of [AIS15])
$$\begin{aligned} &H^1_{\text{ét}}(Y_{\bar{\mathbf{Q}}}, \boldsymbol{\mathcal{D}}^{\cdot}_{\kappa,m}) \cong \mathtt{H}^1_{\text{ét}}(Y_{\bar{\mathbf{Q}}}, \boldsymbol{\mathcal{D}}^{\cdot}_{\kappa,m}) \cong H^1(\Gamma, \mathcal{D}^{\cdot}_{\kappa,m}) \\ \text{and} \quad &H^1_{\text{ét,c}}(Y_{\bar{\mathbf{Q}}}, \boldsymbol{\mathcal{D}}^{\cdot}_{\kappa,m}) \cong \mathtt{H}^1_{\text{ét,c}}(Y_{\bar{\mathbf{Q}}}, \boldsymbol{\mathcal{D}}^{\cdot}_{\kappa,m}) \cong H^1_{\text{c}}(\Gamma, \mathcal{D}^{\cdot}_{\kappa,m}) \end{aligned} \tag{77}$$
of B-modules compatible with the action of the Hecke operators and with the specialisation maps ρ_r. Here $H^j_c(\Gamma, \cdot) = H^{j-1}(\Gamma, I(\cdot))$ is defined to be the $(j-1)$-th cohomology group of Γ with values in the Γ-module
$$I(\cdot) = \mathrm{Hom}_{\mathbf{Z}}(\mathrm{Div}^0(\mathbf{P}^1(\mathbf{Q})), \cdot)$$
(cf. Proposition 4.2 of [AS86b]).
- There are natural maps $H^1_{\text{ét}}(Y_{\bar{\mathbf{Q}}}, \boldsymbol{\mathcal{A}}^{\cdot}_{\kappa,m,j}) \longrightarrow H^1(\Gamma, \mathcal{A}^{\cdot}_{\kappa,m,j})$, inducing an isomorphism of B-modules (cf. Lemma 4.3 below and the discussion preceding it)
$$\mathtt{H}^1_{\text{ét}}(Y_{\bar{\mathbf{Q}}}, \boldsymbol{\mathcal{A}}^{\cdot}_{\kappa,m}) \cong H^1(\Gamma, \mathcal{A}^{\cdot}_{\kappa,m}) \tag{78}$$
compatible with the action of the Hecke operators and with the specialisation maps. In light of the exact sequence (75), the isomorphism (78) yields a Hecke equivariant short exact sequence of B-modules
$$0 \longrightarrow R^1\lim_{\leftarrow j} H^0(Y_{\bar{\mathbf{Q}}}, \boldsymbol{\mathcal{A}}^{\cdot}_{\kappa,m,j}) \longrightarrow H^1_{\text{ét}}(Y_{\bar{\mathbf{Q}}}, \boldsymbol{\mathcal{A}}^{\cdot}_{\kappa,m}) \longrightarrow H^1(\Gamma, \mathcal{A}^{\cdot}_{\kappa,m}) \longrightarrow 0. \tag{79}$$
- The B-modules $\mathtt{H}^1_{\text{ét}}(Y_{\bar{\mathbf{Q}}}, \boldsymbol{\mathcal{D}}^{\cdot}_{\kappa,m})$ and $\mathtt{H}^1_{\text{ét}}(Y_{\bar{\mathbf{Q}}}, \boldsymbol{\mathcal{A}}^{\cdot}_{\kappa,m})$ are equipped with natural continuous actions of $G_{\mathbf{Q}}$ which commute with the Hecke operators and the specialisation maps. Moreover as $G_{\mathbf{Q}}$-modules
$$\begin{aligned} &\mathtt{H}^1_{\text{ét}}(Y_{\bar{\mathbf{Q}}}, \boldsymbol{\mathcal{D}}^{\cdot}_{\kappa,m}(\chi)) = \mathtt{H}^1_{\text{ét}}(Y_{\bar{\mathbf{Q}}}, \boldsymbol{\mathcal{D}}^{\cdot}_{\kappa,m})(\chi_{\mathbf{Q}}) \\ \text{and} \quad &\mathtt{H}^1_{\text{ét}}(Y_{\bar{\mathbf{Q}}}, \boldsymbol{\mathcal{A}}^{\cdot}_{\kappa,m}(\chi)) = \mathtt{H}^1_{\text{ét}}(Y_{\bar{\mathbf{Q}}}, \boldsymbol{\mathcal{A}}^{\cdot}_{\kappa,m})(\chi_{\mathbf{Q}}), \end{aligned} \tag{80}$$

where $\chi_{\mathbf{Q}} = \chi \circ \chi_{\mathrm{cyc}}^{-1} : G_{\mathbf{Q}} \to B^*$ and $\chi_{\mathrm{cyc}} : G_{\mathbf{Q}} \to \mathbf{Z}_p^*$ is the p-adic cyclotomic character. A similar statement holds for the compactly supported cohomology $\mathtt{H}^1_{\text{ét},\mathrm{c}}(Y_{\bar{\mathbf{Q}}}, \boldsymbol{\mathcal{D}}^{\cdot}_{\kappa,m})$.

- We equip $H^1(\Gamma, \mathcal{D}^{\cdot}_{\kappa,m}), H^1_{\mathrm{c}}(\Gamma, \mathcal{D}^{\cdot}_{\kappa,m})$ and $H^1(\Gamma, \mathcal{A}^{\cdot}_{\kappa,m})$ with the structures of continuous $G_{\mathbf{Q}}$-modules via the isomorphisms (77) and (78) respectively. If $h \in \mathbf{Q}_{\geqslant 0}$ (and U is sufficiently small) the slope $\leqslant h$ submodules $H^1(\Gamma, D^{\cdot}_{\kappa,m})^{\leqslant h}$, $H^1_{\mathrm{c}}(\Gamma, D^{\cdot}_{\kappa,m})^{\leqslant h}$ and $H^1(\Gamma, A^{\cdot}_{\kappa,m})^{\leqslant h}$ of $H^1(\Gamma, \mathcal{D}^{\cdot}_{\kappa,m})_{\mathbf{Q}_p}$, $H^1_{\mathrm{c}}(\Gamma, \mathcal{D}^{\cdot}_{\kappa,m})_{\mathbf{Q}_p}$ and $H^1(\Gamma, \mathcal{A}^{\cdot}_{\kappa,m})_{\mathbf{Q}_p}$ respectively (cf. Proposition 4.2) are preserved by the action of $G_{\mathbf{Q}}$.
- Set $\boldsymbol{\Lambda}_{U,j} = (\Lambda_U/\mathfrak{m}^j)^{\text{ét}}$ and $\boldsymbol{\Lambda}_U = (\boldsymbol{\Lambda}_{U,j})_{j\in\mathbf{N}} \in \mathbf{S}(Y_{\text{ét}})$. There are canonical isomorphisms of Λ_U-modules

(81) $$\mathrm{trace}_U : H^2_{\mathrm{c}}(\Gamma, \Lambda_U) \cong H^2_{\text{ét},\mathrm{c}}(Y_{\bar{\mathbf{Q}}}, \boldsymbol{\Lambda}_U) \cong \Lambda_U.$$

  The evaluation morphism $\mathcal{A}^{\cdot}_{U,m} \otimes_{\Lambda_U} \mathcal{D}^{\cdot}_{U,m} \longrightarrow \Lambda_U$ and the trace trace_U induce a cup-product

$$H^1(\Gamma, \mathcal{A}^{\cdot}_{U,m}) \otimes_{\Lambda_U} H^1_{\mathrm{c}}(\Gamma, \mathcal{D}^{\cdot}_{U,m}) \longrightarrow H^2_{\mathrm{c}}(\Gamma, \Lambda_U) \cong \Lambda_U,$$

  under which the Hecke operator $U^{\cdot}_p$ acting on $H^1(\Gamma, \mathcal{A}^{\cdot}_{U,m})$ is adjoint to $U^{\cdot}_p$ acting on $H^1_{\mathrm{c}}(\Gamma, \mathcal{D}^{\cdot}_{U,m})$. This in turn induces for $h \in \mathbf{Q}_{\geqslant 0}$ (and U sufficiently small) morphisms of $\Lambda_U[1/p]$-modules

$$\xi^{\cdot}_{U,m} : H^1(\Gamma, A^{\cdot}_{U,m})^{\leqslant h} \longrightarrow \mathrm{Hom}_{\Lambda_U[1/p]}\big(H^1_{\mathrm{c}}(\Gamma, D^{\cdot}_{U,m})^{\leqslant h}, \Lambda_U[1/p]\big).$$

- Define $\det : \mathsf{T} \times \mathsf{T}' \longrightarrow \mathbf{Z}_p^*$ by $\det((x_1, x_2), (y_1, y_2)) = x_1 y_2 - x_2 y_1$, and denote by $\det_U : \mathsf{T} \times \mathsf{T}' \longrightarrow \Lambda_U^*$ the composition of det with $\kappa_U : \mathbf{Z}_p^* \longrightarrow \Lambda_U^*$. Evaluation at $\det_U$ defines a Γ-equivarint bilinear form $\mathcal{D}_{U,m} \otimes_{\Lambda_U} \mathcal{D}'_{U,m} \longrightarrow \Lambda_U$. Together with trace_U (cf. Equation (81)) this induces a cup-product pairing

(82) $$\det{}^*_U : H^1(\Gamma, \mathcal{D}_{U,m}) \otimes_{\Lambda_U} H^1_{\mathrm{c}}(\Gamma, \mathcal{D}'_{U,m}) \longrightarrow H^2_{\mathrm{c}}(\Gamma, \Lambda_U) \cong \Lambda_U$$

  under which the Hecke operators U_p and U'_p are adjoint to each other. For every $h \in \mathbf{Q}_{\geqslant 0}$ the (inverse of the) adjoint of $\det^*_U$ induces an isomorphism of $\Lambda_U[1/p]$-modules

$$\zeta'_{U,m} : \mathrm{Hom}_{\Lambda_U[1/p]}\big(H^1_{\mathrm{c}}(\Gamma, D'_{U,m})^{\leqslant h}, \Lambda_U[1/p]\big) \cong H^1(\Gamma, D_{U,m})^{\leqslant h}.$$

  Similarly one defines an isomorphism

$$\zeta_{U,m} : \mathrm{Hom}_{\Lambda_U[1/p]}\big(H^1_{\mathrm{c}}(\Gamma, D_{U,m})^{\leqslant h}, \Lambda_U[1/p]\big) \cong H^1(\Gamma, D'_{U,m})^{\leqslant h}.$$

- Let $h \in \mathbf{Q}_{\geqslant 0}$. If U is sufficiently small the composition of $\zeta_{U,m}$ with $\xi_{U,m}$ gives a morphism of $G_{\mathbf{Q}}$-modules

(83) $$\mathtt{s}_{U,h} : H^1(\Gamma, A_{U,m})^{\leqslant h}(\boldsymbol{\kappa}_U) \longrightarrow H^1(\Gamma, D'_{U,m})^{\leqslant h},$$

  where $\boldsymbol{\kappa}_U : G_{\mathbf{Q}} \longrightarrow \Lambda_U^*$ is defined by $\boldsymbol{\kappa}_U(g) = \kappa_U(\chi_{\mathrm{cyc}}(g))$ for every $g \in G_{\mathbf{Q}}$. For every integer $k = r + 2$ in $U \cap \mathbf{Z}$ such that $h < k - 1$, the following diagram of $L[G_{\mathbf{Q}}]$-modules commutes.

$$\begin{array}{ccc} H^1(\Gamma, A_{U,m})^{\leqslant h}(\boldsymbol{\kappa}_U) & \xrightarrow{\mathtt{s}_{U,h}} & H^1(\Gamma, D'_{U,m})^{\leqslant h} \\ \downarrow \rho_k & & \downarrow \rho_k \\ H^1_{\text{ét}}(Y_{\bar{\mathbf{Q}}}, \mathscr{S}_r)_L^{\leqslant' h}(r) & \xrightarrow{\mathtt{s}_r} & H^1_{\text{ét}}(Y_{\bar{\mathbf{Q}}}, \mathscr{L}_r)_L^{\leqslant' h} \end{array} \tag{84}$$

By a slight abuse of notation, here one writes again ρ_k for the composition of the specialisation map $\rho_k : H^1(\Gamma, A_{U,m})^{\leqslant h} \to H^1(\Gamma, A_{r,m})^{\leqslant h}$ (resp., $\rho_k : H^1(\Gamma, D'_{U,m})^{\leqslant h} \to H^1(\Gamma, D'_{r,m})^{\leqslant h}$) with the comparison isomorphism $H^1(\Gamma, A_{r,m})^{\leqslant h} \cong H^1_{\text{ét}}(Y_{\bar{\mathbf{Q}}}, \mathscr{S}_r)^{\leqslant' h}$ (resp., $H^1(\Gamma, D'_{r,m})^{\leqslant h} \cong H^1(\Gamma, \mathscr{L}_r)_L^{\leqslant' h}$) defined in Equation (73). Similarly the composition of $\zeta'_{U,m}$ with $\xi'_{U,m}$ gives a morphism of $G_{\mathbf{Q}}$-modules

$$\mathtt{s}'_{U,h} : H^1(\Gamma, A'_{U,m})^{\leqslant h}(\boldsymbol{\kappa}_U) \longrightarrow H^1(\Gamma, D_{U,m})^{\leqslant h}$$

and the diagram of $G_{\mathbf{Q}}$-modules obtained by replacing $A_{U,m}, D'_{U,m}$ and $\mathtt{s}_{U,h}$ with $A'_{U,m}, D_{U,m}$ and $\mathtt{s}'_{U,h}$ respectively in Equation (84) commutes.

- The Atkin–Lehner operators w_p (resp., w_{Np}) defined in Equations (66) and (67) are $G_{\mathbf{Q}}$-equivariant (resp., $G_{\mathbf{Q}(\mu_N)}$-equivariant).

Due to the lack of a reference, we explain how to construct the crucial isomorphism (78). Let $\cdot$ denote either the empty symbol or $\prime$, and let $\mathrm{Fil}_{i,j}\boldsymbol{\mathcal{A}}^{\cdot}_{\kappa,m} = (\mathrm{Fil}_{i,j}\mathcal{A}^{\cdot}_{\kappa,m})^{\text{ét}}$ be the étale sheaf on Y associated with the finite $B/\mathfrak{m}^i B[\Gamma]$-module $\mathrm{Fil}_{i,j}\mathcal{A}^{\cdot}_{\kappa,m}$. The comparison isomorphisms between étale and Betti cohomology yields isomorphisms

$$\mathrm{comp}_{i,j} : H^1_{\text{ét}}(Y_{\bar{\mathbf{Q}}}, \mathrm{Fil}_{i,j}\boldsymbol{\mathcal{A}}^{\cdot}_{\kappa,m}) \cong H^1(\Gamma, \mathrm{Fil}_{i,j}\mathcal{A}^{\cdot}_{\kappa,m}).$$

The étale cohomology of the affine scheme $Y_{\bar{\mathbf{Q}}}$ commutes with filtered direct limits. Moreover, since the group Γ is finitely generated, the cohomology functor $H^1(\Gamma, \cdot)$ commutes with filtered direct limits (cf. Exercises 1 and 4 on page 196 of [**Bro94**]). Taking the direct limit for $j \to \infty$ of the isomorphisms $\mathrm{comp}_{i,j}$ then gives isomorphisms of $B/\mathfrak{m}^i B$-modules

$$\mathrm{comp}_i : H^1(\Gamma, \mathcal{A}^{\cdot}_{\kappa,m,i}) \cong H^1_{\text{ét}}(Y_{\bar{\mathbf{Q}}}, \boldsymbol{\mathcal{A}}^{\cdot}_{\kappa,m,i}),$$

which in turn entail an isomorphism of B-modules

$$\mathrm{comp} : \varprojlim_i H^1(\Gamma, \mathcal{A}^{\cdot}_{\kappa,m,i}) \cong \mathtt{H}^1_{\text{ét}}(Y_{\bar{\mathbf{Q}}}, \boldsymbol{\mathcal{A}}^{\cdot}_{\kappa,m}).$$

The sought for isomorphism (78) is defined as the composition of the comparison isomorphism comp and the natural map $H^1(\Gamma, \mathcal{A}^{\cdot}_{\kappa,m}) \longrightarrow \varprojlim_i H^1(\Gamma, \mathcal{A}^{\cdot}_{\kappa,m,i})$, which is an isomorphism by Lemma 4.3 below. The Hecke equivariance of the isomorphism (78) is proved precisely as in Sections 3.2 and 3.3 of [**AIS15**].

Lemma 4.3. — *The natural maps*

$$H^1(\Gamma, \mathcal{A}^{\cdot}_{\kappa,m}) \longrightarrow \varprojlim_i H^1(\Gamma, \mathcal{A}^{\cdot}_{\kappa,m,i})$$

are isomorphisms of B-modules.

Proof. — We adapt the proof of [**AIS15**, Lemma 3.13] to our setting. To ease notation, set $\mathcal{A}_i = \mathcal{A}^{\cdot}_{\kappa,m,i}$ and $\mathcal{A} = \mathcal{A}^{\cdot}_{\kappa,m}$. For each Γ-module M, let

$$C^\bullet(\Gamma, M) : 0 \longrightarrow M \xrightarrow{d^0} C^1(\Gamma, M) \xrightarrow{d^1} C^2(\Gamma, M) \longrightarrow \cdots$$

be the usual complex of inhomogeneous cochains computing the cohomology groups $H^j(\Gamma, M) = Z^j(\Gamma, M)/\mathrm{im}(d^{j-1})$, where $C^j(\Gamma, M)$ is the group of maps from Γ^j to M and $Z^j(\Gamma, M) = \ker(d^j)$. Denote by $d^\bullet$ (resp., $d_i^\bullet$) the differentials in $C^\bullet(\Gamma, \mathcal{A})$ (resp., $C^\bullet(\Gamma, \mathcal{A}_i)$), so that one has the following commutative diagram with exact rows. (Recall that by definition $\mathcal{A}_i$ is a shorthand for $\mathcal{A}/\mathfrak{m}_B^i \cdot \mathcal{A}$.)

$$\begin{array}{ccccccc}
\mathcal{A} & \xrightarrow{d^0} & Z^1(\Gamma, \mathcal{A}) & \longrightarrow & H^1(\Gamma, \mathcal{A}) & \longrightarrow & 0 \\
\Big\| & & \Big\downarrow \zeta & & \Big\downarrow \vartheta & & \\
\varprojlim_i \mathcal{A}_i & \xrightarrow{(d_i^0)} & \varprojlim_i Z^1(\Gamma, \mathcal{A}_i) & \xrightarrow{\varepsilon} & \varprojlim_i H^1(\Gamma, \mathcal{A}_i) & &
\end{array}$$

To prove that ϑ is an isomorphism, it is then sufficient to show that ε is surjective and that ζ is an isomorphism. The cokernel of ε is contained in $R^1 \varprojlim_i (\mathcal{A}_i/H^0(\Gamma, \mathcal{A}_i))$, which vanishes since the maps $\mathcal{A}_{i+1}/H^0(\Gamma, \mathcal{A}_{i+1}) \longrightarrow \mathcal{A}_i/H^0(\Gamma, \mathcal{A}_i)$ are surjective. Moreover, as $\mathcal{A} = \varprojlim_i \mathcal{A}_i$, the natural map $C^\bullet(\Gamma, \mathcal{A}) \longrightarrow \varprojlim_i C^\bullet(\Gamma, \mathcal{A}_i)$ is an isomorphism, hence so is ζ by the left exactness of the inverse limit. □

4.3. The ordinary case. — This section explains the relations between the ordinary (id est slope $\leqslant 0$) parts of the modules $H^1(\Gamma, D^{\cdot}_{U,m})$ and the big ordinary Galois representations considered in [**Hid86, Oht95, Oht00**]. This will be particularly relevant for the study of the eigencurve in a neighbourhood of a classical weight-one eigenform (where the Eichler–Shimura isomorphism of [**AIS15**] does not apply).

Since $H^1(\Gamma, \mathcal{D}^{\cdot}_{\kappa,m})$ is a profinite group (as $\mathcal{D}^{\cdot}_{\kappa,m}$ is), the limit $e^{\cdot}_{\mathrm{ord}} = \lim_{n\to\infty} U_p^{\cdot n!}$ defines an idempotent in the B-endomorphism ring of $H^1(\Gamma, \mathcal{D}^{\cdot}_{\kappa,m})$. (Here as usual (B, κ) denotes either (Λ_U, κ_U) or $(\mathscr{O}, r)$ with r in $\mathcal{W}(L)$, and $\cdot$ denotes either the empty symbol or $\prime$.) Set

$$H^1(\Gamma, \mathcal{D}^{\cdot}_{\kappa,m})^{\leqslant 0} = e^{\cdot}_{\mathrm{ord}} \cdot H^1(\Gamma, \mathcal{D}^{\cdot}_{\kappa,m}).$$

This is a finite Λ_B-module, which recasts $H^1(\Gamma, D^{\cdot}_{\kappa,m})^{\leqslant 0}$ after inverting p.

Following [**Hid86, Oht95**], define

$$\mathbf{T} = \varprojlim_r H^1_{\text{ét}}(Y_1(Np^r)_{\bar{\mathbf{Q}}}, \mathbf{Z}_p(1)),$$

where $r \in \mathbf{Z}_{\geqslant 1}$ and the transition maps are given by the traces pr_{1*} induced in cohomology by the degeneracy maps $\mathrm{pr}_1 : Y_1(Np^{r+1}) \longrightarrow Y_1(Np^r)$ introduced in Equation (8). As the maps pr_{1*} are Hecke-equivariant, the module $\mathbf{T}$ is equipped with the action of Hecke operators $T^{\cdot}_\ell$ (resp., $U^{\cdot}_\ell$), for each prime ℓ not dividing (resp., dividing) Np. Moreover, the action of $(\mathbf{Z}/p^r\mathbf{Z})^*$ on $H^1_{\text{ét}}(Y_1(Np^r)_{\bar{\mathbf{Q}}}, \mathbf{Z}_p(1))$ via diamond operators makes $\mathbf{T}$ a module over $\diamond = \mathbf{Z}_p[\![\mathbf{Z}_p^*]\!]$. Let

$$\mathbf{D}' = \mathrm{Hom}_{\mathbf{Z}_p}(\mathrm{Step}(\mathsf{T}'), \mathbf{Z}_p)$$

be the right $\Sigma_0'(p)$-module of measures on T', where $\mathrm{Step}(\mathsf{T}')$ is the set of $\mathbf{Z}_p$-valued step functions on T'. Section 4.1.1 equips $H^1(\Gamma, \mathbf{D}')$ with the action of Hecke operators U_p' and $T_\ell^{\cdot}$, for $\cdot = \emptyset, \prime$ and ℓ a rational prime different from p. A slight variant of Lemma 6.8 of [**GS93**] yields a Hecke-equivariant isomorphism of $\diamond$-modules

$$\mathbf{T} \cong H^1(\Gamma, \mathbf{D}'), \tag{85}$$

where the action of the Iwasawa algebra $\diamond$ on the right hand side arises from that of the group $\mathbf{Z}_p^* = \mathbf{Z}_p^* \cdot \left(\begin{smallmatrix} 1 & \\ & 1 \end{smallmatrix}\right) \hookrightarrow \Sigma_0'(p)$ on $\mathbf{D}'$.

Each measure μ in $\mathbf{D}'$ extends to a Λ_U-linear morphism $\mu_U : \mathscr{C}(\mathsf{T}', \Lambda_U) \longrightarrow \Lambda_U$ on the space $\mathscr{C}(\mathsf{T}', \Lambda_U)$ of Λ_U-valued continuous functions on T'. The map sending μ to the restriction of μ_U to $\mathcal{A}_{U,m}' \hookrightarrow \mathscr{C}(\mathsf{T}', \Lambda_U)$ defines a morphism of $\Sigma_0'(p)$-modules

$$\mathbf{D}' \longrightarrow \mathcal{D}_{U,m}',$$

which in turn induces a Hecke-equivariant morphism of Λ_U-modules

$$H^1(\Gamma, \mathbf{D}') \otimes_\diamond \Lambda_U \longrightarrow H^1(\Gamma, \mathcal{D}_{U,m}'), \tag{86}$$

where Λ_U has the structure of $\diamond$-algebra arising from $\kappa_U : \mathbf{Z}_p^* \longrightarrow \Lambda_U^*$.

After setting

$$\mathbf{T}_U^{\leqslant 0} = e_{\mathrm{ord}}' \cdot \mathbf{T} \otimes_\diamond \Lambda_U,$$

the composition of the maps (85) and (86) yields an isomorphism of Λ_U-modules

$$\mathtt{Sh}_{U,m} : \mathbf{T}_U^{\leqslant 0} \cong H^1(\Gamma, \mathcal{D}_{U,m}')^{\leqslant 0}(1), \tag{87}$$

which is Hecke-equivariant and $G_{\mathbf{Q}}$-equivariant. In order to prove this, let r be a positive integer in U. Since $H^2(\Gamma, \cdot)$ vanishes for each Γ-module $\cdot$ of finite cardinality (and $\mathcal{D}_{U,m}'$ is profinite), evaluation at $k = r + 2$ on Λ_U induces an isomorphism

$$H^1(\Gamma, \mathcal{D}_{U,m}')^{\leqslant 0} \otimes_{\Lambda_U} \Lambda_U/\pi_k \cong H^1(\Gamma, \mathcal{D}_{r,m}')^{\leqslant 0}. \tag{88}$$

Moreover, for each $j \geqslant 0$, the natural map $\mathcal{D}_{r,m}' \longrightarrow L_r(\mathscr{O})$ induces an isomorphism

$$H^j(\Gamma, \mathcal{D}_{r,m}')^{\leqslant 0} \cong H^j(\Gamma, L_r(\mathscr{O}))^{\leqslant' 0}, \tag{89}$$

which for $j = 1$ recasts the isomorphism displayed in Part 2 of Proposition 4.2 after inverting p. (Indeed a direct computation shows that $\left(\begin{smallmatrix} p & 0 \\ pNi & 1 \end{smallmatrix}\right) \in \Sigma_0'(p)$ maps the kernel $\mathcal{K}_{r,m}'$ of $\mathcal{D}_{r,m}' \longrightarrow L_r(\mathscr{O})$ into $p^{r+1} \cdot \mathcal{K}_{r,m}'$ for each $0 \leqslant i \leqslant p-1$, from which one deduces that the anti-ordinary projector e_{ord}' kills $H^j(\Gamma, \mathcal{K}_{r,m}')$ for each $j \geqslant 0$.) On the other hand, the inclusion $S_r(\mathbf{Z}_p) \hookrightarrow \mathscr{C}(\mathsf{T}', \mathbf{Z}_p)$ dualises to a specialisation map $\rho_k : \mathbf{D}' \longrightarrow L_r(\mathbf{Z}_p)$, and Hida's control theorem (cf. [**Hid86, Oht95**]) shows that the isomorphism (85) and ρ_k induce a Hecke-equivariant isomorphism

$$e_{\mathrm{ord}}' \cdot \mathbf{T} \otimes_\diamond \diamond/I_k \cong H^1(\Gamma, L_r(\mathbf{Z}_p))^{\leqslant' 0}, \tag{90}$$

where I_k is the ideal of $\diamond$ generated by $[1+p] - (1+p)^r$ and $[\mu] - \mu^r$, with μ a generator of $\mathbf{F}_p^*$ and $[\cdot] : \mathbf{Z}_p^* \longrightarrow \diamond^*$ the tautological map. It follows from Equations (88)–(90) that the base change of $\mathtt{Sh}_{U,m}$ along the projection $\Lambda_U \longrightarrow \Lambda_U/\pi_k$ is an isomorphism. Together with Nakayama's Lemma, this implies that $\mathtt{Sh}_{U,m}$ is surjective, and that $\ker(\mathtt{Sh}_{U,m}) \otimes_{\Lambda_U} \Lambda_U/\pi_k$ is a quotient of the π_k-torsion submodule of $H^1(\Gamma, \mathcal{D}_{U,m}')^{\leqslant 0}$. The latter is in turn a quotient of $H^0(\Gamma, \mathcal{D}_{r,m}')^{\leqslant 0}$, which vanishes by Equation (89).

Another application of Nakayama's Lemma then proves that $\mathtt{Sh}_{U,m}$ is injective, thus concluding the proof of the claim (87).

Set $\mathscr{O}_U = \Lambda_U[1/p]$ and denote by

$$\mathfrak{h}(U) = \mathfrak{h}(N, U) \hookrightarrow \mathrm{End}_{\Lambda_U}(H^1(\Gamma, \mathcal{D}'_{U,m})^{\leqslant 0})[1/p]$$

the Hecke algebra generated over $\mathscr{O}_U$ by the dual Hecke operators $(U'_q)_{q|Np}$, $(T'_\ell)_{\ell\nmid Np}$ and $(\langle d\rangle)_{d\in(\mathbf{Z}/N\mathbf{Z})^*}$ acting on $H^1(\Gamma, D'_{U,m})^{\leqslant 0}$. For each positive integer r and $\cdot = \emptyset, \prime$, let $h^{\cdot}(Np^r)$ be the ring generated by the Hecke operators $(U^{\cdot}_q)_{q|Np}$, $(T^{\cdot}_\ell)_{\ell\nmid Np}$ and $(\langle d\rangle)_{d\in(\mathbf{Z}/N\mathbf{Z})^*}$ acting on the space $M_2(Np^r)$ of complex modular forms of weight 2. Conjugation by the Atkin–Lehner isomorphism $w_{Np^r} \in \mathrm{Iso}_{\mathbf{C}}(M_2(Np^r))$ restricts to an isomorphism $h(Np^r) \cong h'(Np^r)$, sending U_q and T_ℓ to U'_q and T'_ℓ respectively. Set

$$h^{\cdot}_{Np^\infty} = e^{\cdot}_{\mathrm{ord}} \cdot \varprojlim_{r} (h^{\cdot}(Np^r) \otimes_{\mathbf{Z}} \mathbf{Z}_p) \quad \text{and} \quad h^{\cdot}_{Np^\infty}(U) = h^{\cdot}_{Np^\infty} \otimes_{\diamond} \mathscr{O}_U, \tag{91}$$

where the transition maps in the inverse limit defining h_{Np^∞} (resp., h'_{Np^∞}) are induced by the inclusions $M_2(Np^r) \subset M_2(Np^{r+1})$ (resp., the maps $M_2(Np^r) \hookrightarrow M_2(Np^{r+1})$ sending $f(z)$ to $f(pz)$). The Atkin–Lehner operators $(w_{Np^r})_{r\geqslant 1}$ induce an isomorphism of Λ_U-modules between $h_{Np^\infty}(U)$ and $h'_{Np^\infty}(N)$, and since $h(Np^r)$ acts faithfully on $H^1_{\text{ét}}(Y_1(Np^r)_{\bar{\mathbf{Q}}}, \mathbf{Z}_p(1))$ (cf. Equation (19)), the Shapiro isomorphism $\mathtt{Sh}_{U,m}$ defined in Equation (87) yields an isomorphisms of $\mathscr{O}_U$-modules

$$h_{Np^\infty}(U) \cong \mathfrak{h}(N, U). \tag{92}$$

sending the Hecke operators T_ℓ and U_q to the corresponding duals T'_ℓ and U'_q.

Denote by $\mathcal{C} = \mathcal{C}(N) = \mathrm{Spf}(h_{Np^\infty})_{\mathbf{Q}_p}$ Berthelot's rigid fibre of the formal spectrum of h_{Np^∞} (cf. Section 7 of [dJ95]). The structural maps $\diamond \longrightarrow h_{Np^\infty}$ yield a finite and flat morphism $\kappa : \mathcal{C} \longrightarrow \mathcal{W}$, and Equation (92) gives an isomorphism of $\mathscr{O}_U$-modules

$$\mathfrak{h}(U) \cong \mathscr{O}(\mathcal{C} \times_{\mathcal{W}} U) \tag{93}$$

mapping the dual Hecke operators T'_ℓ ($\ell \nmid Np$) and U'_q ($q|Np$) in the left hand side to the corresponding Hecke operators T_ℓ and U_q in the right hand side, where $\mathscr{O}(\cdot)$ denotes the ring of bounded analytic functions on $\cdot$.

Section 6 of [Pil13] gives an isomorphism between $\mathcal{C}$ and the ordinary locus $\mathscr{C}^{\mathrm{ord}} = \mathscr{C}^{\mathrm{ord}}(N)$ of the Buzzard–Coleman–Mazur eigencurve $\mathscr{C} = \mathscr{C}(N)$ of tame level N, mapping the Hecke operators in h_{Np^∞} to the corresponding Hecke operators in $\mathscr{O}(\mathscr{C}^{\mathrm{ord}})$. In light of Equation (93), this gives isomorphisms

$$\mathfrak{h}(U) \cong \mathscr{O}(\mathscr{C}^{\mathrm{ord}} \times_{\mathcal{W}} U) \tag{94}$$

mapping the dual Hecke operators in the left hand side to the corresponding Hecke operators in the right hand side.

Remark 4.4. — If U is a sufficiently small open disc in $\mathcal{W}$ centred at an integer $k_o \geqslant 2$, and h is a non-negative rational number satisfying $h < k_o - 2$, then the overconvergent Eichler–Shimura isomorphism [AIS15, Theorem 1.3] implies that the isomorphism (94) holds after replacing $\mathscr{C}^{\mathrm{ord}}$ with the slope $\leqslant h$ locus of $\mathscr{C}$, and $\mathfrak{h}(U)$ with the Hecke algebra acting on the slope $\leqslant h$ subspace of $H^1(\Gamma, D'_{U,m})$. On the other hand, their result does not apply when U is centred at $k_o = 1$ (and $h = 0$), a crucial scenario for the applications of the main results of this paper to the arithmetic of elliptic curves (cf. [**BSV20a**]).

5. Hida families

As explained in Section 6 of [**AIS15**] (see also Section 6 of [**GS93**]), the *big* Galois representations associated to p-adic Coleman–Hida families (generically) appear as direct factors of the cohomology groups $H^1(\Gamma, \mathcal{D}_{U,m}^{\cdot})$. This section recalls these results, paying particular attention to the case (not covered in loc. cit.) where the open disc U is centred at weight 1 in $\mathcal{W}(\mathbf{Q}_p)$. To simplify the exposition we limit the discussion to Hida families. This suffices for the applications we have in mind (and requires no mention of the theory of (φ, Γ)-modules and trianguline representations).

Let M be a positive integer coprime to p, let $U \subset \mathcal{W}$ be an L-rational open disc centred at a positive integer $k_o \in \mathbf{Z}_{\geqslant 1}$, and let χ be a Dirichlet character modulo M. Let $\mathscr{O}_U = \Lambda_U[1/p]$ be the ring of bounded analytic functions on U, and let

$$U^{\mathrm{cl}} = \big\{ k \in U \cap \mathbf{Z} \;\big|\; k \geqslant 2 \text{ and } k \equiv k_o \bmod 2 \cdot (p-1) \big\}$$

be the set of *classical points of* U. An *$\mathscr{O}_U$-adic cusp form* of tame level M and tame character χ is a formal q-expansion

$$\boldsymbol{f} = \sum_{n \geqslant 1} a_n(\boldsymbol{f}; \boldsymbol{k}) \cdot q^n \in \mathscr{O}_U[\![q]\!]$$

such that, for each classical weight $k \in U^{\mathrm{cl}}$, the *weight-k specialisation*

$$\boldsymbol{f}_k = \sum_{n \geqslant 1} a_n(\boldsymbol{f}; k) \cdot q^n \in S_k^{\mathrm{ord}}(Mp, \chi)_L$$

is the q-expansion of a *p-ordinary* cusp form in $S_k^{\mathrm{ord}}(Mp, \chi)_L$. Here

$$S_k^{\mathrm{ord}}(Mp, \chi)_L = e_{\mathrm{ord}} \cdot S_k(Mp, \chi)_L,$$

where $e_{\mathrm{ord}} = \lim_{n \to \infty} U_p^{n!}$ is Hida's ordinary projector acting on the L-module $S_k(Mp, \chi)_L$ of cusp forms of weight k, level $\Gamma_1(M) \cap \Gamma_0(p)$, character χ and Fourier coefficients in $\bar{\mathbf{Q}} \cap L$ (under the fixed embedding $\bar{\mathbf{Q}} \hookrightarrow \bar{\mathbf{Q}}_p$). Denote by $S_U^{\mathrm{ord}}(M, \chi)$ the $\mathscr{O}_U$-module of $\mathscr{O}_U$-adic cusp forms of tame level M and character χ. It is equipped with the action of Hecke operators T_ℓ, for primes $\ell \nmid Mp$, and U_ℓ, for primes $\ell | Mp$, which are compatible with the usual Hecke operators on $S_k^{\mathrm{ord}}(Mp, \chi)$ for each $k \in U^{\mathrm{cl}}$. A cusp form $\boldsymbol{f}$ in $S_U^{\mathrm{ord}}(M, \chi)$ is *normalised* if $a_1(\boldsymbol{f}; \boldsymbol{k})$ is the constant function with value one on U. A *(L-rational) Hida family* of tame level M, tame character χ and centre $k_o \in \mathbf{Z}_{\geqslant 1}$ is an $\mathscr{O}_U$-adic cusp form $\boldsymbol{f} \in S_U^{\mathrm{ord}}(M, \chi)$, for some U as above, which is an eigenvector for the Hecke operators U_p and T_ℓ, for each prime $\ell \nmid Mp$ (equivalently such that, for each $k \in U^{\mathrm{cl}}$, the weight-k specialisation $\boldsymbol{f}_k$ is an eigenvector for the Hecke operators U_p and T_ℓ, for all primes $\ell \nmid Mp$.) A normalised Hida family $\boldsymbol{f} \in S_U^{\mathrm{ord}}(M, \chi)$ is *new* (or *primitive*) of tame level M if the conductor of the eigenform $\boldsymbol{f}_k$ is equal to M for all $k > 2$ in U^{cl}. To each Hida family $\boldsymbol{f} \in S_U^{\mathrm{ord}}(M, \chi)$ is associated a unique pair $(M_{\boldsymbol{f}}, \boldsymbol{f}^\sharp)$, where $M_{\boldsymbol{f}}$ is a positive divisor of M and $\boldsymbol{f}^\sharp = \sum_{n \geqslant 1} a_n(\boldsymbol{k}) \cdot q^n$ in $S_U^{\mathrm{ord}}(M_{\boldsymbol{f}}, \chi)$ is a new Hida family of tame level $M_{\boldsymbol{f}}$

such that $U_p(\boldsymbol{f}) = a_p(\boldsymbol{k}) \cdot \boldsymbol{f}$ and $T_\ell(\boldsymbol{f}) = a_\ell(\boldsymbol{k}) \cdot \boldsymbol{f}$ for all primes $\ell \nmid M$. We call $M_{\boldsymbol{f}}$ the *conductor of* $\boldsymbol{f}$ and $\boldsymbol{f}^\sharp$ the *primitive Hida family associated with* $\boldsymbol{f}$. Moreover, we denote by

$$S_U^{\mathrm{ord}}(M, \chi_{\boldsymbol{f}})[\boldsymbol{f}^\sharp] \hookrightarrow S_U^{\mathrm{ord}}(M, \chi_{\boldsymbol{f}})$$

the $\mathscr{O}_U$-module of Hida families in $S_U^{\mathrm{ord}}(M, \chi_{\boldsymbol{f}})$ having $\boldsymbol{f}^\sharp$ as associated primitive Hida family. A *level-N test vector* for $\boldsymbol{f}^\sharp$ is an element of $S_U^{\mathrm{ord}}(M, \chi_{\boldsymbol{f}})[\boldsymbol{f}^\sharp]$ of the form

$$\boldsymbol{f} = \sum_{0<d|M/M_{\boldsymbol{f}}} r_d \cdot \boldsymbol{f}^\sharp(q^d), \tag{95}$$

for analytic functions $(r_d)_d$ in $\mathscr{O}_U$ without common zeros in U.

Fix in the rest of this section a positive divisor $N_{\boldsymbol{f}}$ of N and a normalised eigenform

$$\boldsymbol{f}_{k_o}^\sharp = \sum_{n\geqslant 1} a_n \cdot q^n \in M_{k_o}(\Gamma_1(N_{\boldsymbol{f}}) \cap \Gamma_0(p), \chi_{\boldsymbol{f}})_L$$

of weight $k_o \geqslant 1$, level $N_{\boldsymbol{f}}p$, character $\chi_{\boldsymbol{f}} : (\mathbf{Z}/N_{\boldsymbol{f}}\mathbf{Z})^* \longrightarrow L^*$ and Fourier coefficients in L, satisfying the following (cf. Assumption 1.1)

Assumption 5.1. — *One of the following statements* 1–2 *holds true.*

1. *The form $\boldsymbol{f}_{k_o}^\sharp$ is cuspidal of weight $k_o \geqslant 2$, p-ordinary (id est a_p is a p-adic unit under the fixed embedding $\bar{\mathbf{Q}} \hookrightarrow \bar{\mathbf{Q}}_p$) and its conductor is divisible by $N_{\boldsymbol{f}}$.*
2. *The form $\boldsymbol{f}_{k_o}^\sharp$ is a p-stabilisation of a cuspidal and p-regular weight-one newform of level $N_{\boldsymbol{f}}$, without real multiplication by a quadratic field in which p splits.*

The previous assumption guarantees that the eigencurve $\kappa : \mathscr{C}(N_{\boldsymbol{f}}) \longrightarrow \mathcal{W}$ (cf. Section 4.3) is étale at (the L-rational point corresponding to) $\boldsymbol{f}_{k_o}^\sharp$. In case 5.1(1) (resp., case 5.1(2)) this follows from Corollary 1.4 of [**Hid86**] and Section 6 of [**Pil13**] (resp., Theorems 1.1 and 7.2 of [**BD16**]). As a consequence, there exists an open connected disc $U_{\boldsymbol{f}}$ in $\mathcal{W}_L$ centred at k_o, and a section $U_{\boldsymbol{f}} \hookrightarrow \mathscr{C}(N_{\boldsymbol{f}}) \otimes_{\mathbf{Q}_p} L$ of $\kappa \otimes_{\mathbf{Q}_p} L$ mapping $U_{\boldsymbol{f}}$ isomorphically onto an open admissible neighbourhood of $\boldsymbol{f}_{k_o}^\sharp$. In light of Equation (94), this yields an idempotent $e_{\boldsymbol{f}^\sharp}$ in the Hecke algebra (cf. Section 4.3)

$$\mathcal{H} \stackrel{\mathrm{def}}{=} \mathfrak{h}(N_{\boldsymbol{f}}, U_{\boldsymbol{f}}),$$

and an isomorphism of $\mathscr{O}_{U_{\boldsymbol{f}}}$-algebras between $e_{\boldsymbol{f}^\sharp} \cdot \mathcal{H}$ and $\mathscr{O}_{U_{\boldsymbol{f}}}$. Let

$$\varphi : \mathcal{H} \longrightarrow \mathscr{O}_{U_{\boldsymbol{f}}} \tag{96}$$

be the composition of this isomorphism with the projection onto $e_{\boldsymbol{f}^\sharp} \cdot \mathcal{H}$.

For each positive integer n, denote by $\Delta'_n \subset \Sigma'_0(p) \cap M_2(\mathbf{Z})$ the set of integral matrices $\alpha = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ satisfying $\det(\alpha) = n$, $d \equiv 1 \mod N$, $p \nmid d$ and $c \equiv 0 \mod Np$. Define $T'_n = \sum_{\alpha \in \Delta'_n} T_\alpha$, where T_α is the endomorphism of $H^1(\Gamma_1(N_{\boldsymbol{f}}) \cap \Gamma_0(p), \mathcal{D}'_{U_{\boldsymbol{f}},m})^{\leqslant 0}$ introduced in Section 4.1.1 (and $m = m(U_{\boldsymbol{f}})$ is sufficiently large). The dual Hecke operator T'_n belongs to $\mathcal{H}$ (cf. [**Shi71**, Chapter 3]), and after setting

$$a_n(\boldsymbol{k}) = a_n(\boldsymbol{f}^\sharp, \boldsymbol{k}) = \varphi(T'_n),$$

the formal q-expansion

$$\boldsymbol{f}^\sharp = \sum_{n\geqslant 1} a_n(\boldsymbol{k}) \cdot q^n \in \mathscr{O}_{U_{\boldsymbol{f}}}[\![q]\!]$$

is the (unique) cuspidal primitive Hida family in $S^{\mathrm{ord}}_{U_{\boldsymbol{f}}}(N_{\boldsymbol{f}},\chi_{\boldsymbol{f}})$ of tame level $N_{\boldsymbol{f}}$ and character $\chi_{\boldsymbol{f}}$ specialising to $\boldsymbol{f}^{\sharp}_{k_o}$ at k_o. For each positive integer n, it is an eigenvector for the Hecke operator T_n with eigenvalue $a_n(\boldsymbol{k})$.

The rest of this section summarises the main result from Hida theory needed in the sequel of the paper. Fix a level-N test vector

$$\boldsymbol{f} \in S^{\mathrm{ord}}_{U_{\boldsymbol{f}}}(N,\chi_{\boldsymbol{f}})[\boldsymbol{f}^{\sharp}]$$

for $\boldsymbol{f}^{\sharp}$. To ease notation, set $\Lambda_{\boldsymbol{f}}=\Lambda_{U_{\boldsymbol{f}}}$, $\mathscr{O}_{\boldsymbol{f}}=\mathscr{O}_{U_{\boldsymbol{f}}}$, $\mathcal{D}^{\cdot}_{\boldsymbol{f},m}=\mathcal{D}^{\cdot}_{U_{\boldsymbol{f}},m}$ and $D^{\cdot}_{\boldsymbol{f},m}=D^{\cdot}_{U_{\boldsymbol{f}},m}$ (where as usual $\cdot$ denotes either the empty symbol or $\prime$). Denote by $\boldsymbol{k}-k_o$ a uniformiser at k_o in $\Lambda_{\boldsymbol{f}}$, so that $\mathscr{O}_{\boldsymbol{f}}$ is a module of power series in $L[\![\boldsymbol{k}-k_o]\!]$ which converge for any $\boldsymbol{k}$ in $U_{\boldsymbol{f}}$. One has $\kappa_{U_{\boldsymbol{f}}}(t)=\omega(t)^{k_o-2}\cdot\langle t\rangle^{\boldsymbol{k}-2}$ for all $t\in\mathbf{Z}_p^*$, and

$$\boldsymbol{\kappa}_{U_{\boldsymbol{f}}}=\omega_{\mathrm{cyc}}^{k_o-2}\cdot\kappa_{\mathrm{cyc}}^{\boldsymbol{k}-2}:G_{\mathbf{Q}}\longrightarrow\Lambda_{\boldsymbol{f}}^*. \tag{97}$$

Here ω_{cyc} and κ_{cyc} denote the composition of the p-adic cyclotomic character

$$\chi_{\mathrm{cyc}}:G_{\mathbf{Q}}\longrightarrow\mathbf{Z}_p^*$$

with the projections $\omega:\mathbf{Z}_p^*\longrightarrow\mathbf{F}_p^*$ and $\langle\cdot\rangle:\mathbf{Z}_p\longrightarrow 1+p\mathbf{Z}_p$ respectively, and $\kappa_{\mathrm{cyc}}^{\boldsymbol{k}-2}$ is the $\Lambda_{\boldsymbol{f}}^*$-valued character which on $g\in G_{\mathbf{Q}}$ takes the value $\kappa_{\mathrm{cyc}}(g)^{\boldsymbol{k}-2}$.

- For every classical weight $k>2$ in $U^{\mathrm{cl}}_{\boldsymbol{f}}$ the weight-k specialisation $\boldsymbol{f}_k$ is old at p. Indeed $\boldsymbol{f}_k=f_\alpha$ is the ordinary p-stabilisation of an eigenform $f=f_k$ in $S_k(N,\chi_{\boldsymbol{f}})$ (cf. Equation (54)), hence $a_p(k)=\alpha_f$ is the unit root of

  $$X^2-\frac{a_p(f)}{a_1(f)}\cdot X+\chi_{\boldsymbol{f}}(p)p^{k-1}=(X-\alpha_f)\cdot(X-\beta_f).$$

  (We refer the reader to [**Hid86**] for more details.)
- To ease notation, set

  $$\boldsymbol{\mathcal{V}}=H^1(\Gamma_1(N_{\boldsymbol{f}})\cap\Gamma_0(p),D'_{\boldsymbol{f},m})^{\leqslant 0}(1)\quad\text{and}\quad\mathcal{H}=\mathfrak{h}(N_{\boldsymbol{f}},U_{\boldsymbol{f}}).$$

  According to the main results of [**Oht00**] and the isomorphism (92), there is a short exact sequence of $\mathcal{H}[G_{\mathbf{Q}_p}]$-modules

  $$0\longrightarrow\boldsymbol{\mathcal{V}}^+\longrightarrow\boldsymbol{\mathcal{V}}\longrightarrow\boldsymbol{\mathcal{V}}^-\longrightarrow 0, \tag{98}$$

  where $\boldsymbol{\mathcal{V}}^{\pm}$ are finite free $\mathscr{O}_{\boldsymbol{f}}$-modules. The $G_{\mathbf{Q}_p}$-module $\boldsymbol{\mathcal{V}}^-$ is the maximal unramified $\mathscr{O}_{\boldsymbol{f}}$-quotient of $\boldsymbol{\mathcal{V}}$, and an arithmetic Frobenius acts on it as multiplication by the p-th Fourier coefficient $a_p(k)$ of $\boldsymbol{f}^{\sharp}$. Moreover, there are canonical isomorphisms of $\mathcal{H}$-modules $\boldsymbol{\mathcal{V}}^+\cong\mathcal{H}_{\mathrm{par}}$ and $\boldsymbol{\mathcal{V}}^-\cong\mathrm{Hom}_{\mathscr{O}_U}(\mathcal{H},\mathscr{O}_U)$, where $\mathcal{H}_{\mathrm{par}}$ is the quotient of $\mathcal{H}$ acting faithfully on the parabolic subspace $H^1_{\mathrm{par}}(\Gamma_{\boldsymbol{f}},D'_{\boldsymbol{f},m})^{\leqslant 0}(1)$ of the cohomology group $\boldsymbol{\mathcal{V}}$.

  Applying the idempotent $e_{\boldsymbol{f}^{\sharp}}$ (defined before Equation (96)) to the short exact sequence (98) gives a short exact sequence of $\mathscr{O}_{\boldsymbol{f}}[G_{\mathbf{Q}_p}]$-modules

  $$0\longrightarrow V(\boldsymbol{f}^{\sharp})^+\longrightarrow V(\boldsymbol{f}^{\sharp})\longrightarrow V(\boldsymbol{f}^{\sharp})^-\longrightarrow 0, \tag{99}$$

  where (for $\cdot$ equal to one of the symbols $\emptyset,+$ and $-$)

  $$V(\boldsymbol{f}^{\sharp})^{\cdot}=e_{\boldsymbol{f}^{\sharp}}\cdot\boldsymbol{\mathcal{V}}^{\cdot}$$

is a free $\mathscr{O}_{\boldsymbol{f}}$-direct summand of $\boldsymbol{\mathcal{V}}^{\cdot}$.

- The $\mathscr{O}_{\boldsymbol{f}}[G_{\mathbf{Q}}]$-module $V(\boldsymbol{f}^{\sharp})$ has rank two over $\mathscr{O}_{\boldsymbol{f}}$, and is unramified outside $N_{\boldsymbol{f}}p$. For every prime ℓ not dividing $N_{\boldsymbol{f}}p$, the characteristic polynomial of an arithmetic Frobenius Frob_ℓ in $G_{\mathbf{Q}}$ at ℓ is given by (cf. Equation (106) below)

$$\det\bigl(1 - \mathrm{Frob}_\ell | V(\boldsymbol{f}^{\sharp}) \cdot X\bigr) = 1 - a_\ell(\boldsymbol{k}) \cdot X + \chi_{\boldsymbol{f}}(\ell) \cdot \kappa_{U_{\boldsymbol{f}}}(\ell) \cdot \ell \cdot X^2.$$

In particular the determinant of $V(\boldsymbol{f}^{\sharp})$ is given by (cf. Equation (97))

$$\det{}_{\mathscr{O}_{\boldsymbol{f}}} V(\boldsymbol{f}) = \chi_{\boldsymbol{f}} \cdot \chi_{\mathrm{cyc}} \cdot \boldsymbol{\kappa}_{U_{\boldsymbol{f}}} = \chi_{\boldsymbol{f}} \cdot \omega_{\mathrm{cyc}}^{k_o - 1} \cdot \kappa_{\mathrm{cyc}}^{\boldsymbol{k}-1}. \tag{100}$$

As the arithmetic Frobenius $\mathrm{Frob}_p \in G_{\mathbf{Q}_p}$ acts on $\boldsymbol{\mathcal{V}}^{-}$ as multiplication by $a_p(\boldsymbol{k})$, one deduces isomorphisms of $\mathscr{O}_{\boldsymbol{f}}[G_{\mathbf{Q}_p}]$-modules

$$V(\boldsymbol{f}^{\sharp})^{+} \cong \mathscr{O}_{\boldsymbol{f}}\bigl(1 + \boldsymbol{\kappa}_{U_{\boldsymbol{f}}} + \chi_{\boldsymbol{f}} - \breve{a}_p(\boldsymbol{k})\bigr) \quad \text{and} \quad V(\boldsymbol{f}^{\sharp})^{-} \cong \mathscr{O}_{\boldsymbol{f}}(\breve{a}_p(\boldsymbol{k})), \tag{101}$$

where for every $a \in \Lambda_{\boldsymbol{f}}^{*}$ one writes $\breve{a} : G_{\mathbf{Q}_p} \to \Lambda_{\boldsymbol{f}}^{*}$ for the continuous unramified character satisfying $\breve{a}(\mathrm{Frob}_p) = a$.

- Recall the level-N test vector $\boldsymbol{f}$ for $\boldsymbol{f}^{\sharp}$ fixed above, and define

$$H^1_{\cdot}(\Gamma, D'_{\boldsymbol{f},m})^{\leqslant 0}(1) \longrightarrow\!\!\!\!\rightarrow V(\boldsymbol{f})$$

to be the maximal $\mathscr{O}_{\boldsymbol{f}}$-quotient of $H^1_{\cdot}(\Gamma, D'_{\boldsymbol{f},m})^{\leqslant 0}(1)$ on which the dual Hecke operators T'_ℓ, U'_p, and $\langle d \rangle'$ act respectively as multiplication by $a_\ell(\boldsymbol{k})$, $a_p(\boldsymbol{k})$ and $\chi_{\boldsymbol{f}}(d)$, for each prime ℓ not dividing Np and each unit d in $(\mathbf{Z}/N\mathbf{Z})^{*}$. This is equal to the $G_{\mathbf{Q}}$-modules $V(\boldsymbol{f}^{\sharp}) = e_{\boldsymbol{f}^{\sharp}} \cdot \boldsymbol{\mathcal{V}}$ introduced above when $N = N_{\boldsymbol{f}}$ and $\boldsymbol{f} = \boldsymbol{f}^{\sharp}$. In general, the $\mathscr{O}_U[G_{\mathbf{Q}}]$-module $V(\boldsymbol{f})$ is (non-canonically) isomorphic to the direct sum of a finite number of copies of $V(\boldsymbol{f}^{\sharp})$. In particular, $V(\boldsymbol{f})$ is a free $\mathscr{O}_U$-module, and there is a short exact sequence of $\mathscr{O}_{\boldsymbol{f}}[G_{\mathbf{Q}_p}]$-modules

$$0 \longrightarrow V(\boldsymbol{f})^{+} \longrightarrow V(\boldsymbol{f}) \longrightarrow V(\boldsymbol{f})^{-} \longrightarrow 0 \tag{102}$$

with $V(\boldsymbol{f})^{\pm}$ free of finite rank over $\mathscr{O}_{\boldsymbol{f}}$, and $V(\boldsymbol{f}) \longrightarrow\!\!\!\!\rightarrow V(\boldsymbol{f})^{-}$ the maximal unramified $\mathscr{O}_{\boldsymbol{f}}$-quotient of $V(\boldsymbol{f})$.

Dually, define

$$V^{*}(\boldsymbol{f}) \hookrightarrow H^1_{\mathrm{c}}(\Gamma, D_{\boldsymbol{f},m})^{\leqslant 0}(-\boldsymbol{\kappa}_{U_{\boldsymbol{f}}})$$

be the maximal $\mathscr{O}_{\boldsymbol{f}}$-submodule of $H^1_{\mathrm{c}}(\Gamma, D_{\boldsymbol{f},m})^{\leqslant 0}(-\boldsymbol{\kappa}_{U_{\boldsymbol{f}}})$ on which the Hecke operators T_ℓ, U_p and $\langle d \rangle$ act respectively as multiplication by $a_\ell(\boldsymbol{k})$, $a_p(\boldsymbol{k})$ and $\chi_{\boldsymbol{f}}(d)$, for every prime $\ell \nmid Np$ and every unit d in $(\mathbf{Z}/N\mathbf{Z})^{*}$. Then $V^{*}(\boldsymbol{f})$ is an $\mathscr{O}_{\boldsymbol{f}}[G_{\mathbf{Q}}]$-direct summand of $H^1_{\mathrm{c}}(\Gamma, D_{\boldsymbol{f},m})^{\leqslant 0}(-\boldsymbol{\kappa}_{U_{\boldsymbol{f}}})$, isomorphic to the $\mathscr{O}_{\boldsymbol{f}}$-dual of $V(\boldsymbol{f})$. Indeed the bilinear form $\det^{*}_{U_{\boldsymbol{f}}}$ defined in Equation (82) induces a perfect pairing of $\mathscr{O}_{\boldsymbol{f}}[G_{\mathbf{Q}}]$-modules (cf. [**Oht00**] and Section 4.3)

$$\langle \cdot, \cdot \rangle_{\boldsymbol{f}} : V(\boldsymbol{f}) \otimes_{\mathscr{O}_{\boldsymbol{f}}} V^{*}(\boldsymbol{f}) \longrightarrow \mathscr{O}_{\boldsymbol{f}}. \tag{103}$$

Let $V^{*}(\boldsymbol{f})^{+} \hookrightarrow V^{*}(\boldsymbol{f})$ be the maximal unramified submodule of the restriction of $V^{*}(\boldsymbol{f})$ to $G_{\mathbf{Q}_p}$, and let $V^{*}(\boldsymbol{f})^{-}$ be the quotient of $V^{*}(\boldsymbol{f})$ by $V^{*}(\boldsymbol{f})^{+}$. There is then a short exact sequence of $\mathscr{O}_{\boldsymbol{f}}[G_{\mathbf{Q}_p}]$-modules

$$0 \longrightarrow V^{*}(\boldsymbol{f})^{+} \longrightarrow V^{*}(\boldsymbol{f}) \longrightarrow V^{*}(\boldsymbol{f})^{-} \longrightarrow 0,$$

and the bilinear form $\langle\cdot,\cdot\rangle_{\boldsymbol{f}}$ induces perfect, $G_{\mathbf{Q}_p}$-equivariant pairings

$$\langle\cdot,\cdot\rangle_{\boldsymbol{f}} : V(\boldsymbol{f})^{\pm} \otimes_{\mathscr{O}_{\boldsymbol{f}}} V^*(\boldsymbol{f})^{\mp} \longrightarrow \mathscr{O}_{\boldsymbol{f}}. \tag{104}$$

Because $H^1_{\cdot}(\Gamma, D^{\cdot}_{\boldsymbol{f},m})^{\leqslant 0}$ is an $\mathscr{O}_{\boldsymbol{f}}$-direct summand of $H^1_{\cdot}(\Gamma, D^{\cdot}_{\boldsymbol{f},m})$, there are natural $\mathscr{O}_{\boldsymbol{f}}[G_{\mathbf{Q}}]$-projections

$$\mathrm{pr}_{\boldsymbol{f}} : H^1(\Gamma, D'_{\boldsymbol{f},m})(1) \longrightarrow\!\!\!\!\to V(\boldsymbol{f}) \quad \text{and} \quad \mathrm{pr}^*_{\boldsymbol{f}} : H^1_{\mathrm{c}}(\Gamma, D_{\boldsymbol{f},m})(-\boldsymbol{\kappa}_{U_{\boldsymbol{f}}}) \longrightarrow\!\!\!\!\to V^*(\boldsymbol{f}). \tag{105}$$

- For all classical points k in $U^{\mathrm{cl}}_{\boldsymbol{f}}$ the specialisation map ρ_k in the right column of Equation (84) gives rise to an isomorphism of $L[G_{\mathbf{Q}}]$-modules

$$\rho_k : V(\boldsymbol{f}) \otimes_{\Lambda_{\boldsymbol{f}}} \Lambda_{\boldsymbol{f}}/(\pi_k) \cong H^1_{\text{ét}}(Y_1(N,p)_{\bar{\mathbf{Q}}}, \mathscr{L}_{k-2}(1))_{\boldsymbol{f}^*_k} \cong V(\boldsymbol{f}_k). \tag{106}$$

Here

$$H^1_{\text{ét}}(Y_1(N,p)_{\bar{\mathbf{Q}}}, \mathscr{L}_{k-2}(1))_L \longrightarrow H^1_{\text{ét}}(Y_1(N,p)_{\bar{\mathbf{Q}}}, \mathscr{L}_{k-2}(1))_{\boldsymbol{f}^*_k}$$

is the maximal quotient on which T'_ℓ, U'_p and $\langle d\rangle'$ act respectively as multiplication by $a_\ell(k), a_p(k)$ and $\chi_{\boldsymbol{f}}(p)$ for any prime $\ell \nmid Np$ and any unit d in $(\mathbf{Z}/Np\mathbf{Z})^*$. If $t : Y_1(Np) \to Y_1(N,p)$ is the natural projection (viz. the one induced by the identity on $\mathbf{H}$ under (6)), the second isomorphism in Equation (106) is the one induced by the pull-back

$$t^* : H^1_{\text{ét}}(Y_1(N,p)_{\bar{\mathbf{Q}}}, \mathscr{L}_{k-2}(1)) \longrightarrow H^1_{\text{ét}}(Y_1(Np)_{\bar{\mathbf{Q}}}, \mathscr{L}_{k-2}(1)).$$

If $k_o = 1$, so that $\boldsymbol{f}_1 = \sum_{n\geqslant 1} a_n(1)\cdot q^n$ is a classical, cuspidal weight-one Hecke eigenform (cf. Assumption 5.1), then the weight-one specialisation

$$V(\boldsymbol{f}^\sharp_1) = V(\boldsymbol{f}^\sharp) \otimes_{\Lambda_{\boldsymbol{f}}} \Lambda_{\boldsymbol{f}}/(\pi_1)$$

of $V(\boldsymbol{f}^\sharp)$ yields a canonical model of the dual of the Deligne–Serre representation attached to $\boldsymbol{f}^\sharp_1$. More generally, if $\boldsymbol{f}_1$ is classical, set $V(\boldsymbol{f}_1) = V(\boldsymbol{f}) \otimes_{\Lambda_{\boldsymbol{f}}} \Lambda_{\boldsymbol{f}}/\pi_1$ (which is non-canonically isomorphism to the direct sum of a finite number of $V(\boldsymbol{f}^\sharp_1)$.) In order to have coherent notation and terminology, we still denote by

$$\rho_1 : V(\boldsymbol{f}) \otimes_{\Lambda_{\boldsymbol{f}}} \Lambda_{\boldsymbol{f}}/(\pi_1) \longrightarrow V(\boldsymbol{f}_1) \tag{107}$$

the identity map, and refer to it as the specialisation map at weight one.

Similarly for each classical weight k in $U^{\mathrm{cl}}_{\boldsymbol{f}}$ there are natural isomorphisms of $L[G_{\mathbf{Q}_p}]$-modules

$$\rho_k : V^*(\boldsymbol{f}) \otimes_{\Lambda_{\boldsymbol{f}}} \Lambda_{\boldsymbol{f}}/(\pi_k) \cong V^*(\boldsymbol{f}_k) \tag{108}$$

(cf. the discussion following Equation (84)). Moreover for each $x \in V(\boldsymbol{f})$ and $y \in V^*(\boldsymbol{f})$ one has

$$\langle x, y\rangle_{\boldsymbol{f}}(k) = \langle \rho_k(x), \rho_k(y)\rangle_{\boldsymbol{f}_k}, \tag{109}$$

where $\langle\cdot,\cdot\rangle_{\boldsymbol{f}_k}$ is the perfect bilinear form defined in Equation (24).

- For each k in $U^{\mathrm{cl}}_{\boldsymbol{f}}$ and $\cdot = \emptyset, *$, one has short exact sequences of $L[G_{\mathbf{Q}_p}]$-modules

$$0 \longrightarrow V^{\cdot}(\boldsymbol{f}_k)^+ \longrightarrow V^{\cdot}(\boldsymbol{f}_k) \longrightarrow V^{\cdot}(\boldsymbol{f}_k)^- \longrightarrow 0, \tag{110}$$

where $V(\boldsymbol{f}_k)^-$ is the maximal $G_{\mathbf{Q}_p}$-unramified L-quotient of $V(\boldsymbol{f}_k)$, and $V^*(\boldsymbol{f}_k)^+$ is the maximal $G_{\mathbf{Q}_p}$-unramified L-submodule of $V^*(\boldsymbol{f}_k)$. The specialisation maps (106) and (108) induce isomorphisms

$$\rho_k : V^{\cdot}(\boldsymbol{f})^{\pm} \otimes_k L \cong V^{\cdot}(\boldsymbol{f}_k)^{\pm}. \tag{111}$$

According to Equation (101) the inertia subgroup $I_{\mathbf{Q}_p}$ of $G_{\mathbf{Q}_p}$ acts on $V(\boldsymbol{f}_k)^+$ via $\chi_{\mathrm{cyc}}^{k-1}$, and trivially on $V(\boldsymbol{f}_k)^-$. If $k \geqslant 2$, applying $D_{\mathrm{dR}}(\cdot)$ to the previous exact sequence and using Equation (28) gives natural isomorphisms

$$D_{\mathrm{cris}}(V(\boldsymbol{f}_k)^+) \cong V_{\mathrm{dR}}(\boldsymbol{f}_k)/\mathrm{Fil}^0 \quad \text{and} \quad \mathrm{Fil}^0 V_{\mathrm{dR}}(\boldsymbol{f}_k) \cong D_{\mathrm{cris}}(V(\boldsymbol{f}_k)^-). \tag{112}$$

Similarly $I_{\mathbf{Q}_p}$ acts trivially on $V^*(\boldsymbol{f}_k)^+$ and via $\chi_{\mathrm{cyc}}^{1-k}$ on $V^*(\boldsymbol{f}_k)^-$, hence Equations (28) and (110) give

$$D_{\mathrm{cris}}(V^*(\boldsymbol{f}_k)^+) \cong V^*_{\mathrm{dR}}(\boldsymbol{f}_k)/\mathrm{Fil}^1 \quad \text{and} \quad \mathrm{Fil}^1 V^*_{\mathrm{dR}}(\boldsymbol{f}_k) \cong D_{\mathrm{cris}}(V^*(\boldsymbol{f}_k)^-). \tag{113}$$

- The Atkin–Lehner operator w_{Np} introduced in Equation (67) induces an isomorphism of $\mathscr{O}_{\boldsymbol{f}}[G_{\mathbf{Q}(\zeta_N)}]$-modules (cf. Equation (68))

$$w_{Np} : H^1(\Gamma, D_{U,m})^{\leqslant 0} \cong H^1(\Gamma, D'_{U,m})^{\leqslant 0},$$

intertwining the action of the dual Hecke operators U'_p, T'_ℓ and $\langle d \rangle$ on the left hand side with that of the Hecke operators U_p, T_ℓ and $\langle d \rangle^{-1}$ on the right hand side, for each prime ℓ not dividing Np and each unit d modulo N. Since the form $\boldsymbol{f}^{\sharp}_{k_o}$ is cuspidal, it induces Galois equivariant isomorphisms

$$w^{\cdot}_{Np} : V^*(\boldsymbol{f})^{\cdot}(1 + \boldsymbol{\kappa}_{U_{\boldsymbol{f}}} + \chi_{\boldsymbol{f}}) \cong V(\boldsymbol{f})^{\cdot}, \tag{114}$$

for $\cdot$ equal to one of the symbols $\emptyset, +$ and $-$.
- Set

$$D^*(\boldsymbol{f})^- = \big(\mathtt{V}^*(\boldsymbol{f})^-(1 + \boldsymbol{\kappa}_{U_{\boldsymbol{f}}} + \chi_{\boldsymbol{f}}) \hat{\otimes}_{\mathbf{Z}_p} \hat{\mathbf{Z}}_p^{\mathrm{nr}}\big)^{G_{\mathbf{Q}_p}}[1/p], \tag{115}$$

where $\mathtt{V}^*(\boldsymbol{f})^-$ is a $G_{\mathbf{Q}_p}$-stable $\Lambda_{\boldsymbol{f}}$-lattice in $V^*(\boldsymbol{f})^-$, and $\hat{\mathbf{Z}}_p^{\mathrm{nr}}$ is the ring of integers of the p-adic completion $\hat{\mathbf{Q}}_p^{\mathrm{nr}}$ of the maximal unramified extension of $\mathbf{Q}_p$. (Note that $V^*(\boldsymbol{f})^-(1 + \boldsymbol{\kappa}_{U_{\boldsymbol{f}}} + \chi_{\boldsymbol{f}})$ is an unramified $G_{\mathbf{Q}_p}$-module, cf. Equations (101) and (104).) It is a free finite $\mathscr{O}_{\boldsymbol{f}}$-module (of rank one if $\boldsymbol{f} = \boldsymbol{f}^{\sharp}$ is primitive). For each classical point k in $U_{\boldsymbol{f}}^{\mathrm{cl}}$, the isomorphism (111) and the second isomorphism in Equation (113) induce a specialisation isomorphism

$$\rho_k : D^*(\boldsymbol{f})^- \otimes_k L \cong \Big(V^*(\boldsymbol{f}_k)^-(k - 1 + \chi_{\boldsymbol{f}}) \otimes_{\mathbf{Q}_p} \hat{\mathbf{Q}}_p^{\mathrm{nr}}\Big)^{G_{\mathbf{Q}_p}} \cong \mathrm{Fil}^1 V^*_{\mathrm{dR}}(\boldsymbol{f}_k). \tag{116}$$

As $V^*(\boldsymbol{f}_k)^-(k-1)$ is unramified, in the previous equation one identifies the middle term with the tensor product of $D_{\mathrm{cris}}(V^*(\boldsymbol{f}_k)^-)$, $D_{\mathrm{cris}}(\mathbf{Q}_p(k-1))$ and $D_{\mathrm{cris}}(L(\chi_{\boldsymbol{f}}))$. The second isomorphism then arises from Equation (113), the canonical isomorphism $D_{\mathrm{cris}}(\mathbf{Q}_p(k-1)) \cong \mathbf{Q}_p$, and the isomorphism between $D_{\mathrm{cris}}(L(\chi_{\boldsymbol{f}}))$ and L sending the Gauß sum $\sum_{a \in (\mathbf{Z}/c(\chi_{\boldsymbol{f}})\mathbf{Z})^*} \check{\chi}_{\boldsymbol{f}}(a) \otimes \zeta^a_{c(\chi_{\boldsymbol{f}})}$ of the primitive character $\check{\chi}_{\boldsymbol{f}}$ attached to $\chi_{\boldsymbol{f}}$ to the identity, where $c(\chi_{\boldsymbol{f}})$ is the conductor of $\chi_{\boldsymbol{f}}$ and $\zeta_{c(\chi_{\boldsymbol{f}})}$ is a primitive $c(\chi_{\boldsymbol{f}})$-th root of unity.

In light of the isomorphisms (87) and (114), the main result of [**Oht00**] and Theorem 9.5.2 of [**KLZ17**] yield an *Eichler–Shimura isomorphism*

$$\mathtt{ES}_{\boldsymbol{f}}^{-} : D^{*}(\boldsymbol{f})^{-} \cong S_{U_{\boldsymbol{f}}}^{\mathrm{ord}}(N, \chi_{\boldsymbol{f}})[\boldsymbol{f}^{\sharp}], \tag{117}$$

whose base change along evaluation at a classical point $k \in U_{\boldsymbol{f}}^{\mathrm{cl}}$ is equal to the composition of the specialisation isomorphism (116) with the isomorphism $\mathrm{Fil}^{1} V_{\mathrm{dR}}^{*}(\boldsymbol{f}_k) \cong S_k(Np, L)_{\boldsymbol{f}_k}$ defined in Equation (27). One defines

$$\omega_{\boldsymbol{f}} \in D^{*}(\boldsymbol{f})^{-} \tag{118}$$

to be the image of the Hida family $\boldsymbol{f}$ under the inverse of $\mathtt{ES}_{\boldsymbol{f}}^{-}$, so that

$$\rho_k(\omega_{\boldsymbol{f}}) = \omega_{\boldsymbol{f}_k} \tag{119}$$

for each classical point k in $U_{\boldsymbol{f}}^{\mathrm{cl}}$ (cf. Equation (30)). (When $k_o \geqslant 2$, the overconvergent Eichler–Shimura isomorphism proved in [**AIS15**] extends these results to Coleman families of slope at most $k_o - 2$.)

- Set

$$D^{*}(\boldsymbol{f})^{+} = \left(\mathtt{V}^{*}(\boldsymbol{f})^{+} \hat{\otimes}_{\mathbf{Z}_p} \hat{\mathbf{Z}}_p^{\mathrm{nr}}\right)^{G_{\mathbf{Q}_p}}[1/p], \tag{120}$$

where $\mathtt{V}^{*}(\boldsymbol{f})^{+}$ is a $G_{\mathbf{Q}_p}$-stable $\Lambda_{\boldsymbol{f}}$-lattice in $V^{*}(\boldsymbol{f})^{+}$. The perfect duality $\langle \cdot, \cdot \rangle_{\boldsymbol{f}}$ (cf. Equation (104)), the Atkin–Lehner isomorphism w_{Np}^{+} (cf. Equation (114)) and the Eichler–Shimura isomorphism $\mathtt{ES}_{\boldsymbol{f}}^{-}$ give rise to an isomorphism

$$\mathtt{ES}_{\boldsymbol{f}}^{+} : D^{*}(\boldsymbol{f})^{+} \cong \mathrm{Hom}_{\mathscr{O}_{\boldsymbol{f}}}(S_{U_{\boldsymbol{f}}}^{\mathrm{ord}}(N, \chi_f)[\boldsymbol{f}^{\sharp}], \mathscr{O}_{\boldsymbol{f}}),$$

whose base change along evaluation at $k \in U_{\boldsymbol{f}}^{\mathrm{cl}}$ on $\mathscr{O}_{\boldsymbol{f}}$ equals the composition of the specialisation isomorphism

$$\rho_k : D^{*}(\boldsymbol{f})^{+} \otimes_k L \cong \left(V^{*}(\boldsymbol{f}_k)^{+} \otimes_{\mathbf{Q}_p} \hat{\mathbf{Q}}_p^{\mathrm{nr}}\right)^{G_{\mathbf{Q}_p}} \cong V_{\mathrm{dR}}^{*}(\boldsymbol{f}_k)/\mathrm{Fil}^{1} \tag{121}$$

arising from Equations (111) and (113), and the isomorphism

$$V_{\mathrm{dR}}^{*}(\boldsymbol{f}_k)/\mathrm{Fil}^{1} \cong \mathrm{Hom}_L(S_k(Np, L)_{\boldsymbol{f}_k^{*}}, L) \cong \mathrm{Hom}_L(S_k(Np, L)_{\boldsymbol{f}_k}, L),$$

where the first map is the adjoint of the perfect duality $\langle \cdot, \cdot \rangle_{\boldsymbol{f}_k}$ defined in Equation (32) (cf. Equation (109)), and the second is the dual of

$$(-1)^{k_o - 2} \cdot w_{Np} : S_k(Np, L)_{\boldsymbol{f}_k} \cong S_k(Np, L)_{\boldsymbol{f}_k^{*}}.$$

We claim that (shrinking $U_{\boldsymbol{f}}$ if necessary) there exists

$$\eta_{\boldsymbol{f}} \in D^{*}(\boldsymbol{f})^{+} \tag{122}$$

such that, for each classical point k in $U_{\boldsymbol{f}}^{\mathrm{cl}}$, one has (cf. Equation (34))

$$\rho_k(\eta_{\boldsymbol{f}}) = (p-1) a_p(k) \cdot \eta_{\boldsymbol{f}_k}. \tag{123}$$

Indeed, write $\boldsymbol{f} = \sum_d r_d \cdot \boldsymbol{f}^{\sharp}(q^d)$, with functions $(r_d)_{d|(N/N_{\boldsymbol{f}})}$ in $\mathscr{O}_{\boldsymbol{f}}$ without common zeros. For each positive divisor d of $N/N_{\boldsymbol{f}}$, the $\mathbf{Q}$-rational morphism $v_d : Y_1(N, p)_{\mathbf{Q}} \longrightarrow Y_1(N_{\boldsymbol{f}}, p)_{\mathbf{Q}}$ arising from multiplication by d on $\mathbf{H}$ (cf. Equation (6)) induces a $G_{\mathbf{Q}}$-equivariant morphism $v_{d*} : V^{*}(\boldsymbol{f}) \longrightarrow V^{*}(\boldsymbol{f}^{\sharp})$ (cf.

Equation (77)), which in turn induces $v_{d*}: D^*(\boldsymbol{f})^- \longrightarrow D^*(\boldsymbol{f}^\sharp)^-$. Under the Eichler–Shimura isomorphism $\mathtt{ES}_{\boldsymbol{f}}^-$, the latter gives rise to a map

$$v_{d_*}: S_{U_{\boldsymbol{f}}}^{\mathrm{ord}}(N, \chi_{\boldsymbol{f}})[\boldsymbol{f}^\sharp] \longrightarrow S_{U_{\boldsymbol{f}}}^{\mathrm{ord}}(N_{\boldsymbol{f}}, \chi_{\boldsymbol{f}})[\boldsymbol{f}^\sharp] = \mathscr{O}_{\boldsymbol{f}} \cdot \boldsymbol{f}^\sharp.$$

Set $\mathrm{Trace}_{\boldsymbol{f}} = \sum_d r_d \cdot v_{d*}$, and define the big differential $\check{\eta}_{\boldsymbol{f}} \in D^*(\boldsymbol{f})^+$ to be the image under the inverse of $\mathtt{ES}_{\boldsymbol{f}}^+$ of the linear form sending the Hida family $\boldsymbol{f}'$ in $S_{U_{\boldsymbol{f}}}^{\mathrm{ord}}(N, \chi_{\boldsymbol{f}})[\boldsymbol{f}^\sharp]$ to the first Fourier coefficient of $\mathrm{Trace}_{\boldsymbol{f}}(\boldsymbol{f}')$:

$$\mathtt{ES}_{\boldsymbol{f}}^+(\check{\eta}_{\boldsymbol{f}})(\boldsymbol{f}') = a_1(\mathrm{Trace}_{\boldsymbol{f}}(\boldsymbol{f}')).$$

It follows from the definitions and Equation (109) that

$$\rho_k(\check{\eta}_{\boldsymbol{f}}) = (-1)^{k_o - 2} \cdot \frac{(\boldsymbol{f}_k, \boldsymbol{f}_k)_{Np}}{(\boldsymbol{f}_k^\sharp, \boldsymbol{f}_k^\sharp)_{N_{\boldsymbol{f}}p}} \cdot \eta_{\boldsymbol{f}_k}$$

for each classical point k in $U_{\boldsymbol{f}}^{\mathrm{cl}}$. As explained in the proof of Lemma 2.19 of [**DR14**], the elements $(-1)^{k_o-2} \cdot \frac{(\boldsymbol{f}_k, \boldsymbol{f}_k)_{Np}}{(\boldsymbol{f}_k^\sharp, \boldsymbol{f}_k^\sharp)_{N_{\boldsymbol{f}}p}}$ are interpolated by an analytic function $\mathscr{E}_{\boldsymbol{f}}$ on $U_{\boldsymbol{f}}$, which does not vanish at k_o (as $\boldsymbol{f}_{k_o}$ is non-zero by the definition of level-N test vector for $\boldsymbol{f}^\sharp$). Shrinking $U_{\boldsymbol{f}}$ if necessary, one can then assume that $\mathscr{E}_{\boldsymbol{f}}$ is a unit in $\mathscr{O}_{\boldsymbol{f}}$, and define the sought-for $\mathscr{O}_{\boldsymbol{f}}$-adic differential $\eta_{\boldsymbol{f}}$ to be $(p-1) \cdot \mathscr{E}_{\boldsymbol{f}}^{-1} \cdot a_p(\boldsymbol{k})$ times $\check{\eta}_{\boldsymbol{f}}$.

- Similarly as in Equations (115) and (120), for $\cdot = \pm$, define the $\mathscr{O}_{\boldsymbol{f}}$-module

(124) $$D(\boldsymbol{f})^\cdot = \left(\mathtt{V}(\boldsymbol{f})^\cdot(\boldsymbol{\nu}^\cdot)\hat{\otimes}_{\mathbf{Z}_p}\hat{\mathbf{Z}}_p^{\mathrm{nr}}\right)^{G_{\mathbf{Q}_p}}[1/p],$$

  where $\mathtt{V}(\boldsymbol{f})^\cdot$ is a $G_{\mathbf{Q}_p}$-stable $\mathscr{O}_{\boldsymbol{f}}$-lattice in $V(\boldsymbol{f})^\cdot$, $\boldsymbol{\nu}^-$ is the trivial character and $\boldsymbol{\nu}^+ = -1 - \boldsymbol{\kappa}_{U_{\boldsymbol{f}}}$ (so that the twist of $V(\boldsymbol{f})^\cdot$ by $\boldsymbol{\nu}^\cdot$ is unramified, cf. Equation (101)). The pairings $\langle \cdot, \cdot \rangle_{\boldsymbol{f}}$ defined in Equation (104) and the isomorphism $D_{\mathrm{cris}}(L(\chi_{\boldsymbol{f}})) \cong L$ sending the Gauß sum $G(\chi_{\boldsymbol{f}})$ to the identity induce perfect dualities of $\mathscr{O}_{\boldsymbol{f}}$-modules (denoted again by the same symbols)

(125) $$\langle \cdot, \cdot \rangle_{\boldsymbol{f}} : D(\boldsymbol{f})^\pm \otimes_{\mathscr{O}_{\boldsymbol{f}}} D^*(\boldsymbol{f})^\mp \longrightarrow \mathscr{O}_{\boldsymbol{f}}.$$

  Similarly as in Equations (116) and (121), for each classical point $k \in U_{\boldsymbol{f}}^{\mathrm{cl}}$, the specialisation maps (111) and the isomorphisms (112) give rise to specialisation isomorphisms of L-modules

(126) $$\rho_k : D(\boldsymbol{f})^+ \otimes_k L \cong V_{\mathrm{dR}}(\boldsymbol{f}_k)/\mathrm{Fil}^0 \quad \text{and} \quad \rho_k : D(\boldsymbol{f})^- \otimes_k L \cong \mathrm{Fil}^0 V_{\mathrm{dR}}(\boldsymbol{f}_k).$$

  Under the isomorphisms (116), (121) and (126), the base change of (125) along evaluation at k on $\mathscr{O}_{\boldsymbol{f}}$ is compatible with the perfect duality (31).
- If $k_o = 1$, the representations $V(\boldsymbol{f}_1)$ and $V^*(\boldsymbol{f}_1)$ are Artin representations unramified at p. After setting $V^\cdot(\boldsymbol{f}_1)^\pm = V^\cdot(\boldsymbol{f})^\pm \otimes_1 L$ (for $\cdot = \emptyset, *$), one has a decomposition of $G_{\mathbf{Q}_p}$-modules

$$V^\cdot(\boldsymbol{f}_1) \cong V^\cdot(\boldsymbol{f}_1)^+ \oplus V^\cdot(\boldsymbol{f}_1)^-.$$

  Indeed, according to Assumption 5.1(2) one has

$$V(\boldsymbol{f}_1)^+ = V(\boldsymbol{f}_1)^{\mathrm{Frob}_p = \beta_{\boldsymbol{f}_1}} \quad \text{and} \quad V(\boldsymbol{f}_1)^- = V(\boldsymbol{f}_1)^{\mathrm{Frob}_p = \alpha_{\boldsymbol{f}_1}},$$

where Frob_p is an arithmetic Frobenius, $\alpha_{\boldsymbol{f}_1} = a_p(1)$ and $\alpha_{\boldsymbol{f}_1} \cdot \beta_{\boldsymbol{f}_1} = \chi_{\boldsymbol{f}}(p)$.

In order to have a uniform notation (cf. Equation (112)), if $k_o = 1$ one sets $V^{\cdot}_{\mathrm{dR}}(\boldsymbol{f}_1) = D_{\mathrm{cris}}(V^{\cdot}(\boldsymbol{f}_1))$ and defines

$$V_{\mathrm{dR}}(\boldsymbol{f}_1)/\mathrm{Fil}^0 = D_{\mathrm{cris}}(V(\boldsymbol{f}_1)^+) \quad \text{and} \quad \mathrm{Fil}^0 V_{\mathrm{dR}}(\boldsymbol{f}_1) = D_{\mathrm{cris}}(V(\boldsymbol{f}_1)^-). \tag{127}$$

Similarly set $\mathrm{Fil}^1 V^*_{\mathrm{dR}}(\boldsymbol{f}_1) = D_{\mathrm{cris}}(V^*(\boldsymbol{f}_1)^-)$ and $V^*_{\mathrm{dR}}(\boldsymbol{f}_1)/\mathrm{Fil}^1 = D_{\mathrm{cris}}(V^*(\boldsymbol{f}_1)^+)$. The pairing (103) then induces a perfect and $G_{\mathbf{Q}}$-equivariant duality

$$V(\boldsymbol{f}_1) \otimes_L V^*(\boldsymbol{f}_1) \longrightarrow L,$$

under which $V(\boldsymbol{f}_1)^+$ is the orthogonal complement of $V^*(\boldsymbol{f}_1)^+$. This in turn induces on the crystalline Dieudonné modules a perfect pairing

$$\langle \cdot, \cdot \rangle_{\boldsymbol{f}_1} : V_{\mathrm{dR}}(\boldsymbol{f}_1) \otimes_L V^*_{\mathrm{dR}}(\boldsymbol{f}_1) \longrightarrow L, \tag{128}$$

which identifies $\mathrm{Fil}^0 V_{\mathrm{dR}}(\boldsymbol{f}_1)$ and $V_{\mathrm{dR}}(\boldsymbol{f}_1)/\mathrm{Fil}^0$ with the duals of $V^*_{\mathrm{dR}}(\boldsymbol{f}_1)/\mathrm{Fil}^1$ and $\mathrm{Fil}^1 V^*_{\mathrm{dR}}(\boldsymbol{f}_1)$ respectively. One finally defines

$$\omega_{\boldsymbol{f}_1} = \rho_1(\omega_{\boldsymbol{f}}) \in \mathrm{Fil}^1 V^*_{\mathrm{dR}}(\boldsymbol{f}_1) \quad \text{and} \quad \eta_{\boldsymbol{f}_1} = \rho_1(\eta_{\boldsymbol{f}}) \in V^*_{\mathrm{dR}}(\boldsymbol{f}_1)/\mathrm{Fil}^1 \tag{129}$$

as the specialisations of $\omega_{\boldsymbol{f}}$ and $\eta_{\boldsymbol{f}}$ respectively at weight one.

6. Garrett–Rankin p-adic L-functions

Fix three primitive L-rational Hida families

$$\begin{aligned} \boldsymbol{f}^\sharp &= \sum_{n \geqslant 1} a_n(\boldsymbol{k}) \cdot q^n \in S^{\mathrm{ord}}_{U_{\boldsymbol{f}}}(N_{\boldsymbol{f}}, \chi_{\boldsymbol{f}}), \\ \boldsymbol{g}^\sharp &= \sum_{n \geqslant 1} b_n(\boldsymbol{l}) \cdot q^n \in S^{\mathrm{ord}}_{U_{\boldsymbol{g}}}(N_{\boldsymbol{g}}, \chi_{\boldsymbol{g}}) \\ \text{and} \quad \boldsymbol{h}^\sharp &= \sum_{n \geqslant 1} c_n(\boldsymbol{m}) \cdot q^n \in S^{\mathrm{ord}}_{U_{\boldsymbol{h}}}(N_{\boldsymbol{h}}, \chi_{\boldsymbol{h}}). \end{aligned}$$

Let N be the least common multiple of $N_{\boldsymbol{f}}, N_{\boldsymbol{g}}$ and $N_{\boldsymbol{h}}$, and let

$$\boldsymbol{f} \in S^{\mathrm{ord}}_{U_{\boldsymbol{f}}}(N, \chi_{\boldsymbol{f}}), \quad \boldsymbol{g} \in S^{\mathrm{ord}}_{U_{\boldsymbol{g}}}(N, \chi_{\boldsymbol{g}}) \ \text{ and } \ \boldsymbol{h} \in S^{\mathrm{ord}}_{U_{\boldsymbol{h}}}(N, \chi_{\boldsymbol{h}})$$

be Hida families with associated primitive forms $\boldsymbol{f}^\sharp, \boldsymbol{g}^\sharp$ and $\boldsymbol{h}^\sharp$ respectively. Suppose that Assumption 1.2 holds true, namely $\chi_{\boldsymbol{f}} \cdot \chi_{\boldsymbol{g}} \cdot \chi_{\boldsymbol{h}}$ is the trivial character modulo N. Denote by $\Sigma^{\mathrm{gen}}_{\boldsymbol{f}}$ the set of classical triples $w = (k, l, m)$ in $\Sigma_{\boldsymbol{f}}$ such that p does not divide the conductor of $\boldsymbol{f}_k, \boldsymbol{g}_l$ and $\boldsymbol{h}_m$.

For any $w \in \Sigma^{\mathrm{gen}}_{\boldsymbol{f}}$ one has $\boldsymbol{f}_k = (f_k)_\alpha, \boldsymbol{g}_l = (g_l)_\alpha$ and $\boldsymbol{h}_m = (h_m)_\alpha$ for (unique) p-ordinary eigenforms f_k, g_l and h_m of common level N (cf. Equation (54)). Similarly $\boldsymbol{f}^\sharp_k, \boldsymbol{g}^\sharp_l$ and $\boldsymbol{h}^\sharp_m$ are the ordinary p-stabilisations of newforms $f^\sharp_k, g^\sharp_l$ and $h^\sharp_m$ of levels $N_{\boldsymbol{f}}$, $N_{\boldsymbol{g}}$ and $N_{\boldsymbol{h}}$ respectively.

Lemma 6.1. — *There exists a Hida family $w_N(\boldsymbol{f})$ in $S^{\mathrm{ord}}_{U_{\boldsymbol{f}}}(N, \bar{\chi}_{\boldsymbol{f}})$ such that, for any $k \in U^{\mathrm{cl}}_{\boldsymbol{f}}$ with p not dividing the conductor of $\boldsymbol{f}_k$, the weight-k specialisation $w_N(\boldsymbol{f})_k$ is the ordinary p-stabilisation of $f^w_k = w_N(f_k)$.*

Proof. — A direct computation (see Proposition 1.5 of [**AL78**]) shows that

$$w_N \circ \mathrm{pr}_p^* = \langle (p,1) \rangle \cdot \mathrm{pr}_p^* \circ w_N \quad \text{and} \quad w_N \circ \mathrm{pr}_1^* = \mathrm{pr}_1^* \circ w_N$$

as morphisms from $H^1_{\mathrm{dR}}(Y_1(N)_{\mathbf{Q}_p}, \mathscr{S}_{\mathrm{dR},k-2})_L$ to $H^1_{\mathrm{dR}}(Y_1(Np)_{\mathbf{Q}_p}, \mathscr{S}_{\mathrm{dR},k-2})_L$, where $\langle (p,1) \rangle$ is the diamond operator associated with $(p,1)$ under the identification $\mathbf{Z}/Np\mathbf{Z} = \mathbf{Z}/N\mathbf{Z} \times \mathbf{F}_p$. As a consequence

$$(f_k^w)_\alpha = \left(\mathrm{pr}_1^* \circ w_N - \frac{\bar{\chi}_f(p)\beta_{f_k}}{p^{k-1}} \cdot \mathrm{pr}_p^* \circ w_N \right) f_k \tag{130}$$
$$= w_N \circ \left(\mathrm{pr}_1^* - \frac{\beta_{f_k}}{p^{k-1}} \cdot \mathrm{pr}_p^* \right) f_k = w_N(\boldsymbol{f}_k).$$

The lemma follows from the previous equation and [**KLZ17**, Proposition 10.1.2], namely the existence of a morphism $w_N : S^{\mathrm{ord}}_{U_{\boldsymbol{f}}}(N, \chi_{\boldsymbol{f}}) \to S^{\mathrm{ord}}_{U_{\boldsymbol{f}}}(N, \bar{\chi}_{\boldsymbol{f}})$ which specialises to the Atkin–Lehner operator w_N on the ordinary part of $S_k(\Gamma_1(N,p), \chi_{\boldsymbol{f}})$ for each classical weight k in $U^{\mathrm{cl}}_{\boldsymbol{f}}$ (cf. Equations (69) and (117)). □

According to the previous lemma and the results of [**HT01, DR14, Hid85**] Hida's method (cf. [**Hid85**]) can be applied to construct a *square-root Garrett–Rankin p-adic L-function*

$$\mathscr{L}_p^f(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h}) \in \mathscr{O}_{\boldsymbol{fgh}}$$

such that, for each classical triple $w = (k,l,m)$ in $\Sigma^{\mathrm{gen}}_{\boldsymbol{f}}$, one has

$$\mathscr{L}_p^f(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})(w) = \mathscr{L}_p^f(\boldsymbol{f}_k, \boldsymbol{g}_l, \boldsymbol{h}_m), \tag{131}$$

where $\mathscr{L}_p^f(\boldsymbol{f}_k, \boldsymbol{g}_l, \boldsymbol{h}_m)$ is the p-adic period associated in Equation (55) to the p-stabilisation of the triple (f_k, g_l, h_m).

Remark 6.2. — The p-adic L-function $\mathscr{L}_p^f(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})$ slightly differs from the one denoted by the same symbol in [**DR14**]. Precisely our $\mathscr{L}_p^f(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})$ is equal to their $\mathscr{L}_p^f(w_N(\boldsymbol{f}^*), \boldsymbol{g}, \boldsymbol{h})$, where $\boldsymbol{f}^*$ is the Hida family which specialises to the dual of $\boldsymbol{f}_k$ for each k in $U^{\mathrm{cl}}_{\boldsymbol{f}}$.

6.1. Test vectors and special value formulae. — In this section assume the following hypotheses (cf. [**Hsi20**]).

Assumption 6.3. —

1. *There is a triple (k,l,m) in Σ such that the local sign $\varepsilon_q(\boldsymbol{f}_k^\sharp, \boldsymbol{g}_l^\sharp, \boldsymbol{h}_m^\sharp)$ is equal to $+1$ for all primes $q|N$.*
2. *The greatest common divisor of $N_{\boldsymbol{f}}, N_{\boldsymbol{g}}$ and $N_{\boldsymbol{h}}$ is squarefree.*
3. *There is a classical point k in $U^{\mathrm{cl}}_{\boldsymbol{f}}$ such that $V(f_k^\sharp)$ is residually irreducible and p-distinguished.*

Under these assumptions, Section 3.5 of [**Hsi20**] implies the existence of an explicit level-N test vector $(\boldsymbol{f}^\star, \boldsymbol{g}^\star, \boldsymbol{h}^\star)$ for $(\boldsymbol{f}^\sharp, \boldsymbol{g}^\sharp, \boldsymbol{h}^\sharp)$ such that the *Garrett–Rankin triple product p-adic L-function*

$$L_p(\boldsymbol{f}^\sharp, \boldsymbol{g}^\sharp, \boldsymbol{h}^\sharp) = \mathscr{L}_p^f(\boldsymbol{f}^\star, \boldsymbol{g}^\star, \boldsymbol{h}^\star)^2$$

satisfies the following interpolation property (see Theorem A of loc. cit.). For all $w = (k, l, m)$ in $\Sigma_{\boldsymbol{f}}^{\mathrm{gen}}$

(132)
$$L_p(\boldsymbol{f}_k^\sharp, \boldsymbol{g}_l^\sharp, \boldsymbol{h}_m^\sharp) = \frac{\Gamma(k,l,m)}{2^{\alpha(k,l,m)}} \cdot \frac{\mathcal{E}(\boldsymbol{f}_k^\sharp, \boldsymbol{g}_l^\sharp, \boldsymbol{h}_m^\sharp)^2}{\mathcal{E}_0(\boldsymbol{f}_k^\sharp)^2 \cdot \mathcal{E}_1(\boldsymbol{f}_k^\sharp)^2} \cdot \prod_{q|N} \mathrm{Loc}_q \cdot \frac{L(f_k^\sharp \otimes g_l^\sharp \otimes h_m^\sharp, \frac{k+l+m-2}{2})}{\pi^{2(k-2)} \cdot (f_k^\sharp, f_k^\sharp)_{N_f}^2},$$

where the notations are as follows.

- $\alpha(\boldsymbol{k}, \boldsymbol{l}, \boldsymbol{m}) \in \mathscr{O}_{\boldsymbol{fgh}}$ is a linear form in the variables $\boldsymbol{k}, \boldsymbol{l}$ and $\boldsymbol{m}$ and

(133)
$$\Gamma(k,m,l) = \big((k+l+m-4)/2\big)! \cdot \big((k+l-m-2)/2\big)! \cdot \big((k+m-l-2)/2\big)! \cdot \big((k-l-m)/2\big)!.$$

- Set $c_w = (k+l+m-2)/2$, $\alpha_k = a_p(k)$, $\beta_k = \chi_{\boldsymbol{f}}(p)p^{k-1}/\alpha_k$, $\alpha_l = b_p(l)$ et cetera. Then

(134)
$$\mathcal{E}(\boldsymbol{f}_k^\sharp, \boldsymbol{g}_l^\sharp, \boldsymbol{h}_m^\sharp) = \left(1 - \frac{\beta_k\alpha_l\alpha_m}{p^{c_w}}\right)\left(1 - \frac{\beta_k\beta_l\alpha_m}{p^{c_w}}\right)\left(1 - \frac{\beta_k\alpha_l\beta_m}{p^{c_w}}\right)\left(1 - \frac{\beta_k\beta_l\beta_m}{p^{c_w}}\right),$$

(135)
$$\mathcal{E}_0(\boldsymbol{f}_k^\sharp) = 1 - \frac{\beta_k}{\alpha_k} \quad \text{and} \quad \mathcal{E}_1(\boldsymbol{f}_k^\sharp) = 1 - \frac{\beta_k}{p \cdot \alpha_k}.$$

- For each rational prime q dividing N, Loc_q is an explicit non-zero rational number, independent of w.
- Let $\pi(f_k^\sharp), \pi(g_l^\sharp)$ and $\pi(h_m^\sharp)$ be the cuspidal automorphic representations of GL_2 attached to $f_k^\sharp, g_l^\sharp$ and $h_m^\sharp$ respectively, and set $\Pi_x = \pi(f_k^\sharp) \otimes \pi(g_l^\sharp) \otimes \pi(h_m^\sharp)$. Then
$$L(f_k^\sharp \otimes g_l^\sharp \otimes h_m^\sharp, s) = L(\Pi_w, s + (3-k-l-m)/2).$$
Thanks to the results of Garrett and Harris–Kudla [**Gar87, HK91**] one knows that $L(f_k^\sharp \otimes g_l^\sharp \otimes h_m^\sharp, s)$ admits an analytic continuation to all of $\mathbf{C}$ and satisfies a functional equation with global epsilon factor $\varepsilon(\Pi_x, 1/2)$ equal to $+1$ relating its values at s and $k+l+m-2-s$.

This is proved by Hsieh in Theorem A of [**Hsi20**] relying on the special value formulae of Garrett, Harris–Kudla and Ichino [**Gar87, HK91, Ich08**].

7. Selmer groups and big logarithms

Let $(\boldsymbol{f}^\sharp, \boldsymbol{g}^\sharp, \boldsymbol{h}^\sharp)$ and $(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})$ be as in Section 6.

7.1. A four-variable big logarithm. — Let (cf. Section 5, in particular Equations (97), (102) and (101))
$$M(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})_f = V(\boldsymbol{f})^- \hat{\otimes}_L V(\boldsymbol{g})^+ \hat{\otimes}_L V(\boldsymbol{h})^+ \left(\omega_{\mathrm{cyc}}^{2-l_o-m_o} \cdot \kappa_{\mathrm{cyc}}^{2-\boldsymbol{l}-\boldsymbol{m}}\right).$$
This is a free $\mathscr{O}_{\boldsymbol{fgh}}$-module on which $G_{\mathbf{Q}_p}$ acts via the *unramified* character
$$\Psi : G_{\mathbf{Q}_p} \twoheadrightarrow G_{\mathbf{Q}_p}^{\mathrm{ur}} \longrightarrow \mathscr{O}_{\boldsymbol{fgh}}^*$$

defined by

$$\Psi(\mathrm{Frob}_p) = \frac{\chi_{\boldsymbol{g}}\chi_{\boldsymbol{h}}(p)\cdot a_p(\boldsymbol{k})}{b_p(\boldsymbol{l})\cdot c_p(\boldsymbol{m})} \tag{136}$$

(cf. Equation (101)). Let $\mathscr{O}_{\mathrm{cyc}} \subset \mathbf{Q}_p[\![\boldsymbol{j}-j_o]\!]$ be the ring of bounded analytic functions on an open disc U_{cyc} centred at $j_o = (k_o - l_o - m_o)/2$, and let $\kappa_{\mathrm{cyc}}^{-\boldsymbol{j}} : G_{\mathbf{Q}} \to \mathscr{O}_{\mathrm{cyc}}^*$ be defined by $\kappa_{\mathrm{cyc}}^{-\boldsymbol{j}}(g) = \exp_p(-\boldsymbol{j}\cdot\log_p(\chi_{\mathrm{cyc}}(g)))$. Denote by $\bar{\mathscr{O}}_{\boldsymbol{fgh}}$ the tensor product $\mathscr{O}_{\boldsymbol{fgh}}\hat{\otimes}_{\mathbf{Q}_p}\mathscr{O}_{\mathrm{cyc}}$ and define the $\bar{\mathscr{O}}_{\boldsymbol{fgh}}[G_{\mathbf{Q}_p}]$-module

$$\bar{M}(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})_f = M(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})_f\hat{\otimes}_{\mathbf{Q}_p}\mathscr{O}_{\mathrm{cyc}}\left(\omega_{\mathrm{cyc}}^{-j_o}\cdot\kappa_{\mathrm{cyc}}^{-\boldsymbol{j}}\right). \tag{137}$$

Denote by $\mathcal{Z} = \mathcal{Z}_{\boldsymbol{fgh}}$ the set of integers such that $j \equiv j_o \pmod{p-1}$ and set $\bar{\Sigma} = \Sigma\times\mathcal{Z}$. For all $w = (k,l,m) \in \Sigma$ let $\Psi_w : G_{\mathbf{Q}_p} \to L^*$ be the composition of Ψ with evaluation at w on $\mathscr{O}_{\boldsymbol{fgh}}$ and define $M(\boldsymbol{f}_k,\boldsymbol{g}_l,\boldsymbol{h}_m)_f = M(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})_f\otimes_w L$ as the base change of $M(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})$ under evaluation at x on $\mathscr{O}_{\boldsymbol{fgh}}$, which is isomorphic to $L(\Psi_w)^a$ for some positive integer $a \geqslant 1$. If $x = (w,j) \in \bar{\Sigma}$ then evaluation at x on $\bar{\mathscr{O}}_{\boldsymbol{fgh}}$ induces a natural isomorphism of $L[G_{\mathbf{Q}_p}]$-modules

$$\rho_x : \bar{M}(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})_f\otimes_x L \cong M(\boldsymbol{f}_k,\boldsymbol{g}_l,\boldsymbol{h}_m)_f(-j).$$

If

$$\Lambda_{\boldsymbol{fgh}} = \Lambda_{\boldsymbol{f}}\hat{\otimes}_{\mathscr{O}}\Lambda_{\boldsymbol{g}}\hat{\otimes}_{\mathscr{O}}\Lambda_{\boldsymbol{h}}$$

then

$$M(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})_f = \mathtt{M}(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})_f[1/p]$$

for a $\Lambda_{\boldsymbol{fgh}}[G_{\mathbf{Q}_p}]$-module $\mathtt{M}(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})_f$, free of finite rank over $\Lambda_{\boldsymbol{fgh}}$. Let $\hat{\mathbf{Z}}_p^{\mathrm{nr}} = W(\bar{\mathbf{F}}_p)$ be the ring of Witt vectors of an algebraic closure of $\mathbf{F}_p$ and define

$$D(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})_f = \left(\mathtt{M}(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})_f\hat{\otimes}_{\mathbf{Z}_p}\hat{\mathbf{Z}}_p^{\mathrm{nr}}\right)^{G_{\mathbf{Q}_p}}[1/p]$$

and

$$\bar{D}(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})_f = D(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})_f\hat{\otimes}_{\mathbf{Q}_p}\mathscr{O}_{\mathrm{cyc}}.$$

(Note that $D(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})_f$ is naturally isomorphic to $D(\boldsymbol{f})^-\hat{\otimes}_L D(\boldsymbol{g})^+\hat{\otimes}_L D(\boldsymbol{h})^+$, cf. Equation (124).) As $\mathtt{M}(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})_f$ is unramified and free over $\Lambda_{\boldsymbol{fgh}}$, $D(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})_f$ is a free $\mathscr{O}_{\boldsymbol{fgh}}$-module of the same rank as $M(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})_f$. For all classical triples $w = (k,l,m)$ in Σ the specialisation maps (106) induce a natural isomorphism

$$\rho_w : D(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})_f\otimes_w L \cong D_{\mathrm{cris}}(M(\boldsymbol{f}_k,\boldsymbol{g}_l,\boldsymbol{h}_m)_f)$$

Let t_o denote Fontaine's p-adic analogue of $2\pi i$, which depends on a fixed choice of a compatible sequence ζ_{p^∞} of p^n-th roots of unit for $n \geqslant 0$. The element $t = t_o^{-1}\otimes\zeta_{p^\infty}$ is a canonical generator of $D_{\mathrm{cris}}(\mathbf{Q}_p(1))$, and gives rise to a generator t^i of $D_{\mathrm{cris}}(\mathbf{Q}_p(i))$ for each $i \in \mathbf{Z}$. For any $x = (w,j)$ in $\bar{\Sigma}$ define the isomorphism

$$\rho_x : \bar{D}(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})_f\otimes_x L \cong D_{\mathrm{cris}}\big(M(\boldsymbol{f}_k,\boldsymbol{g}_l,\boldsymbol{h}_m)_f(-j)\big). \tag{138}$$

by the formulae $\rho_x(\alpha\hat{\otimes}\beta) = \beta(j)\cdot\rho_w(\alpha)\otimes t^{-j}$, for each $\alpha \in D(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})_f$ and $\beta \in \mathscr{O}_{\mathrm{cyc}}$.

If $j < 0$ then the Bloch–Kato exponential map gives an isomorphism

$$\exp_x : D_{\mathrm{cris}}\big(M(\boldsymbol{f}_k,\boldsymbol{g}_l,\boldsymbol{h}_m)_f(-j)\big) \cong H^1(\mathbf{Q}_p, M(\boldsymbol{f}_k,\boldsymbol{g}_l,\boldsymbol{h}_m)_f(-j)),$$

and one writes $\log_x$ for its inverse. If $j \geqslant 0$ denote by

$$\exp_x^* : H^1(\mathbf{Q}_p, M(\boldsymbol{f}_k, \boldsymbol{g}_l, \boldsymbol{h}_m)_f(-j)) \longrightarrow D_{\mathrm{cris}}(M(\boldsymbol{f}_k, \boldsymbol{g}_l, \boldsymbol{h}_m)_f(-j))$$

the Bloch–Kato dual exponential map. The following proposition is a consequence of the work of Ochiai [**Och03**] and Loeffler–Zerbes [**LZ14**], which extends previous work of Coleman–Perrin-Riou [**Col79, PR94**] (see also Theorem 8.2.3 of [**KLZ17**]).

Proposition 7.1. — *There exists a unique morphism of $\bar{\mathscr{O}}_{\boldsymbol{fgh}}$-modules*

$$\bar{\mathcal{L}}_{\boldsymbol{f}} : H^1(\mathbf{Q}_p, \bar{M}(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})_f) \longrightarrow \bar{D}(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})_f$$

such that for any $x = (w, j)$ in $\bar{\Sigma}$ with $\Psi_w(\mathrm{Frob}_p) \neq p^{1+j}$ and any $\mathscr{Z}$ in $H^1(\mathbf{Q}_p, \bar{M}(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})_f)$ one has

$$\bar{\mathcal{L}}_{\boldsymbol{f}}(\mathscr{Z})_x = \left(1 - \frac{p^j}{\Psi_w(\mathrm{Frob}_p)}\right)\left(1 - \frac{\Psi_w(\mathrm{Frob}_p)}{p^{1+j}}\right)^{-1} \cdot \begin{cases} \frac{(-1)^{j+1}}{(-j-1)!} \log_x(\mathscr{Z}_x) & \text{if } j < 0 \\ j! \exp_x^*(\mathscr{Z}_x) & \text{if } j \geqslant 0 \end{cases},$$

where $\bar{\mathcal{L}}_{\boldsymbol{f}}(\mathscr{Z})_x$ and $\mathscr{Z}_x$ are shorthands for $\rho_x \circ \bar{\mathcal{L}}_{\boldsymbol{f}}(\mathscr{Z})$ and $\rho_{x*}(\mathscr{Z})$ respectively.

7.1.1. *$\mathscr{O}_{\boldsymbol{fgh}}$-adic differentials.* — Recall the $\mathscr{O}_{\boldsymbol{f}}$-modules $D^*(\boldsymbol{f})^{\pm}$ (resp., $D(\boldsymbol{f})^{\pm}$) introduced in Equations (115) and (120) (resp., Equation (124)), and define similarly $D^*(\boldsymbol{\xi})^{\pm}$ and $D(\boldsymbol{\xi})^{\pm}$ for $\boldsymbol{\xi} = \boldsymbol{g}, \boldsymbol{h}$. Then (cf. Section 7.1)

$$\bar{D}(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})_f = D(\boldsymbol{f})^{-} \hat{\otimes}_L D(\boldsymbol{g})^{+} \hat{\otimes}_L D(\boldsymbol{h})^{+} \hat{\otimes}_{\mathbf{Q}_p} \mathscr{O}_{\mathrm{cyc}},$$

and one defines dually

$$\bar{D}^*(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})_f = D^*(\boldsymbol{f})^{+} \hat{\otimes}_L D^*(\boldsymbol{g})^{-} \hat{\otimes}_L D^*(\boldsymbol{h})^{-} \hat{\otimes}_{\mathbf{Q}_p} \mathscr{O}_{\mathrm{cyc}},$$

so that the perfect dualities $\langle \cdot, \cdot \rangle_{\boldsymbol{\xi}}$, for $\boldsymbol{\xi} = \boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h}$ (cf. Equation (125)) yield a pairing

$$\langle \cdot, \cdot \rangle_{\boldsymbol{fgh}} : \bar{D}(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})_f \otimes_{\bar{\mathscr{O}}_{\boldsymbol{fgh}}} \bar{D}^*(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})_f \longrightarrow \bar{\mathscr{O}}_{\boldsymbol{fgh}}. \tag{139}$$

Moreover, identifying $D_{\mathrm{cris}}(\mathbf{Q}_p(i)) = \mathbf{Q}_p \cdot t^i$ with $\mathbf{Q}_p$ ($i \in \mathbf{Z}$), the isomorphisms (116), (121), (126) (and their analogues for $\boldsymbol{g}$ and $\boldsymbol{h}$) give specialisation isomorphisms

$$\rho_x : \bar{D}(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})_f \otimes_x L \cong \mathrm{Fil}^0 V_{\mathrm{dR}}(\boldsymbol{f}_k) \otimes_L V_{\mathrm{dR}}(\boldsymbol{g}_l)/\mathrm{Fil}^0 \otimes_L V_{\mathrm{dR}}(\boldsymbol{h}_m)/\mathrm{Fil}^0 \tag{140}$$

and

$$\rho_x : \bar{D}^*(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})_f \otimes_x L \cong V_{\mathrm{dR}}^*(\boldsymbol{f}_k)/\mathrm{Fil}^1 \otimes_L \mathrm{Fil}^1 V_{\mathrm{dR}}^*(\boldsymbol{g}_l) \otimes_L \mathrm{Fil}^1 V_{\mathrm{dR}}^*(\boldsymbol{h}_m), \tag{141}$$

for each classical 4-tuple $x = (k, l, m, j)$ in $\bar{\Sigma}$ with $k, l, m \geqslant 2$.

Define the $\bar{\mathscr{O}}_{\boldsymbol{fgh}}$*-adic differential* (cf. Equations (118) and (122))

$$\eta_{\boldsymbol{f}} \omega_{\boldsymbol{g}} \omega_{\boldsymbol{h}} = \eta_{\boldsymbol{f}} \otimes \omega_{\boldsymbol{g}} \otimes \omega_{\boldsymbol{h}} \otimes 1 \in \bar{D}^*(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})_f. \tag{142}$$

According to Equation (119), Equation (123), and the discussion following Equation (126), for each $x = (k, l, m, j) \in \bar{\Sigma}$ with $k, l, m \geqslant 2$ and each $\boldsymbol{\mu}$ in $\bar{D}(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})_f$ one has

$$\langle \boldsymbol{\mu}, \eta_{\boldsymbol{f}} \omega_{\boldsymbol{g}} \omega_{\boldsymbol{h}} \rangle_{\boldsymbol{fgh}}(x) = (p-1) a_p(k) \cdot \langle \rho_x(\boldsymbol{\mu}), \eta_{\boldsymbol{f}_k} \otimes \omega_{\boldsymbol{g}_l} \otimes \omega_{\boldsymbol{h}_m} \rangle_{\boldsymbol{f}_k \boldsymbol{g}_l \boldsymbol{h}_m}, \tag{143}$$

where $\langle \cdot, \cdot \rangle_{\boldsymbol{f}_k \boldsymbol{g}_l \boldsymbol{h}_m}$ is the product of the perfect dualities $\langle \cdot, \cdot \rangle_{\xi}$ introduced in Equation (32), for ξ equal to $\boldsymbol{f}_k$, $\boldsymbol{g}_l$ and $\boldsymbol{h}_m$.

Define the *four-variable $\boldsymbol{f}$-big logarithm*

$$\bar{\mathscr{L}}_{\boldsymbol{f}} = \bar{\mathscr{L}}og(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h}) \stackrel{\text{def}}{=} \left\langle \bar{\mathcal{L}}_{\boldsymbol{f}}(\cdot), \eta_{\boldsymbol{f}}\omega_{\boldsymbol{g}}\omega_{\boldsymbol{h}} \right\rangle_{\boldsymbol{fgh}} : H^1(\mathbf{Q}_p, \bar{M}(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})_f) \longrightarrow \bar{\mathscr{O}}_{\boldsymbol{fgh}} \tag{144}$$

to be the composition of $\bar{\mathcal{L}}_{\boldsymbol{f}}$ with the linear form $\langle \cdot, \eta_{\boldsymbol{f}}\omega_{\boldsymbol{g}}\omega_{\boldsymbol{h}}\rangle_{\boldsymbol{fgh}}$ on $\bar{D}(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})_f$.

Mutatis mutandis the previous constructions apply after replacing $\boldsymbol{f}$ with $\boldsymbol{a} = \boldsymbol{g},\boldsymbol{h}$. One obtains four-variable *$\boldsymbol{a}$-big logarithms* $\bar{\mathscr{L}}_{\boldsymbol{a}} : H^1(\mathbf{Q}_p, \bar{M}(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})_a) \longrightarrow \bar{\mathscr{O}}_{\boldsymbol{fgh}}$.

7.1.1.1. *Weight-one specialisations.* — With the notations introduced in the last part of Section 5 (cf. Equations (127)–(129)), the isomorphisms (140) and (141) and the definition of the pairing $\langle \cdot,\cdot\rangle_{\boldsymbol{f}_k\boldsymbol{g}_l\boldsymbol{h}_m}$ extend to all classical 4-tuples $x = (k,l,m,j)$ in $\bar{\Sigma}$, independently on whether the weights k,l and m are geometric or not (id est equal to 1). Moreover, if $k \geqslant 2$, Equation (143) still holds.

7.2. The balanced Selmer group.

— Define the continuous character

$$\Xi_{\boldsymbol{fgh}} : G_{\mathbf{Q}} \longrightarrow \mathscr{O}^*_{\boldsymbol{fgh}}$$

by the formula

$$\Xi_{\boldsymbol{fgh}}(g) = \omega_{\text{cyc}}(g)^{(4-k_o-l_o-m_o)/2} \cdot \kappa_{\text{cyc}}(g)^{(4-\boldsymbol{k}-\boldsymbol{l}-\boldsymbol{m})/2},$$

for every g in $G_{\mathbf{Q}}$, and the $\mathscr{O}_{\boldsymbol{fgh}}[G_{\mathbf{Q}}]$-representation

$$V(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h}) = V(\boldsymbol{f})\hat{\otimes}_L V(\boldsymbol{g})\hat{\otimes}_L V(\boldsymbol{h}) \otimes_{\mathscr{O}_{\boldsymbol{fgh}}} \Xi_{\boldsymbol{fgh}}.$$

Equations (103) and (114) imply that $V(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})$ is Kummer self-dual: the product of the perfect dualities $[\cdot,\cdot]_{\boldsymbol{\xi}} : V(\boldsymbol{\xi}) \otimes_{\mathscr{O}_{\boldsymbol{\xi}}} V(\boldsymbol{\xi}) \longrightarrow \mathscr{O}_{\boldsymbol{\xi}}(1 + \boldsymbol{\kappa}_{U_{\boldsymbol{\xi}}} + \chi_{\boldsymbol{\xi}})$ defined by $[x,y]_{\boldsymbol{\xi}} = \langle x, w_{Np}^{-1}(y)\rangle_{\boldsymbol{\xi}}$ yields a perfect, skew-symmetric duality (cf. Assumption 1.2)

$$[\cdot,\cdot]_{\boldsymbol{fgh}} : V(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h}) \otimes_{\mathscr{O}_{\boldsymbol{fgh}}} V(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h}) \longrightarrow \mathscr{O}_{\boldsymbol{fgh}}(1),$$

whose adjoint identifies $V(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})$ with its own Kummer dual. Moreover, for all $w = (k,l,m)$ in Σ the specialisation maps (106) induce isomorphisms

$$\rho_w : V(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h}) \otimes_w L \cong V(\boldsymbol{f}_k,\boldsymbol{g}_l,\boldsymbol{h}_m) \tag{145}$$

(cf. Equation (47)), where $\cdot \otimes_w L$ denotes the base change under evaluation at w.

Define a decreasing filtration $\mathscr{F}^{\cdot}V(\boldsymbol{f})$ on $V(\boldsymbol{f})$ by $\mathscr{F}^jV(\boldsymbol{f}) = V(\boldsymbol{f})$ for every $j \leqslant 0$, $\mathscr{F}^1V(\boldsymbol{f}) = V(\boldsymbol{f})^+$ and $\mathscr{F}^jV(\boldsymbol{f}) = 0$ for $j \geqslant 2$, and similarly $\mathscr{F}^{\cdot}V(\boldsymbol{g})$ and $\mathscr{F}^{\cdot}V(\boldsymbol{h})$. Let $\mathscr{F}^{\cdot}V(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})$ be the tensor product filtration:

$$\mathscr{F}^nV(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h}) = \left[\sum_{p+q+r=n} \mathscr{F}^pV(\boldsymbol{f})\hat{\otimes}_L\mathscr{F}^qV(\boldsymbol{g})\hat{\otimes}_L\mathscr{F}^rV(\boldsymbol{h})\right]\otimes_{\mathscr{O}_{\boldsymbol{fgh}}} \Xi_{\boldsymbol{fgh}}.$$

This is a decreasing filtration of $V(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})$ by $\mathscr{O}_{\boldsymbol{fgh}}[G_{\mathbf{Q}_p}]$-submodules, satisfying $\mathscr{F}^4V(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h}) = 0$ and $\mathscr{F}^0V(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h}) = V(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})$. The annihilator of $\mathscr{F}^iV(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})$ under the duality $[\cdot,\cdot]_{\boldsymbol{fgh}}$ is equal to $\mathscr{F}^{4-i}V(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})$, hence the adjoint of $[\cdot,\cdot]_{\boldsymbol{fgh}}$ induces isomorphisms of $\mathscr{O}_{\boldsymbol{fgh}}[G_{\mathbf{Q}_p}]$-modules

$$\text{gr}^iV(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h}) \cong \text{Hom}_{\mathscr{O}_{\boldsymbol{fgh}}}(\text{gr}^{3-i}V(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h}), \mathscr{O}_{\boldsymbol{fgh}}(1)) \tag{146}$$

(where $\text{gr}^iV(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h}) = \mathscr{F}^iV(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})/\mathscr{F}^{i+1}$). If one sets

$$V(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})_f = V(\boldsymbol{f})^-\hat{\otimes}_LV(\boldsymbol{g})^+\hat{\otimes}_LV(\boldsymbol{h})^+ \otimes_{\mathscr{O}_{\boldsymbol{fgh}}} \Xi_{\boldsymbol{fgh}},$$

and defines similarly $V(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})_g$ and $V(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})_h$, then

$$\mathrm{gr}^2V(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h}) = V(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})_f \oplus V(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})_g \oplus V(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})_h \tag{147}$$

as $\mathscr{O}_{\boldsymbol{fgh}}[G_{\mathbf{Q}_p}]$-modules. It follows form Equation (146) and the definitions that the inertia subgroup $I_{\mathbf{Q}_p(\mu_p)}$ of the absolute Galois group of $\mathbf{Q}(\mu_p)$ acts on $\mathrm{gr}^3V(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})$ and $\mathrm{gr}^0V(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})$ via the characters $\kappa_{\mathrm{cyc}}^{(\boldsymbol{k}+\boldsymbol{l}+\boldsymbol{m}-2)/2}$ and $\kappa_{\mathrm{cyc}}^{(4-\boldsymbol{k}-\boldsymbol{l}-\boldsymbol{m})/2}$ respectively. In addition, Equations (146) and (147) show that $\mathrm{gr}^2V(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})$ and $\mathrm{gr}^1V(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})$ are isomorphic respectively to the direct sum of a finite number of copies of

$$\kappa_{\mathrm{cyc}}^{\frac{\boldsymbol{l}+\boldsymbol{m}-\boldsymbol{k}}{2}} \oplus \kappa_{\mathrm{cyc}}^{\frac{\boldsymbol{l}+\boldsymbol{k}-\boldsymbol{m}}{2}} \oplus \kappa_{\mathrm{cyc}}^{\frac{\boldsymbol{k}+\boldsymbol{m}-\boldsymbol{l}}{2}} \quad \text{and} \quad \kappa_{\mathrm{cyc}}^{\frac{\boldsymbol{k}-\boldsymbol{l}-\boldsymbol{m}+2}{2}} \oplus \kappa_{\mathrm{cyc}}^{\frac{\boldsymbol{m}-\boldsymbol{l}-\boldsymbol{k}+2}{2}} \oplus \kappa_{\mathrm{cyc}}^{\frac{\boldsymbol{l}-\boldsymbol{k}-\boldsymbol{m}+2}{2}}$$

as $I_{\mathbf{Q}(\mu_p)}$-modules (where $\kappa_{\mathrm{cyc}}^{\bullet} = \mathscr{O}_{\boldsymbol{fgh}}(\kappa_{\mathrm{cyc}}^{\bullet})$). In particular, for each $i \in \mathbf{Z}$ one has

$$H^0(\mathbf{Q}_p, \mathrm{gr}^iV(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})) = 0. \tag{148}$$

Define the *balanced local condition*

$$H^1_{\mathrm{bal}}(\mathbf{Q}_p, V(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})) = H^1(\mathbf{Q}_p, \mathscr{F}^2V(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})).$$

In light of Equation (148), the morphism induced on the first $G_{\mathbf{Q}_p}$-cohomology groups by the inclusion $\mathscr{F}^2V(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h}) \hookrightarrow V(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})$ is injective, hence we can, and will, identify the balanced local condition with a submodule of $H^1(\mathbf{Q}_p, V(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h}))$, namely

$$H^1_{\mathrm{bal}}(\mathbf{Q}_p, V(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})) = \mathrm{Im}\big(H^1(\mathbf{Q}_p, \mathscr{F}^2V(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})) \longrightarrow H^1(\mathbf{Q}_p, V(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h}))\big).$$

For $\cdot = f, g, h$, one denotes by $p_\cdot$ both the natural $G_{\mathbf{Q}_p}$-equivariant projection

$$p_\cdot : \mathscr{F}^2V(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h}) \twoheadrightarrow V(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})_\cdot$$

arising from Equation (147) and the morphism

$$p_\cdot : H^1_{\mathrm{bal}}(\mathbf{Q}_p, V(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})) \longrightarrow H^1(\mathbf{Q}_p, V(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})_\cdot)$$

it induces in cohomology.

For all morphisms of L-algebras $\varphi : \mathscr{O}_{\boldsymbol{fgh}} \longrightarrow \mathscr{O}_\varphi$, set

$$V_\varphi(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})_\cdot = V(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})_\cdot \otimes_\varphi \mathscr{O}_\varphi \quad \text{and} \quad \mathscr{F}^\cdot V_\varphi(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h}) = \mathscr{F}^\cdot V(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h}) \otimes_\varphi \mathscr{O}_\varphi,$$

denote again by by $p_\cdot : V_\varphi(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h}) \twoheadrightarrow V_\varphi(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})_\cdot$ the natural projections, and define

$$H^1_{\mathrm{bal}}(\mathbf{Q}_p, V_\varphi(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})) = \mathrm{Im}\big(H^1(\mathbf{Q}_p, \mathscr{F}^2V_\varphi(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})) \longrightarrow H^1(\mathbf{Q}_p, V_\varphi(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h}))\big).$$

If $w = (k,l,m)$ is a triple in Σ and φ is evaluation at w, we identify $V_\varphi(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})$ with $V(\boldsymbol{f}_k,\boldsymbol{g}_l,\boldsymbol{h}_m)$ under the specialisation isomorphism ρ_w (cf. Equation (145)).

One has the following crucial lemma.

Lemma 7.2. — *If $w = (k,m,l) \in \Sigma_{\mathrm{bal}}$ is a* balanced *classical triple, then*

$$H^1_{\mathrm{bal}}(\mathbf{Q}_p, V(\boldsymbol{f}_k,\boldsymbol{g}_l,\boldsymbol{h}_m)) = H^1_{\mathrm{fin}}(\mathbf{Q}_p, V(\boldsymbol{f}_k,\boldsymbol{g}_l,\boldsymbol{h}_m)), \tag{149}$$

where $H^1_{\mathrm{fin}}(\mathbf{Q}_p,\cdot)$ is the Bloch–Kato finite local condition (cf. Lemma 3.5). As a consequence, the Bloch–Kato exponential map gives an isomorphism

$$\exp_p : V_{\mathrm{dR}}(\boldsymbol{f}_k,\boldsymbol{g}_l,\boldsymbol{h}_m)/\mathrm{Fil}^0 \cong H^1_{\mathrm{bal}}(\mathbf{Q}_p, V(\boldsymbol{f}_k,\boldsymbol{g}_l,\boldsymbol{h}_m)).$$

Proof. — Set $V = V(\boldsymbol{f}_k, \boldsymbol{g}_l, \boldsymbol{h}_m)$, and consider the exact sequence of $G_{\mathbf{Q}_p}$-modules

$$0 \longrightarrow \mathscr{F}^2V \longrightarrow V \longrightarrow V/\mathscr{F}^2 \longrightarrow 0.$$

The discussion preceding Equation (148) shows that $\mathscr{F}^2V$ has Hodge–Tate weights

$$\frac{k+l+m-2}{2}, \quad \frac{k+l-m}{2}, \quad \frac{k+m-l}{2} \quad \text{and} \quad \frac{l+m-k}{2},$$

while $V/\mathscr{F}^2$ has Hodge–Tate weights

$$\frac{k-l-m+2}{2}, \quad \frac{l-k-m+2}{2}, \quad \frac{m-k-l+2}{2} \quad \text{and} \quad \frac{4-k-l-m}{2}.$$

Since w is a balanced classical triple, it follows that all the Hodge–Tate weights of $\mathscr{F}^2V$ (resp., $V/\mathscr{F}^2$) are positive (resp., non-positive), hence

$$\mathrm{tg}_{\mathrm{dR}}(\mathscr{F}^2V) = D_{\mathrm{dR}}(\mathscr{F}^2V) \quad \text{and} \quad \mathrm{Fil}^0 D_{\mathrm{dR}}(V/\mathscr{F}^2) = D_{\mathrm{dR}}(V/\mathscr{F}^2) \tag{150}$$

(where $\mathrm{tg}_{\mathrm{dR}}(\cdot) = D_{\mathrm{dR}}(\cdot)/\mathrm{Fil}^0$). The second equality implies that $H^1_{\mathrm{exp}}(\mathbf{Q}_p, V/\mathscr{F}^2)$ vanishes (cf. Corollary 3.8.4 of [**BK90**]), and since $\mathscr{F}^2V$ is isomorphic to the Kummer dual of $V/\mathscr{F}^2$, this in turn implies $H^1(\mathbf{Q}_p, \mathscr{F}^2V) = H^1_{\mathrm{geo}}(\mathbf{Q}_p, \mathscr{F}^2V)$ (cf. Proposition 3.8 of [**BK90**]). As $H^1_{\mathrm{fin}}(\mathbf{Q}_p, V) = H^1_{\mathrm{geo}}(\mathbf{Q}_p, V)$ by Lemma 3.5, one deduces that $H^1_{\mathrm{fin}}(\mathbf{Q}_p, V)$ contains the balanced subspace $H^1_{\mathrm{bal}}(\mathbf{Q}_p, V)$. On the other hand, Equation (150) shows that the inclusion $\mathscr{F}^2V \hookrightarrow V$ induces an isomorphism between the tangent space of $\mathscr{F}^2V$ and that of V. It follows that $H^1_{\mathrm{exp}}(\mathbf{Q}_p, V)$ is contained in the image of $H^1_{\mathrm{exp}}(\mathbf{Q}_p, \mathscr{F}^2V)$, hence a fortiori in the balanced subspace $H^1_{\mathrm{bal}}(\mathbf{Q}_p, V)$. Since $H^1_{\mathrm{exp}}(\mathbf{Q}_p, V) = H^1_{\mathrm{fin}}(\mathbf{Q}_p, V)$ by Lemma 3.5, this concludes the proof of the first statement. The second statement follows from the first and Lemma 3.5. □

7.3. The three-variable big logarithms. — Let $w = (k, l, m)$ be a classical triple in Σ. If $w \in \Sigma_{\mathrm{bal}}$ is *balanced*, then the differential $\eta^{\alpha}_{\boldsymbol{f}_k} \otimes \omega_{\boldsymbol{g}_l} \otimes \omega_{\boldsymbol{h}_m}$ belongs to $\mathrm{Fil}^0 V^*_{\mathrm{dR}}(\boldsymbol{f}_k, \boldsymbol{g}_l, \boldsymbol{h}_m)$ by Equation (53). In this case denote by

$$\log_p : H^1_{\mathrm{bal}}(\mathbf{Q}_p, V(\boldsymbol{f}_k, \boldsymbol{g}_l, \boldsymbol{h}_m)) \cong V_{\mathrm{dR}}(\boldsymbol{f}_k, \boldsymbol{g}_l, \boldsymbol{h}_m)/\mathrm{Fil}^0$$

the inverse of the Bloch–Kato exponential (cf. Lemma 7.2), and define

$$\log_p(\cdot)_f = \log_p(\cdot)\big(\eta^{\alpha}_{\boldsymbol{f}_k} \otimes \omega_{\boldsymbol{g}_l} \otimes \omega_{\boldsymbol{h}_m}\big) : H^1_{\mathrm{bal}}(\mathbf{Q}_p, V(\boldsymbol{f}_k, \boldsymbol{g}_l, \boldsymbol{h}_m)) \longrightarrow L$$

to be the composition of $\log_p$ with evaluation on $\eta^{\alpha}_{\boldsymbol{f}_k} \otimes \omega_{\boldsymbol{g}_l} \otimes \omega_{\boldsymbol{h}_m}$. Here one identifies $V_{\mathrm{dR}}(\boldsymbol{f}_k, \boldsymbol{g}_l, \boldsymbol{h}_m)/\mathrm{Fil}^0$ with the dual of $\mathrm{Fil}^0 V^*_{\mathrm{dR}}(\boldsymbol{f}_k, \boldsymbol{g}_l, \boldsymbol{h}_m)$ under the product of the perfect dualities $\langle \cdot, \cdot \rangle_{\boldsymbol{\xi}_u}$ introduced in Equation (31), for $\boldsymbol{\xi}_u = \boldsymbol{f}_k, \boldsymbol{g}_l, \boldsymbol{h}_m$.

If w belongs to $\Sigma_{\boldsymbol{f}}$ denote by

$$\exp^*_p : H^1(\mathbf{Q}_p, V(\boldsymbol{f}_k, \boldsymbol{g}_l, \boldsymbol{h}_m)) \longrightarrow \mathrm{Fil}^0 V_{\mathrm{dR}}(\boldsymbol{f}_k, \boldsymbol{g}_l, \boldsymbol{h}_m)$$

the Bloch–Kato dual exponential map, and by

$$\exp^*_p(\cdot)_f = \exp^*_p(\cdot)\big(\eta^{\alpha}_{\boldsymbol{f}_k} \otimes \omega_{\boldsymbol{g}_l} \otimes \omega_{\boldsymbol{h}_m}\big) : H^1(\mathbf{Q}_p, V(\boldsymbol{f}_k, \boldsymbol{g}_l, \boldsymbol{h}_m)) \longrightarrow L$$

its composition with evaluation at $\eta^{\alpha}_{\boldsymbol{f}_k} \otimes \omega_{\boldsymbol{g}_l} \otimes \omega_{\boldsymbol{h}_m}$. As above, here one identifies $\mathrm{Fil}^0 V_{\mathrm{dR}}(\boldsymbol{f}_k, \boldsymbol{g}_l, \boldsymbol{h}_m)$ with a subspace of the dual of $V^*_{\mathrm{dR}}(\boldsymbol{f}_k, \boldsymbol{g}_l, \boldsymbol{h}_m)$ under the tensor product of the pairings $\langle \cdot, \cdot \rangle_{\boldsymbol{\xi}_u}$ defined in (31) and (128). (If either l or m is equal to

1, the definitions of $V_{\mathrm{dR}}(\boldsymbol{f}_k, \boldsymbol{g}_l, \boldsymbol{h}_m)$ and $V^*_{\mathrm{dR}}(\boldsymbol{f}_k, \boldsymbol{g}_l, \boldsymbol{h}_m)$ given in Equations (50) and (51) are understood in light of the conventions of Section 5, cf. Equation (127).)

To ease notation set $\alpha_k = a_p(k), \beta_k = \chi_{\boldsymbol{f}}(p)p^{k-1}/\alpha_k, \alpha_l = b_p(l)$ et cetera. Recall that for each classical triple $w = (k, l, m)$ in Σ one writes $c_w = (k + l + m - 2)/2$ (which belongs to $\mathbf{N}$ by Assumption 1.2).

Proposition 7.3. — *There is a unique morphism of $\mathscr{O}_{\boldsymbol{fgh}}$-modules*

$$\mathscr{L}_{\boldsymbol{f}} = \mathscr{L}og(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h}) : H^1_{\mathrm{bal}}(\mathbf{Q}_p, V(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})) \longrightarrow \mathscr{O}_{\boldsymbol{fgh}}$$

such that, for all $w = (k, l, m) \in \Sigma$ with $\alpha_k\beta_l\beta_m \neq p^{c_w}$ and $\mathfrak{Z} \in H^1_{\mathrm{bal}}(\mathbf{Q}_p, V(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h}))$

$$\mathscr{L}_{\boldsymbol{f}}(\mathfrak{Z})(w) = (p-1)\alpha_k \cdot \frac{\left(1 - \frac{\beta_k\alpha_l\alpha_m}{p^{c_w}}\right)}{\left(1 - \frac{\alpha_k\beta_l\beta_m}{p^{c_w}}\right)} \cdot \begin{cases} \frac{(-1)^{c_w-k}}{(c_w-k)!}\log_p(\mathfrak{Z}_w)_f & \text{if } \ w \in \Sigma_{\mathrm{bal}} \\ (k - c_w - 1)!\exp^*_p(\mathfrak{Z}_w)_f & \text{if } \ w \in \Sigma_{\boldsymbol{f}} \end{cases},$$

where $\mathfrak{Z}_w = \rho_{w*}(\mathfrak{Z})$. Moreover $\mathscr{L}_{\boldsymbol{f}}$ factors through

$$p_{f*} : H^1_{\mathrm{bal}}(\mathbf{Q}_p, V(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})) \to H^1(\mathbf{Q}_p, V(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})_f).$$

Proof. — Set $\bar{M}_f = \bar{M}(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})_f$, $V = V(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})$ and $V_f = V(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})_f$. Let

$$\vartheta : \bar{\mathscr{O}}_{\boldsymbol{fgh}} \longrightarrow \mathscr{O}_{\boldsymbol{fgh}}$$

be the surjective morphism of L-algebras which sends the analytic function $F(\boldsymbol{k}, \boldsymbol{l}, \boldsymbol{m}, \boldsymbol{j})$ to its restriction $F(\boldsymbol{k}, \boldsymbol{l}, \boldsymbol{m}, (\boldsymbol{k} - \boldsymbol{l} - \boldsymbol{m})/2)$ to the hyperplane defined by the equation $2\boldsymbol{j} = \boldsymbol{k} - \boldsymbol{l} - \boldsymbol{m}$. (Here we implicitly shrink the discs $U_{\boldsymbol{f}}, U_{\boldsymbol{g}}$ and $U_{\boldsymbol{h}}$ if necessary, in order to guarantee that $(\boldsymbol{k} - \boldsymbol{l} - \boldsymbol{m})/2$ takes values in the disc U_{cyc} fixed in Section 7.1.) Unwinding the definitions one finds that ϑ induces an isomorphism of $\mathscr{O}_{\boldsymbol{fgh}}[G_{\mathbf{Q}_p}]$-modules (denoted by the same symbol)

$$\vartheta : \bar{M}_f \otimes_\vartheta \mathscr{O}_{\boldsymbol{fgh}} \cong V_f. \tag{151}$$

We claim that this map entails an isomorphism

$$\vartheta_* : H^1(\mathbf{Q}_p, \bar{M}_f) \otimes_\vartheta \mathscr{O}_{\boldsymbol{fgh}} \cong H^1(\mathbf{Q}_p, V_f). \tag{152}$$

Granting this, one can define $\mathscr{L}_{\boldsymbol{f}}$ by the composition

$$\begin{aligned} \mathscr{L}_{\boldsymbol{f}} : H^1_{\mathrm{bal}}(\mathbf{Q}_p, V(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})) &\xrightarrow{p_{f*}} H^1(\mathbf{Q}_p, V(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})_f) \\ &\xrightarrow{\vartheta_*^{-1}} H^1(\mathbf{Q}_p, \bar{M}(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})_f) \otimes_\vartheta \mathscr{O}_{\boldsymbol{fgh}} \xrightarrow{\bar{\mathscr{L}}_{\boldsymbol{f}} \otimes \mathrm{id}} \mathscr{O}_{\boldsymbol{fgh}}, \end{aligned}$$

where $\bar{\mathscr{L}}_{\boldsymbol{f}}$ is the four-variable $\boldsymbol{f}$-big logarithm defined in Equation (144). Unravelling the definitions, one checks that the interpolation property satisfied by $\mathscr{L}_{\boldsymbol{f}}$ is a direct consequence of Proposition 7.1. It then remains to prove the claim (152).

As $\bar{M}_f$ is a free module over the domain $\bar{\mathscr{O}}_{\boldsymbol{fgh}}$, the claim (152) is equivalent to the vanishing of the $(2\boldsymbol{j} - \boldsymbol{k} + \boldsymbol{l} + \boldsymbol{m})$-torsion submodule of $H^2(\mathbf{Q}_p, \bar{M}_f)$. Set

$$\bar{\mathtt{\Lambda}} = \Lambda_{\boldsymbol{fgh}} \hat{\otimes}_{\mathbf{Z}_p} \Lambda_{\mathrm{cyc}},$$

where Λ_{cyc} is the $\mathbf{Z}_p$-module of functions in $\mathscr{O}_{\mathrm{cyc}}$ bounded by one. The $\mathscr{O}$-algebra $\bar{\Lambda}_{\boldsymbol{fgh}}$ is isomorphic to a power series ring in four variables with coefficients in $\mathscr{O}$. In particular, it is a regular local complete Noetherian ring with finite residue field (hence a UFD). Write $\bar{M}_f = \bar{\mathtt{M}}_f[1/p]$ for a $\bar{\mathtt{\Lambda}}[G_{\mathbf{Q}_p}]$-module $\bar{\mathtt{M}}_f$ free of finite rank over

$\bar{\Lambda}$. For every discrete or compact $\bar{\Lambda}$-module $\cdot$ write $\mathscr{D}(\cdot) = \mathrm{Hom}_{\mathrm{cont}}(\cdot, \mathbf{Q}_p/\mathbf{Z}_p)$ for its Pontrjagin dual. According to the local Tate duality and the Pontrjagin duality

$$H^2(\mathbf{Q}_p, \bar{M}_f)[2\boldsymbol{j} - \boldsymbol{k} + \boldsymbol{l} + \boldsymbol{m}] = \mathscr{D}\Big(\mathscr{D}(\bar{\mathtt{M}}_f(-1))^{G_{\mathbf{Q}_p}}/(2\boldsymbol{j} - \boldsymbol{k} + \boldsymbol{l} + \boldsymbol{m})\Big)[1/p]. \tag{153}$$

Let Frob_p be the arithmetic Frobenius in $G_p^{\mathrm{nr}} = \mathrm{Gal}(\mathbf{Q}_p^{\mathrm{nr}}/\mathbf{Q}_p)$ and let γ be a topological generator of $G_p^{\mathrm{tr}} = \mathrm{Gal}(\mathbf{Q}_p(\mu_{p^\infty})/\mathbf{Q}_p)$ (recall that p is odd). By construction (after identifying $G_{\mathbf{Q}_p}^{\mathrm{ab}}$ with the product of G_p^{nr} and G_p^{tr}) Frob_p acts on $\bar{\mathtt{M}}_f$ as multiplication by $\Psi_{\boldsymbol{o}} = \Psi(\mathrm{Frob}_p)$ and γ acts on $\bar{\mathtt{M}}_f(-1)$ as multiplication by the inverse of $\Gamma_o = \omega_o^{1+j_o} \cdot \gamma_o^{1+\boldsymbol{j}}$, where $\omega_o = \omega_{\mathrm{cyc}}(\gamma)$ and $\gamma_o = \kappa_{\mathrm{cyc}}(\gamma)$. This yields

$$\mathscr{D}(\bar{\mathtt{M}}_f(-1))^{G_{\mathbf{Q}_p}}/(2\boldsymbol{j} - \boldsymbol{k} + \boldsymbol{l} + \boldsymbol{m}) \cong \bigoplus_{i=0}^{a} \mathscr{D}\left(\frac{\bar{\Lambda}}{(\Psi_o - 1, \Gamma_o - 1)}[2\boldsymbol{j} - \boldsymbol{k} + \boldsymbol{l} + \boldsymbol{m}]\right)$$

for some positive integer a (cf. Equation (137)). We prove that the module

$$\frac{\bar{\Lambda}}{(\Psi_o - 1, \Gamma_o - 1)}[2\boldsymbol{j} - \boldsymbol{k} + \boldsymbol{l} + \boldsymbol{m}]$$

is killed by a power of p, which together with Equation (153) proves the claim (152). If $j_o \neq -1$, the function $\Gamma_o - 1$ is a unit in $\Lambda_{\mathrm{cyc}}[1/p]$, hence $\bar{\Lambda}/(\Psi_o - 1, \Gamma_o - 1)$ is killed by a power of p. Assume then $j_o = -1$ and let $F = F(\boldsymbol{w}, \boldsymbol{j})$ be an element of $\bar{\Lambda}$ whose image in $\bar{\Lambda}/(\Psi_o - 1, \Gamma_o - 1)$ is killed by $2\boldsymbol{j} - \boldsymbol{k} + \boldsymbol{l} + \boldsymbol{m}$. This implies that

$$(\boldsymbol{l} + \boldsymbol{m} - \boldsymbol{k} - 2) \cdot F(\boldsymbol{w}, -1) = (\Psi_o(\boldsymbol{w}) - 1) \cdot G(\boldsymbol{w})$$

for some $G(\boldsymbol{w})$ in $\Lambda_{\boldsymbol{fgh}}$. As $j_o = -1$ there is a classical triple $w = (k, l, m) \in \Sigma$ such that $l + m - k - 2 = 0$ and such that p does not divide the conductor of $\boldsymbol{f}_k, \boldsymbol{g}_l$ and $\boldsymbol{h}_m$. According to the Ramanujan–Petersson conjecture the inverse of $\Psi_o(w)$ has complex absolute value $\sqrt{p}$ for every such w (cf. Equation (136)). As a consequence $\boldsymbol{l} + \boldsymbol{m} - \boldsymbol{k} - 2$ is not an irreducible factor of $\Psi_o - 1$, hence the latter divides $F(\boldsymbol{w}, -1)$ by the previous equation. This proves that F belongs to the ideal generated by $\Psi_o - 1$ and $\boldsymbol{j} + 1$. As $(\Gamma_o - 1)/(1 + \boldsymbol{j})$ is a unit in $\Lambda_{\mathrm{cyc}}[1/p]$, it follows that $p^{N(\gamma_o)} \cdot F$ maps to zero in $\bar{\Lambda}/(\Psi_o - 1, \Gamma_o - 1)$ for a non-negative integer $N(\gamma_o)$ independent of F, as was to be shown. □

We call $\mathscr{L}_{\boldsymbol{f}}$ the *three variable $\boldsymbol{f}$-big logarithm.* Mutatis mutandis, for $\boldsymbol{a} = \boldsymbol{g}, \boldsymbol{h}$ one defines *$\boldsymbol{a}$-big logarithms*

$$\mathscr{L}_{\boldsymbol{a}} : H^1_{\mathrm{bal}}(\mathbf{Q}_p, V(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})) \longrightarrow \mathscr{O}_{\boldsymbol{fgh}},$$

which factor through $p_{a*} : H^1_{\mathrm{bal}}(\mathbf{Q}_p, V(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})) \longrightarrow H^1(\mathbf{Q}_p, V(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})_a)$ and satisfy similar interpolation properties.

8. Proof of Theorem A

This section proves Theorem A stated in the Introduction.

8.1. Construction of $\kappa(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})$. — Fix a nonnegative integer $\imath \geqslant 1$, which will be made sufficiently large below. For $\boldsymbol{\xi} = \boldsymbol{f},\boldsymbol{g},\boldsymbol{h}$ and $\cdot = \emptyset, \prime$ set $\mathcal{A}^{\cdot}_{\boldsymbol{\xi}} = \mathcal{A}^{\cdot}_{U_{\boldsymbol{\xi}},\imath}$, $\boldsymbol{\mathcal{A}}^{\cdot}_{\boldsymbol{\xi}} = \boldsymbol{\mathcal{A}}^{\cdot}_{U_{\boldsymbol{\xi}},\imath}$ $\mathcal{D}^{\cdot}_{\boldsymbol{\xi}} = \mathcal{D}^{\cdot}_{U_{\boldsymbol{f}},\imath}$ and $\boldsymbol{\mathcal{D}}^{\cdot}_{\boldsymbol{\xi}} = \boldsymbol{\mathcal{D}}^{\cdot}_{U_{\boldsymbol{\xi}},\imath}$ (cf. Section 4 for the relevant definitions). Similarly, for any $u \in U_{\boldsymbol{\xi}} \cap \mathbf{Z}$, set $\mathcal{A}^{\cdot}_{u} = \mathcal{A}^{\cdot}_{u,\imath}$, $\mathcal{D}^{\cdot}_{u} = \mathcal{D}^{\cdot}_{u,\imath}$, $\boldsymbol{\mathcal{A}}^{\cdot}_{u} = \boldsymbol{\mathcal{A}}^{\cdot}_{u,\imath}$ and $\boldsymbol{\mathcal{D}}^{\cdot}_{u} = \boldsymbol{\mathcal{D}}^{\cdot}_{u,\imath}$.

Set

$$(\mathsf{T}\times\mathsf{T})_0 = \big\{(t_1,t_2) \in \mathsf{T}\times\mathsf{T} \mid \det(t_1,t_2) \in \mathbf{Z}_p^*\big\},$$

where $\det((x_1,x_2),(y_1,y_2)) = x_1y_2 - x_2y_1$. Let $(\mathsf{T}\times\mathsf{T})^0$ be the complement of $(\mathsf{T}\times\mathsf{T})_0$ in $\mathsf{T}\times\mathsf{T}$. Note that $(\mathsf{T}\times\mathsf{T})_0$ and $(\mathsf{T}\times\mathsf{T})^0$ are open compact subsets of $\mathsf{T}\times\mathsf{T}$, preserved by the diagonal action of $\Gamma_0(p\mathbf{Z}_p)$. Identify $\mathcal{A}_{\boldsymbol{g}}\hat{\otimes}\mathcal{A}_{\boldsymbol{h}} = \mathcal{A}_{\boldsymbol{g}}\hat{\otimes}_{\mathscr{O}}\mathcal{A}_{\boldsymbol{h}}$ with a space of locally analytic functions on $\mathsf{T}\times\mathsf{T}$, homogeneous of weights $\kappa_{\boldsymbol{g}} = \kappa_{U_{\boldsymbol{g}}}$ and $\kappa_{\boldsymbol{h}} = \kappa_{U_{\boldsymbol{h}}}$ in the first and second variable respectively. The orthonormal basis of $\mathcal{A}_{\boldsymbol{g}}\hat{\otimes}\mathcal{A}_{\boldsymbol{h}}$ arising from Remark 4.1 gives a decomposition of $\Gamma_0(p\mathbf{Z}_p)$-modules

$$\mathcal{A}_{\boldsymbol{g}}\hat{\otimes}\mathcal{A}_{\boldsymbol{h}} = (\mathcal{A}_{\boldsymbol{g}}\hat{\otimes}\mathcal{A}_{\boldsymbol{h}})_0 \oplus (\mathcal{A}_{\boldsymbol{g}}\hat{\otimes}\mathcal{A}_{\boldsymbol{h}})^0,$$

where $(\mathcal{A}_{\boldsymbol{g}}\hat{\otimes}\mathcal{A}_{\boldsymbol{h}})_0$ and $(\mathcal{A}_{\boldsymbol{g}}\hat{\otimes}\mathcal{A}_{\boldsymbol{h}})^0$ consist in locally analytic functions supported on $(\mathsf{T}\times\mathsf{T})_0$ and $(\mathsf{T}\times\mathsf{T})^0$ respectively. Let $\Lambda_{\boldsymbol{fgh}} = \Lambda_{\boldsymbol{f}}\hat{\otimes}_{\mathscr{O}}\Lambda_{\boldsymbol{g}}\hat{\otimes}_{\mathscr{O}}\Lambda_{\boldsymbol{h}}$ and define the characters $\kappa^*_{\boldsymbol{f}} : \mathbf{Z}_p^* \to \Lambda^*_{\boldsymbol{fgh}}$ and $\kappa^*_{\boldsymbol{fgh}} : \mathbf{Z}_p^* \to \Lambda^*_{\boldsymbol{fgh}}$ by

$$\kappa^*_{\boldsymbol{f}}(u) = \omega(u)^{(l_o+m_o-k_o-2)/2}\cdot\langle u\rangle^{(\boldsymbol{l}+\boldsymbol{m}-\boldsymbol{k}-2)/2}$$

$$\text{and}\quad \kappa^*_{\boldsymbol{fgh}}(u) = \omega(u)^{(k_o+l_o+m_o-6)/2}\cdot\langle u\rangle^{(\boldsymbol{k}+\boldsymbol{l}+\boldsymbol{m}-6)/2}$$

for every $u = \omega(u)\cdot\langle u\rangle$ in $\mathbf{Z}_p^* = \mathbf{F}_p^*\times 1 + p\mathbf{Z}_p$. (Recall by the discussion preceding Equation (97) that $\kappa_{\boldsymbol{f}}(u)$ is equal to $\omega(u)^{k_o-2}\cdot\langle u\rangle^{\boldsymbol{k}-2}$, and similarly for $\kappa_{\boldsymbol{g}}$ and $\kappa_{\boldsymbol{h}}$.) Here one uses Assumption 1.2, which guarantees that the quantity $k_o+l_o+m_o$ is an even integer. Define similarly $\kappa^*_{\boldsymbol{g}}$ and $\kappa^*_{\boldsymbol{h}}$, so that $\kappa^*_{\boldsymbol{fgh}} = \kappa^*_{\boldsymbol{f}} + \kappa^*_{\boldsymbol{g}} + \kappa^*_{\boldsymbol{h}}$ (again with additive notation). After noting that $\det : \mathbf{Z}_p^2\times\mathbf{Z}_p^2 \to \mathbf{Z}_p$ maps $\mathsf{T}'\times\mathsf{T}$ to $\mathbf{Z}_p^*$, let

$$\mathbf{Det} = \mathbf{Det}^{\boldsymbol{fgh}}_{N,p} : \mathsf{T}'\times\mathsf{T}\times\mathsf{T} \longrightarrow \Lambda_{\boldsymbol{fgh}}$$

be the function which vanishes identically on $\mathsf{T}'\times(\mathsf{T}\times\mathsf{T})^0$ and on an element $(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z})$ in $\mathsf{T}'\times(\mathsf{T}\times\mathsf{T})_0$ takes the value

$$\mathbf{Det}(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z}) = \det(\boldsymbol{x},\boldsymbol{y})^{\kappa^*_{\boldsymbol{h}}}\cdot\det(\boldsymbol{x},\boldsymbol{z})^{\kappa^*_{\boldsymbol{g}}}\cdot\det(\boldsymbol{y},\boldsymbol{z})^{\kappa^*_{\boldsymbol{f}}}.$$

Because $\kappa^*_{\boldsymbol{g}}+\kappa^*_{\boldsymbol{h}} = \kappa_{\boldsymbol{f}}$, one has $\mathbf{Det}(u\cdot\boldsymbol{x},\boldsymbol{y},\boldsymbol{z}) = \kappa_{\boldsymbol{f}}(u)\cdot\mathbf{Det}(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z})$ for every $u \in \mathbf{Z}_p^*$, hence for $\imath$ big enough $\mathbf{Det}(\boldsymbol{x},\boldsymbol{y}_o,\boldsymbol{z}_o)$ belongs to $\mathcal{A}'_{\boldsymbol{f}}$ for every $(\boldsymbol{y}_o,\boldsymbol{z}_o) \in \mathsf{T}\times\mathsf{T}$. Similarly $\mathbf{Det}(\boldsymbol{x}_o,\boldsymbol{y},\boldsymbol{z}_o) \in \mathcal{A}_{\boldsymbol{g}}$ and $\mathbf{Det}(\boldsymbol{x}_o,\boldsymbol{y}_o,\boldsymbol{z}) \in \mathcal{A}_{\boldsymbol{h}}$ for every $(\boldsymbol{x}_o,\boldsymbol{z}_o) \in \mathsf{T}'\times\mathsf{T}$ and $(\boldsymbol{x}_o,\boldsymbol{y}_o) \in \mathsf{T}'\times\mathsf{T}$ respectively. Moreover

$$\mathbf{Det}(\boldsymbol{x}\cdot\gamma,\boldsymbol{y}\cdot\gamma,\boldsymbol{z}\cdot\gamma) = \det(\gamma)^{\kappa^*_{\boldsymbol{fgh}}}\cdot\mathbf{Det}(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z})$$

for every $\gamma \in \Gamma_0(p\mathbf{Z}_p)$. As a consequence $\mathbf{Det}$ can be identified with an element of $\mathcal{A}'_{\boldsymbol{f}}\hat{\otimes}\mathcal{A}_{\boldsymbol{g}}\hat{\otimes}\mathcal{A}_{\boldsymbol{h}}(-\kappa^*_{\boldsymbol{fgh}})$, which is invariant under the diagonal action of $\Gamma_0(p\mathbf{Z}_p)$ (cf. Section 4.2). Since the $\Gamma_0(p\mathbf{Z}_p)$-representation $\mathcal{A}'_{\boldsymbol{f}}\hat{\otimes}\mathcal{A}_{\boldsymbol{g}}\hat{\otimes}\mathcal{A}_{\boldsymbol{h}}$ corresponds to the pro-sheaf $\boldsymbol{\mathcal{A}}'_{\boldsymbol{f}}\otimes\boldsymbol{\mathcal{A}}_{\boldsymbol{g}}\otimes\boldsymbol{\mathcal{A}}_{\boldsymbol{h}}$ on $Y = Y_1(N,p)$ under the functor $\cdot^{\text{ét}}$ (cf. loco citato) this yields

$$\mathbf{Det}^{\boldsymbol{fgh}}_{N,p} \in H^0_{\text{ét}}(Y,\boldsymbol{\mathcal{A}}'_{\boldsymbol{f}}\otimes\boldsymbol{\mathcal{A}}_{\boldsymbol{g}}\otimes\boldsymbol{\mathcal{A}}_{\boldsymbol{h}}(-\kappa^*_{\boldsymbol{fgh}})). \tag{154}$$

Let $\Gamma = \Gamma_1(N,p)$ and let $d : Y \longrightarrow Y^3$ be the diagonal embedding. Define

$$\kappa(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h}) = \frac{1}{a_p(\boldsymbol{k})} \cdot \kappa(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})^o \in H^1(\mathbf{Q}, V(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})), \tag{155}$$

where

$$\kappa(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})^o = \mathrm{AJ}_{\text{ét}}^{\boldsymbol{fgh}}\big(\mathbf{Det}_{N,p}^{\boldsymbol{fgh}}\big)$$

is the image of the *big* invariant $\mathbf{Det}_{N,p}^{\boldsymbol{fgh}}$ under the *big Abel–Jacobi map* $\mathrm{AJ}_{\text{ét}}^{\boldsymbol{fgh}}$ defined by the following composition.

$$\begin{aligned}
H^0_{\text{ét}}(Y, \boldsymbol{\mathcal{A}}'_{\boldsymbol{f}} \otimes \boldsymbol{\mathcal{A}}_{\boldsymbol{g}} \otimes \boldsymbol{\mathcal{A}}_{\boldsymbol{h}}(-\kappa^*_{\boldsymbol{fgh}})) &\xrightarrow{d_*} H^4_{\text{ét}}(Y^3, \boldsymbol{\mathcal{A}}'_{\boldsymbol{f}} \boxtimes \boldsymbol{\mathcal{A}}_{\boldsymbol{g}} \boxtimes \boldsymbol{\mathcal{A}}_{\boldsymbol{h}}(-\kappa^*_{\boldsymbol{fgh}}) \otimes_{\mathbf{Z}_p} \mathbf{Z}_p(2)) \\
&\xrightarrow{\mathtt{HS}} H^1\big(\mathbf{Q}, H^3_{\text{ét}}\big(Y^3_{\bar{\mathbf{Q}}}, \boldsymbol{\mathcal{A}}'_{\boldsymbol{f}} \boxtimes \boldsymbol{\mathcal{A}}_{\boldsymbol{g}} \boxtimes \boldsymbol{\mathcal{A}}_{\boldsymbol{h}}\big)(2+\boldsymbol{\kappa}^*_{\boldsymbol{fgh}})\big) \\
&\xrightarrow{\mathtt{K}} H^1\big(\mathbf{Q}, H^1(\Gamma, A'_{\boldsymbol{f}}) \hat{\otimes}_L H^1(\Gamma, A_{\boldsymbol{g}}) \hat{\otimes}_L H^1(\Gamma, A_{\boldsymbol{h}})(2+\boldsymbol{\kappa}^*_{\boldsymbol{fgh}})\big) \\
&\xrightarrow{(w_p \otimes \mathrm{id} \otimes \mathrm{id})_*} H^1\big(\mathbf{Q}, H^1(\Gamma, A_{\boldsymbol{f}}) \hat{\otimes}_L H^1(\Gamma, A_{\boldsymbol{g}}) \hat{\otimes}_L H^1(\Gamma, A_{\boldsymbol{h}})(2+\boldsymbol{\kappa}^*_{\boldsymbol{fgh}})\big) \\
&\xrightarrow{\mathtt{s}_{\boldsymbol{fgh}*}} H^1\big(\mathbf{Q}, H^1(\Gamma, D'_{\boldsymbol{f}})^{\leqslant 0} \hat{\otimes}_L H^1(\Gamma, D'_{\boldsymbol{g}})^{\leqslant 0} \hat{\otimes}_L H^1(\Gamma, D'_{\boldsymbol{h}})^{\leqslant 0}(2-\boldsymbol{\kappa}^*_{\boldsymbol{fgh}})\big) \\
&\xrightarrow{\mathrm{pr}_{\boldsymbol{fgh}*}} H^1\big(\mathbf{Q}, V(\boldsymbol{f}) \hat{\otimes}_L V(\boldsymbol{g}) \hat{\otimes}_L V(\boldsymbol{h})(-1-\boldsymbol{\kappa}^*_{\boldsymbol{fgh}})\big) = H^1(\mathbf{Q}, V(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})).
\end{aligned} \tag{156}$$

Here $\boldsymbol{\kappa}^*_{\boldsymbol{fgh}} : G_{\mathbf{Q}} \to \Lambda^*_{\boldsymbol{fgh}}$ denotes the composition of $\kappa^*_{\boldsymbol{fgh}}$ with the p-adic cyclotomic character χ_{cyc}. The first arrow is the push-forward by the diagonal embedding d. The morphism $\mathtt{HS}$ arises from the Hochschild–Serre spectral sequence and Equation (80). (Note that $H^4_{\text{ét}}(Y^3_{\bar{\mathbf{Q}}}, \mathscr{F})$ vanishes for every pro-sheaf $\mathscr{F} \in \mathbf{S}(Y^3_{\text{ét}})$, as follows easily from Equation (75) and [Mil80, Chapter VI, Theorem 7.2].) The map $\mathtt{K}$ comes from the Künneth decomposition and the projection in Equation (79). The morphism $(w_p \otimes \mathrm{id} \otimes \mathrm{id})_*$ is the one induced in cohomology by the $G_{\mathbf{Q}}$-equivariant Atkin–Lehner operator $w_p : H^1(\Gamma, \mathcal{A}'_{\boldsymbol{f}}) \to H^1(\Gamma, \mathcal{A}_{\boldsymbol{f}})$ (cf. Sections 4.1.2 and 4.2). The penultimate arrow $\mathtt{s}_{\boldsymbol{fgh}*}$ is induced by the tensor product of the morphisms of $G_{\mathbf{Q}}$-modules

$$H^1(\Gamma, A_{\boldsymbol{a}}) \twoheadrightarrow H^1(\Gamma, A_{\boldsymbol{a}})^{\leqslant 0} \xrightarrow{\mathtt{s}_{\boldsymbol{a}}} H^1(\Gamma, D'_{\boldsymbol{a}})^{\leqslant 0}(-\boldsymbol{\kappa}_{U_{\boldsymbol{a}}})$$

for $\boldsymbol{a} = \boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h}$, where the first map is the projection to the slope $\leqslant 0$ part and $\mathtt{s}_{\boldsymbol{a}} = \mathtt{s}_{U_{\boldsymbol{a}},0}$ is defined in Equation (83). Finally $\mathrm{pr}_{\boldsymbol{fgh}}$ denotes the tensor product of the $G_{\mathbf{Q}}$-equivariant projections $\mathrm{pr}_{\boldsymbol{a}}$ defined in Equation (105).

8.2. Balanced specialisations of $\kappa(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})$. — Let $w = (k,l,m) \in \Sigma_{\mathrm{bal}}$ be a *balanced* triple of classical weights, let $\boldsymbol{r} = (k-2, l-2, m-2) = w - \mathbf{2}$, and let $r = (r_1 + r_2 + r_3)/2$. Recall the diagonal classes

$$\tilde{\kappa}_{Np,\boldsymbol{r}} \in H^1_{\mathrm{geo}}(\mathbf{Q}, W_{Np,\boldsymbol{r}}) \quad \text{and} \quad \kappa_{Np,\boldsymbol{r}} = \mathtt{s}_{\boldsymbol{r}*}(\tilde{\kappa}_{Np,\boldsymbol{r}}) \in H^1_{\mathrm{geo}}(\mathbf{Q}, V_{Np,\boldsymbol{r}})$$

introduced in Equations (43) and (46), and define the *twisted diagonal class*

$$\kappa^\dagger(\boldsymbol{f}_k, \boldsymbol{g}_l, \boldsymbol{h}_m) = \mathrm{pr}_{\boldsymbol{f}_k \boldsymbol{g}_l \boldsymbol{h}_m *}\big(\mathtt{s}_{\boldsymbol{r}*}\big((w'_p \otimes \mathrm{id} \otimes \mathrm{id})_*\big(\tilde{\kappa}_{Np,\boldsymbol{r}}\big)\big)\big) \in H^1_{\mathrm{geo}}(\mathbf{Q}, V(\boldsymbol{f}_k, \boldsymbol{g}_l, \boldsymbol{h}_m)). \tag{157}$$

Here $\mathrm{pr}_{\boldsymbol{f}_k \boldsymbol{g}_l \boldsymbol{h}_m}$ is the projection defined in Equation (48) and

$$(w'_p \otimes \mathrm{id} \otimes \mathrm{id})_* : H^1(\mathbf{Q}(\mu_p), W_{Np,\boldsymbol{r}}) \longrightarrow H^1(\mathbf{Q}(\mu_p), W_{Np,\boldsymbol{r}})$$

is the map induced by the dual Atkin–Lehner operator

$$w'_p : H^1_{\text{ét}}(Y_1(Np)_{\bar{\mathbf{Q}}}, \mathscr{S}_{r_1}) \cong H^1_{\text{ét}}(Y_1(Np)_{\bar{\mathbf{Q}}}, \mathscr{S}_{r_1})$$

(cf. Section 2.3.1) and the Künneth decomposition on $W_{Np,\boldsymbol{r}}$. A priori the class $\kappa^\dagger(\boldsymbol{f}_k, \boldsymbol{g}_l, \boldsymbol{h}_m)$ then lives in the geometric subgroup of $H^1(\mathbf{Q}(\mu_p), V(\boldsymbol{f}_k, \boldsymbol{g}_l, \boldsymbol{h}_m))$. On the other hand the forms $\boldsymbol{f}_k$, $\boldsymbol{g}_l$ and $\boldsymbol{h}_m$ have level $\Gamma_1(N,p) = \Gamma_1(N) \cap \Gamma_0(p)$, hence the cohomology class $\tilde{\kappa}_{Np,\boldsymbol{r}}$ is in the image of the map induced in $G_{\mathbf{Q}}$-cohomology by the pull-back $H^3_{\text{ét}}(Y_1(N,p)^3_{\bar{\mathbf{Q}}}, \mathscr{S}_{[\boldsymbol{r}]})(c_w) \longrightarrow H^3_{\text{ét}}(Y_1(Np)^3_{\bar{\mathbf{Q}}}, \mathscr{S}_{[\boldsymbol{r}]})(c_w) = W_{Np,\boldsymbol{r}}$. Because the Atkin–Lehner operator w'_p acting on $H^1_{\text{ét}}(Y_1(N,p)_{\bar{\mathbf{Q}}}, \mathscr{S}_{k-2})$ is $G_{\mathbf{Q}}$-equivariant, this implies that $\kappa^\dagger(\boldsymbol{f}_k, \boldsymbol{g}_l, \boldsymbol{h}_m)$ is fixed by the action of the Galois group of $\mathbf{Q}(\mu_p)$ over $\mathbf{Q}$, hence can naturally be viewed as a geometric class in $H^1(\mathbf{Q}, V(\boldsymbol{f}_k, \boldsymbol{g}_l, \boldsymbol{h}_m))$.

With the notations already introduced one has the following

Theorem 8.1. — *For each balanced triple $w = (k,l,m)$ in Σ_{bal} one has*

$$(p-1)\alpha_{\boldsymbol{f}_k} \cdot \rho_w\big(\kappa(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})\big) = \left(1 - \frac{\alpha_{\boldsymbol{f}_k}\beta_{\boldsymbol{g}_l}\beta_{\boldsymbol{h}_m}}{p^{r+2}}\right) \cdot \kappa^\dagger(\boldsymbol{f}_k, \boldsymbol{g}_l, \boldsymbol{h}_m).$$

Before giving the proof of Theorem 8.1 we deduce the following

Corollary 8.2. — *$\kappa(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})$ lies in the balanced Selmer group $H^1_{\text{bal}}(\mathbf{Q}, V(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h}))$.*

Proof. — By definition one has to prove that the class

$$\text{res}_{\mathscr{F},p}(\kappa(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})) \in H^1(\mathbf{Q}, V(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})/\mathscr{F}^2 V(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h}))$$

is zero, where $\text{res}_{\mathscr{F},p}$ is the composition of the restriction at p and the map induced by $V(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h}) \twoheadrightarrow V(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})/\mathscr{F}^2$. According to Proposition 3.2 for every balanced triple $w = (k,l,m)$ in Σ_{bal} one has

$$\text{res}_p\big(\kappa^\dagger(\boldsymbol{f}_k, \boldsymbol{g}_l, \boldsymbol{h}_m)\big) \in H^1_{\text{geo}}(\mathbf{Q}_p, V(\boldsymbol{f}_k, \boldsymbol{g}_l, \boldsymbol{h}_m)).$$

Let Σ^o_{bal} be the set of (k,l,m) in Σ_{bal} such that p does not divide the conductors of $\boldsymbol{f}_k, \boldsymbol{g}_l$ and $\boldsymbol{h}_m$. One has

$$H^1_{\text{geo}}(\mathbf{Q}_p, V(\boldsymbol{f}_k, \boldsymbol{g}_k, \boldsymbol{h}_m)) = \ker\big(H^1(\mathbf{Q}_p, V(\boldsymbol{f}_k, \boldsymbol{g}_l, \boldsymbol{h}_m)) \to H^1(\mathbf{Q}_p, V(\boldsymbol{f}_k, \boldsymbol{g}_l, \boldsymbol{h}_m)/\mathscr{F}^2)\big)$$

and

$$\alpha_{\boldsymbol{f}_k}\beta_{\boldsymbol{g}_l}\beta_{\boldsymbol{h}_m} \neq p^{r+2}$$

for all $w = (k,l,m)$ in Σ^o_{bal} (by the Ramunajan–Petersson conjecture). The previous two equations and Theorem 8.1 imply that the class $\text{res}_{\mathscr{F},p}(\kappa(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h}))$ specialises to zero in $H^1(\mathbf{Q}_p, V(\boldsymbol{f}_k, \boldsymbol{g}_l, \boldsymbol{h}_m)/\mathscr{F}^2)$ at every w in Σ^o_{bal}. Because Σ^o_{bal} is dense in $U_{\boldsymbol{f}} \times U_{\boldsymbol{g}} \times U_{\boldsymbol{h}}$, to conclude the proof it is then sufficient to show that $H^1(\mathbf{Q}_p, V(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})/\mathscr{F}^2)$ is $\mathscr{O}_{\boldsymbol{fgh}}$-torsion free (hence a submodule of a reflexive $\mathscr{O}_{\boldsymbol{fgh}}$-module), which implies that $\bigcap_{w \in \Sigma^o_{\text{bal}}} (\boldsymbol{k}-k, \boldsymbol{l}-l, \boldsymbol{m}-m) \cdot H^1(\mathbf{Q}_p, V(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})/\mathscr{F}^2) = 0$. This is a consequence of the following claim. If $\wp \in \mathscr{O}_{\boldsymbol{fgh}}$ is irreducible and one sets $\mathscr{O}_\wp = \mathscr{O}_{\boldsymbol{fgh}}/(\wp)$, then

$$H^0(\mathbf{Q}_p, V(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})/\mathscr{F}^2 \otimes_{\mathscr{O}_{\boldsymbol{fgh}}} \mathscr{O}_\wp) = 0. \tag{158}$$

The rest of the proof is then devoted to the proof of this claim.

Section 7.2 shows that there is a short exact sequence of $G_{\mathbf{Q}_p(\mu_p)}$-modules

$$\mathscr{O}_\wp(\theta_{\boldsymbol{f}})^{\oplus a} \oplus \mathscr{O}_\wp(\theta_{\boldsymbol{g}})^{\oplus a} \oplus \mathscr{O}_\wp(\theta_{\boldsymbol{h}})^{\oplus a} \hookrightarrow V(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})/\mathscr{F}^2 \otimes_{\mathscr{O}_{\boldsymbol{fgh}}} \mathscr{O}_\wp \twoheadrightarrow \mathscr{O}_\wp(\theta_{\boldsymbol{fgh}})^{\oplus a},$$

where a is a positive integer and the characters $\theta_\cdot : G_{\mathbf{Q}_p(\mu_p)} \to \mathscr{O}_\wp^*$ are defined by

$$\begin{aligned}\theta_{\boldsymbol{fgh}} &= \kappa_{\mathrm{cyc}}^{(4-\boldsymbol{k}-\boldsymbol{l}-\boldsymbol{m})/2} \cdot \check{a}_p(\boldsymbol{k}) \cdot \check{b}_p(\boldsymbol{l}) \cdot \check{c}_p(\boldsymbol{m}),\\ \theta_{\boldsymbol{f}} &= \kappa_{\mathrm{cyc}}^{(\boldsymbol{k}-\boldsymbol{l}-\boldsymbol{m}+2)/2} \cdot \chi_{\boldsymbol{f}} \cdot \check{b}_p(\boldsymbol{l}) \cdot \check{c}_p(\boldsymbol{m}) \cdot \check{a}_p(\boldsymbol{k})^{-1}\end{aligned}$$

and similarly for $\theta_{\boldsymbol{g}}$ and $\theta_{\boldsymbol{h}}$. Set $\wp_{\boldsymbol{fgh}} = 4-\boldsymbol{k}-\boldsymbol{l}-\boldsymbol{m}$, set $\wp_{\boldsymbol{f}} = \boldsymbol{k}-\boldsymbol{l}-\boldsymbol{m}+2$ and define similarly $\wp_{\boldsymbol{g}}$ and $\wp_{\boldsymbol{h}}$. Denote by $\wp_{\boldsymbol{a}}$ and $\theta_{\boldsymbol{a}}$ one of $\wp_\cdot$ and $\theta_\cdot$ respectively. If $\wp \cdot \mathscr{O}_{\boldsymbol{fgh}}$ is different from one of the ideals $\wp_{\boldsymbol{a}} \cdot \mathscr{O}_{\boldsymbol{fgh}}$, then $H^0(I_{\mathbf{Q}_p(\mu_p)}, V(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})/\mathscr{F}^2 \otimes_{\mathscr{O}_{\boldsymbol{fgh}}} \mathscr{O}_\wp)$ is trivial and (158) holds true. Assume now $\wp = u \cdot \wp_{\boldsymbol{a}}$ for a unit u in $\mathscr{O}_{\boldsymbol{fgh}}$, so that $\theta_{\boldsymbol{a}}$ is an unramified character of $G_{\mathbf{Q}_p(\mu_p)}$. According to the Ramanujan–Petersson conjecture one has

$$|\theta_{\boldsymbol{a}}(\mathrm{Frob}_p)(w)| = \sqrt{p}$$

for all $w \in \Sigma_{\mathrm{bal}}^o \cap V(\wp)$ (where $|\cdot|$ is the complex absolute value and $V(\wp)$ is the zero locus of $\wp$). Shrinking the discs $U_\cdot$ if necessary, we can assume that $\Sigma_{\mathrm{bal}}^o \cap V(\wp)$ is non-empty (otherwise $\wp$ would be a unit). The previous equation then implies that the characters $\theta_\cdot$ are non-trivial and (158) follows. □

Proof of Theorem 8.1. — According to [Mil06, Section II.7] for every $n, i \geqslant 1$ there is a trace isomorphism

$$\mathrm{Trace}_{Y^n} : H^{2n+3}_{\text{ét,c}}(Y^n, \mathscr{O}/\mathfrak{m}^i(n+1)) \cong \mathscr{O}/\mathfrak{m}^i.$$

(See Chapter II, Section 2 of loc. cit. for the definition of $H^\cdot_{\text{ét,c}}(Y^n,\cdot)$, denoted $H^\cdot_c(Y^n,\cdot)$ there.) For all finite smooth sheaves $\mathscr{F}$ of $\mathscr{O}/\mathfrak{m}^i$-modules on $Y^n_{\text{ét}}$, Trace_{Y^n} and the cup-product define perfect pairings

$$(159)\qquad (\cdot,\cdot)_{Y^n} = \mathrm{Trace}_{Y^n} \circ \cup : H^j_{\text{ét}}(Y^n, \mathscr{F}) \otimes_L H^{2n+3-j}_{\text{ét,c}}(Y^n, \mathscr{G}(n+1)) \longrightarrow \mathscr{O}/\mathfrak{m}^i,$$

where $\mathscr{G}$ is the dual of $\mathscr{F}$ (cf. Chapter II, Corollary 7.7 of [Mil06]). Denote by $\mathscr{F}^\cdot_u$ in $\mathbf{S}_f(Y_{\text{ét}})$ the sheaf associated to $\mathrm{Fil}_{i,j}\mathcal{A}^\cdot_{u,\imath}$ for $u \geqslant 0$ and fixed $j \geqslant i \geqslant 0$, and by $\mathscr{G}^\cdot_u$ the $\mathscr{O}/\mathfrak{m}^i$-dual of $\mathscr{F}^\cdot_u$. One has a Hecke equivariant diagram of adjoint morphisms, where the Hecke operators are defined by constructions similar to those of Section 2.3.
(160)

$$\begin{array}{ccccc} H^0_{\text{ét}}(Y, \mathscr{F}'_{r_1} \otimes \mathscr{F}_{r_2} \otimes \mathscr{F}_{r_3}(r)) & \times & H^5_{\text{ét,c}}(Y, \mathscr{G}'_{r_1} \otimes \mathscr{G}_{r_2} \otimes \mathscr{G}_{r_3}(2-r)) & \xrightarrow{(\cdot,\cdot)_Y} & \mathscr{O}/\mathfrak{m}^i \\ \downarrow d_* & & \uparrow d^* & & \| \\ H^4_{\text{ét}}(Y^3, \mathscr{F}'_{r_1} \boxtimes \mathscr{F}_{r_2} \boxtimes \mathscr{F}_{r_3}(r+2)) & \times & H^5_{\text{ét,c}}(Y^3, \mathscr{G}'_{r_1} \boxtimes \mathscr{G}_{r_2} \boxtimes \mathscr{G}_{r_3}(2-r)) & \xrightarrow{(\cdot,\cdot)_{Y^3}} & \mathscr{O}/\mathfrak{m}^i \end{array}$$

Let $\mathcal{A}^\cdot$ and $\boldsymbol{\mathcal{A}}^\cdot$ be shorthands for $\mathcal{A}^\cdot_{\cdot,\imath}$ and $\boldsymbol{\mathcal{A}}^\cdot_{\cdot,\imath}$ respectively. Similarly as above, the orthonormal basis of $\mathcal{A}_u \hat{\otimes} \mathcal{A}_v$ arising from Remark 4.1 gives a decomposition of $\Gamma_0(p\mathbf{Z}_p)$-modules

$$\mathcal{A}_u \hat{\otimes} \mathcal{A}_v = (\mathcal{A}_u \hat{\otimes} \mathcal{A}_v)_0 \oplus (\mathcal{A}_u \hat{\otimes} \mathcal{A}_v)^0,$$

where $(\mathcal{A}_r\hat{\otimes}\mathcal{A}_s)_0$ (resp., $(\mathcal{A}_r\hat{\otimes}\mathcal{A}_s)^0$) can be identified with a space of locally analytic functions on $\mathsf{T}\times\mathsf{T}$ supported on $(\mathsf{T}\times\mathsf{T})_0$ (resp., $(\mathsf{T}\times\mathsf{T})^0$). This in turn induces similar decompositions

$$\mathscr{F}_u\otimes\mathscr{F}_v=(\mathscr{F}_u\otimes\mathscr{F}_v)_0\oplus(\mathscr{F}_u\otimes\mathscr{F}_v)^0 \quad \text{and} \quad \mathscr{G}_u\otimes\mathscr{G}_v=(\mathscr{G}_u\otimes\mathscr{G}_v)_0\oplus(\mathscr{G}_u\otimes\mathscr{G}_v)^0.$$

Let $t:Y_1(Np)\to Y_1(N,p)=Y$ be the natural projection. To ease notations, let $\mathtt{Det}\in H^0_{\text{ét}}(Y,\boldsymbol{\mathcal{A}}'_{r_1}\otimes\boldsymbol{\mathcal{A}}_{r_2}\otimes\boldsymbol{\mathcal{A}}_{r_3}(r))$ denote the image of $\mathtt{Det}^{\boldsymbol{r}}_{Np}$ under the composition of the push-forward t_* with the natural map

$$H^0_{\text{ét}}(Y,\mathscr{S}_{r_1}\otimes\mathscr{S}_{r_2}\otimes\mathscr{S}_{r_3}(r))\longrightarrow H^0_{\text{ét}}(Y,\boldsymbol{\mathcal{A}}'_{r_1}\otimes\boldsymbol{\mathcal{A}}_{r_2}\otimes\boldsymbol{\mathcal{A}}_{r_3}(r)).$$

For $j=j(i)$ large enough, let $\mathtt{D}=\mathtt{D}^{\boldsymbol{r}}_{i,j}\in H^0_{\text{ét}}(Y,\mathscr{F}'_{r_1}\otimes\mathscr{F}_{r_2}\otimes\mathscr{F}_{r_3}(r))$ be a representative of $\mathtt{Det}\,(\mathrm{mod}\ \mathfrak{m}^i)$ (cf. Section 4.2), and let $\mathtt{D}_0=\mathtt{D}^{\boldsymbol{r}}_{i,j,0}$ be its projection to the cohomology group $H^0_{\text{ét}}(Y,\mathscr{F}'_{r_1}\otimes(\mathscr{F}_{r_2}\otimes\mathscr{F}_{r_3})_0(r))$. By construction

$$(p-1)\cdot\rho_w(\mathbf{Det})=\varprojlim_i \mathtt{D}^{\boldsymbol{r}}_{i,j,0}. \tag{161}$$

For all z in $H^5_{\text{ét,c}}(Y^3,\mathscr{G}'_{r_1}\boxtimes\mathscr{G}_{r_2}\boxtimes\mathscr{G}_{r_3}(2-r))$ one has the equalities (cf. Equation (160))

$$\begin{aligned}\big(d_*(\mathtt{D}-\mathtt{D}_0),1\otimes U_p^{\otimes 2}(z)\big)_{Y^3}&=\big(\mathtt{D}-\mathtt{D}_0,d^*\big(1\otimes U_p^{\otimes 2}(z)\big)\big)_Y\\&=\big(\mathtt{D}-\mathtt{D}_0,\delta^*\big(1\otimes\delta^*\big(1\otimes U_p^{\otimes 2}(z)\big)\big)\big)_Y\\&=(\mathtt{D},\delta^*(1\otimes U_p(1\otimes\delta^*(z))))_Y\\&=p^{r-r_1}\cdot\big(\mathtt{D},\delta^*\big(U'_p\otimes 1\,(1\otimes\delta^*(z))\big)\big)_Y\\&=p^{r-r_1}\cdot\big(\mathtt{D},d^*\big(U'_p\otimes 1\otimes 1(z)\big)\big)_Y\\&=p^{r-r_1}\cdot\big(d_*(\mathtt{D}),U'_p\otimes 1\otimes 1(z)\big)_{Y^3},\end{aligned} \tag{162}$$

where $\delta:Y\to Y^2$ is the diagonal embedding. To justify the third equality one notes that

$$1\otimes\delta^*\circ 1\otimes U_p^{\otimes 2}-1\otimes U_p\circ 1\otimes\delta^*$$

(resp., $1\otimes U_p\circ 1\otimes\delta^*$) takes values in the submodule $H^5_{\text{ét,c}}(Y,\mathscr{G}'_{r_1}\otimes(\mathscr{G}_{r_2}\otimes\mathscr{G}_{r_3})_0(2-r))$ (resp., in $H^5_{\text{ét,c}}(Y,\mathscr{G}'_{r_1}\otimes(\mathscr{G}_{r_2}\otimes\mathscr{G}_{r_3})^0(2-r))$), and that $H^5_{\text{ét,c}}(Y,\mathscr{G}'_{r_1}\otimes(\mathscr{G}_{r_2}\otimes\mathscr{G}_{r_3})_0(2-r))$ is orthogonal to $H^0_{\text{ét,c}}(Y,\mathscr{F}'_{r_1}\otimes(\mathscr{F}_{r_2}\otimes\mathscr{F}_{r_3})^0(r))$. (Compare with the proof of Proposition 5.4 of [**GS20**].)

All the other equalities in Equation (162) but the fourth are standard. To prove the remaining equality, let $\pi:Y\to\mathrm{Spec}(\mathbf{Z}[1/Np])$ and $\boldsymbol{\pi}=\pi\times\pi:Y^2\to\mathrm{Spec}(\mathbf{Z}[1/Np])$ be the structural maps. Let $R\pi_!$ and $R\boldsymbol{\pi}_!$ be the δ-functors associated in [**FK88**, Chapter I, Definition 8.6] with the compactifiable maps π and $\boldsymbol{\pi}$, so that by definition $H^q_{\text{ét,c}}(Y,\cdot)=H^q_{\text{ét,c}}(\mathbf{Z}[1/Np],R\pi_!\cdot)$ and $H^q_{\text{ét,c}}(Y^2,\cdot)=H^q_{\text{ét,c}}(\mathbf{Z}[1/Np],R\boldsymbol{\pi}_!\cdot)$ for any $q\geqslant 0$ (cf. Section II.7 of [**Mil06**]). If $\mathscr{G}$ denotes the étale sheaf $\mathscr{G}'_{r_1}\boxtimes(\mathscr{G}_{r_2}\otimes\mathscr{G}_{r_3})(2-r)$ on Y^2, one can lift the Hecke operators $1\otimes U_p$ and $U'_p\otimes 1$ on $H^{\cdot}_{\text{ét,c}}(Y^2,\mathscr{G})$ to morphisms (denoted by the same symbols) $R\boldsymbol{\pi}_!\mathscr{G}\longrightarrow R\boldsymbol{\pi}_!\mathscr{G}$ (cf. Section 2.3). The diagonal embedding $\delta^*:Y\longrightarrow Y^2$, the morphism of sheaves

$$\beta:\delta^*\mathscr{G}=\mathscr{G}'_{r_1}\otimes\mathscr{G}_{r_2}\otimes\mathscr{G}_{r_3}\,(2-r)\longrightarrow\mathscr{O}/\mathfrak{m}^i(2)$$

defined by the cup product with $\mathtt{D}$, and the trace morphism

$$\mathrm{tr}_Y : R\pi_!\mathscr{O}/\mathfrak{m}^i(2) \longrightarrow \mathscr{O}/\mathfrak{m}^i[-2]$$

(see the discussion preceding Theorem 7.6 in [**Mil06**, Chapter II, Section 7]) induce a map $\vartheta = \mathrm{tr}_Y \circ \beta \circ \delta^* : R\boldsymbol{\pi}_!\mathscr{G} \longrightarrow \mathscr{O}/\mathfrak{m}^i[-2]$. In order to prove the forth equality in Equation (162) it is then sufficient to prove that the composition $\Xi = \vartheta \circ 1 \otimes U_p$ agrees with $\Psi = \bar{\chi}_{\boldsymbol{f}}(p)p^{r-r_1} \cdot \vartheta \circ U_p' \otimes 1$. By using the Künneth isomorphism

$$R\boldsymbol{\pi}_!\mathscr{G} \cong R\pi_!\mathscr{G}_{r_1}' \otimes^{\mathbf{L}}_{\mathscr{O}} R\pi_!(\mathscr{G}_{r_2} \otimes \mathscr{G}_{r_3}(2-r)),$$

the sought for equality $\Xi = \Psi$ follows from the same formal computation as in the proof of Proposition 2.9 of [**GS20**].

Since the operators $1 \otimes U_p^{\otimes 2}$ and $U_p' \otimes 1 \otimes 1$ acting on $H^5_{\text{ét},c}(Y^3, \mathscr{G}_{r_1}' \boxtimes \mathscr{G}_{r_2} \boxtimes \mathscr{G}_{r_3}(2-r))$ are the adjoints under $(\cdot,\cdot)_{Y^3}$ of the operators $1 \otimes U_p^{\otimes 2}$ and $U_p' \otimes 1 \otimes 1$ acting on $H^4_{\text{ét}}(Y^3, \mathscr{F}_{r_1}' \boxtimes \mathscr{F}_{r_2} \boxtimes \mathscr{F}_{r_3}(r+2))$, and since $(\cdot,\cdot)_{Y^3}$ is perfect, Equation (162) yields

$$(1 \otimes U_p \otimes U_p) \circ d_*(\mathtt{D} - \mathtt{D}_0) = p^{r-r_1} \cdot (U_p' \otimes 1 \otimes 1) \circ d_*(\mathtt{D}).$$

In light of Equation (161), this implies

$$\begin{aligned}(163) \qquad & (p-1) \cdot (1 \otimes U_p \otimes U_p) \circ \mathtt{K} \circ \mathtt{HS} \circ d_* \circ \rho_w(\mathbf{Det}) \\ & = \big(1 \otimes U_p \otimes U_p - p^{r-r_1} \cdot U_p' \otimes 1 \otimes 1\big) \circ \mathtt{K} \circ \mathtt{HS} \circ d_*(\mathtt{Det})\end{aligned}$$

in $H^1_{\text{ét}}\big(\mathbf{Q}, H^1(\Gamma, A_{r_1}')\hat{\otimes}_L H^1(\Gamma, A_{r_2})\hat{\otimes}_L H^1(\Gamma, A_{r_3})(r+2)\big)$, where $A_u^{\cdot}$ is a shorthand for $A_{u,\imath}^{\cdot}$, and the morphisms $\mathtt{K}, \mathtt{HS}$ and d_* are defined as in Equation (156), after replacing the big étale sheaf $\boldsymbol{\mathcal{A}}_{\boldsymbol{f}}' \otimes \boldsymbol{\mathcal{A}}_{\boldsymbol{g}} \otimes \boldsymbol{\mathcal{A}}_{\boldsymbol{h}}$ with $\boldsymbol{\mathcal{A}}_{r_1}' \otimes \boldsymbol{\mathcal{A}}_{r_2} \otimes \boldsymbol{\mathcal{A}}_{r_3}$. To ease notations write $\heartsuit$ (resp., $\spadesuit$) for the left (resp., right) hand side of Equation (163).

For each nonnegative integer u and $\mathscr{F}_u = \mathscr{S}_u, \mathscr{L}_u$, let

$$H^1_{\text{ét}}(Y_1(Np)_{\bar{\mathbf{Q}}}, \mathscr{F}_u)_o \hookrightarrow H^1_{\text{ét}}(Y_1(Np)_{\bar{\mathbf{Q}}}, \mathscr{F}_u)_L$$

be the L-direct summand on which the diamond operator $\langle d \rangle$ acts trivially for each integer d coprime to p and congruent to one modulo N, so that the pull-back t^* yields an isomorphism between $H^1_{\text{ét}}(Y_{\bar{\mathbf{Q}}}, \mathscr{F}_u)_L$ and $H^1_{\text{ét}}(Y_1(Np)_{\bar{\mathbf{Q}}}, \mathscr{F}_u)^o$, with inverse $\frac{1}{p-1}$ times the push-forward t_*. For $\cdot = \emptyset, \prime$ denote by

$$c_u^{\cdot} : H^1_{\text{ét}}(Y_1(Np)_{\bar{\mathbf{Q}}}, \mathscr{S}_u)_o \longrightarrow H^1(\Gamma, A_u^{\cdot})$$

the composition of t_* with the comparison morphism introduced in Equation (72). By construction

$$(c_{r_1}'\hat{\otimes}c_{r_2}\hat{\otimes}c_{r_3})_* \circ \mathtt{K}\,(\tilde{\kappa}_{Np,\boldsymbol{r}}) = \mathtt{K} \circ \mathtt{HS} \circ d_*(\mathtt{Det})$$

(where the morphism $\mathtt{K}$ which appear in the left hand side refers to the Künneth decomposition of $W_{Np,\boldsymbol{r}} = H^3_{\text{ét}}(Y_1(Np)_{\bar{\mathbf{Q}}}, \mathscr{S}_{[\boldsymbol{r}]})(r+2))$, hence

$$\spadesuit = (c_{r_1}'\hat{\otimes}c_{r_2}\hat{\otimes}c_{r_3})_* \circ \big(1 \otimes U_p' \otimes U_p' - p^{r-r_1} \cdot U_p \otimes 1 \otimes 1\big) \circ \mathtt{K}\,(\tilde{\kappa}_{Np,\boldsymbol{r}})$$

(cf. the discussion following Equation (72)). Since $w_p \circ c_u' = c_u \circ w_p'$, where w_p' is the Atkin–Lehner operator defined in Section 2.3.1 and w_p is the one defined in

Equation (66), and since $w'_p U_p = \langle p\rangle_N\, U'_p w'_p$ as endomorphisms of $H^1_{\text{ét}}(Y_1(Np)_{\bar{\mathbf{Q}}}, \mathscr{S}_u)$, one deduces

$$(164)\quad w_{p,f*}(\spadesuit) = c_{\boldsymbol{r}*} \circ \left(1 \otimes U'_p \otimes U'_p - p^{r-r_1} \cdot \langle p\rangle_N\, U'_p \otimes 1 \otimes 1\right) \circ w'_{p,f*} \circ \mathtt{K}\,(\tilde{\kappa}_{Np,\boldsymbol{r}}),$$

where $w_{p,f} = w_p \otimes \mathrm{id} \otimes \mathrm{id}$, $w'_{p,f} = w'_p \otimes \mathrm{id} \otimes \mathrm{id}$ and $c_{\boldsymbol{r}} = c_{r_1} \hat{\otimes} c_{r_2} \hat{\otimes} c_{r_3}$.

Taking $h = 0$ and replacing A_U and D'_U with A_u and D'_u (for $u \in \mathbf{N}$) respectively in the definition of the map $\mathtt{s}_{U,h}$ (cf. Equation (83)) yields a $G_{\mathbf{Q}}$-equivariant morphism

$$\mathtt{s}_{u,0} : H^1(\Gamma, A_u)^{\leqslant 0}(u) \longrightarrow H^1(\Gamma, D'_u)^{\leqslant' 0},$$

which intertwines the action of U_p on the source with that of U'_p on the target. If

$$\mathrm{comp}_u : H^1(\Gamma, D'_u)^{\leqslant' 0} \longrightarrow H^1_{\text{ét}}(Y_1(Np)_{\bar{\mathbf{Q}}}, \mathscr{L}_u)_o^{\leqslant' 0}$$

denotes the composition of $t^* : H^1_{\text{ét}}(Y_{\bar{\mathbf{Q}}}, \mathscr{L}_u)_L \longrightarrow H^1_{\text{ét}}(Y_1(Np)_{\bar{\mathbf{Q}}}, \mathscr{L}_u)_o$ with the comparison isomorphism defined in Equation (73), then (cf. Equation (44))

$$(165)\qquad \mathrm{comp}_u \circ \mathtt{s}_{u,0} \circ c_u = \frac{1}{p-1} \cdot \mathtt{s}_{u*}$$

as maps from $H^1_{\text{ét}}(Y_1(Np), \mathscr{S}_u)_L^{\leqslant' 0}(u)$ to $H^1(\Gamma, \mathscr{L}_u)_L^{\leqslant' 0}$. Set $\mathtt{s}_{\boldsymbol{r},0} = \mathtt{s}_{r_1,0} \otimes \mathtt{s}_{r_2,0} \otimes \mathtt{s}_{r_3,0}$ and $\mathrm{comp}_{\boldsymbol{r}} = \mathrm{comp}_{r_1} \otimes \mathrm{comp}_{r_2} \otimes \mathrm{comp}_{r_3}$. It then follows from Equation (164) and the definition of the twisted diagonal class $\kappa^\dagger(\boldsymbol{f}_k, \boldsymbol{g}_l, \boldsymbol{h}_m)$ that the equality

(166)

$$\mathrm{pr}_{\boldsymbol{f}_k \boldsymbol{g}_l \boldsymbol{h}_m *} \circ \mathrm{comp}_{\boldsymbol{r}*} \circ \mathtt{s}_{\boldsymbol{r},0*} \circ w_{p,f*}(\spadesuit) = \frac{\alpha_{\boldsymbol{g}_l}\alpha_{\boldsymbol{h}_m}}{p-1}\left(1 - \frac{\bar{\chi}_{\boldsymbol{f}}(p) p^{r-r_1} \alpha_{\boldsymbol{f}_k}}{\alpha_{\boldsymbol{g}_l}\alpha_{\boldsymbol{h}_m}}\right) \cdot \kappa^\dagger(\boldsymbol{f}_k, \boldsymbol{g}_l, \boldsymbol{h}_m)$$

holds in $H^1_{\text{ét}}(\mathbf{Z}[1/Np], V(\boldsymbol{f}_k, \boldsymbol{g}_l, \boldsymbol{h}_m))$. (Here $\mathrm{pr}_{\boldsymbol{f}_k \boldsymbol{g}_l \boldsymbol{h}_m}$ is the tensor product of the projections $\mathrm{pr}_\cdot$ defined in Equation (23), for $\cdot$ equal to $\boldsymbol{f}_k, \boldsymbol{g}_l$ and $\boldsymbol{h}_m$.)

By construction, one has

$$\mathtt{K} \circ \mathtt{HS} \circ d_* \circ \rho_w(\mathbf{Det}) = \rho_w \circ \mathtt{K} \circ \mathtt{HS} \circ d_*(\mathbf{Det}),$$

where the maps $\mathtt{K}, \mathtt{HS}$ and d_* which appear in the right hand side are the ones introduced in Equation (156). Since the maps ρ_w and $\mathrm{comp}_{\boldsymbol{r}}$ are Hecke-equivariant, and since $\mathtt{s}_{u,0}$ intertwines the action of U_p on $H^1(\Gamma, A_u)^{\leqslant 0}$ with that of U'_p on $H^1(\Gamma, D'_u)^{\leqslant' 0}$ (for each nonnegative integer u), it follows that

$$(167)\quad \diamondsuit = (p-1)\alpha_{\boldsymbol{g}_l}\alpha_{\boldsymbol{h}_m} \cdot \mathrm{pr}_{\boldsymbol{f}_k \boldsymbol{g}_l \boldsymbol{h}_m} \circ \mathrm{comp}_{\boldsymbol{r}*} \circ \mathtt{s}_{\boldsymbol{r},0*} \circ w_{p,f*} \circ \rho_w \circ \mathtt{K} \circ \mathtt{HS} \circ d_*(\mathbf{Det}),$$

where one defines

$$\diamondsuit = \mathrm{pr}_{\boldsymbol{f}_k \boldsymbol{g}_l \boldsymbol{h}_m *} \circ \mathrm{comp}_{\boldsymbol{r}*} \circ \mathtt{s}_{\boldsymbol{r},0*} \circ w_{p,f*}(\heartsuit).$$

One has $w_{p,f*} \circ \rho_w = \rho_w \circ w_{p,f*}$. Moreover the diagram (84) and Equation (165) yield

$$\mathrm{comp}_u \circ \mathtt{s}_{u,0} \circ \rho_{u+2} = \frac{1}{p-1} \cdot \mathtt{s}_{u*} \circ c_u^{-1} \circ \rho_{u+2} = \frac{1}{p-1} \cdot \mathrm{comp}_u \circ \rho_{u+2} \circ \mathtt{s}_{U_{\boldsymbol{\xi}},0}$$

as morphisms from $H^1(\Gamma, A_{\boldsymbol{\xi}})^{\leqslant 0}(\kappa_{\boldsymbol{\xi}}) \longrightarrow H^1_{\text{ét}}(Y_1(Np)_{\bar{\mathbf{Q}}}, \mathscr{L}_u)_o^{\leqslant' 0}$, for $(\boldsymbol{\xi}, u)$ equal to one of the pairs $(\boldsymbol{f}, k-2)$, $(\boldsymbol{g}, l-2)$ and $(\boldsymbol{h}, m-2)$, (cf. the discussion following the diagram (84)). (With a slight abuse of notation, in the previous equation one writes c_u^{-1} for the inverse of the isomorphism between $H^1_{\text{ét}}(Y_1(Np)_{\bar{\mathbf{Q}}}, \mathscr{S}_u)_o^{\leqslant' 0}$ and $H^1(\Gamma, A_u)^{\leqslant 0}$ induced by c_u.) Finally, with the notations introduced in Equations (105) and (106), one has the following equality of $G_{\mathbf{Q}}$-equivariant maps from $H^1(\Gamma, D'_{\boldsymbol{\xi}})^{\leqslant 0}(1)$ to $V(\boldsymbol{f}_k)$:

$$\mathrm{pr}_{\boldsymbol{\xi}_u} \circ \mathrm{comp}_u \circ \rho_{u+2} = \rho_{u+2} \circ \mathrm{pr}_{\boldsymbol{\xi}}.$$

It then follows from Equation (167) and the definitions of ($\diamondsuit$ and) $\kappa(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})^o$ that

$$\mathrm{pr}_{\boldsymbol{f}_k \boldsymbol{g}_l \boldsymbol{h}_m *} \circ \mathrm{comp}_{\boldsymbol{r}*} \circ \mathbf{s}_{\boldsymbol{r},0*} \circ w_{p,f*}(\heartsuit) = \alpha_{\boldsymbol{g}_l} \alpha_{\boldsymbol{h}_m} \cdot \rho_w\big(\kappa(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})^o\big). \tag{168}$$

As $\chi_{\boldsymbol{f}} \chi_{\boldsymbol{g}} \chi_{\boldsymbol{h}} = 1$ by Assumption 1.2, and by definition $\alpha_{\boldsymbol{g}_l} \beta_{\boldsymbol{g}_l} = \chi_{\boldsymbol{g}}(p) p^{r_2+1}$, $\alpha_{\boldsymbol{h}_m} \beta_{\boldsymbol{h}_m} = \chi_{\boldsymbol{h}}(p) p^{r_3+1}$ and $2r = r_1 + r_2 + r_3$, the theorem follows from Equations (163), (166) and (168). (Recall that $\kappa(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})^o = a_p(\boldsymbol{k}) \cdot \kappa(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})$.)

□

8.3. p-stabilisation of diagonal classes. — Write in this section

$$Y_1(M) = Y_1(M)_{\mathbf{Q}},$$

for every integer $M \geqslant 3$. Recall the degeneracy maps $\mathrm{pr}_i : Y_1(Np) \to Y_1(N)$, for $i = 1, p$, defined in Section 2.2.

Let $w \in \Sigma_{\mathrm{bal}}$ and $\boldsymbol{r} = w - \mathbf{2}$ be as in the previous section. Assume $k, l, m \geqslant 3$ and that p does not divide the conductors of $\boldsymbol{f}_k, \boldsymbol{g}_l$ and $\boldsymbol{h}_m$. As in Section 6 let $f = f_k$ (resp., $g = g_l$ and $h = h_m$) be the cusp form of weight k (resp., l, m), level $\Gamma_1(N)$ and character $\chi_{\boldsymbol{f}}$ (resp., $\chi_{\boldsymbol{g}}$, $\chi_{\boldsymbol{h}}$) whose ordinary p-stabilisation is $\boldsymbol{f}_k$ (resp., $\boldsymbol{g}_l$, $\boldsymbol{h}_m$). It is an eigenvector for the Hecke operator T_ℓ, with the same eigenvalue as $\boldsymbol{f}_k$ (resp., $\boldsymbol{g}_l$, $\boldsymbol{h}_m$), for every prime $\ell \nmid Np$, and an eigenvector for T_p with eigenvalue $a_p(f) = \alpha_{\boldsymbol{f}_k} + \beta_{\boldsymbol{f}_k}$ (resp., $a_p(g) = \alpha_{\boldsymbol{g}_l} + \beta_{\boldsymbol{g}_l}$, $a_p(h) = \alpha_{\boldsymbol{h}_m} + \beta_{\boldsymbol{h}_m}$). Assume without loss of generality that $\beta_{\boldsymbol{a}}$ belongs to L for $\boldsymbol{a} = \boldsymbol{f}_k, \boldsymbol{g}_l, \boldsymbol{h}_m$, and denote by

$$\Pi^{\alpha}_{\boldsymbol{r}*} : V_{Np,\boldsymbol{r}} \otimes_{\mathbf{Q}_p} L \longrightarrow V_{N,\boldsymbol{r}} \otimes_{\mathbf{Q}_p} L$$

the morphism (cf. Equations (20) and (45))

$$\Pi^{\alpha}_{\boldsymbol{r}*} = \left(\mathrm{pr}_{1*} - \frac{\beta_{\boldsymbol{f}_k}}{p^{k-1}} \cdot \mathrm{pr}_{p*}\right) \otimes \left(\mathrm{pr}_{1*} - \frac{\beta_{\boldsymbol{g}_l}}{p^{l-1}} \cdot \mathrm{pr}_{p*}\right) \otimes \left(\mathrm{pr}_{1*} - \frac{\beta_{\boldsymbol{h}_m}}{p^{m-1}} \cdot \mathrm{pr}_{p*}\right). \tag{169}$$

A direct computation shows that the composition $\mathrm{pr}_{fgh} \circ \Pi^{\alpha}_{\boldsymbol{r}*}$ factors through the projection $\mathrm{pr}_{\boldsymbol{f}_k \boldsymbol{g}_l \boldsymbol{h}_m}$, hence $\Pi^{\alpha}_{\boldsymbol{r}*}$ induces a morphism

$$\Pi^{\alpha}_{f_k g_l h_m *} : V(\boldsymbol{f}_k, \boldsymbol{g}_l, \boldsymbol{h}_m) \longrightarrow V(f_k, g_l, h_m)$$

of $L[G_{\mathbf{Q}}]$-modules, which is indeed an isomorphism (see Equation (48) for the definition of the projections pr_{fgh} and $\mathrm{pr}_{\boldsymbol{f}_k \boldsymbol{g}_l \boldsymbol{h}_m}$). Note that $\boldsymbol{r} = (r_1, r_2, r_3)$ and (f_k, g_l, h_m) satisfy Assumption 3.1 and Assumption 3.4 respectively, hence the class $\kappa(f_k, g_l, h_m)$ in $H^1(\mathbf{Q}, V(f_k, g_l, h_m))$ is defined. Denote again by

$$\Pi^{\alpha}_{f_k g_l h_m *} : H^1(\mathbf{Q}, V(\boldsymbol{f}_k, \boldsymbol{g}_l, \boldsymbol{h}_m)) \to H^1(\mathbf{Q}, V(f_k, g_l, h_m))$$

the morphism induced in Galois cohomology by $\Pi^{\alpha}_{f_k g_l h_m *}$.

Proposition 8.3. — *Assume $k, l, m \geqslant 3$ and that p does not divide the conductors of $\boldsymbol{f}_k, \boldsymbol{g}_l$ and $\boldsymbol{h}_m$. Then*

$$\Pi^{\alpha}_{f_k g_l h_m *}\big(\kappa^{\dagger}(\boldsymbol{f}_k, \boldsymbol{g}_l, \boldsymbol{h}_m)\big)$$

is equal to

$$(p-1)\alpha_{\boldsymbol{f}_k}\left(1-\frac{\beta_{\boldsymbol{f}_k}\alpha_{\boldsymbol{g}_l}\beta_{\boldsymbol{h}_m}}{p^{r+2}}\right)\left(1-\frac{\beta_{\boldsymbol{f}_k}\beta_{\boldsymbol{g}_l}\alpha_{\boldsymbol{h}_m}}{p^{r+2}}\right)\left(1-\frac{\beta_{\boldsymbol{f}_k}\beta_{\boldsymbol{g}_l}\beta_{\boldsymbol{h}_m}}{p^{r+2}}\right)\cdot\kappa(f_k, g_l, h_m).$$

Proof. — Fix a geometric point $\eta : \mathrm{Spec}(\mathbf{C}) \to Y(1, N(p))$, corresponding to the class of z in $\mathbf{H}$ under the isomorphism (6). With a slight abuse of notation denote again by η the complex point $\nu_p \circ \eta : \mathrm{Spec}(\mathbf{C}) \to Y(1, N)$, and by $\check{\eta}$ both the complex points $\varphi_p \circ \eta : \mathrm{Spec}(\mathbf{C}) \to Y(1(p), N)$ and $\check{\nu}_p \circ \varphi_p \circ \eta : \mathrm{Spec}(\mathbf{C}) \to Y(1, N)$. Then η and $\check{\eta}$ correspond respectively to the classes of z and $p \cdot z$ under the analytic isomorphisms (6). With the notations of Section 2.3 (see in particular the diagram (9)) write

$$\mathscr{T}_{(p)} = R^1 v_{1,N(p)*}\mathbf{Z}_p(1), \quad \mathscr{T}^{(p)} = R^1 v_{1(p),N*}\mathbf{Z}_p(1) \ \text{ and } \ \mathscr{T} = R^1 v_{1,N*}\mathbf{Z}_p(1)$$

for the relative Tate modules of $E(1, N(p)) \to Y(1, N(p))$, $E(1(p), N) \to Y(1(p), N)$ and $E(1, N) \to Y(1, N)$ respectively (cf. Section 2.3). There are then natural isomorphisms

$$\mathscr{T}_{(p),\eta} \cong \mathbf{Z}_p \oplus \mathbf{Z}_p \cdot z \cong \mathscr{T}_{\eta} \quad \text{and} \quad \mathscr{T}^{(p)}_{\check{\eta}} \cong \mathbf{Z}_p \oplus \mathbf{Z}_p \cdot pz \cong \mathscr{T}_{\check{\eta}}. \tag{170}$$

Here the subscripts η and $\check{\eta}$ denote the stalks at η and $\check{\eta}$ respectively, and for each ω in $\mathbf{H}$ one writes

$$\mathbf{Z}_p \oplus \mathbf{Z}_p \cdot \omega = H_1(\mathbf{C}/\Lambda_\omega, \mathbf{Z}) \otimes_{\mathbf{Z}} \mathbf{Z}_p$$

for the p-adic completion of the integral homology of the complex elliptic curve $\mathbf{C}/\Lambda_\omega$, where $\Lambda_\omega = \mathbf{Z} \oplus \mathbf{Z} \cdot \omega$. As in Sections 3 and 4.2, after identifying $\mathscr{T}_{(p),\eta}$ with $\mathbf{Z}_p \oplus \mathbf{Z}_p$ under the $\mathbf{Z}_p$-basis $\{1, z\}$, the natural action of the étale fundamental group $\mathcal{G}_{(p)} = \pi_1^{\text{ét}}(Y(1, N(p)), \eta)$ (resp., $\mathcal{G}^{(p)} = \pi_1^{\text{ét}}(Y(1(p), N), \check{\eta})$) on $\mathscr{T}_{(p),\eta}$ (resp., $\mathscr{T}^{(p)}_{\check{\eta}}$) gives a continuous representation $\varrho_{(p)} : \mathcal{G}_{(p)} \to \Gamma(1, N(p)) \otimes_{\mathbf{Z}} \mathbf{Z}_p \hookrightarrow \mathrm{GL}_2(\mathbf{Z}_p)$ ($\varrho^{(p)} : \mathcal{G}^{(p)} \to \Gamma(1(p), N) \otimes_{\mathbf{Z}} \mathbf{Z}_p \hookrightarrow \mathrm{GL}_2(\mathbf{Z}_p)$), where $\Gamma(1, N(p))$ (resp., $\Gamma(1(p), N)$) is the subgroup of matrices in $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ in $\mathrm{SL}_2(\mathbf{Z})$ with $c \equiv 0$, $d \equiv 1 \pmod N$ and $c \equiv 0 \pmod p$ (resp., $b \equiv 0 \pmod p$). For each $i \geqslant 0$ set

$$\mathscr{S}_{(p),i} = \mathrm{Symm}^i_{\mathbf{Z}_p} \mathscr{T}_{(p)}(-1) \quad \text{and} \quad \mathscr{S}^{(p)}_i = \mathrm{Symm}^i_{\mathbf{Z}_p} \mathscr{T}^{(p)}(-1),$$

where as in Section 2.3 the Tate twists $\mathscr{T}_{(p)}(-1)$ and $\mathscr{T}^{(p)}(-1)$ are identified with the duals of $\mathscr{T}_{(p)}$ and $\mathscr{T}^{(p)}$ under the Weil pairings on $E(1, N(p))$ and $E(1(p), N)$ respectively. Then the stalks of $\mathscr{S}_{(p),i}$ and $\mathscr{S}^{(p)}_i$ at η and $\check{\eta}$, viewed as representations of $\mathcal{G}_{(p)}$ and $\mathcal{G}^{(p)}$ respectively, correspond via $\varrho_{(p)}$ and $\varrho^{(p)}$ to the $\Gamma(1, N(p))$-module $S_i = S_i(\mathbf{Z}_p)$ and the $\Gamma(1(p), N)$-module S_i (cf. Section 3). As a consequence, for each

$j \geqslant 0$ and $u \in \mathbf{Z}$ there is a natural inclusion (cf. Section 4.2)

$$H^0(\Gamma(1,N(p)), S_i \otimes \det{}^{-u}) \longrightarrow H^0(\mathcal{G}_{(p)}, S_i \otimes \det{}^{-u}) = H^0_{\text{ét}}(Y(1,N(p)), \mathscr{S}_{(p),i} \otimes_{\mathbf{Z}_p} \mathbf{Z}_p(u)), \tag{171}$$

and an isomorphism

$$H^j_{\text{ét}}(Y(1,N(p))_{\bar{\mathbf{Q}}}, \mathscr{S}_{(p),i}) \cong H^j(\Gamma(1,N(p)), S_i),$$

and similarly for the data $(\Gamma(1(p),N), \mathcal{G}^{(p)}, \mathscr{S}^{(p)}_i)$ in place of $(\Gamma(1,N(p)), \mathcal{G}_{(p)}, \mathscr{S}_{(p),i})$. As already explained in Section 3, there are similar isomorphisms after replacing $\varrho_{(p)}$ with the representations $\varrho : \mathcal{G} \to \mathrm{GL}_2(\mathbf{Z}_p)$ (resp., $\check{\varrho} : \check{\mathcal{G}} \to \mathrm{GL}_2(\mathbf{Z}_p)$) arising from the action of $\mathcal{G} = \pi_1^{\text{ét}}(Y(1,N),\eta)$ (resp., $\check{\mathcal{G}} = \pi_1^{\text{ét}}(Y(1,N),\check{\eta})$) on the stalk at η (resp., $\check{\eta}$) of $\mathscr{S}_i = \mathscr{S}_i(\mathbf{Z}_p)$. Under these isomorphisms, the maps

$$\begin{aligned} \lambda^i_{p*} &= (\lambda^i_{p*})_{\check{\eta}} : S_i \cong (\mathscr{S}_{(p),i})_\eta \longrightarrow (\mathscr{S}^{(p)}_i)_{\check{\eta}} \cong S_i \\ \text{and} \quad \lambda^{i*}_p &= (\lambda^{i*}_p)_\eta : S_i \cong (\mathscr{S}^{(p)}_i)_{\check{\eta}} \longrightarrow (\mathscr{S}_{(p),i})_\eta \cong S_i \end{aligned} \tag{172}$$

induced respectively on the stalks at $\check{\eta}$ and η by the morphisms (16) are given by

$$\lambda^i_{p*}(P) = \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} \cdot P \quad \text{and} \quad \lambda^{i*}_p(P) = \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} \cdot P, \tag{173}$$

for P in S_i. Indeed the base change $\lambda_{\check{\eta}} : \mathbf{C}/\Lambda_z = E(1,N(p)) \times_\eta \mathbf{C} \longrightarrow E(1(p),N) \times_{\check{\eta}} \mathbf{C} \cong \mathbf{C}/\Lambda_{pz}$ of the p-isogeny λ_p along $\check{\eta}$ is induced by multiplication by p on $\mathbf{C}$, hence the map $\lambda_{\check{\eta}*} : \mathscr{T}^{(p)} \longrightarrow \mathscr{T}_{(p)}$ it induces on the Tate modules is represented by $\left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right)$, once one identifies $\mathscr{T}_{(p)}$ and $\mathscr{T}^{(p)}$ with $\mathbf{Z}_p^2$ under the $\mathbf{Z}_p$-bases $\{1,z\}$ and $\{1,pz\}$ (cf. Equation (170)). Because the dual isogeny $\lambda'_{\check{\eta}}$ of $\lambda_{\check{\eta}}$ is the map $\mathbf{C}/\Lambda_{pz} \to \mathbf{C}/\Lambda_z$ induced by the identity on $\mathbf{C}$, and $\lambda_{\check{\eta}*}$ and $\lambda'_{\check{\eta}*}$ are adjoint to each other under the Weil pairings on $\mathbf{C}/\Lambda_z$ and $\mathbf{C}/\Lambda_{pz}$, Equation (173) follows.

After this preliminary discussion, we divide the proof into three steps. For each triple i,j,k of elements of $\{1,p\}$ write

$$\mathrm{pr}_{ijk*} = \mathrm{pr}_{i*} \otimes \mathrm{pr}_{j*} \otimes \mathrm{pr}_{k*} : Z_{Np,\boldsymbol{r}}(n) \to Z_{Np,\boldsymbol{r}}(n),$$

for $n \in \mathbf{Z}$ and $Z = V$ or $Z = W$, and denote by the same symbol the map they induce in $G_{\mathbf{Q}}$-cohomology. For any curve X over $\mathbf{Q}$ write $d : X \longrightarrow X^3$ for the diagonal embedding.

Step 1. One has the identities in $H^1(\mathbf{Q}, V_{N,\boldsymbol{r}}(r+2))$:

$$\mathrm{pr}_{111*}(\kappa_{Np,\boldsymbol{r}}) = (p^2-1) \cdot \kappa_{N,\boldsymbol{r}} \quad \text{and} \quad \mathrm{pr}_{ppp*}(\kappa_{Np,\boldsymbol{r}}) = (p^2-1)p^r \cdot \kappa_{N,\boldsymbol{r}}. \tag{174}$$

As the element $\mathrm{Det}^{\boldsymbol{r}} = \mathrm{Det}^{\boldsymbol{r}}_N$ is invariant under $\mathrm{GL}_2(\mathbf{Z}_p)$, it defines under the inclusion (171) an element $\mathtt{Det}^{\boldsymbol{r}}$ in $H^0_{\text{ét}}(Y(1,N(p)), \mathscr{S}_{(p),i}(r))$, and similarly elements (denoted by the same symbol) in $H^0_{\text{ét}}(Y(1(p),N), \mathscr{S}_{(p),i}(r))$ and $H^0_{\text{ét}}(Y(1,N), \mathscr{S}_i(r))$. According to Equation (173) and the definition of $\mathrm{Det}^{\boldsymbol{r}}$ in Equation (41) one has

$$\lambda^{\boldsymbol{r}}_{p*}(\mathrm{Det}^{\boldsymbol{r}}) = p^r \cdot \mathrm{Det}^{\boldsymbol{r}}, \tag{175}$$

where $\lambda_{p*}^{\boldsymbol{r}} = \lambda_{p*}^{r_1} \otimes \lambda_{p*}^{r_2} \otimes \lambda_{p*}^{r_3} \otimes \mathrm{id} : S_{\boldsymbol{r}} \otimes \det^{-r} \to S_{\boldsymbol{r}} \otimes \det^{-r}$, hence (since $\check{\nu}_p$ has degree $p+1$)

$$\check{\nu}_{p*} \circ \varphi_{p*} \circ \lambda_{p*}^{\boldsymbol{r}}(\mathtt{Det}^{\boldsymbol{r}}) = (p+1)p^r \cdot \mathtt{Det}^{\boldsymbol{r}} \in H^0_{\text{ét}}(Y(1,N), \mathscr{S}_{\boldsymbol{r}}(r)).$$

Retracing the definitions of Section 2.3 and using Equation (21) this gives

$$\mathrm{pr}_{p*}(\mathtt{Det}^{\boldsymbol{r}}) = (p^2-1)p^r \cdot \mathtt{Det}^{\boldsymbol{r}}.$$

The previous equation and the functoriality of the Hochschild–Serre spectral sequence implies (cf. Section 3)

$$\mathrm{pr}_{ppp*}(\kappa_{Np,\boldsymbol{r}}) = \mathtt{s}_{\boldsymbol{r}*} \circ \mathtt{HS} \circ \mathrm{pr}_{ppp*} \circ d_*(\mathtt{Det}^{\boldsymbol{r}}) = \mathtt{s}_{\boldsymbol{r}*} \circ \mathtt{HS} \circ d_* \circ \mathrm{pr}_{p*}(\mathtt{Det}^{\boldsymbol{r}}) = (p^2-1)p^r \cdot \kappa_{N,\boldsymbol{r}}.$$

This proves the second identity in Equation (174). The first one is proved by a similar (and simpler) argument.

Step 2. The following identities hold in $H^1(\mathbf{Q}, V_{N,\boldsymbol{r}}(r+2))$:

(176)

$$\mathrm{pr}_{p11*}(\kappa_{Np,\boldsymbol{r}}) = (p-1) \cdot T_p \otimes \mathrm{id} \otimes \mathrm{id}(\kappa_{N,\boldsymbol{r}}); \qquad \mathrm{pr}_{1pp*}(\kappa_{Np,\boldsymbol{r}}) = (p-1)p^{r-r_1} \cdot T'_p \otimes \mathrm{id} \otimes \mathrm{id}(\kappa_{N,\boldsymbol{r}});$$

$$\mathrm{pr}_{1p1*}(\kappa_{Np,\boldsymbol{r}}) = (p-1) \cdot \mathrm{id} \otimes T_p \otimes \mathrm{id}(\kappa_{N,\boldsymbol{r}}); \qquad \mathrm{pr}_{p1p*}(\kappa_{Np,\boldsymbol{r}}) = (p-1)p^{r-r_2} \cdot \mathrm{id} \otimes T'_p \otimes \mathrm{id}(\kappa_{N,\boldsymbol{r}});$$

$$\mathrm{pr}_{11p*}(\kappa_{Np,\boldsymbol{r}}) = (p-1) \cdot \mathrm{id} \otimes \mathrm{id} \otimes T_p(\kappa_{N,\boldsymbol{r}}); \qquad \mathrm{pr}_{pp1*}(\kappa_{Np,\boldsymbol{r}}) = (p-1)p^{r-r_3} \cdot \mathrm{id} \otimes \mathrm{id} \otimes T'_p(\kappa_{N,\boldsymbol{r}}).$$

We prove the second identity in the first line. Note that the finite étale cover $\check{\nu}_p$ is not Galois. To remedy this let $\vartheta : \mathcal{Y} \longrightarrow Y(1,N)$ be a finite étale Galois morphism which factors through $\check{\nu}_p \circ \varphi_p : Y(1,N(p)) \longrightarrow Y(1,N)$, say $\vartheta = \check{\nu}_p \circ \varphi_p \circ \alpha$ with $\alpha : \mathcal{Y} \longrightarrow Y(1,N(p))$. Denote by $G = \mathrm{Gal}(\vartheta)$ its Galois group. For each $u \geqslant 1$ denote by $\pi_{1*}^u = \nu_{p*} : H^1(Y(1,N(p)), \mathscr{S}_{(p),u}) \to H^1(Y(1,N), \mathscr{S}_u)$, and similarly set $\pi_1^{u*} = \nu_p^*$. Set

$$\begin{aligned} \pi_{p*}^u &= \check{\nu}_{p*} \circ \varphi_{p*} \circ \lambda_{p*}^u, \\ \pi_p^{u*} &= \lambda_p^{u*} \circ \varphi_p^* \circ \check{\nu}_p^*, \\ \pi_{ijk}^{\boldsymbol{r}*} &= \pi_i^{r_1*} \otimes \pi_j^{r_2*} \otimes \pi_k^{r_3*} \\ \text{and} \quad \pi_{ijk*}^{\boldsymbol{r}} &= \pi_{i*}^{r_1} \otimes \pi_{j*}^{r_2} \otimes \pi_{k*}^{r_3}, \end{aligned}$$

where i, j, k is any triple of elements of $\{1, p\}$. Moreover for each morphism $a : X \to Y$ of curves over $\mathbf{Q}$ write $\boldsymbol{a} = a \times_{\mathbf{Q}} a \times_{\mathbf{Q}} a : X^3 \to Y^3$. With these notations it follows directly from the definitions that

$$\pi_{1pp*}^{\boldsymbol{r}} \circ \pi_{ppp}^{\boldsymbol{r}*} = (p+1)^2 p^{r_2+r_3} \cdot T'_p \otimes \mathrm{id} \otimes \mathrm{id}. \tag{177}$$

On the other hand, after setting

$$\kappa_{Np,\boldsymbol{r}}^{\star} = \mathtt{s}_{\boldsymbol{r}*} \circ \mathtt{HS} \circ d_* \circ \vartheta^*(\mathtt{Det}^{\boldsymbol{r}}),$$

one has $(p+1)\deg(\alpha) \cdot \kappa_{N,\boldsymbol{r}} = \boldsymbol{\vartheta}_*(\kappa_{Np,\boldsymbol{r}}^{\star})$, hence

$$\begin{aligned} (p+1)\deg(\alpha)^4 \cdot \pi_{ppp}^{\boldsymbol{r}*}(\kappa_{N,\boldsymbol{r}}) &= \lambda_p^{\boldsymbol{r}*} \circ \boldsymbol{\alpha}_* \circ \boldsymbol{\vartheta}^* \circ \boldsymbol{\vartheta}_*(\kappa_{Np,\boldsymbol{r}}^{\star}) \\ &= \sum_{(g_1,g_2,g_3) \in G^3} \lambda_p^{\boldsymbol{r}*} \circ \boldsymbol{\alpha}_* \circ (g_1 \times g_2 \times g_3)_*(\kappa_{Np,\boldsymbol{r}}^{\star}). \end{aligned}$$

For each $g, h \in G$ one has $\pi^{r_i}_{p*} \circ \lambda^{r_i *}_p \circ \alpha_* \circ g_* = p^{r_i} \cdot \vartheta_* = p^{r_i} \cdot \vartheta_* \circ h_*$, hence the previous equation yields

(178)

$$(p+1)\deg(\alpha)^4 \cdot \pi^{\boldsymbol{r}}_{1pp*} \circ \pi^{\boldsymbol{r}*}_{ppp}(\kappa_{N,\boldsymbol{r}})$$
$$= p^{r_2+r_3} \sum_{(g_1,g_2,g_3)\in G^3} \big(\nu_{p*} \circ \lambda^{r_1 *}_p \otimes \check{\nu}_{p*} \circ \varphi_{p*} \otimes \check{\nu}_{p*} \circ \varphi_{p*}\big) \circ \boldsymbol{\alpha}_* \circ \boldsymbol{g}_{1*}(\kappa^{\star}_{Np,\boldsymbol{r}})$$
$$= (p+1)^3 p^{r_2+r_3} \deg(\alpha)^4 \cdot (\nu_{p*} \otimes \check{\nu}_{p*} \circ \varphi_{p*} \otimes \check{\nu}_{p*} \circ \varphi_{p*}) \circ (\lambda^{r_1 *}_p \otimes \mathrm{id} \otimes \mathrm{id})(\kappa^{\bullet}_{Np,\boldsymbol{r}}),$$

where $\kappa^{\bullet}_{Np,\boldsymbol{r}} = \mathtt{s}_{\boldsymbol{r}*} \circ \mathtt{HS} \circ d_* \circ (\check{\nu}_p \circ \varphi_p)^*(\mathtt{Det}^{\boldsymbol{r}})$. According to Equations (41) and (173)

$$\lambda^{\boldsymbol{r}}_{p*}(\kappa^{\bullet}_{Np,\boldsymbol{r}}) = \lambda^{r_1}_{p*} \otimes \lambda^{r_2}_{p*} \otimes \lambda^{r_3}_{p*}(\kappa^{\bullet}_{Np,\boldsymbol{r}}) = p^r \cdot \kappa^{\bullet}_{Np,\boldsymbol{r}}$$
$$\text{and} \quad \lambda^{r_1 *}_p \circ \lambda^{r_1}_{p*}(P) = \begin{pmatrix} p & 0 \\ 0 & p \end{pmatrix} \cdot P = p^{r_1} \cdot P,$$

for P in S_{r_1}, hence (since $2r = r_1 + r_2 + r_3$) one can rewrite Equation (178) as

$$\pi^{\boldsymbol{r}}_{1pp*} \circ \pi^{\boldsymbol{r}*}_{ppp}(\kappa_{N,\boldsymbol{r}}) = (p+1)^2 p^r \cdot \pi^{\boldsymbol{r}}_{1pp*}(\kappa^{\bullet}_{Np,\boldsymbol{r}}). \tag{179}$$

(Note that, regarding the natural isomorphism of Equation (171) and its analogue for $Y(1, N(p))$ as equalities, the pullback by $\check{\nu}_p \circ \varphi_p$ is identified with the identity.) In addition Equation (8) gives

$$\mathrm{pr}_{1pp*}(\kappa_{Np,\boldsymbol{r}}) = \pi^{\boldsymbol{r}}_{1pp*} \circ \boldsymbol{\mu}_{p*}(\kappa_{Np,\boldsymbol{r}}) = (p-1) \cdot \pi^{\boldsymbol{r}}_{1pp*}(\kappa^{\bullet}_{Np,\boldsymbol{r}}). \tag{180}$$

Equations (177), (179) and (180) finally give

$$(p+1)^2 p^r \cdot \mathrm{pr}_{1pp*}(\kappa_{Np,\boldsymbol{r}}) = (p-1)(p+1)^2 p^{r_2+r_3} \cdot T'_p \otimes \mathrm{id} \otimes \mathrm{id}(\kappa_{N,\boldsymbol{r}}).$$

This proves the second identity in the first line of Equation (176). The other equalities in the second column (resp., the equalities in the first column) are proved by a similar (resp., similar and simpler) argument.

Step 3. We can now conclude the proof of the proposition.

Applying the projector $\mathrm{pr}_{f_k g_l h_m}$ (see Equation (48)) to the identities (174) and (176) gives

$$\begin{aligned}
\mathrm{pr}_{111*}(\kappa_{Np,\boldsymbol{r}})_{fgh} &= (p^2-1) \cdot \kappa(f,g,h);\\
\mathrm{pr}_{ppp*}(\kappa_{Np,\boldsymbol{r}})_{fgh} &= p^r(p^2-1) \cdot \kappa(f,g,h);\\
\mathrm{pr}_{p11*}(\kappa_{Np,\boldsymbol{r}})_{fgh} &= (p-1)\bar{\chi}_{\boldsymbol{f}}(p) a_p(f) \cdot \kappa(f,g,h);\\
\mathrm{pr}_{1pp*}(\kappa_{Np,\boldsymbol{r}})_{fgh} &= (p-1)p^{r-r_1} a_p(f) \cdot \kappa(f,g,h);\\
\mathrm{pr}_{1p1*}(\kappa_{Np,\boldsymbol{r}})_{fgh} &= (p-1)\bar{\chi}_{\boldsymbol{g}}(p) a_p(g) \cdot \kappa(f,g,h);\\
\mathrm{pr}_{p1p*}(\kappa_{Np,\boldsymbol{r}})_{fgh} &= (p-1)p^{r-r_2} a_p(g) \cdot \kappa(f,g,h);\\
\mathrm{pr}_{11p*}(\kappa_{Np,\boldsymbol{r}})_{fgh} &= (p-1)\bar{\chi}_{\boldsymbol{h}}(p) a_p(h) \cdot \kappa(f,g,h);\\
\mathrm{pr}_{pp1*}(\kappa_{Np,\boldsymbol{r}})_{fgh} &= (p-1)p^{r-r_3} a_p(h) \cdot \kappa(f,g,h).
\end{aligned} \tag{181}$$

Here $(f,g,h) = (f_k, g_l, h_m)$, $\mathrm{pr}_{ijk*}(\kappa_{Np,\boldsymbol{r}})_{fgh}$ is a shorthand for the image of $\mathrm{pr}_{ijk*}(\kappa_{Np,\boldsymbol{r}})$ under $\mathrm{pr}_{fgh*} = \mathrm{pr}_{f_k g_l h_m *}$, and we used the identity $T'_p = T_p \circ \langle p \rangle'$ as endomorphisms of $H^1_{\text{ét}}(Y_1(N)_{\bar{\mathbf{Q}}}, \mathscr{L}_i(j))_{\mathbf{Q}_p}$. Because the map

$$\mathtt{s}_i : H^1_{\text{ét}}(Y_1(Np)_{\bar{\mathbf{Q}}}, \mathscr{S}_i) \to H^1_{\text{ét}}(Y_1(Np)_{\bar{\mathbf{Q}}}, \mathscr{L}_i)(-i)$$

intertwines the action of the dual Atkin–Lehner operators w'_p on both sides, it follows from the definitions that

$$\Pi^{\alpha}_{fgh*}\big(\kappa^{\dagger}(\boldsymbol{f}_k, \boldsymbol{g}_l, \boldsymbol{h}_m)\big) = \mathrm{pr}_{fgh*}\Big(\Pi^{\alpha}_{\boldsymbol{r}*}\Big(\big(w'_p \otimes \mathrm{id} \otimes \mathrm{id}\big)_*(\kappa_{Np,\boldsymbol{r}})\Big)\Big). \tag{182}$$

It it easily checked that

$$\mathrm{pr}_{p*} \circ w'_p = p^i \cdot \mathrm{pr}_{1*} \quad \text{and} \quad \mathrm{pr}_{1*} \circ w'_p = \langle p\rangle' \cdot \mathrm{pr}_{p*}$$

as morphisms from $H^1_{\text{ét}}(Y_1(Np)_{\bar{\mathbf{Q}}}, \mathscr{L}_i)$ to $H^1_{\text{ét}}(Y_1(N)_{\bar{\mathbf{Q}}}, \mathscr{L}_i)$. As a consequence, setting $\langle p\rangle'_f = \langle p\rangle' \otimes \mathrm{id} \otimes \mathrm{id}$ and writing $\alpha_f = \alpha_{\boldsymbol{f}_k}, \beta_f = \beta_{\boldsymbol{f}_k}, \alpha_g = \alpha_{\boldsymbol{g}_l}$ et cetera, one has

$$\begin{aligned}
&\Pi^{\alpha}_{\boldsymbol{r}*} \circ \big(w'_p \otimes \mathrm{id} \otimes \mathrm{id}\big) \\
&\quad = \left(\langle p\rangle' \cdot \mathrm{pr}_{p*} - \frac{\beta_f}{p} \cdot \mathrm{pr}_{1*}\right) \otimes \left(\mathrm{pr}_{1*} - \frac{\beta_g}{p^{r_2+1}} \cdot \mathrm{pr}_{p*}\right) \otimes \left(\mathrm{pr}_{1*} - \frac{\beta_h}{p^{r_3+1}} \cdot \mathrm{pr}_{p*}\right) \\
&\quad = \langle p\rangle'_f \cdot \mathrm{pr}_{p11*} - \frac{\beta_f}{p} \cdot \mathrm{pr}_{111*} - \frac{\beta_g \,\langle p\rangle'_f}{p^{r_2+1}} \cdot \mathrm{pr}_{pp1*} - \frac{\beta_h \,\langle p\rangle'_f}{p^{r_3+1}} \cdot \mathrm{pr}_{p1p*} + \frac{\beta_f\beta_g}{p^{r_2+2}} \cdot \mathrm{pr}_{1p1*} \\
&\qquad + \frac{\beta_f\beta_h}{p^{r_3+2}} \cdot \mathrm{pr}_{11p*} + \frac{\beta_g\beta_h \,\langle p\rangle'_f}{p^{r_2+r_3+2}} \cdot \mathrm{pr}_{ppp*} - \frac{\beta_f\beta_g\beta_h}{p^{r_2+r_3+3}} \cdot \mathrm{pr}_{1pp*}.
\end{aligned}$$

Together with Equations (181) and (182) this yields

$$\Pi^{\alpha}_{fgh*}\big(\kappa^{\dagger}(\boldsymbol{f}_k, \boldsymbol{g}_l, \boldsymbol{h}_m)\big) = (p-1) \cdot \mathscr{E}_f(f,g,h) \cdot \kappa(f,g,h),$$

where (recalling that $a_p(\xi) = \alpha_\xi + \beta_\xi$ and $\alpha_\xi\beta_\xi = \chi_{\boldsymbol{\xi}}(p)p^{s-1}$ for $\xi \in S_s(N, \chi_{\boldsymbol{\xi}})$, that $2r = r_1 + r_2 + r_3$ and that $\chi_{\boldsymbol{f}}\chi_{\boldsymbol{g}}\chi_{\boldsymbol{h}}(p) = 1$ by Assumption 1.2)

$$\begin{aligned}
\mathscr{E}_f(f,g,h) &= \alpha_f + \beta_f - \beta_f - \frac{\beta_f}{p} - \frac{\chi_{\boldsymbol{f}}(p)\beta_g\alpha_h}{p^{r_2+r_3-r+1}} - \frac{\chi_{\boldsymbol{f}}(p)\beta_g\beta_h}{p^{r_2+r_3-r+1}} - \frac{\chi_{\boldsymbol{f}}(p)\alpha_g\beta_h}{p^{r_3+r_2-r+1}} \\
&\quad - \frac{\chi_{\boldsymbol{f}}(p)\beta_g\beta_h}{p^{r_3+r_2-r+1}} + \frac{\bar{\chi}_{\boldsymbol{g}}(p)\beta_f\alpha_g\beta_g}{p^{r_2+2}} + \frac{\bar{\chi}_{\boldsymbol{g}}(p)\beta_f\beta_g^2}{p^{r_2+2}} + \frac{\bar{\chi}_{\boldsymbol{h}}(p)\beta_f\beta_h\alpha_h}{p^{r_3+2}} + \frac{\bar{\chi}_{\boldsymbol{h}}(p)\beta_f\beta_h^2}{p^{r_3+2}} \\
&\quad + \frac{\chi_{\boldsymbol{f}}(p)\beta_g\beta_h}{p^{r_2+r_3-r+1}} + \frac{\chi_{\boldsymbol{f}}(p)\beta_g\beta_h}{p^{r_2+r_3-r+2}} - \frac{\alpha_f\beta_f\beta_g\beta_h}{p^{r_1+r_2+r_3-r+3}} - \frac{\beta_f^2\beta_g\beta_h}{p^{r_1+r_2+r_3-r+3}} \\
&= \alpha_f \cdot \left(1 - \frac{\beta_f\beta_g\alpha_h}{p^{r+2}} - \frac{\beta_f\alpha_g\beta_h}{p^{r+2}} - \frac{\beta_f\beta_g\beta_h}{p^{r+2}} + \frac{\chi_{\boldsymbol{h}}(p)\beta_f^2\beta_g^2}{p^{r_1+r_2+3}} \right. \\
&\qquad \left. + \frac{\bar{\chi}_{\boldsymbol{f}}(p)\beta_f^2}{p^{r_1+2}} + \frac{\chi_{\boldsymbol{g}}(p)\beta_f^2\beta_h^2}{p^{r_1+r_3+3}} - \frac{\bar{\chi}_{\boldsymbol{f}}(p)\beta_f^3\beta_g\beta_h}{p^{r+r_1+4}}\right) \\
&= \alpha_f \cdot \left(1 - \frac{\beta_f\alpha_g\beta_h}{p^{r+2}}\right)\left(1 - \frac{\beta_f\beta_g\alpha_h}{p^{r+2}}\right)\left(1 - \frac{\beta_f\beta_g\beta_h}{p^{r+2}}\right).
\end{aligned} \tag{183}$$

This concludes the proof of the proposition. ∎

8.4. p-stabilisation of de Rham classes. — Let $w = (k, l, m)$ be a classical triple in Σ, such that p does not divide the conductors of $\boldsymbol{f}_k, \boldsymbol{g}_l$ and $\boldsymbol{h}_m$. As in the previous section denote by f_k, g_l and h_m the modular forms of level $\Gamma_1(N)$ with ordinary p-stabilisations $\boldsymbol{f}_k$, $\boldsymbol{g}_l$ and $\boldsymbol{h}_m$ respectively. For each integer $M \geqslant 3$ denote by $V^*_{\mathrm{dR},\boldsymbol{r}}(M)$ the $(k+l+m-2)/2$-th Tate twist of the tensor product of the de Rham cohomology groups $H^1_{\mathrm{dR}}(Y_1(M)_{\mathbf{Q}_p}, \mathscr{S}_{\mathrm{dR},r_j})_L$, for $j = 1, 2, 3$. Then the restriction of the morphism

$$V^*_{\mathrm{dR},\boldsymbol{r}}(N) \longrightarrow V^*_{\mathrm{dR},\boldsymbol{r}}(Np)$$

defined by

$$\left(\mathrm{pr}_1^* - \frac{\beta_{\boldsymbol{f}_k}}{p^{k-1}} \cdot \mathrm{pr}_p^*\right) \otimes \left(\mathrm{pr}_1^* - \frac{\beta_{\boldsymbol{g}_l}}{p^{l-1}} \cdot \mathrm{pr}_p^*\right) \otimes \left(\mathrm{pr}_1^* - \frac{\beta_{\boldsymbol{h}_m}}{p^{m-1}} \cdot \mathrm{pr}_p^*\right)$$

to the (f, g, h)-isotypic component of $V^*_{\mathrm{dR},\boldsymbol{r}}(N)$ gives a *p-stabilisation* isomorphism

$$\Pi^{\alpha *}_{f_k g_l h_m} : V^*_{\mathrm{dR}}(f_k, g_l, h_m) \cong V^*_{\mathrm{dR}}(\boldsymbol{f}_k, \boldsymbol{g}_l, \boldsymbol{h}_m).$$

Lemma 8.4. — *Assume that p does not divide the conductors of $\boldsymbol{f}_k, \boldsymbol{g}_l$ and $\boldsymbol{h}_m$. Then*

$$\Pi^{\alpha *}_{f_k g_l h_m}\big(\eta^{\alpha}_{f_k} \otimes \omega_{g_l} \otimes \omega_{h_m}\big) = (p-1)\alpha_{\boldsymbol{f}_k}\left(1 - \frac{\beta_{\boldsymbol{f}_k}}{\alpha_{\boldsymbol{f}_k}}\right)\left(1 - \frac{\beta_{\boldsymbol{f}_k}}{p\alpha_{\boldsymbol{f}_k}}\right) \cdot \eta^{\alpha}_{\boldsymbol{f}_k} \otimes \omega_{\boldsymbol{g}_l} \otimes \omega_{\boldsymbol{h}_m}.$$

Proof. — Set $\Pi^{\alpha *}_k = \mathrm{pr}_1^* - \frac{\beta_{\boldsymbol{f}_k}}{p^{k-1}} \cdot \mathrm{pr}_p^*$, set $\Pi^{\alpha}_{k*} = \mathrm{pr}_{1*} - \frac{\beta_{\boldsymbol{f}_k}}{p^{k-1}} \cdot \mathrm{pr}_{p*}$ and define similarly $\Pi^{\alpha *}_l$ and $\Pi^{\alpha *}_m$. By the definition of p-stabilisation (cf. Equation (54)), one has $\Pi^{\alpha *}_k(\omega_\xi) = \omega_{\xi_\alpha}$ for any $\xi \in S_k(N, L)_{f_k}$, and similarly for $\Pi^{\alpha *}_l$ and $\Pi^{\alpha *}_m$. In particular

$$\Pi^{\alpha *}_l(\omega_{g_l}) = \omega_{\boldsymbol{g}_l} \quad \text{and} \quad \Pi^{\alpha *}_m(\omega_{h_m}) = \omega_{\boldsymbol{h}_m}. \tag{184}$$

According to Equation (3.4.5) on Page 76 of [Shi71], one has

$$(a^w, b^w)_M = M^{n-2} \cdot (a, b)_M$$

for any cuspidal forms a and b of weight n and level $\Gamma_1(M)$, where we recall that $\cdot^w = w_M(\cdot)$ is a shorthand for the image of $\cdot$ under the Atkin–Lehner operator w_M defined in Equation (33), and $(\cdot, \cdot)_M$ is the Petersson product on $S_n(M, \mathbf{C})$ defined after Equation (35). It follows that (cf. Equation (34) and the discussion following it)

$$\big\langle \eta_{\boldsymbol{f}_k}, w_{Np} \circ \Pi^{\alpha *}_k(\omega_\xi)\big\rangle_{\boldsymbol{f}_k} = \frac{(\boldsymbol{f}^w_k, \xi^w_\alpha)_{Np}}{(\boldsymbol{f}^w_k, \boldsymbol{f}^w_k)_{Np}} = \frac{(\boldsymbol{f}_k, \xi_\alpha)_{Np}}{(\boldsymbol{f}_k, \boldsymbol{f}_k)_{Np}} = \frac{(f_k, \xi)_N}{(f_k, f_k)_N}. \tag{185}$$

for each ξ in $S_k(N, L)_{f_k}$, where $\xi^w_\alpha = w_{Np}(\xi_\alpha)$.

The (easily verified) relations $w_{Np} \circ \mathrm{pr}_1^* = \mathrm{pr}_p^* \circ w_N$ and $w_{Np} \circ \mathrm{pr}_p^* = p^{k-2} \cdot \mathrm{pr}_1^* \circ w_N$ yield

$$\begin{aligned}\Pi^{\alpha}_{k*} \circ w_{Np} \circ \Pi^{\alpha *}_k &= \left(\mathrm{pr}_{1*} - \frac{\beta_{\boldsymbol{f}_k}}{p^{k-1}} \cdot \mathrm{pr}_{p*}\right) \circ \left(\mathrm{pr}_p^* - \frac{\beta_{\boldsymbol{f}_k}}{p} \cdot \mathrm{pr}_1^*\right) \circ w_N \\ &= (p-1)\left(T'_p - \frac{2(p+1)\beta_{\boldsymbol{f}_k}}{p} + \frac{\beta^2_{\boldsymbol{f}_k}}{p^k} \cdot T_p\right) \circ w_N.\end{aligned}$$

As $a_p(f_k) = \alpha_{\boldsymbol{f}_k} + \beta_{\boldsymbol{f}_k}$ and $T_p' \circ w_N$ and $T_p \circ w_N$ act respectively as $a_p(f_k) \cdot w_N$ and $\bar{\chi}_{\boldsymbol{f}}(p) a_p(f_k) \cdot w_N$ on $V_{\mathrm{dR}}^*(f_k)$, a direct computation then gives (cf. Equation (183))

$$\Pi_{k*}^{\alpha} \circ w_{Np} \circ \Pi_k^{\alpha *} = (p-1)\alpha_{\boldsymbol{f}_k} \left(1 - \frac{\beta_{\boldsymbol{f}_k}}{\alpha_{\boldsymbol{f}_k}}\right)\left(1 - \frac{\beta_{\boldsymbol{f}_k}}{p\alpha_{\boldsymbol{f}_k}}\right) \cdot w_N$$

as morphisms from $V_{\mathrm{dR}}^*(f_k)$ to $V_{\mathrm{dR}}^*(f_k^*)$. Because $\Pi_k^{\alpha *}$ and Π_{k*}^{α} are adjoint to each other under the pairings $\langle \cdot, \cdot \rangle_{f_k}$ and $\langle \cdot, \cdot \rangle_{\boldsymbol{f}_k}$, this implies

$$\frac{\langle \Pi_k^{\alpha *}(\eta_{f_k}), w_{Np} \circ \Pi_k^{\alpha *}(\omega_\xi) \rangle_{\boldsymbol{f}_k}}{(p-1)\alpha_{\boldsymbol{f}_k}\left(1 - \frac{\beta_{\boldsymbol{f}_k}}{\alpha_{\boldsymbol{f}_k}}\right)\left(1 - \frac{\beta_{\boldsymbol{f}_k}}{p\alpha_{\boldsymbol{f}_k}}\right)} = \langle \eta_{f_k}, w_N(\omega_\xi) \rangle_{f_k} = \frac{(f_k^w, \xi^w)_N}{(f_k^w, f_k^w)_N} = \frac{(f_k, \xi)_N}{(f_k, f_k)_N} \tag{186}$$

for each ξ in $S_k(N, L)_{f_k} = \mathrm{Fil}^1 V_{\mathrm{dR}}^*(f_k)$. As the composition $w_{Np} \circ \Pi_k^{\alpha *}$ gives an isomorphism between $S_k(N, L)_{f_k}$ and $S_k(Np, L)_{\boldsymbol{f}_k^*}$, and the isomorphism

$$\Pi_k^{\alpha *} : V_{\mathrm{dR}}^*(f_k) \cong V_{\mathrm{dR}}^*(\boldsymbol{f}_k)$$

commutes with the action of the Frobenius endomorphism on both sides, comparing Equation (185) with Equation (186) yields the identity

$$\Pi_k^{\alpha *}(\eta_{f_k}^{\alpha}) = (p-1)\alpha_{\boldsymbol{f}_k}\left(1 - \frac{\beta_{\boldsymbol{f}_k}}{\alpha_{\boldsymbol{f}_k}}\right)\left(1 - \frac{\beta_{\boldsymbol{f}_k}}{p\alpha_{\boldsymbol{f}_k}}\right) \cdot \eta_{\boldsymbol{f}_k}^{\alpha}$$

(cf. Equation (37) for the definition of the differential $\eta_{\boldsymbol{f}_k}^{\alpha}$). The lemma follows from the previous equation and Equation (184). □

8.5. Conclusion of the proof. — This section concludes the proof of Theorem A.

According to Corollary 8.2 the class $\kappa(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})$ belongs to $H^1_{\mathrm{bal}}(\mathbf{Q}, V(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h}))$. Let Σ_{bal}^o be the set of balanced triples (k, l, m) such that $k, l, m \geqslant 3$ and p does not divide the conductors of $\boldsymbol{f}_k, \boldsymbol{g}_l$ and $\boldsymbol{h}_m$. Let $\boldsymbol{\xi}$ denote one of $\boldsymbol{f}, \boldsymbol{g}$ and $\boldsymbol{h}$. Because Σ_{bal}^o is dense in $U_{\boldsymbol{f}} \times U_{\boldsymbol{g}} \times U_{\boldsymbol{h}}$, in order to prove Theorem A it is sufficient to show that

$$\mathscr{L}_{\boldsymbol{\xi}}\big(\kappa(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})\big)(w) = \mathscr{L}_p^{\xi}(\boldsymbol{f}_k, \boldsymbol{g}_l, \boldsymbol{h}_m) \tag{187}$$

for every $w = (k, l, m)$ in Σ_{bal}^o, where to ease the notation one writes

$$\mathscr{L}_{\boldsymbol{\xi}}(\kappa(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})) = \mathscr{L}_{\boldsymbol{\xi}}(\mathrm{res}_p(\kappa(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h}))).$$

Fix such a triple w and to ease notation set $\alpha_f = \alpha_{\boldsymbol{f}_k}, \beta_f = \beta_{\boldsymbol{f}_k}, \alpha_g = \alpha_{\boldsymbol{g}_l}$ et cetera.

Consider first the case $\boldsymbol{\xi} = \boldsymbol{f}$. Write as usual $\boldsymbol{r} = (r_1, r_2, r_3) = (k-2, l-2, m-2)$. Since p does not divide the conductor of $\boldsymbol{f}_k, \boldsymbol{g}_l$ and $\boldsymbol{h}_m$, the Ramanujan–Petersson conjecture gives

$$\left(1 - \frac{\beta_f}{\alpha_f}\right)\left(1 - \frac{\beta_f}{p\alpha_f}\right)\left(1 - \frac{\alpha_f \beta_g \beta_h}{p^{r+2}}\right) \neq 0.$$

Moreover $\boldsymbol{f}_k = f_\alpha$ (resp., $\boldsymbol{g}_l = g_\alpha$, $\boldsymbol{h}_m = h_\alpha$) is the ordinary p-stabilisation of a cusp form $f = f_k$ (resp., $g = g_l$, $h = h_m$) of level $\Gamma_1(N)$. Proposition 7.3, the definition of $\log_p(\cdot)_f$ and Lemma 8.4 then prove that

$$(-1)^{r-r_1}(r-r_1)! \cdot \mathscr{L}_{\boldsymbol{f}}\big(\kappa(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})\big)(w)$$

is equal to

$$\frac{\left(1-\frac{\beta_f\alpha_g\alpha_h}{p^{r+2}}\right)}{\left(1-\frac{\beta_f}{\alpha_f}\right)\left(1-\frac{\beta_f}{p\alpha_f}\right)\left(1-\frac{\alpha_f\beta_g\beta_h}{p^{r+2}}\right)}\cdot \log_p\big(\kappa(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})_w\big)\big(\Pi^{\alpha *}_{fgh}\big(\eta_f^\alpha\otimes\omega_g\otimes\omega_h\big)\big),$$

where $\kappa(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})_w\in H^1_{\rm bal}(\mathbf{Q}_p,V(\boldsymbol{f}_k,\boldsymbol{g}_l,\boldsymbol{h}_m))$ is the image of $\kappa(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})$ under the specialisation map ρ_w (and as usual $\log_p(\cdot)$ is a shorthand for $\log_p(\mathrm{res}_p(\cdot))$ for all global classes $\cdot$ in $H^1_{\rm bal}(\mathbf{Q},V(\boldsymbol{f}_k,\boldsymbol{g}_l,\boldsymbol{h}_m)))$. As $\Pi^{\alpha *}_{fgh}$ is the transpose of Π^{α}_{fgh*}, the functoriality under correspondences of the Faltings comparison isomorphism for $E_1^{\cdot}(N)$ and of the Leray spectral sequence (from which Equation (26) is deduced) imply that

$$\log_p\big(\kappa(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})_w\big)\circ\Pi^{\alpha *}_{fgh}=\log_p\Big(\Pi^{\alpha}_{fgh*}\big(\kappa(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})_w\big)\Big) \tag{188}$$

as functionals on $\mathrm{Fil}^0V^*_{\rm dR}(f,g,h)$. According to Theorem 8.1 and Proposition 8.3

$$\Pi^{\alpha}_{fgh*}\big(\kappa(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})_w\big)$$

equals

$$\left(1-\frac{\alpha_f\beta_g\beta_h}{p^{r+2}}\right)\left(1-\frac{\beta_f\alpha_g\beta_h}{p^{r+2}}\right)\left(1-\frac{\beta_f\beta_g\alpha_h}{p^{r+2}}\right)\left(1-\frac{\beta_f\beta_g\beta_h}{p^{r+2}}\right)\cdot\kappa(f,g,h). \tag{189}$$

The previous three equations show that $\mathscr{L}_{\boldsymbol{f}}\big(\kappa(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})\big)(w)$ is equal to the product of

$$\frac{(-1)^{r-r_1}}{(r-r_1)!}\frac{\left(1-\frac{\beta_f\alpha_g\alpha_h}{p^{r+2}}\right)\left(1-\frac{\beta_f\alpha_g\beta_h}{p^{r+2}}\right)\left(1-\frac{\beta_f\beta_g\alpha_h}{p^{r+2}}\right)\left(1-\frac{\beta_f\beta_g\beta_h}{p^{r+2}}\right)}{\left(1-\frac{\beta_f}{\alpha_f}\right)\left(1-\frac{\beta_f}{p\alpha_f}\right)}$$

and

$$\log_p\big(\kappa(f,g,h)\big)(\eta_f^\alpha\otimes\omega_g\otimes\omega_h),$$

which in turn is equal to $\mathscr{L}_p^f(\boldsymbol{f}_k,\boldsymbol{g}_l,\boldsymbol{h}_m)$ by the explicit reciprocity law Proposition 3.6. This proves Equation (187), and with it Theorem A, for $\boldsymbol{\xi}=\boldsymbol{f}$.

The proofs of Equation (187) for $\boldsymbol{\xi}=\boldsymbol{g},\boldsymbol{h}$ are similar. We give the details for $\boldsymbol{\xi}=\boldsymbol{g}$. Exchanging the roles of $\boldsymbol{f}$ and $\boldsymbol{g}$ in the constructions of Sections 7.1, 7.3, and 8.4, (the resulting) Propositions 7.3 and 8.4 proves that

$$(-1)^{r-r_2}(r-r_2)!\cdot\mathscr{L}_{\boldsymbol{g}}\big(\kappa(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})\big)(w)$$

is equal to

$$\frac{\left(1-\frac{\alpha_f\beta_g\alpha_h}{p^{r+2}}\right)}{\left(1-\frac{\beta_g}{\alpha_g}\right)\left(1-\frac{\beta_g}{p\alpha_g}\right)\left(1-\frac{\beta_f\alpha_g\beta_h}{p^{r+2}}\right)}\cdot \log_p\big(\kappa(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})_w\big)\big(\Pi^{\alpha *}_{fgh}\big(\omega_f\otimes\eta_g^\alpha\otimes\omega_h\big)\big).$$

Equations (188)–(189) (which are symmetric in $(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})$) then prove that the special value $\mathscr{L}_{\boldsymbol{g}}(\kappa(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h}))(w)$ is the product of

$$\frac{(-1)^{r-r_2}}{(r-r_2)!}\cdot\frac{\left(1-\frac{\alpha_f\beta_g\alpha_h}{p^{r+2}}\right)\left(1-\frac{\alpha_f\beta_g\beta_h}{p^{r+2}}\right)\left(1-\frac{\beta_f\beta_g\alpha_h}{p^{r+2}}\right)\left(1-\frac{\beta_f\beta_g\beta_h}{p^{r+2}}\right)}{\left(1-\frac{\beta_g}{\alpha_g}\right)\left(1-\frac{\beta_g}{p\alpha_g}\right)}.$$

and

$$\log_p\big(\kappa(f,g,h)\big)(\omega_f\otimes\eta_g^\alpha\otimes\omega_h)$$

This is precisely the formula for $\mathscr{L}_p^g(\boldsymbol{f}_k,\boldsymbol{g}_l,\boldsymbol{h}_m)$ obtained by replacing the triple $(\boldsymbol{f}_k,\boldsymbol{g}_l,\boldsymbol{h}_m)$ with $(\boldsymbol{g}_l,\boldsymbol{f}_k,\boldsymbol{h}_m)$ in the statement of the explicit reciprocity law Proposition 3.6, thus concluding the proof of Theorem A.

9. Proof of Theorem B

This section proves Theorem B stated in the Introduction. The notations and assumptions are as in Section 1.2. Then $(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})$ is a level-N test vector for $(\boldsymbol{f}^\sharp,\boldsymbol{g}^\sharp,\boldsymbol{h}^\sharp)$ and $w_o=(k,l,m)$ is an *unbalanced* triple in $\Sigma_{\boldsymbol{f}}$.

For the convenience of the reader, we briefly describe the contents of the different subsections. Section 9.1 proves Theorem B assuming that w_o is not exceptional in the sense of Section 1.2. Section 9.2 proves an exceptional zero formula for the big logarithm $\mathscr{L}_{\boldsymbol{f}}$ when w_o is *exceptional of type* (5), viz. in the exceptional case arising from the vanishing at w_o of the analytic f-Euler factor $\mathcal{E}_f^*(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})$ introduced in Equation (4). Section 9.3 constructs the improved diagonal classes $\kappa_g^*(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})$ and $\kappa_h^*(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})$ introduced in Section 1.2. Their construction is nontrivial only when the g-Euler factor $\mathcal{E}_g(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})$ defined in Equation (1) vanishes at w_o, that is when w_o is exceptional of type (3) (cf. Section 1.2). Section 9.4 finally proves Theorem B when w_o is exceptional.

9.1. Proof in the non-exceptional case. — This section proves Theorem B when w_o is not exceptional.

Lemma 9.1. — *The Bloch–Kato finite, exponential and geometric subspaces of the local cohomology group $H^1(\mathbf{Q}_p,V(\boldsymbol{f}_k,\boldsymbol{g}_l,\boldsymbol{h}_m))$ are all equal.*

Proof. — We use the notations introduced in the proof of Lemma 3.5. As in loco citato, it is sufficient to prove that $D_{\mathrm{st}}^{\varphi=1,\mathrm{N}=0}$ vanishes.

Since $k\geqslant l+m$, one has $\mathrm{ord}_p(\alpha_w^f)\leqslant-1$ and $\mathrm{ord}_p(\beta_w^\cdot)\leqslant-1$ for $\cdot=\emptyset,g,h$, hence $D_{\mathrm{st}}^{\varphi=1}$ is contained in the L-module generated by $\mathbf{a}_w,\mathbf{a}_w^g,\mathbf{a}_w^h$ and $\mathbf{b}_w^f$. Moreover

$$|\alpha_w|_\infty=p^{(\varepsilon_w-1)/2},\quad|\alpha_w^\xi|_\infty=p^{(\varepsilon_w-2\cdot\varepsilon_\xi-1)/2}\quad\text{and}\quad|\beta_w^f|_\infty=p^{(2\cdot\varepsilon_f-\varepsilon_w-1)/2}$$

for $\xi=g,h$ (cf. loco citato for the notation). It follows that $D_{\mathrm{st}}^{\varphi=1}$ is equal to zero if $\varepsilon_w=0$ or $\varepsilon_w=2$. If $\varepsilon_w=3$, then $D_{\mathrm{st}}^{\varphi=1}$ is contained in $L\cdot\mathbf{a}_w^g\oplus L\cdot\mathbf{a}_w^h$ and

$$\mathrm{N}(r\cdot\mathbf{a}_w^g+s\cdot\mathbf{a}_w^h)=(r+s)\cdot\mathbf{b}_w^f+r\cdot\mathbf{b}_w^h+s\cdot\mathbf{b}_w^g,$$

for each r,s in L, hence $D_{\mathrm{st}}^{\varphi=1,\mathrm{N}=0}=0$. If $\varepsilon_w=\varepsilon_\xi=1$ for $\xi=g,h$ and $\{\xi,\zeta\}=\{g,h\}$, then $D_{\mathrm{st}}^{\varphi=1}$ is contained in the L-module generated by $\mathbf{a}_w$ and $\mathbf{a}_w^\zeta$, and

$$\mathrm{N}(r\cdot\mathbf{a}_w+s\cdot\mathbf{a}_w^\zeta)=r\cdot\mathbf{a}_w^\xi+s\cdot\mathbf{b}_w^f,$$

hence $D_{\mathrm{st}}^{\varphi=1,\mathrm{N}=0}=0$. Finally, if $\varepsilon_w=\varepsilon_f=1$, one has

$$\mathrm{N}(r\cdot\mathbf{a}_w+s\cdot\mathbf{a}_w^g+t\cdot\mathbf{a}_w^h+u\cdot\mathbf{b}_w^f)=r\cdot\mathbf{a}_w^f+s\cdot\mathbf{b}_w^h+t\cdot\mathbf{b}_w^g+u\cdot\mathbf{b}_w,$$

hence $D^{\varphi=1,\mathrm{N}=0}$ vanishes also in this case, concluding the proof of the lemma. □

In light of Lemma 9.1, in order to prove Theorem B it is sufficient to show that

$$\exp_p^*(\kappa(\boldsymbol{f}_k,\boldsymbol{g}_l,\boldsymbol{h}_m)) = 0 \quad \text{if and only if} \quad L(f_k^\sharp \otimes g_l^\sharp \otimes h_m^\sharp, (k+l+m-2)/2) = 0, \tag{190}$$

where $\exp_p^*$ is the Bloch–Kato dual exponential and $\exp_p^*(\cdot) = \exp_p^*(\mathrm{res}_p(\cdot))$ for any $\cdot$ in the global cohomology group $H^1(\mathbf{Q}, V(\boldsymbol{f}_k,\boldsymbol{g}_l,\boldsymbol{h}_m))$.

Set

$$V^{\cdot}(\boldsymbol{f}_k,\boldsymbol{g}_l,\boldsymbol{h}_m)^{\pm} = V^{\cdot}(\boldsymbol{f}_k)^{\pm} \otimes_L V^{\cdot}(\boldsymbol{g}_l) \otimes_L V^{\cdot}(\boldsymbol{h}_m)(c.), \tag{191}$$

where $c. = (4-k-l-m)/2$ and $c. = (k+l+m-2)/2$ if $\cdot = \emptyset$ and $\cdot = *$ respectively. Because $k \geqslant l+m$ the inclusion $V^*(\boldsymbol{f}_k,\boldsymbol{g}_l,\boldsymbol{h}_m)^+ \hookrightarrow V^*(\boldsymbol{f}_k,\boldsymbol{g}_l,\boldsymbol{h}_m)$ and the projection $V(\boldsymbol{f}_k,\boldsymbol{g}_l,\boldsymbol{h}_m) \twoheadrightarrow V(\boldsymbol{f}_k,\boldsymbol{g}_l,\boldsymbol{h}_m)^-$ induce isomorphisms

$$\begin{aligned} D_{\mathrm{st}}(V^*(\boldsymbol{f}_k,\boldsymbol{g}_l,\boldsymbol{h}_m)^+) &\cong V^*_{\mathrm{dR}}(\boldsymbol{f}_k,\boldsymbol{g}_l,\boldsymbol{h}_m)/\mathrm{Fil}^0 \\ \text{and} \qquad \mathrm{Fil}^0 V_{\mathrm{dR}}(\boldsymbol{f}_k,\boldsymbol{g}_l,\boldsymbol{h}_m) &\cong D_{\mathrm{st}}(V(\boldsymbol{f}_k,\boldsymbol{g}_l,\boldsymbol{h}_m)^-) \end{aligned} \tag{192}$$

respectively. (If $\boldsymbol{g}_l$ or $\boldsymbol{h}_m$ is a weight-one modular form, the modules $V_{\mathrm{dR}}(\boldsymbol{f}_k,\boldsymbol{g}_l,\boldsymbol{h}_m)$ and $V^*_{\mathrm{dR}}(\boldsymbol{f}_k,\boldsymbol{g}_l,\boldsymbol{h}_m)$ are defined using the conventions introduced in the last item of Sections 5, cf. Equations (127) and (129) and Section 7.1.1.1.) Let

$$\langle\cdot,\cdot\rangle_{\boldsymbol{f}_k\boldsymbol{g}_l\boldsymbol{h}_m} : \mathrm{Fil}^0 V_{\mathrm{dR}}(\boldsymbol{f}_k,\boldsymbol{g}_l,\boldsymbol{h}_m) \otimes_L V^*_{\mathrm{dR}}(\boldsymbol{f}_k,\boldsymbol{g}_l,\boldsymbol{h}_m)/\mathrm{Fil}^0 \longrightarrow L$$

be the perfect pairing induced on the de Rham modules by the specialisation at w_o (cf. Equations (106)–(109)) of the tensor product of the pairings $\langle\cdot,\cdot\rangle_{\boldsymbol{\xi}}$ defined in Equation (103), for $\boldsymbol{\xi} = \boldsymbol{f},\boldsymbol{g},\boldsymbol{h}$. (According to Equation (109), if k, l and m are all geometric this is also induced by the tensor product of the pairings $\langle\cdot,\cdot\rangle_{\xi}$ introduced in Equation (31), for $\xi = \boldsymbol{f}_k,\boldsymbol{g}_l,\boldsymbol{h}_m$.) By construction $V(\boldsymbol{f}_k,\boldsymbol{g}_l,\boldsymbol{h}_m)_f$ is a $G_{\mathbf{Q}_p}$-submodule of $V(\boldsymbol{f}_k,\boldsymbol{g}_l,\boldsymbol{h}_m)^-$, and the image of

$$D_{\mathrm{cris}}(V(\boldsymbol{f}_k,\boldsymbol{g}_l,\boldsymbol{h}_m)_f) \hookrightarrow D_{\mathrm{st}}(V(\boldsymbol{f}_k,\boldsymbol{g}_l,\boldsymbol{h}_m)^-) \cong \mathrm{Fil}^0 V_{\mathrm{dR}}(\boldsymbol{f}_k,\boldsymbol{g}_l,\boldsymbol{h}_m)$$

(cf. Equation (192)) is orthogonal under $\langle\cdot,\cdot\rangle_{\boldsymbol{f}_k\boldsymbol{g}_l\boldsymbol{h}_m}$ to the kernel of the projection

$$V^*_{\mathrm{dR}}(\boldsymbol{f}_k,\boldsymbol{g}_l,\boldsymbol{h}_m)/\mathrm{Fil}^0 \cong D_{\mathrm{st}}(V^*(\boldsymbol{f}_k,\boldsymbol{g}_l,\boldsymbol{h}_m)^+) \twoheadrightarrow D_{\mathrm{cris}}(V^*(\boldsymbol{f}_k,\boldsymbol{g}_l,\boldsymbol{h}_m)_f),$$

where $V^*(\boldsymbol{f}_k,\boldsymbol{g}_l,\boldsymbol{h}_m)_f$ is the c_*-th Tate twist of $V^*(\boldsymbol{f}_k)^+ \otimes_L V^*(\boldsymbol{g}_l)^- \otimes_L V^*(\boldsymbol{h}_m)^-$. Moreover, after setting $x_o = (w_o, (k-l-m)/2)$ (and identifying $D_{\mathrm{cris}}(\mathbf{Q}_p(i))$ with $\mathbf{Q}_p \cdot t^i$), one has by definition (cf. Section 7)

$$\begin{aligned} D_{\mathrm{cris}}(V(\boldsymbol{f}_k,\boldsymbol{g}_l,\boldsymbol{h}_m)_f) &= \bar{D}(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})_f \otimes_{x_o} L \\ \text{and} \quad D_{\mathrm{cris}}(V^*(\boldsymbol{f}_k,\boldsymbol{g}_l,\boldsymbol{h}_m)_f) &= \bar{D}^*(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})_f \otimes_{x_o} L. \end{aligned}$$

By Corollary 8.2 the class $\kappa(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})$ is balanced, viz. its restriction at p is the image of a (unique) class $\breve{\kappa}(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})$ in $H^1(\mathbf{Q}_p, \mathscr{F}^2 V(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h}))$. Let $\breve{\kappa}(\boldsymbol{f}_k,\boldsymbol{g}_l,\boldsymbol{h}_m)$ be the specialisation of $\breve{\kappa}(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})$ at w_o, and let $\kappa(\boldsymbol{f}_k,\boldsymbol{g}_l,\boldsymbol{h}_m)_f$ be its image in $H^1(\mathbf{Q}_p, V(\boldsymbol{f}_k,\boldsymbol{g}_l,\boldsymbol{h}_m)_f)$ under the morphism p_{f*} (cf. Section 7.2). As the diagram

$$\begin{array}{ccc} H^1(\mathbf{Q}_p, \mathscr{F}^2 V(\boldsymbol{f}_k,\boldsymbol{g}_l,\boldsymbol{h}_m)) & \longrightarrow & H^1(\mathbf{Q}_p, V(\boldsymbol{f}_k,\boldsymbol{g}_l,\boldsymbol{h}_m)) \\ \downarrow{\scriptstyle p_{f*}} & & \downarrow \\ H^1(\mathbf{Q}_p, V(\boldsymbol{f}_k,\boldsymbol{g}_l,\boldsymbol{h}_m)_f) & \longrightarrow & H^1(\mathbf{Q}_p, V(\boldsymbol{f}_k,\boldsymbol{g}_l,\boldsymbol{h}_m)^-) \end{array} \tag{193}$$

commutes, the previous paragraph reduces the proof of Equation (190) to the following claim.

($\boldsymbol{\alpha}$) The Garrett L-function $L(f_k^\sharp \otimes g_l^\sharp \otimes h_m^\sharp, s)$ vanishes at $s = (k+l+m-2)/2$ if and only if

$$\left\langle \exp_p^*(\kappa(\boldsymbol{f}_k, \boldsymbol{g}_l, \boldsymbol{h}_m)_f), \mu \right\rangle_{\boldsymbol{f}_k \boldsymbol{g}_l \boldsymbol{h}_m} = 0$$

for all differentials μ in $\bar{D}(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})_f \otimes_{x_o} L$. Here $\exp_p^*$ is the Bloch–Kato dual exponential on $H^1(\mathbf{Q}_p, V(\boldsymbol{f}_k, \boldsymbol{g}_l, \boldsymbol{h}_m)_f)$ and $\langle \cdot, \cdot \rangle_{\boldsymbol{f}_k \boldsymbol{g}_l \boldsymbol{h}_m}$ is the specialisation at x_o of the bilinear form $\langle \cdot, \cdot \rangle_{\boldsymbol{fgh}}$ defined in Equation (139).

As $(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})$ varies through the level-N test vectors for $(\boldsymbol{f}^\sharp, \boldsymbol{g}^\sharp, \boldsymbol{h}^\sharp)$, the specialisations at x_o of the associated $\bar{\mathscr{O}}_{\boldsymbol{fgh}}$-adic differentials $\eta_{\boldsymbol{f}} \omega_{\boldsymbol{g}} \omega_{\boldsymbol{h}}$ (cf. Equation (142)) generate $\bar{D}^*(\boldsymbol{f}_k, \boldsymbol{g}_l, \boldsymbol{h}_m)_f \otimes_{x_o} L$. This follows from the results of Sections 2.5, 5 and 7.1.1. As a consequence the claim ($\boldsymbol{\alpha}$) is equivalent to

($\boldsymbol{\beta}$) The Garrett L-function $L(f_k^\sharp \otimes g_l^\sharp \otimes h_m^\sharp, s)$ vanishes at $s = (k+l+m-2)/2$ if and only if

$$\left\langle \exp_p^*(\kappa(\boldsymbol{f}_k, \boldsymbol{g}_l, \boldsymbol{h}_m)_f), \eta_{\boldsymbol{f}_k} \omega_{\boldsymbol{g}_l} \omega_{\boldsymbol{h}_m} \right\rangle_{\boldsymbol{f}_k \boldsymbol{g}_l \boldsymbol{h}_m} = 0$$

for all level-N test vectors $(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})$ for $(\boldsymbol{f}^\sharp, \boldsymbol{g}^\sharp, \boldsymbol{h}^\sharp)$, where $\eta_{\boldsymbol{f}_k} \omega_{\boldsymbol{g}_l} \omega_{\boldsymbol{h}_m}$ in $D_{\mathrm{cris}}(V^*(\boldsymbol{f}_k, \boldsymbol{g}_l, \boldsymbol{h}_m)_f)$ is the specialisation of $\eta_{\boldsymbol{f}} \omega_{\boldsymbol{g}} \omega_{\boldsymbol{h}}$ at x_o (cf. Section 7.1.1).

Remark 9.2. — As explained in Remark 1.3(3), the class $\kappa(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})$, hence $\breve{\kappa}(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})$ and a fortiori $\kappa(\boldsymbol{f}_k, \boldsymbol{g}_l, \boldsymbol{h}_m)_f$, is independent of the choice of the level-N test vector $(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})$ for $(\boldsymbol{f}^\sharp, \boldsymbol{g}^\sharp, \boldsymbol{h}^\sharp)$.

Assume in the rest of this section that w_o is *not* exceptional. This implies that

$$\beta_{\boldsymbol{f}_k} \alpha_{\boldsymbol{g}_l} \alpha_{\boldsymbol{h}_m} \neq p^{(k+l+m-2)/2}$$

for each test vector $(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})$. (As usual $\beta_{\boldsymbol{f}_k} = \chi_{\boldsymbol{f}}(p) p^{k-1}/a_p(k)$, hence the previous equation is a consequence of Equation (5) and the Ramanujan–Petersson conjecture.) According to Theorem A, (the proof of) Proposition 7.3 and the previous equation, for each level-N test vector $(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})$ one has

$$\mathscr{L}_p^f(\boldsymbol{f}_k, \boldsymbol{g}_l, \boldsymbol{h}_m) = \mathscr{E}_{w_o} \cdot \left\langle \exp_p^*(\kappa(\boldsymbol{f}_k, \boldsymbol{g}_l, \boldsymbol{h}_m)_f), \eta_{\boldsymbol{f}_k} \omega_{\boldsymbol{g}_l} \omega_{\boldsymbol{h}_m} \right\rangle_{\boldsymbol{f}_k \boldsymbol{g}_l \boldsymbol{h}_m}$$

for a *non-zero* algebraic number $\mathscr{E}_{w_o}$. The statement ($\boldsymbol{\beta}$) can then be rephrased as

($\boldsymbol{\gamma}$) $L(f_k^\sharp \otimes g_l^\sharp \otimes h_m^\sharp, (k+l+m-2)/2) = 0$ if and only if $\mathscr{L}_p^f(\boldsymbol{f}_k, \boldsymbol{g}_l, \boldsymbol{h}_m) = 0$ for all level-N test vectors $(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})$ for $(\boldsymbol{f}^\sharp, \boldsymbol{g}^\sharp, \boldsymbol{h}^\sharp)$.

Under the current Assumption 1.7 on the local signs $\varepsilon_\ell(f_k^\sharp, g_l^\sharp, h_m^\sharp)$, the claim ($\boldsymbol{\gamma}$) is a consequence of Jacquet's conjecture proved by Harris–Kudla in [**HK91**]. Indeed, as w_o is not exceptional, there exist test vectors $(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})$ such that $\mathscr{L}_p^f(\boldsymbol{f}_k, \boldsymbol{g}_l, \boldsymbol{h}_m)$ is a non-zero multiple of the complex central value $L(f_k^\sharp \otimes g_l^\sharp \otimes h_m^\sharp, (k+l+m-2)/2)$ (cf. Section 6 and [**DR14**, Theorems 4.2 and 4.7]).

9.2. Derivatives of big logarithms I. — Assume in this section that the unbalanced classical triple w_o in $\Sigma_{\boldsymbol{f}}$ satisfies the conditions displayed in Equation (5) of Section 1.2. In particular $w_o = (2, 1, 1)$.

Denote by $\mathscr{I} = \mathscr{I}_{w_o}$ the ideal of functions in $\mathscr{O}_{\boldsymbol{fgh}}$ which vanish at w_o. The exceptional zero condition (5) and Proposition 7.3 imply that the big logarithm $\mathscr{L}_{\boldsymbol{f}}$ takes values in $\mathscr{I}$. According to loc. cit. $\mathscr{L}_{\boldsymbol{f}}$ factors through the morphism induced by the projection $p_f : \mathscr{F}^2 V(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})) \longrightarrow V(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})_f$ and we write again

$$\mathscr{L}_{\boldsymbol{f}} : H^1(\mathbf{Q}_p, V(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})_f) \longrightarrow \mathscr{I}$$

for the resulting map. The aim of this section is to prove Proposition 9.3 below, which gives a formula for the derivative of $\mathscr{L}_{\boldsymbol{f}}$ at w_o, namely for the the composition of $\mathscr{L}_{\boldsymbol{f}}$ with the projection $\mathscr{I} \to \mathscr{I}/\mathscr{I}^2$. In order to state it we need to introduce further notations.

Since $\bar{\chi}_{\boldsymbol{f}}(p) = \chi_{\boldsymbol{g}}\chi_{\boldsymbol{h}}(p)$ and $\bar{\chi}_{\boldsymbol{f}}(p) \cdot a_p(2) = b_p(1) \cdot c_p(1)$ under the current assumptions, the $G_{\mathbf{Q}_p}$-representation

$$V(\boldsymbol{f}_2)^-_{\beta\beta} \stackrel{\text{def}}{=} V(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})_f \otimes_{w_o} L = V(\boldsymbol{f}_2)^- \otimes_L V(\boldsymbol{g}_1)^+ \otimes_L V(\boldsymbol{h}_1)^+$$

is isomorphic to the direct sum of a finite number of copies of the trivial p-adic representation of $G_p = G_{\mathbf{Q}_p}$ (cf. Section 7.2). Let G_p^{ab} be the Galois group of the maximal abelian extension of $\mathbf{Q}_p$, and let

$$\text{rec}_p : \mathbf{Q}_p^* \hat{\otimes} \mathbf{Q}_p \cong G_p^{\text{ab}} \hat{\otimes} \mathbf{Q}_p$$

be the reciprocity map of local class field theory, normalised by requiring that $\text{rec}_p(p^{-1})$ is an arithmetic Frobenius. Identify $H^1(\mathbf{Q}_p, \mathbf{Q}_p) = \text{Hom}_{\text{cont}}(G_p^{\text{ab}}, \mathbf{Q}_p)$ with $\text{Hom}_{\text{cont}}(\mathbf{Q}_p^*, \mathbf{Q}_p)$ under rec_p, so that

$$\begin{aligned} &H^1(\mathbf{Q}_p, V(\boldsymbol{f}_2)^-_{\beta\beta}) = \text{Hom}_{\text{cont}}(\mathbf{Q}_p^*, \mathbf{Q}_p) \otimes_{\mathbf{Q}_p} V(\boldsymbol{f}_2)^-_{\beta\beta} \\ \text{and} \qquad &D_{\text{cris}}(V(\boldsymbol{f}_2)^-_{\beta\beta}) = V(\boldsymbol{f}_2)^-_{\beta\beta}. \end{aligned} \tag{194}$$

Under these identifications the Bloch–Kato dual exponential $\exp_p^*$ on $H^1(\mathbf{Q}_p, V(\boldsymbol{f}_2)^-_{\beta\beta})$ satisfies

$$\exp_p^*(\psi \otimes v) = \psi(e(1)) \cdot v \in V(\boldsymbol{f}_2)^-_{\beta\beta} \tag{195}$$

for all $\psi \otimes v$ in $\text{Hom}_{\text{cont}}(\mathbf{Q}_p^*, \mathbf{Q}_p) \otimes_{\mathbf{Q}_p} V(\boldsymbol{f}_2)^-_{\beta\beta}$, where

$$e(1) = (1+p) \hat{\otimes} \log_p(1+p)^{-1} \in \mathbf{Z}_p^* \hat{\otimes} \mathbf{Q}_p.$$

Similarly the $G_{\mathbf{Q}_p}$-module

$$V^*(\boldsymbol{f}_2)^+_{\beta\beta} \stackrel{\text{def}}{=} V^*(\boldsymbol{f}_2)^+ \otimes_L V^*(\boldsymbol{g}_1)^- \otimes_L V^*(\boldsymbol{h}_1)^-$$

is isomorphic to the direct sum of several copies of the trivial representation of $G_{\mathbf{Q}_p}$, hence $D_{\text{cris}}(V^*(\boldsymbol{f}_2)^+_{\beta\beta}) = V^*(\boldsymbol{f}_2)^+_{\beta\beta}$ and Paragraph 7.1.1.1 give a perfect pairing

$$\langle \cdot, \cdot \rangle_{\boldsymbol{f}_2\boldsymbol{g}_1\boldsymbol{h}_1} : V(\boldsymbol{f}_2)^-_{\beta\beta} \otimes_L V^*(\boldsymbol{f}_2)^+_{\beta\beta} \longrightarrow L.$$

For each $\mathfrak{z} = \psi \otimes v$ in $H^1(\mathbf{Q}_p, V(\boldsymbol{f}_2)^-_{\beta\beta})$, with $\psi \in \mathrm{Hom}_{\mathrm{cont}}(\mathbf{Q}_p^*, \mathbf{Q}_p)$ and $v \in V(\boldsymbol{f}_2)^-_{\beta\beta}$, and each q in $\mathbf{Q}^*$, define (cf. Equation (129) and the discussion preceding it)

$$\mathfrak{z}(q) = \psi(q) \cdot v \in V(\boldsymbol{f}_2)^-_{\beta\beta}$$

and

$$\mathfrak{z}(q)_f = (p-1)a_p(2) \cdot \big\langle \mathfrak{z}(q), \eta_{\boldsymbol{f}_2} \otimes \omega_{\boldsymbol{g}_1} \otimes \omega_{\boldsymbol{h}_1} \big\rangle_{\boldsymbol{f}_2\boldsymbol{g}_1\boldsymbol{h}_1} \in L.$$

Let $\mathfrak{z}$ in $H^1(\mathbf{Q}_p, V(\boldsymbol{f}_2, \boldsymbol{g}_1, \boldsymbol{h}_1))$ be the specialisation at w_o of a balanced class $\mathfrak{Z}$ in $H^1_{\mathrm{bal}}(\mathbf{Q}_p, V(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h}))$, that is $\mathfrak{z} = \rho_{w_o *}(\mathfrak{Z})$. Then $\mathfrak{Z}$ is the natural image of a unique class $\mathfrak{Y}$ in $H^1(\mathbf{Q}_p, \mathscr{F}^2 V(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h}))$. Define

$$\mathfrak{y}_f = p_{f*}\big(\rho_{w_o *}(\mathfrak{Y})\big) \in H^1(\mathbf{Q}_p, V(\boldsymbol{f}_2)^-_{\beta\beta}) \tag{196}$$

$$\text{and} \quad \exp_p^*(\mathfrak{z})_f = (p-1)a_p(2) \cdot \big\langle \exp_p^*(\mathfrak{y}_f), \eta_{\boldsymbol{f}_2} \otimes \omega_{\boldsymbol{g}_1} \otimes \omega_{\boldsymbol{h}_1} \big\rangle_{\boldsymbol{f}_2\boldsymbol{g}_1\boldsymbol{h}_1}.$$

The following key proposition studies the derivatives of the logarithm $\mathscr{L}_{\boldsymbol{f}}$, extending some of the results of [**Ven16**]. Its proof exploits the existence of an *improved* big logarithm for the restriction of $\mathscr{L}_{\boldsymbol{f}}$ to the *improving plane* $\mathcal{H}_{\boldsymbol{f}}$ defined by the equation $\boldsymbol{k} = \boldsymbol{l} + \boldsymbol{m}$. Part 1 of the proposition is a crucial ingredient in the proof of the main result of our contribution [**BSV20a**], and Part 3 is essential for the ongoing proof of Theorem B in the exceptional case (cf. Section 9.4). Part 2 is not used elsewhere in the paper and is stated for completeness (and with future applications of this work in mind). Before stating the proposition, we introduce some notation.

For the proof of Theorem B, we are especially interested in the *improving line* $\mathcal{H}_{\boldsymbol{fg}}$ in $U_{\boldsymbol{f}} \times U_{\boldsymbol{g}} \times U_{\boldsymbol{h}}$ defined by the equations $\boldsymbol{k} = \boldsymbol{l}+1$ and $\boldsymbol{m} = 1$; it is the intersection of the improving planes $\mathcal{H}_{\boldsymbol{g}}$ (introduced in Section 1.2) and $\mathcal{H}_{\boldsymbol{f}}$. Let $\mathrm{res}_{\boldsymbol{fg}} : \mathscr{O}_{\boldsymbol{fgh}} \longrightarrow \mathscr{O}_{\boldsymbol{g}}$ be the morphism sending the analytic function $F(\boldsymbol{k}, \boldsymbol{l}, \boldsymbol{m})$ to its restriction $F(\boldsymbol{l}+1, \boldsymbol{l}, 1)$ to the improving line $\mathcal{H}_{\boldsymbol{fg}}$. For each $\mathscr{O}_{\boldsymbol{fgh}}$-module M, denote by $M|_{\mathcal{H}_{\boldsymbol{fg}}} = M \otimes_{\mathrm{res}_{\boldsymbol{fg}}} \mathscr{O}_{\boldsymbol{g}}$ the base chance of M along $\mathrm{res}_{\boldsymbol{fg}}$, and for each m in M denote by $m|_{\mathcal{H}_{\boldsymbol{fg}}}$ the image of m under the projection $M \longrightarrow M|_{\mathcal{H}_{\boldsymbol{fg}}}$. Set

$$V(\boldsymbol{fg}, \boldsymbol{h}_1) = V(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})|_{\mathcal{H}_{\boldsymbol{fg}}} \quad \text{and} \quad V(\boldsymbol{fg}, \boldsymbol{h}_1)_f = V(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})_f|_{\mathcal{H}_{\boldsymbol{fg}}}.$$

Shrinking $U_{\boldsymbol{g}}$ and $U_{\boldsymbol{h}}$ if necessary, assume that $l+m$ belongs to $U_{\boldsymbol{f}}$ for each (l, m) in $U_{\boldsymbol{g}} \times U_{\boldsymbol{h}}$, and recall the analytic f-Euler factor

$$\mathcal{E}_f^*(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h}) = 1 - \frac{b_p(\boldsymbol{l}) \cdot c_p(\boldsymbol{m})}{\bar{\chi}_{\boldsymbol{f}}(p) \cdot a_p(\boldsymbol{l}+\boldsymbol{m})} \in \mathscr{O}_{\boldsymbol{g}} \hat{\otimes}_L \mathscr{O}_{\boldsymbol{h}} \tag{197}$$

introduced in Equation (4). (We also recall that $a_p(\boldsymbol{k})$, $b_p(\boldsymbol{l})$ and $c_p(\boldsymbol{m})$ are the p-th Fourier coefficients of the primitive Hida families $\boldsymbol{f}^\sharp$, $\boldsymbol{g}^\sharp$ and $\boldsymbol{h}^\sharp$ associated respectively with $\boldsymbol{f}$, $\boldsymbol{g}$ and $\boldsymbol{h}$.) In the present exceptional zero scenario (cf. Equation (5)) it vanishes at $(\boldsymbol{l}, \boldsymbol{m}) = (1, 1)$. Denote by

$$\mathcal{E}_f^*(\boldsymbol{fg}, \boldsymbol{h}_1) = \mathcal{E}_f^*(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})|_{\mathcal{H}_{\boldsymbol{fg}}} \in \mathscr{O}_{\boldsymbol{g}}$$

the restriction of $\mathcal{E}_f^*(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})$ to $\mathcal{H}_{\boldsymbol{fg}}$. Finally define the *analytic $\mathscr{L}$-invariants*

$$\mathfrak{L}_{\boldsymbol{f}}^{\mathrm{an}} = -2 \cdot d \log a_p(\boldsymbol{k})|_{\boldsymbol{k}=2}, \quad \mathfrak{L}_{\boldsymbol{g}}^{\mathrm{an}} = -2 \cdot d \log b_p(\boldsymbol{l})|_{\boldsymbol{l}=1} \quad \text{and} \quad \mathfrak{L}_{\boldsymbol{h}}^{\mathrm{an}} = -2 \cdot d \log c_p(\boldsymbol{m})|_{\boldsymbol{m}=1}.$$

We can now state the main result of this section.

Proposition 9.3. —

1. *Let* $\mathfrak{Z} \in H^1(\mathbf{Q}_p, V(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})_f)$ *and let* $\mathfrak{z} = \rho_{w_o}(\mathfrak{Z}) \in H^1(\mathbf{Q}_p, V(\boldsymbol{f}_2)^-_{\beta\beta})$. *Then*
$$\begin{aligned} 2(1-1/p)\cdot \mathscr{L}_{\boldsymbol{f}}(\mathfrak{Z}) \equiv &\Big(\mathfrak{z}(p^{-1})_f - \mathfrak{L}^{\mathrm{an}}_{\boldsymbol{f}}\cdot \mathfrak{z}(e(1))_f\Big)\cdot(\boldsymbol{k}-2)\\ &+\Big(\mathfrak{L}^{\mathrm{an}}_{\boldsymbol{g}}\cdot \mathfrak{z}(e(1))_f - \mathfrak{z}(p^{-1})_f\Big)\cdot(\boldsymbol{l}-1)\\ &+\Big(\mathfrak{L}^{\mathrm{an}}_{\boldsymbol{h}}\cdot \mathfrak{z}(e(1))_f - \mathfrak{z}(p^{-1})_f\Big)\cdot(\boldsymbol{m}-1) \qquad \big(\mathrm{mod}\ \mathscr{I}^2\big).\end{aligned}$$
2. *Let* $\mathfrak{Z}$ *be a local balanced class in* $H^1_{\mathrm{bal}}(\mathbf{Q}_p, V(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h}))$ *and let* $\mathfrak{z} = \rho_{w_o}(\mathfrak{Z})$ *be its* w_o*-specialisation in* $H^1_{\mathrm{bal}}(\mathbf{Q}_p, V(\boldsymbol{f}_2,\boldsymbol{g}_1,\boldsymbol{h}_1))$. *Then*
$$2(1-1/p)\cdot \mathscr{L}_{\boldsymbol{f}}(\mathfrak{Z})$$
is congruent modulo $\mathscr{I}^2$ *to*
$$\Big(\big(\mathfrak{L}^{\mathrm{an}}_{\boldsymbol{g}} - \mathfrak{L}^{\mathrm{an}}_{\boldsymbol{f}}\big)\cdot(\boldsymbol{l}-1) + \big(\mathfrak{L}^{\mathrm{an}}_{\boldsymbol{h}} - \mathfrak{L}^{\mathrm{an}}_{\boldsymbol{f}}\big)\cdot(\boldsymbol{m}-1)\Big)\cdot \exp^*_p(\mathfrak{z})_f.$$
3. *There exists a morphism*
$$\mathscr{L}^*_{V(\boldsymbol{fg},\boldsymbol{h}_1)_f} : H^1(\mathbf{Q}_p, V(\boldsymbol{fg},\boldsymbol{h}_1)_f) \longrightarrow \mathscr{O}_{\boldsymbol{g}}$$
such that, for each local class $\mathfrak{Z}$ *in* $H^1(\mathbf{Q}_p, V(\boldsymbol{fg},\boldsymbol{h}_1)_f)$ *and each positive integer* $l \geqslant 1$ *in* $U_{\boldsymbol{g}}$ *congruent to* 1 *modulo* $p-1$*, one has*
$$\mathscr{E}(l)\cdot \mathscr{L}^*_{V(\boldsymbol{fg},\boldsymbol{h}_1)_f}(\mathfrak{Z})(l) = (p-1)a_p(l+1)\cdot \langle \exp^*_p(\mathfrak{z}), \eta_{\boldsymbol{f}_{l+1}}\omega_{\boldsymbol{g}_l}\omega_{\boldsymbol{h}_1}\rangle_{\boldsymbol{f}_{l+1}\boldsymbol{g}_l\boldsymbol{h}_1},$$
where $\mathscr{E}(l) = 1 - \frac{\bar{\chi}_{\boldsymbol{f}}(p)\cdot a_p(l+1)}{p\cdot b_p(l)\cdot c_p(1)}$ *and* $\mathfrak{z} = \rho_l(\mathfrak{Z})$ *in* $H^1(\mathbf{Q}_p, V(\boldsymbol{f}_{l+1},\boldsymbol{g}_l,\boldsymbol{h}_1)_f)$ *is the weight-*l *specialisation of* $\mathfrak{Z}$. *Moreover, the following diagram commutes.*
$$\begin{array}{ccc} H^1(\mathbf{Q}_p, V(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})_f) & \xrightarrow{\quad \mathscr{L}_{\boldsymbol{f}} \quad} & \mathscr{O}_{\boldsymbol{fgh}} \\ \Big\downarrow {\scriptstyle \mathrm{res}_{\boldsymbol{fg}*}} & & \Big\downarrow {\scriptstyle \mathrm{res}_{\boldsymbol{fg}}} \\ H^1(\mathbf{Q}_p, V(\boldsymbol{fg},\boldsymbol{h}_1)_f) & \xrightarrow{\ \mathcal{E}^*_f(\boldsymbol{fg},\boldsymbol{h}_1)\cdot \mathscr{L}^*_{V(\boldsymbol{fg},\boldsymbol{h}_1)_f}\ } & \mathscr{O}_{\boldsymbol{g}} \end{array}$$

Proof. — Let $\varepsilon : \bar{\mathscr{O}}_{\boldsymbol{fgh}} \longrightarrow \mathscr{O}_{\boldsymbol{fgh}}$ be the map which sends the analytic function $F(\boldsymbol{k},\boldsymbol{l},\boldsymbol{m},\boldsymbol{j})$ in $\bar{\mathscr{O}}_{\boldsymbol{fgh}}$ to its restriction $F(\boldsymbol{k},\boldsymbol{l},\boldsymbol{m},0) \in \mathscr{O}_{\boldsymbol{fgh}}$ to the hyperplane $\boldsymbol{j} = 0$ (see Section 7.1 and note that $j_o = 0$). Because $M(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})_f$ is equal (by definition) to the base change $\bar{M}(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})_f \otimes_\varepsilon \mathscr{O}_{\boldsymbol{fgh}}$, this induces in cohomology
$$\varepsilon_* : H^1(\mathbf{Q}_p, \bar{M}(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})_f) \longrightarrow H^1(\mathbf{Q}_p, M(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})_f).$$
A slight generalisation of [**Ven16**, Proposition 3.8] stated in Lemma 9.4 below gives an *improved big dual exponential*
$$\mathcal{L}^*_{\boldsymbol{f}} : H^1(\mathbf{Q}_p, M(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})_f) \longrightarrow D(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})_f$$
such that, for all classes $\mathfrak{Z}$ in $H^1(\mathbf{Q}_p, M(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})_f)$ and all $w = (k,l,m) \in \Sigma$, one has
$$\big(1 - p^{-1}\cdot \Psi_w(\mathrm{Frob}_p)\big)\cdot \mathcal{L}^*_{\boldsymbol{f}}(\mathfrak{Z})(w) = \exp^*(\mathfrak{Z}_w), \tag{198}$$
where Ψ_w is the composition of the unramified character $\Psi : G_{\mathbf{Q}_p} \longrightarrow \mathscr{O}^*_{\boldsymbol{fgh}}$ introduced in Equation (136) with evaluation at w, $\exp^*$ is the Bloch–Kato dual exponential on $H^1(\mathbf{Q}_p, M(\boldsymbol{f}_k,\boldsymbol{g}_l,\boldsymbol{h}_m)_f)$, and $\mathfrak{Z}_w$ is a shorthand for $\rho_{w*}(\mathfrak{Z})$. (Precisely, after setting

$\mathscr{R} = \mathscr{O}_{\boldsymbol{fgh}}$, $\mathscr{M} = M(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})_f$ and $\Phi = \Psi$, then one has $\mathcal{L}^*_{\boldsymbol{f}} = \mathscr{E}xp^*_\Psi$ with the notations of Lemma 9.4.) Recall the big logarithm $\bar{\mathscr{L}}_{\boldsymbol{f}}$ introduced in Equation (144), and let

$$\mathscr{L}^*_{\boldsymbol{f}} : H^1(\mathbf{Q}_p, M(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})_f) \longrightarrow \mathscr{O}_{\boldsymbol{fgh}}$$

be the composition of $\mathcal{L}^*_{\boldsymbol{f}}$ with the base change

$$\langle \cdot, \eta_{\boldsymbol{f}}\omega_{\boldsymbol{g}}\omega_{\boldsymbol{h}}\rangle_{\boldsymbol{fgh}} \otimes_\varepsilon \mathscr{O}_{\boldsymbol{fgh}} : D(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})_f \to \mathscr{O}_{\boldsymbol{fgh}}$$

of the linear form $\langle \cdot, \eta_{\boldsymbol{f}}\omega_{\boldsymbol{g}}\omega_{\boldsymbol{h}}\rangle_{\boldsymbol{fgh}}$ along ε. Equation (198) and Proposition 7.1 yield

$$\varepsilon \circ \bar{\mathscr{L}}_{\boldsymbol{f}} = \big(1 - \Psi(\mathrm{Frob}_p)^{-1}\big) \cdot \mathscr{L}^*_{\boldsymbol{f}} \circ \varepsilon_*. \tag{199}$$

Define $\varrho = \rho_{w_o} : \bar{\mathscr{O}}_{\boldsymbol{fgh}} \longrightarrow \mathscr{O}_{\mathrm{cyc}}$ by $\varrho(F(\boldsymbol{k},\boldsymbol{l},\boldsymbol{m},\boldsymbol{j})) = F(w_o,\boldsymbol{j})$ and denote by $\bar{M}(\boldsymbol{f}_2,\boldsymbol{g}_1,\boldsymbol{h}_1)_f$ the base change $\bar{M}(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})_f \otimes_\varrho \mathscr{O}_{\mathrm{cyc}}$. Note that in the present setting $G_{\mathbf{Q}_p}$ acts on $\bar{M}(\boldsymbol{f}_2,\boldsymbol{g}_1,\boldsymbol{h}_1)_f$ via the character $\kappa_{\mathrm{cyc}}^{-\boldsymbol{j}}$, and for all integers j divisible by $p-1$, evaluation at j on $\mathscr{O}_{\mathrm{cyc}}$ induces a natural isomorphism (cf. Sect. 7.1)

$$V(\boldsymbol{f}_2)^-_{\beta\beta}(-j) = \bar{M}(\boldsymbol{f}_2,\boldsymbol{g}_1,\boldsymbol{h}_1)_f \otimes_j L. \tag{200}$$

The results of Coleman and Perrin-Riou (see Section 4 of [PR94]) then give a morphism of $\mathscr{O}_{\mathrm{cyc}}$-modules

$$\mathscr{L}_{\mathrm{cyc}} : H^1(\mathbf{Q}_p, \bar{M}(\boldsymbol{f}_2,\boldsymbol{g}_1,\boldsymbol{h}_1)_f) \longrightarrow \mathscr{O}_{\mathrm{cyc}}$$

such that, for all classes $\mathfrak{Z}$ in $H^1(\mathbf{Q}_p, \bar{M}(\boldsymbol{f}_2,\boldsymbol{g}_1,\boldsymbol{h}_1)_f)$ and all integers $j \geqslant 0$ satisfying $j \equiv 0 \pmod{p-1}$, one has

$$\mathscr{L}_{\mathrm{cyc}}(\mathfrak{Z})(j) = j!\frac{(1-p^j)}{(1-p^{-j-1})}\exp^*(\mathfrak{Z}_j)_f. \tag{201}$$

Here $\mathfrak{Z}_j$ is the image of $\mathfrak{Z}$ in $H^1(\mathbf{Q}_p, V(\boldsymbol{f}_2)^-_{\beta\beta}(-j))$ under the morphism induced by (200) and one writes again

$$\exp^*(\cdot)_f = (p-1)a_p(2) \cdot \big\langle \exp^*(\cdot), \eta_{\boldsymbol{f}_2}\omega_{\boldsymbol{g}_1}\omega_{\boldsymbol{h}_1}\big\rangle_{\boldsymbol{f}_2\boldsymbol{g}_1\boldsymbol{h}_1}$$

for the composition of the linear form $(p-1)a_p(2) \cdot \big\langle \cdot, \eta_{\boldsymbol{f}_2}\omega_{\boldsymbol{g}_1}\omega_{\boldsymbol{h}_1}\big\rangle_{\boldsymbol{f}_2\boldsymbol{g}_1\boldsymbol{h}_1}$ on $V(\boldsymbol{f}_2)^-_{\beta\beta}$ with the Bloch–Kato dual exponential map

$$\exp^* : H^1(\mathbf{Q}_p, V(\boldsymbol{f}_2)^-_{\beta\beta}(-j)) \longrightarrow V(\boldsymbol{f}_2)^-_{\beta\beta} \otimes_{\mathbf{Q}_p} \mathbf{Q}_p \cdot t^{-j} \cong V(\boldsymbol{f}_2)^-_{\beta\beta}$$

(cf. Section 7.1 and Equation (194)). According to Proposition 3.6 of [Ven16] (see also [Ben14, Proposition 2.2.2]), for all classes $\mathfrak{Z}$ in $H^1(\mathbf{Q}_p, \bar{M}(\boldsymbol{f}_2,\boldsymbol{g}_1,\boldsymbol{h}_1)_f)$ one has

$$\frac{d}{d\boldsymbol{j}}\mathscr{L}_{\mathrm{cyc}}(\mathfrak{Z})_{\boldsymbol{j}=0} = (1-1/p)^{-1} \cdot \mathfrak{z}(p^{-1})_f, \tag{202}$$

where $\mathfrak{z}$ is a shorthand for $\mathfrak{Z}_0$. Moreover Proposition 7.1 and Equation (201) yield the identity

$$\varrho \circ \bar{\mathscr{L}}_{\boldsymbol{f}} = \mathscr{L}_{\mathrm{cyc}} \circ \varrho_*. \tag{203}$$

Let $\mathfrak{Z}$ be a class in $H^1(\mathbf{Q}_p, V(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})_f)$ and let $\mathfrak{z} = \rho_{w_o}(\mathfrak{Z}) \in H^1(\mathbf{Q}_p, V(\boldsymbol{f}_2)^-_{\beta\beta})$ be its specialisation at w_o. As explained in the proof of Proposition 7.3 (see in particular Equations (151) and (152)), the class $\mathfrak{Z}$ can by lifted to an element $\mathscr{Z}$ in $H^1(\mathbf{Q}_p, \bar{M}(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})_f)$ via the map induced in cohomology by the isomorphism

$$\bar{M}(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})_f/(2\boldsymbol{j}-\boldsymbol{k}+\boldsymbol{l}+\boldsymbol{m})\cdot\bar{M}(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h}) \cong V(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})_f,$$

and one has

$$\mathscr{L}_{\boldsymbol{f}}(\mathfrak{Z})(\boldsymbol{k},\boldsymbol{l},\boldsymbol{m}) = \bar{\mathscr{L}}_{\boldsymbol{f}}(\mathscr{Z})(\boldsymbol{k},\boldsymbol{l},\boldsymbol{m},(\boldsymbol{k}-\boldsymbol{l}-\boldsymbol{m})/2), \tag{204}$$

for any such lift $\mathscr{Z}$. As (cf. Equation (136))

$$2\cdot\big(1-\Psi(\mathrm{Frob}_p)^{-1}\big) = \mathfrak{L}^{\mathrm{an}}_{\boldsymbol{g}}\cdot(\boldsymbol{l}-1)+\mathfrak{L}^{\mathrm{an}}_{\boldsymbol{h}}\cdot(\boldsymbol{m}-1)-\mathfrak{L}^{\mathrm{an}}_{\boldsymbol{f}}\cdot(\boldsymbol{k}-2)+\cdots,$$

where the dots denote the terms of higher degree in the Taylor expansion at w_o, Equations (199) and (203) yield that $2(1-1/p)\cdot\bar{\mathscr{L}}_{\boldsymbol{f}}(\mathscr{Z})$ is equal to

$$2\left(1-\Psi(\mathrm{Frob}_p)^{-1}\right)(1-1/p)\cdot\mathscr{L}^*_{\boldsymbol{f}}(\varepsilon_*(\mathscr{Z}))+2(1-1/p)\cdot\mathscr{L}_{\mathrm{cyc}}(\varrho_*(\mathscr{Z}))+\cdots,$$

which in turn agrees with

$$\mathfrak{z}(e(1))_f\cdot\left(\mathfrak{L}^{\mathrm{an}}_{\boldsymbol{g}}\cdot(\boldsymbol{l}-1)+\mathfrak{L}^{\mathrm{an}}_{\boldsymbol{h}}\cdot(\boldsymbol{m}-1)-\mathfrak{L}^{\mathrm{an}}_{\boldsymbol{f}}\cdot(\boldsymbol{k}-2)\right)+2\cdot\mathfrak{z}(p^{-1})_f\cdot\boldsymbol{j}+\cdots$$

by Equations (195), (198) and (202). This proves Part 1 in the statement.

To prove Part 2 let $\mathfrak{Z},\mathfrak{Y},\mathfrak{z}$ and $\mathfrak{y}_f$ be as in Equation (196), so that

$$\exp^*_p(\mathfrak{z})_f = \mathfrak{y}_f(e(1))_f \tag{205}$$

(cf. Equation (195)). Note that the $L[G_p]$-module $\mathscr{F}^2V(\boldsymbol{f}_2,\boldsymbol{g}_1,\boldsymbol{h}_1)$ splits as the direct sum of its submodules $V(\boldsymbol{f}_2)^+_{\alpha\beta} = V(\boldsymbol{f}_2,\boldsymbol{g}_1,\boldsymbol{h}_1)_g$, $V(\boldsymbol{f}_2)^+_{\beta\alpha} = V(\boldsymbol{f}_2,\boldsymbol{g}_1,\boldsymbol{h}_1)_h$ and

$$V(\boldsymbol{f}_2)_{\beta\beta} = V(\boldsymbol{f}_2)\otimes_L V(\boldsymbol{g}_1)^+\otimes_L V(\boldsymbol{h}_1)^+$$

(cf. Section 7.2). Moreover, if $V(\boldsymbol{f}_2)^+_{\beta\beta}$ denotes the tensor product of $V(\boldsymbol{f}_2)^+, V(\boldsymbol{g}_1)^+$ and $V(\boldsymbol{h}_1)^+$ (that is $\mathscr{F}^3V(\boldsymbol{f}_2,\boldsymbol{g}_1,\boldsymbol{h}_1)$ with the notations of Section 7.2), the projection $V(\boldsymbol{f}_2)_{\beta\beta}\twoheadrightarrow V(\boldsymbol{f}_2)^-_{\beta\beta}$ gives rise to a short exact sequence of $G_{\mathbf{Q}_p}$-modules

$$0\longrightarrow V(\boldsymbol{f}_2)^+_{\beta\beta}\xrightarrow{i^+}V(\boldsymbol{f}_2)_{\beta\beta}\xrightarrow{\pi^-}V(\boldsymbol{f}_2)^-_{\beta\beta}\longrightarrow 0. \tag{206}$$

It follows that the image of $H^1(\mathbf{Q}_p,\mathscr{F}^2V(\boldsymbol{f}_2,\boldsymbol{g}_1,\boldsymbol{h}_1))$ under p_{f*} equals that of $H^1(\mathbf{Q}_p,V(\boldsymbol{f}_2)_{\beta\beta})$ under π^-, hence

$$\mathfrak{y}_f\in\pi^-_*\left(H^1(\mathbf{Q}_p,V(\boldsymbol{f}_2)_{\beta\beta})\right). \tag{207}$$

The short exact sequence (206) defines an extension class q_f in

$$\mathrm{Ext}^1_{L[G_p]}\big(V(\boldsymbol{f}_2)^-_{\beta\beta},V(\boldsymbol{f}_2)^+_{\beta\beta}\big)\cong H^1(\mathbf{Q}_p,L(1))\otimes_L\mathrm{Hom}_L\big(V(\boldsymbol{f}_2)^-_{\beta\beta},V(\boldsymbol{f}_2)^+_{\beta\beta}(-1)\big).$$

After identifying $H^1(\mathbf{Q}_p,L(1))$ with $\mathbf{Q}^*_p\hat{\otimes}L$ under the Kummer isomorphism, this defines a morphism

$$\begin{aligned}\mathrm{L}_{q_f}: H^1(\mathbf{Q}_p,V(\boldsymbol{f}_2)^-_{\beta\beta}) &\cong \mathrm{Hom}_{\mathrm{cont}}(\mathbf{Q}^*_p,L)\otimes_L V(\boldsymbol{f}_2)^-_{\beta\beta}\\ &\longrightarrow V(\boldsymbol{f}_2)^+_{\beta\beta}(-1)\cong H^2(\mathbf{Q}_p,V(\boldsymbol{f}_2)^+_{\beta\beta}),\end{aligned}$$

where the last isomorphism arises from the invariant map $H^2(\mathbf{Q}_p,L(1))\cong L$ of local class field theory. A direct computation, carried out in Lemma 9.5 below, shows that $\mathtt{L}_{q_f}$ is equal to the connecting morphism $H^1(\mathbf{Q}_p, V(\boldsymbol{f}_2)^-_{\beta\beta}) \longrightarrow H^2(\mathbf{Q}_p, V(\boldsymbol{f}_2)^+_{\beta\beta})$ associated with the exact sequence (206). It then follows from Equation (207) that

$$\mathtt{L}_{q_f}(\mathfrak{y}_f) = 0. \tag{208}$$

According to Theorem 3.18 of [GS93] q_f is of the form $\mathfrak{q}_f \otimes \delta_f$ for some linear form $\delta_f : V(\boldsymbol{f}_2)^-_{\beta\beta} \longrightarrow V(\boldsymbol{f}_2)^+_{\beta\beta}$ and $\mathfrak{q}_f$ in $\mathbf{Q}_p^* \hat{\otimes} L$ such that $\mathrm{ord}_p(\mathfrak{q}_f) \neq 0$ and

$$\mathfrak{L}^{\mathrm{an}}_{\boldsymbol{f}} = \log_p(\mathfrak{q}_f)/\mathrm{ord}_p(\mathfrak{q}_f).$$

Then

$$\log_{\mathfrak{q}_f} = \log_p - \mathfrak{L}^{\mathrm{an}}_{\boldsymbol{f}} \cdot \mathrm{ord}_p \in \mathrm{Hom}_{\mathrm{cont}}(\mathbf{Q}_p^*, L)$$

is the branch of the p-adic logarithm which vanishes at $\mathfrak{q}_f$ and $L \cdot \log_{\mathfrak{q}_f} \otimes_L V(\boldsymbol{f}_2)^-_{\beta\beta}$ is contained in the kernel of $\mathtt{L}_{q_f}$. Taking the long exact sequence associated with (206) one easily checks that the kernel of $\mathtt{L}_{q_f}$ has the same dimension as $V(\boldsymbol{f}_2)^-_{\beta\beta}$, hence $L \cdot \log_{\mathfrak{q}_f} \otimes_L V(\boldsymbol{f}_2)^-_{\beta\beta}$ is equal to the kernel of $\mathtt{L}_{q_f}$. Equation (208) then yields $\mathfrak{y}_f = \log_{\mathfrak{q}_f} \otimes v_f$ for some v_f in $V(\boldsymbol{f}_2)^-_{\beta\beta}$, hence

$$\mathfrak{y}_f(p^{-1}) = \mathfrak{L}^{\mathrm{an}}_{\boldsymbol{f}} \cdot v_f = \mathfrak{L}^{\mathrm{an}}_{\boldsymbol{f}} \cdot \mathfrak{y}_f(e(1)). \tag{209}$$

Part 1 of the proposition and Equations (205) and (209) give

$$\begin{aligned}
&2(1-1/p) \cdot \mathscr{L}_{\boldsymbol{f}}(\mathfrak{Z}) = 2(1-1/p) \cdot \mathscr{L}_{\boldsymbol{f}} \circ p_{f*}(\mathfrak{Y}) \\
&\overset{\text{Part 1}}{\equiv} \Big(\mathfrak{y}_f(p^{-1})_f - \mathfrak{L}^{\mathrm{an}}_{\boldsymbol{f}} \cdot \mathfrak{y}_f(e(1))_f\Big) \cdot (\boldsymbol{k}-2) \\
&+ \Big(\mathfrak{L}^{\mathrm{an}}_{\boldsymbol{g}} \cdot \mathfrak{y}_f(e(1))_f - \mathfrak{y}_f(p^{-1})_f\Big) \cdot (\boldsymbol{l}-1) + \Big(\mathfrak{L}^{\mathrm{an}}_{\boldsymbol{h}} \cdot \mathfrak{y}_f(e(1))_f - \mathfrak{y}_f(p^{-1})_f\Big) \cdot (\boldsymbol{m}-1) \\
&\overset{\text{Eq. (209)}}{\equiv} \mathfrak{y}_f(e(1))_f \cdot \Big(\big(\mathfrak{L}^{\mathrm{an}}_{\boldsymbol{g}} - \mathfrak{L}^{\mathrm{an}}_{\boldsymbol{f}}\big) \cdot (\boldsymbol{l}-1) + \big(\mathfrak{L}^{\mathrm{an}}_{\boldsymbol{h}} - \mathfrak{L}^{\mathrm{an}}_{\boldsymbol{f}}\big) \cdot (\boldsymbol{m}-1)\Big) \\
&\overset{\text{Eq. (205)}}{\equiv} \exp^*(\mathfrak{z})_f \cdot \Big(\big(\mathfrak{L}^{\mathrm{an}}_{\boldsymbol{g}} - \mathfrak{L}^{\mathrm{an}}_{\boldsymbol{f}}\big) \cdot (\boldsymbol{l}-1) + \big(\mathfrak{L}^{\mathrm{an}}_{\boldsymbol{h}} - \mathfrak{L}^{\mathrm{an}}_{\boldsymbol{f}}\big) \cdot (\boldsymbol{m}-1)\Big) \pmod{\mathscr{I}^2},
\end{aligned}$$

as was to be shown.

We finally prove Part 3. Taking $\mathscr{R} = \mathscr{O}_{\boldsymbol{g}}$, $\mathscr{M} = V(\boldsymbol{fg}, \boldsymbol{h}_1)_f$ and $\Phi = \mathrm{res}_{\boldsymbol{fg}} \circ \Psi$ in Lemma 9.4 gives an improved big dual exponential

$$\mathscr{E}xp^*_{V(\boldsymbol{fg},\boldsymbol{h}_1)_f} : H^1(\mathbf{Q}_p, V(\boldsymbol{fg}, \boldsymbol{h}_1)_f) \longrightarrow D(\boldsymbol{fg}, \boldsymbol{h}_1),$$

where $D(\boldsymbol{fg}, \boldsymbol{h}_1)_f = (\mathtt{V}(\boldsymbol{fg}, \boldsymbol{h}_1)_f \hat{\otimes}_{\mathbf{Z}_p} \hat{\mathbf{Z}}_p^{\mathrm{nr}})^{G_{\mathbf{Q}_p}}[1/p]$ and $\mathtt{V}(\boldsymbol{fg}, \boldsymbol{h}_1)_f$ is a $G_{\mathbf{Q}_p}$-invariant $\Lambda_{\boldsymbol{g}}$-lattice in $V(\boldsymbol{fg}, \boldsymbol{h}_1)_f$. Note that $D(\boldsymbol{fg}, \boldsymbol{h}_1)_f$ is naturally isomorphic to the base change of $D(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})_f$ along $\mathrm{res}_{\boldsymbol{fg}} : \mathscr{O}_{\boldsymbol{fgh}} \longrightarrow \mathscr{O}_{\boldsymbol{g}}$, and define

$$\mathscr{L}^*_{V(\boldsymbol{fg},\boldsymbol{h}_1)_f} : H^1(\mathbf{Q}_p, V(\boldsymbol{fg}, \boldsymbol{h}_1)_f) \longrightarrow \mathscr{O}_{\boldsymbol{g}}$$

to be the composition of $\mathscr{E}xp^*_{V(\boldsymbol{fg},\boldsymbol{h}_1)_f}$ with the base change

$$\langle \cdot, \eta_{\boldsymbol{f}} \omega_{\boldsymbol{g}} \omega_{\boldsymbol{h}} \rangle \otimes_{\mathrm{res}_{\boldsymbol{fg}}} \mathscr{O}_{\boldsymbol{g}} : D(\boldsymbol{fg}, \boldsymbol{h}_1)_f \longrightarrow \mathscr{O}_{\boldsymbol{g}}$$

along $\mathrm{res}_{\boldsymbol{fg}}$ of the linear form $\langle \cdot, \eta_{\boldsymbol{f}} \omega_{\boldsymbol{g}} \omega_{\boldsymbol{h}} \rangle_{\boldsymbol{fgh}}$ on $D(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})_f$. After noting that

$$1 - \Psi(\mathrm{Frob}_p)^{-1}(\boldsymbol{l}+\boldsymbol{m}, \boldsymbol{l}, \boldsymbol{m}) = \mathcal{E}^*_f(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h}) \quad \text{and} \quad 1 - p^{-1} \cdot \Psi_w(\mathrm{Frob}_p) = \mathscr{E}(l)$$

for each positive integer $l \geqslant 1$ in $U_{\boldsymbol{g}}$ congruent to 1 modulo $p-1$, where $w = (l+1, l, 1)$ in $\mathcal{H}_{\boldsymbol{fg}}$, the interpolation property satisfied by $\mathscr{L}_{V(\boldsymbol{fg},\boldsymbol{h}_1)_f}$ and the commutativity of the diagram in the statement follow directly from Equation (143) (cf. Section 7.1.1.1 for the case $l = 1$), Proposition 7.3 (and its proof) and Lemma 9.4. □

The following two lemmas have been invoked in the proof of Proposition 9.3.

Lemma 9.4. — *Let R be a complete local Noetherian ring with finite residue field of characteristic p, and let $\mathscr{R} = R[1/p]$. Let M be a free R-module of finite rank, equipped with the action of $G_{\mathbf{Q}_p}$ given by a continuous* unramified *character $\Phi : G_{\mathbf{Q}_p} \longrightarrow R^*$. Set $\mathscr{M} = M[1/p]$. Then there exists a morphism of $\mathscr{R}$-modules*

$$\mathscr{E}xp^*_{\Phi} : H^1(\mathbf{Q}_p, \mathscr{M}) \longrightarrow (M\hat{\otimes}_{\mathbf{Z}_p}\hat{\mathbf{Z}}_p^{\mathrm{nr}})^{G_{\mathbf{Q}_p}}[1/p]$$

such that, for each continuous morphism of $\mathbf{Z}_p$-algebras $\nu : R \to \bar{\mathbf{Q}}_p$ and each class $\mathfrak{Z} \in H^1(\mathbf{Q}_p, \mathscr{M})$, one has

$$\nu\big(\mathscr{E}xp^*_{\Phi}(\mathfrak{Z})\big) = \big(1 - p^{-1}\cdot\Phi_\nu(\mathrm{Frob}_p)\big)^{-1}\cdot \exp^*_p(\mathfrak{Z}_\nu),$$

where the notations are as follows. Set $\mathscr{O}_\nu = \nu(R)$ and $L_\nu = \mathrm{Frac}(\mathscr{O}_\nu)$. The unramified character $\Phi_\nu : G_{\mathbf{Q}_p} \longrightarrow \mathscr{O}_\nu^*$ is the composition of Φ with ν, the class $\mathfrak{Z}_\nu$ in $H^1(\mathbf{Q}_p, L_\nu(\Phi_\nu))$ is the image of $\mathfrak{Z}$ under the map induced in cohomology by ν, and

$$\exp^*_p : H^1(\mathbf{Q}_p, L_\nu(\Phi_\nu)) \longrightarrow D_{\mathrm{cris}}(L_\nu(\Phi)) = (\mathscr{O}_\nu(\Phi_\nu)\hat{\otimes}_{\mathbf{Z}_p}\hat{\mathbf{Z}}_p^{\mathrm{nr}})^{G_{\mathbf{Q}_p}}[1/p]$$

is the Bloch–Kato dual exponential.

Proof. — When $\mathscr{R} = \mathscr{O}_{\boldsymbol{f}}$ and $\mathscr{M} = \mathscr{O}_{\boldsymbol{f}}(\check{a}_p(\boldsymbol{k}))$, this is [**Ven16**, Proposition 3.8]. Mutatis mutandis, the proof of loco citato works in this more general setting. □

Lemma 9.5. — *Let M and N be two finite dimensional L-vector spaces, equipped with the trivial action of the absolute Galois group G_p of $\mathbf{Q}_p$, let*

$$0 \longrightarrow M(1) \xrightarrow{\alpha} V \xrightarrow{\beta} N \longrightarrow 0 \tag{210}$$

be a short exact sequence of (continuous) $L[G_p]$-modules, and let

$$q_V \in \mathrm{Ext}^1_{L[G_p]}(N, M(1)) \cong \hat{\mathbf{Q}}_p^* \otimes_{\mathbf{Z}_p} \mathrm{Hom}_L(N, M)$$

be the corresponding extension class (where one identifies $H^1(\mathbf{Q}_p, \mathbf{Z}_p(1))$ with the p-adic completion $\hat{\mathbf{Q}}_p^*$ of $\mathbf{Q}_p$ via the Kummer map). Then the connecting morphism

$$\delta_V : H^1(\mathbf{Q}_p, N) \longrightarrow H^1(\mathbf{Q}_p, M(1))$$

associated with the short exact sequence is equal to the composition

$$\mathtt{L}_V : H^1(\mathbf{Q}_p, N) \cong \mathrm{Hom}_{\mathrm{cont}}(\hat{\mathbf{Q}}_p^*, \mathbf{Z}_p) \otimes_{\mathbf{Z}_p} N \xrightarrow{e_V} M \cong H^2(\mathbf{Q}_p, M(1)),$$

where the first isomorphism arises from the local Artin map $\mathrm{rec}_p : \mathbf{Q}_p^* \longrightarrow G_p^{\mathrm{ab}}$ (sending p^{-1} to an arithmetic Frobenius), the second isomorphism arises from the invariant map $\mathrm{inv}_p : H^2(\mathbf{Q}_p, \mathbf{Z}_p(1)) \cong \mathbf{Z}_p$, and e_V is evaluation at q_V (under the product of the natural dualities $\hat{\mathbf{Q}}_p^* \otimes_{\mathbf{Z}_p} \mathrm{Hom}_{\mathrm{cont}}(\hat{\mathbf{Q}}_p^*, \mathbf{Z}_p) \longrightarrow \mathbf{Z}_p$ and $\mathrm{Hom}_L(N, M) \otimes_L N \longrightarrow M$).

Proof. — Identify $M(1)$ with a subspace of V via the injective morphism α, and fix an L-linear section $\sigma : N \longrightarrow V$ of β. Under the natural isomorphisms

$$\mathrm{Ext}^1_{L[G_p]}(N, M(1)) = \mathrm{Ext}^1_{L[G_p]}(L, \mathrm{Hom}_L(N, M)(1)) = H^1(\mathbf{Q}_p, \mathrm{Hom}_L(N, M)(1)),$$

the extension class of (210) is represented by the 1-cocylce

$$\xi_V = \xi_{V,\sigma} : G_p \longrightarrow \mathrm{Hom}_L(M, N)(1)$$

defined by the formulae

$$g(\sigma(n)) - \sigma(n) = \xi_V(g)(n)$$

for each g in G_p and each n in N.

For each 1-cocycle (id est continuous morphism of groups) $\varphi : G_p \longrightarrow N$, the image of φ under the connecting map δ_V is represented by the 2-cocycle $\delta_V^o(\varphi)$ defined by

$$\delta_V^o(\varphi)(g, h) = g\big(\sigma(\varphi(h))\big) - \sigma\big(\varphi(gh)\big) + \sigma(\varphi(g)) = \xi_V(g)(\varphi(h)) = \xi_V \cup_{\mathrm{ev}} \varphi\,(g, h),$$

where $\cup_{\mathrm{ev}} : C^\bullet_{\mathrm{cont}}(G_p, \mathrm{Hom}_L(N, M)(1)) \otimes_L C^\bullet_{\mathrm{cont}}(G_p, N) \longrightarrow C^\bullet_{\mathrm{cont}}(G_p, M(1))$ denotes the cup-product induced on continuous cochains by the evaluation pairing

$$\mathrm{ev} : \mathrm{Hom}_L(N, M) \otimes_L N \longrightarrow M$$

(cf. Sections 3.4.1.2 and 3.4.5.1 of [**Nek06**]). If $\langle\cdot,\cdot\rangle_{\mathrm{ev}}$ denotes the composition of the cup-product pairing induced in $(1, 1)$-cohomology by $\cup_{\mathrm{ev}}$ with the M-linear extension

$$\mathrm{inv}_M : H^2(\mathbf{Q}_p, M(1)) = H^2(\mathbf{Q}_p, \mathbf{Z}_p(1)) \otimes_{\mathbf{Z}_p} M \cong M$$

of the local invariant map inv_p, it follows that

$$\mathrm{inv}_M(\delta_V(\varphi)) = \langle cl(\xi_V), \varphi\rangle_{\mathrm{ev}}\,, \tag{211}$$

where $cl(\cdot)$ denotes the class represented by $\cdot$. Under the natural isomorphisms

$$H^1(\mathbf{Q}_p, \mathrm{Hom}_L(N, M)(1)) = H^1(\mathbf{Q}_p, \mathbf{Z}_p(1)) \otimes_{\mathbf{Z}_p} \mathrm{Hom}_L(N, M)$$

and $H^1(\mathbf{Q}_p, N) = H^1(\mathbf{Q}_p, \mathbf{Z}_p) \otimes_{\mathbf{Z}_p} N$, the pairing $\langle\cdot,\cdot\rangle_{\mathrm{ev}}$ corresponds to the product of ev and the local Tate duality

$$\langle\cdot,\cdot\rangle : H^1(\mathbf{Q}_p, \mathbf{Z}_p(1)) \otimes_{\mathbf{Z}_p} H^1(\mathbf{Q}_p, \mathbf{Z}_p) \xrightarrow{\cup} H^2(\mathbf{Q}_p, \mathbf{Z}_p(1)) \xrightarrow{\mathrm{inv}_p} \mathbf{Z}_p$$

associated with the multiplication pairing $\mathbf{Z}_p(1) \otimes_{\mathbf{Z}_p} \mathbf{Z}_p \longrightarrow \mathbf{Z}_p$. Finally one has

$$\langle \kappa(q), \chi\rangle = \chi(\mathrm{rec}_p(q))$$

for each χ in $H^1(\mathbf{Q}_p, \mathbf{Z}_p)$ and each q in $\mathbf{Q}_p^*$, where $\kappa : \mathbf{Q}_p^* \longrightarrow H^1(\mathbf{Q}_p, \mathbf{Z}_p(1))$ denotes the Kummer map (cf. Proposition 1 in Section 2.3 of [**Ser67**]), hence

$$\langle cl(\xi_V), \varphi\rangle_{\mathrm{ev}} = e_V(\varphi),$$

which combined with Equation (211) concludes the proof. □

9.3. Improved diagonal classes. — This section proves the existence of the big g-improved diagonal class introduced in Equation (2) of Section 1.2.

Section 8.1 associates to the *ordered* triple of Hida families $(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})$ the big diagonal class $\kappa(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})$ (which is symmetric in the forms $\boldsymbol{g}$ and $\boldsymbol{h}$). After identifying the big $G_{\mathbf{Q}}$-representations $V(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h}), V(\boldsymbol{g},\boldsymbol{f},\boldsymbol{h})$ and $V(\boldsymbol{h},\boldsymbol{f},\boldsymbol{g})$ under the natural isomorphisms, a priori the three classes

$$\kappa(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h}), \quad \kappa(\boldsymbol{g},\boldsymbol{f},\boldsymbol{h}) \quad \text{and} \quad \kappa(\boldsymbol{h},\boldsymbol{f},\boldsymbol{g})$$

in $H^1(\mathbf{Q},V(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h}))$ may be different. This is indeed *not* the case.

Lemma 9.6. — *The classes $\kappa(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})$, $\kappa(\boldsymbol{g},\boldsymbol{f},\boldsymbol{h})$ and $\kappa(\boldsymbol{h},\boldsymbol{f},\boldsymbol{g})$ are equal.*

Proof. — Let Σ^o_{bal} be the set of balanced triples $w=(k,l,m)$ such that p does not divide the conductors of $\boldsymbol{f}_k$, $\boldsymbol{g}_l$ and $\boldsymbol{h}_m$. Since $H^1(\mathbf{Q},V(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h}))$ is a torsion-free $\mathscr{O}_{\boldsymbol{fgh}}$-module and Σ^o_{bal} is dense in $U_{\boldsymbol{f}}\times U_{\boldsymbol{g}}\times U_{\boldsymbol{h}}$, one has

$$\bigcap_{w\in\Sigma^o_{\mathrm{bal}}}(\boldsymbol{k}-k,\boldsymbol{l}-l,\boldsymbol{m}-m)\cdot H^1(\mathbf{Q},V(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h}))=0.$$

It is then sufficient to prove that the three classes in the statement have the same specialisation at each balanced classical triple w in Σ^o_{bal}. Because the map $\Pi^{\alpha}_{f_k g_l h_m *}$ (defined after Equation (169)) is an isomorphism at each point (k,l,m) of Σ^o_{bal}, this is a consequence of Theorem 8.1 and Proposition 8.3. □

We now construct the g-improved balanced diagonal class

$$\kappa^*_g(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})\in H^1_{\mathrm{bal}}(\mathbf{Q},V(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})|_{\mathcal{H}_{\boldsymbol{g}}}) \tag{212}$$

satisfying Equation (2) of Section 1.2.

Set $\Lambda_{\boldsymbol{gh}}=\Lambda_{\boldsymbol{g}}\hat{\otimes}_{\mathscr{O}}\Lambda_{\boldsymbol{h}}$, so that $\mathscr{O}_{\boldsymbol{gh}}=\Lambda_{\boldsymbol{gh}}[1/p]$. For every $\Lambda_{\boldsymbol{gfh}}$-module M, define

$$M|_{\mathcal{H}_{\boldsymbol{g}}}=M\otimes_{\nu_{\boldsymbol{g}}}\Lambda_{\boldsymbol{gh}}$$

to be the base change of the $\Lambda_{\boldsymbol{gfh}}$-module M under the morphism $\nu_{\boldsymbol{g}}:\Lambda_{\boldsymbol{gfh}}\longrightarrow\Lambda_{\boldsymbol{gh}}$ sending the analytic function $F(\boldsymbol{k},\boldsymbol{l},\boldsymbol{m})$ to its restriction $F(\boldsymbol{l}-\boldsymbol{m}+2,\boldsymbol{l},\boldsymbol{m})$ to the g-improving plane $\mathcal{H}_{\boldsymbol{g}}$ (cf. Section 1.2). A similar notation applies to $\mathscr{O}_{\boldsymbol{gfh}}$-modules and sheaves of $\Lambda_{\boldsymbol{gfh}}$ or $\mathscr{O}_{\boldsymbol{gfh}}$-modules.

Remark 9.7. — The space $\mathcal{A}'_{\boldsymbol{g}}\hat{\otimes}\mathcal{A}_{\boldsymbol{f}}\hat{\otimes}\mathcal{A}_{\boldsymbol{h}}|_{\mathcal{H}_{\boldsymbol{g}}}$ is identified with a subspace of the $\Lambda_{\boldsymbol{gh}}$-valued functions f on $\mathsf{T}'\times\mathsf{T}\times\mathsf{T}$ that are locally analytic and such that

$$f\left(t_x\cdot x,t_y\cdot y,t_z\cdot z\right)=\nu_{\boldsymbol{g}}(t_x^{\kappa_{\boldsymbol{f}}}t_y^{\kappa_{\boldsymbol{g}}}t_z^{\kappa_{\boldsymbol{h}}})\cdot f\left(x,y,z\right).$$

(This can be seen by applying [**GS16**, Lemma 7.3] with $X=\mathsf{T}'\times\mathsf{T}\times\mathsf{T}$ to reduce the statement to the fact that the formation of locally analytic function - without the homogeneity property imposed - is compatible with base change.) Conversely, such a function f can be assumed to be in the image of $\mathcal{A}'_{\boldsymbol{f}}\hat{\otimes}\mathcal{A}_{\boldsymbol{g}}\hat{\otimes}\mathcal{A}_{\boldsymbol{h}}|_{\mathcal{H}_{\boldsymbol{g}}}$, by increasing the radius of convergence in the definition of $\mathcal{A}^{\cdot}_{\boldsymbol{f}}=\mathcal{A}^{\cdot}_{U_{\boldsymbol{f}},\imath}$, $\mathcal{A}^{\cdot}_{\boldsymbol{g}}=\mathcal{A}^{\cdot}_{U_{\boldsymbol{g}},\imath}$ and $\mathcal{A}^{\cdot}_{\boldsymbol{h}}=\mathcal{A}^{\cdot}_{U_{\boldsymbol{h}},\imath}$. .

Consider the analytic function $\mathtt{D}^*_{\boldsymbol{g}} : \mathsf{T}' \times \mathsf{T} \times \mathsf{T} \longrightarrow \Lambda_{\boldsymbol{gfh}}$ defined by the formula

$$\mathtt{D}^*_{\boldsymbol{g}}(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z}) = \det(\boldsymbol{x},\boldsymbol{y})^{\kappa^*_{\boldsymbol{h}}} \cdot \det(\boldsymbol{x},\boldsymbol{z})^{\kappa^*_{\boldsymbol{f}}} \cdot \det(\boldsymbol{y},\boldsymbol{z})^{(k+m-l-2)/2}$$

for each $(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z})$ in $\mathsf{T}' \times \mathsf{T} \times \mathsf{T}$ with $\boldsymbol{a} = (a_1, a_2)$ for $\boldsymbol{a} = \boldsymbol{x},\boldsymbol{y},\boldsymbol{z}$. (Because we apply an integer power to the last determinant, there is no need to restrict to the domain $\mathsf{T}' \times (\mathsf{T} \times \mathsf{T})_0$ as we did in the definition of **Det** in Section 8.1.) Then $\mathbf{Det}^*_g := \nu_{\boldsymbol{g}} \circ \mathtt{D}^*_{\boldsymbol{g}} : \mathsf{T}' \times \mathsf{T} \times \mathsf{T} \longrightarrow \Lambda_{\boldsymbol{gh}}$ is a locally analytic function satisfying the homogeneity property of Remark 9.7. It also satisfies the invariance property

$$\mathbf{Det}^*_g(\boldsymbol{x}\cdot\gamma,\boldsymbol{y}\cdot\gamma,\boldsymbol{z}\cdot\gamma) = \det(\gamma)^{\nu_{\boldsymbol{g}}\circ\kappa^*_{\boldsymbol{gfh}}} \cdot \mathbf{Det}^*_g(\boldsymbol{x}\cdot\gamma,\boldsymbol{y}\cdot\gamma,\boldsymbol{z}\cdot\gamma).$$

Applying Remark 9.7 and recalling that $\kappa_{\boldsymbol{g}} = \nu_{\boldsymbol{g}} \circ \kappa^*_{\boldsymbol{gfh}}$, we have thus defined

$$\mathbf{Det}^*_{\boldsymbol{g}} \in H^0(\Gamma_0(p\mathbf{Z}_p), \mathcal{A}'_{\boldsymbol{g}}\hat{\otimes}\mathcal{A}_{\boldsymbol{f}}\hat{\otimes}\mathcal{A}_{\boldsymbol{h}}|_{\mathcal{H}_{\boldsymbol{g}}}(-\kappa_{\boldsymbol{g}})). \tag{213}$$

With the notations of Sections 4.2 and 8.1, let

$$\boldsymbol{\mathcal{A}}'_{\boldsymbol{g}}\boxtimes\boldsymbol{\mathcal{A}}_{\boldsymbol{f}}\boxtimes\boldsymbol{\mathcal{A}}_{\boldsymbol{h}}|_{\mathcal{H}_{\boldsymbol{g}}} = \mathcal{A}'_{\boldsymbol{g}}\hat{\otimes}\mathcal{A}_{\boldsymbol{f}}\hat{\otimes}\mathcal{A}_{\boldsymbol{h}}|^{\text{ét}}_{\mathcal{H}_{\boldsymbol{g}}} \quad\text{and}\quad \boldsymbol{\mathcal{A}}'_{\boldsymbol{g}}\otimes\boldsymbol{\mathcal{A}}_{\boldsymbol{f}}\otimes\boldsymbol{\mathcal{A}}_{\boldsymbol{h}}|_{\mathcal{H}_{\boldsymbol{g}}} = d^*\big(\boldsymbol{\mathcal{A}}'_{\boldsymbol{g}}\boxtimes\boldsymbol{\mathcal{A}}_{\boldsymbol{f}}\boxtimes\boldsymbol{\mathcal{A}}_{\boldsymbol{h}}|_{\mathcal{H}_{\boldsymbol{g}}}\big)$$

be the étale sheaf on Y^3 associated with the representation $\mathcal{A}_{\boldsymbol{f}}\hat{\otimes}\mathcal{A}'_{\boldsymbol{g}}\hat{\otimes}\mathcal{A}_{\boldsymbol{h}}|_{\mathcal{H}_{\boldsymbol{g}}}$ in $\mathbf{M}(\Gamma_0(p\mathbf{Z}_p)^3)$ and its pull back under the diagonal embedding $d : Y \longrightarrow Y^3$ respectively, so that one has a natural inclusion

$$H^0(\Gamma_0(p\mathbf{Z}_p), \mathcal{A}'_{\boldsymbol{g}}\hat{\otimes}\mathcal{A}_{\boldsymbol{f}}\hat{\otimes}\mathcal{A}_{\boldsymbol{h}}|_{\mathcal{H}_{\boldsymbol{g}}}(-\kappa_{\boldsymbol{g}})) \hookrightarrow H^0_{\text{ét}}(Y, \boldsymbol{\mathcal{A}}'_{\boldsymbol{g}}\otimes\boldsymbol{\mathcal{A}}_{\boldsymbol{f}}\otimes\boldsymbol{\mathcal{A}}_{\boldsymbol{h}}|_{\mathcal{H}_{\boldsymbol{g}}}(-\kappa_{\boldsymbol{g}})). \tag{214}$$

On the other hand, consider the following composition.

$$\begin{aligned} &H^0_{\text{ét}}(Y, \boldsymbol{\mathcal{A}}'_{\boldsymbol{g}}\otimes\boldsymbol{\mathcal{A}}_{\boldsymbol{f}}\otimes\boldsymbol{\mathcal{A}}_{\boldsymbol{h}}|_{\mathcal{H}_{\boldsymbol{g}}}(-\kappa_{\boldsymbol{g}})) \\ &\xrightarrow{d_*} H^4_{\text{ét}}(Y^3, \boldsymbol{\mathcal{A}}'_{\boldsymbol{g}}\boxtimes\boldsymbol{\mathcal{A}}_{\boldsymbol{f}}\boxtimes\boldsymbol{\mathcal{A}}_{\boldsymbol{h}}|_{\mathcal{H}_{\boldsymbol{g}}}(-\kappa_{\boldsymbol{g}})\otimes_{\mathbf{Z}_p}\mathbf{Z}_p(2)) \\ &\xrightarrow{\mathtt{HS}} H^1\big(\mathbf{Q}, H^3_{\text{ét}}\big(Y^3_{\bar{\mathbf{Q}}}, \boldsymbol{\mathcal{A}}'_{\boldsymbol{g}}\boxtimes\boldsymbol{\mathcal{A}}_{\boldsymbol{f}}\boxtimes\boldsymbol{\mathcal{A}}_{\boldsymbol{h}}|_{\mathcal{H}_{\boldsymbol{g}}}\big)(2+\boldsymbol{\kappa}_{\boldsymbol{g}})\big) \end{aligned} \tag{215}$$

Because $H^4_{\text{ét}}(Y^3_{\bar{\mathbf{Q}}}, \mathscr{F})$ vanishes for every pro-sheaf $\mathscr{F} \in \mathbf{S}(Y^3_{\text{ét}})$ (cf. the discussion following Equation (156)), one has a natural isomorphism

$$H^3_{\text{ét}}\big(Y^3_{\bar{\mathbf{Q}}}, \boldsymbol{\mathcal{A}}'_{\boldsymbol{g}}\boxtimes\boldsymbol{\mathcal{A}}_{\boldsymbol{f}}\boxtimes\boldsymbol{\mathcal{A}}_{\boldsymbol{h}}|_{\mathcal{H}_{\boldsymbol{g}}}\big) = H^3_{\text{ét}}\big(Y^3_{\bar{\mathbf{Q}}}, \boldsymbol{\mathcal{A}}'_{\boldsymbol{f}}\boxtimes\boldsymbol{\mathcal{A}}_{\boldsymbol{g}}\boxtimes\boldsymbol{\mathcal{A}}_{\boldsymbol{h}}\big)|_{\mathcal{H}_{\boldsymbol{g}}}.$$

Moreover, as in Equation (156), the base change along $\nu_{\boldsymbol{g}}$ of the projection arising from the Künneth decomposition et cetera induce a map

$$H^1\big(\mathbf{Q}, H^3_{\text{ét}}\big(Y^3_{\bar{\mathbf{Q}}}, \boldsymbol{\mathcal{A}}'_{\boldsymbol{f}}\boxtimes\boldsymbol{\mathcal{A}}_{\boldsymbol{g}}\boxtimes\boldsymbol{\mathcal{A}}_{\boldsymbol{h}}\big)|_{\mathcal{H}_{\boldsymbol{g}}}(2+\boldsymbol{\kappa}_{\boldsymbol{g}})\big) \longrightarrow H^1(\mathbf{Q}, V(\boldsymbol{g},\boldsymbol{f},\boldsymbol{h})|_{\mathcal{H}_{\boldsymbol{g}}}), \tag{216}$$

and we denote by

$$\mathrm{AJ}^{\boldsymbol{gfh}}_{\text{ét}} : H^0_{\text{ét}}(Y, \boldsymbol{\mathcal{A}}'_{\boldsymbol{g}}\otimes\boldsymbol{\mathcal{A}}_{\boldsymbol{f}}\otimes\boldsymbol{\mathcal{A}}_{\boldsymbol{h}}|_{\mathcal{H}_{\boldsymbol{g}}}(-\kappa_{\boldsymbol{g}})) \longrightarrow H^1(\mathbf{Q}, V(\boldsymbol{g},\boldsymbol{f},\boldsymbol{h})|_{\mathcal{H}_{\boldsymbol{g}}}) \tag{217}$$

the composition of the maps (215) and (216).

Identifying $V(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})|_{\mathcal{H}_{\boldsymbol{g}}}$ and $V(\boldsymbol{g},\boldsymbol{f},\boldsymbol{h})|_{\mathcal{H}_{\boldsymbol{g}}}$, one defines the sought for g-improved diagonal class (212) to be the image of $\mathbf{Det}^*_{\boldsymbol{g}}$ under the big Abel–Jacobi map defined in Equation (217), multiplied by the normalising factor $\frac{1}{b_p(\boldsymbol{l})}$ (cf. Equation (155)):

$$\kappa^*_g(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h}) = \frac{1}{b_p(\boldsymbol{l})} \cdot \mathrm{AJ}^{\boldsymbol{gfh}}_{\text{ét}}\big(\mathbf{Det}^*_{\boldsymbol{g}}\big).$$

(Here one views $\mathbf{Det}_{\boldsymbol{g}}^*$ as a global section of the étale sheaf $\boldsymbol{\mathcal{A}}'_{\boldsymbol{g}} \otimes \boldsymbol{\mathcal{A}}_{\boldsymbol{f}} \otimes \boldsymbol{\mathcal{A}}_{\boldsymbol{h}}|_{\mathcal{H}_{\boldsymbol{g}}}(-\kappa_{\boldsymbol{g}})$ via the inclusion (214).) The balancedness of $\kappa_{\boldsymbol{g}}^*(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})$ follows from a similar argument as the one in the proof of Corollary 8.2.

We now verify that $\kappa_g^*(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})$ satisfies the identity displayed in Equation (2):

$$\kappa(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})|_{\mathcal{H}_{\boldsymbol{g}}} = \mathcal{E}_g(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h}) \cdot \kappa_g^*(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h}). \tag{218}$$

Let $\mathcal{H}_{\boldsymbol{g}}^{\mathrm{cl}}$ be the intersection of $\mathcal{H}_{\boldsymbol{g}}$ with $U_{\boldsymbol{f}}^{\mathrm{cl}} \times U_{\boldsymbol{g}}^{\mathrm{cl}} \times U_{\boldsymbol{h}}^{\mathrm{cl}}$. As $H^1(\mathbf{Q}, V(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})|_{\mathcal{H}_{\boldsymbol{g}}})$ is a torsion-free $\mathscr{O}_{\boldsymbol{gh}}$-module, in order to prove the previous equation it is sufficient to show that

$$\rho_{w*}(\kappa(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})) = \mathcal{E}_g(\boldsymbol{f}_k,\boldsymbol{g}_l,\boldsymbol{h}_m) \cdot \rho_{w*}(\kappa_g^*(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})) \tag{219}$$

for each classical triple $w = (k,l,m)$ in the subset

$$\mathcal{H}_{\boldsymbol{g}}^{\mathrm{bal}} = \{(k,l,m) \in \mathcal{H}_{\boldsymbol{g}}^{\mathrm{cl}} \mid m \geqslant 3\}$$

of $\mathcal{H}_{\boldsymbol{g}}^{\mathrm{cl}}$, where $\rho_w : V(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h}) \longrightarrow V(\boldsymbol{f}_l,\boldsymbol{g}_l,\boldsymbol{h}_m)$ is the specialisation map (cf. Equation (145)) and $\mathcal{E}_g(\boldsymbol{f}_k,\boldsymbol{g}_l,\boldsymbol{h}_m)$ is the value of $\mathcal{E}_g(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})$ at (l,m). The set $\mathcal{H}_{\boldsymbol{g}}^{\mathrm{bal}}$ is the intersection of $\mathcal{H}_{\boldsymbol{g}}$ with the balanced region Σ_{bal}. Moreover Lemma 9.6 and Theorem 8.1 yield

$$(p-1)b_p(l) \cdot \varrho_{w*}(\kappa(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})) = \mathcal{E}_g(\boldsymbol{f}_k,\boldsymbol{g}_l,\boldsymbol{h}_m) \cdot \kappa^\dagger(\boldsymbol{g}_l,\boldsymbol{f}_k,\boldsymbol{h}_m)$$

for each $w = (k,l,m)$ in $\mathcal{H}_{\boldsymbol{g}}^{\mathrm{bal}}$. (Recall from Equation (157) that the definition of the twisted diagonal class $\kappa^\dagger(\boldsymbol{g}_l,\boldsymbol{f}_k,\boldsymbol{h}_m)$ is *not* symmetric in the forms $\boldsymbol{f}_k, \boldsymbol{g}_l$ and $\boldsymbol{h}_m$. Indeed, after identifying $V(\boldsymbol{g}_l,\boldsymbol{f}_k,\boldsymbol{h}_m)$ with $V(\boldsymbol{f}_k,\boldsymbol{g}_l,\boldsymbol{h}_m)$, it follows from Theorem 8.1 and Lemma 9.6 that the class $\kappa^\dagger(\boldsymbol{g}_l,\boldsymbol{f}_k,\boldsymbol{h}_m)$ is in general *not* equal to $\kappa^\dagger(\boldsymbol{f}_k,\boldsymbol{g}_l,\boldsymbol{h}_m)$.) To prove Equation (219), and with it Equation (218), it then remains to prove that

$$(p-1)b_p(l) \cdot \rho_{w*}\big(\kappa_g^*(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})\big) = \kappa^\dagger(\boldsymbol{g}_l,\boldsymbol{f}_k,\boldsymbol{h}_m)$$

for each $w = (k,l,m)$ in $\mathcal{H}_{\boldsymbol{g}}^{\mathrm{bal}}$. After unwinding the definition, this is in turn a direct consequence of the identity

$$\rho_w(\mathbf{Det}_g^*) = \mathrm{Det}_{Np}^{\boldsymbol{r}(w)},$$

where $\boldsymbol{r}(w) = (l-2, k-2, m-2)$, which holds true in $S_{\boldsymbol{r}(w)} \hookrightarrow \mathcal{A}'_{l-2} \hat{\otimes} \mathcal{A}_{k-2} \hat{\otimes} \mathcal{A}_{m-2}$ for each balanced triple $w = (k,l,m)$ in $\mathcal{H}_{\boldsymbol{g}}^{\mathrm{bal}}$ by the very definitions of the invariants $\mathbf{Det}_g^*$ and $\mathrm{Det}_{Np}^{\boldsymbol{r}}$ (cf. Equations (213) and (41)).

9.4. Conclusion of the proof. — Assume that $w_o = (2,1,1)$ is exceptional. As in Section 9.2, denote by $\mathcal{H}_{\boldsymbol{fg}}$ the intersection of the improving planes $\mathcal{H}_{\boldsymbol{g}}$ and $\mathcal{H}_{\boldsymbol{f}}$, that is the set of triples in $U_{\boldsymbol{f}} \times U_{\boldsymbol{g}} \times U_{\boldsymbol{h}}$ of the form $(\boldsymbol{l}+1, \boldsymbol{l}, 1)$. Denote by

$$\mathscr{L}_p^f(\boldsymbol{fg},\boldsymbol{h}_1) = \mathscr{L}_p^f(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})|_{\mathcal{H}_{\boldsymbol{fg}}} \in \mathscr{O}_{\boldsymbol{g}}$$

the analytic function on $U_{\boldsymbol{g}}$ which on $\boldsymbol{l}$ takes the value $\mathscr{L}_p^f(\boldsymbol{f}_{l+1},\boldsymbol{g}_l,\boldsymbol{h}_1)$ (cf. Equation (55)). Define similarly

$$\mathcal{E}_f^*(\boldsymbol{fg},\boldsymbol{h}_1) = \mathcal{E}_f^*(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})|_{\mathcal{H}_{\boldsymbol{fg}}} \in \mathscr{O}_{\boldsymbol{g}} \quad \text{and} \quad \mathcal{E}_g(\boldsymbol{fg},\boldsymbol{h}_1) = \mathcal{E}_g(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})|_{\mathcal{H}_{\boldsymbol{fg}}} \in \mathscr{O}_{\boldsymbol{g}}.$$

Lemma 9.8. — *Let h_1 be the modular form of weight one and level $\Gamma_1(N)$ with p-stabilisation $\boldsymbol{h}_1$. One has*

$$\mathscr{L}_p^f(\boldsymbol{f}\boldsymbol{g},\boldsymbol{h}_1)=\mathcal{E}_f^*(\boldsymbol{f}\boldsymbol{g},\boldsymbol{h}_1)\cdot\mathcal{E}_g(\boldsymbol{f}\boldsymbol{g},\boldsymbol{h}_1)\cdot\mathscr{L}_p^{f*}(\boldsymbol{f}\boldsymbol{g},h_1),$$

where $\mathscr{L}_p^{f*}(\boldsymbol{f}\boldsymbol{g},h_1)$ is the analytic function in $\mathscr{O}_{\boldsymbol{g}}$ which on the classical point $l\geqslant 1$ in $U_{\boldsymbol{g}}^{\mathrm{cl}}$ takes the value

$$\mathscr{L}_p^{f*}(\boldsymbol{f}_{l+1},\boldsymbol{g}_l,h_1)=\frac{(w_N(\boldsymbol{f})_{l+1},h_1\cdot\boldsymbol{g}_l)_{Np}}{(w_N(\boldsymbol{f})_{l+1},w_N(\boldsymbol{f})_{l+1})_{Np}}.$$

Moreover, the following two conditions are equivalent.

1. *$\mathscr{L}_p^{f*}(\boldsymbol{f}_2,\boldsymbol{g}_1,h_1)$ is zero for all level-N test vectors $(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})$ for $(\boldsymbol{f}^\sharp,\boldsymbol{g}^\sharp,\boldsymbol{h}^\sharp)$.*
2. *The complex central value $L(f_2^\sharp\otimes g_1^\sharp\otimes h_1^\sharp,1)$ vanishes.*

Proof. — Set $U=U_{\boldsymbol{g}}$, denote by $(\cdot,\cdot)_U: S_U^{\mathrm{ord}}(N,\bar{\chi}_{\boldsymbol{f}})\otimes_{\mathcal{O}(U)}S_U^{\mathrm{ord}}(N,\bar{\chi}_{\boldsymbol{f}})\longrightarrow\mathcal{O}(U)$ the $\mathcal{O}(U)$-adic Petersson product (cf. Section 7 of [**Hid93**]) and define

$$\mathscr{L}_p^{f*}(\boldsymbol{f}\boldsymbol{g},h_1)=\frac{(w_N(\boldsymbol{f})_{+1},e_{\mathrm{ord}}(h_1\cdot\boldsymbol{g}))_U}{(w_N(\boldsymbol{f})_{+1},w_N(f)_{+1})_U}.$$

Here $w_N(\boldsymbol{f})$ is the Hida family introduced in Lemma 6.1, $w_N(\boldsymbol{f})_{+1}$ is the family in $S_U^{\mathrm{ord}}(N,\bar{\chi}_{\boldsymbol{f}})$ whose specialisation at the classical point $m\geqslant 2$ equals $w_N(\boldsymbol{f}_{m+1})$ and e_{ord} is Hida's ordinary projector from the space of $\mathcal{O}(U)$-adic cusp forms of tame level N and character $\bar{\chi}_{\boldsymbol{f}}$ onto $S_U^{\mathrm{ord}}(N,\bar{\chi}_{\boldsymbol{f}})$, cf. [**Hid93**]. (Concretely $e_{\mathrm{ord}}(h_1\cdot\boldsymbol{g})_l$ equals $e_{\mathrm{ord}}(h_1\cdot\boldsymbol{g}_l)$ for each classical point l in U^{cl}, where the idempotent e_{ord} occurring in the right hand side is equal to $\lim_{n\to\infty}U_p^{n!}$.) By construction the value of $\mathscr{L}_p^{f*}(\boldsymbol{f}\boldsymbol{g},h_1)$ at a classical point $m\geqslant 1$ equals $\mathscr{L}_p^{f*}(\boldsymbol{f}_{l+1},\boldsymbol{g}_l,h_1)$.

Recall the operator $V=V_p$ on $L[\![q]\!]$ defined by $V\left(\sum c_nq^n\right)=\sum c_nq^{np}$. Then

$$\boldsymbol{h}_1=(1-\beta_{\boldsymbol{h}_1}\cdot V)h_1\quad\text{and}\quad\boldsymbol{h}_1^{[p]}=(1-\alpha_{\boldsymbol{h}_1}\cdot V)\boldsymbol{h}_1$$

with $\alpha_{\boldsymbol{h}_1}\cdot\beta_{\boldsymbol{h}_1}=\chi_{\boldsymbol{h}}(p)$, and similarly $\boldsymbol{g}_l^{[p]}=(1-\alpha_{\boldsymbol{g}_l}\cdot V)\boldsymbol{g}_l$. Since $\boldsymbol{g}_l^{[p]}\cdot V(\boldsymbol{h}_1)$ is p-depleted (viz. its n-th Fourier coefficient is zero if $p|n$), it is killed by e_{ord}, hence

$$\begin{aligned}\big(w_N(\boldsymbol{f}_{l+1}),\boldsymbol{g}_l\cdot V(\boldsymbol{h}_1)\big)_{Np}&=\alpha_{\boldsymbol{g}_l}\cdot\big(w_N(\boldsymbol{f}_{l+1}),V(\boldsymbol{g}_l\cdot\boldsymbol{h}_1)\big)_{Np}\\&=\frac{\alpha_{\boldsymbol{g}_l}}{\bar{\chi}_{\boldsymbol{f}}(p)\alpha_{\boldsymbol{f}_{l+1}}}\cdot\big(w_N(\boldsymbol{f}_{l+1}),\boldsymbol{g}_l\cdot\boldsymbol{h}_1\big)_{Np}.\end{aligned}$$

(To justify the last equality, note that $e_{\mathrm{ord}}\circ V=U_p^{-1}\cdot e_{\mathrm{ord}}$ and U_p acts on $w_N(\boldsymbol{f}_{l+1})$ as $\bar{\chi}_{\boldsymbol{f}}(p)\cdot\alpha_{\boldsymbol{f}_{l+1}}$.) Then

$$\big(w_N(\boldsymbol{f}_{l+1}),e_{\mathrm{ord}}\big(\boldsymbol{g}_l\cdot\boldsymbol{h}_1^{[p]}\big)\big)_{Np}=\left(1-\frac{\alpha_{\boldsymbol{g}_l}\alpha_{\boldsymbol{h}_1}}{\bar{\chi}_{\boldsymbol{f}}(p)\alpha_{\boldsymbol{f}_{l+1}}}\right)\cdot\big(w_N(\boldsymbol{f}_{l+1}),\boldsymbol{g}_l\cdot\boldsymbol{h}_1\big)_{Np}.$$

Similarly the vanishing of $e_{\mathrm{ord}}(\boldsymbol{g}_l^{[p]}\cdot V(h_1))$ yields

$$\big(w_N(\boldsymbol{f}_{l+1}),\boldsymbol{g}_l\cdot\boldsymbol{h}_1\big)_{Np}=\left(1-\frac{\bar{\chi}_{\boldsymbol{g}}(p)\alpha_{\boldsymbol{g}_l}}{\alpha_{\boldsymbol{h}_1}\alpha_{\boldsymbol{f}_{l+1}}}\right)\cdot\big(w_N(\boldsymbol{f}_{l+1}),\boldsymbol{g}_l\cdot h_1\big)_{Np}.$$

Using once again the identity $e_{\mathrm{ord}}(\boldsymbol{g}_l^{[p]} \cdot V(\boldsymbol{h}_1)) = 0$ one deduces that $\boldsymbol{g}^{[p]} \cdot \boldsymbol{h}_1 - \boldsymbol{g}_l \cdot \boldsymbol{h}_1^{[p]}$ is killed by e_{ord}, hence the previous two equations give (cf. Equations (55) and (131))

$$\begin{aligned}\mathscr{L}_p^f(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})(w) &= \frac{(w_N(\boldsymbol{f}_{l+1}), e_{\mathrm{ord}}(\boldsymbol{g}_l^{[p]}\cdot \boldsymbol{h}_1))_{Np}}{(w_N(\boldsymbol{f}_{l+1}), w_N(\boldsymbol{f}_{l+1}))} \\ &= \left(1 - \frac{\alpha_{\boldsymbol{g}_l}\alpha_{\boldsymbol{h}_1}}{\bar{\chi}_{\boldsymbol{f}}(p)\alpha_{\boldsymbol{f}_{l+1}}}\right)\left(1 - \frac{\bar{\chi}_{\boldsymbol{g}}(p)\alpha_{\boldsymbol{g}_l}}{\alpha_{\boldsymbol{h}_1}\alpha_{\boldsymbol{f}_{l+1}}}\right)\cdot \frac{(w_N(\boldsymbol{f}_{l+1}), \boldsymbol{g}_l\cdot h_1)_{Np}}{(w_N(\boldsymbol{f}_{l+1}), w_N(\boldsymbol{f}_{l+1}))_{Np}} \\ &= \mathcal{E}_f^*(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})(w)\cdot \mathcal{E}_g(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})(w)\cdot \mathscr{L}_p^{f*}(\boldsymbol{f}_{l+1},\boldsymbol{g}_l,h_1)\end{aligned}$$

for each $l \geqslant 1$, where $w = (l+1, l, 1)$. (See Equations (1) and (197) for the definitions of $\mathcal{E}_g(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})$ and $\mathcal{E}_f^*(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})$ respectively.) This proves the first statement.

The second statement follows from the main result of [HK91] and Theorem 3 of [DN10]. (Note that $(w_N(\boldsymbol{f}_2), g_1\cdot h_1)_{Np} = 0$ for each level-N test vectors $(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})$ for $(\boldsymbol{f}^\sharp,\boldsymbol{g}^\sharp,\boldsymbol{h}^\sharp)$, cf. the discussion preceding the statement of [DN10, Theorem 3].) □

As in Section 9.2, for each $\mathscr{O}_{\boldsymbol{fgh}}$-module M denote by $M|_{\mathcal{H}_{\boldsymbol{fg}}} = M\otimes_{\mathrm{res}_{\boldsymbol{fg}}}\mathscr{O}_{\boldsymbol{g}}$ the base change of M along the morphism $\mathrm{res}_{\boldsymbol{fg}} : \mathscr{O}_{\boldsymbol{fgh}} \longrightarrow \mathscr{O}_{\boldsymbol{g}}$ sending $F(\boldsymbol{k},\boldsymbol{l},\boldsymbol{m})$ to $F(\boldsymbol{l}+1,\boldsymbol{l},1)$, and for each m in M denote by $m|_{\mathcal{H}_{\boldsymbol{fg}}}$ the natural image of m in the quotient $M|_{\mathcal{H}_{\boldsymbol{fg}}}$ of M. Finally, if ξ is equal to one of f, g and h, define

$$\mathscr{F}^\bullet V(\boldsymbol{fg},\boldsymbol{h}_1) = \mathscr{F}^\bullet V(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})|_{\mathcal{H}_{\boldsymbol{fg}}} \quad\text{and}\quad V(\boldsymbol{fg},\boldsymbol{h}_1)_\xi = V(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})_\xi|_{\mathcal{H}_{\boldsymbol{fg}}}.$$

Lemma 9.9. — *The map*

$$H^1(\mathbf{Q}_p, \mathscr{F}^2 V(\boldsymbol{fg},\boldsymbol{h}_1)) \longrightarrow H^1(\mathbf{Q}_p, V(\boldsymbol{fg},\boldsymbol{h}_1))$$

induced by the inclusion $\mathscr{F}^2 V(\boldsymbol{fg},\boldsymbol{h}_1) \hookrightarrow V(\boldsymbol{fg},\boldsymbol{h}_1)$ *is injective.*

Proof. — Set $M = V(\boldsymbol{fg},\boldsymbol{h}_1)$ and $M_\xi = V(\boldsymbol{fg},\boldsymbol{h}_1)_\xi$. The statement follows from the vanishing of $H^0(\mathbf{Q}_p, V(\boldsymbol{fg},\boldsymbol{h}_1)/\mathscr{F}^2)$, which in turn follows from the claim:

$$H^0(\mathbf{Q}_p, \mathrm{gr}^0 M) = H^0(\mathbf{Q}_p, \mathrm{gr}^1 M) = 0. \tag{220}$$

To prove the claim, recall from Section 7.2 that the inertia subgroup of $G_{\mathbf{Q}(\mu_p)}$ acts on $\mathrm{gr}^0 M = M/\mathscr{F}^1 M$ via the character $\kappa_{\mathrm{cyc}}^{1-\boldsymbol{l}}$, hence $H^0(\mathbf{Q}_p, \mathrm{gr}^0 M) = 0$. Moreover, denote by Φ_f, Φ_g and Φ_h the $\mathscr{O}_{\boldsymbol{g}}$-valued unramified characters of $G_{\mathbf{Q}_p}$ sending an arithmetic Frobenius to $\frac{\bar{\chi}_{\boldsymbol{f}}(p)\cdot a_p(\boldsymbol{l}+1)}{b_p(\boldsymbol{l})\cdot c_p(1)}$, $\frac{\bar{\chi}_{\boldsymbol{g}}(p)\cdot b_p(\boldsymbol{l})}{a_p(\boldsymbol{l}+1)\cdot c_p(1)}$ and $\frac{\bar{\chi}_{\boldsymbol{h}}(p)\cdot c_p(1)}{a_p(\boldsymbol{l}+1)\cdot b_p(\boldsymbol{l})}$ respectively. Then $G_{\mathbf{Q}_p(\mu_p)}$ acts on M_f, M_g and M_h via the characters Φ_f, $\Phi_g\cdot\kappa_{\mathrm{cyc}}^{\boldsymbol{l}}$ and $\Phi_h\cdot\kappa_{\mathrm{cyc}}$ respectively (cf. Section 7.2). According to the Ramanujan–Petersson conjecture the complex numbers $a_p(l+1)$ and $b_p(l)$ have absolute values $p^{l/2}$ and $p^{(l-1)/2}$ respectively for each classical point $l \geqslant 3$ in $U_{\boldsymbol{g}}$, hence $H^0(\mathbf{Q}_p, M_\xi(j)) = 0$ for $\xi = f, g, h$ and each integer j. Since $\mathrm{gr}^2 M$ is isomorphic to the direct sum of M_f, M_g and M_h, and since $\mathrm{gr}^1 M$ is isomorphic to the Kummer $\mathscr{O}_{\boldsymbol{g}}$-dual of $\mathrm{gr}^2 M$ (cf. Section 7.2), the claim follows. □

We can now conclude the proof of Theorem B in the exceptional case.

Recall the g-improved balanced class $\kappa_g^*(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})$ in $H^1_{\mathrm{bal}}(\mathbf{Q}, V(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})|_{\mathcal{H}_{\boldsymbol{g}}})$ constructed in Section 9.3. By the definition of the balanced condition (cf. Section 7.2), the restrictions at p of the classes $\kappa(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})$ and $\kappa_g^*(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})$ are the images of classes

$$\breve{\kappa}(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h}) \in H^1(\mathbf{Q}_p, \mathscr{F}^2V(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})) \quad \text{and} \quad \breve{\kappa}_g^*(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h}) \in H^1(\mathbf{Q}_p, \mathscr{F}^2V(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})|_{\mathcal{H}_g})$$

respectively. Denote by

$$\breve{\kappa}(\boldsymbol{fg},\boldsymbol{h}_1) = \breve{\kappa}(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})|_{\mathcal{H}_{\boldsymbol{fg}}} \quad \text{and} \quad \breve{\kappa}_g^*(\boldsymbol{fg},\boldsymbol{h}_1) = \breve{\kappa}_g^*(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})|_{\mathcal{H}_{\boldsymbol{fg}}}$$

their restrictions to the improving line $\mathcal{H}_{\boldsymbol{fg}}$, and set

$$\kappa(\boldsymbol{fg},\boldsymbol{h}_1)_f = p_{f*}(\breve{\kappa}(\boldsymbol{fg},\boldsymbol{h}_1)) \quad \text{and} \quad \kappa_g^*(\boldsymbol{fg},\boldsymbol{h}_1) = p_{f*}(\breve{\kappa}_g^*(\boldsymbol{fg},\boldsymbol{h}_1)),$$

where $p_f : \mathscr{F}^2V(\boldsymbol{fg},\boldsymbol{h}_1) \longrightarrow V(\boldsymbol{fg},\boldsymbol{h}_1)_f$ is the natural projection (cf. Section 7.2). According to Equation (218) and Lemma 9.9 one has

$$\kappa(\boldsymbol{fg},\boldsymbol{h}_1)_f = \mathcal{E}_g(\boldsymbol{fg},\boldsymbol{h}_1) \cdot \kappa_g^*(\boldsymbol{fg},\boldsymbol{h}_1)_f.$$

It then follows from Theorem A, Part 3 of Proposition 9.3 and Lemma 9.8 that

$$\mathscr{L}_p^{f*}(\boldsymbol{fg},h_1) = \mathscr{L}^*_{V(\boldsymbol{fg},\boldsymbol{h}_1)_f}(\kappa_g^*(\boldsymbol{fg},\boldsymbol{h}_1)).$$

Evaluating both sides of the previous equation at $\boldsymbol{l}=1$ and using once again Part 3 of Proposition 9.3 one gets the identity

$$\mathscr{L}_p^{f*}(\boldsymbol{f}_2,\boldsymbol{g}_1,h_1) = p \cdot a_p(2) \cdot \left\langle \exp_p^*(\kappa_g^*(\boldsymbol{f}_2,\boldsymbol{g}_1,\boldsymbol{h}_1)_f), \eta_{\boldsymbol{f}_2}\omega_{\boldsymbol{g}_1}\omega_{\boldsymbol{h}_1} \right\rangle_{\boldsymbol{f}_2\boldsymbol{g}_1\boldsymbol{h}_1} \tag{221}$$

where $\kappa_g^*(\boldsymbol{f}_2,\boldsymbol{g}_1,\boldsymbol{h}_1)_f$ is the weight-1 specialisation of $\kappa_g^*(\boldsymbol{fg},\boldsymbol{h}_1)_f$:

$$\kappa_g^*(\boldsymbol{f}_2,\boldsymbol{g}_1,\boldsymbol{h}_1)_f = \rho_{1*}(\kappa_g^*(\boldsymbol{fg},\boldsymbol{h}_1)_f) \in H^1(\mathbf{Q}_p, V(\boldsymbol{f}_2)^-_{\beta\beta}).$$

Similarly as in Section 9.1, we claim that the following statements are equivalent.

(a) The complex central value $L(f_2^\sharp \otimes g_1^\sharp \otimes h_1^\sharp, 1)$ vanishes.
(b) $\mathscr{L}_p^{f*}(\boldsymbol{f}_2,\boldsymbol{g}_1,h_1) = 0$ for all level-N test vectors $(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})$ for $(\boldsymbol{f}^\sharp,\boldsymbol{g}^\sharp,\boldsymbol{h}^\sharp)$.
(c) $\exp_p^*(\kappa_g^*(\boldsymbol{f}_2,\boldsymbol{g}_1,\boldsymbol{h}_1)_f) = 0$.
(d) $\exp_p^*(\mathrm{res}_p(\kappa_g^*(\boldsymbol{f}_2,\boldsymbol{g}_1,\boldsymbol{h}_1))) = 0$.
(e) $\kappa_g^*(\boldsymbol{f}_2,\boldsymbol{g}_1,\boldsymbol{h}_1)$ is crystalline at p.

(As usual, here $\kappa_g^*(\boldsymbol{f}_2,\boldsymbol{g}_1,\boldsymbol{h}_1)$ in $H^1(\mathbf{Q}_p, V(\boldsymbol{f}_2,\boldsymbol{g}_1,\boldsymbol{h}_1))$ denotes the specialisation of $\kappa_g^*(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})$ at w_o.) The equivalence between (a) and (b) is proved in Lemma 9.8.

As $(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})$ varies through the level-N test vectors for $(\boldsymbol{f}^\sharp,\boldsymbol{g}^\sharp,\boldsymbol{h}^\sharp)$, the differentials $\eta_{\boldsymbol{f}_2}\omega_{\boldsymbol{g}_1}\omega_{\boldsymbol{h}_1}$ generate the L-module $V^*(\boldsymbol{f}_2)^+_{\beta\beta} = D_{\mathrm{dR}}(V^*(\boldsymbol{f}_2)^+_{\beta\beta})$ (cf. Section 9.2). Equation (221) then proves that (b) and (c) are equivalent to each other. (Recall that $\kappa(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})$, hence $\kappa_g^*(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})$, is independent of the choice of the level-N test vectors $(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})$ for $(\boldsymbol{f}^\sharp,\boldsymbol{g}^\sharp,\boldsymbol{h}^\sharp)$, cf. Remark 1.3(3).)

The equivalence between (c) and (d) follows, as in Section 9.1, from the balancedness of the improved diagonal class. More precisely, the projection

$$p^- : V(\boldsymbol{f}_2,\boldsymbol{g}_1,\boldsymbol{h}_1) \longrightarrow V(\boldsymbol{f}_2,\boldsymbol{g}_1,\boldsymbol{h}_1)^-$$

induces an isomorphism between $\mathrm{Fil}^0V_{\mathrm{dR}}(\boldsymbol{f}_2,\boldsymbol{g}_1,\boldsymbol{h}_1)$ and $D_{\mathrm{dR}}(V(\boldsymbol{f}_2,\boldsymbol{g}_1,\boldsymbol{h}_1)^-)$, hence ($d$) is equivalent to the vanishing of the dual exponential of $p_*^-(\mathrm{res}_p(\kappa(\boldsymbol{f}_2,\boldsymbol{g}_1,\boldsymbol{h}_1)))$. In addition, since $V(\boldsymbol{f}_2)^-_{\beta\beta} = V(\boldsymbol{f}_2,\boldsymbol{g}_1,\boldsymbol{h}_1)_f$ is a $G_{\mathbf{Q}_p}$-direct summand of $V(\boldsymbol{f}_2,\boldsymbol{g}_1,\boldsymbol{h}_1)^-$ (cf. Section 9.2), and since $\kappa_g^*(\boldsymbol{f}_2,\boldsymbol{g}_1,\boldsymbol{h}_1)$ is balanced at p, the diagram (193) yields

$$p_*^-\big(\mathrm{res}_p\big(\kappa_g^*(\boldsymbol{f}_2,\boldsymbol{g}_1,\boldsymbol{h}_1)\big)\big) = \kappa_g^*(\boldsymbol{f}_2,\boldsymbol{g}_1,\boldsymbol{h}_1)_f,$$

thus proving the equivalence between (c) and (d).

Finally, the equivalence between (d) and (e) follows from Lemma 9.1. This concludes the proof of Theorem B in the exceptional case.

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Massimo Bertolini, Marco Adamo Seveso, and Rodolfo Venerucci, Massimo Bertolini: Essen, Germany • *E-mail :* massimo.bertolini@uni-due.de • Marco Seveso: Milano, Italy • *E-mail :* seveso.marco@gmail.com • Rodolfo Venerucci: Essen, Germany *E-mail :* rodolfo.venerucci@uni-due.de

BALANCED DIAGONAL CLASSES AND RATIONAL POINTS ON ELLIPTIC CURVES

by

Massimo Bertolini, Marco Adamo Seveso, and Rodolfo Venerucci

Abstract. — Let A be an elliptic curve over the rationals with multiplicative reduction at a prime p, and let K be a quadratic field in which p is inert. Under a generalised Heegner assumption, our previous contribution [BSV20] to this volume attaches to (A, p, K) balanced diagonal classes in the Selmer groups of the p-adic Tate module of A over certain ring class fields of K. These classes are obtained as p-adic limits of geometric classes in the cohomology of higher-dimensional Kuga–Sato varieties. The main result of this paper relates these diagonal classes to p-adic logarithms of Heegner or Stark–Heegner points, depending on whether K is complex or real respectively.

To Bernadette Perrin-Riou on her 65th birthday

Contents

1. Description and statement of results

Let $(\boldsymbol{f}, \boldsymbol{g}_\alpha, \boldsymbol{h}_\alpha)$ be a triple of p-adic Hida families of common tame level N. Assume that f interpolates the weight 2 cusp form attached to an elliptic curve $A/\mathbf{Q}$ with multiplicative reduction at p, and that $\boldsymbol{g}_\alpha$ and $\boldsymbol{h}_\alpha$ respectively specialise in weight 1 to (p-stabilised) theta-series g_α and h_α associated to the same quadratic extension $K/\mathbf{Q}$, having good reduction at p and inverse characters. Let $\kappa(\boldsymbol{f}, \boldsymbol{g}_\alpha, \boldsymbol{h}_\alpha)$ be the diagonal class constructed in our previous contribution [BSV20] to this volume. This article builds on the main results of loc. cit. to relate (a component of) the Bloch–Kato

2000 ***Mathematics Subject Classification.*** — 11G05 (11G40).

logarithm of the specialisation at $(2,1,1)$ of $\kappa(\boldsymbol{f},\boldsymbol{g}_\alpha,\boldsymbol{h}_\alpha)$ to the product of the formal group logarithms of two Heegner points, respectively Stark–Heegner points when K is imaginary, respectively real. See Theorem A below for the precise statement, holding under Assumption 1.1.

Our strategy goes along the following lines. Let $\mathscr{L}_p^f(\boldsymbol{f},g_\alpha,h_\alpha)$ denote the restriction to the line $(\boldsymbol{k},1,1)$ of the triple product p-adic L-function $\mathscr{L}_p^f(\boldsymbol{f},\boldsymbol{g}_\alpha,\boldsymbol{h}_\alpha)$ defined in loc. cit.. Section 3 shows that $\mathscr{L}_p^f(\boldsymbol{f},g_\alpha,h_\alpha)^2$ factors as a product of two Hida-Rankin p-adic L-functions attached to A/K. A suitable extension of main result of [BD07], resp. [BD09] for K imaginary quadratic, resp. real quadratic shows that the second derivative at $\boldsymbol{k}=2$ of the above mentioned Hida–Rankin p-adic L-functions is equal to the square of the formal group logarithm of a Heegner point, resp. Stark–Heegner point. Theorem A of [BSV20] describes $\mathscr{L}_p^f(\boldsymbol{f},g_\alpha,h_\alpha)$ as the image by a branch of the Perrin–Riou logarithm of the restriction of $\kappa(\boldsymbol{f},\boldsymbol{g}_\alpha,\boldsymbol{h}_\alpha)$ to the line $(\boldsymbol{k},1,1)$. Theorem A of this paper then follows from Proposition 2.2, which extends results of [Ven16] to obtain a formula for the second derivative of the Perrin-Riou logarithm of the above class at $\boldsymbol{k}=2$.

More precisely, let $A/\mathbf{Q}$ be an elliptic curve of conductor $N_f p$, having multiplicative reduction at a prime $p>3$ (hence $p\nmid N_f$). Let $K/\mathbf{Q}$ be a quadratic extension of discriminant d_K coprime with $N_f p$ and quadratic character $\varepsilon_K : (\mathbf{Z}/d_K\mathbf{Z})^* \to \mu_2$. Let

$$f=\sum_{n\geqslant 1} a_n(A)\cdot q^n \in S_2(N_f p,\mathbf{Z})^{\mathrm{new}}$$

be the weight-two newform associated with A by the modularity theorem of Wiles, Taylor–Wiles et al., and let

$$\nu_g : G_K \longrightarrow \bar{\mathbf{Q}}^* \quad \text{and} \quad \nu_h : G_K \longrightarrow \bar{\mathbf{Q}}^*$$

be two ray class characters of K. Write $N_f = N_f^+\cdot N_f^-$, where N_f^- is the product of the prime divisors of N_f which are inert in $K/\mathbf{Q}$. We make the following

Assumption 1.1. —

1. *(Heegner hypothesis) p is* inert *in $K/\mathbf{Q}$, N_f^- is square-free and $\varepsilon_K(-N_f^-)=+1$.*
2. *(Modularity) When $K/\mathbf{Q}$ is real, both ν_g and ν_h have mixed signature.*
3. *(Cuspidality) The characters ν_g and ν_h are not induced by Dirichlet characters.*
4. *(Self-duality) The central characters of ν_g and ν_h are inverse to each other.*
5. *(Local signs) The conductors of ν_g and ν_h are coprime to $p\cdot d_K\cdot N_f$.*
6. *(Residual irreducibility) The $\mathbf{F}_p[G_\mathbf{Q}]$-module $A_p(\bar{\mathbf{Q}})$ of p-torsion points of A is irreducible.*

Let ν_ξ denote either ν_g or ν_h and let $L/\mathbf{Q}_p$ be a finite extension containing the Fourier coefficients of f and the values of ν_ξ. In light of Assumption 1.1, the two-dimensional L-representation $\mathrm{Ind}_\mathbf{Q}^K(\nu_\xi)$ of $G_\mathbf{Q}$ induced by $\nu_\xi : G_K \longrightarrow L^*$ is *odd* and *irreducible*. Thanks to the work of Hecke [Miy06, Section 4.8], it arises from the cuspidal weight-one theta series

$$\xi=\sum_{\mathfrak{a}} \nu_\xi(\mathfrak{a})\cdot q^{\mathbf{N}\mathfrak{a}} \in S_1(N_\xi,\chi_\xi).$$

Here the sum runs over the ideals $\mathfrak{a}$ of $\mathcal{O}_K$ which are coprime to the conductor $\mathfrak{f}_\xi$ of ν_ξ, $\mathbf{N}\mathfrak{a}$ denotes the norm of $\mathfrak{a}$, $N_\xi = d_K \cdot \mathbf{N}\mathfrak{f}_\xi$ and $\chi_\xi = \varepsilon_K \cdot \nu_\xi^{\mathrm{cen}}$, where $\nu_\xi^{\mathrm{cen}} : G_{\mathbf{Q}} \longrightarrow \bar{\mathbf{Q}}^*$ is the central character of ν_ξ. The form ξ is primitive of conductor N_ξ and the dual of its Deligne–Serre L-representation is isomorphic to $\mathrm{Ind}_{\mathbf{Q}}^K(\nu_\xi)$.

Since p is inert in $K/\mathbf{Q}$, one has $a_p(\xi) = 0$ so that the p-th Hecke polynomial of ξ is equal to

$$X^2 + \chi_\xi(p).$$

Let $\alpha_\xi \in \mathscr{O}^*$ be a fixed square root of $-\chi_\xi(p)$, and write

$$\xi_\alpha = \xi(q) - \beta_\xi \cdot \xi(q^p) \in S_1(N_\xi p, \chi_\xi), \qquad \text{with } \beta_\xi = \frac{\chi_\xi(p)}{\alpha_\xi} = -\alpha_\xi \tag{1}$$

for the corresponding p-stabilisation. (Here we assume that L contains α_ξ.) Since $\chi_g \cdot \chi_h$ is the trivial character, without loss of generality we may assume that the roots $\alpha_g, \beta_g, \alpha_h, \beta_h$ are ordered in such a way that

$$\alpha_g \cdot \alpha_h = \beta_g \cdot \beta_h = a_p(A) = \pm 1. \tag{2}$$

As explained in Section 5 of our contribution [BSV20], the work of Hida and Wiles implies the existence of a unique triple $(\boldsymbol{f}^\sharp, \boldsymbol{g}_\alpha^\sharp, \boldsymbol{h}_\alpha^\sharp)$ of L-rational primitive Hida families of tame conductors (N_f, N_g, N_h) and tame characters (χ_f, χ_g, χ_h) which specialises to the triple (f, g_α, h_α) at w_o. Note that the triple $(\boldsymbol{f}^\sharp, \boldsymbol{g}^\sharp, \boldsymbol{h}^\sharp)$ satisfies Assumptions 1.1 and 1.2 stated in Section 1 of [BSV20] (cf. Equation (1) and Assumption 1.1.3), and that $w_o = (2, 1, 1)$ is *exceptional* in the sense of Section 1.2 of loc. cit. (cf. Equation (2)).

With notations as in Section 1.1 of loc. cit., denote by N the least common multiple of N_f, N_g and N_h, by $V(\boldsymbol{f}, \boldsymbol{g}_\alpha, \boldsymbol{h}_\alpha)$ the big Galois representation attached to any choice of level-N test vector for $(\boldsymbol{f}^\sharp, \boldsymbol{g}_\alpha^\sharp, \boldsymbol{h}_\alpha^\sharp)$ (cf. Remark 1.3(3) of loc. cit.), and by

$$\kappa(\boldsymbol{f}, \boldsymbol{g}_\alpha, \boldsymbol{h}_\alpha) \in H^1_{\mathrm{bal}}(\mathbf{Q}, V(\boldsymbol{f}, \boldsymbol{g}_\alpha, \boldsymbol{h}_\alpha))$$

the corresponding diagonal class. In [Hsi20] Hsieh constructs a distinguished level-N test vector $(\boldsymbol{f}, \boldsymbol{g}_\alpha, \boldsymbol{h}_\alpha)$ (denoted $(\boldsymbol{f}^\star, \boldsymbol{g}_\alpha^\star, \boldsymbol{h}_\alpha^\star)$ in [BSV20, Section 6.1]) for $(\boldsymbol{f}^\sharp, \boldsymbol{g}_\alpha^\sharp, \boldsymbol{h}_\alpha^\sharp)$, and computes explicitly the local constants which appear in the interpolation formulae satisfied by the p-adic L-function $\mathscr{L}_p^f(\boldsymbol{f}, \boldsymbol{g}_\alpha, \boldsymbol{h}_\alpha)$ (cf. Sections 1.1 and 6.1 of loc. cit.).

Let $V_p(A) = \mathrm{Ta}_p(A) \otimes_{\mathbf{Z}} \mathbf{Q}$ be the p-adic Tate module of A with $\mathbf{Q}_p$-coefficients, let $Y_1(N_f p)$ be the open modular curve over $\mathbf{Q}$ of level $\Gamma_1(N_f p)$, and let $V(f)$ be the f-isotypic quotient of $H^1_{\text{ét}}(Y_1(N_f p)_{\bar{\mathbf{Q}}}, \mathbf{Q}_p(1))$ (cf. Sections 2.1 and 2.4 of [BSV20]). Fix a modular parametrisation

$$\wp_\infty : Y_1(N_f p) \longrightarrow A.$$

This induces an isomorphism of $G_{\mathbf{Q}}$-modules

$$\wp_{\infty *} : V(f) \cong V_p(A) \tag{3}$$

which we often consider as an equality in what follows. Set

$$V(f, g, h) = V_p(A) \otimes_{\mathbf{Q}_p} V(g) \otimes_L V(h),$$

where $V(\xi) = V(\xi_\alpha)$ is the canonical model of the dual of the Deligne–Serre representation of $\xi = g, h$ arising from the specialisation of $V(\boldsymbol{\xi}_\alpha)$ at weight one (cf. Section 5 of [**BSV20**]). The fixed test vector $(\boldsymbol{f}, \boldsymbol{g}_\alpha, \boldsymbol{h}_\alpha)$ and modular parametrisation $\wp_\infty$ determine a projection $V(\boldsymbol{f}_2, \boldsymbol{g}_{\alpha 1}, \boldsymbol{h}_{\alpha 1}) \twoheadrightarrow V(f,g,h)$ (denoted $\varpi_\star$ in Section 2 below), mapping the specialisation at w_o of $\kappa(\boldsymbol{f}, \boldsymbol{g}_\alpha, \boldsymbol{h}_\alpha)$ to a global class

$$\kappa_{\alpha\alpha}(f,g,h) \in H^1(\mathbf{Q}, V(f,g,h)).$$

Let c be the non-trivial element of $\mathrm{Gal}(K/\mathbf{Q})$ and let $\nu_\xi^c : G_K \to L^*$ be the conjugate of ν_ξ by c. By Assumption 1.1(4) the characters

$$\varphi = \nu_g \cdot \nu_h \quad \text{and} \quad \psi = \nu_g \cdot \nu_h^c$$

are *ring class characters* of K (i.e., $\varphi^c = \varphi^{-1}$ and $\psi^c = \psi^{-1}$). Note the factorisation of $G_{\mathbf{Q}}$-representations

$$V(f,g,h) \cong V_p(A) \otimes \mathrm{Ind}_{\mathbf{Q}}^K(\varphi) \oplus V_p(A) \otimes \mathrm{Ind}_{\mathbf{Q}}^K(\psi). \tag{4}$$

In particular the Bloch–Kato Selmer group $\mathrm{Sel}(\mathbf{Q}, V(f,g,h))$ decomposes as

$$\mathrm{Sel}(\mathbf{Q}, V(f,g,h)) \cong \mathrm{Sel}(K_\varphi, V_p(A))^\varphi \oplus \mathrm{Sel}(K_\psi, V_p(A))^\psi, \tag{5}$$

where $K_\cdot/K$ denotes the ring class field having the same conductor as $\cdot$ and $\mathrm{Sel}(K_\cdot, V_p(A))^\cdot$ is the submodule of the Selmer group $\mathrm{Sel}(K_\cdot, V_p(A)) \otimes_{\mathbf{Q}_p} L$ of $V_p(A) \otimes_{\mathbf{Q}_p} L$ over $K_\cdot$ on which $\mathrm{Gal}(K_\cdot/K)$ acts via the inverse of $\cdot$.

It follows from Equation (4) and the Artin formalism that the Garrett triple product L-function $L(f \otimes g \otimes h, s) = L(V(f,g,h), s)$ factors as the product of the Rankin L-functions $L(A/K, \varphi, s)$ and $L(A/K, \psi, s)$, which have both sign -1 in their functional equation by Assumption 1.1.1. In particular $L(f \otimes g \otimes h, s)$ vanishes to order at least two at $s = 1$. Theorem B of [**BSV20**] in the exceptional case then proves that the diagonal class $\kappa_{\alpha\alpha}(f,g,h)$ is crystalline at p, hence belongs to the Bloch–Kato Selmer group $\mathrm{Sel}(\mathbf{Q}, V(f,g,h))$ of the representation $V(f,g,h)$ of $G_{\mathbf{Q}}$:

$$\kappa_{\alpha\alpha}(f,g,h) \in \mathrm{Sel}(\mathbf{Q}, V(f,g,h)).$$

Write ϱ for either φ or ψ. The articles [**BD07**] and [**BD09**] (see also [**GSS16**]) associate to $\boldsymbol{f}$ and ϱ a p-adic L-function

$$L_p(\boldsymbol{f}/K, \varrho) \in \mathscr{O}_{\boldsymbol{f}},$$

interpolating the central values of the L-series $L(f_k/K, \varrho, s)$ of the base change of f_k to K twisted by ϱ. Their definition, which depends only on the primitive family $\boldsymbol{f}^\sharp$, is recalled in Section 3.2 below.

Write K_p for the completion of K at the inert prime p. Noting that p splits completely in K_ϱ/K, let Frob_p in $\mathrm{Gal}(K_\varrho/\mathbf{Q})$ be the Frobenius element determined by the fixed embedding of $\bar{\mathbf{Q}}$ into $\bar{\mathbf{Q}}_p$, mapping K_ϱ to K_p. Denote by

$$\log_{\omega_f} : A(K_p)_L = A(K_p) \hat{\otimes} L \longrightarrow K_p \otimes_{\mathbf{Q}_p} L$$

the L-linear extension of the composition

$$A(K_p) \hat{\otimes} \mathbf{Q}_p \cong H^1_{\mathrm{fin}}(K_p, V(f)) \xrightarrow{\log_p} \tan_{K_p}(f) \cong K_p,$$

where H^1_{fin} is the finite subspace of H^1, $\tan_{K_p}(f)$ is the tangent space of the de Rham module $H^0(K_p, V(f) \otimes_{\mathbf{Q}_p} B_{\mathrm{dR}})$, the first isomorphism arises from the map $\wp_{\infty *}$ and Kummer theory, $\log_p$ is the Bloch–Kato logarithm and the second isomorphism is evaluation at the canonical differential ω_f in the dual of $\tan_{K_p}(f)$ associated with f (see Section 2.5 of [BSV20], in particular Equations (29), (30) and (32)). Under our running assumptions, the p-adic L-function $L_p(\boldsymbol{f}/K,\varrho)$ vanishes at $\boldsymbol{k}=2$ to order at least two. An extension of the main results of [BD07] and [BD09] in the imaginary quadratic and real quadratic setting respectively – see in particular [GSS16, LMH20, LV14, Mok11] – prove the existence of a non-zero algebraic constant $\mathcal{Q}\in\bar{\mathbf{Q}}^*$ such that

$$(6)\qquad c_f^2\cdot\frac{d^2}{d\boldsymbol{k}^2}L_p(\boldsymbol{f}/K,\varrho)_{\boldsymbol{k}=2}=\mathcal{Q}\cdot\log^2_{\omega_f}(P_\varrho^\varepsilon),$$

where $c_f=c_f(\wp_\infty)\in K_p^*$ is an explicit non-zero p-adic constant (depending on $\wp_\infty$) introduced in Section 2.2 below (see also Remark 1.2), and the point P_ϱ^ε in $A(K_p)_L$ are defined as follows.

If K is imaginary quadratic, choose a primitive Heegner point P in $A(K_\varrho)$ and let

$$P_\varrho=\sum_{\sigma\in\mathrm{Gal}(K_\varrho/K)}\varrho(\sigma)^{-1}\cdot P^\sigma\quad\text{and}\quad P_\varrho^\varepsilon=P_\varrho+\varepsilon\cdot P_\varrho^{\mathrm{Frob}_p}\quad\text{for }\varepsilon=a_p(A).$$

Note that the *global* point P_ϱ^ε is viewed in Equation (6) as a local point via our fixed embedding of $\bar{\mathbf{Q}}$ into $\bar{\mathbf{Q}}_p$. When ϱ is quadratic one checks that Frob_p acts on P_ϱ via a sign ε_ϱ (see for example the discussion in Section 4 of [BD07]).

If K is real quadratic, the *local point* P_ϱ in $A(K_p)$ is defined as in the above formula, by exploiting the action of $\mathrm{Pic}(\mathcal{O}_\varrho)$ on a *Stark–Heegner point* $P\in A(K_p)$ attached to K_ϱ, where $\mathrm{Pic}(\mathcal{O}_\varrho)\cong\mathrm{Gal}(K_\varrho/K)$ is the Picard group of the order $\mathcal{O}_\varrho$ of K corresponding to K_ϱ via class field theory.

Remark 1.2. — The main results of [BD07, BD09] are stated in terms of the logarithm

$$\log_A=\log_{q_A}\circ\varphi_{\mathrm{Tate}}^{-1}:A(K_p)\longrightarrow K_p,$$

where q_A is the Tate period of $A_{\mathbf{Q}_p}$, $\varphi_{\mathrm{Tate}}:K_p^*/q_A^{\mathbf{Z}}\cong A(K_p)$ is the Tate parametrisation and $\log_{q_A}:K_p^*\longrightarrow K_p$ is the branch of the p-adic logarithm which vanishes at q_A (see Section 2.2 below for more details). The p-adic constant $c_f\in K_p^*$ (defined in Equation (14) below) accounts for the discrepancy between $\log_A$ and the logarithm $\log_{\omega_f}$ introduced above (cf. Lemma 2.1 below). The nontrivial element of $\mathrm{Gal}(K_p/\mathbf{Q}_p)$ acts on c_f as multiplication by $\varepsilon=a_p(A)$, hence c_f^2 belongs to $\mathbf{Q}_p^*$. Similarly $\log^2_{\omega_f}(P_\varrho^\varepsilon)$ belongs to L, so that the identity (6) takes place in L.

Denote by

$$\mathscr{L}_p^f(\boldsymbol{f},g_\alpha,h_\alpha)\in\mathscr{O}_{\boldsymbol{f}}$$

the restriction of $\mathscr{L}_p^f(\boldsymbol{f},\boldsymbol{g}_\alpha,\boldsymbol{h}_\alpha)$ to the line $(\boldsymbol{k},1,1)$. Theorem 3.1 below shows the factorisation formula

$$(7)\qquad \mathscr{L}_p^f(\boldsymbol{f},g_\alpha,h_\alpha)^2=\mathscr{A}\cdot L_p(\boldsymbol{f}/K,\varphi)\cdot L_p(\boldsymbol{f}/K,\psi),$$

where $\mathscr{A}$ is a bounded analytic function on $U_{\boldsymbol{f}}$ such that $\mathscr{A}(2)$ is an element of $\bar{\mathbf{Q}}^*$.

Under the assumptions of this section, Proposition 2.2 gives a formula for the second derivative of the Perrin-Riou big logarithm of a balanced class along the line

$(\boldsymbol{k}, 1, 1)$ at the point $\boldsymbol{k} = 2$. Combined with [**BSV20**, Theorem A], this gives the equality

$$c_f^2 \cdot \frac{d^2}{d\boldsymbol{k}^2} \mathscr{L}_p^f(\boldsymbol{f}, g_\alpha, h_\alpha)_{\boldsymbol{k}=2} = \mathcal{Q} \cdot \log_{\beta\beta}\big(\mathrm{res}_p(\kappa_{\alpha\alpha}(f,g,h))\big), \tag{8}$$

where $\mathcal{Q}$ is an explicit constant in $\mathbf{Q}^*$ and $\log_{\beta\beta}\big(\mathrm{res}_p(\kappa_{\alpha\alpha}(f,g,h))\big)$ is the evaluation of the p-adic Bloch–Kato logarithm of $\mathrm{res}_p(\kappa_{\alpha\alpha}(f,g,h))$ at a canonical differential $\omega_f \otimes \omega_{g_\alpha} \otimes \omega_{h_\alpha}$ (see Section 2 for details).

Combining Equations (6), (7) and (8) yields

Theorem A. — *For $\mathcal{Q}$ in $\bar{\mathbf{Q}}^*$ one has the equality*

$$\log_{\beta\beta}\big(\mathrm{res}_p\big(\kappa_{\alpha\alpha}(f,g,h)\big)\big) = \mathcal{Q} \cdot \log_{\omega_f}(P_\varphi^\varepsilon) \cdot \log_{\omega_f}(P_\psi^\varepsilon).$$

Recall that the complex L-function $L(f \otimes g \otimes h, s)$ attached to $V(f,g,h)$ vanishes to order at least 2 at $s = 1$ by Assumption 1.1.

Corollary B. — *Let K be imaginary quadratic. If $\varrho = \varphi$ or ψ is quadratic, assume that $\varepsilon = \varepsilon_\varrho$. Then*

$$\frac{d^2}{ds^2} L(f \otimes g \otimes h, s)_{s=1} \neq 0 \qquad \Longleftrightarrow \qquad \log_{\beta\beta}\big(\mathrm{res}_p\big(\kappa_{\alpha\alpha}(f,g,h)\big)\big) \neq 0.$$

Proof. — Under the current assumptions P_ϱ^ε is non-zero whenever P_ϱ is non-zero. Corollary B then follows from Theorem A combined with S.-W. Zhang's proof of the Gross–Zagier formula for Shimura curves [**Zha01**]. □

Remark C. — Theorem A and a suitable converse to the Gross–Zagier–Kolyvagin theorem show that the equivalent statements of Corollary B are also equivalent to the equality

$$\mathrm{Sel}(\mathbf{Q}, V(f,g,h)) = L \cdot \kappa_{\alpha\alpha}(f,g,h) \oplus L \cdot \kappa_{\beta\beta}(f,g,h), \tag{9}$$

that is the Selmer group $\mathrm{Sel}(\mathbf{Q}, V(f,g,h))$ is generated by the global class $\kappa_{\alpha\alpha}(f,g,h)$ and its counterpart $\kappa_{\beta\beta}(f,g,h)$ defined by replacing the pair $(\boldsymbol{g}_\alpha, \boldsymbol{h}_\alpha)$ with $(\boldsymbol{g}_\beta, \boldsymbol{h}_\beta)$ (cf. Equation (2)).

To show that the equality (9) follows from the non-vanishing of the second derivative of $L(f \otimes g \otimes h, s)$, one notes that this condition implies that $\mathrm{Sel}(\mathbf{Q}, V(f,g,h))$ is two-dimensional by the Gross–Zagier–Kolyvagin theorem. The classes $\kappa_{\alpha\alpha}(f,g,h)$ and $\kappa_{\beta\beta}(f,g,h)$ are both non-trivial by Corollary B, hence one is reduced to prove that they are linearly independent. This follows again from Corollary B, noting that

$$\log_{\beta\beta}\big(\mathrm{res}_p(\kappa_{\beta\beta}(f,g,h))\big) = 0$$

since the Selmer class $\kappa_{\beta\beta}(f,g,h)$ arises from the *balanced* class $\kappa(\boldsymbol{f}, \boldsymbol{g}_\beta, \boldsymbol{h}_\beta)$.

Conversely, assume that the classes $\kappa_{\alpha\alpha}(f,g,h)$ and $\kappa_{\beta\beta}(f,g,h)$ generate the Selmer group $\mathrm{Sel}(\mathbf{Q}, V(f,g,h))$, so that

$$\dim_L \mathrm{Sel}(\mathbf{Q}, V(f,g,h)) \leqslant 2. \tag{10}$$

Granting a *converse of the Gross–Zagier–Kolyvagin theorem* of the form

$$\dim_L \mathrm{Sel}(K_\varrho, V_p(A))^\varrho \leqslant 1 \quad \Longrightarrow \quad \mathrm{ord}_{s=1} L(f/K, \varrho, s) = \dim_L \mathrm{Sel}(K_\varrho, V_p(A))^\varrho \tag{11}$$

for ϱ equal to φ and ψ as above, one concludes readily as follows. Since the sign of the functional equation of $L(f/K, \varrho, s)$ is -1, Equations (10) and (11) imply that $L(f/K, \varrho, s)$ has a simple zero at $s = 1$ for $\varrho = \varphi$ and ψ, hence $L(f \otimes g \otimes h, s)$ has a double zero at $s = 1$. The above converse theorem may be approached by an extension of the methods of the forthcoming work [**BLV17**], which prove Birch and Swinnerton-Dyer formulae for general families of anticyclotomic characters of p-power conductor and are suited to extend such formulae to arbitrary ring class characters.

In the real quadratic setting, the next result relates the (local) Stark–Heegner points to the (global) Selmer group $\mathrm{Sel}(\mathbf{Q}, V(f, g, h))$.

Corollary D. — *Assume that K is real quadratic. If the Stark–Heegner points $P_{\varphi}^{\varepsilon}$ and P_{ψ}^{ε} are both non-trivial, then* $\dim_L \mathrm{Sel}(\mathbf{Q}, V(f, g, h)) \geqslant 2$.

Proof. — Theorem A implies that $\kappa_{\alpha\alpha}(f, g, h)$ and $\kappa_{\beta\beta}(f, g, h)$ are non-zero. The same argument as in Remark C shows that these classes are linearly independent. □

Remark E. — Under the assumptions of Corollary D, the definition of $\kappa_{\alpha\alpha}(f, g, h)$ and $\kappa_{\beta\beta}(f, g, h)$ combined with Theorem A imply that the Stark–Heegner point $P_{\varrho}^{\varepsilon}$ ($\varrho = \varphi, \psi$) arises as the restriction at p of a Selmer class in $\mathrm{Sel}(K_{\varrho}, V_p(A))^{\varrho}$. We refer the reader to the contribution [**DR20**] by Darmon–Rotger to this volume for an extensive discussion of this application (see in particular Theorem A of loc. cit.).

2. Derivatives of big logarithms II

This section should be regarded as a continuation of [**BSV20**, Section 6], where a study of multivariable Perrin-Riou logarithms is undertaken. After the preliminary Sections 2.1 and 2.2, Proposition 2.2 in Section 2.3 establishes a formula for the second derivative of the Perrin-Riou big logarithm of a balanced class along the line $(\boldsymbol{k}, 1, 1)$ at the point $\boldsymbol{k} = 2$, which constitutes a crucial ingredient in the proof of Theorem A.

Let (f, g, h) and $(\boldsymbol{f}^{\sharp}, \boldsymbol{g}_{\alpha}^{\sharp}, \boldsymbol{h}_{\alpha}^{\sharp})$ be as in Section 1. Denote by $(\boldsymbol{f}, \boldsymbol{g}_{\alpha}, \boldsymbol{h}_{\alpha})$, or more simply $(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})$, any level-$N$ test vector for $(\boldsymbol{f}^{\sharp}, \boldsymbol{g}_{\alpha}^{\sharp}, \boldsymbol{h}_{\alpha}^{\sharp})$ (where N is as in Section 1). Throughout this section Assumption 1.1 is in force. In particular Assumption 6.3 of loc. cit. is satisfied (as $A_p(\bar{\mathbf{Q}})$ is p-distinguished by Tate's theory, since $p \geqslant 5$, cf. Section 2.2 below), hence one can consider the distinguished level-N test vector $(\boldsymbol{f}^{\star}, \boldsymbol{g}_{\alpha}^{\star}, \boldsymbol{h}_{\alpha}^{\star})$ introduced in Section 6.1 of loc. cit.. (To ease notations, the latter was simply denoted $(\boldsymbol{f}, \boldsymbol{g}_{\alpha}, \boldsymbol{h}_{\alpha})$ in Section 1).

2.1. The projection $\varpi_{\boldsymbol{fgh}}$ and the class $\kappa_{\alpha\alpha}(f, g, h)$. — Associated with the choice of a test vector $(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h}) = (\boldsymbol{f}, \boldsymbol{g}_{\alpha}, \boldsymbol{h}_{\alpha})$ we define a $G_{\mathbf{Q}}$-equivariant projection

$$\varpi_{\boldsymbol{fgh}} : V(\boldsymbol{f}_2, \boldsymbol{g}_1, \boldsymbol{h}_1) \longrightarrow V(f, g_{\alpha}, h_{\alpha}) \tag{12}$$

by the following recipe. Let $\boldsymbol{\xi}$ denote one of $\boldsymbol{f}, \boldsymbol{g}_{\alpha}$ or $\boldsymbol{h}_{\alpha}$. For each positive integer d dividing N/N_{ξ} denote by

$$v_d : Y_1(N, p) \longrightarrow Y_1(N_{\xi}, p)$$

the degeneracy map corresponding to multiplication by d on $\mathbf{H}$ under the analytic isomorphism defined in Equation (6) of loc. cit.. The $\mathbf{Q}$-rational map v_d induces pull-backs $v_d^* : V^*(\boldsymbol{\xi}^\sharp)^\cdot \longrightarrow V^*(\boldsymbol{\xi})^\cdot$ (for $\cdot = \emptyset, \pm$), which in turn induce morphisms $v_d^* : D^*(\boldsymbol{\xi}^\sharp)^\pm \longrightarrow D^*(\boldsymbol{\xi})^\pm$ and $v_d^* : H^1(\mathbf{Q}_p, V^*(\boldsymbol{\xi}^\sharp)^\cdot) \longrightarrow H^1(\mathbf{Q}_p, V^*(\boldsymbol{\xi})^\cdot)$ between the associated period rings and Galois cohomology groups. As d runs over the positive divisors of N/N_ξ, the images of $D^*(\boldsymbol{\xi}^\sharp)^\pm$ under the operators v_d^* generate $D^*(\boldsymbol{\xi})^\pm$ over $\mathscr{O}_{\boldsymbol{\xi}}$. As a consequence, if $\omega_{\boldsymbol{\xi}}^\cdot$ and $\eta_{\boldsymbol{\xi}}^\cdot$ (for $\cdot = \emptyset, \sharp$) denote the $\mathscr{O}_{\boldsymbol{\xi}}$-adic differentials associated to $\boldsymbol{\xi}^\cdot$ in Equations (118) and (122) of loc. cit. respectively, one has

$$\eta_{\boldsymbol{f}} = v_{\boldsymbol{f}}^*(\eta_{\boldsymbol{f}}^\sharp), \quad \omega_{\boldsymbol{g}} = v_{\boldsymbol{g}}^*(\omega_{\boldsymbol{g}}^\sharp) \quad \text{and} \quad \omega_{\boldsymbol{h}} = v_{\boldsymbol{h}}^*(\omega_{\boldsymbol{h}}^\sharp)$$

with $\mathscr{O}_{\boldsymbol{\xi}}$-linear combinations $v_{\boldsymbol{\xi}}^*$ of the operators v_d^*. (See Section 5 of [BSV20], especially Equation (95), Equations (117)–(123) and the discussion following them, for more details.) Denote by $v_{\boldsymbol{\xi}*} : V(\boldsymbol{\xi}) \longrightarrow V(\boldsymbol{\xi}^\sharp)$ the dual of $v_{\boldsymbol{\xi}}^*$ under the perfect pairing (103) of loc. cit. and set

$$\varpi_{\boldsymbol{fgh}} = v_{\boldsymbol{f}*} \otimes v_{\boldsymbol{g}*} \otimes v_{\boldsymbol{h}*} : V(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h}) \longrightarrow V(\boldsymbol{f}^\sharp, \boldsymbol{g}_\alpha^\sharp, \boldsymbol{h}_\alpha^\sharp).$$

With a slight abuse of notation, the map (12) is defined as the base change of $\varpi_{\boldsymbol{fgh}}$ under evaluation at $w_o = (2,1,1)$ on $\mathscr{O}_{\boldsymbol{fgh}}$ (cf. Equations (106) and (107) of [BSV20]).

Recall the modular parametrisation

$$\wp_\infty : Y_1(N_f p) \longrightarrow A$$

fixed in Section 1 (cf. Equation (3)) and set

$$\varpi_\star = \wp_{\infty*} \otimes \mathrm{id} \circ \varpi_{\boldsymbol{f}^\star \boldsymbol{g}_\alpha^\star \boldsymbol{h}_\alpha^\star} : V(\boldsymbol{f}_2, \boldsymbol{g}_1, \boldsymbol{h}_1) \longrightarrow V(f, g_\alpha, h_\alpha) \cong V(f,g,h),$$

(where id denotes the identity on $V(g_\alpha) \otimes_L V(h_\alpha) = V(g) \otimes_L V(h)$.) Then with the notation of Section 1 (cf. Remark 1.3(3) and Theorem B of [BSV20])

$$\kappa_{\alpha\alpha}(f,g,h) = \varpi_\star(\kappa(\boldsymbol{f}_2, \boldsymbol{g}_1, \boldsymbol{h}_1)) \in \mathrm{Sel}(\mathbf{Q}, V(f,g,h)).$$

For each local crystalline class $\mathfrak{z}$ in $H^1_{\mathrm{fin}}(\mathbf{Q}_p, V(f, g_\alpha, h_\alpha))$ define the *$\beta\beta$-component* of its p-adic logarithm by

$$\log_{\beta\beta}(\mathfrak{z}) = \big\langle \log_p(\mathfrak{z}), \omega_f \otimes \omega_{g_\alpha} \otimes \omega_{h_\alpha} \big\rangle_{f g_\alpha h_\alpha},$$

where ω_f is the differential associated with f in Equation (30) of [BSV20], the weight-one differentials ω_{g_α} and ω_{h_α} are the specialisations of $\omega_{\boldsymbol{g}_\alpha}^\sharp$ and $\omega_{\boldsymbol{h}_\alpha}^\sharp$ at weight one (cf. Equation (129) of [BSV20]), and the pairing $\langle \cdot, \cdot \rangle_{f g_\alpha h_\alpha}$ arises from the product of perfect dualities $\langle \cdot, \cdot \rangle_\xi$ introduced in Equations (31) and (128) of [BSV20], for $\xi = f, g_\alpha, h_\alpha$. Finally for any global Selmer class κ in $\mathrm{Sel}(\mathbf{Q}, V(f,g,h))$ define (cf. Equation (8))

$$\log_{\beta\beta}(\mathrm{res}_p(\kappa)) = \log_{\beta\beta}(\kappa_p),$$

where $\kappa_p \in H^1_{\mathrm{fin}}(\mathbf{Q}_p, V(f, g_\alpha, h_\alpha))$ is defined by $\wp_{\infty*} \otimes \mathrm{id}(\kappa_p) = \mathrm{res}_p(\kappa)$.

2.2. Tate's theory and the constant c_f. — The Tate parametrisation (cf. Chapter V of [Sil94]) yields a rigid analytic isomorphism

$$\varphi_{\mathrm{Tate}} : E_{q_A} \longrightarrow A_{K_p}$$

between the Tate curve

$$E_{q_A} = \mathbf{G}_{m,K_p}^{\mathrm{rig}}/q_A^{\mathbf{Z}}$$

over K_p and the base change A_{K_p} of A to K_p. Here $\mathbf{G}_{m,K_p}^{\mathrm{rig}}$ is the rigid multiplicative group over K_p and $q_A \in p\mathbf{Z}_p$ is the Tate period of $A_{\mathbf{Q}_p}$ (cf. loc. cit.).

Denote again by

$$\varphi_{\mathrm{Tate}} : V_p(E_{q_A}) \cong V_p(A)$$

the isomorphism of G_{K_p}-modules induced by the Tate parametrisation on the p-adic Tate modules with $\mathbf{Q}_p$-coefficients, and define

$$\wp_{\mathrm{Tate}} = \varphi_{\mathrm{Tate}}^{-1} \circ \wp_{\infty *} : V(f) \cong V_p(E_{q_A})$$

as the composition of its inverse with $\wp_{\infty *} : V(f) \cong V_p(A)$ (cf. Equation (3)). It induces a morphism of filtered modules (denoted by the same symbol)

$$\wp_{\mathrm{Tate}} : D_{\mathrm{dR},K_p}(V(f)) \cong D_{\mathrm{dR},K_p}(V_p(E_{q_A})),$$

where $D_{\mathrm{dR},K_p}(\cdot) = H^0(K_p, \cdot \otimes_{\mathbf{Q}_p} B_{\mathrm{dR}})$ is Fontaine's de Rham functor.

The projection $\mathbf{G}_{m,K_p}^{\mathrm{rig}} \longrightarrow E_{q_A}$ gives rise to an exact sequence of G_{K_p}-modules

$$0 \longrightarrow \mathbf{Q}_p(1) \longrightarrow V_p(E_{q_A}) \longrightarrow \mathbf{Q}_p \longrightarrow 0. \tag{13}$$

Applying Fontaine's de Rham functor $D_{\mathrm{dR},K_p}(\cdot) = H^0(K_p, \cdot \otimes_{\mathbf{Q}_p} B_{\mathrm{dR}})$ to the previous exact sequence yields a morphism $D_{\mathrm{dR},K_p}(V_p(E_{q_A})) \longrightarrow D_{\mathrm{dR},K_p}(\mathbf{Q}_p) = K_p$, which restricts to an isomorphism $\mathrm{Fil}^0 D_{\mathrm{dR},K_p}(V_p(E_{q_A})) \cong K_p$. Define

$$\mathbf{1}_A \in \mathrm{Fil}^0 D_{\mathrm{dR},K_p}(V_p(E_{q_A}))$$

for the generator corresponding to the identity of K_p under this isomorphism. On the other hand, the newform f corresponds (under Faltings' comparison isomorphism) to a canonical generator ω_f of $\mathrm{Fil}^0 D_{\mathrm{dR},K_p}(V(f)) = \mathrm{Fil}^1 V_{\mathrm{dR}}^*(f) \otimes_{\mathbf{Q}_p} K_p$ (cf. Equations (29) and (30) of [BSV20], noting that $V(f)(-1) = V^*(f)$). The non-zero p-adic constant

$$c_f \in K_p^*$$

which appears in Equation (6) of Section 1 is defined by the identity

$$\wp_{\mathrm{Tate}}(\omega_f) = c_f \cdot \mathbf{1}_A. \tag{14}$$

With the notations of Section 1, the following lemma shows that Equation (6) is a restatement of the main results of [BD07, BD09] (cf. Remark 1.2).

Lemma 2.1. — *Up to sign, one has the identity*

$$\log_{\omega_f} = \frac{c_f}{\deg(\wp_\infty)} \cdot \log_A .$$

Proof. — Let $u \in \mathcal{O}_{K_p}^*$ be a p-adic unit and let $P = \varphi_{\mathrm{Tate}}(u)$ be its image in $A(K_p)$ under the Tate parametrisation, so that

$$\log_A(P) = \log_p(u), \tag{15}$$

where $\log_p : K_p^* \longrightarrow K_p$ is the p-adic logarithm.

For V equal to one of $\mathbf{Q}_p(1), V_p(A), V_p(E_{q_A})$ and $V(f)$, denote by $\mathrm{tang}_{K_p}(V)$ the tangent space of $D_{\mathrm{dR},K_p}(V)$ and by

$$\log_V : H^1_{\mathrm{fin}}(K_p, V) \longrightarrow \mathrm{tang}_{K_p}(V)$$

the Bloch–Kato logarithm (viz. the inverse of the Bloch–Kato exponential map for V, which is an isomorphism). After identifying $\mathcal{O}_{K_p}^* \hat{\otimes} \mathbf{Q}_p$, resp. $A(K_p) \hat{\otimes} \mathbf{Q}_p$ with the finite subspace of $H^1(K_p, \mathbf{Q}_p(1))$, resp. $H^1(K_p, V_p(A))$ via Kummer theory, one has

$$\log_p(u) = \big\langle \log_{\mathbf{Q}_p(1)}(u), 1 \big\rangle_m = \big\langle \log_{V_p(E_{q_A})}(u), \mathbf{1}_A \big\rangle_W = \big\langle \log_{V_p(A)}(P), \varphi_{\mathrm{Tate}}(\mathbf{1}_A) \big\rangle_W, \tag{16}$$

where

$$\langle \cdot, \cdot \rangle_m : D_{\mathrm{dR},K_p}(\mathbf{Q}_p(1)) \otimes_{K_p} D_{\mathrm{dR},K_p}(\mathbf{Q}_p) \longrightarrow D_{\mathrm{dR},K_p}(\mathbf{Q}_p(1)) = K_p$$

is the pairing associated with the multiplication $m : \mathbf{Q}_p(1) \otimes_{\mathbf{Q}_p} \mathbf{Q}_p \longrightarrow \mathbf{Q}_p(1)$, and for $\mathcal{A}$ equal to either A_{K_p} or E_{q_A}, the morphism

$$\langle \cdot, \cdot \rangle_W : \mathrm{tang}_{K_p}(V_p(\mathcal{A})) \otimes_{K_p} \mathrm{Fil}^0 D_{\mathrm{dR},K_p}(V_p(\mathcal{A})) \longrightarrow D_{\mathrm{dR},K_p}(\mathbf{Q}_p(1)) = K_p$$

is the one induced by the Weil pairing $W : V_p(\mathcal{A}) \otimes_{\mathbf{Q}_p} V_p(\mathcal{A}) \longrightarrow \mathbf{Q}_p(1)$. (The first identity in Equation (16) is well known, while the others follow from the functoriality of the Bloch–Kato logarithm and of the Weil pairing, after noting that the Weil pairing on E_{q_A} and the multiplication map m are compatible via the exact sequence (13).)

Under the natural isomorphism between $V_p(A)$ and $H^1_{\text{ét}}(A_{\bar{\mathbf{Q}}}, \mathbf{Q}_p(1))$, the Weil pairing agrees (up to sign) with the cup-product pairing

$$H^1_{\text{ét}}(A_{\bar{\mathbf{Q}}}, \mathbf{Q}_p(1)) \otimes_{\mathbf{Q}_p} H^1_{\text{ét}}(A_{\bar{\mathbf{Q}}}, \mathbf{Q}_p(1)) \longrightarrow H^2_{\text{ét}}(A_{\bar{\mathbf{Q}}}, \mathbf{Q}_p(2)) \cong \mathbf{Q}_p(1)$$

associated with the multiplication map $\mathbf{Q}_p(1) \otimes_{\mathbf{Q}_p} \mathbf{Q}_p(1) \longrightarrow \mathbf{Q}_p(2)$, hence

$$\big\langle \log_{V_p(A)}(P), \varphi_{\mathrm{Tate}}(\mathbf{1}_A) \big\rangle_W = \deg(\wp_\infty) \cdot \big\langle \log_{V(f)}(\wp_{\infty *}^{-1}(P)), \wp_{\infty *}^{-1} \circ \varphi_{\mathrm{Tate}}(\mathbf{1}_A) \big\rangle_f.$$

By the definitions of $\log_{\omega_f}$ and c_f, the right hand side of the previous equation equals

$$\frac{\deg(\wp_\infty)}{c_f} \cdot \log_{\omega_f}(P).$$

Together with Equations (15)–(16), this prove that $\log_{\omega_f}(P)$ and $\frac{c_f}{\deg(\wp_\infty)} \cdot \log_A(P)$ are equal for each point $P \in A(K_p)$ in the image of $\mathcal{O}_{K_p}^*$ under the Tate parametrisation. Since $\mathcal{O}_{K_p}^*$ has finite index in $E_{q_A}(K_p)$, this concludes the proof. □

2.3. An exceptional zero formula and Equation (8). — As above, denote by $(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})=(\boldsymbol{f},\boldsymbol{g}_\alpha,\boldsymbol{h}_\alpha)$ a level-N test vector for $(\boldsymbol{f}^\sharp,\boldsymbol{g}_\alpha^\sharp,\boldsymbol{h}_\alpha^\sharp)$. Let

$$\mathfrak{Z}\in H^1_{\rm bal}(\mathbf{Q}_p,V(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h}))$$

be a local balanced class such that

$$\mathfrak{z}\stackrel{\rm def}{=}\rho_{w_o}(\mathfrak{Z})\in H^1_{\rm fin}(\mathbf{Q}_p,V(\boldsymbol{f}_2,\boldsymbol{g}_1,\boldsymbol{h}_1)).$$

In other words we assume that the specialisation $\mathfrak{z}$ of $\mathfrak{Z}$ at $w_o=(2,1,1)$ belongs to the Bloch–Kato Selmer finite subspace of $H^1(\mathbf{Q}_p,V(\boldsymbol{f}_2,\boldsymbol{g}_1,\boldsymbol{h}_1))$. The aim of this section is to prove the following *exceptional zero formula* for the analytic function

$$\mathscr{L}_{\boldsymbol{f}}(\mathfrak{Z};\boldsymbol{k},1,1)=\mathscr{L}og(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})(\mathfrak{Z})|_{(\boldsymbol{k},\boldsymbol{l},\boldsymbol{m})=(\boldsymbol{k},1,1)}\in\mathscr{O}_{\boldsymbol{f}},$$

viz. the restriction to the line $(\boldsymbol{k},1,1)$ of the image of $\mathfrak{Z}$ under the Perrin-Riou logarithm $\mathscr{L}_{\boldsymbol{f}}=\mathscr{L}og(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})$ (cf. [**Ven16**]). In light of Theorems A and B of our article [**BSV20**], taking $(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})=(\boldsymbol{f}^\star,\boldsymbol{g}_\alpha^\star,\boldsymbol{h}_\alpha^\star)$ and $\mathfrak{Z}=\mathrm{res}_p(\kappa(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h}))$ in its statement yields the key Equation (8) used in Section 1 to derive Theorem A.

Proposition 2.2. — *One has* $\mathrm{ord}_{\boldsymbol{k}=2}\mathscr{L}_{\boldsymbol{f}}(\mathfrak{Z};k,1,1)\geqslant 2$ *and (up to sign)*

$$c_{\boldsymbol{f}}^2\cdot\frac{d^2}{d\boldsymbol{k}^2}\mathscr{L}_{\boldsymbol{f}}(\mathfrak{Z};\boldsymbol{k},1,1)_{\boldsymbol{k}=2}=\frac{\deg(\wp_\infty)}{2\mathrm{ord}_p(q_A)}\left(1-\frac{1}{p}\right)^{-1}\cdot\log_{\beta\beta}(\varpi_{\boldsymbol{fgh}}(\mathfrak{z})).$$

We first prove a simple lemma. As in Section 1.1 of [**BSV20**], denote by $\Lambda_{\boldsymbol{f}}$ the ring of analytic functions on $U_{\boldsymbol{f}}$ bounded by one, so that $\mathscr{O}_{\boldsymbol{f}}=\Lambda_{\boldsymbol{f}}[1/p]$. Let

$$\Phi:G_{\mathbf{Q}_p}\longrightarrow\Lambda_{\boldsymbol{f}}^*$$

be a continuous character such that $\Phi(\cdot)_{\boldsymbol{k}=2}$ is the trivial character, and let $\boldsymbol{V}$ be a free $\mathscr{O}_{\boldsymbol{f}}$-module of finite rank on which $G_{\mathbf{Q}_p}$ acts via $\Phi\cdot\chi_{\rm cyc}$. Let $V=\boldsymbol{V}\otimes_2 L$ be the base change of $\boldsymbol{V}$ under evaluation at $\boldsymbol{k}=2$ on $\mathscr{O}_{\boldsymbol{f}}$. Multiplication by $\boldsymbol{k}-2$ on $\boldsymbol{V}$ gives rise to an exact sequence

$$(17)\quad\cdots\longrightarrow H^i(\mathbf{Q}_p,\boldsymbol{V})\stackrel{\boldsymbol{k}-2}{\longrightarrow}H^i(\mathbf{Q}_p,\boldsymbol{V})\longrightarrow H^i(\mathbf{Q}_p,V)\stackrel{\delta}{\longrightarrow}H^{i+1}(\mathbf{Q}_p,\boldsymbol{V})\longrightarrow\cdots.$$

As $\Phi(\cdot)_{\boldsymbol{k}=2}$ is the trivial character of $G_{\mathbf{Q}_p}$ the representation V is the direct sum of a finite number of copies of $L(1)$, hence there are natural isomorphisms

$$H^1(\mathbf{Q}_p,V)\cong\mathbf{Q}_p^*\hat{\otimes}V(-1)\quad\text{and}\quad H^2(\mathbf{Q}_p,V)\cong V(-1)$$

arising from Kummer's theory and the invariant map $\mathrm{inv}_p:H^2(\mathbf{Q}_p,\mathbf{Q}_p(1))\cong\mathbf{Q}_p$ respectively. One considers the previous isomorphisms as identities in the rest of this section. Define

$$\beta_{\boldsymbol{V}}:\mathbf{Q}_p^*\hat{\otimes}V(-1)\stackrel{\delta}{\longrightarrow}H^2(\mathbf{Q}_p,\boldsymbol{V})\longrightarrow H^2(\mathbf{Q}_p,\boldsymbol{V})\otimes_2 L\cong V(-1),$$

where the second map is the natural projection (and the isomorphism comes from the exact sequence (17), since $H^3(\mathbf{Q}_p,\boldsymbol{V})$ vanishes). Because $\Phi(\cdot)_{\boldsymbol{k}=2}$ is the trivial character its derivative defines a morphism

$$\frac{d}{d\boldsymbol{k}}\Phi(\cdot)_{\boldsymbol{k}=2}\in H^1(\mathbf{Q}_p,L)\cong\mathrm{Hom}_{\rm cont}(\mathbf{Q}_p^*,L),$$

where the isomorphism is induced by the reciprocity map

$$\mathrm{rec}_p : \mathbf{Q}_p^* \hat{\otimes} \mathbf{Q}_p \cong G_{\mathbf{Q}_p}^{\mathrm{ab}} \hat{\otimes} \mathbf{Q}_p$$

(normalised as in [**BSV20**, Section 9.2]). Taking the tensor product over L with $V(-1)$ this induces a morphism (denoted by the same symbol)

$$\frac{d}{d\boldsymbol{k}} \Phi(\cdot)_{\boldsymbol{k}=2} : \mathbf{Q}_p^* \hat{\otimes} V(-1) \longrightarrow V(-1).$$

Lemma 2.3. — $\beta_{\boldsymbol{V}} = \frac{d}{d\boldsymbol{k}} \Phi(\cdot)_{\boldsymbol{k}=2}$.

Proof. — Without loss of generality one can assume that $\boldsymbol{V}$ is equal to $\mathscr{O}_{\boldsymbol{f}}(\Phi \cdot \chi_{\mathrm{cyc}})$, hence $V = L(1)$. Let $x = q \hat{\otimes} v$ be an element of $\mathbf{Q}_p^* \hat{\otimes} L$ and let $c_x : G_{\mathbf{Q}_p} \to L(1)$ be a 1-cocycle representing it. Let $\tilde{c}_x : G_{\mathbf{Q}_p} \to \mathscr{O}_{\boldsymbol{f}}(\Phi \cdot \chi_{\mathrm{cyc}})$ be the 1-cochain defined by viewing c_x as a function with values in $\mathscr{O}_{\boldsymbol{f}}$. Clearly $\tilde{c}_x(\cdot)_{\boldsymbol{k}=2} = c_x$. If d denotes the differential in the complex $\mathrm{C}^{\bullet}_{\mathrm{cont}}(\mathbf{Q}_p, \mathscr{O}_{\boldsymbol{f}}(\Phi \cdot \chi_{\mathrm{cyc}}))$ of inhomogeneous continuous cochains of $G_{\mathbf{Q}_p}$ with values in $\mathscr{O}_{\boldsymbol{f}}(\Phi \cdot \chi_{\mathrm{cyc}})$, then

$$d\tilde{c}_x(\sigma, \tau) = (\Phi(\sigma) - 1) \cdot \chi_{\mathrm{cyc}}(\sigma) \cdot c_x(\tau) = \frac{d}{d\boldsymbol{k}} \Phi(\sigma)_{\boldsymbol{k}=2} \cdot \big(\chi_{\mathrm{cyc}}(\sigma) \cdot c_x(\tau)\big) \cdot (\boldsymbol{k} - 2) + \cdots,$$

where the dots denote higher terms in the Taylor expansion at $\boldsymbol{k} = 2$. This and local class field theory yield

$$\beta_{\boldsymbol{V}}(x) = \mathrm{inv}_p\Big(\frac{d}{d\boldsymbol{k}} \Phi(\cdot)_{\boldsymbol{k}=2} \cup cl(c_x)\Big) = \frac{d}{d\boldsymbol{k}} \Phi(q)_{\boldsymbol{k}=2} \cdot v,$$

where $\cup$ is the cup-product associated with the multiplication map $L \otimes_L L(1) \longrightarrow L(1)$. The lemma follows. □

Proof of Proposition 2.2. — By assumption $\mathfrak{Z} = \imath_*(\mathfrak{Y})$ is the image of a (unique) cohomology class $\mathfrak{Y}$ in $H^1(\mathbf{Q}_p, \mathscr{F}^2 V(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h}))$ under the map induced by the inclusion $\imath : \mathscr{F}^2 V(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h}) \to V(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})$. Set

$$\mathfrak{y} = \rho_{w_o *}(\mathfrak{Y}) \in H^1(\mathbf{Q}_p, \mathscr{F}^2 V(\boldsymbol{f}_2, \boldsymbol{g}_1, \boldsymbol{h}_1)),$$

so that $\mathfrak{z} = \rho_{w_o *}(\mathfrak{Z})$ is the image of $\mathfrak{y}$ under the natural map. By construction (cf. [**BSV20**, Proposition 7.3])

$$\mathscr{L}_{\boldsymbol{f}}(\mathfrak{Z}) = \mathscr{L}_{\boldsymbol{f}}(p_{f*}(\mathfrak{Y})). \tag{18}$$

If $\bullet$ and $\circ$ denote either α or β, define as in Section 9.2 of loc. cit. (cf. the proof of Proposition 9.3 of loc. cit.)

$$V(\boldsymbol{f}_2)^{\cdot}_{\bullet\circ} = V(\boldsymbol{f}_2)^{\cdot} \otimes_L V(\boldsymbol{g}_1)_{\bullet} \otimes_L V(\boldsymbol{h}_1)_{\circ},$$

where $\cdot = \emptyset, \pm$ and $V(\boldsymbol{\xi}_1)_\beta = V(\boldsymbol{\xi}_1)^+$ and $V(\boldsymbol{\xi}_1)_\alpha = V(\boldsymbol{\xi}_1)^-$ for $\boldsymbol{\xi} = \boldsymbol{g}, \boldsymbol{h}$. In the present setting the form $\boldsymbol{\xi}_1$ is *regular*, viz. $\alpha_{\boldsymbol{\xi}_1}$ and $\beta_{\boldsymbol{\xi}_1} = -\alpha_{\boldsymbol{\xi}_1}$ are distinct, hence $V(\boldsymbol{\xi}_1)_\bullet$ is equal to the subspace $V(\boldsymbol{\xi}_1)^{\mathrm{Frob}_p = \bullet}$ of $V(\boldsymbol{\xi}_1)$ on which an arithmetic Frobenius Frob_p acts as multiplication by $\bullet_{\boldsymbol{\xi}_1}$ (cf. Section 9.2 of loc. cit.). It follows that for $\cdot = \emptyset$ and $\cdot = \pm$ there are *canonical* direct sum decompositions

$$V(\boldsymbol{f}_2, \boldsymbol{g}_1, \boldsymbol{h}_1)^{\cdot} = V(\boldsymbol{f}_2)^{\cdot}_{\alpha\alpha} \oplus V(\boldsymbol{f}_2)^{\cdot}_{\alpha\beta} \oplus V(\boldsymbol{f}_2)^{\cdot}_{\beta\alpha} \oplus V(\boldsymbol{f}_2)^{\cdot}_{\beta\beta} \tag{19}$$

of $L[G_{\mathbf{Q}_p}]$-modules. In particular $V(\boldsymbol{f}_2, \boldsymbol{g}_1, \boldsymbol{h}_1)_f = V(\boldsymbol{f}_2)^-_{\beta\beta}$ is a direct summand of $V(\boldsymbol{f}_2, \boldsymbol{g}_1, \boldsymbol{h}_1)^-$ (cf. Equation (191) of loc. cit.), hence

$$p_{f*}(\mathfrak{y}) = 0$$

since by assumption $\mathfrak{z}$ is crystalline (cf. Section 9.1 of loc. cit., in particular Equation (193)). As a consequence

$$p_{f*}(\mathfrak{Y}) = (\boldsymbol{k} - 2) \cdot \mathfrak{Y}_{\boldsymbol{k}} + (\boldsymbol{l} - 1) \cdot \mathfrak{Y}_{\boldsymbol{l}} + (\boldsymbol{m} - 1) \cdot \mathfrak{Y}_{\boldsymbol{m}} \tag{20}$$

for classes $\mathfrak{Y}_\cdot$ in $H^1(\mathbf{Q}_p, V(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})_f)$ (cf. the proof of Proposition 7.3 of loc. cit. or [**Ven16**, Lemma 5.6]). Set

$$\mathfrak{y}_{\boldsymbol{k}} = \rho_{w_o *}(\mathfrak{Y}_{\boldsymbol{k}}) \in H^1(\mathbf{Q}_p, V(\boldsymbol{f}_2, \boldsymbol{g}_1, \boldsymbol{h}_1)_f).$$

Because $\mathscr{L}_{\boldsymbol{f}}$ is $\mathscr{O}_{\boldsymbol{fgh}}$-linear, Equation (18), Proposition 9.3(1) of loco citato and Theorem 3.14 of [**GS93**] give

$$\left(1 - \frac{1}{p}\right) \cdot \frac{d^2}{d\boldsymbol{k}^2} \mathscr{L}_{\boldsymbol{f}}(\mathfrak{Z}, \boldsymbol{k}, 1, 1)_{\boldsymbol{k}=2} = \mathfrak{y}_{\boldsymbol{k}}(p^{-1})_f - \mathfrak{L}^{\mathrm{an}}_{\boldsymbol{f}} \cdot \mathfrak{y}_{\boldsymbol{k}}(e(1))_f = \frac{-1}{\mathrm{ord}_p(q_A)} \cdot \mathfrak{y}_{\boldsymbol{k}}(q_A)_f, \tag{21}$$

where

$$-\frac{1}{2} \cdot \mathfrak{L}^{\mathrm{an}}_{\boldsymbol{f}} = d\mathrm{log} a_p(\boldsymbol{k})_{\boldsymbol{k}=2}$$

is the logarithmic derivative at $\boldsymbol{k} = 2$ of the p-th Fourier coefficient $a_p(\boldsymbol{k})$ of $\boldsymbol{f}^\sharp$ (cf. Section 9.2 of [**BSV20**]). In particular this implies that the quantity $\mathfrak{y}_{\boldsymbol{k}}(q_A)_f$ is independent of the choice of $\mathfrak{Y}_{\boldsymbol{k}}$ satisfying Equation (20).

As shown in the proof of Proposition 9.3 of loc. cit. the class of the extension

$$0 \longrightarrow V(\boldsymbol{f}_2)^+_{\beta\beta} \longrightarrow V(\boldsymbol{f}_2)_{\beta\beta} \longrightarrow V(\boldsymbol{f}_2)^-_{\beta\beta} \longrightarrow 0 \tag{22}$$

in

$$\mathrm{Ext}^1_{L[G_{\mathbf{Q}_p}]}(V(\boldsymbol{f}_2)^-_{\beta\beta}, V(\boldsymbol{f}_2)^+_{\beta\beta}) \cong \mathbf{Q}_p^* \hat{\otimes}_{\mathbf{Q}_p} \mathrm{Hom}_L(V(\boldsymbol{f}_2)^-_{\beta\beta}, V(\boldsymbol{f}_2)^+_{\beta\beta}(-1))$$

is equal to

$$q_{\boldsymbol{f}_2} = q_A \hat{\otimes} \delta_{\boldsymbol{f}_2}$$

for an isomorphism $\delta_{\boldsymbol{f}_2} : V(\boldsymbol{f}_2)^-_{\beta\beta} \to V(\boldsymbol{f}_2)^+_{\beta\beta}(-1)$, and the connecting morphisms $\partial^i_{\boldsymbol{f}_2}$ associated to (22) satisfy

$$\partial^0_{\boldsymbol{f}_2}(v) = q_A \hat{\otimes} \delta_{\boldsymbol{f}_2}(v) = q_{\boldsymbol{f}_2} \cup v \quad \text{and} \quad \partial^1_{\boldsymbol{f}_2}(\varphi \otimes v) = -\varphi(q_A) \cdot \delta_{\boldsymbol{f}_2}(v) = -q_{\boldsymbol{f}_2} \cup (\varphi \otimes v) \tag{23}$$

for all φ in $\mathrm{Hom}_{\mathrm{cont}}(\mathbf{Q}_p^*, L)$ and v in $V(\boldsymbol{f}_2)^-_{\beta\beta}$, where $\cup$ is the cup-product induced by the evaluation map. Define

$$V(\boldsymbol{f})^\cdot_{\beta\beta} = \left(V(\boldsymbol{f})^\cdot \otimes_{\mathscr{O}_{\boldsymbol{f}}} \kappa_{\mathrm{cyc}}^{1-\boldsymbol{k}/2}\right) \otimes_L V(\boldsymbol{g}_1)^+ \otimes_L V(\boldsymbol{h}_1)^+.$$

These are $\mathscr{O}_{\boldsymbol{f}}[G_{\mathbf{Q}_p}]$-modules, sitting in a short exact sequence

$$0 \longrightarrow V(\boldsymbol{f})^+_{\beta\beta} \longrightarrow V(\boldsymbol{f})_{\beta\beta} \longrightarrow V(\boldsymbol{f})^-_{\beta\beta} \longrightarrow 0$$

which specialises to (22) under evaluation at $\boldsymbol{k} = 2$ on $\mathscr{O}_{\boldsymbol{f}}$. Identify the $\mathscr{O}_{\boldsymbol{f}}$-module $V(\boldsymbol{f})_{\beta\beta}$ with the direct sum of $V(\boldsymbol{f})^+_{\beta\beta}$ and $V(\boldsymbol{f})^-_{\beta\beta}$ under a fixed $\mathscr{O}_{\boldsymbol{f}}$-splitting of the previous exact sequence. There is then a continuous map

$$q_{\boldsymbol{f}} : G_{\mathbf{Q}_p} \longrightarrow \mathrm{Hom}_{\mathscr{O}_{\boldsymbol{f}}}(V(\boldsymbol{f})^-_{\beta\beta}, V(\boldsymbol{f})^+_{\beta\beta})$$

satisfying the following properties. For all $\boldsymbol{v}^{\pm} \in V(\boldsymbol{f})^{\pm}_{\beta\beta}$ and $\sigma \in G_{\mathbf{Q}_p}$ (cf. Equation (101) of loc. cit.)

$$\sigma(\boldsymbol{v}^+) = \frac{\omega_{\mathrm{cyc}}(\sigma)\cdot\kappa_{\mathrm{cyc}}^{\boldsymbol{k}/2}(\sigma)}{\psi_{\boldsymbol{f}}\psi_{\boldsymbol{g}_1}\psi_{\boldsymbol{h}_1}(\sigma)}\cdot\boldsymbol{v}^+ \quad\text{and}\quad \sigma(\boldsymbol{v}^-) = \frac{\psi_{\boldsymbol{f}}(\sigma)\kappa_{\mathrm{cyc}}^{1-\boldsymbol{k}/2}(\sigma)}{\psi_{\boldsymbol{g}_1}\psi_{\boldsymbol{h}_1}(\sigma)}\cdot\boldsymbol{v}^- + q_{\boldsymbol{f}}(\sigma,\boldsymbol{v}^-), \tag{24}$$

where $\psi_{\boldsymbol{f}} : G^{\mathrm{nr}}_{\mathbf{Q}_p} \longrightarrow \Lambda^*_{\boldsymbol{f}}$ is the unramified character of $G_{\mathbf{Q}_p}$ which sends an arithmetic Frobenius Frob_p to $a_p(\boldsymbol{k})$, and similarly $\psi_{\boldsymbol{g}_1}, \psi_{\boldsymbol{h}_1} : G^{\mathrm{nr}}_{\mathbf{Q}_p} \longrightarrow \mathscr{O}^*$ are defined by $\psi_{\boldsymbol{g}_1}(\mathrm{Frob}_p) = b_p(1)$ and $\psi_{\boldsymbol{h}_1}(\mathrm{Frob}_p) = c_p(1)$ respectively. (Here one uses that both $\chi_{\boldsymbol{f}}$ and $\chi_{\boldsymbol{g}}\cdot\chi_{\boldsymbol{h}}$ are equal to the trivial character.) Moreover the specialisation

$$q_{\boldsymbol{f}}(\cdot)_{\boldsymbol{k}=2} : G_{\mathbf{Q}_p} \longrightarrow \mathbf{Q}_p(1)\otimes_{\mathbf{Q}_p}\mathrm{Hom}_{\mathscr{O}_f}(V(\boldsymbol{f}_2)^-_{\beta\beta}, V(\boldsymbol{f}_2)^+_{\beta\beta}(-1))$$

of $q_{\boldsymbol{f}}$ at $\boldsymbol{k}=2$ (via $\mathrm{Hom}_{\mathscr{O}_{\boldsymbol{f}}}(V(\boldsymbol{f})^-_{\beta\beta}, V(\boldsymbol{f})^+_{\beta\beta})\otimes_2 L \cong \mathrm{Hom}_L(V(\boldsymbol{f}_2)^-_{\beta\beta}, V(\boldsymbol{f}_2)^+_{\beta\beta})$) is a 1-cocycle satisfying

$$cl\big(q_{\boldsymbol{f}}(\cdot)_{\boldsymbol{k}=2}\big) = q_{\boldsymbol{f}_2}. \tag{25}$$

For future reference denote by $\Phi_{\boldsymbol{f}} : G_{\mathbf{Q}_p} \longrightarrow \Lambda^*_{\boldsymbol{f}}$ the character

$$\Phi_{\boldsymbol{f}} = \kappa_{\mathrm{cyc}}^{\boldsymbol{k}/2-1}\cdot\psi_{\boldsymbol{f}}^{-1}\cdot\psi_{\boldsymbol{g}_1}^{-1}\cdot\psi_{\boldsymbol{h}_1}^{-1}, \tag{26}$$

so that $\Phi_{\boldsymbol{f}}(\cdot)_{\boldsymbol{k}=2}$ is the trivial character and $G_{\mathbf{Q}_p}$ acts on $V(\boldsymbol{f})^+_{\beta\beta}$ via $\chi_{\mathrm{cyc}}\cdot\Phi_{\boldsymbol{f}}$.

Denote by

$$\mathfrak{Y}_{\beta\beta} \in H^1(\mathbf{Q}_p, V(\boldsymbol{f})_{\beta\beta}) \quad\text{and}\quad \mathfrak{Y}_{\boldsymbol{k},\beta\beta} \in H^1(\mathbf{Q}_p, V(\boldsymbol{f})^-_{\beta\beta})$$

the images of $\mathfrak{Y}$ and $\mathfrak{Y}_{\boldsymbol{k}}$ under the maps induced by the projections

$$\mathscr{F}^2 V(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h}) \twoheadrightarrow \mathscr{F}^2 V(\boldsymbol{f},\boldsymbol{g}_1,\boldsymbol{h}_1) \twoheadrightarrow V(\boldsymbol{f})_{\beta\beta}$$

and

$$V(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})_f \twoheadrightarrow V(\boldsymbol{f},\boldsymbol{g}_1,\boldsymbol{h}_1)_f = V(\boldsymbol{f})^-_{\beta\beta}$$

respectively. (Here $V(\boldsymbol{f},\boldsymbol{g}_1,\boldsymbol{h}_1) = V(\boldsymbol{f})^{\cdot}\otimes_L V(\boldsymbol{g}_1)\otimes_L V(\boldsymbol{h}_1)(\kappa_{\mathrm{cyc}}^{1-\boldsymbol{k}/2})$. Note that the discussion leading to Equation (19) yields a similar canonical decomposition of the $\mathscr{O}_{\boldsymbol{f}}[G_{\mathbf{Q}}]$-module $V(\boldsymbol{f},\boldsymbol{g}_1,\boldsymbol{h}_1)$.) According to Equation (20) the cohomology class $\mathfrak{Y}_{\beta\beta}$ is represented by a 1-cocycle of the form

$$Y_{\beta\beta} = Y^+_{\beta\beta}\oplus(\boldsymbol{k}-2)\cdot Y^-_{\beta\beta} : G_{\mathbf{Q}_p} \longrightarrow V(\boldsymbol{f})_{\beta\beta},$$

for 1-cochains $Y^{\cdot}_{\beta\beta} : G_{\mathbf{Q}_p} \to V(\boldsymbol{f})^{\cdot}_{\beta\beta}$. Using Equation (24) the cocycle relation for $Y_{\beta\beta}$ gives

$$dY^+_{\beta\beta}(\sigma,\tau) = -(\boldsymbol{k}-2)\cdot q_{\boldsymbol{f}}(\sigma, Y^-_{\beta\beta}(\tau)) \quad\text{and}\quad dY^-_{\beta\beta} = 0. \tag{27}$$

In particular the specialisations $y^{\cdot}_{\beta\beta} : G_{\mathbf{Q}_p} \to V(\boldsymbol{f}_2)^{\cdot}_{\beta\beta}$ of $Y^{\cdot}_{\beta\beta}$ at $\boldsymbol{k}=2$ are both 1-cocycles and by construction

$$i^+_*(\mathfrak{y}^+_{\beta\beta}) = \mathfrak{y}_{\beta\beta} \quad\text{and}\quad (\boldsymbol{k}-2)\cdot cl(Y^-_{\beta\beta}) = (\boldsymbol{k}-2)\cdot\mathfrak{Y}_{\boldsymbol{k},\beta\beta}, \tag{28}$$

where $\mathfrak{y}^{\pm}_{\beta\beta} = cl(y^{\pm}_{\beta\beta}) \in H^1(\mathbf{Q}_p, V(\boldsymbol{f}_2)^{\pm}_{\beta\beta})$ are the classes represented by $y^{\pm}_{\beta\beta}$, the map i^+_* is the one induced by the inclusion $i^+ : V(\boldsymbol{f}_2)^+_{\beta\beta} \hookrightarrow V(\boldsymbol{f}_2)_{\beta\beta}$ and $\mathfrak{y}_{\beta\beta}$ in $H^1(\mathbf{Q}_p, V(\boldsymbol{f}_2)_{\beta\beta})$ is the image of $\mathfrak{y}$ under the map induced by the projection onto the direct summand $V(\boldsymbol{f}_2)_{\beta\beta}$ of $\mathscr{F}^2V(\boldsymbol{f}_2,\boldsymbol{g}_1,\boldsymbol{h}_1)$. The second identity in Equation (28) implies

$$\mathfrak{y}_{\boldsymbol{k}}(q_A)_f = \mathfrak{y}^-_{\beta\beta}(q_A)_f$$

(cf. the remark after Equation (21)), hence Equation (21) can be rephrased as

$$\left(1-\frac{1}{p}\right)\cdot\frac{d^2}{d\boldsymbol{k}^2}\mathscr{L}_{\boldsymbol{f}}(\mathfrak{z},\boldsymbol{k},1,1)_{\boldsymbol{k}=2} = \frac{-1}{\mathrm{ord}_p(q_A)}\cdot\mathfrak{y}^-_{\beta\beta}(q_A)_f. \tag{29}$$

In light of Equations (24)–(26) and Lemma 2.3, the first equalities in Equations (27) and (28) yield

$$\begin{aligned}-\partial^1_{\boldsymbol{f}_2}(\mathfrak{y}^-_{\beta\beta}) &= \mathrm{inv}_p\big(cl\big(q_{\boldsymbol{f}_2}(\sigma,y^-_{\beta\beta}(\tau))\big)\big)\\ &= -\beta_{V(\boldsymbol{f})^+_{\beta\beta}}(\mathfrak{y}^+_{\beta\beta}) = -\frac{d}{d\boldsymbol{k}}\Phi_{\boldsymbol{f}}(\mathfrak{y}^+_{\beta\beta})_{\boldsymbol{k}=2} = -\frac{1}{2}\cdot\log_{q_A}(\mathfrak{y}^+_{\beta\beta}).\end{aligned} \tag{30}$$

More precisely, the first equality follows from Equation (23), the second from Equations (25) and (27) and the definition of $\beta_{V(\boldsymbol{f})^+_{\beta\beta}}$, and the third from Lemma 2.3. Finally, for each unit u in $\mathbf{Z}_p^*$, one has (cf. Equation (26))

$$\frac{d}{d\boldsymbol{k}}\Phi_{\boldsymbol{f}}(u)_{\boldsymbol{k}=2} = \frac{d}{d\boldsymbol{k}}\kappa_{\mathrm{cyc}}^{\boldsymbol{k}/2-1}(\mathrm{rec}_p(u))_{\boldsymbol{k}=2} = \frac{d}{d\boldsymbol{k}}\big(u^{\boldsymbol{k}/2-1}\big)_{\boldsymbol{k}=2} = \frac{1}{2}\cdot\log_p(u)$$

and

$$\frac{d}{d\boldsymbol{k}}\Phi_{\boldsymbol{f}}(p)_{\boldsymbol{k}=2} = \alpha_g\cdot\alpha_h\cdot\frac{d}{d\boldsymbol{k}}a_p(\boldsymbol{k})_{\boldsymbol{k}=2} = -\frac{1}{2}\cdot\mathfrak{L}^{\mathrm{an}}_{\boldsymbol{f}},$$

which in light of the identity $\mathfrak{L}^{\mathrm{an}}_{\boldsymbol{f}} = \frac{\log_p(q_A)}{\mathrm{ord}_p(q_A)}$ proved in [GS93, Theorem 3.14] yields the last equality in Equation (30). (Here one denotes again by

$$\log_{q_A}:\mathbf{Q}_p^*\hat{\otimes}V(\boldsymbol{f}_2)^+_{\beta\beta}(-1)\longrightarrow V(\boldsymbol{f}_2)^+_{\beta\beta}(-1)\cong D_{\mathrm{cris}}(V(\boldsymbol{f}_2)^+_{\beta\beta})$$

the morphism induced by $\log_{q_A} = \log_p - \frac{\log_p(q_A)}{\mathrm{ord}_p(q_A)}\cdot\mathrm{ord}_p:\mathbf{Q}_p^*\to\mathbf{Q}_p$).

As the connecting morphisms $\partial^0_{\boldsymbol{f}_2}$ and $-\partial^1_{\boldsymbol{f}_2}$ are adjoint to each other under the cup-product induced by $\langle\cdot,\cdot\rangle_{\boldsymbol{f}_2\boldsymbol{g}_1\boldsymbol{h}_1}$, Equations (23), (29) and (30) combine to give

$$\left(1-\frac{1}{p}\right)\cdot\frac{d^2}{d\boldsymbol{k}^2}\mathscr{L}_{\boldsymbol{f}}(\mathfrak{z},\boldsymbol{k},1,1)_{\boldsymbol{k}=2} = \frac{1}{2\mathrm{ord}_p(q_A)}\cdot\big\langle\log_{q_A}(\mathfrak{y}^+_{\beta\beta}),\delta^{-1}_{\boldsymbol{f}_2}\big(\eta_{\boldsymbol{f}_2}\otimes\omega_{\boldsymbol{g}_1}\otimes\omega_{\boldsymbol{h}_1}\big)\big\rangle_{\boldsymbol{f}_2\boldsymbol{g}_1\boldsymbol{h}_1}. \tag{31}$$

Since f has trivial character, one has $V^*(f)^{\cdot} = V(f)^{\cdot}(-1)$ for $\cdot = \emptyset,\pm$ (cf. Sections 2.5 and 5 of [BSV20]). There are then natural $\mathrm{Gal}(K_p/\mathbf{Q}_p)$-equivariant isomorphisms

$$\mathrm{Fil}^1D_{\mathrm{dR},K_p}(V^*(f))\cong\mathrm{Fil}^0D_{\mathrm{dR},K_p}(V(f))\cong D_{\mathrm{cris},K_p}(V(f)^-) = V(f)^-\otimes_{\mathbf{Q}_p}K_p,$$

under which we identify the differential (cf. Section 2.5 of loco citato)

$$\omega_f\in\mathrm{Fil}^1V^*_{\mathrm{dR}}(f) = \mathrm{Fil}^1D_{\mathrm{dR},K_p}(V^*(f))^{\mathrm{Gal}(K_p/\mathbf{Q}_p)}$$

with an element of $V(f)^-$. Lemma 2.4 below proves that

$$\delta_f(\omega_f\otimes\omega_{g_\alpha}\otimes\omega_{h_\alpha}) = \pm\frac{c_f^2}{\deg(\wp_\infty)}\cdot\eta_f\otimes\omega_{g_\alpha}\otimes\omega_{h_\alpha}$$

in $V^*(f)^+_{\beta\beta} = V(f)^+(-1) \otimes_{\mathbf{Q}_p} V^*(g)^- \otimes_L V(h)^-$, hence by construction

$$\delta^{-1}_{\boldsymbol{f}_2}\big(\eta_{\boldsymbol{f}_2} \otimes \omega_{\boldsymbol{g}_1} \otimes \omega_{\boldsymbol{h}_1}\big) = \pm \frac{\deg(\wp_\infty)}{c_f^2} \cdot \varpi^*_{\boldsymbol{fgh}}\big(\omega_f \otimes \omega_{g_\alpha} \otimes \omega_{h_\alpha}\big), \tag{32}$$

where $\varpi^*_{\boldsymbol{fgh}} = v^*_{\boldsymbol{f}} \otimes v^*_{\boldsymbol{g}} \otimes v^*_{\boldsymbol{h}}$ is the adjoint of $\varpi_{\boldsymbol{fgh}}$ under the Poicaré dualities $\langle \cdot, \cdot \rangle_{fg_\alpha h_\alpha}$ and $\langle \cdot, \cdot \rangle_{\boldsymbol{f}_2 \boldsymbol{g}_1 \boldsymbol{h}_1}$. Finally, the first identity in Equation (28) gives

$$\log_{q_A}(\mathfrak{y}^+_{\beta\beta}) = \pi_{\beta\beta}(\log_p(\mathfrak{z})), \tag{33}$$

where $\pi_{\beta\beta}$ is the composition

$$D_{\mathrm{dR}}(V(\boldsymbol{f}_2, \boldsymbol{g}_1, \boldsymbol{h}_1))/\mathrm{Fil}^0 \cong D_{\mathrm{st}}(V(\boldsymbol{f}_2, \boldsymbol{g}_1, \boldsymbol{h}_1)^+) \twoheadrightarrow D_{\mathrm{cris}}(V(\boldsymbol{f}_2)^+_{\beta\beta})$$

arising from Equations (191) and (192) of [BSV20] and Equation (19). Since by construction the $\beta\beta$-logarithm $\log_{\beta\beta}$ factors through the projection $\pi_{\beta\beta}$, the proposition is a direct consequence of Equations (31)–(33). □

Lemma 2.4. — *Let*

$$\partial_f : V(f)^- \longrightarrow K_p^* \hat{\otimes} V(f)^+(-1)$$

be the connecting morphism associated with the exact sequence of G_{K_p}-modules

$$0 \longrightarrow V(f)^+ \longrightarrow V(f) \longrightarrow V(f)^- \longrightarrow 0.$$

Then $\partial_f = q_A \hat{\otimes} \delta_f$ for an isomorphism

$$\delta_f : V(f)^- \longrightarrow V(f)^+(-1)$$

satisfying, up to sign, the following identity in $V(f)^+(-1)$:

$$\delta_f(\omega_f) = \frac{c_f^2}{\deg(\wp_\infty)} \cdot \eta_f.$$

Proof. — Consider the following diagram of $\mathbf{Q}_p[G_{K_p}]$-modules with exact rows, in which all the vertical maps are isomorphisms.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbf{Q}_p(1) & \longrightarrow & V_p(E_{q_A}) & \longrightarrow & \mathbf{Q}_p & \longrightarrow & 0 \\ & & \downarrow {\scriptstyle \varphi^+_{\mathrm{Tate}}} & & \downarrow {\scriptstyle \varphi_{\mathrm{Tate}}} & & \downarrow {\scriptstyle \varphi^-_{\mathrm{Tate}}} & & \\ 0 & \longrightarrow & V_p(A)^+ & \longrightarrow & V_p(A) & \longrightarrow & V_p(A)^- & \longrightarrow & 0 \\ & & \uparrow {\scriptstyle \wp^+_{\infty *}} & & \uparrow {\scriptstyle \wp_{\infty *}} & & \uparrow {\scriptstyle \wp^-_{\infty *}} & & \\ 0 & \longrightarrow & V(f)^+ & \longrightarrow & V(f) & \longrightarrow & V(f)^- & \longrightarrow & 0 \end{array} \tag{34}$$

Here φ_{Tate} is the map induced on the p-adic Tate modules by the Tate uniformisation $E_{q_A} \cong A_{K_p}$, and the first row is the short exact sequence induced by the natural projection $\mathbf{G}^{\mathrm{rig}}_{m,K_p} \longrightarrow E_{q_A}$ (cf. Introduction).

The class in

$$\mathrm{Ext}^1_{\mathbf{Q}_p[G_{K_p}]}(\mathbf{Q}_p, \mathbf{Q}_p(1)) = H^1(K_p, \mathbf{Q}_p(1)) \cong K_p^* \hat{\otimes} \mathbf{Q}_p$$

represented by the first row equals $q_A\hat{\otimes}1$, hence the associated connecting morphism

$$\partial_{\mathrm{Tate}} : \mathbf{Q}_p \longrightarrow K_p^*\hat{\otimes}\mathbf{Q}_p$$

satisfies

$$\partial_{\mathrm{Tate}}(1) = q_A\hat{\otimes}1. \tag{35}$$

After setting

$$\gamma_{q_A} = \frac{-1}{\mathrm{ord}_p(q_A)} \cdot \mathrm{ord}_p \in \mathrm{Hom}_{\mathrm{cont}}(K_p^*, \mathbf{Q}_p) \cong H^1(K_p, \mathbf{Q}_p),$$

this implies

$$\langle \gamma_{q_A}, \partial_{\mathrm{Tate}}(1)\rangle_m = 1, \tag{36}$$

where

$$\langle \cdot, \cdot \rangle_m : H^1(K_p, \mathbf{Q}_p) \otimes_{\mathbf{Q}_p} H^1(K_p, \mathbf{Q}_p(1)) \longrightarrow K_p$$

is the local Tate pairing attached to the multiplication $m : \mathbf{Q}_p \otimes_{\mathbf{Q}_p} \mathbf{Q}_p(1) \longrightarrow \mathbf{Q}_p(1)$. Moreover, the Diagram (34) and Equation (35) imply that the connecting morphisms

$$\partial_A : V_p(A)^- \longrightarrow K_p^*\hat{\otimes}V_p(A)^+(-1) \quad \text{and} \quad \partial_f : V(f)^- \longrightarrow K_p^*\hat{\otimes}V(f)^+(-1)$$

associated respectively to the second and third rows of Diagram (34) are of the form

$$\partial_A = q_A \otimes \delta_A \quad \text{and} \quad \partial_f = q_A \otimes \delta_f \tag{37}$$

for isomorphisms $\delta_A : V_p(A)^- \longrightarrow V_p(A)^+(-1)$ and $\delta_f : V(f)^- \longrightarrow V(f)^+(-1)$.

Up to sign, one has the identities

$$\begin{aligned} \langle \omega_f, \delta_f(\omega_f)\rangle_f &= \langle \gamma_{q_A} \otimes \omega_f, \partial_f(\omega_f)\rangle_f \\ &= \frac{1}{\deg(\wp_\infty)} \cdot \left\langle \gamma_{q_A} \otimes \wp^-_{\infty *}(\omega_f), \partial_A(\wp^-_{\infty *}(\omega_f))\right\rangle_{\mathrm{Weil}} \\ &= \frac{c_f^2}{\deg(\wp_\infty)} \cdot \left\langle \gamma_{q_A} \otimes \varphi^-_{\mathrm{Tate}}(1), \partial_A(\varphi^-_{\mathrm{Tate}}(1))\right\rangle_{\mathrm{Weil}} \\ &= \frac{c_f^2}{\deg(\wp_\infty)} \cdot \left\langle \gamma_{q_A} \otimes \varphi^-_{\mathrm{Tate}}(1), \varphi^+_{\mathrm{Tate}}(\partial_{\mathrm{Tate}}(1))\right\rangle_{\mathrm{Weil}} \\ &= \frac{c_f^2}{\deg(\wp_\infty)} \cdot \langle \gamma_{q_A}, \partial_{\mathrm{Tate}}(1)\rangle_m , \end{aligned} \tag{38}$$

where $\langle \cdot, \cdot \rangle_{\mathrm{Weil}} : H^1(K_p, V_p(A)^+) \otimes_{\mathbf{Q}_p} H^1(K_p, V_p(A)^-) \longrightarrow K_p$ is the local Tate pairing associated with the Weil paring on $V_p(A)$. Indeed, the first equality follows from Equation (37). The second equality follows (up to sign) from the functoriality of Poincaré duality under finite morphisms of curves and its compatibility with the Weil pairing on elliptic curves. The third equality follows from the definition of c_f (cf. Equation (14)). The fourth equality follows from Diagram (34). The fifth and last equality follows from the functoriality of the Weil paring under isogenies, after noting that the Kummer duality between $\mathbf{Q}_p(1)$ and $\mathbf{Q}_p$ induced by the Weil pairing on $V_p(E_{q_A})$ is equal (up to sign) to the multiplication map m.

Since $V(f)^+(-1) = D_{\mathrm{cris}}(V(f)^+)$ is a one-dimensional $\mathbf{Q}_p$-vector space generated by η_f and $\langle \omega_f, \eta_f\rangle_f = 1$, the lemma follows from Equations (36) and (38). □

3. Factorisations of p-adic L-functions

This section is devoted to the proof of Theorem 3.1 below, viz. the crucial factorisation formula (7) of Section 1, under the assumptions listed therein. In light of the discussion of Section 1 (see Equations (7) and (8)) and of Section 2, this is the last step in our proof of Theorem A.

The reader is cautioned that the notations for p-adic L-functions in force here are consistent with those of [**BSV20**, Section 6] and differ slightly from those of Section 1. Thus $L_p(\boldsymbol{f}^\sharp, \boldsymbol{g}^\sharp, \boldsymbol{h}^\sharp)$ denotes the square of the triple product square-root p-adic L-function $\mathscr{L}_p^f(\boldsymbol{f}^\star, \boldsymbol{g}^\star, \boldsymbol{h}^\star)$ attached to our fixed choice of test vector $(\boldsymbol{f}^\star, \boldsymbol{g}^\star, \boldsymbol{h}^\star)$, and the restriction of $L_p(\boldsymbol{f}^\sharp, \boldsymbol{g}^\sharp, \boldsymbol{h}^\sharp)$ to the line $(\boldsymbol{k}, 1, 1)$ is denoted

$$L_p(\boldsymbol{f}^\sharp, \boldsymbol{g}_1^\sharp, \boldsymbol{h}_1^\sharp) = L_p(\boldsymbol{f}^\sharp, g_\alpha, h_\alpha)$$

(recall that $\boldsymbol{g}^\sharp$ and $\boldsymbol{h}^\sharp$ interpolate the chosen p-stabilisations g_α and h_α respectively). Accordingly, the Hida–Rankin p-adic L-functions associated to the ring class characters φ and ψ are denoted by $L_p(\boldsymbol{f}^\sharp, \varphi)$ and $L_p(\boldsymbol{f}^\sharp, \psi)$ (as observed in Section 1, they depend only on the primitive family $\boldsymbol{f}^\sharp$).

Theorem 3.1. — *Up to shrinking $U_{\boldsymbol{f}}$ if necessary, there is a factorisation*

$$L_p(\boldsymbol{f}^\sharp, \boldsymbol{g}_1^\sharp, \boldsymbol{h}_1^\sharp) = \mathscr{A} \cdot L_p(\boldsymbol{f}^\sharp/K, \varphi) \cdot L_p(\boldsymbol{f}^\sharp/K, \psi),$$

where $\mathscr{A} \in \mathscr{O}_{\boldsymbol{f}}^*$ is a bounded analytic function on $U_{\boldsymbol{f}}$ such that

$$\mathscr{A}(2) \in \mathbf{Q}(\boldsymbol{g}_1^\sharp, \boldsymbol{h}_1^\sharp)^*,$$

$\mathbf{Q}(\boldsymbol{g}_1^\sharp, \boldsymbol{h}_1^\sharp)$ *being the field generated by the Fourier coefficients of $\boldsymbol{g}_1^\sharp$ and $\boldsymbol{h}_1^\sharp$.*

3.1. The Mazur–Kitagawa p-adic L-function. — Let χ be a Dirichlet character of conductor coprime to $N_f p$. For every classical point $k \in U_{\boldsymbol{f}}^{\mathrm{cl}}$ let $L(f_k^\sharp, \chi, s)$ be the Hecke L-series of $f_k^\sharp \otimes \chi$, defined as the analytic continuation of the Dirichlet series $\sum_{n\geqslant 1} \chi(n) a_n(f_k^\sharp) \cdot n^{-s}$ converging absolutely for $\Re(s) > (k+1)/2$. A result of Shimura gives *complex periods* $\Omega_\infty(f_k^\sharp)^+$ and $\Omega_\infty(f_k^\sharp)^-$ in $\mathbf{C}^*$ satisfying the following properties. One has

$$\Omega_\infty(f_k^\sharp)^+ \cdot \Omega_\infty(f_k^\sharp)^- = (f_k^\sharp, f_k^\sharp)_{N_f p^{r(k)}},$$

where $r(k)$ is equal to one if $k = 2$ and to zero otherwise. Upon setting

$$\Omega_\infty(f_k^\sharp, \chi) = \Omega_\infty(f_k^\sharp)^{\mathrm{sign}(\chi)}$$

($\mathrm{sign}(\chi)$ being the sign of $\chi(-1)$) the quantity

$$L(f_k^\sharp, \chi, k/2)_{\mathrm{alg}} = \frac{(k/2-1)! \cdot \mathfrak{g}(\bar{\chi}) \cdot L(f_k^\sharp, \chi, k/2)}{(-2\pi i)^{k/2-1} \cdot \Omega_\infty(f_k^\sharp, \chi)} \in \mathbf{Q}(f_k^\sharp, \chi) \tag{39}$$

belongs to the number field $\mathbf{Q}(f_k^\sharp, \chi)$ generated over $\mathbf{Q}$ by the Fourier coefficients of $f_k^\sharp$ and the values of χ. Here $\mathfrak{g}(\bar{\chi}) = \sum_{a \in (\mathbf{Z}/c_\chi \mathbf{Z})^*} \bar{\chi}(a) \cdot \zeta_{c_\chi}^a$ is the Gauß sum of $\bar{\chi} = \chi^{-1}$, where c_χ is the conductor of χ and $\zeta_{c_\chi} = e^{2\pi i/c_\chi}$. One calls $L(f_k^\sharp, \chi, k/2)_{\mathrm{alg}}$ the *algebraic part* of the central critical value $L(f_k^\sharp, \chi, k/2)$.

According to a result of Mazur and Kitagawa (cf. [**Kit94, GS93, BD07**]) the algebraic central values $L(f_k^\sharp, \chi, k/2)_{\mathrm{alg}}$, defined for $k \in U_{\boldsymbol{f}}^{\mathrm{cl}}$, can be interpolated by an analytic function

$$L_p(\boldsymbol{f}^\sharp, \chi) \in \mathscr{O}_{\boldsymbol{f}},$$

which we call the *Mazur–Kitagawa* p-adic L-function of $(\boldsymbol{f}^\sharp, \chi)$. More precisely, up to shrinking $U_{\boldsymbol{f}}$ if necessary, there exist for every $k \in U_{\boldsymbol{f}}^{\mathrm{cl}}$ non-zero p-adic periods

$$\lambda_k^+, \lambda_k^- \in \bar{\mathbf{Q}}_p^*, \quad \text{with} \quad \lambda_2^\pm = 1,$$

such that
(40)

$$L_p(\boldsymbol{f}^\sharp, \chi)(k) = \lambda_k^{\mathrm{sign}(\chi)} \cdot \left(1 - \frac{p^{k/2-1}\chi(p)}{a_p(k)}\right) \cdot \left(1 - \varepsilon_k(p) \cdot \frac{p^{k/2-1}\bar{\chi}(p)}{a_p(k)}\right) \cdot L(f_k^\sharp, \chi, k/2)_{\mathrm{alg}},$$

where $\varepsilon_k(p) = 0$ if $k = 2$ (i.e. if $f_k^\sharp$ is p-new) and $\varepsilon_k(p) = 1$ otherwise (i.e. if $f_k^\sharp$ is p-old).

Remark 3.2. — 1. The p-adic L-function $L_p(\boldsymbol{f}^\sharp, \chi)$ is the restriction to the *central critical line* $s = k/2$ of a two-variable p-adic L-function

$$L_p^{\mathrm{MK}}(\boldsymbol{f}^\sharp, \chi) = L_p^{\mathrm{MK}}(\boldsymbol{f}^\sharp, \chi)(\boldsymbol{k}, \boldsymbol{j}) \in \mathscr{O}_{\boldsymbol{f}} \hat{\otimes} \mathscr{O}_{\mathrm{cyc}}$$

of the weight variable $\boldsymbol{k} \in U_{\boldsymbol{f}}$ and *cyclotomic* variable $\boldsymbol{j}$ (cf. [**BSV20**, Section 7.1]). For every classical point $k \in U_{\boldsymbol{f}}^{\mathrm{cl}}$ one has

$$L_p^{\mathrm{MK}}(\boldsymbol{f}^\sharp, \chi)(k, \boldsymbol{j}) = \lambda_k^{\mathrm{sign}(\chi)} \cdot L_p(f_k^\sharp, \chi)(\boldsymbol{j}),$$

where $L_p(f_k^\sharp, \chi) = L_p(f_k^\sharp, \chi)(\boldsymbol{j}) \in \mathscr{O}_{\mathrm{cyc}}$ is the cyclotomic p-adic L-function of $f_k^\sharp \otimes \chi$ (cf. [**MTT86**]) defined as the Mellin transform of a measure on $\mathbf{Z}_p^* \times (\mathbf{Z}/c_\chi \mathbf{Z})^*$ associated to the sign(χ)-modular symbol attached to $f_k^\sharp$. In order to construct $L_p^{\mathrm{MK}}(\boldsymbol{f}^\sharp, \chi)$ one interpolates these modular symbols, and the p-adic periods $\lambda_k^\pm$ are the *error terms* arising from the p-adic interpolation.

2. If $k = 2$ and

$$\chi(p) = a_p(2)$$

(with $a_p(2) = a_p(A) = \pm 1$), the Euler factor $1 - \frac{p^{k/2-1}\chi(p)}{a_p(k)}$ which appears in Equation (40) vanishes. In this *exceptional zero* situation (cf. [**MTT86**]) $L_p(\boldsymbol{f}^\sharp, \chi)$ vanishes at $k = 2$ independently of whether the complex L-series $L(f, \chi, s)$ vanishes at $s = 1$ or not.

3.2. Hida–Rankin p-adic L-functions attached to quadratic fields. — Let $K/\mathbf{Q}$ be a quadratic field of discriminant coprime to $N_f p$, satisfying the Heegner hypothesis given in Assumption 1.1(1). To lighten notations, assume in the real quadratic case that $N_f^- = 1$ (so that one works with forms on GL_2).

The Hida–Rankin p-adic L-function attached to the pair $(\boldsymbol{f}^\sharp, \varrho)$ ($\varrho = \varphi$ or ψ) introduced in [**BD07**] and [**BD09**] is an analytic function

$$L_p(\boldsymbol{f}^\sharp/K, \varrho) \in \mathscr{O}_{\boldsymbol{f}}$$

satisfying the following interpolation property. For every classical point $k \in U_{\boldsymbol{f}}^{\mathrm{cl}}$

$$L_p(\boldsymbol{f}^\sharp/K, \varrho)(k) = \Omega_p(f_k^\sharp, \varrho)^2 \left(1 - \frac{p^{k-2}}{a_p(k)^2}\right)^2 L(f_k^\sharp/K, \varrho, k/2)_{\mathrm{alg}}, \tag{41}$$

where the *algebraic part* of $L(f_k^\sharp/K, \varrho, k/2)$ is defined by

$$L(f_k^\sharp/K, \varrho, k/2)_{\mathrm{alg}} = \frac{(k/2-1)!^2 \cdot d_K^{(k-1)/2}}{(2\pi i)^{k-2} \cdot \Omega_\infty(f_k^\sharp, \varrho)} \cdot L(f_k^\sharp/K, \varrho, k/2) \in L. \tag{42}$$

Here $L(f_k^\sharp/K, \varrho, s) = L(f_k^\sharp \otimes \vartheta_\varrho, s)$ is the Rankin–Selberg convolution of $f_k^\sharp$ and the weight-one theta series ϑ_ϱ associated to ϱ, and the complex and p-adic *periods* $\Omega_\infty(f_k^\sharp, \varrho)$ and $\Omega_p(f_k^\sharp, \varrho)$ are defined as follows.

When K is real quadratic, then

$$\Omega_\infty(f_k^\sharp, \varrho) = \left(\Omega_\infty(f_k^\sharp)^{\mathrm{sign}(\varrho)}\right)^2, \qquad \Omega_p(f_k^\sharp, \varrho) = \left(\lambda_k^{\mathrm{sign}(\varrho)}\right)^2.$$

When $K/\mathbf{Q}$ is imaginary quadratic, one sets

$$\Omega_\infty(f_k^\sharp, \varrho) = (f_k^\sharp, f_k^\sharp)_{N_f p^{r(k)}},$$

where $r(k) = 1$ if $k = 2$ and $r(k) = 0$ otherwise.

We finally recall the definition of the p-adic periods $\Omega_p(f_k^\sharp, \varrho)$ in the imaginary case. With the notations of Assumption 1.1 let $B_{/\mathbf{Q}}$ be the definite quaternion algebra with discriminant $N_f^- \infty$. As explained in Section 2 of [BD07] the form $f_k^\sharp$ gives rise, via the Jacquet–Langlands correspondence, to a weight-k eigenform ϕ_k on $\hat{B}^*$ of level $\Sigma_0(pN^+, N^-) \subset \hat{B}^*$, having the same system of Hecke eigenvalues as $f_k^\sharp$. This form is unique up to multiplication by a non-zero scalar. As in loc. cit., for every $k > 2$ (resp., $k = 2$) normalise ϕ_k by requiring that its Petersson norm is equal to 1 (resp., that it takes values in $\mathbf{Z}$). This characterises ϕ_k up to sign for $k > 2$. According to Theorem 2.5 of loc. cit. (up to shrinking $U_{\boldsymbol{f}}$ if necessary) there exists an $\mathscr{O}_{\boldsymbol{f}}$-adic family ϕ_∞ of eigenforms on $\hat{B}^*$ whose specialisation at a classical point $k \in U^{\mathrm{cl}}$ is equal to $\lambda_B(k) \cdot \phi_k$, for some

$$\lambda_B(k) \in L^* \quad \text{with} \quad \lambda_B(2) = 1$$

(see Section 2 of loc. cit. for the details). The definition of $L_p(\boldsymbol{f}^\sharp/K)$ given in Section 3 of loc. cit. depends on ϕ_∞, and one sets $\Omega_p(f_k^\sharp, \varrho) = \lambda_B(k)$. In particular $\Omega_p(f, \varrho) = 1$.

3.3. Proof of Theorem 3.1. — The decomposition of Galois representations

$$V(g) \otimes_L V(h) = \mathrm{Ind}_{\mathbf{Q}}^K(\nu_g) \otimes_L \mathrm{Ind}_{\mathbf{Q}}^K(\nu_h) = \mathrm{Ind}_{\mathbf{Q}}^K(\varphi) \oplus \mathrm{Ind}_{\mathbf{Q}}^K(\psi)$$

yields for every $k \in U_{\boldsymbol{f}}^{\mathrm{cl}}$ a factorisation of complex L-functions

$$L(f_k^\sharp \otimes g \otimes h, s) = L(f_k^\sharp/K, \varphi, s) \cdot L(f_k^\sharp/K, \psi, s). \tag{43}$$

The imaginary case. Assume that $K/\mathbf{Q}$ is imaginary quadratic and let k be a classical point in $U_{\boldsymbol{f}}^{\mathrm{cl}} \cap \mathbf{Z}_{>2}$. Then the complex period $\Omega_\infty(f_k^\sharp, \varrho)$ is equal to the

Petersson norm $\langle f_k^\sharp, f_k^\sharp\rangle_{N_f p^{r(k)}}$, hence Equations (42), (43) and [**BSV20**, (133)], give

(44)
$$\frac{\Gamma(k,1,1)}{2^{\alpha(k,1,1)}}\cdot\frac{L(f_k^\sharp\otimes g\otimes h,k/2)}{\pi^{2(k-2)}\cdot(f_k^\sharp,f_k^\sharp)_{N_f}^2}=\frac{2^{2k-4-\alpha(k,1,1)}}{d_K^{k-1}}\cdot L(f_k^\sharp/K,\varphi,k/2)_{\mathrm{alg}}\cdot L(f_k^\sharp/K,\psi,k/2)_{\mathrm{alg}}.$$

With notations as in [**BSV20**, Section 6], one finds from Equations (1) and (2)

$$\mathcal{E}(\boldsymbol{f}_k^\sharp,\boldsymbol{g}_1^\sharp,\boldsymbol{h}_1^\sharp)=\left(1-\frac{p^{k/2-1}}{a_p(k)}\right)^2\left(1+\frac{p^{k/2-1}}{a_p(k)}\right)^2=\left(1-\frac{p^{k-2}}{a_p(k)^2}\right)^2. \tag{45}$$

Since $\Omega_p(f_k^\sharp,\varrho)$ is equal to the quaternionic period $\lambda_B(k)$ for both $\varrho=\varphi$ and $\varrho=\psi$ (cf. the discussion following Equation (41)), Equations (42), (41), (44), (45) and [**BSV20**, (132), (135)] yield

$$L_p(\boldsymbol{f}^\sharp,\boldsymbol{g}_1^\sharp,\boldsymbol{h}_1^\sharp)(k)=\mathscr{A}_{B,k}^2\cdot\mathscr{A}_k^o\cdot L_p(\boldsymbol{f}^\sharp/K,\varphi)(k)\cdot L_p(\boldsymbol{f}^\sharp/K,\psi)(k) \tag{46}$$

for every $k\in U_{\boldsymbol{f}}^{\mathrm{cl}}\cap\mathbf{Z}_{>2}$, where one writes

$$\mathscr{A}_{B,k}=\frac{1}{\lambda_B(k)^2\cdot\mathcal{E}_0(\boldsymbol{f}_k^\sharp)\cdot\mathcal{E}_1(\boldsymbol{f}_k^\sharp)}\quad\text{and}\quad\mathscr{A}_k^o=\frac{2^{2k-4-\alpha(k,1,1)}}{d_K^{k-1}}\prod_{v|N}\mathrm{Loc}_v.$$

Since Loc_v is a non-zero constant in $\mathbf{Q}^*$ for every $v|N$, and p does not divide d_K, the values $\mathscr{A}_k^o\in\mathbf{Q}^*$ for $k\in U_{\boldsymbol{f}}^{\mathrm{cl}}$ are interpolated by a unit in $\mathscr{O}_{\boldsymbol{f}}^*$. Equation (46) then reduces the proof of Theorem 3.1 to the following statement.

Lemma 3.3. — *There exists a bounded analytic function $\mathscr{A}_B\in\mathscr{O}_{\boldsymbol{f}}$ satisfying the following properties.*

1. *$\mathscr{A}_B(k)=\mathscr{A}_{B,k}$ for infinitely many classical points $k\in U_{\boldsymbol{f}}^{\mathrm{cl}}$.*
2. *$\mathscr{A}_B(2)$ is a non-zero element in $\mathbf{Q}^*$.*

We defer the proof of Lemma 3.3 to Section 3.4 below.

The real case. Assume that K is real quadratic and let $k\in U_{\boldsymbol{f}}^{\mathrm{cl}}\cap\mathbf{Z}_{>2}$. Define the quantity

$$\mathscr{A}_{\mathrm{GL}_2,k}=\frac{1}{\lambda_k^+\cdot\lambda_k^-\cdot\mathcal{E}_0(\boldsymbol{f}_k)\cdot\mathcal{E}_1(\boldsymbol{f}_k)}. \tag{47}$$

By a similar argument as in the imaginary case, one reduces the proof of Theorem 3.1 to the following statement.

Lemma 3.4. — *There exists a bounded analytic function $\mathscr{A}_{\mathrm{GL}_2}\in\mathscr{O}_{\boldsymbol{f}}$ satisfying the following properties.*

1. *$\mathscr{A}_{\mathrm{GL}_2}(k)=\mathscr{A}_{\mathrm{GL}_2,k}$ for infinitely many classical points $k\in U_{\boldsymbol{f}}^{\mathrm{cl}}$.*
2. *$\mathscr{A}_{\mathrm{GL}_2}(2)$ is a non-zero element in $\mathbf{Q}^*$.*

3.4. Proofs of Lemma 3.3 and Lemma 3.4. — According to Proposition 5.2 of [**BD07**] there exists an analytic function $\mathscr{A}_{\mathrm{GL}_2}^B\in\mathscr{O}_{\boldsymbol{f}}$ (denoted η in loc. cit.) such that, for every $k\in U_{\boldsymbol{f}}^{\mathrm{cl}}\cap\mathbf{Z}_{>2}$

$$\mathscr{A}_{\mathrm{GL}_2}^B(k)=\frac{\lambda_B(k)^2}{\lambda_k^+\cdot\lambda_k^-}=\frac{\mathscr{A}_{\mathrm{GL}_2,k}}{\mathscr{A}_{B,k}}\quad\text{and}\quad\mathscr{A}_{\mathrm{GL}_2}^B(2)\in\mathbf{Q}^*.$$

In particular, after shrinking $U_{\boldsymbol{f}}$ if necessary, the analytic function $\mathscr{A}^B_{\mathrm{GL}_2}$ is a unit in $\mathscr{O}_{\boldsymbol{f}}$. This implies that Lemma 3.3 follows from Lemma 3.4, hence to conclude the proof of Theorem 3.1 it is sufficient to prove the latter.

To prove Lemma 3.4 we consider triple product p-adic L-functions associated to $\boldsymbol{f}^\sharp$ and two weight one Eisenstein series attached to the characters which appear in the following lemma.

Lemma 3.5. — *There exists two Dirichlet characters χ and ψ satisfying the following properties.*

1. *The conductors c_χ and c_ψ of χ and ψ are coprime to each other and coprime to $N_f p$.*
2. *χ is* even *and $\chi(p)$ is different from ± 1.*
3. *ψ is* odd *and $\psi(p) = -a_p(f)$.*
4. *Both $L(f,\chi,s)$ and $L(f,\psi,s)$ do not vanish at $s=1$.*

Proof. — Let ℓ be a prime which does not divide $N_f p$. According to the main result of [**Roh84**] there exists $n_o \in \mathbf{N}$ such that $L(f,\chi,1) \neq 0$ for every primitive Dirichlet character χ of $\mathrm{Gal}(\mathbf{Q}(\mu_{\ell^n})^+/\mathbf{Q}) = (\mathbf{Z}/\ell^n\mathbf{Z})^*/\{\pm 1\}$ with $n \geqslant n_o$, where $\mathbf{Q}(\mu_{\ell^n})^+$ is the maximal totally real subfield of the ℓ^n-th cyclotomic extension of $\mathbf{Q}$. If $n \geqslant n_o$ is such that $\ell^n \nmid p^4 - 1$, this shows that there exists a character χ such that

(a) the conductor $c_\chi = \ell^n$ of χ is coprime to $N_f p$.
(b) $\chi(-1) = +1$ and $\chi(p) \neq \pm 1$.
(c) $L(f,\chi,s)$ does not vanish at $s=1$.

Let q be a fixed prime which divides N_f exactly, whose existence is guaranteed by Assumption 1.1. For every quadratic character σ denote by $\mathrm{sign}(f \otimes \sigma)$ the sign at $s=1$ in the functional equation satisfied by the Hecke L-function $L(f,\sigma,s)$. Choose any quadratic Dirichlet character ψ_1 satisfying the following properties.

(d) The conductor $c(\psi_1)$ of ψ_1 is coprime with $\ell \cdot N_f p$.
(e) $\psi_1(-1) = +1$ and $\psi_1(t) = +1$ for every prime t which divides N_f/q.
(f) $\psi_1(p) = -a_p(f)$ and $\psi_1(q) = a_p(f) \cdot \mathrm{sign}(f)$.

One has (cf. Theorem 3.66 of [**Shi71**])

$$\mathrm{sign}(f \otimes \psi_1) = \mathrm{sign}(f) \cdot \psi_1(-N_f p) = -1,$$

hence the main result of [**BFH90**] shows that there exists a quadratic Dirichlet character ψ_2 such that

(g) the conductor of ψ_2 is coprime to $\ell \cdot c(\psi_2) \cdot N_f p$.
(h) $\psi_2(-1) = -1$ and $\psi_2(t) = +1$ for every prime divisor t of $N_f p$.
(i) $L(f, \psi_1 \cdot \psi_2, s)$ does not vanish at $s=1$.

According to (a)–(i) the characters χ and $\psi = \psi_1 \cdot \psi_2$ satisfy the required properties. □

Fix two characters χ and ψ satisfying the conclusions of the previous lemma, and set $N = N_f c_\chi c_\psi$ and

$$\xi = \chi^{-1} \cdot \psi^{-1}.$$

Since χ, ψ and ξ are non-trivial and ξ is odd, one can consider the weight one Eisenstein series

$$E(\chi,\psi) = \sum_{n=1}^{\infty} \sigma(\chi,\psi)(n) \cdot q^n \in M_1(N, \xi^{-1})$$

and

$$E(\xi) = E(\mathbf{1},\xi) = \frac{L(\xi,0)}{2} + \sum_{n\geqslant 1} \sigma(\mathbf{1},\xi)\cdot q^n \in M_1(N,\xi),$$

where $\sigma(\alpha,\beta)(n) = \sum_{d|n} \alpha(n/d)\cdot\beta(d)$ for every Dirichlet characters α and β, and $\mathbf{1}$ is the trivial character. Following Section 3 of [BD14], for every classical point $k \in U_{\boldsymbol{f}}^{\mathrm{cl}}$ define

$$L_p(\boldsymbol{f}_k^\sharp, E(\chi,\psi)) = \frac{\left(\boldsymbol{f}_k^\sharp, e_{\mathrm{ord}}\left(d^{k/2-1}\breve{E}(\xi)\times\breve{E}(\chi,\psi)\right)\right)_{Np}}{(\boldsymbol{f}_k^\sharp,\boldsymbol{f}_k^\sharp)_{Np}}, \tag{48}$$

where $\breve{E}(\xi) = E(\xi)^{[p]} \in \mathbf{M}_1(N,\xi)$ and $\breve{E}(\chi,\psi) = E(\chi,\psi)^{[p]} \in \mathbf{M}_1(N,\xi^{-1})$ are the p-depletions of $E(\xi)$ and $E(\chi,\psi)$ (cf. [BSV20, Section 3.1]). The article [BD14] shows that the function which to $k \in U_{\boldsymbol{f}}^{\mathrm{cl}}$ associates $L_p(\boldsymbol{f}_k^\sharp, E(\chi,\psi))$ extends to an analytic function

$$L_p(\boldsymbol{f}^\sharp, E(\chi,\psi)) \in \mathscr{O}_{\boldsymbol{f}}.$$

(The notation is justified by the following lemma, cf. Remark 3.7.) For all $k \in U_{\boldsymbol{f}}^{\mathrm{cl}}$ define

$$C_{\chi,\psi}(k) = \frac{-iN_f}{2^{k-2}\cdot\chi(c_\psi)\cdot\psi(c_\chi)\cdot[\Gamma_1(N_f):\Gamma_1(N)]}.$$

For $\cdot = \chi,\psi$ Section 3.1 associates to $(\boldsymbol{f}^\sharp,\cdot)$ the Mazur–Kitagawa p-adic L-function $L_p(\boldsymbol{f}^\sharp,\cdot) \in \mathscr{O}_{\boldsymbol{f}}$.

Lemma 3.6. — 1. *Let $\mathbf{Q}(\chi,\psi)$ be the field generated over $\mathbf{Q}$ by the values of χ and ψ. Then*

$$L_p(\boldsymbol{f}^\sharp, E(\chi,\psi))(2) = (p+1)\cdot C_{\chi,\psi}(2)\cdot L_p(\boldsymbol{f}^\sharp,\chi)(2)\cdot L_p(\boldsymbol{f}^\sharp,\psi)(2) \in \mathbf{Q}(\chi,\psi)^*.$$

In particular the p-adic L-function $L_p(\boldsymbol{f}^\sharp, E(\chi,\psi))$ does not vanish at $k=2$.

2. *(cf. [BD14]) For every classical point $k \in U_{\boldsymbol{f}}^{\mathrm{cl}}$ (strictly) greater than 2 one has*

$$L_p(\boldsymbol{f}^\sharp, E(\chi,\psi))(k) = \mathscr{A}_{\mathrm{GL}_2,k}\cdot C_{\chi,\psi}(k)\cdot L_p(\boldsymbol{f}^\sharp,\chi)(k)\cdot L_p(\boldsymbol{f}^\sharp,\psi)(k), \tag{49}$$

Proof. — 1. Write for simplicity $g = E(\xi)$ and $h = E(\chi,\psi)$, and consider the p-stabilisations

$$g_\alpha(q) = g(q) - \xi(p)\cdot g(q^p), \quad g_\beta(q) = g(q) - g(q^p) \quad\text{and}\quad h_\alpha(q) = h(q) - \psi(p)\cdot h(q^p).$$

Then f (resp., g_α, g_β, h_α) is an eigenvector for the U_p-operator with eigenvalue $\alpha_f = a_p(2) = \pm 1$ (resp., 1, $\xi(p)$, $\chi(p)$), hence Lemma 3.5 and the same computations as in the proof of [DR14, Lemma 4.10] show that

$$2\cdot(f, g_\beta\cdot h_\alpha)_{Np} = (1-\chi(p)/a_p(2))\cdot(f, g_\alpha\cdot h_\alpha)_{Np}.$$

As $\xi(p) \neq 1$ by Lemma 3.5, one can write $g = (g_\alpha - \xi(p) \cdot g_\beta)/(1 - \xi(p))$, which together with the previous equation and a direct computation gives the identity

$$L_p(\boldsymbol{f}^\sharp, E(\chi,\psi))(2) = 2\left(1 - \frac{\chi(p)}{a_p(2)}\right) \cdot \frac{(f, g\cdot h_\alpha)_{Np}}{(f,f)_{Np}}. \tag{50}$$

The L-series of the forms f and h_α admit Euler product expansions, hence the Rankin method (see the argument leading to Equation (18) of [BD14], or [Shi76, Theorem 2 and Lemma 1]) gives

$$(f, g\cdot h_\alpha)_{Np} = -i\mathfrak{g}(\xi) N_f p \cdot L(f\otimes h_\alpha, 1), \tag{51}$$

where $\mathfrak{g}(\cdot)$ is the Gauß sum of the character $\cdot$. (Note that $(\cdot,\cdot)_{Np}$ equals $8\pi^2$ times the Petersson product defined in Equation 9 of [BD14].) Since the characters χ and ψ have opposite parity, one has

$$\Omega_\infty(f,\chi)\cdot\Omega_\infty(f,\psi) = (f,f)_{N_f p} = [\Gamma_1(N_f):\Gamma_1(N)]^{-1}\cdot (f,f)_{Np}. \tag{52}$$

Moreover a direct comparison of Euler factors (cf. [Shi76, Lemma 1]) and Lemma 3.5 give

$$L(f\otimes h_\alpha, 1) = \left(1 - \frac{a_p(2)\psi(p)}{p}\right) L(f\otimes h, 1) = \left(1+\frac{1}{p}\right) L(f,\chi,1)\cdot L(f,\psi,1). \tag{53}$$

As $\mathfrak{g}(\xi) = \mathfrak{g}(\chi^{-1})\cdot\mathfrak{g}(\psi^{-1})\cdot\chi^{-1}(c_\psi)\psi^{-1}(c_\chi)$ (since $(c_\chi, c_\psi) = 1$), the statement is a direct consequence of Equations (39)–(40), Equations (50)–(53) and Lemma 3.5.

2. This is proved in Proposition 3.3 of [BD14]. Since the setting of loc. cit. is slightly different from ours, for the convenience of the reader we briefly review the argument. Equations (35) and (41) and Proposition 3.2 of [BD14], together with Proposition 4.6 of [DR14], show that for every classical point $k > 2$ one has

$$L_p(\boldsymbol{f}^\sharp, E(\chi,\psi))(k) = \frac{\mathcal{E}(\boldsymbol{f}_k^\sharp,\chi,\psi)}{\mathcal{E}_0(\boldsymbol{f}_k^\sharp)\cdot\mathcal{E}_1(\boldsymbol{f}_k^\sharp)} \cdot \frac{\left(f_k^\sharp, \delta^{k/2-1}E(\xi)\cdot E(\chi,\psi)\right)_N}{(f_k^\sharp, f_k^\sharp)_N},$$

where

$$\delta^{k/2-1} : M_1(N,\xi) \longrightarrow M_{k-1}^{\mathrm{an}}(N,\xi)$$

is the $(k/2-1)$-th iterate of the Shimura–Maaß derivative operator. Here $\mathcal{E}_0(\boldsymbol{f}_k^\sharp)$ and $\mathcal{E}_1(\boldsymbol{f}_k^\sharp)$ are as in Equation [BSV20, (135)], and

$$\mathcal{E}(\boldsymbol{f}_k^\sharp,\chi,\psi) = \left(1 - \frac{p^{k/2-1}\chi(p)}{a_p(k)}\right)\left(1 - \frac{p^{k/2-1}\bar{\chi}(p)}{a_p(k)}\right)\left(1 - \frac{p^{k/2-1}\psi(p)}{a_p(k)}\right)^2.$$

(Recall that $\psi = \psi^{-1}$ is a quadratic character, cf. Lemma 3.5, and that $\mathcal{E}_i(\boldsymbol{f}_k^\sharp)$ is non-zero for $k > 2$.) The Rankin method (see Equations (18) and (19) of [BD14]) yields

$$\left(f_k^\sharp, \delta^{k/2-1}E(\xi)\cdot E(\chi,\psi)\right)_N = \frac{-iN_f\mathfrak{g}(\xi)\cdot(k/2-1)!^2}{2^{k-2}\cdot(-2\pi i)^{k-2}} \cdot L(f_k^\sharp,\chi,k/2)\cdot L(f_k^\sharp,\psi,k/2).$$

As in the proof of Part 1 the statement follows easily from the definitions and the previous three equations. □

Since the analytic functions $L_p(\boldsymbol{f}^\sharp, E(\chi,\psi))$, $L_p(\boldsymbol{f}^\sharp,\chi)$ and $L_p(\boldsymbol{f}^\sharp,\psi)$ do not vanish at $k=2$ by Lemma 3.6(1), and since $C_{\chi,\psi}(k)$ is clearly an invertible element of $\mathscr{O}_{\boldsymbol{f}}$, Lemma 3.6(2) implies that the values $\mathscr{A}_{\mathrm{GL}_2,k}$, defined for $k \in U^{\mathrm{cl}} \cap \mathbf{Z}_{>2}$, are interpolated by an analytic function $\mathscr{A}_{\mathrm{GL}_2}(k)$ which does not vanish at $k=2$. In addition, the explicit formula for the value of $L_p(\boldsymbol{f}^\sharp, E(\chi,\psi))$ at $k=2$ displayed in Lemma 3.6(1) gives

$$\mathscr{A}_{\mathrm{GL}_2}(2) = p+1.$$

This concludes the proof of Lemma 3.4, and with it the proofs of Lemma 3.3 and Theorem 3.1.

Remark 3.7. — 1. The previous lemma (or better its proof) shows that $L_p(\boldsymbol{f}^\sharp, E(\chi,\psi))$ can be though of as a p-adic Rankin–Selberg convolution, which interpolates the critical values $L(f_k^\sharp \otimes E(\chi,\psi), k/2)$ of the convolution of $f_k^\sharp$ with $E(\chi,\psi)$. One can also think of $L_p(\boldsymbol{f}^\sharp, E(\chi,\psi)) = \mathscr{L}_p(\boldsymbol{f}^\sharp, E(\xi), E(\chi,\psi))$ as a square-root triple-product p-adic L-function (cf. Equations (48) and [**BSV20**, (55)]), whose square interpolates the complex central values $L(f_k^\sharp \otimes E(\xi) \otimes E(\chi,\psi), k/2)$.

2. Note that the Euler factor $\mathcal{E}_1(\boldsymbol{f}_k^\sharp) = 1 - \frac{p^{k-2}}{a_p(k)^2}$ vanishes at $k=2$, as a manifestation of the presence of an exceptional zero for $L_p(\boldsymbol{f}^\sharp, s)$ and $L_p(\boldsymbol{f}^\sharp, \boldsymbol{g}_1^\sharp, \boldsymbol{h}_1^\sharp)$ in the sense of [**MTT86**] (cf. Remark 3.2(2)).

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Massimo Bertolini, Marco Adamo Seveso, and Rodolfo Venerucci, Massimo Bertolini: Essen, Germany • *E-mail :* massimo.bertolini@uni-due.de • Marco Seveso: Milano, Italy • *E-mail :* seveso.marco@gmail.com • Rodolfo Venerucci: Essen, Germany *E-mail :* rodolfo.venerucci@uni-due.de