DIAGONAL CYCLES AND EULER SYSTEMS I: A p-ADIC GROSS-ZAGIER FORMULA

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Abstract. This article is the first in a series devoted to studying *generalised Gross-Kudla-Schoen diagonal cycles* in the product of three Kuga-Sato varieties and the Euler system properties of the associated Selmer classes, with special emphasis on their application to the Birch–Swinnerton-Dyer conjecture and the theory of Stark-Heegner points. The basis for the entire study is a p-adic formula of Gross-Zagier type which relates the images of these diagonal cycles under the p-adic Abel-Jacobi map to special values of certain p-adic L-functions attached to the Garrett-Rankin triple convolution of three Hida families of modular forms. The main goal of this article is to describe and prove this formula.

Cet article est le premier d'une série consacrée aux cycles de Gross-Kudla-Schoen généralisés *appartenant aux groupes de Chow de produits de trois variétés de Kuga-Sato, et aux* systèmes d'Euler *qui leur sont associés. La série au complet repose sur une variante p-adique de la formule de Gross-Zagier qui relie l'image des cycles de Gross-Kudla-Schoen par l'application d'Abel-Jacobi p-adique aux valeurs spéciales de certaines fonctions L p-adiques attachées à la convolution de Garrett-Rankin de trois familles de Hida de formes modulaires cuspidales. L'objectif principal de cet article est de décrire et de démontrer cette variante.*

MSC: 11F12, 11G05, 11G35, 11G40.

Keywords: Gross-Kudla-Schoen cycle, Garrett-Rankin p-adic L-function, p-adic Abel-Jacobi map, Chow group, Coleman integration.

Contents

1. Introduction

This article is the first in a series devoted to studying *generalised diagonal cycles* in the product of three Kuga-Sato varieties and the Euler system properties of the associated Selmer classes, with special emphasis on their application to the Birch–Swinnerton-Dyer conjecture and the theory of Stark-Heegner points. The basis for the entire study is a p-adic Gross-Zagier formula relating

- the image under the p-adic Abel-Jacobi map of certain *generalised Gross-Kudla-Schoen cycles* in the product of three Kuga-Sato varieties, to
- the special value of the p-adic L-function of [HaTi] attached to the Garrett-Rankin triple convolution of three Hida families of modular forms, at a point lying outside its region of interpolation.

In order to precisely state the main result, let

$$f = \sum a_n(f)q^n \in S_k(N_f, \chi_f),$$

$$g = \sum a_n(g)q^n \in S_\ell(N_g, \chi_g),$$

$$h = \sum a_n(h)q^n \in S_m(N_h, \chi_h)$$

be three normalized primitive cuspidal eigenforms of weights $k, \ell, m \geq 2$, levels $N_f, N_g, N_h \geq 1$, and Nebentypus characters χ_f, χ_g, and χ_h, respectively. Let $N := \mathrm{lcm}(N_f, N_g, N_h)$ and assume that

$$\chi_f \cdot \chi_g \cdot \chi_h = 1,$$

so that in particular $k + \ell + m$ is even.

The triple (k, ℓ, m) is said to be *balanced* if the largest weight is strictly smaller than the sum of the other two. A triple of weights which is not balanced will be called *unbalanced*, and the largest weight in an unbalanced triple will be referred to as the *dominant weight*.

Section 4.1 recalls the definition of the Garrett-Rankin L-function $L(f, g, h; s)$ attached to the triple tensor product

$$V_p(f, g, h) := V_p(f) \otimes V_p(g) \otimes V_p(h)$$

of the (compatible systems of) p-adic Galois representations $V_p(f)$, $V_p(g)$ and $V_p(h)$ attached to f, g and h respectively. This L-function satisfies a functional equation relating its values at s and $k+\ell+m-2-s$. In particular, the parity of the order of vanishing of $L(f, g, h; s)$ at the central critical point $c := \frac{k+\ell+m-2}{2}$ is controlled by the sign $\varepsilon \in \{\pm 1\}$ in this functional equation, a quantity that can be expressed as a product $\varepsilon = \prod_{v|N\infty} \varepsilon_v$, $\varepsilon_v \in \{\pm 1\}$, of local root numbers indexed by the places dividing $N\infty$. The following hypothesis is assumed throughout:

H: The local root numbers ε_v at all the finite primes $v|N$ are equal to $+1$.

This assumption holds in a broad collection of settings of arithmetic interest. For instance, it is satisfied in either of the following two cases:

- $\gcd(N_f, N_g, N_h) = 1$, or,
- $N = N_f = N_g = N_h$ is square-free and $a_v(f)a_v(g)a_v(h) = -1$ for all primes $v \mid N$.

Assumption H implies that $\varepsilon = \varepsilon_\infty$ depends only on the local sign at ∞, which in turn depends only on whether the weights of (f, g, h) are balanced or not:

$$\varepsilon = \varepsilon_\infty = \begin{cases} -1 & \text{if } (k, \ell, m) \text{ is balanced;} \\ 1 & \text{if } (k, \ell, m) \text{ is unbalanced.} \end{cases}$$

In particular, the L-function $L(f, g, h, s)$ necessarily vanishes (to odd order) at its central point c when (k, ℓ, m) is balanced.

Let $\mathcal{E}$ denote the universal generalised elliptic curve fibered over $X = X_1(N)$. For any $n \geq 0$, let $\mathcal{E}^n$ be the n-th Kuga-Sato variety over $X_1(N)$. It is an $n+1$-dimensional variety obtained by desingularising the n-fold fiber product of $\mathcal{E}$ over $X_1(N)$. (Cf. [Sc] for a more detailed account of its construction.) The p-adic Galois representation $V_p(f,g,h)$ occurs in the middle cohomology of the triple product

$$W := \mathcal{E}^{k-2} \times \mathcal{E}^{\ell-2} \times \mathcal{E}^{m-2}. \tag{1}$$

When (k,ℓ,m) is balanced and assumption H is satisfied, the conjectures of Bloch-Kato and Beilinson-Bloch predict (because of the vanishing of $L(f,g,h,c)$) that there should then exist a non-trivial cycle in the Chow group $\mathbb{Q} \otimes \mathrm{CH}^c(W)_0$ of rational equivalence classes of null-homologous cycles of codimension c on the variety W of (1). Section 3.1 introduces cycles $\Delta_{f,g,h} \in \mathbb{Q} \otimes \mathrm{CH}^c(W)_0$ which are natural candidates to fulfill these expectations, and whose construction we now briefly summarize.

Set $r = \frac{k+\ell+m-6}{2}$. As explained in §3.1, there exists an essentially unique, natural way of embedding the Kuga-Sato variety $\mathcal{E}^r$ in the variety W. Its image gives rise to an element in the Chow group $\mathrm{CH}^{r+2}(W)$ which, suitably modified, becomes homologically trivial. In this way, we obtain a cycle

$$\Delta_{k,\ell,m} \in \mathrm{CH}^{r+2}(W)_0 := \ker(\mathrm{CH}^{r+2}(W) \xrightarrow{\mathrm{cl}} H^{2r+4}_{\mathrm{dR}}(W/\mathbb{C})).$$

In the special case where $k = \ell = m = 2$, the cycle $\Delta_{2,2,2}$ is just the modified diagonal considered by Gross–Kudla [GrKu] and Gross–Schoen [GrSc].

The cycles $\Delta_{f,g,h}$ alluded to above are defined as the (f,g,h)-isotypical component of the null-homologous cycle $\Delta_{k,\ell,m}$ with respect to the action of the Hecke operators.

It is natural to conjecture that the heights of these cycles in the sense of Beilinson and Bloch are well-defined (cf. [GrKu] and [GrSc] for more details on the necessary definitions), and can be directly related to the first derivative of the triple product L-function $L(f,g,h,s)$ at the central point:

$$h(\Delta_{f,g,h}) \stackrel{?}{=} (\text{Explicit non-zero factor}) \times L'(f,g,h,r+2). \tag{2}$$

When $(k,\ell,m) = (2,2,2)$, this was predicted in [GrKu] and has recently been proved by X. Yuan, S. Zhang and W. Zhang in [YZZ].

Remark 1.1. It would be natural to relax assumption H to the weaker condition

$$H_{\mathrm{even}}\text{: The set of primes } v|N \text{ for which } \varepsilon_v = -1 \text{ is of } \textit{even cardinality}. \tag{3}$$

This is sufficient to guarantee that $\varepsilon = \varepsilon_\infty$, and can be dealt with at the cost of replacing Kuga-Sato varieties with more general objects arising from the self-fold products of certain families of abelian surfaces (or genus two curves) fibered over Shimura curves rather than classical modular curves. Hypothesis H may thus be regarded as analogous to the classical Heegner or Gross-Zagier hypothesis imposed in the study of the Rankin-Selberg L-function $L(f \otimes \theta_K, s)$ attached to a single eigenform f and the weight one theta series of an imaginary quadratic field K. Both are meant to avoid having to deal with Shimura curves associated with a quaternion division algebra, and make it possible to confine one's attention to classical modular curves. Much of our study extends to the setting of H_{even} by appealing to the work of P. Kassaei [Kas99] and R. Brasca [Br]; in our exposition we have tried to present our results in a way that suggests the modifications necessary to deal with arbitrary Shimura curves.

In this work we do not focus on (2), but rather on a p-adic analogue. Our main result relates the image of $\Delta_{f,g,h}$ under the p-adic Abel-Jacobi map

$$\mathrm{AJ}_p : \mathrm{CH}^{r+2}(W)_0(\mathbb{Q}_p) \longrightarrow \mathrm{Fil}^{r+2} H^{2r+3}_{\mathrm{dR}}(W/\mathbb{Q}_p)^\vee \tag{4}$$

to the special value of a triple product p-adic L-function attached to three Hida families of modular forms, which we now describe in more detail.

Fix an odd prime number $p \nmid N$ at which f, g and h are ordinary. Let

$$\mathbf{f} : \Omega_f \longrightarrow \mathbb{C}_p[[q]], \qquad \mathbf{g} : \Omega_g \longrightarrow \mathbb{C}_p[[q]], \quad \mathbf{h} : \Omega_h \longrightarrow \mathbb{C}_p[[q]]$$

denote the Hida families of overconvergent p-adic modular forms passing though f, g and h, respectively, as constructed in [Hi86a] and [Hi86b], and briefly reviewed in §2.6 below. The spaces Ω_f, Ω_g and Ω_h are finite rigid analytic coverings of suitable subsets of the *weight space*

$$\Omega := \mathrm{Hom}_{\mathrm{cts}}(\mathbb{Z}_p^\times, \mathbb{C}_p^\times),$$

which contains the integers $\mathbb{Z}$ as a dense subset via the natural inclusion $k \mapsto (x \mapsto x^k)$. A point $x \in \Omega_f$ is said to be *classical* if its image in Ω, denoted $\kappa(x)$, belongs to $\mathbb{Z}^{\geq 2}$, and the set of classical points in Ω_f is denoted by $\Omega_{f,\mathrm{cl}}$. Part of the requirement that $\mathbf{f}$ be a Hida family is that the formal q-series $f_x^{(p)} := \mathbf{f}(x)$ should correspond to a normalised eigenform of weight $\kappa(x)$ on $\Gamma_1(N) \cap \Gamma_0(p)$, for almost all $x \in \Omega_{f,\mathrm{cl}}$. For all but finitely many such x, the form $f_x^{(p)}$ is the ordinary p-stabilisation of a normalised eigenform on $\Gamma_1(N)$, denoted f_x.

The natural domain of definition of the triple product p-adic L-functions is the p-adic analytic space

$$\Sigma := \Omega_f \times \Omega_g \times \Omega_h.$$

Let $\Sigma_{\mathrm{cl}} := \Omega_{f,\mathrm{cl}} \times \Omega_{g,\mathrm{cl}} \times \Omega_{h,\mathrm{cl}} \subset \Sigma$ denote its subset of "classical points". This set is naturally partitioned into four disjoint subsets:

$$\begin{aligned}
\Sigma_f &= \{(x,y,z) \in \Sigma_{\mathrm{cl}}, \quad \text{such that } \kappa(x) \geq \kappa(y) + \kappa(z)\};\\
\Sigma_g &= \{(x,y,z) \in \Sigma_{\mathrm{cl}}, \quad \text{such that } \kappa(y) \geq \kappa(x) + \kappa(z)\};\\
\Sigma_h &= \{(x,y,z) \in \Sigma_{\mathrm{cl}}, \quad \text{such that } \kappa(z) \geq \kappa(x) + \kappa(y)\};\\
\Sigma_{\mathrm{bal}} &= \{(x,y,z) \in \Sigma_{\mathrm{cl}}, \quad \text{such that } (\kappa(x), \kappa(y), \kappa(z)) \text{ is balanced.}\}.
\end{aligned}$$

Section 4 exploits the strategy pioneered by Hida [Hi88b] and subsequently extended by Harris and Tilouine [HaTi] to construct *three a priori distinct* p-adic L-functions of three variables, denoted

$$\mathscr{L}_p{}^f(\mathbf{f},\mathbf{g},\mathbf{h}), \quad \mathscr{L}_p{}^g(\mathbf{f},\mathbf{g},\mathbf{h}), \quad \mathscr{L}_p{}^h(\mathbf{f},\mathbf{g},\mathbf{h}) : \Sigma \longrightarrow \mathbb{C}_p,$$

which interpolate the *square-roots* of the *central critical values* of the classical L-function $L(f_x, g_y, h_z, s)$, as (x,y,z) ranges over Σ_f, Σ_g, and Σ_h respectively. The precise interpolation property defining the three p-adic L-functions is spelled out in Theorem 4.7 of Section 4.2.

Given $(x,y,z) \in \Sigma_{\mathrm{bal}}$, the Heegner assumption H can be used to show that the classical L-function $L(f_x, g_y, h_z, s)$ vanishes at its central point for reasons of sign. The *central critical derivative* $L'(f_x, g_y, h_z, \frac{\kappa(x)+\kappa(y)+\kappa(z)-2}{2})$ is then a natural object of arithmetic interest. In the p-adic realm, the three distinct p-adic avatars of the classical L-function, namely, $\mathscr{L}_p{}^f(\mathbf{f},\mathbf{g},\mathbf{h})$, $\mathscr{L}_p{}^g(\mathbf{f},\mathbf{g},\mathbf{h})$, and $\mathscr{L}_p{}^h(\mathbf{f},\mathbf{g},\mathbf{h})$, need not vanish at the balanced point (x,y,z), since this point lies outside the region of classical interpolation. The corresponding p-adic special values can be viewed as different p-adic avatars of the complex leading term, and one might expect them to encode similar information related to the motive of $V_{f_x} \otimes V_{g_y} \otimes V_{h_z}$.

Remark 1.2. One can also envisage a fourth p-adic L-function

$$L_p^{\mathrm{bal}}(\mathbf{f},\mathbf{g},\mathbf{h})(x,y,z,s) : \Omega_f \times \Omega_g \times \Omega_h \times \Omega \longrightarrow \mathbb{C}_p$$

involving a further "cyclotomic" variable s and interpolating the classical special values of $L(f_x, g_y, h_z, s)$–but *not* their square roots–in the critical range

$$(x,y,z) \in \Sigma_{\mathrm{bal}}, \qquad 1 \leq s \leq \kappa(x) + \kappa(y) + \kappa(z) - 3.$$

The construction of such an L-function is described in [BoPa] (see also the references therein). Under hypothesis H, the function $L_p^{\mathrm{bal}}(\mathbf{f},\mathbf{g},\mathbf{h})$ should vanish identically on the "central critical hyperplane" $2s = \kappa(x) + \kappa(y) + \kappa(z) - 2$, and the presence of the cyclotomic variable is therefore key to ensuring its non-triviality. This "balanced" p-adic L-function plays no role in this article.

In stating our main result, it will be convenient to assume for simplicity that $N_f = N_g = N_h$, i.e., that the three eigenforms f, g and h are new of the same level N. (This assumption, which is too restrictive for most interesting arithmetic applications, will be relaxed in the body of the text.) To any classical newform ϕ of weight $r+2$ on $\Gamma_1(N)$ which is ordinary at p, there corresponds a cohomology class

$$\omega_\phi \in \mathrm{Fil}^{r+1} H^{r+1}_{\mathrm{dR}}(\mathcal{E}^r/\bar{\mathbb{Q}}) \subset \mathrm{Fil}^{r+1} H^{r+1}_{\mathrm{dR}}(\mathcal{E}^r/\mathbb{C}_p),$$

where the inclusion is induced from our fixed embedding $\bar{\mathbb{Q}} \subset \mathbb{C}_p$. The ϕ-isotypic component of $H^{r+1}_{\mathrm{dR}}(\mathcal{E}^r/\mathbb{C}_p)$ is two-dimensional over $\mathbb{C}_p$ and (because ϕ is ordinary) it admits a one-dimensional *unit root subspace*, denoted $H^{r+1}_{\mathrm{dR}}(\mathcal{E}^r/\mathbb{C}_p)^{\text{u-r}}$, on which the Frobenius endomorphism acts as multiplication by a p-adic unit. This unit root subspace is complementary to the middle step in the Hodge filtration, and hence, there is a unique element $\eta_\phi^{\text{u-r}} \in H^{r+1}_{\mathrm{dR}}(\mathcal{E}^r/\mathbb{C}_p)^{\text{u-r}}$ satisfying

$$\langle \omega_\phi, \eta_\phi^{\text{u-r}} \rangle = 1,$$

where $\langle\ ,\ \rangle$ denotes the non-degenerate Poincaré pairing on $H^{r+1}_{\mathrm{dR}}(\mathcal{E}^r/\mathbb{C}_p)$. One thus obtains a natural basis $(\omega_\phi, \eta_\phi^{\text{u-r}})$ of the ϕ-isotypic component of $H^{r+1}_{\mathrm{dR}}(\mathcal{E}^r/\mathbb{C}_p)$, for any ordinary classical newform $\phi \in S_{r+2}(\Gamma_1(N))$.

If (f,g,h) is a triple of newforms of level N and balanced weights $(k,\ell,m) = (r_1+2, r_2+2, r_3+2)$, then

$$\begin{aligned} \eta_f^{\text{u-r}} \otimes \omega_g \otimes \omega_h &\in H^{r_1+1}_{\mathrm{dR}}(\mathcal{E}^{r_1}) \otimes \mathrm{Fil}^{r_2+1} H^{r_2+1}_{\mathrm{dR}}(\mathcal{E}^{r_2}) \otimes \mathrm{Fil}^{r_3+1} H^{r_3+1}_{\mathrm{dR}}(\mathcal{E}^{r_3}) \\ &\subset \mathrm{Fil}^{r+2}\left(H^{r_1+1}_{\mathrm{dR}}(\mathcal{E}^{r_1}) \otimes H^{r_2+1}_{\mathrm{dR}}(\mathcal{E}^{r_2}) \otimes H^{r_3+1}_{\mathrm{dR}}(\mathcal{E}^{r_3})\right) \\ &\subset \mathrm{Fil}^{r+2} H^{2r+3}_{\mathrm{dR}}(W/\bar{\mathbb{Q}}_p), \end{aligned} \tag{5}$$

where the last inclusion arises from the Künneth decomposition. In particular, the class $\eta_f^{\text{u-r}} \otimes \omega_g \otimes \omega_h$ lies in the domain of $\mathrm{AJ}_p(\Delta)$ when (k,ℓ,m) is balanced.

For any $f \in S_k(N,\chi)$, we shall always write

$$(x^2 - a_p(f)x + \chi_f(p)p^{k-1}) = (x-\alpha_f)(x-\beta_f), \qquad \text{with } \mathrm{ord}_p(\alpha_f) \le \mathrm{ord}_p(\beta_f),$$

so that in particular α_f is a p-adic unit when f is ordinary.

The main result of this article is

Theorem 1.3. *Given $(x,y,z) \in \Sigma_{\mathrm{bal}}$, let*

$$(f,g,h) := (f_x, g_y, h_z), \qquad (k,\ell,m) := (\kappa(x), \kappa(y), \kappa(z)),$$

$$c := (k+\ell+m-2)/2, \qquad k = \ell + m - 2 - 2t \quad (\textit{with } t \ge 0),$$

and let $\Delta := \Delta_{k,\ell,m}$ be the generalised diagonal cycle in $\mathcal{E}^{k-2} \times \mathcal{E}^{\ell-2} \times \mathcal{E}^{m-2}$. Then

$$\mathscr{L}_p{}^f(\mathbf{f},\mathbf{g},\mathbf{h})(x,y,z) = (-1)^t \frac{\mathcal{E}(f,g,h)}{t! \cdot \mathcal{E}_0(f)\mathcal{E}_1(f)} \times \mathrm{AJ}_p(\Delta)(\eta_f^{\text{u-r}} \otimes \omega_g \otimes \omega_h),$$

where

$$\begin{aligned} \mathcal{E}(f,g,h) &:= \left(1 - \beta_f\alpha_g\alpha_h p^{-c}\right) \times \left(1 - \beta_f\alpha_g\beta_h p^{-c}\right) \\ &\quad\times \left(1 - \beta_f\beta_g\alpha_h p^{-c}\right) \times \left(1 - \beta_f\beta_g\beta_h p^{-c}\right), \end{aligned} \tag{6}$$

$$\mathcal{E}_0(f) := (1 - \beta_f^2 \chi_f^{-1}(p) p^{1-k}), \tag{7}$$

$$\mathcal{E}_1(f) := (1 - \beta_f^2 \chi_f^{-1}(p) p^{-k}). \tag{8}$$

See Theorem 5.1 for a more general statement involving newforms of possibly different levels. Theorem 1.3 can be viewed as a p-adic analogue of the archimedean formula suggested in (2), and in fact the Euler factor $\mathcal{E}(f,g,h)$ that arises in the formula fits within the general conjectural description of the p-adic L-function of an arbitrary motive given by Panciskin in [Pan, p. 285].

One could also envisage a more direct p-adic analogue, relating the cyclotomic p-adic height of Δ_{f_x,g_y,h_z} to the derivative in the cyclotomic direction of the p-adic L-function $L_p^{\mathrm{bal}}(\mathbf{f},\mathbf{g},\mathbf{h})(x,y,z,s)$ alluded to in Remark 1.2. Such a formula, which is currently unavailable in the literature, would be closer in spirit to the archimedean formulae conjectured in [GrKu] and proved in [YZZ], and to the p-adic analogues of the Gross-Zager formula proved in [PR] and [Ne1], while Theorem 1.3 is better adapted to the arithmetic applications that the authors wish to pursue by exploiting the *Euler system properties* of p-adic families of diagonal cycles and associated explicit reciprocity laws in the spirit of Coates–Wiles and Kato–Perrin-Riou. Let us close this introduction by describing some of these applications.

I. The Euler system of diagonal cycles. Theorem 1.3 is available when f is replaced by g or h if the latter two forms are also assumed to be ordinary. The Abel-Jacobi image $\mathrm{AJ}_p(\Delta)$ therefore encodes the values of the three distinct p-adic L-functions $\mathscr{L}_p{}^f(\mathbf{f},\mathbf{g},\mathbf{h})$, $\mathscr{L}_p{}^g(\mathbf{f},\mathbf{g},\mathbf{h})$ and $\mathscr{L}_p{}^h(\mathbf{f},\mathbf{g},\mathbf{h})$ at $(x,y,z)\in\Sigma_{\mathrm{bal}}$. This suggests that these L-functions should be viewed as the different projections of a common *Euler system* obtained by p-adically interpolating the diagonal cycles themselves as (x,y,z) ranges over Σ_{bal}. (More precisely: their images $\kappa(f_x,g_y,h_z)\in H^1(\mathbb{Q},V_{f_x,g_y,h_z})$ under the p-adic étale Abel-Jacobi map, where V_{f_x,g_y,h_z} denotes the self-dual Tate twist of the triple tensor product $V_{f_x}\otimes V_{g_y}\otimes V_{h_z}$ of the Deligne representations attached to the eigenforms f_x, g_y and h_z.) The sequel [DR] to this paper pieces these global classes together into an element

$$\underline{\kappa}(\mathbf{f},\mathbf{g},\mathbf{h})\in H^1(\mathbb{Q},\mathbf{V}_{\mathbf{f},\mathbf{g},\mathbf{h}}),$$

where $\mathbf{V}_{\mathbf{f},\mathbf{g},\mathbf{h}}$ is a twist of the tensor product $\mathbf{V}_{\mathbf{f}}\otimes\mathbf{V}_{\mathbf{g}}\otimes\mathbf{V}_{\mathbf{h}}$ of Hida's Λ-adic representations attached to $\mathbf{f}$, $\mathbf{g}$, and $\mathbf{h}$ respectively, interpolating the representations V_{f_x,g_y,h_z}. The three Garrett-Rankin p-adic L-functions are then obtained from the image of $\underline{\kappa}(\mathbf{f},\mathbf{g},\mathbf{h})$ under a homomorphism interpolating the Bloch-Kato logarithms attached to the specialisations $V_{f_x}\otimes V_{g_y}\otimes V_{h_z}$ in the range $(x,y,z)\in\Sigma_{\mathrm{bal}}$. The resulting construction of the Garrett-Rankin p-adic L-functions in terms of diagonal cycles is directly analogous to the construction of the Kubota-Leopoldt (resp. Katz) p-adic L-function from the p-adic logarithms of circular (resp. elliptic) units. An application of this point of view to the Birch-Swinnerton-Dyer conjecture, obtained by considering weight one specialisations of $\mathbf{g}$ and $\mathbf{h}$, is the implication

$$L(E,\rho_1\otimes\rho_2,1)\neq 0 \implies \mathrm{Hom}_{G_{\mathbb{Q}}}(\rho_1\otimes\rho_2,E(\bar{\mathbb{Q}})\otimes\mathbb{C})=0,$$

where E is an elliptic curve over $\mathbb{Q}$ and ρ_1 and ρ_2 are odd irreducible two-dimensional Artin representations for which $\rho_1\otimes\rho_2$ has determinant one.

II. Beilinson-Flach elements. The methods of the present article, transposed to the setting where $\mathbf{h}$ is a Hida family of Eisenstein series, lead to a proof [BDR] of a p-adic Beilinson formula relating the p-adic regulators of certain Flach elements in the Higher Chow group $\mathrm{CH}^2(X_0(N)\times X_0(N),1)$ to values of Hida's three-variable p-adic L-function attached to the Rankin convolution of two Hida families $\mathbf{f}$ and $\mathbf{g}$ of cusp forms. A notable application of this result (when made to vary in p-adic families, as in the previous paragraph) is the implication

$$L(E,\rho,1)\neq 0 \implies \mathrm{Hom}_{G_{\mathbb{Q}}}(\rho,E(\bar{\mathbb{Q}})\otimes\mathbb{C})=0, \tag{9}$$

where ρ is an odd, irreducible two-dimensional Artin representation.

III. Beilinson elements and Kato's Euler system. The point of view sketched in the two previous Remarks is also consistent with the Kato-Perrin-Riou approach to the Mazur-Swinnerton-Dyer p-adic L-function attached to a cusp form of weight 2, which one recovers when both $\mathbf{g}$ and $\mathbf{h}$ are taken to be Hida families of Eisenstein series, the role of the diagonal cycles being played by Beilinson elements in the K_2 of modular curves, or in the higher Chow group $\mathrm{CH}^2(X_0(N), 2)$. Guided by this analogy and relying crucially on the techniques of the present work, the article [BD] gives a new proof of the p-adic Beilinson formula relating the p-adic regulators of the Beilinson elements to the special values at $s = 2$ of the Mazur-Swinnerton-Dyer p-adic L-functions. The authors' study of diagonal cycles and Flach elements is thus a direct generalisation of the approach of Kato which underlies the proof of (9) when ρ is replaced by a Dirichlet character.

IV. The Euler system of Heegner points, revisited. In part, the authors were led to Theorem 1.3 by the analogy with the main result of [BDP], in which the images of Heegner points (or more general Heegner cycles) under p-adic Abel-Jacobi maps are related to the values of certain anticyclotomic p-adic L-functions at classical points lying outside the range of p-adic interpolation defining them. In his forthcoming PhD thesis, F. Castella uses this to construct these anticyclotomic p-adic L-functions in terms of p-adic logarithms of Heegner points, leading to a treatment of the Heegner point Euler system entirely parallel to the other examples alluded to above (namely, circular units, elliptic units, Beilinson-Kato elements, Flach elements, and diagonal cycles.)

V. p-adic calculations of Chow-Heegner points. The article [La2] and the forthcoming Ph. D thesis of M. Daub [Da] combine Theorem 1.3 with Alan Lauder's fast algorithm [La1] for calculating ordinary projections to calculate the "Chow-Heegner points" attached to Gross-Schoen diagonal cycles (as described in [YZZ] and studied in [DRS]) by p-adic analytic methods, thus supplying the p-adic counterpart of the complex calculations carried out in [DDLR].

Notations: Throughout the article, given a power $q = p^d$ of a prime p, the symbols $\mathbb{F}_q$, $\mathbb{Z}_q$, and $\mathbb{Q}_q$ are reserved for the finite field with q elements, its ring of Witt vectors, and the finite unramified extension of $\mathbb{Q}_p$ of degree d respectively. For any field extension $F/\mathbb{Q}$, we will write $G_F = \mathrm{Gal}\,(\bar{F}/F)$ for the absolute Galois group of an algebraic closure of F, and if X is any variety over F, we let $\bar{X} := X \times_{\mathrm{Spec}(F)} \mathrm{Spec}(\bar{F})$ denote the base change of X to the algebraic closure. We adopt the usual conventions regarding motives and their Tate twists: for any integer j we write $\mathbb{Z}(j) = H^2(\mathbb{P}_1)^{\otimes -j}$, so that a geometric frobenius element at $\ell \neq p$ (resp. at $\ell = p$) acts on its p-adic étale (resp. cristalline) realisation as multiplication by ℓ^{-j}. If M is a motive over $\mathbb{Q}$, we let $M(j) := M \otimes \mathbb{Z}(j)$ denote its j-th Tate twist, so that $L(M(j), s) = L(M, s + j)$, where $L(M, s)$ is the L-function attached to M.

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2. Cohomology and modular forms

2.1. **The de Rham cohomology of curves over p-adic rings.** Let $\mathcal{X}$ be an arbitrary smooth proper curve over $\mathrm{Spec}(\mathbb{Z}_p)$ and write $\tilde{X}$ and X for its special and generic fiber, respectively. Let $\{P_1, \ldots, P_s\} \subset \tilde{X}(\bar{\mathbb{F}}_p)$ be a non-empty $G_{\mathbb{F}_p}$-stable collection of closed points in its special fiber. Since $\tilde{X}$ is smooth, these points admit lifts $\tilde{P}_1, \ldots, \tilde{P}_s \in \mathcal{X}(\mathcal{O}_{\mathbb{C}_p})$ to characteristic zero. It will be convenient (albeit not indispensable) to fix a choice of such lifts which is stable under the natural action of $G_{\mathbb{Q}_p}$. This determines the affine scheme

$$\mathcal{X}' = \mathcal{X} - \{\tilde{P}_1, \dots, \tilde{P}_s\}$$

over $\mathrm{Spec}(\mathbb{Z}_p)$, whose special and generic fiber are denoted $\tilde{X}'$ and X' respectively. Because $\mathcal{X}$ is proper over $\mathrm{Spec}(\mathbb{Z}_p)$, there is a natural identification $X(\mathbb{C}_p) = \mathcal{X}(\mathcal{O}_{\mathbb{C}_p})$ and a resulting *reduction map*

$$\mathrm{red} : X(\mathbb{C}_p) \longrightarrow \tilde{X}(\bar{\mathbb{F}}_p).$$

The *standard affinoid* in $X(\mathbb{C}_p)$ attached to X' is defined to be

$$\mathcal{A} := \mathrm{red}^{-1}(\tilde{X}'(\bar{\mathbb{F}}_p)).$$

It is a connected affinoid region obtained by deleting from $X(\mathbb{C}_p)$ a collection of s disjoint residue discs.

For each $j = 1, \dots, s$, choose a local coordinate λ_j at $\tilde{P}_j$, i.e., a rigid analytic isomorphism from the residue disc of $\tilde{P}_j$ in $X(\mathbb{C}_p)$ to the open unit disc in $\mathbb{C}_p$, sending $\tilde{P}_j$ to the origin. These local parameters give rise to a family of *wide open neighborhoods* of $\mathcal{A}$, indexed by a real parameter $\epsilon > 0$ and defined by

$$\mathcal{W}_\epsilon := \mathcal{A} \cup \bigcup_{j=1}^{s} \{x \in \mathrm{red}^{-1}(P_j) \text{ with } \mathrm{ord}_p \lambda_j(x) < \epsilon\}.$$

The region $\mathcal{W}_\epsilon$ is obtained by adjoining open annuli $\mathcal{V}_1, \dots, \mathcal{V}_s$ of "width ϵ" to $\mathcal{A}$ around the boundaries of each of the deleted residue discs. Note the obvious inclusions

$$\mathcal{A} \subset \mathcal{W}_\epsilon \subset X'(\mathbb{C}_p), \qquad \mathcal{W}_{\epsilon_1} \subset \mathcal{W}_{\epsilon_2} \text{ if } \epsilon_1 < \epsilon_2.$$

One of the main reasons for working with the wide open neighborhoods $\mathcal{W}_\epsilon$ rather than with the affinoid $\mathcal{A}$ itself is that their cohomology is better behaved and a more faithful reflection of algebraic de Rham cohomology over $\mathbb{C}_p$. More precisely, let $\mathcal{O}$ and Ω^1 denote the structure sheaf and the sheaf of differentials on X. The restrictions of these sheaves to $\mathcal{A}$ and to $\mathcal{W}_\epsilon$, viewed as rigid analytic sheaves, are denoted by the same symbols by a slight abuse of notation. If K is any complete subfield of $\mathbb{C}_p$ we will denote by $\Omega^1(\mathcal{W}_\epsilon/K)$ the rigid differentials on $\mathcal{W}_\epsilon$ that are defined over K. Set

$$H^1_{\mathrm{rig}}(\mathcal{W}_\epsilon/K) := \frac{\Omega^1(\mathcal{W}_\epsilon/K)}{d\mathcal{O}_{\mathcal{W}_\epsilon/K}}.$$

Note that, because X' is affine, we have $H^1_{\mathrm{dR}}(X'/K) = \frac{\Omega^1(X'/K)}{d\mathcal{O}_{X'}}$. The natural restriction map $\Omega^1(X'/K) \longrightarrow \Omega^1(\mathcal{W}_\epsilon/K)$ sends exact forms to exact forms and therefore induces a map

$$\mathrm{comp}_\epsilon : H^1_{\mathrm{dR}}(X'/K) \longrightarrow H^1_{\mathrm{rig}}(\mathcal{W}_\epsilon/K).$$

For each annulus $\mathcal{V}_1, \dots, \mathcal{V}_s$ appearing in the definition of $\mathcal{W}_\epsilon$, let

$$\mathrm{res}_{\mathcal{V}_j} : \Omega^1(\mathcal{W}_\epsilon/K) \longrightarrow K(-1)$$

denote the p-adic annular residue, as it is described in [Cole94, Ch. 7] for example.

The annular residue vanishes on $d\mathcal{O}_{\mathcal{W}_\epsilon}$, and is therefore well-defined on cohomology. It is related to the usual residue of algebraic differential forms through the following commutative diagram:

$$\begin{tikzcd} H^1_{\mathrm{dR}}(X'/K) \arrow[r, "\oplus_j \mathrm{res}_{\tilde{P}_j}"] \arrow[d, "\mathrm{comp}_\epsilon"] & K(-1)^s \arrow[r, "\Sigma"] \arrow[d, equal] & K(-1) \arrow[r] \arrow[d, equal] & 0 \\ H^1_{\mathrm{rig}}(\mathcal{W}_\epsilon/K) \arrow[r, "\oplus_j \mathrm{res}_{\mathcal{V}_j}"] & K(-1)^s \arrow[r, "\Sigma"] & K(-1) \arrow[r] & 0. \end{tikzcd} \tag{10}$$

Proposition 2.1. *For any $\epsilon > 0$, the map comp_ϵ is an isomorphism of K-vector spaces.*

Proof. This follows as in [Cole94] from the long exact Mayer-Vietoris sequence (cf. A.2. of loc.cit. for instance) associated to the admissible covering of X by wide open subsets consisting of $\mathcal{W}_\epsilon$ together with the p-adic residue discs in the complement of $\mathcal{A}$. □

Thanks to this proposition, the de Rham cohomology $H^1_{\mathrm{dR}}(X_K)$ of the base change of X to K can be identified with the space of classes of rigid analytic forms on $\mathcal{W}_\epsilon$ over K with vanishing annular residues (cf. e.g. [BDP, Prop. 4.11]). This description of $H^1_{\mathrm{dR}}(X_K)$ is the basis for a concrete description of the action of the Frobenius operator on de Rham cohomology. Let $\sigma \in \mathrm{Gal}\,(\bar{K}/\mathbb{Q}_p)$ be a Frobenius automorphism and let

$$\Phi : \mathcal{A} \longrightarrow \mathcal{A}$$

be any characteristic zero lift of the Frobenius morphism on the special fiber $\tilde{X}'$. This rigid analytic morphism extends to a morphism $\Phi : \mathcal{W}_\epsilon \longrightarrow \mathcal{W}_{\epsilon'}$, for suitable $0 < \epsilon < \epsilon'$, and induces linear maps

$$\Phi : \mathcal{O}(\mathcal{W}_{\epsilon'}/K) \longrightarrow \mathcal{O}(\mathcal{W}_\epsilon/K), \qquad \Phi : \Omega^1(\mathcal{W}_{\epsilon'}/K) \longrightarrow \Omega^1(\mathcal{W}_\epsilon/K).$$

Although the operator Φ need not preserve any of the spaces $\Omega^1(\mathcal{W}_\epsilon/K)$ for a given fixed ϵ, it does give rise to an endomorphism on $H^1_{\mathrm{dR}}(X'/K)$, which shall also be denoted Φ by abuse of notation. This Frobenius morphism is thus the unique endomorphism of $H^1_{\mathrm{dR}}(X'/K)$ that completes the top row in the following commutative diagram:

$$\begin{tikzcd} H^1_{\mathrm{dR}}(X'/K) \arrow[r, dotted, "\Phi"] \arrow[d, equal, "\mathrm{comp}_{\epsilon'}"] & H^1_{\mathrm{dR}}(X'/K) \arrow[d, equal, "\mathrm{comp}_{\epsilon}"] \\ H^1_{\mathrm{rig}}(\mathcal{W}_{\epsilon'}/K) \arrow[r, "\Phi"] & H^1_{\mathrm{rig}}(\mathcal{W}_\epsilon/K). \end{tikzcd} \tag{11}$$

The endomorphism Φ, which is compatible with the annular residue in the obvious sense, preserves the subspace $H^1_{\mathrm{dR}}(X_K)$.

The K-vector subspace of $H^1_{\mathrm{dR}}(X_K)$ (resp. of $H^1_{\mathrm{dR}}(X'/K)$) spanned by the vectors on which Φ acts via multiplication by a p-adic unit is called the *unit root subspace* and is denoted by $H^1_{\mathrm{dR}}(X_K)^{\text{u-r}}$ (resp. by $H^1_{\mathrm{dR}}(X'/K)^{\text{u-r}}$). More generally, the subspaces spanned by vectors on which Φ acts with slope $t \in \mathbb{Q}$ (i.e., as multiplication by a scalar $\lambda \in \mathbb{C}_p^\times$ with $\mathrm{ord}_p\lambda = t$) is called the *slope t subspace* of $H^1_{\mathrm{dR}}(X_K)$ and is denoted $H^1_{\mathrm{dR}}(X_K)^{\Phi,t}$.

The de Rham cohomology $H^1_{\mathrm{dR}}(X_K)$ is equipped with the usual alternating Poincaré duality

$$\langle\ ,\ \rangle : H^1_{\mathrm{dR}}(X_K) \times H^1_{\mathrm{dR}}(X_K) \longrightarrow H^2_{\mathrm{dR}}(X_K) = K(-1),$$

which in terms of representatives $\omega_1, \omega_2 \in \Omega^1_{\mathrm{rig}}(\mathcal{W}_\epsilon)$ for cohomology classes ξ_1 and ξ_2 is described by the formula

$$\langle \xi_1, \xi_2 \rangle = \sum_{j=1}^{s} \mathrm{res}_{\mathcal{V}_j}(F^{(j)}_{\omega_1} \cdot \omega_2),$$

where $\mathrm{res}_{\mathcal{V}_j}$ is the p-adic annular residue and $F^{(j)}_{\omega_1}$ denotes a local analytic primitive of ω_1 on the annulus $\mathcal{V}_j$.

Poincaré duality is compatible with the Frobenius endomorphism in the sense that

$$\langle \Phi\xi_1, \Phi\xi_2 \rangle = \Phi\langle \xi_1, \xi_2 \rangle = p\langle \xi_1, \xi_2 \rangle.$$

In particular, Poincaré duality descends to a well-defined non-degenerate pairing

$$\langle\ ,\ \rangle : H^1_{\mathrm{dR}}(X_K)^{\text{u-r}} \times H^1_{\mathrm{dR}}(X_K)^{\Phi,1} \longrightarrow K(-1). \tag{12}$$

2.2. **p-adic modular forms.** Denote by $S_k^{(p)}(N)$ the space of p-adic modular forms of weight k and tame level N (as defined in [Se] or [Kat73] for example) and let $S_k^{\mathrm{oc}}(N)$ denote its subspace of *overconvergent forms*. These spaces are equipped with two non-commuting operators U and V, defined on q-expansions by the rule

$$(Uf)(q) = \sum_{n=1}^{\infty} a_{pn} q^n, \qquad (Vf)(q) = \sum_{n=1}^{\infty} a_n q^{pn}. \tag{13}$$

These operators satisfy

$$UVf = f, \qquad VUf(q) = \sum_{n=1}^{\infty} a_{np} q^{np},$$

so that

$$f^{[p]} := (1 - VU)f(q) = \sum_{p \nmid n} a_n q^n \tag{14}$$

has fourier coefficients supported on the integers prime to p. Note that the operators U and V do not commute–although, as we will explain later, the operators induced on the cohomology of modular curves (with coefficients in the relevant local systems) *do* commute, and are inverse to each other.

The operator U on the p-adic Banach space of overconvergent modular forms is completely continuous and gives rise to a slope decomposition on this infinite-dimensional vector space. A p-adic modular form which belongs to the slope 0 subspace for U is said to be *ordinary*, and the space of all such p-adic modular forms is denoted by $S_k^{\mathrm{ord}}(N)$. The ordinary subspace (like all the finite slope subspaces) is finite-dimensional over $\mathbb{C}_p$. More precisely, Coleman's classicality theorem (cf. [Cole95]) asserts that any ordinary overconvergent modular form of weight $k \geq 2$ is (the p-stabilisation of) a classical modular form of weight k on $\Gamma_1(N)$. In particular, $S_k^{\mathrm{ord}}(N)$ is naturally contained in the space $S_k(\Gamma_1(N) \cap \Gamma_0(p))$ of classical modular forms (with fourier coefficients in $\mathbb{C}_p$). Hida's ordinary projector

$$e_{\mathrm{ord}} := \lim_n U^{n!} \tag{15}$$

gives a Hecke-equivariant projection from $S_k^{\mathrm{oc}}(N)$ to $S_k^{\mathrm{ord}}(N)$.

Let $\mathcal{X}_1(N)$ denote the modular curve over $\mathrm{Spec}(\mathbb{Z}[1/N])$ classifying generalised elliptic curves equipped with an embedding of the finite flat group scheme μ_N of N-th roots of unity. (See [Kas99] for a more general scenario.)

Recalling that the prime p does not divide N, set

$$\mathcal{X} := \mathcal{X}_1(N) \times_{\mathrm{Spec}\, \mathbb{Z}[1/N]} \mathbb{Z}_p, \quad X = \mathcal{X} \times \mathrm{Spec}(\mathbb{Q}_p)$$

for the smooth curve over $\mathbb{Z}_p$ (resp. over $\mathbb{Q}_p$) obtained by change of base.

Let $P_1, \ldots, P_s \in \tilde{X}(\mathbb{F}_{p^2})$ denote the supersingular points of the special fiber of $\mathcal{X}$. These points are the zeroes of a distinguished mod p modular form of weight $(p-1)$—the so-called *Hasse invariant*. As in the previous section, we may choose lifts $\tilde{P}_1, \ldots, \tilde{P}_s \in \mathcal{X}(\mathbb{Z}_{p^2})$ of the supersingular points to characteristic zero. For example, when $p \geq 5$, one may do this by taking the zeroes of the Eisenstein series E_{p-1}, which is the customary lift of the Hasse invariant to characteristic 0. We continue to write

$$\mathcal{X}' = \mathcal{X} - \{\tilde{P}_1, \ldots, \tilde{P}_s\}$$

for the resulting affine scheme over $\mathrm{Spec}(\mathbb{Z}_p)$, whose special and generic fiber are denoted $\tilde{X}'$ and X' respectively. The connected affinoid region obtained by deleting from $X(\mathbb{C}_p)$ the s disjoint supersingular residue discs:

$$\mathcal{A} = \mathcal{A}_{\mathrm{ord}} := \mathrm{red}^{-1}(\tilde{X}'(\bar{\mathbb{F}}_p))$$

is called the *ordinary locus*, or *Hasse domain*, in $X(\mathbb{C}_p)$. Note that

$$\mathcal{A} = \{x \in X(\mathbb{C}_p) \text{ with } \mathrm{ord}_p E_{p-1}(x) = 0\}, \tag{16}$$

where

$$\mathrm{ord}_p E_{p-1}(x) := \mathrm{ord}_p E_{p-1}(A_x, \omega_x),$$

and

1. A_x is the generalised elliptic curve with $\Gamma_1(N)$-structure attached to x by the moduli interpretation of X;
2. $\omega_x \in \Omega^1(A_x/\mathbb{C}_p)$ is a regular differential on A_x, chosen so that it extends to a regular differential over $\mathcal{O}_{\mathbb{C}_p}$ if A_x has good reduction at p, or corresponds to the canonical differential on the Tate curve if x lies in the residue disc of a cusp.
3. The notation $E_{p-1}(A_x, \omega_x)$ follows Katz's geometric definition of modular forms as functions on such pairs, as described in [Kat73, Ch. 1], for example.

While $E_{p-1}(A_x, \omega_x)$ genuinely depends on the choice of ω_x, its p-adic valuation is independent of it since any two such choices differ by multiplication by an element of $\mathcal{O}_{\mathbb{C}_p}^\times$.

Define a system of wide open neighborhoods of $\mathcal{A}$ by choosing a real parameter $\epsilon > 0$ and setting

$$\mathcal{W}_\epsilon := \{x \in X(\mathbb{C}_p) \text{ with } \mathrm{ord}_p E_{p-1}(x) < \epsilon\}.$$

When $0 < \epsilon < 1$, the region $\mathcal{W}_\epsilon$ does not depend on the choice of lift of the Hasse invariant to characteristic zero. As suggested by (16), the quantity $\mathrm{ord}_p E_{p-1}(x)$ is (at least, when it is not too large) a sensible measure of the "degree of supersingularity" of the elliptic curve A_x associated to x. If $\mathrm{ord}_p E_{p-}(x) < \frac{p}{p+1}$, Katz has shown that the elliptic curve A_x, although supersingular at p, continues to admit a *canonical subgroup* Z_x, a connected subgroup scheme of A_x of order p generalising the canonical subgroup on an ordinary elliptic curve. This makes it possible to choose a *canonical* lift to characteristic zero of the Frobenius morphism, by setting, for all $x \in \mathcal{W}_\epsilon$ with $\epsilon < \frac{p}{p+1}$,

$$\Phi(x) = \text{ Point corresponding to } A_x/Z_x. \tag{17}$$

The affinoid $\mathcal{A}$ and the wide opens $\mathcal{W}_\epsilon$ play a key role in Katz's geometric description of p-adic and overconvergent modular forms, which we will now briefly recall.

Modular forms of weight two. We begin with the somewhat simpler case of forms of weight $k = 2$, which can be treated by specialising the discussion of Section 2.1 to the case where $X = X_1(N)$ with $p \nmid N$, so that X has good reduction at p. For any complete subfield K of $\mathbb{C}_p$ we have

$$S_2^{(p)}(N; K) = \Omega^1_{\mathrm{rig}}(\mathcal{A}/K), \qquad S_2^{\mathrm{oc}}(N; K) = \bigcup_{\epsilon > 0} \Omega^1_{\mathrm{rig}}(\mathcal{W}_\epsilon/K), \tag{18}$$

that is to say, the space of rigid sections of $\Omega^1(\mathcal{W}_\epsilon/K)$ is identified with the p-adic Banach space of overconvergent modular forms of weight two with "annuli of convergence of width ϵ", and fourier coefficients in K. Under this identification, a weight two modular form f with q-expansion $f(q) = \sum_n a_n(f) q^n$ corresponds to the differential

$$\omega_f = f(q)\frac{dq}{q} = \left(\sum_{n=1}^{\infty} a_n(f) q^n\right) \frac{dq}{q}.$$

By definition of the operators U and V,

$$f^{[p]}(q) = (1 - VU) f(q) = q\frac{d}{dq}F, \qquad \text{where} \qquad F(q) = \sum_{p \nmid n} \frac{a_n(f)}{n} q^n. \tag{19}$$

The modular form $F(q)$ is an overconvergent modular form of weight 0 and level N, i.e., an element of $\mathcal{O}_{\mathcal{W}_\epsilon}$ for a suitable $\epsilon > 0$. It corresponds to the rigid analytic primitive of $\omega_{f^{[p]}}$ which vanishes at the cusp ∞, i.e., it is determined by the properties

$$dF = f^{[p]}(q)\frac{dq}{q}, \qquad F(\infty) = 0.$$

Define $S_2^{\mathrm{oc}}(N;K)_0$ to be the subspace of $S_2^{\mathrm{oc}}(N;K)$ consisting of overconvergent modular forms with coefficients in K and vanishing residues at the supersingular annuli. Set also $S_2^{\mathrm{ord}}(N;K)_0 = S_2^{\mathrm{oc}}(N;K)_0 \cap S_2^{\mathrm{ord}}(N;K)$ and write $\phi^{\mathrm{ord}} := e_{\mathrm{ord}}\phi$ for the ordinary projection of an overconvergent modular form.

Lemma 2.2. *If ϕ is an overconvergent p-adic modular form of weight two on $\Gamma_1(N)$, then the class of ω_ϕ belongs to $H^1_{\mathrm{rig}}(\mathcal{W}_\epsilon)^{\Phi,1}$. Furthermore, the assignment $\phi \mapsto [\omega_\phi]$ induces isomorphisms*

$$S_2^{\mathrm{ord}}(N;K) \xrightarrow{\sim} H^1_{\mathrm{rig}}(\mathcal{W}_\epsilon/K)^{\Phi,1}, \qquad S_2^{\mathrm{ord}}(N;K)_0 \xrightarrow{\sim} H^1_{\mathrm{dR}}(X_K)^{\Phi,1}.$$

Proof. The Frobenius morphism $\Phi : \Omega^1(\mathcal{W}_\epsilon) \longrightarrow \Omega^1(\mathcal{W}_{\epsilon/p})$ is related to the operator V on overconvergent modular forms of weight two by $\Phi(\omega_f) = p\omega_{Vf}$. The relation between the operators U and Φ on cohomology (relative to the identifications described above between differentials and weight two modular forms) is therefore given by

$$\Phi = pV = pU^{-1}.$$

Hence, if ϕ belongs to the slope zero subspace for the action of U, the class of the rigid differential ω_ϕ lies in the slope one subspace $H^1_{\mathrm{rig}}(\mathcal{W}_\epsilon)^{\Phi,1}$ for the action of Φ. The first statement follows. The second is a well-known result of Coleman: cf. the cases $k = 0$ of Cor. 6.3.1. and Prop. 6.6 of [Cole95] for the injectivity and surjectivity respectively of the maps induced by restriction. $\square$

Proposition 2.3. *For any class $\eta \in H^1_{\mathrm{dR}}(X_K)^{\mathrm{u\text{-}r}}$ and any overconvergent modular form $\phi \in S_2^{\mathrm{oc}}(N;K)_0$, we have*

$$\langle \eta, \omega_\phi\rangle = \langle \eta, \omega_{\phi^{\mathrm{ord}}}\rangle,$$

and Poincaré duality induces a well-defined non-degenerate pairing

$$\langle\ ,\ \rangle : H^1_{\mathrm{dR}}(X_K)^{\mathrm{u\text{-}r}} \times S_2^{\mathrm{ord}}(N;K)_0 \longrightarrow K.$$

Proof. This follows directly from Lemma 2.2 in light of (12). $\square$

Modular forms of higher weight. Turning now to the case of forms of general weight $k \geq 2$, let $\mathcal{E} \longrightarrow Y$ denote the universal elliptic curve over the affine modular curve $Y = Y_1(N)/\mathbb{Q}$. Let $\underline{\omega} := \pi_*\Omega^1_{\mathcal{E}/Y}$ denote the line bundle of relative differentials on $\mathcal{E}$ over Y. It extends to a sheaf over $X = X_1(N)$, also denoted by $\underline{\omega}$, by setting

$$H^0(\operatorname{Spec}\mathbb{Q}[[q]], \underline{\omega}) = \mathbb{Q}[[q]] \cdot \omega_{\mathrm{can}},$$

where $\omega_{\mathrm{can}} := \frac{dt}{t}$ is the canonical differential on the Tate curve $\mathbb{G}_m/q^{\mathbb{Z}}$.

Let $\Omega^1_X(\log \mathrm{cusps})$ be the sheaf of differentials 1-forms on X with logarithmic poles at the cusps, for which

$$H^0(\operatorname{Spec}\mathbb{Q}[[q]], \Omega^1(\log \mathrm{cusps})) = \mathbb{Q}[[q]] \cdot \frac{dq}{q}.$$

A modular form ϕ on X of weight $k = r+2$ with fourier coefficients in a field K (viewed as a function on "test objects" (E, ω_E), following the point of view adopted in [Kat73, Ch. 1] for example) corresponds to a global section of the sheaf $\underline{\omega}^{r+2}$ over X_K, by sending ϕ to the global section $\phi(E, \omega_E)\omega_E^{r+2}$, and hence we can define

$$S_k(N;K) = H^0(X_1(N)_K, \underline{\omega}^k).$$

Let

$$\mathcal{L} := \mathbb{R}^1\pi_*\Omega^\bullet_{\mathcal{E}/Y}$$

be the relative de Rham cohomology sheaf on Y. It is equipped with a filtration

$$0 \longrightarrow \underline{\omega} \longrightarrow \mathcal{L} \longrightarrow \underline{\omega}^{-1} \longrightarrow 0 \tag{20}$$

arising from the Hodge filtration on the fibers. The sheaf $\mathcal{L}$ and its r-th symmetric power

$$\mathcal{L}_r := \mathrm{sym}^r\,\mathcal{L}$$

are vector bundles over Y (of rank 2 and r+1 respectively) endowed with a canonical integrable connection:

$$\nabla : \mathcal{L}_r \longrightarrow \mathcal{L}_r \otimes \Omega^1_X(\log \text{cusps}),$$

the so-called *Gauss-Manin connection.* The sheaf $\mathcal{L}_r$ extends to a sheaf on X by setting

$$H^0(\mathrm{Spec}\,\mathbb{Q}[[q]], \mathcal{L}_r) = \mathbb{Q}[[q]]\eta^r_{\mathrm{can}} + \mathbb{Q}[[q]]\eta^{r-1}_{\mathrm{can}}\omega_{\mathrm{can}} + \cdots + \mathbb{Q}[[q]]\omega^r_{\mathrm{can}}, \tag{21}$$

where $\eta_{\mathrm{can}} := \nabla(q\frac{d}{dq})(\omega_{\mathrm{can}})$ is a complementary vector to ω_{can} in $H^1_{\mathrm{dR}}(\mathcal{E}/\mathbb{Z}[[q]])$, and similarly at the other cusps of X. The connection ∇ extends by setting

$$\nabla\omega_{\mathrm{can}} = \eta_{\mathrm{can}} \otimes \frac{dq}{q}, \qquad \nabla\eta_{\mathrm{can}} = 0. \tag{22}$$

The filtration (20) gives rise to an $(r+1)$-step decreasing filtration on $\mathcal{L}_r$, with successive quotients

$$\mathcal{L}_r/\mathrm{Fil}^1\mathcal{L}_r \simeq \underline{\omega}^{-r}, \cdots, \mathrm{Fil}^{r-j}\mathcal{L}_r/\mathrm{Fil}^{r-j+1}\mathcal{L}_r \simeq \underline{\omega}^{r-2j}, \cdots, \mathrm{Fil}^r\mathcal{L}_r \simeq \underline{\omega}^r. \tag{23}$$

The connection ∇ obeys Griffiths transversality:

$$\nabla\mathrm{Fil}^{r-j}\mathcal{L}_r \subset \mathrm{Fil}^{r-j-1}\mathcal{L}_r \otimes \Omega^1_X(\log \text{cusps}).$$

Furthermore, it induces an isomorphism of $\mathcal{O}_X$-modules:

$$\nabla : \frac{\mathrm{Fil}^{r-j}\mathcal{L}_r}{\mathrm{Fil}^{r-j+1}\mathcal{L}_r} \simeq \frac{\mathrm{Fil}^{r-j-1}\mathcal{L}_r}{\mathrm{Fil}^{r-j}\mathcal{L}_r} \otimes \Omega^1_X(\log \text{cusps}), \tag{24}$$

which gives rise (by setting $r = 1$ and $j = 0$ for example) to the *Kodaira-Spencer isomorphism*

$$\mathrm{KS} : \underline{\omega}^{\otimes 2} \xrightarrow{\sim} \Omega^1_X(\log \text{cusps}).$$

It follows that a modular form ϕ of weight $r+2$ can be interpreted, via this isomorphism, as a global section of $\underline{\omega}^r \otimes \Omega^1_X(\log \text{cusps})$, while a cusp form can be interpreted as a global section of $\underline{\omega}^r \otimes \Omega^1_X$, i.e.,

$$S_k(N; K) = H^0(X_K, \underline{\omega}^r \otimes \Omega^1_X).$$

At a geometric point of X attached to an elliptic curve E with $\Gamma_1(N)$-level structure, the section ω_ϕ is given by

$$\omega_\phi = \phi(E, \omega_E)\omega^r_E\,\mathrm{KS}(\omega^2_E).$$

At the test object $(\mathbb{G}_m/q^{\mathbb{Z}}, \omega_{\mathrm{can}})$ corresponding to the Tate curve with its canonical differential, this leads to the formula for ω_ϕ in terms of the q-expansion of ϕ:

$$\omega_\phi(q) = \phi(q)\omega^r_{\mathrm{can}}\left(\frac{dq}{q}\right).$$

Let $(\mathcal{L}_r \otimes \Omega^1_X)_{\mathrm{par}}$ denote the subsheaf of $\mathcal{L}_r \otimes \Omega^1(\log \text{cusps})$ defined in the neighborhood of the cusps by

$$\begin{aligned} H^0(\mathrm{Spec}\,\mathbb{Q}[[q]], (\mathcal{L}_r \otimes \Omega^1_X)_{\mathrm{par}}) &= \nabla H^0(\mathrm{Spec}\,\mathbb{Q}[[q]], \mathcal{L}_r) \\ &= (\mathbb{Q}[[q]]\eta^r_{\mathrm{can}} + \mathbb{Q}[[q]]\eta^{r-1}_{\mathrm{can}}\omega_{\mathrm{can}} + \cdots + q\mathbb{Q}[[q]]\omega^r_{\mathrm{can}})\frac{dq}{q}. \end{aligned}$$

Note that for $r=0$ we have $(\mathcal{L}_r \otimes \Omega^1_X)_{\rm par} = \Omega^1_X$. A standard inductive argument using (24) shows that the natural inclusion $\underline{\omega}^r \longrightarrow \mathcal{L}_r$ induces an isomorphism

$$H^0(X_K, \underline{\omega}^r \otimes \Omega^1_X) \simeq \frac{H^0(X_K, (\mathcal{L}_r \otimes \Omega^1_X)_{\rm par})}{\nabla H^0(X_K, \mathcal{L}_r)}. \tag{25}$$

We write

$$H^j_{\rm dR}(X_K, \mathcal{L}_r, \nabla) := \mathbb{H}^j(0 \to \mathcal{L}_r \xrightarrow{\nabla} \mathcal{L}_r \otimes \Omega^1_X(\log \text{cusps}) \to 0)$$

for the j-th hypercohomology of the complex of sheaves on X over K associated to ∇, and $H^j_{\rm par}(X_K, \mathcal{L}_r, \nabla)$ for the *parabolic cohomology*:

$$H^j_{\rm par}(X_K, \mathcal{L}_r, \nabla) := \mathbb{H}^j(0 \to \mathcal{L}_r \xrightarrow{\nabla} (\mathcal{L}_r \otimes \Omega^1_X)_{\rm par} \to 0).$$

(Cf. also Chapter 2.1 of [BDP] for example.) These algebraic de Rham cohomology groups are finite-dimensional K-vector spaces. The group $H^1_{\rm par}(X_K, \mathcal{L}_r, \nabla)$ is equipped with a two-step Hodge filtration given by

$$\mathrm{Fil}^j H^1_{\rm par}(X_K, \mathcal{L}_r, \nabla) = \begin{cases} H^1_{\rm par}(X_K, \mathcal{L}_r, \nabla) & \text{if } j \le 0; \\ H^0(X_K, \underline{\omega}^r \otimes \Omega^1_X) & \text{if } 1 \le j \le r+1; \\ 0 & \text{if } j \ge r+2, \end{cases} \tag{26}$$

giving rise to the exact sequence

$$0 \longrightarrow H^0(X_K, \underline{\omega}^r \otimes \Omega^1_X) \longrightarrow H^1_{\rm par}(X_K, \mathcal{L}_r, \nabla) \longrightarrow H^1(X_K, \underline{\omega}^{-r}) \longrightarrow 0. \tag{27}$$

We note the Poincaré duality pairing

$$\langle \ , \ \rangle : H^1_{\rm par}(X_K, \mathcal{L}_r, \nabla) \times H^1_{\rm par}(X_K, \mathcal{L}_r, \nabla) \longrightarrow K(-1-r). \tag{28}$$

Since $H^0(X_K, \underline{\omega}^r \otimes \Omega^1_X)$ is its own orthogonal complement under (28), Poincaré duality descends to a perfect pairing

$$\langle \ , \ \rangle : H^1(X_K, \underline{\omega}^{-r}) \times H^0(X_K, \underline{\omega}^r \otimes \Omega^1_X) \longrightarrow K(-1-r), \tag{29}$$

which is denoted by the same symbol by a slight abuse of notation.

In the sequel, if $\mathcal{L}$ is a local system on a variety V, endowed with an integrable connection ∇ that is clear from the context, we shall write $H^\star(V, \mathcal{L})$ instead of $H^\star(V, \mathcal{L}, \nabla)$, and likewise for similar cohomology groups.

2.3. **Nearly holomorphic modular forms and the Shimura-Maass derivative.** Suppose now that $K = \mathbb{C}$. Hodge theory gives a canonical (real-analytic, but non-holomorphic) splitting

$$\mathrm{Spl}_{\rm hdg} : \mathcal{L} \longrightarrow \underline{\omega}$$

of the exact sequence (20) over the affine modular curve Y. This map can be viewed as a homomorphism of $\mathcal{O}_{Y(\mathbb{C})_{\rm an}}$–modules, where $\mathcal{O}_{Y(\mathbb{C})_{\rm an}}$ is the structure sheaf of real analytic functions on $Y(\mathbb{C})$. We will also denote by the same symbol the associated map $\mathcal{L}^k \longrightarrow \underline{\omega}^k$, as well as the resulting map

$$\mathrm{Spl}_{\rm hdg} : H^0(X_{\mathbb{C}}, (\mathcal{L}_r \otimes \Omega^1_X)_{\rm par}) \longrightarrow H^0(Y(\mathbb{C})_{\rm an}, \underline{\omega}^r \otimes \Omega^1_X) \tag{30}$$

restricted to regular (i.e., holomorphic) sections over X of the sheaf $(\mathcal{L}_r \otimes \Omega^1_X)_{\rm par}$.

Definition 2.4. The image of $\mathrm{Spl}_{\rm hdg}$ is called the space of *nearly holomorphic cusp forms* of weight $k = r+2$ on $\Gamma_1(N)$.

Nearly holomorphic cusp forms were introduced in [Sh86]; see also [Hi93, Ch. 10] for a more elementary description. The space of nearly holomorphic cusp forms on $\Gamma_1(N)$ is denoted by $S_k^{\rm nh}(N; \mathbb{C})$. It is contained in the space of real analytic functions on the upper half plane which satisfy the same transformation property under $\Gamma_1(N)$ as (holomorphic) modular forms of weight k. The following basic facts about nearly holomorphic modular forms, which we recall without proof, will be useful in the sequel:

(i) The map $\mathrm{Spl}_{\mathrm{hdg}}$ of (30) is injective, and hence induces an isomorphism of finite-dimensional complex vector spaces:

$$\mathrm{Spl}_{\mathrm{hdg}} : H^0(X_{\mathbb{C}}, (\mathcal{L}_r \otimes \Omega^1_X)_{\mathrm{par}}) \xrightarrow{\sim} S_k^{\mathrm{nh}}(N; \mathbb{C}).$$

(Cf. equation (5a) in §10.1 of [Hi93].)

(ii) If K is any subfield of $\mathbb{C}$, the image of $H^0(X_K, (\mathcal{L}_r \otimes \Omega^1_X)_{\mathrm{par}})$ under $\mathrm{Spl}_{\mathrm{hdg}}$ yields a natural K-structure on $S_k^{\mathrm{nh}}(N; \mathbb{C})$, and is denoted by $S_k^{\mathrm{nh}}(N; K)$.

(iii) If $\phi = \mathrm{Spl}_{\mathrm{hdg}}(\omega_\phi)$ belongs to $S_k^{\mathrm{nh}}(N; K)$, then equation (25) allows us to write

$$\omega_\phi = \Pi_N^{\mathrm{hol}}(\phi) + \nabla s, \quad \text{with } \begin{cases} \Pi_N^{\mathrm{hol}}(\phi) \in H^0(X_K, \underline{\omega}^r \otimes \Omega^1_X) = S_k(N; K), \\ s \in H^0(X_K, \mathcal{L}_r). \end{cases} \tag{31}$$

The modular form $\Pi_N^{\mathrm{hol}}(\phi)$ is called the *holomorphic projection* of the nearly holomorphic modular form ϕ. (Cf. equation (8a) in §10.1 of [Hi93]; the fact that Π_N^{hol}, which is denoted by H in loc. cit., preserves K-rational structures is stated in (8b).)

(iv) The inverse of the Kodaira-Spencer isomorphism followed by the Gauss-Manin connection gives a well-defined map

$$\tilde{\nabla} : H^0(X_K, (\mathcal{L}_r \otimes \Omega^1_X)_{\mathrm{par}}) \longrightarrow H^0(X_K, (\mathcal{L}_{r+2} \otimes \Omega^1_X)_{\mathrm{par}}). \tag{32}$$

This map corresponds (under the identification $\mathrm{Spl}_{\mathrm{hdg}}$) to the weight k Shimura-Maass derivative operator $\delta_k = \frac{1}{2\pi i}(\frac{d}{dz} + \frac{k}{z-\bar{z}})$, i.e., the following diagram commutes:

$$\begin{tikzcd} H^0(X_K, (\mathcal{L}_r \otimes \Omega^1_X)_{\mathrm{par}}) \arrow[r, "\mathrm{Spl}_{\mathrm{hdg}}"] \arrow[d, "\tilde{\nabla}"'] & S_k^{\mathrm{nh}}(N; K) \arrow[d, "\delta_k"] \\ H^0(X_K, (\mathcal{L}_{r+2} \otimes \Omega^1_X)_{\mathrm{par}}) \arrow[r, "\mathrm{Spl}_{\mathrm{hdg}}"] & S_{k+2}^{\mathrm{nh}}(N; K). \end{tikzcd}$$

This follows from a direct calculation based on the identities (in terms of the standard complex coordinates $\tau \in \mathcal{H}$ and $z \in \mathbb{C}/\langle 1, \tau\rangle$)

$$\nabla(2\pi i dz) = 2\pi i \left(\frac{dz - \bar{d}z}{\tau - \bar{\tau}}\right) \otimes d\tau, \qquad \nabla \bar{d}z = 0, \qquad \mathrm{KS}((2\pi i dz)^{\otimes 2}) = 2\pi i d\tau.$$

Since nearly holomorphic modular forms are closed under taking products and under applying the Shimura-Maass derivative, it follows that, if $g \in S_\ell(N; K)$ and $h \in S_m(N; K)$ are classical cusp forms with fourier coefficients in K, then for all $t \geq 0$, the product $\delta_\ell^t g \times h$ belongs to $S_{\ell+m+2t}^{\mathrm{nh}}(N; K)$. More precisely, let

$$\tilde{\nabla}^t \omega_g \otimes \tilde{\omega}_h \in H^0(X_K, (\mathcal{L}_{\ell+m+2t-2} \otimes \Omega^1_X)_{\mathrm{par}}) \tag{33}$$

be the global section obtained by tensoring the sections

$$\tilde{\nabla}^t \omega_g \in H^0(X_K, (\mathcal{L}_{\ell+2t-2} \otimes \Omega^1_X)_{\mathrm{par}}) \text{ and } \tilde{\omega}_h := \mathrm{KS}^{-1}(\omega_h) \in H^0(X_K, \mathcal{L}_m).$$

Then

$$\delta_\ell^t g \times h = \mathrm{Spl}_{\mathrm{hdg}}(\tilde{\nabla}^t \omega_g \otimes \tilde{\omega}_h). \tag{34}$$

Hodge theory gives a canonical splitting of the exact sequence (27):

$$H^1_{\mathrm{par}}(X_{\mathbb{C}}, \mathcal{L}_r) = H^0(X_{\mathbb{C}}, \underline{\omega}^r \otimes \Omega^1_X) \oplus \overline{H^0(X_{\mathbb{C}}, \underline{\omega}^r \otimes \Omega^1_X)}.$$

The Petersson scalar product $(\ ,\ )_N$ of level N on $S_k^{\rm nh}(N;\mathbb{C})$ is defined by the familiar rule

$$(35)\qquad (\ ,\ )_N : S_k^{\rm nh}(N;\mathbb{C})\times S_k^{\rm nh}(N;\mathbb{C})\longrightarrow \mathbb{C},\qquad (f_1,f_2)_N=\int_{\Gamma_1(N)\backslash\mathfrak{H}}\overline{f_1(z)}\,f_2(z)\,y^k\frac{dxdy}{y^2},$$

where the integration is performed relative to the variable $z=x+iy$ on any fundamental domain in the upper-half plane $\mathfrak{H}$ under the action of $\Gamma_1(N)$. Note that this pairing is Hermitian-linear in the first argument and $\mathbb{C}$-linear in the second, in contrast with the more customary conventions.

Lemma 2.5. *For all $\eta\in S_{r+2}(N;\mathbb{C})$, and all $\phi\in S_{r+2}^{\rm nh}(N;\mathbb{C})$,*

$$(\eta,\phi)_N=(\eta,\Pi_N^{\rm hol}(\phi))_N.$$

The equation of Lemma 2.5 gives an independent definition of the holomorphic projection, since the Petersson product is a perfect pairing on the space $S_k(N;\mathbb{C})$ of cusp forms.

2.4. **Nearly overconvergent modular forms and the d operator.** Recall the ordinary locus $\mathcal{A}$ and its system of wide open neighborhoods $\mathcal{W}_\epsilon\supset\mathcal{A}$ that were introduced in Section 2.2. We define

$$(36)\qquad H^1_{\rm rig}(\mathcal{W}_\epsilon,\mathcal{L}_r):=\frac{H^0(\mathcal{W}_\epsilon,(\mathcal{L}_r\otimes\Omega^1_X)_{\rm par})}{\nabla H^0(\mathcal{W}_\epsilon,\mathcal{L}_r)},$$

where $H^0(\mathcal{W}_\epsilon,-)$ designates a space of rigid sections. The quotient on the right of (36) is related to overconvergent cusp forms of weight $k=r+2$ by noting that any rigid section $\eta\in H^0(\mathcal{W}_\epsilon,(\mathcal{L}_r\otimes\Omega^1_X)_{\rm par})$ can be written as

$$(37)\qquad \eta=\eta_0+\nabla s,\qquad \text{with } \eta_0\in H^0(\mathcal{W}_\epsilon,\underline{\omega}^r\otimes\Omega^1_X),\quad s\in H^0(\mathcal{W}_\epsilon,\mathcal{L}_r).$$

This fact follows from an inductive argument based on (24).

Similarly as in (18), the spaces of overconvergent modular (cusp) forms of weight k are

$$(38)\qquad S_k^{\rm oc}(N;K)=\bigcup_{\epsilon>0}H^0(\mathcal{W}_\epsilon/K,\underline{\omega}^r\otimes\Omega^1_{\rm rig})\subseteq M_k^{\rm oc}(\Gamma_1(N),K)=\bigcup_{\epsilon>0}H^0(\mathcal{W}_\epsilon/K,\underline{\omega}^k).$$

It is also known (cf. [Cole95]) that any overconvergent modular form of weight $-r$, viewed as a section $s\in H^0(\mathcal{W}_\epsilon,\underline{\omega}^{-r})$, admits a unique lift $\tilde{s}$ under the projection $H^0(\mathcal{W}_\epsilon,\mathcal{L}_r)\longrightarrow H^0(\mathcal{W}_\epsilon,\underline{\omega}^{-r})$ induced by (23) satisfying

$$\nabla\tilde{s}\in H^0(\mathcal{W}_\epsilon,\underline{\omega}^r\otimes\Omega^1_X),$$

and that $\nabla\tilde{s}$ corresponds to the overconvergent modular form $d^{r+1}s$, where $d:=q\frac{d}{dq}$ is Serre's operator sending p-adic modular forms of weight m to p-adic modular forms of weight $m+2$. (Note that the operator d does not preserve overconvergence in general, even though d^{r+1} maps $S_{-r}^{\rm oc}(N)$ to $S_{r+2}^{\rm oc}(N)$.) Thanks to (37), equation (36) can be re-written as

$$(39)\qquad H^1_{\rm rig}(\mathcal{W}_\epsilon,\mathcal{L}_r)=\frac{H^0(\mathcal{W}_\epsilon,\underline{\omega}^r\otimes\Omega^1_X)}{\nabla H^0(\mathcal{W}_\epsilon,\mathcal{L}_r)\cap H^0(\mathcal{W}_\epsilon,\underline{\omega}^r\otimes\Omega^1_X)}=\frac{S_{r+2}^{\rm oc}(N)}{d^{r+1}S_{-r}^{\rm oc}(N)}.$$

Just as when $r=0$, there is a canonical isomorphism

$$(40)\qquad H^1_{\rm par}(X'_{\mathbb{C}_p},\mathcal{L}_r)=H^1_{\rm rig}(\mathcal{W}_\epsilon,\mathcal{L}_r)$$

between the algebraic (parabolic) de Rham cohomology over $\mathbb{C}_p$ and the rigid de Rham cohomology. As explained in [BDP, § 4.5], for each supersingular annulus $\mathcal{V}_j$, $j=1,...,s$, there is a residue map

$$(41)\qquad {\rm res}: H^1_{\rm rig}(\mathcal{W}_\epsilon,\mathcal{L}_r)\longrightarrow (H^0(\mathcal{V}_j,\mathcal{L}_r)^{\nabla=0})^\vee\simeq\mathcal{L}_r(P_j)\simeq\left({\rm Sym}^r\,H^1_{\rm dR}(E_j)\right)(-1),$$

where $E_j/\mathbb{C}_p$ stands for the supersingular elliptic curve corresponding to the point $\tilde{P}_j$.

By [BDP, Prop. 4.11], the image of $H^1_{\mathrm{par}}(X_{\mathbb{C}_p}, \mathcal{L}_r)$ in $H^1_{\mathrm{par}}(X'_{\mathbb{C}_p}, \mathcal{L}_r)$ under the restriction map consists of those classes represented by $\mathcal{L}_r$-valued differential forms ω on $\mathcal{W}_\epsilon$ whose residues at the supersingular annuli $\mathcal{V}_j$, $j = 1, ..., s$, are all zero.

Over $\mathcal{A}$, the slope decomposition arising from the action of Frobenius gives a canonical splitting

$$\mathrm{Spl}_{\text{u-r}} : \mathcal{L} \longrightarrow \underline{\omega} \tag{42}$$

of the exact sequence (20). This map can be viewed as a homomorphism of $\mathcal{O}_{\mathcal{A}}$-modules, where $\mathcal{O}_{\mathcal{A}}$ is the structure sheaf of rigid analytic functions on $\mathcal{A}$. We will also denote by the same symbol the associated map $\mathcal{L}^r \longrightarrow \underline{\omega}^r$, as well as the resulting map

$$\mathrm{Spl}_{\text{u-r}} : \bigcup_{\epsilon>0} H^0(W_\epsilon, (\mathcal{L}_r \otimes \Omega^1_X)_{\mathrm{par}}) \longrightarrow H^0(\mathcal{A}, \underline{\omega}^r \otimes \Omega^1_X) \tag{43}$$

on overconvergent sections. The image of $\mathrm{Spl}_{\text{u-r}}$ is contained in the space of p-adic modular forms of weight $k = r + 2$, and contains the space of overconvergent modular forms, but is not equal to it in general, because the splitting (42) does not extend to a rigid analytic splitting over any of the wide opens $\mathcal{W}_\epsilon$. The following definition arises naturally from our parallel discussion of the complex-analytic setting.

Definition 2.6. The image of the map $\mathrm{Spl}_{\text{u-r}}$ in (43), denoted $S_k^{\text{n-oc}}(N; \mathbb{C}_p)$, is called the space of *nearly overconvergent* modular forms of weight k on $\Gamma_1(N)$.

We refer the reader to the forthcoming work [Ur] of Urban, where this notion has also been introduced independently.

Note that the weight of a nearly overconvergent modular form always belongs to $\mathbb{Z}^{\geq 2}$, by definition. The following basic facts about nearly overconvergent modular forms are analogous to those that were observed in the complex setting:

(i) The map $\mathrm{Spl}_{\text{u-r}}$ of (43) is injective, and induces an isomorphism of p-adic Fréchet spaces:

$$\mathrm{Spl}_{\text{u-r}} : \bigcup_{\epsilon>0} H^0(\mathcal{W}_\epsilon, (\mathcal{L}_r \otimes \Omega^1_X)_{\mathrm{par}}) \xrightarrow{\sim} S_k^{\text{n-oc}}(N; \mathbb{C}_p).$$

This is a consequence of the main theorem of [CGJ], which asserts that the p-adic modular form E_2 is transcendental over the ring of overconvergent modular forms.

(ii) If K is any subfield of $\mathbb{C}_p$, the image of $\bigcup_{\epsilon>0} H^0(W_\epsilon/K, (\mathcal{L}_r \otimes \Omega^1_X)_{\mathrm{par}})$ under $\mathrm{Spl}_{\text{u-r}}$ yields a natural K-vector space $S_k^{\text{n-oc}}(N; K) \subset S_k^{\text{n-oc}}(N; \mathbb{C}_p)$.

(iii) Let $\phi = \mathrm{Spl}_{\text{u-r}}(\omega_\phi)$ be an element of $S_k^{\text{n-oc}}(N; \mathbb{C}_p)$, where ω_ϕ is a global section of $(\mathcal{L}_r \otimes \Omega^1_X)_{\mathrm{par}}$ over some $\mathcal{W}_\epsilon$. Equation (37) then allows us to write

$$\omega_\phi = \Pi_N^{\mathrm{oc}}(\phi) + \nabla s, \quad \text{with} \left\{ \begin{array}{l} \Pi_N^{\mathrm{oc}}(\phi) \in H^0(\mathcal{W}_\epsilon, \underline{\omega}^r \otimes \Omega^1) = S_k^{\mathrm{oc}}(N; K), \\ s \in H^0(\mathcal{W}_\epsilon, \mathcal{L}_r). \end{array} \right. \tag{44}$$

The overconvergent modular form $\Pi_N^{\mathrm{oc}}(\phi)$ of weight k and level N is called the *overconvergent projection* of the nearly overconvergent modular form ϕ. Note that $\Pi_N^{\mathrm{oc}}(\phi)$ is only well-defined modulo $d^{r+1}(S_{-r}^{\mathrm{oc}}(N))$, by (39).

(iv) If K is a field equipped with simultaneous embeddings into $\mathbb{C}$ and $\mathbb{C}_p$, then there are natural identifications

$$S_k^{\mathrm{nh}}(N; K) \xleftarrow{\mathrm{Spl}_{\mathrm{hdg}}} H^0(X_K, (\mathcal{L}_r \otimes \Omega^1_X)_{\mathrm{par}}) \xrightarrow{\mathrm{Spl}_{\text{u-r}}} S_k^{\text{n-oc}}(N; K).$$

It follows directly from the definitions that the holomorphic and overconvergent projections, restricted to $S_k^{\mathrm{nh}}(N; K)$ and $S_k^{\text{n-oc}}(N; K)$ respectively, take values in $S_k(N; K)$ and are equal (under the above identification of $S_k^{\mathrm{nh}}(N; K)$ and $S_k^{\text{n-oc}}(N; K)$).

(v) The map $\tilde{\nabla}$ of (32) corresponds (under the identification $\mathrm{Spl}_{\text{u-r}}$) to the operator $d = q\frac{d}{dq}$ on p-adic modular forms, i.e., the following diagram commutes:

$$\begin{array}{ccc} \bigcup_{\epsilon>0} H^0(\mathcal{W}_\epsilon, (\mathcal{L}_r \otimes \Omega^1_X)_{\mathrm{par}}) & \xrightarrow{\mathrm{Spl}_{\text{u-r}}} & S_k^{\text{n-oc}}(N;\mathbb{C}_p) \\ \downarrow{\scriptstyle \tilde{\nabla}} & & \downarrow{\scriptstyle d} \\ \bigcup_{\epsilon>0} H^0(\mathcal{W}_\epsilon, (\mathcal{L}_{r+2} \otimes \Omega^1_X)_{\mathrm{par}}) & \xrightarrow{\mathrm{Spl}_{\text{u-r}}} & S_{k+2}^{\text{n-oc}}(N;\mathbb{C}_p). \end{array}$$

A nearly overconvergent modular form admits a q-expansion, and hence Hida's ordinary projector e_{ord} of (15) extends formally to the space $S_k^{\text{n-oc}}(N;\mathbb{C}_p)$. The following lemma relates this ordinary projection to the overconvergent projection Π_N^{oc}.

Lemma 2.7. *Let ϕ be a nearly overconvergent modular form on $\Gamma_1(N)$. Its image under Hida's ordinary projector is overconvergent, and thus classical on $\Gamma_1(N) \cap \Gamma_0(p)$. More precisely,*

$$e_{\mathrm{ord}}\phi = e_{\mathrm{ord}}\Pi_N^{\mathrm{oc}}(\phi). \tag{45}$$

Proof. If s is any overconvergent section of $\mathcal{L}_r$, then a direct calculation using the relations (22) shows that

$$\mathrm{Spl}_{\text{u-r}}(\nabla s) \text{ belongs to } d(\mathbb{C}_p \otimes \mathcal{O}_{\mathbb{C}_p}[[q]]) \cdot \omega_{\mathrm{can}}^r \otimes \frac{dq}{q}.$$

But e_{ord} annihilates the image under d of any q-series with bounded denominators, so

$$e_{\mathrm{ord}}(\mathrm{Spl}_{\text{u-r}}(\nabla s)) = 0. \tag{46}$$

Now write $\phi = \mathrm{Spl}_{\text{u-r}}(\omega_\phi)$, with ω_ϕ a rigid section of $(\mathcal{L}_r \otimes \Omega^1_X)_{\mathrm{par}}$ over some $\mathcal{W}_\epsilon$. By definition of the overconvergent projection,

$$\omega_\phi = \phi_0 + \nabla s \quad \text{ with } \phi_0 \in H^0(\mathcal{W}_\epsilon, \underline{\omega}^r \otimes \Omega^1_X) \text{ and } s \in H^0(\mathcal{W}_\epsilon, \mathcal{L}_r).$$

Applying the operator $\mathrm{Spl}_{\text{u-r}}$ to this last identity gives $\phi = \Pi_N^{\mathrm{oc}}(\phi) + \mathrm{Spl}_{\text{u-r}}(\nabla s)$. The result now follows by applying e_{ord} and invoking (46). □

Let $g \in S_\ell(N;K)$ and $h \in S_m(N;K)$ be classical cusp forms defined over K, and fix embeddings of K into $\mathbb{C}$ and $\mathbb{C}_p$. The forms g and h can then be regarded simultaneously as complex and overconvergent modular forms.

Proposition 2.8. *For all $t \geq 0$, the modular form $d^t g \times h$ belongs to $S_{\ell+m+2t}^{\text{n-oc}}(N;K)$ and*

$$e_{\mathrm{ord}}(d^t g \times h) = e_{\mathrm{ord}}\Pi_N^{\mathrm{hol}}(\delta_\ell^t g \times h).$$

Proof. Recall the global section $\tilde{\nabla}^t \omega_g \otimes \tilde{\omega}_h \in H^0(X_K, (\mathcal{L}_{\ell+m+2t-2} \otimes \Omega^1_X)_{\mathrm{par}})$ that was introduced in (33). Since

$$d^t g \times h = \mathrm{Spl}_{\text{u-r}}(\tilde{\nabla}^t \omega_g \otimes \tilde{\omega}_h), \qquad \delta_\ell^t g \times h = \mathrm{Spl}_{\mathrm{hdg}}(\tilde{\nabla}^t \omega_g \otimes \tilde{\omega}_h),$$

it follows that

$$\Pi_N^{\mathrm{oc}}(d^t g \times h) = \Pi_N^{\mathrm{hol}}(\delta_\ell^t g \times h).$$

The proposition follows by applying the projector e_{ord} to this identity and invoking (45). □

We next turn to the case where the exponent appearing in Proposition 2.8 is *strictly negative*. After replacing g by its p-depletion $g^{[p]} \in S_k^{\mathrm{oc}}(N;\mathbb{C}_p)$, the form $d^{-1-t}g^{[p]} \times h$ (with $t \geq 0$) is still a p-adic modular form of weight $k := \ell + m - 2t - 2$. (Cf. for instance Théorème 5 of §2 of [Se].) The following proposition shows that it is nearly overconvergent, at least in certain cases where $k \geq 2$.

Proposition 2.9. *Assume that $0 \le t \le \min(\ell-2, m-2)$, so that $k := \ell+m-2t-2 \ge 2$. Then the p-adic modular form $d^{-1-t}g^{[p]} \times h$ belongs to $S_k^{\text{n-oc}}(N;\mathbb{C}_p)$, and in particular*

$$e_{\text{ord}}(d^{-1-t}g^{[p]} \times h) \in S_k^{\text{ord}}(N;\mathbb{C}_p) \subset S_k(\Gamma_1(N)\cap\Gamma_0(p);\mathbb{C}_p).$$

Proof. Set $r_2 = \ell-2$ and $r_3 = m-2$. Since $1-VU$ annihilates $H^1_{\text{rig}}(\mathcal{W}_\epsilon, \mathcal{L}_{r_2})$, it follows that the overconvergent section $\omega_{g^{[p]}}$ of $\underline{\omega}^{r_2} \otimes \Omega^1_X$ is ∇-exact, i.e., there exists $G^{[p]} \in H^0(\mathcal{W}_\epsilon, \mathcal{L}_{r_2})$ satisfying $\omega_{g^{[p]}} = \nabla G^{[p]}$. The q-expansion of the section $G^{[p]}$ can be written down explicitly in terms of the differentials ω_{can} and η_{can} on the Tate curve. Using (22), one checks that it is equal to

$$G^{[p]}(q) = \sum_{j=0}^{r_2} (-1)^j j! \binom{r_2}{j} d^{-j-1} g^{[p]}(q) \omega_{\text{can}}^{r_2-j} \eta_{\text{can}}^j. \tag{47}$$

Set $r = r_2 + r_3 - t$, and let $\mathcal{L}^{(1)}, \ldots, \mathcal{L}^{(r)}$ denote r copies of the sheaf $\mathcal{L}_1$ over X, numbered consecutively. If $S = \{i_1, \ldots, i_s\}$ with $i_1 < i_2 < \ldots < i_s$ is any subset of $\{1, \ldots, r\}$, we set

$$\mathcal{L}^S := \mathcal{L}^{(i_1)} \otimes \cdots \otimes \mathcal{L}^{(i_s)}.$$

Let $\epsilon_S := \frac{1}{a!}\sum_{\sigma\in\Sigma_s} \text{sgn}(\sigma)\sigma$ denote the symmetrisation projector and write $\mathcal{L}_S := \epsilon_S \mathcal{L}^S$, viewed as a subsheaf of $\mathcal{L}^S$ in the obvious way.

Choose now two subsets B and C of $\{1, \ldots, r\}$ of cardinalities r_2 and r_3 respectively, whose union is equal to $\{1, \ldots, r\}$. Such a choice is possible, since $r = r_2 + r_3 - t \le r_2 + r_3$. Put $A' = B \cap C$ and $A = \{1, \ldots, r\} \setminus A'$.

Poincaré duality on the fibers of $\mathcal{E}$ gives rise to a duality $\mathcal{L}_1 \times \mathcal{L}_1 \longrightarrow \mathcal{O}_X(-1)$ of sheaves. Since the cardinality of A' is t, this in turn induces a map of sheaves

$$\mathcal{L}^B \otimes \mathcal{L}^C \longrightarrow \mathcal{L}^A(-t). \tag{48}$$

The natural inclusions $\mathcal{L}_{r_2} \subset \mathcal{L}^{r_2} \simeq \mathcal{L}^B$ and $\mathcal{L}_{r_3} \subset \mathcal{L}^{r_3} \simeq \mathcal{L}^C$ allow us to build certain rigid sections of $\mathcal{L}^B$ and $(\mathcal{L}^C \otimes \Omega^1_X)_{\text{par}}$ out of $G^{[p]}$ and ω_h respectively, denoted

$$G^{[p]}(B) \in H^0(\mathcal{W}_\epsilon, \mathcal{L}^B), \qquad \omega_h(C) \in H^0(\mathcal{W}_\epsilon, (\mathcal{L}^C \otimes \Omega^1_X)_{\text{par}}).$$

Taking the tensor product of these two sections and applying (48) gives an overconvergent section $G^{[p]}(B) \otimes \omega_h(C) \in H^0(\mathcal{W}_\epsilon, (\mathcal{L}^A \otimes \Omega^1_X)_{\text{par}})$, whose symmetrisation can be viewed as an element $\epsilon_A(G^{[p]}(B) \otimes \omega_h(C)) \in H^0(\mathcal{W}_\epsilon, (\mathcal{L}_{r-t} \otimes \Omega^1_X)_{\text{par}})$. A direct calculation using (47) reveals that

$$\text{Spl}_{\text{u-r}}(\epsilon_A(G^{[p]}(B) \otimes \omega_h(C))) = (-1)^t t! (d^{-1-t} g^{[p]} \times h) \omega_{\text{can}}^{r_1} \left(\frac{dq}{q}\right), \tag{49}$$

which implies that $d^{-1-t}g^{[p]} \times h$ belongs to $S_k^{\text{n-oc}}(N;\mathbb{C}_p)$, as desired. $\square$

The sheaf $\mathcal{L}_r$ equipped with the Gauss-Manin connection is an overconvergent F-crystal in the sense of [Cole94, Sec. 10], i.e., the action of Frobenius on the relative de Rham cohomology $\mathcal{L}_r$ induces a horizontal morphism

$$\Phi_{\mathcal{L}_r,\epsilon,\epsilon'} : \Phi^*(\mathcal{L}_r)_{|\mathcal{W}_\epsilon} \longrightarrow (\mathcal{L}_r)_{|\mathcal{W}_{\epsilon'}},$$

for suitable $\epsilon, \epsilon' > 0$. These give rise to the Frobenius endomorphism

$$\Phi_r := \text{comp}_{\epsilon'}^{-1} \circ \Phi_{\mathcal{L}_r,\epsilon,\epsilon'} \circ \Phi^* \circ \text{comp}_\epsilon$$

of $H^1_{\text{dR}}(X, \mathcal{L}_r)$, which is equal to Φ when $r = 0$. By abuse of notation, we will continue to write Φ instead of Φ_r, since the context will make it clear which Frobenius endomorphism is being referred to.

The action of Φ on $H^1_{\mathrm{dR}}(X_{\mathbb{C}_p}, \mathcal{L}_r)$ induces a decomposition of its f-isotypic part, for all ordinary modular forms f of weight $k = r + 2$:

$$H^1_{\mathrm{dR}}(X_{\mathbb{C}_p}, \mathcal{L}_r, \nabla)^f = H^0(X_{\mathbb{C}_p}, \underline{\omega}^r \otimes \Omega^1_X)^f \oplus H^1_{\mathrm{dR}}(X_{\mathbb{C}_p}, \mathcal{L}_r)^{f,\text{u-r}},$$

where the superscript u-r denotes the unit-root subspace, i.e., the part of the cohomology on which the Frobenius endomorphism acts with slope zero. This decomposition plays a role somewhat analogous to that of the Hodge decomposition in the complex setting.

The following lemma generalizes Lemma 2.2 to higher weight.

Lemma 2.10. *If $\phi = \mathrm{Spl}_{\text{u-r}}(\omega_\phi) \in S_k^{\text{n-oc}}(N)_0$ is an ordinary overconvergent p-adic modular form of weight $k = r + 2 > 2$ on $\Gamma_1(N)$, then the class of ω_ϕ belongs to $H^1_{\mathrm{dR}}(X_K, \mathcal{L}_r)^{\Phi,k-1}$. Furthermore, the assignment $\phi \mapsto [\omega_\phi]$ induces an isomorphism*

$$S_k^{\mathrm{ord}}(N) \xrightarrow{\sim} H^1_{\mathrm{dR}}(X_K, \mathcal{L}_r)^{\Phi,k-1}.$$

Proof. The Frobenius morphism Φ is related to the operator V on overconvergent modular forms of weight k by

$$\Phi(\omega_f) = p^{k-1}\omega_{Vf},$$

as can be seen from a computation on the Tate object $(\mathbb{G}_m/q^{\mathbb{Z}}, \frac{dt}{t})$ after noting that $\Phi(\frac{dt}{t}) = p\frac{dt}{t}$. The relation between the operators U and Φ on cohomology (relative to the identifications described above between differentials and weight two modular forms) is therefore given by

$$\Phi = p^{k-1}V = p^{k-1}U^{-1}.$$

If ϕ belongs to the slope zero subspace for the action of U, the class of the rigid differential ω_ϕ lies in the slope $k-1$ subspace of $H^1_{\mathrm{rig}}(\mathcal{W}_\epsilon, \mathcal{L}_r)^{\Phi,k-1}$ for the action of Φ. Since Φ acts with slope $k/2$ on the target of the residue map, the class ω_ϕ automatically has vanishing residues at the supersingular annuli when $k > 2$, and the class of ω_ϕ can therefore be viewed as an element of $H^1_{\mathrm{dR}}(X_K, \mathcal{L}_r)$. The injectivity and surjectivity assertions in the second statement follow from Cor. 6.3.1. and Prop. 6.6 of [Cole95], just as in the proof of Lemma 2.2. ☐

We also record the generalisation of Prop. 2.3 for $k \geq 2$.

Proposition 2.11. *Let η be any class in $H^1_{\mathrm{dR}}(X_K, \mathcal{L}_r)^{\text{u-r}}$, and let ϕ be a nearly overconvergent modular form of weight $k > 2$ on $\Gamma_1(N)$ with vanishing residues at the supersingular annuli. Then*

$$\langle \eta, \omega_\phi \rangle = \langle \eta, \omega_{\phi^{\mathrm{ord}}} \rangle, \qquad \text{where } \phi^{\mathrm{ord}} := e_{\mathrm{ord}}\phi.$$

In other words the expression $\langle \eta, \omega_\phi \rangle$ depends only on the ordinary projection of ϕ, and Poincaré duality induces a well-defined pairing

$$\langle \ , \ \rangle : H^1_{\mathrm{dR}}(X_{\mathbb{C}_p}, \mathcal{L}_r)^{\text{u-r}} \times e_{\mathrm{ord}} S_k^{\text{n-oc}}(N) \longrightarrow \mathbb{C}_p.$$

Proof. The Poincaré pairing of (28) is compatible with the Frobenius endomorphism Φ and hence gives rise to a well-defined pairing

$$\langle \ , \ \rangle : H^1_{\mathrm{dR}}(X_K, \mathcal{L}_r)^{\text{u-r}} \times H^1_{\mathrm{dR}}(X_K, \mathcal{L}_r)^{\Phi,k-1} \longrightarrow \mathbb{C}_p(-1-r). \tag{50}$$

The proposition therefore follows from Lemma 2.10. ☐

2.5. **Periods of modular forms.** The newform $f \in S_k(N_f, \chi_f; K_f) \subset S_k(N_f; K_f)$ generates an automorphic representation of $\mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}})$, denoted π_f. For any multiple N of N_f and any field $K \supset K_f$, let $S_k(N; K)[\pi_f]$ denote the f-isotypic subspace of $S_k(N; K)$, attached to the automorphic representation π_f. The space $S_k(N; K)[\pi_f]$ is a finite-dimensional K-vector space of dimension $\sigma_0(N/N_f)$, where $\sigma_0(m)$ denotes as usual the number of divisors of m, and a basis for $S_k(N; K)[\pi_f]$ is given by

$$\{f(dz) = f(q^d)\}_{d|(N/N_f)}. \tag{51}$$

Similar remarks and notations apply to the modular forms g and h.

Fix from now on a field $K \supset K_f$, endowed with a complex embedding $K \subset \mathbb{C}$.

Lemma 2.12. *For all $\breve{f} \in S_k(N;K)[\pi_f]$ and all $\phi \in S_k(N;K)$, the Petersson scalar product $(\breve{f}, \phi)_N$ is a K-rational multiple of the period $(f, f)_N$.*

Proof. Let $\mathbb{T}_N$ be the Hecke algebra of level N generated over $\mathbb{Q}$ by the "good" Hecke operators T_n with $\gcd(n, N) = 1$. The eigenform f corresponds to an idempotent $e_f \in \mathbb{T}_N \otimes K_f$ which induces a projection of $S_k(N;K)$ onto $S_k(N;K)[\pi_f]$. The vector $\breve{f}$ is orthogonal to the kernel of e_f and therefore $(\breve{f}, \phi)_N = (\breve{f}, e_f\phi)_N$ depends only on the projection of ϕ to $S_k(N;K)[\pi_f]$. By the remark preceeding equation (51), it is thererefore enough to show that, for all divisors d_1 and d_2 of N/N_f,

$$(f(d_1z), f(d_2z))_N = \varrho(f, d_1, d_2) \cdot (f, f)_N, \text{ with } \varrho(f, d_1, d_2) \in K_f.$$

If $d_1d_2 = 1$, this is clear. Otherwise, let q be a prime dividing d_1d_2. The compatibility beween the weight k slash operator and the Petersson scalar product shows that

$$(f(d_1z), f(d_2z))_N = q^{-k}(f(d_1/qz), f(d_2/qz))_N, \qquad \text{if } q|d_1 \text{ and } q|d_2. \tag{52}$$

Otherwise, assume without loss of generality that q divides d_1 but not d_2. We then have

$$(f(d_1z), f(d_2z))_N = \begin{cases} \frac{q^{1-k}a_q(f)}{q+1}(f(d_1/qz), f(d_2z))_N & \text{if } q||d_1, \\ \frac{q^{1-k}a_q(f)}{q}(f(d_1/qz), f(d_2z))_N & \\ \qquad - \chi_f(q)q^{-k-1}(f(d_1/q^2z), f(d_2z))_N, & \text{if } q^2|d_1. \end{cases} \tag{53}$$

These relations imply that

$$\varrho(f, d_1, d_2) = \begin{cases} q^{-k}\varrho(f, d_1/q, d_2/q) & \text{if } q|d_1 \text{ and } q|d_2; \\ \frac{q^{1-k}a_q(f)}{q+1}\varrho(f, d_1/q, d_2) & \text{if } q||d_1 \text{ and } q \nmid d_2; \\ \frac{q^{1-k}a_q(f)}{q}\varrho(f, d_1/q, d_2) - \chi_f(q)q^{-1-k}\varrho(f, d_1/q^2, d_2), & \text{if } q^2|d_1 \text{ and } q \nmid d_2. \end{cases} \tag{54}$$

Equation (54), together with the fact that $\varrho(f, 1, 1) = 1$ and $\varrho(f, d_1, d_2) = \bar{\varrho}(f, d_2, d_1)$ make it clear (by induction on d_1d_2) that $\varrho(f, d_1, d_2)$ belongs to K_f for all d_1, d_2 dividing N/N_f. $\square$

The above proposition allows us to associate to any (not necessarily new) eigenform $\breve{f} \in S_k(N;K_f)$, and to any modular form $\phi \in S_k(N;K)$, a K-rational period

$$J(\breve{f}, \phi) := \frac{(\breve{f}, \phi)_N}{(f, f)_N} \in K.$$

Let

$$\eta^{\text{a-h}}_{\breve{f}} \in \overline{H^0(X_{\mathbb{C}}, \underline{\omega}^r \otimes \Omega^1_X)} \subset H^1_{\text{dR}}(X_{\mathbb{C}}, \mathcal{L}_r) \tag{55}$$

denote the class of the anti-holomorphic form $\frac{1}{\langle \overline{\omega_{\breve{f}}}, \omega_{\breve{f}} \rangle} \cdot \overline{\omega_{\breve{f}}}$, and let $\eta_{\breve{f}}$ denote its natural image in $H^1(X_{\mathbb{C}}, \underline{\omega}^{-r})$ under the projection in (27).

Corollary 2.13. *The class $\eta_{\breve{f}}$ belongs to $H^1(X_{K_f}, \underline{\omega}^{-r})$.*

Proof. By Lemma 2.12, for any $\omega_\phi \in H^0(X_{K_f}, \underline{\omega}^r \otimes \Omega^1_X)$ associated to a cusp form $\phi \in S_k(N;K_f)$ with K_f-rational fourier coefficients, we have

$$\langle \eta_{\breve{f}}, \omega_\phi \rangle = \langle \eta^{\text{a-h}}_{\breve{f}}, \omega_\phi \rangle = \frac{(\breve{f}, \phi)_N}{(f, f)_N} \text{ belongs to } K_f. \tag{56}$$

(Where the first occurence of $\langle \ , \ \rangle$ designates the pairing of (29) induced from Poincaré duality, with $K = \mathbb{C}$.) But $H^1(X_{K_f}, \underline{\omega}^{-r})$ can be characterized as the set of $\eta \in H^1(X_{\mathbb{C}}, \underline{\omega}^{-r})$ satisfying (56) for all $\phi \in S_k(N;K_f)$. Corollary 2.13 follows. $\square$

2.6. **Hida families.** Fix a rational prime $p > 2$ and assume that the field $K \supset K_f$ of the previous section is furnished with a p-adic embedding $K \hookrightarrow \mathbb{C}_p$. Let $\mathcal{O}$ denote the ring of integers of the p-adic closure of K in $\mathbb{C}_p$.

For every $p \nmid N$ write the characteristic polynomial of a Frobenius element at p acting on the two-dimensional Galois representation over $K_f \otimes \mathbb{Q}_p$ associated by Deligne to f as

$$L^{(p)}(f;T) = (1 - a_p(f)T + \chi_f(p)p^{k-1}T) = (1 - \alpha_{f,p}T)(1 - \beta_{f,p}T), \tag{57}$$

where χ is the nebentype character of f and $\alpha_{f,p}$ and $\beta_{f,p} \in \bar{\mathbb{Q}}$ are the reciprocal roots of the polynomial $L^{(p)}(f;T)$.

The modular form f is said to be *ordinary* at p if the two reciprocal roots of (57) can be labelled in such a way that $\alpha_{f,p}$ is a p-adic unit, i.e., belongs to $\mathcal{O}^\times$. The modular form given by

$$f^{(p)}(q) := f(q) - \beta_{f,p} f(q^p)$$

is called the *ordinary p-stabilization of f*; it has level pN, although it is only new at the primes dividing N, and it is again an eigenform for all the Hecke operators. To be precise,

$$f^{(p)}|T_\ell = a_\ell(f) f^{(p)}, \quad \forall \ell \neq p, \qquad f^{(p)}|U_p = \alpha_{f,p} f^{(p)}.$$

Set $\Gamma = 1 + pN\mathbb{Z}_p$ and let $\Lambda = \mathcal{O}[[\Gamma]]$ be the completed group ring of Γ. The *weight space* is defined to be

$$\Omega = \mathrm{Spf}(\Lambda)(\mathcal{O}) = \mathrm{Hom}_{\mathcal{O}-\mathrm{alg}}(\Lambda, \mathcal{O}),$$

which may naturally be identified with the space $\mathrm{Hom}_{\mathrm{cts}}(\Gamma, \mathcal{O}^\times)$ of continuous characters of Γ. The subset of classical characters of Ω is defined to be

$$\Omega_{\mathrm{cl}} = \{\chi_k := (\gamma \mapsto \gamma^k), \qquad \text{with } k \in \mathbb{Z}^{\geq 2}\}.$$

Given any finite flat extension Λ_f of Λ, let

$$\Omega_f := \mathrm{Spf}(\Lambda_f)(\mathcal{O}) = \mathrm{Hom}(\Lambda_f, \mathcal{O}).$$

This space is endowed with a natural p-adic topology and is equipped with a natural projection

$$\kappa : \Omega_f \to \Omega$$

to weight space induced by the inclusion $\Lambda \subseteq \Lambda_f$ of $\mathcal{O}$-algebras. A point $x \in \Omega_f$ for which $\kappa(x)$ belongs to Ω_{cl} will be referred to as a *classical point* of Ω_f, and the set of all such classical points will be denoted $\Omega_{f,\mathrm{cl}}$.

We will mostly work with the following definition of a *Hida family* of p-adic modular forms, which is slightly more restrictive than what can sometimes be found in the literature.

Definition 2.14. Let $N_f \geq 1$ be an integer and let p be a prime not dividing N_f. A *Hida family* of tame level N_f is a quadruple $(\Lambda_f, \Omega_f, \Omega_{f,\mathrm{cl}}, \mathbf{f})$, where

(i) Λ_f is a finite flat extension of Λ;

(ii) Ω_f is a non-empty open subset of $X_f := \mathrm{Hom}(\Lambda_f, \mathbb{C}_p)$ and $\Omega_{f,\mathrm{cl}}$ is a p-adically dense subset of Ω_f whose image under κ lies in Ω_{cl};

(iii) $\mathbf{f} := \sum \mathbf{a}_n q^n \in \Lambda_f[[q]]$ is a formal q-series with coefficients in Λ_f such that, for all $x \in \Omega_{f,\mathrm{cl}}$, the power series

$$f_x^{(p)} := \sum_{n=1}^{\infty} \mathbf{a}_n(x) q^n$$

is the q-expansion of the ordinary p-stabilization of a normalised newform (denoted f_x) of weight $\kappa(x)$ on $\Gamma_1(N_f)$.

The collection of $\{f_x\}_{x\in\Omega_{f,\mathrm{cl}}}$ arising from Hida theory can be thought of as a p-adically coherent collection of eigenforms on $\Gamma_1(N_f)$ of varying weights. In particular, the fourier coefficients $a_n(f_x)$ are analytic functions of x when $p\nmid n$, (but not when $p|n$, in general). The following theorem of Hida reveals the ubiquity of Hida families in the above sense.

Theorem 2.15 (Hida)**.** [Hi86b] *Let f be an ordinary newform in $S_k(N_f;K_f)$. There exists a Hida family $(\Lambda_f,\Omega_f,\Omega_{f,\mathrm{cl}},\mathbf{f})$ of tame level N_f and a classical point $x_0\in\Omega_f$ satisfying*

$$\kappa(x_0)=k, \qquad f_{x_0}=f.$$

By shrinking Ω_f if necessary, we can and will assume that $\kappa(x)=k \pmod{p-1}$ for all $x\in\Omega_f$; in particular the integers $\kappa(x)$ and k have the same parity for all classical $x\in\Omega_f$.

It will be also convenient, for later purposes, to dispose of a somewhat more flexible notion of p-adic families of modular forms, interpolating classical modular forms which are not necessarily new, or even Hecke eigenvectors; as well, it will be convenient to allow the fourier coefficients to belong to more general coefficients rings which are not necessarily finite over Λ. This leads to the following definition:

Definition 2.16. A Λ-adic modular form of tame level N is a quadruple $(R,\Omega_\phi,\Omega_{\phi,\mathrm{cl}},\boldsymbol{\phi})$, where

(i) R is a complete, finitely generated (but not necessarily finite), flat extension of Λ;
(ii) Ω_ϕ is an open subset of $\mathrm{Hom}(R,\mathbb{C}_p)$ and $\Omega_{\phi,\mathrm{cl}}$ is a dense subset of Ω_ϕ;
(ii) $\boldsymbol{\phi}:=\sum \mathbf{a}_nq^n\in R[[q]]$ is a formal q-series with coefficients in R such that, for all $x\in\Omega_{\phi,\mathrm{cl}}$, the power series

$$\phi_x^{(p)}:=\sum_{n=1}^{\infty}\mathbf{a}_n(x)q^n\in\mathbb{C}_p[[q]]$$

is the q-expansion of a classical ordinary cusp form in $S_{\kappa(x)}(\Gamma_1(N)\cap\Gamma_0(p);\mathbb{C}_p):=S_{\kappa(x)}(\Gamma_1(N)\cap\Gamma_0(p);\mathbb{Q})\otimes\mathbb{C}_p$.

The following examples of Λ-adic families of modular forms are of importance in our discussion.

1. Families of old forms. Let $f\in S_k(N_f;K_f)$ be a newform of level N_f and let N be some multiple of N_f with $p\nmid N$. Let $\breve{f}\in S_k(N;K_f)[\pi_f]$ be an element of the old class of level N associated to f. By the remark preceding equation (51), there are (unique) scalars $\lambda_d\in K$ indexed by the divisors of N/N_f and satisfying

$$\breve{f}=\sum_{d|(N/N_f)}\lambda_d\cdot f(q^d).$$

The p-stabilisation of the modular form $f(q^d)$ is the weight k specialisation of the formal q-series

$$\mathbf{f}(q^d):=\sum_n \mathbf{a}_nq^{dn},$$

where $(\Lambda_f,\Omega_f,\Omega_{f,\mathrm{cl}},\mathbf{f})$ is the Hida family of tame level N_f attached to f via Theorem 2.15. We can then set

$$\breve{\mathbf{f}}(q):=\sum_{d|(N/N_f)}\lambda_d\mathbf{f}(q^d). \tag{58}$$

The triple $(\Lambda_f,\Omega_f,\Omega_{f,\mathrm{cl}},\breve{\mathbf{f}})$ is a Λ-adic modular form whose classical specialisations are eigenvectors for the good Hecke operators T_n with $\gcd(n,N)=1$, and for the U operator, but they are not new at the primes dividing N/N_f and specialise to the old-form $\breve{f}$ at the weight k point $x_0\in\Omega_{f,\mathrm{cl}}$ alluded to in Theorem 2.15.

2. Products of modular forms. Let $(\Lambda_g, \Omega_g, \mathbf{g})$ and $(\Lambda_h, \Omega_h, \mathbf{h})$ be Λ-adic modular forms of tame level N (for example, Hida families of eigenforms arising from Theorem 2.15 or families of oldforms constructed as in the previous paragraph). Let $\Lambda_{gh} := \Lambda_g \otimes_{\mathcal{O}} \Lambda_h$ be the finitely generated (but not finite) Λ-algebra equipped with the natural diagonal embedding $\Lambda \longrightarrow \Lambda_g \otimes \Lambda_h$ sending the group-like element $[a] \in \Lambda$ to $[a] \otimes [a]$. Set

$$\Omega_{gh} := \Omega_g \times \Omega_h, \qquad \Omega_{gh,\mathrm{cl}} := \Omega_{g,\mathrm{cl}} \times \Omega_{h,\mathrm{cl}}.$$

Then the quadruple $(\Lambda_{gh}, \Omega_{gh}, \Omega_{gh,\mathrm{cl}}, e_{\mathrm{ord}}(\mathbf{g} \times \mathbf{h}))$, where the power series $\mathbf{g} \times \mathbf{h}$ is viewed as an element of $\Lambda_{gh}[[q]]$ in the natural way, and e_{ord} is Hida's ordinary projection operator, is an example of a Λ-adic family of modular forms with Fourier coefficients in Λ_{gh}.

3. Derivatives of modular forms. Recall the operators U, V defined in (13), and the differential operator $d := q\frac{d}{dq}$, which induces a map from p-adic modular forms of weight k to p-adic modular forms of weight $k+2$. Recall that

$$\Omega := \mathrm{Spf}(\Lambda)(\mathcal{O}) = \mathrm{Hom}_{\mathcal{O}-\mathrm{alg}}(\Lambda, \mathcal{O}) = \mathrm{Hom}(\Gamma, \mathcal{O}^\times)$$

and notice that for any $n \in \mathbb{Z}$ such that $p \nmid n$, the group-like element $[n]$ gives rise to a function

$$[n] : \Omega \longrightarrow \mathcal{O}$$

whose value at a character $k \in \Omega$ represented by $\gamma \mapsto \gamma^k$, is simply $[n](k) = n^k$.

Let $\mathbf{g}$ and $\mathbf{h}$ be Λ-adic families of eigenforms of tame level N (but not necessarily new of that level). Assume that p is a prime that does not divide N, and let $a_p(\mathbf{g}) \in \Lambda_g$, $a_p(\mathbf{h}) \in \Lambda_h$ denote the Hecke eigenvalues associated to the Hecke operator T_p. Let

$$\mathbf{g}^{[p]} := (1 - VU)\mathbf{g} = (1 - a_p(\mathbf{g})V + [p]p^{-1}V^2)\mathbf{g} = \sum_{p \nmid n} a_n(\mathbf{g})q^n$$

be the Λ-adic counterpart of the modular form defined in (14). The specialisation $g_y^{[p]}$ of $\mathbf{g}^{[p]}$ at a classical point $y \in \Omega_{g,\mathrm{cl}}$ can either be viewed as a p-adic modular form of tame level N as in (14), or as a classical modular form of level Np^2. Note that $\mathbf{g}^{[p]}$ is not ordinary; in fact it lies in the kernel of the U operator. But the fact that $\mathbf{g}^{[p]}$ has fourier coefficients supported on the integers prime to p allows the formal q-series

$$d^\bullet \mathbf{g}^{[p]} := \sum_{p \nmid n} [n] a_n(\mathbf{g}) q^n \tag{59}$$

to be viewed as an element of $\Lambda \otimes_{\mathcal{O}} \Lambda_g[[q]]$. The specialization of this q-series at a classical point $(t, y) \in \Omega_{\mathrm{cl}} \times \Omega_{g,\mathrm{cl}}$ is simply

$$(d^\bullet \mathbf{g}^{[p]})_{(t,y)} = d^t g_y^{[p]}.$$

Define $R_{gh} := \Lambda \otimes_{\mathcal{O}} \Lambda_g \otimes_{\mathcal{O}} \Lambda_h$, regarded as a Λ-algebra by mapping the group-like element $[a] \in \Gamma$ to $[a^2] \otimes [a] \otimes [a]$, so that the map from $\mathrm{Hom}(R_{gh}, \mathbb{C}_p) = \Omega \times \Omega_g \times \Omega_h$ to the weight space sends the classical point $(t, y, z) \in \mathbb{Z}^{\geq 0} \times \Omega_{g,\mathrm{cl}} \times \Omega_{h,\mathrm{cl}}$ to $\kappa(y) + \kappa(z) + 2t$.

Let $d^\bullet \mathbf{g}^{[p]} \times \mathbf{h}$ denote the product of $d^\bullet \mathbf{g}^{[p]}$ and $\mathbf{h}$, viewed as a formal series with coefficients in R_{gh}. Let $e_{\mathrm{ord}}(d^\bullet \mathbf{g} \times \mathbf{h})$ denote its image under the ordinary projection operator. The specialisation of $e_{\mathrm{ord}}(d^\bullet \mathbf{g}^{[p]} \times \mathbf{h})$ at (t, y, z) is equal to

$$e_{\mathrm{ord}}(d^\bullet \mathbf{g}^{[p]} \times \mathbf{h})_{t,y,z} = e_{\mathrm{ord}}(d^t g_y^{[p]} \times h_z^{(p)}) = e_{\mathrm{ord}}(d^t g_y^{[p]} \times h_z), \tag{60}$$

where the last equality is a consequence of the following simple but extremely useful lemma:

Lemma 2.17. *If g and h are p-adic modular forms of tame level N, then $g^{[p]} \times (Vh)$ is in the kernel of the U operator, and in particular*

$$e_{\mathrm{ord}}(g^{[p]} \times Vh) = 0.$$

Proof. This follows from the fact that $a_n(g^{[p]} \times (Vh)) = 0$ whenever $p|n$. $\square$

The above discussion is summarised in the following proposition, which plays a key role in the construction of the triple Garrett-Rankin p-adic L-function described in Chapter 4.

Proposition 2.18. *Let*

$$\Omega_{gh,\mathrm{cl}} := \{(t,y,z) \in \mathbb{Z} \times \Omega_{g,\mathrm{cl}} \times \Omega_{h,\mathrm{cl}}, \quad t > -\min(\kappa(y),\kappa(z)).\}$$

The quadruple

$$e_{\mathrm{ord}}(d^{\bullet}\mathbf{g}^{[p]} \times \mathbf{h}) := (R_{gh}, \Omega \times \Omega_g \times \Omega_h, \Omega_{gh,\mathrm{cl}}, e_{\mathrm{ord}}(d^{\bullet}\mathbf{g}^{[p]} \times \mathbf{h}))$$

is an ordinary Λ-adic modular form of tame level N. For all $(t,y,z) \in \Omega_{gh,\mathrm{cl}}$, the specialisation of this family at (t,y,z) is the classical modular form (with coefficients in $\mathbb{C}_p$)

$$e_{\mathrm{ord}}(d^t g_y^{[p]} \times h_z) \in S_k(\Gamma_1(N) \cap \Gamma_0(p); \mathbb{C}_p).$$

This specialisation has algebraic fourier coefficients (lying in $K = K_{g_y}K_{h_z}$) when $t \geq 0$.

We conclude this section by describing the Λ-adic interpolation of the periods that arise in Lemma 2.12. Write $\mathbf{S}^{\mathrm{ord}}(N;R)$ for the space of Λ-adic modular forms with coefficients in the Λ-algebra R. A Hida family $(\Lambda_f, \Omega_f, \mathbf{f})$ of eigenforms in the sense of definition 2.14 gives rise to a subspace

$$\mathbf{S}^{\mathrm{ord}}(N;\Lambda_f)[\pi_f] := \left\{ \breve{\mathbf{f}} \in \mathbf{S}^{\mathrm{ord}}(N;\Lambda_f) \quad \text{such that } T_n\breve{\mathbf{f}} = \mathbf{a}_n\breve{\mathbf{f}}, \quad \text{for all } (n,N)=1 \right\}.$$

Letting Λ_f' denote the fraction field of the integral domain Λ_f, the vector space $\mathbf{S}^{\mathrm{ord}}(N;\Lambda_f')[\pi_f]$ is finite-dimensional over Λ_f' and has for basis the set $\{\mathbf{f}(q^d)\}_{d|(N/N_f)}$ of Λ-adic forms. Let $\mathbf{f} \in \mathbf{S}^{\mathrm{ord}}(N_f,\Lambda_f)$ be a Hida family of ordinary eigenforms, and let $\breve{\mathbf{f}} \in \mathbf{S}^{\mathrm{ord}}(N,\Lambda_f)[\pi_{\mathbf{f}}]$ be an associated test vector of tame level N.

Let $\boldsymbol{\phi} = (R, \Omega_\phi, \Omega_{\phi,\mathrm{cl}}, \boldsymbol{\phi}) \in \mathbf{S}^{\mathrm{ord}}(N;R)$ be a Λ-adic modular form, and let $(x,y) \in \Omega_{f,\mathrm{cl}} \times \Omega_{\phi,\mathrm{cl}}$ be a pair of points with $\kappa(x) = \kappa(y)$. The specialisation $\phi_y^{(p)}$ of $\boldsymbol{\phi}$ at $y \in \Omega_{\phi,\mathrm{cl}}$ need not be the p-stabilisation of a classical modular form, but its projection $e_{f_x}\phi_y^{(p)}$ to the f_x-isotypic component is the p-stabilisation of a classical modular form, which shall be denoted $\phi_{x,y}$.

If R is any flat Λ-algebra with associated analytic space $\Omega_R = \mathrm{Hom}(R,\mathcal{O})$, the elements of the ring $\Lambda_f' \otimes_\Lambda R$ can be viewed as "rational functions" on the fiber product $\Omega_f \times_\Omega \Omega_R$ with poles at finitely many $x \in \Omega_f$. Given $J \in \Lambda_f' \otimes_\Lambda R$ and $(x,y) \in \Omega_f \times_\Omega \Omega_R$, we write $J(x,y)$ for the value of J at (x,y), when it is defined. Let $(\ ,\ )_{N,p}$ denote the Petersson scalar product on modular forms attached to the group $\Gamma_1(N) \cap \Gamma_0(p)$.

Lemma 2.19. *For all $\breve{\mathbf{f}} \in \mathbf{S}^{\mathrm{ord}}(N;\Lambda_f)[\pi_f]$ and all $\boldsymbol{\phi} = (R, \Omega_\phi, \Omega_{\phi,\mathrm{cl}}, \boldsymbol{\phi}) \in \mathbf{S}^{\mathrm{ord}}(N;R)$, there exists (a unique) $J(\breve{\mathbf{f}},\boldsymbol{\phi}) \in \Lambda_f' \otimes_\Lambda R$ such that, for all classical points $(x,y) \in \Omega_{f,\mathrm{cl}} \times_{\Omega_{\mathrm{cl}}} \Omega_{\phi,\mathrm{cl}}$,*

$$J(\breve{\mathbf{f}},\boldsymbol{\phi})(x,y) = \frac{(\breve{f}_x^{(p)}, \phi_{x,y}^{(p)})_{N,p}}{(f_x^{(p)}, f_x^{(p)})_{N,p}} = \frac{(\breve{f}_x, \phi_{x,y})_N}{(f_x,f_x)_N} = \langle \eta_{f_x}, \phi_{x,y} \rangle, \tag{61}$$

where the last pairing is the Poincaré duality between $H^1(X_{\mathbb{C}_p}, \underline{\omega}^{-r})$ and $H^0(X_{\mathbb{C}_p}, \underline{\omega}^{r} \otimes \Omega^1_X)$.

Proof. Let $\mathbb{T}_N$ be the Λ-adic Hecke algebra of tame level N generated by the Hecke operators T_n with $\gcd(n,Np)=1$ and U acting on $\mathbf{S}^{\mathrm{ord}}(N;\Lambda)$. The Hida family $\mathbf{f}$ corresponds to an idempotent $e_{\mathbf{f}} \in \mathbb{T}_N \otimes_\Lambda \Lambda_f'$ which induces a projection of $\mathbf{S}^{\mathrm{ord}}(N;\Lambda)$ to $\mathbf{S}^{\mathrm{ord}}(N;\Lambda_f')[\pi_f]$. Hence $e_{\mathbf{f}}\boldsymbol{\phi}$ is a $(\Lambda_f' \otimes_\Lambda R)$-linear combination of the forms $\mathbf{f}(q^d)$ with $d|N/N_f$, while the same is of course true for $\breve{\mathbf{f}}$. It is therefore enough to show that, for all divisors d_1 and d_2 of N/N_f,

$$\frac{(f_x^{(p)}(q^{d_1}), f_x^{(p)}(q^{d_2}))_{N,p}}{(f_x^{(p)}, f_x^{(p)})_{N,p}} = \varrho(\mathbf{f},d_1,d_2)(x), \text{ for some } \varrho(\mathbf{f},d_1,d_2) \in \Lambda_f \otimes \mathbb{Q}.$$

But the proof of Lemma 2.12 with f replaced by $f_x^{(p)}$ and $\Gamma_1(N)$ by $\Gamma_1(N)\cap\Gamma_0(p)$ shows that $\varrho(f_x^{(p)}, d_1, d_2)$ (defined in the obvious way) is a polynomial involving the expressions $q^{\kappa(x)}$, $a_q(f_x)$, $\frac{1}{q}$ and $\frac{1}{q+1}$ as q ranges over the divisors of $d_1 d_2$. Since the primes q that arise are different from p, these expressions all belong to $\Lambda_f\otimes\mathbb{Q}$ and hence the same is true of $\varrho(\mathbf{f}, d_1, d_2)$. It follows that the expression $J(\breve{\mathbf{f}}, \boldsymbol{\phi})$ defined by

$$J(\breve{\mathbf{f}}, \boldsymbol{\phi})(x,y) := \frac{(\breve{f}_x^{(p)}, \phi_y^{(p)})_{N,p}}{(f_x^{(p)}, f_x^{(p)})_{N,p}}$$

belongs to $\Lambda'_f\otimes_\Lambda R$, and the first equality in (61) follows. The second equality follows from a direct calculation, and the third from the definition of η_{f_x} and the familiar expression for the Poincaré pairing in terms of the Petersson scalar product. Lemma 2.19 follows. □

3. Diagonal cycles and p-adic Abel-Jacobi maps

3.1. **Generalised Gross-Kudla-Schoen cycles.** The main aim of this section is to introduce, for each triplet (k,ℓ,m) of *balanced* weights, a distinguished algebraic cycle $\Delta_{k,\ell,m}$ on the product of three Kuga-Sato varieties fibered over the modular curve $X = X_1(N)$. When the triplet is $(k,\ell,m) = (2,2,2)$, the cycle $\Delta_{2,2,2}$ is the one introduced by Gross and Kudla in [GrKu] and studied in detail by Gross and Schoen in [GrSc].

Let $\mathcal{E}$ denote the universal elliptic curve fibered over X, as constructed in [Sc]. It is a projective smooth algebraic surface defined over $\mathbb{Q}$ equipped with a proper regular (but not smooth) fibration

$$\pi : \mathcal{E} \longrightarrow X,$$

whose fiber $\mathcal{E}_x$ at a point $x\in Y$ outside the finite set of cusps is the elliptic curve corresponding to x under the moduli interpretation.

Fix a base point $o\in X(\mathbb{Q})$, say the cusp at infinity. Write u for the unique automorphism on $\mathcal{E}$ extending the involution -1 on the fibers with respect to the zero section σ_0 of π.

For any $r\geq 0$, write $\mathcal{E}^r = \mathcal{E}_1\times_X \ldots \times_X \mathcal{E}_r$ for the r-th fibered product of $\mathcal{E}$ over X. A generic point in $\mathcal{E}^r$ is $(x; P_1, \ldots, P_r)$ where $x\in X$ and P_i are points in the fiber $\mathcal{E}_x$. Let

$$\epsilon_{\mathrm{sym}} = \frac{1}{r!}\sum_{\sigma\in\Sigma_r}\mathrm{sgn}(\sigma)\sigma \in \mathrm{Corr}(\mathcal{E}^r)\otimes\mathbb{Q}$$

denote the projector in which a permutation σ acts on $\mathcal{E}^r$ by permuting the factors in the fibration $\pi:\mathcal{E}^r\longrightarrow X$. Let also

$$\epsilon_{\mathrm{inv}} := \left(\frac{1-u_1}{2}\right)\otimes\cdots\otimes\left(\frac{1-u_r}{2}\right)\in \mathrm{Corr}(\mathcal{E}^r)\otimes\mathbb{Q}$$

denote the idempotent in the ring of correspondences from $\mathcal{E}^r$ to itself, in which u_j denotes the involution on the j-th factor in the fibration $\mathcal{E}^r\longrightarrow X$. Write

$$\epsilon_r = \epsilon_{\mathrm{sym}}\cdot\epsilon_{\mathrm{inv}}\in\mathrm{Corr}(\mathcal{E}^r)\otimes\mathbb{Q} \tag{62}$$

for the composition of the two idempotents.

Put $(k,\ell,m) = (r_1+2, r_2+2, r_3+2)$ with $r_3\geq r_2\geq r_1\geq 0$ and set

$$r = \frac{r_1+r_2+r_3}{2}\geq 0.$$

Let us now define a generalized Gross-Kudla-Schoen cycle $\Delta_{k,\ell,m}$ of codimension $r+2$ in the $(2r+3)$-dimensional variety

$$W = \mathcal{E}^{r_1}\times\mathcal{E}^{r_2}\times\mathcal{E}^{r_3}$$

introduced in (1) of the introduction. We shall regard it as an element in the group

$$\mathrm{CH}^{r+2}(W) := \mathcal{Z}^{r+2}(W)\otimes\mathbb{Q}/\mathcal{Z}^{r+2}_{\mathrm{rat}}(W)\otimes\mathbb{Q}$$

of rational equivalence classes of cycles of codimension $r+2$ with coefficients in $\mathbb{Q}$.

We treat the three cases $(k,\ell,m)=(2,2,2)$, $(2,\ell,\ell)$ with $\ell>2$, and $k,\ell,m>2$ separately.

Definition 3.1. Assume $(k,\ell,m)=(2,2,2)$. For any non-empty subset $I\subseteq\{1,2,3\}$, let

$$X_I := \{(P_1,P_2,P_3)\in X\times X\times X : P_i=P_j \text{ for all } \{i,j\}\subset I, P_j=o \text{ for all } j\notin I\}.$$

Then the Gross-Kudla-Schoen diagonal cycle is defined to be

$$\Delta_{2,2,2}=X_{123}-X_{12}-X_{13}-X_{23}+X_1+X_2+X_3\in \mathrm{CH}^2(X_1\times X_2\times X_3).$$

To treat the remaining cases, choose three subsets

$$A=\{a_1,\dots,a_{r_1}\},\qquad B=\{b_1,\dots,b_{r_2}\},\qquad C=\{c_1,\dots,c_{r_3}\} \tag{63}$$

of $\{1,\dots,r\}$ of cardinalities r_1, r_2 and r_3 respectively, such that $A\cap B\cap C=\emptyset$. If some of the r_i is 0, we take the corresponding subset to be the empty set. One readily checks that such subsets exist, and that they satisfy $A=B\cup C\setminus B\cap C$, $B=C\cup A\setminus C\cap A$ and $C=A\cup B\setminus A\cap B$. The choice of the triplet (A,B,C) is unique up to permutations in S_r. Since the union of any two of the sets A, B, C is equal to $\{1,...,r\}$, the maps

$$\begin{aligned}&\varphi_{ABC}:\mathcal{E}^r\longrightarrow\mathcal{E}^{r_1}\times\mathcal{E}^{r_2}\times\mathcal{E}^{r_3},\quad (x;P_1,...,P_r)\mapsto((x;P_{a_i}),(x;P_{b_i}),(x;P_{c_i}))\\ &\text{and}\quad \varphi_{BC}:\mathcal{E}^r\longrightarrow\mathcal{E}^{r_2}\times\mathcal{E}^{r_3},\quad (x;P_1,...,P_r)\mapsto((x;P_{b_i}),(x;P_{c_i}))\end{aligned} \tag{64}$$

are closed embeddings of $\mathcal{E}^r$ into W (and likewise for the analogous maps φ_{AB} and φ_{AC}).

Definition 3.2. If $(k,\ell,m)=(2,\ell,\ell)$ for some $\ell=r+2>2$, the generalised Gross-Kudla-Schoen cycle is defined by

$$\Delta_{2,\ell,\ell}=(\mathrm{Id},\epsilon_{r_2},\epsilon_{r_3})(\varphi_{ABC}(\mathcal{E}^r)-\{o\}\times\varphi_{BC}(\mathcal{E}^r))\in\mathrm{CH}^{r+2}(X\times\mathcal{E}^r\times\mathcal{E}^r).$$

Definition 3.3. If $k,\ell,m>2$, the generalised Gross-Kudla-Schoen cycle is defined by

$$\Delta_{k,\ell,m}=(\epsilon_{r_1},\epsilon_{r_2},\epsilon_{r_3})\varphi_{ABC}(\mathcal{E}^r)\in\mathrm{CH}^{r+2}(W).$$

By examining the image of these cycles by the cycle class map in each of the Künneth components of the complex de Rham cohomology group $H^{2r+4}_{\mathrm{dR}}(W/\mathbb{C})$ of the variety W, it follows from [BDP, Lemma 2.2] that the cycles $\Delta_{k,\ell,m}$ we defined for each triple of balanced weights (k,ℓ,m) are null-homologous, that is to say:

$$\Delta_{k,\ell,m}\in\mathrm{CH}^{r+2}(W)_0:=\ker\left(\mathrm{cl}:\mathrm{CH}^{r+2}(W)\to H^{2r+4}_{\mathrm{dR}}(W)\right).$$

Fix a prime $p\nmid N$ and

$$\mathrm{AJ}_p:\mathrm{CH}^{r+2}(W)_0\longrightarrow\mathrm{Fil}^{r+2}H^{2r+3}_{\mathrm{dR}}(W)^\vee \tag{65}$$

denote the *p-adic Abel-Jacobi map*, as introduced e.g. in [Ne2, (1.2)], [Bes00].

There are several equivalent definitions of (65). Given that $\mathrm{Fil}^{r+2}H^{2r+3}_{\mathrm{dR}}(W)^\vee$ is naturally isomorphic to the group $\mathrm{Ext}^1(\mathbb{Q}_p,H^{2r+3}_{\mathrm{dR}}(W))$ of isomorphism classes of extensions of the trivial filtered frobenius module $\mathbb{Q}_p$ by $H^{2r+3}_{\mathrm{dR}}(W)$, the map AJ_p sends Δ to the extension class given by

$$\begin{tikzcd} 0 \arrow[r] & H^{2r+3}_{\mathrm{dR}}(W) \arrow[r] \arrow[d, equal] & V_\Delta \arrow[r] \arrow[d] & \mathbb{Q}_p \arrow[r] \arrow[d, "\mathrm{cl}_{\mathrm{dR}}"] & 0 \\ 0 \arrow[r] & H^{2r+3}_{\mathrm{dR}}(W) \arrow[r] & H^{2r+3}_{\mathrm{dR}}(W\setminus|\Delta|) \arrow[r] & H^{2r+4}_{|\Delta|}(W) \arrow[r] & 0, \end{tikzcd} \tag{66}$$

where the lower row is the short excision exact sequence associated with the pair (W,Δ) and the upper row is obtained from the lower by pull-back under the cycle class map $\mathrm{cl}_{\mathrm{dR}}$ in de Rham cohomology.

3.2. **A formula for $\mathrm{AJ}_p(\Delta_{k,\ell,m})$ in terms of Coleman integration.** The goal of this section is to give a p-adic analytic description of (a piece of) the functional $\mathrm{AJ}_p(\Delta_{k,\ell,m})$, involving p-adic integration of differential forms, à la Coleman.

Let $(k,\ell,m)=(r_1+2,r_2+2,r_3+2)$ with $r_3\geq r_2\geq r_1\geq 0$ be a triplet of balanced weights and recall that we set $r=\frac{r_1+r_2+r_3}{2}$.

Our description of $\mathrm{AJ}_p(\Delta_{k,\ell,m})$ rests on the p-adic analytic description of $H^1_{\mathrm{dR}}(X_{\mathbb{C}_p},\mathcal{L}_r)$ that was given in Sections 2.1 and 2.2. In order to keep track of the factors it will be convenient to write $X^2=X_2\times X_3$ and likewise write $(X')^2=X'_2\times X'_3$, where X' denotes the affine subvariety of X fixed in Section 2.1. By Proposition 2.1 and equation (40) we have

$$H^1_{\mathrm{dR}}(X',\mathcal{L}_{r_2})\otimes H^1_{\mathrm{dR}}(X',\mathcal{L}_{r_3})\simeq H^1_{\mathrm{rig}}(\mathcal{W}_\epsilon,\mathcal{L}_{r_2})\otimes H^1_{\mathrm{rig}}(\mathcal{W}_\epsilon,\mathcal{L}_{r_3}).$$

We fix a choice of lift of Frobenius to $\mathcal{W}_\epsilon\times\mathcal{W}_\epsilon$ by setting $\Phi:=\Phi_2\times\Phi_3$, where Φ_2 and Φ_3 denote the canonical lift of the Frobenius endomorphism on $\mathcal{W}_\epsilon$ described in (17), viewed as acting on the first and second factors of $X_2\times X_3$ respectively. This choice yields a linear transformation

$$\Phi:=\Phi_2\otimes\Phi_3:\Omega^1_{\mathrm{rig}}(\mathcal{W}_\epsilon,\mathcal{L}_{r_2})\otimes\Omega^1_{\mathrm{rig}}(\mathcal{W}_\epsilon,\mathcal{L}_{r_3})\longrightarrow\Omega^1_{\mathrm{rig}}(\mathcal{W}_\epsilon,\mathcal{L}_{r_2})\otimes\Omega^1_{\mathrm{rig}}(\mathcal{W}_\epsilon,\mathcal{L}_{r_3})$$

and induces an endomorphism of $H^1_{\mathrm{rig}}(\mathcal{W}_\epsilon,\mathcal{L}_{r_2})\otimes H^1_{\mathrm{rig}}(\mathcal{W}_\epsilon,\mathcal{L}_{r_3})$ which we denote with the same letter. View $\mathcal{L}_{r_2}\otimes\mathcal{L}_{r_3}$ as a sheaf on $X\times X$. The Gauss-Manin connection on $\mathcal{L}_{r_1}$ and on $\mathcal{L}_{r_2}$ gives rise to a connection

$$\nabla:\mathcal{L}_{r_2}\otimes\mathcal{L}_{r_2}\longrightarrow\mathcal{L}_{r_2}\otimes\mathcal{L}_{r_3}\otimes\Omega^1_{X\times X}$$

on sheaves on $X\times X$, denoted again by ∇ by abuse of notation, and defined on sections by the rule

$$\nabla(s_2\otimes s_3)=\nabla(s_2)\otimes s_3+s_2\otimes\nabla(s_3).$$

Given a class

$$\begin{aligned}\omega_2\otimes\omega_3 &\in H^0(X,\Omega^1_X\otimes\underline{\omega}^{r_2})\otimes H^0(X,\Omega^1_X\otimes\underline{\omega}^{r_3})\\ &= \mathrm{Fil}^{r_2+r_3+2}(H^1_{\mathrm{dR}}(X,\mathcal{L}_{r_2})\otimes H^1_{\mathrm{dR}}(X,\mathcal{L}_{r_3}))\\ &\subset H^1_{\mathrm{rig}}(\mathcal{W}_\epsilon,\mathcal{L}_{r_2})\otimes H^1_{\mathrm{rig}}(\mathcal{W}_\epsilon,\mathcal{L}_{r_3})\subset H^2_{\mathrm{rig}}(\mathcal{W}_\epsilon\times\mathcal{W}_\epsilon,\mathcal{L}_{r_2}\otimes\mathcal{L}_{r_3}),\end{aligned}$$

choose a polynomial $P\in\mathbb{C}_p[x]$ satisfying

(i) $P(\Phi)=P(\Phi_2\Phi_3)$ annihilates the class of $\omega_2\otimes\omega_3$ in $H^2_{\mathrm{rig}}(\mathcal{W}_\epsilon\times\mathcal{W}_\epsilon,\mathcal{L}_{r_2}\otimes\mathcal{L}_{r_3})$.

(ii) For each supersingular annulus $\mathcal{V}_j$, $j=1,...,s$, $P(\Phi)$ annihilates the target of the residue map (41), namely $(\mathrm{Sym}^r H^1_{\mathrm{dR}}(E_j))(-1)$.

(iii) None of the roots of $P(x)$ are of complex absolute value $p^{\frac{r_2+r_3+1}{2}}$.

The existence of such a polynomial follows from the fact that the eigenvalues of the geometric Frobenius Φ acting on $H^1_{\mathrm{dR}}(X_{\mathbb{C}_p},\mathcal{L}_r)$ (resp. on $(\mathrm{Sym}^r H^1_{\mathrm{dR}}(E_j))(-1)$) are algebraic numbers with complex absolute value $p^{\frac{r+1}{2}}$ (resp. $p^{\frac{r+2}{2}}$), and hence a polynomial P satisfying (i) and (ii) can be chosen so that all its roots have complex absolute value either $p^{\frac{r_2+r_3+2}{2}}$ or $p^{\frac{r+2}{2}}$; such a choice of P automatically satisfies (iii) because the triplet of weights is assumed throughout to be balanced.

See the discussion right after Proposition 3.7 below for a justification of the need of conditions (ii) and (iii). As for (i), note that a direct consequence of it is the existence of a rigid analytic primitive of $P(\Phi)(\omega_2\wedge\omega_3)$, as we quote in the following statement.

Lemma 3.4. *There exists a real $\epsilon>0$ and an $\mathcal{L}_{r_2}\otimes\mathcal{L}_{r_3}$-valued rigid one-form $\rho(P,\omega_2,\omega_3)$ on $\mathcal{W}_\epsilon\times\mathcal{W}_\epsilon$ satisfying*

$$\nabla\rho(P,\omega_2,\omega_3)=P(\Phi)(\omega_2\wedge\omega_3). \tag{67}$$

This one-form is well-defined up to rigid ∇-closed one-forms on $\mathcal{W}_\epsilon\times\mathcal{W}_\epsilon$.

Recall the base point $o \in X$ that was chosen to define the cycle $\Delta_{2,2,2}$. This choice of base point determines the horizontal and vertical inclusions

$$\iota_2 : X = X \times \{o\} \hookrightarrow X \times X, \qquad \iota_3 : X = \{o\} \times X \hookrightarrow X \times X. \tag{68}$$

Let

$$\iota_{23} : X \hookrightarrow X \times X \tag{69}$$

denote the diagonal morphism. By abuse of notation, we will denote by the same symbols the resulting maps

$$\iota_2, \iota_3, \iota_{23} : X' \hookrightarrow X' \times X', \qquad \iota_2, \iota_3, \iota_{23} : \mathcal{W}_\epsilon \hookrightarrow \mathcal{W}_\epsilon \times \mathcal{W}_\epsilon.$$

Finally write

$$\varphi_{23}^* := \iota_{23}^* - \iota_2^* - \iota_3^* : \Omega^1_{\mathrm{rig}}(\mathcal{W}_\epsilon \times \mathcal{W}_\epsilon) \longrightarrow \Omega^1(\mathcal{W}_\epsilon)$$

for the maps obtained by combining the pullbacks of these three morphisms.

Lemma 3.5. *The map $H^1_{\mathrm{rig}}(\mathcal{W}_\epsilon \times \mathcal{W}_\epsilon) \longrightarrow H^1_{\mathrm{rig}}(\mathcal{W}_\epsilon)$ induced by φ_{23}^* is the zero map.*

Proof. This is a direct consequence of the following two facts:

(1) The map

$$\varphi_{23}^* : H^1_{\mathrm{dR}}(X'_2 \times X'_3/\mathbb{Q}_p) \longrightarrow H^1_{\mathrm{dR}}(X'/\mathbb{Q}_p)$$

is 0. This can be checked by choosing a complex embedding of $\mathbb{Q}_p$ and noting that the map $\varphi_{23} : H_1(X'/\mathbb{C}) \longrightarrow H_1(X'_2 \times X'_3/\mathbb{C})$ vanishes, using topological methods.

(2) There is a commutative diagram

$$\begin{array}{ccc} H^1_{\mathrm{dR}}(X'_2 \times X'_3/\mathbb{Q}_p) & \xrightarrow{\varphi_{23}^*} & H^1_{\mathrm{dR}}(X'/\mathbb{Q}_p) \\ \Big\| \mathrm{comp}_\epsilon & & \Big\| \mathrm{comp}_\epsilon \\ H^1_{\mathrm{rig}}(\mathcal{W}_\epsilon \times \mathcal{W}_\epsilon/\mathbb{Q}_p) & \xrightarrow{\varphi_{23}^*} & H^1_{\mathrm{rig}}(\mathcal{W}_\epsilon/\mathbb{Q}_p). \end{array}$$

□

To formulate the counterpart of Lemma 3.5 for general weights (k, ℓ, m), recall the subsets A, B and C of $\{1, \ldots, r\}$ of cardinalities r_1, r_2 and $r_3 \geq 0$ that were chosen in §3.1, and denote by A', B' and C' their respective complements. Note that

$$A' = B \cap C, \quad B' = A \cap C \quad \text{and} \quad C' = A \cap B.$$

Recall also the diagonal inclusions

$$\varphi_{ABC} : \mathcal{E}^r \longrightarrow \mathcal{E}^{r_1} \times \mathcal{E}^{r_2} \times \mathcal{E}^{r_3} \;\text{ and }\; \varphi_{BC} : \mathcal{E}^r \longrightarrow \mathcal{E}^{r_2} \times \mathcal{E}^{r_3}$$

introduced in (64). As in (48), the map φ_{BC} together with Poincaré duality on the fibers of the projection $\mathcal{E}^r \to \mathcal{E}^{r_1}$ induced by A, give rise to a pullback operation on sheaves

$$\mathcal{L}_{r_2} \otimes \mathcal{L}_{r_3} \longrightarrow \mathcal{L}_A(-t) \tag{70}$$

where $t := |A'| = r - r_1$ satisfies $k = \ell + m - 2 - 2t$. We hence obtain a map

$$\varphi_{A,BC}^* : \Omega^1_{\mathrm{rig}}(\mathcal{W}_\epsilon \times \mathcal{W}_\epsilon, \mathcal{L}_{r_2} \otimes \mathcal{L}_{r_3}) \longrightarrow \Omega^1_{\mathrm{rig}}(\mathcal{W}_\epsilon, \mathcal{L}_A(-t)).$$

Set

$$\varphi^* := \begin{cases} \varphi_{23}^* & \text{if } (k, \ell, m) = (2, 2, 2), \\ \varphi_{A,BC}^* & \text{otherwise.} \end{cases} \tag{71}$$

Corollary 3.6. *If $\sigma \in \Omega^1(\mathcal{W}_\epsilon \times \mathcal{W}_\epsilon, \mathcal{L}_{r_2} \otimes \mathcal{L}_{r_3})$ is ∇-closed, then $\varphi^*(\sigma)$ is ∇-exact on $\mathcal{W}_\epsilon$.*

Proof. If $(r_1, r_2, r_3) = (0,0,0)$, this is just Lemma 3.5. If $r_3, r_2 > 0$, by the Künneth decomposition we have

$$\begin{aligned} H^1_{\mathrm{dR}}(X' \times X', \mathcal{L}_{r_2} \otimes \mathcal{L}_{r_3}) \quad \simeq \quad & H^1_{\mathrm{dR}}(X', \mathcal{L}_{r_2}) \otimes H^0_{\mathrm{dR}}(X', \mathcal{L}_{r_3}) \\ & \oplus \;\; H^0_{\mathrm{dR}}(X', \mathcal{L}_{r_2}) \otimes H^1_{\mathrm{dR}}(X', \mathcal{L}_{r_3}), \end{aligned}$$

and these modules vanish because the sheaves $\mathcal{L}_{r_2}$ and $\mathcal{L}_{r_3}$ have no global horizontal sections (cf. [BDP, Lemma 2.1]). In light of (40) we deduce that in fact σ is already exact, and a fortiori so is $\varphi^*(\sigma)$. □

Proposition 3.7. *The element*

$$\xi(P, \omega_2, \omega_3) := \text{class of } \varphi^*\rho(\mathrm{P}, \omega_2, \omega_3) \text{ in } \mathrm{H}^1_{\mathrm{rig}}(\mathcal{W}_\epsilon, \mathcal{L}_{\mathrm{A}}(-\mathrm{t}))$$

does not depend on the choice of rigid differential $\rho(P, \omega_2, \omega_3)$ satisfying (67) *of Lemma 3.4 and has vanishing annular residues. In particular, $\xi(P, \omega_2, \omega_3)$ belongs to $H^1_{\mathrm{dR}}(X, \mathcal{L}_A(-t))$.*

Proof. The first assertion is a direct consequence of Corollary 3.6. In light of (41) and the discussion following it, the second assertion follows the condition (ii) satisfied by P. □

The Frobenius endomorphism Φ acts on $H^1_{\mathrm{dR}}(X, \mathcal{L}_A(-t))$ with eigenvalues of complex absolute value $\sqrt{p}^{1+r_2+r_3}$. Therefore, since the roots of P have absolute value either p^{r+1} or $\sqrt{p}^{2+r_2+r_3}$, the endomorphism $P(\Phi)$ acts invertibly on $H^1_{\mathrm{dR}}(X, \mathcal{L}_A(-t))$. In particular, for all $\eta \in H^1_{\mathrm{dR}}(X, \mathcal{L}_A(-t))$, the class $P(\Phi)^{-1}\eta$ is well-defined. Furthermore, the class

$$\xi(\omega_2, \omega_3) := P(\Phi)^{-1}\xi(P, \omega_2, \omega_3) \in H^1_{\mathrm{rig}}(\mathcal{W}_\epsilon, \mathcal{L}_A(-t)) \tag{72}$$

does not depend on the choice of polynomial P. More precisely, replacing P by a polynomial PQ satisfying the conditions before the statement of Lemma 3.4, one sees that

$$\rho(PQ, \omega_2, \omega_3) = Q(\Phi)\rho(P, \omega_2, \omega_3), \qquad \xi(PQ, \omega_2, \omega_3) = Q(\Phi)\xi(P, \omega_2, \omega_3).$$

We will now describe (part of) the image of the generalized Gross-Kudla-Schoen cycle $\Delta_{k,\ell,m}$ under the p-adic Abel-Jacobi map. More precisely, we will describe the restriction of the functional $\mathrm{AJ}_p(\Delta_{k,\ell,m})$ to the summand

$$H^1_{\mathrm{dR}}(X, \mathcal{L}_{r_1}) \otimes H^0(X, \underline{\omega}^{r_2} \otimes \Omega^1_X) \otimes H^0(X, \underline{\omega}^{r_3} \otimes \Omega^1_X) \subseteq H^{2r+3}_{\mathrm{dR}}(W), \tag{73}$$

where the inclusion arises from the Künneth decomposition as in (5).

Let $\iota_A : \mathcal{E}^A \longrightarrow \mathcal{E}^{r_1}$ denote the natural isomorphism between the varieties $\mathcal{E}^A := \mathcal{E}_{a_1} \times_X \times ... \times_X \mathcal{E}_{a_{r_1}}$ and $\mathcal{E}^{r_1}$, giving rise to an isomorphism of sheaves between $\mathcal{L}_{r_1}$ and $\mathcal{L}_A$.

Theorem 3.8. *Let $\eta \otimes \omega_2 \otimes \omega_3$ be any class in* (73)*. Then*

$$\mathrm{AJ}_p(\Delta_{k,\ell,m})(\eta \otimes \omega_2 \otimes \omega_3) = \langle \iota_A^*\eta, \xi(\omega_2, \omega_3)\rangle, \tag{74}$$

where

$$\langle\ ,\ \rangle : H^1_{\mathrm{dR}}(X, \mathcal{L}_A) \times H^1_{\mathrm{dR}}(X, \mathcal{L}_A(-t)) \longrightarrow \mathbb{C}_p(-1-r)$$

arises from Poincaré duality.

Remark 3.9. The reader mainly interested in the case $(k, \ell, m) = (2,2,2)$ will notice that Theorem 3.8 asserts that the image of the Gross-Kudla-Schoen cycle $\Delta_{2,2,2}$ in X^3 under the p-adic Abel-Jacobi map satisfies

$$\mathrm{AJ}_p(\Delta_{2,2,2})(\eta \otimes \omega_2 \otimes \omega_3) = \langle \eta, \xi(\omega_2, \omega_3)\rangle, \tag{75}$$

where $\langle\ ,\ \rangle$ is the usual Poincaré duality pairing on $H^1_{\mathrm{dR}}(X)$.

3.3. **Proof of the formula: Besser's finite polynomial cohomology.** The goal of this section is proving Theorem 3.8, and to do that we first set some notation. Let V be a smooth, proper, irreducible variety of dimension d over $\mathbb{Q}_p$ which admits a smooth proper, flat model $\mathcal{V}$ over $\mathbb{Z}_p$. There are *syntomic* and *finite polynomial* cohomology groups

$$H^i_{\mathrm{syn}}(\mathcal{V},n) \subseteq H^i_{\mathrm{fp}}(\mathcal{V},n)$$

for every $i, n \geq 0$ (cf. e.g. [Bes00] and the references therein). These groups are related to the p-adic de Rham cohomology of V by the following diagram (cf. [Bes00, Prop. 2.5 (1-2)]):

$$\begin{tikzcd}
0 \arrow[r] & H^{i-1}_{\mathrm{dR}}(V)/\mathrm{Fil}^n H^{i-1}_{\mathrm{dR}}(V) \arrow[r, "\mathrm{i}_{\mathrm{syn}}"] \arrow[d, equal] & H^i_{\mathrm{syn}}(\mathcal{V},n) \arrow[r, "\mathrm{p}_{\mathrm{syn}}"] \arrow[d] & \mathrm{Fil}^n H^i_{\mathrm{dR}}(V) \arrow[d, equal] & \\
0 \arrow[r] & H^{i-1}_{\mathrm{dR}}(V)/\mathrm{Fil}^n H^{i-1}_{\mathrm{dR}}(V) \arrow[r, "\mathrm{i}_{\mathrm{fp}}"] & H^i_{\mathrm{fp}}(\mathcal{V},n) \arrow[r, "\mathrm{p}_{\mathrm{fp}}"] & \mathrm{Fil}^n H^i_{\mathrm{dR}}(V) \arrow[r] & 0,
\end{tikzcd} \tag{76}$$

in which the rows are exact. In addition there are compatible cycle class maps

$$\begin{tikzcd}
\mathrm{CH}^i(\mathcal{V}) \arrow[r, "\mathrm{cl}_{\mathrm{syn}}"] \arrow[d] \arrow[rr, bend left, "\mathrm{cl}_{\mathrm{fp}}"] & H^{2i}_{\mathrm{syn}}(\mathcal{V},i) \arrow[r] \arrow[dr, "\mathrm{p}_{\mathrm{syn}}"] & H^{2i}_{\mathrm{fp}}(\mathcal{V},i) \arrow[d, "\mathrm{p}_{\mathrm{fp}}"] \\
\mathrm{CH}^i(V) \arrow[rr, "\mathrm{cl}_{\mathrm{dR}}"] & & \mathrm{Fil}^i H^{2i}_{\mathrm{dR}}(V)
\end{tikzcd}$$

which give rise (by the formalism of e.g. [Ne2, (1.2)]) to the *p-adic Abel-Jacobi maps*

$$\mathrm{AJ}_p : \mathrm{CH}^i(V)_0 \longrightarrow \ker(\mathrm{p}_{\mathrm{syn}}) = H^{2i-1}_{\mathrm{dR}}(V)/\mathrm{Fil}^i(H^{2i-1}_{\mathrm{dR}}(V)) \simeq \mathrm{Fil}^{d-i+1} H^{2d-2i+1}_{\mathrm{dR}}(V)^\vee \tag{77}$$

that we already introduced in (65). Besides the definition of AJ_p given in (66), Besser proved in [Bes00, Theorem 1,2] that the p-adic Abel-Jacobi map may be expressed purely in terms of Coleman integration.

Namely, let $\Delta = \sum a_j \Delta_j$ be a representative of a class in $\mathrm{CH}^i(\mathcal{V})_0$, with $a_j \in \mathbb{Q}$ and Δ_j irreducible, smooth proper subschemes of $\mathcal{V}$ over $\mathbb{Z}_p$. Besser showed that AJ_p sends the null-homological cycle Δ to the functional

$$\mathrm{AJ}_p(\Delta) : \mathrm{Fil}^{d-i+1} H^{2d-2i+1}_{\mathrm{dR}}(V) \longrightarrow \mathbb{Q}_p,\ \omega \mapsto \int_\Delta \omega := \sum a_j \mathrm{tr}(\iota_j^*(\tilde{\omega})), \tag{78}$$

where $\tilde{\omega} \in H^{2d-2i+1}_{\mathrm{fp}}(\mathcal{V}, d-i+1)$ is a lift of ω under the map p_{fp} appearing in (76), $\iota_j : \Delta_j \hookrightarrow \mathcal{V}$ is the natural inclusion and

$$\mathrm{tr} : H^{2d-2i+1}_{\mathrm{fp}}(\Delta_j, d-i+1) \xrightarrow{\sim} \mathbb{Q}_p$$

is the canonical trace isomorphism ([Bes00, Prop. 2.5 (4)]).

It is Besser's formula (78) which we shall apply in order to prove Theorem 3.8. As a final piece of notation, let

$$\langle\, , \rangle_{\mathrm{fp}} : H^i_{\mathrm{fp}}(\mathcal{V},n) \times H^{2d+1-i}_{\mathrm{fp}}(\mathcal{V}, d+1-n) \longrightarrow \mathbb{Q}_p \tag{79}$$

denote the cup product in finite polynomial cohomology described in [Bes00, Prop. 2.5 (4)].

We first focus in the particular case $(k,\ell,m) = (2,2,2)$. Set

$$\varphi_{\mathrm{dR},*} := \iota_{23,\mathrm{dR},*} - \iota_{2,\mathrm{dR},*} - \iota_{3,\mathrm{dR},*} : H^1_{\mathrm{dR}}(X) \longrightarrow \mathrm{Fil}^1 H^3_{\mathrm{dR}}(X^2),$$

$$\varphi_{\mathrm{fp},*} := \iota_{23,\mathrm{fp}*} - \iota_{2,\mathrm{fp}*} - \iota_{3,\mathrm{fp}*} : H^1_{\mathrm{fp}}(\mathcal{X},0) \longrightarrow H^3_{\mathrm{fp}}(\mathcal{X}^2,1)$$

to be the maps induced by push-forward by the three embeddings ι_2, ι_3, ι_{23} introduced in (68) and (69) on de Rham and finite polynomial cohomology, respectively.

By [Bes00, Proposition 2.5 (1) and Lemma 5.1], there is a commutative diagram

$$\begin{tikzcd} & 0 \arrow[r, "i_1"] \arrow[d] & H^1_{\mathrm{fp}}(\mathcal{X},0) \arrow[r, "\mathrm{p}_1"] \arrow[d, "\varphi_{\mathrm{fp},*}"'] & H^1_{\mathrm{dR}}(X) \arrow[r] \arrow[d, "\varphi_{\mathrm{dR},*}"'] & 0 \\ 0 \arrow[r] & \frac{H^2_{\mathrm{dR}}(X^2)}{\mathrm{Fil}^1 H^2_{\mathrm{dR}}(X^2)} \arrow[r, "i_2"] & H^3_{\mathrm{fp}}(\mathcal{X}^2,1) \arrow[r, "\mathrm{p}_2"] & \mathrm{Fil}^1 H^3_{\mathrm{dR}}(X^2) \arrow[r] & 0 \end{tikzcd} \tag{80}$$

with exact rows arising from (76).

Let $\tilde{\eta} \in H^1_{\mathrm{fp}}(\mathcal{X},0)$ be a preimage of η under p_1 and $\tilde{\omega}_2, \tilde{\omega}_3 \in H^1_{\mathrm{fp}}(\mathcal{X},1)$ be preimages of ω_2 and ω_3 respectively, so that $\tilde{\omega}_2 \otimes \tilde{\omega}_3$ is a preimage in $H^2_{\mathrm{fp}}(\mathcal{X}^2,2)$ of $\omega_2 \otimes \omega_3 \in \mathrm{Fil}^2 H^2_{\mathrm{dR}}(X^2)$.

Lemma 3.10. *The following equality holds in $\mathbb{Q}_p$:*

$$\langle \mathrm{cl}_{\mathrm{fp}}(\Delta_{2,2,2}), \tilde{\eta} \otimes \tilde{\omega}_2 \otimes \tilde{\omega}_3 \rangle_{\mathrm{fp}} = \langle \varphi_{\mathrm{fp}\,*}(\tilde{\eta}), \tilde{\omega}_2 \otimes \tilde{\omega}_3 \rangle_{\mathrm{fp}}.$$

Proof. The class of the cycle $\mathcal{X}_{123} - \mathcal{X}_{12} - \mathcal{X}_{13} - \mathcal{X}_{23} + \mathcal{X}_1 + \mathcal{X}_2 + \mathcal{X}_3$ is an integral, smooth proper model of $\Delta_{2,2,2}$ in $\mathrm{CH}^2(\mathcal{X}^3)_0$. Hence

$$\langle \mathrm{cl}_{\mathrm{fp}}(\Delta_{2,2,2}), \tilde{\eta} \otimes \tilde{\omega}_2 \otimes \tilde{\omega}_3 \rangle_{\mathrm{fp}} = -\sum_I (-1)^{|I|} \mathrm{tr}_{\mathcal{X}}(\iota_I^*(\tilde{\eta} \otimes \tilde{\omega}_2 \otimes \tilde{\omega}_3))$$

where I ranges through the seven non-empty subsets of $\{1,2,3\}$ and $\mathrm{tr}_{\mathcal{X}} : H^3_{\mathrm{fp}}(\mathcal{X},2) \xrightarrow{\sim} \mathbb{Q}_p$ is Besser's trace isomorphism. For $I = 123$, 12 and 13, it follows from the very definitions that the above traces may be recast as the cup-products

$$\begin{aligned} \mathrm{tr}_{\mathcal{X}}(\iota_{123}^*(\tilde{\eta} \otimes \tilde{\omega}_2 \otimes \tilde{\omega}_3)) &= \langle \tilde{\eta}, (\tilde{\omega}_2 \otimes \tilde{\omega}_3)_{|\mathcal{X}_{23}} \rangle_{\mathrm{fp}} \\ \mathrm{tr}_{\mathcal{X}}(\iota_{12}^*(\tilde{\eta} \otimes \tilde{\omega}_2 \otimes \tilde{\omega}_3)) &= \langle \tilde{\eta}, (\tilde{\omega}_2 \otimes \tilde{\omega}_3)_{|\mathcal{X}_2} \rangle_{\mathrm{fp}} \\ \mathrm{tr}_{\mathcal{X}}(\iota_{13}^*(\tilde{\eta} \otimes \tilde{\omega}_2 \otimes \tilde{\omega}_3)) &= \langle \tilde{\eta}, (\tilde{\omega}_2 \otimes \tilde{\omega}_3)_{|\mathcal{X}_3} \rangle_{\mathrm{fp}} \end{aligned}$$

on $\mathcal{X}$, while $\mathrm{tr}_{\mathcal{X}}(\iota_I^*(\tilde{\eta} \otimes \tilde{\omega}_2 \otimes \tilde{\omega}_3))$ vanishes for the remaining four choices of I.

On the other hand, the push-forward map $\varphi_{\mathrm{fp}\,*}$ is dual under cup-product (79) to the pull-back homomorphism $\varphi_{\mathrm{fp}}^* = \iota_{23,\mathrm{fp}}^* - \iota_{2,\mathrm{fp}}^* - \iota_{3,\mathrm{fp}}^* : H^2_{\mathrm{fp}}(\mathcal{X}^2,2) \to H^2_{\mathrm{fp}}(\mathcal{X},2)$, and hence

$$\langle \varphi_{\mathrm{fp}\,*}(\tilde{\eta}), \tilde{\omega}_2 \otimes \tilde{\omega}_3 \rangle_{\mathrm{fp}} = \langle \tilde{\eta}, (\tilde{\omega}_2 \otimes \tilde{\omega}_3)_{|\mathcal{X}_{23}} \rangle_{\mathrm{fp}} - \langle \tilde{\eta}, (\tilde{\omega}_2 \otimes \tilde{\omega}_3)_{|\mathcal{X}_2} \rangle_{\mathrm{fp}} - \langle \tilde{\eta}, (\tilde{\omega}_2 \otimes \tilde{\omega}_3)_{|\mathcal{X}_3} \rangle_{\mathrm{fp}}.$$

This proves the lemma. □

Lemma 3.10 combined with (78) yields

$$\mathrm{AJ}_p(\Delta_{2,2,2})(\eta \otimes \omega_2 \otimes \omega_3) = \langle \varphi_{\mathrm{fp}\,*}(\tilde{\eta}), \tilde{\omega}_2 \otimes \tilde{\omega}_3 \rangle_{\mathrm{fp}}. \tag{81}$$

By the functoriality of [Bes00, (8)], there is another commutative diagram with exact rows

$$\begin{tikzcd} 0 \arrow[r] & H^1_{\mathrm{dR}}(X^2) \arrow[r, "\mathrm{i}"] \arrow[d, "\varphi_{\mathrm{dR}}^*"'] & H^2_{\mathrm{fp}}(\mathcal{X}^2,2) \arrow[r, "\mathrm{p}"] \arrow[d, "\varphi_{\mathrm{fp}}^*"'] & \mathrm{Fil}^2 H^2_{\mathrm{dR}}(X^2) \arrow[r] \arrow[d, "\varphi_{\mathrm{dR}}^*"'] & 0 \\ 0 \arrow[r] & H^1_{\mathrm{dR}}(X) \arrow[r, "\mathrm{i}"] & H^2_{\mathrm{fp}}(\mathcal{X},2) \arrow[r, "\mathrm{p}"] & \mathrm{Fil}^2 H^2_{\mathrm{dR}}(X) = 0. \end{tikzcd} \tag{82}$$

In light of [Bes00, Def. 2.2 and eq. (6)], a pair

$$(\alpha, \beta) \in \Omega^1_{\mathrm{rig}}(\mathcal{W}^2_\epsilon) \oplus \mathrm{Fil}^2 H^2_{\mathrm{dR}}(X^2) = \Omega^1_{\mathrm{rig}}(\mathcal{W}^2_\epsilon) \oplus \Omega^2(X^2)$$

satisfying $d\alpha = P(\Phi)(\beta)$ gives rise to a class in $H^2_{\mathrm{fp}}(\mathcal{X}^2,2)$. Since the map p of the upper row of (82) is the projection $[\alpha,\beta] \mapsto [\beta]$ to the second factor, the class $\tilde{\omega}_2 \otimes \tilde{\omega}_3$ in $H^2_{\mathrm{fp}}(\mathcal{X}^2,2)$ may be represented by the pair

$$(\rho, \omega_2 \otimes \omega_3) \in \Omega^1_{\mathrm{rig}}(\mathcal{W}^2_\epsilon) \oplus \Omega^2(X^2),$$

where $\rho := \rho(P, \omega_2, \omega_3)$ is as in Lemma 3.4. Since $\mathrm{Fil}^2 H^2_{\mathrm{dR}}(X) = 0$, the class $\varphi^*_{\mathrm{fp}}(\tilde{\omega}_2 \otimes \tilde{\omega}_3)$ has a preimage in $H^1_{\mathrm{dR}}(X)$ under the map i of the lower row of (82). Such preimage is the class of $\xi := \xi(\omega_2, \omega_3) \in H^1_{\mathrm{dR}}(X)$, as

$$i(\xi) := [P(\Phi)\xi, 0] = [\varphi^*\rho, 0] = \varphi^*_{\mathrm{fp}}(\tilde{\omega}_2 \otimes \tilde{\omega}_3) \in H^2_{\mathrm{fp}}(\mathcal{X}, 2). \tag{83}$$

From this we find, as desired, that

$$\langle \varphi_{\mathrm{fp}\,*}(\tilde{\eta}), \tilde{\omega}_2 \otimes \tilde{\omega}_3 \rangle_{\mathrm{fp}} = \langle \tilde{\eta}, \varphi^*_{\mathrm{fp}}(\tilde{\omega}_2 \otimes \tilde{\omega}_3)\rangle_{\mathrm{fp}} = \langle \tilde{\eta}, i(\xi)\rangle_{\mathrm{fp}} = \langle \eta, \xi \rangle,$$

where the first equality follows from the functoriality of the cup product, the second is (83) and the third is a direct consequence of [Bes00, (14)].

This concludes the proof of Theorem 3.8 in the case where all three weights are equal to 2. We turn now to show how a similar approach yields Theorem 3.8 in the remaining cases, that is to say, when either $(r_1, r_2, r_3) = (0, r, r)$ with $r > 0$ or (r_1, r_2, r_3) with $r_3 \geq r_2 \geq r_1 > 0$ and $r_3 \leq r_1 + r_2$.

By a slight abuse of notation, let us still denote W the base change of the triple product $\mathcal{E}^{r_1} \times \mathcal{E}^{r_2} \times \mathcal{E}^{r_3}$ to $\mathbb{Q}_p$, which recall has dimension $r_1 + r_2 + r_3 + 3 = 2r + 3$.

Let $\eta \in H^1_{\mathrm{dR}}(X, \mathcal{L}_{r_1}) \subset H^{r_1+1}_{\mathrm{dR}}(\mathcal{E}^{r_1})$ and

$$\omega_2 \in H^1_{\mathrm{dR}}(X, \omega^{r_2} \otimes \Omega^1_X) \subset \mathrm{Fil}^{r_2+1} H^{r_2+1}_{\mathrm{dR}}(\mathcal{E}^{r_2}), \quad \omega_3 \in H^1_{\mathrm{dR}}(X, \omega^{r_3} \otimes \Omega^1_X) \subset \mathrm{Fil}^{r_3+1} H^{r_3+1}_{\mathrm{dR}}(\mathcal{E}^{r_3})$$

be classes as in (73), which we may interchangeably regard either as $\mathcal{L}_{r_i}$-valued 1-forms on X or as $(r_i + 1)$-forms on $\mathcal{E}^{r_i}$. We thus have

$$\eta \otimes \omega_2 \otimes \omega_3 \in \mathrm{Fil}^{r_2+r_3+2} H^3_{\mathrm{dR}}(X^3, \mathcal{L}_{r_1} \otimes \mathcal{L}_{r_2} \otimes \mathcal{L}_{r_3}) \subset \mathrm{Fil}^{r_2+r_3+2} H^{2r+3}_{\mathrm{dR}}(W).$$

Denote $\mathfrak{E}^r$ the canonical integral, smooth and proper model of $\mathcal{E}^r$ over $\mathbb{Z}_p$. Let

$$\tilde{\eta} \in H^{r_1+1}_{\mathrm{fp}}(\mathfrak{E}^{r_1}, -t), \; \tilde{\omega}_2 \in H^{r_2+1}_{\mathrm{fp}}(\mathfrak{E}^{r_2}, r_2 + 1) \;\text{ and }\; \tilde{\omega}_3 \in H^{r_3+1}_{\mathrm{fp}}(\mathfrak{E}^{r_3}, r_3 + 1)$$

be, respectively, lifts of η, ω_2 and ω_3 in finite polynomial cohomology, and write $\tilde{\eta} \otimes \tilde{\omega}_2 \otimes \tilde{\omega}_3 \in H^{2r+3}_{\mathrm{fp}}(\mathcal{W}, r + 2)$ for their tensor product.

Similarly as for φ_{BC} and φ_{ABC}, write $\varphi_A : \mathcal{E}^r \longrightarrow \mathcal{E}^{r_1}$, $(x; P_1, ..., P_r) \mapsto (x; P_{a_1}, ..., P_{a_{r_1}})$ for the natural projection induced by the choice of the subset A of $\{1, ..., r\}$.

Lemma 3.11. *We have*

$$\mathrm{AJ}_p(\Delta_{k,\ell,m})(\eta \otimes \omega_2 \otimes \omega_3) = \langle \varphi^*_{A,\mathrm{fp}}(\tilde{\eta}), \varphi^*_{BC,\mathrm{fp}}(\tilde{\omega}_2 \otimes \tilde{\omega}_3)\rangle_{\mathfrak{E}^r,\mathrm{fp}}. \tag{84}$$

Proof. Cycle $\Delta_{k,\ell,m}$ is a linear combination $\sum_j a_j \Delta_j$, where $a_j \in \mathbb{Q}$ and Δ_j are irreducible subvarieties of W, all of which are the image of $\mathcal{E}^r$ under a closed embedding $\iota_j : \mathcal{E}^r \hookrightarrow W$; see Definitions 3.2 and 3.3 for more details. These embeddings lift in a natural way to proper maps $\iota_j : \mathfrak{E}^r \hookrightarrow \mathcal{W}$, and $\sum a_j \iota_j(\mathfrak{E}^r)$ is thus a representative of $\Delta_{k,\ell,m}$ in $\mathrm{CH}^{r+2}(\mathcal{W})_0$.

By Besser's formula (78),

$$\mathrm{AJ}_p(\Delta_{k,\ell,m})(\eta \otimes \omega_2 \otimes \omega_3) = \langle \mathrm{cl}_{\mathrm{fp}}(\Delta_{k,\ell,m}), \tilde{\eta} \otimes \tilde{\omega}_2 \otimes \tilde{\omega}_3 \rangle_{\mathrm{fp}} = \sum_j a_j \mathrm{tr}_{\mathfrak{E}^r}(\iota^*_j(\tilde{\eta} \otimes \tilde{\omega}_2 \otimes \tilde{\omega}_3)). \tag{85}$$

If $k = 2$ (and thus $\ell = m \geq 3$), according to Definition 3.2 then half of the terms in the above sum correspond to the image of $\{o\} \times \varphi_{BC}(\mathcal{E}^r)$ under some automorphism of $\mathcal{E}^{r_1} \times \mathcal{E}^{r_2} \times \mathcal{E}^{r_3}$ that acts as the identity on the copy of $\mathcal{E}^{r_1}$. Here o is some choice of a base point in $\mathcal{E}^{r_1}$. For such embeddings, it follows that $\iota^*_j(\tilde{\eta} \otimes \tilde{\omega}_2 \otimes \tilde{\omega}_3))$ vanishes, and hence those terms do not contribute to the sum in (85).

In view of this and Definition 3.3, for any balanced triple $(k, \ell, m) \neq (2, 2, 2)$ we conclude that $\mathrm{AJ}_p(\Delta_{k,\ell,m})(\eta \otimes \omega_2 \otimes \omega_3)$ is equal to a linear combination of the form

$$\sum a_{\alpha_1,\alpha_2,\alpha_3} \mathrm{tr}_{\mathfrak{E}^r}(\varphi^*_{ABC} \circ (\alpha_1 \otimes \alpha_2 \otimes \alpha_3)^*(\tilde{\eta} \otimes \tilde{\omega}_2 \otimes \tilde{\omega}_3)) \tag{86}$$

where α_i are automorphisms of $\mathfrak{E}^{r_i}$ for $i = 1, 2, 3$ and the scalars $a_{\alpha_1,\alpha_2,\alpha_3}$ are each equal to $\pm\frac{1}{2^n}$ for some $n \geq 0$. We refer to §3.1 for the explicit description of the automorphisms α_i that intervene here; all them satisfy $\alpha_i^*(\tilde{\omega}) = \pm\tilde{\omega} \in H^{r_i+1}_{\mathrm{fp}}(\mathfrak{E}^{r_i}, \cdot)$ for the classes in play, so that $\mathrm{tr}_{\mathfrak{E}^r}(\varphi^*_{ABC} \circ (\alpha_1 \otimes \alpha_2 \otimes \alpha_3)^*(\tilde{\eta} \otimes \tilde{\omega}_2 \otimes \tilde{\omega}_3)) = \pm\mathrm{tr}_{\mathfrak{E}^r}(\varphi^*_{ABC}(\tilde{\eta} \otimes \tilde{\omega}_2 \otimes \tilde{\omega}_3))$. It is a simple combinatorial exercise to check that the terms in (86) sum up to $\mathrm{tr}_{\mathfrak{E}^r}(\varphi^*_{ABC,\mathrm{fp}}(\tilde{\eta} \otimes \tilde{\omega}_2 \otimes \tilde{\omega}_3))$.

By Besser's definitions of trace and cup-product in finite polynomial cohomology, it directly follows that $\mathrm{tr}_{\mathfrak{E}^r}(\varphi^*_{ABC,\mathrm{fp}}(\tilde{\eta} \otimes \tilde{\omega}_2 \otimes \tilde{\omega}_3)) = \langle \varphi^*_{A,\mathrm{fp}}(\tilde{\eta}), \varphi^*_{BC,\mathrm{fp}}(\tilde{\omega}_2 \otimes \tilde{\omega}_3)\rangle_{\mathfrak{E}^r,\mathrm{fp}}$. □

The class $\varphi^*_{BC,\mathrm{dR}}(\omega_2 \otimes \omega_3)$ in de Rham cohomology vanishes, because it lies in $\epsilon_r H^{r_2+r_3+2}_{\mathrm{dR}}(\mathcal{E}^r)$ and the idempotent ϵ_r annihilates $H^j_{\mathrm{dR}}(\mathcal{E}^r)$ for all $j \neq r+1$; note that indeed $r_2+r_2+2 \neq r+1$ because the triplet (k, ℓ, m) is balanced. Hence the pair $(\varphi^*_{BC}(\rho(P, \omega_2, \omega_3)), 0)$ is a representative of $\varphi^*_{BC,\mathrm{fp}}(\tilde{\omega}_2 \otimes \tilde{\omega}_3)$ in finite polynomial cohomology.

It follows exactly as in (83) that $P(\Phi)^{-1}\varphi^*_{BC}(\rho(P, \omega_2, \omega_3))$ is a preimage of the class $\varphi^*_{BC,\mathrm{fp}}(\tilde{\omega}_2 \otimes \tilde{\omega}_3)$ under Besser's map

$$\mathrm{i} : H^{r_2+r_3+1}_{\mathrm{dR}}(\mathcal{E}^r)/\mathrm{Fil}^{r_2+r_3+2}(H^{r_2+r_3+1}_{\mathrm{dR}}(\mathcal{E}^r)) \longrightarrow H^{r_2+r_3+2}_{\mathrm{fp}}(\mathfrak{E}^r, r_2 + r_3 + 2).$$

It thus again follows that

$$\langle \varphi^*_{A,\mathrm{fp}}(\tilde{\eta}), \varphi^*_{BC,\mathrm{fp}}(\tilde{\omega}_2 \otimes \tilde{\omega}_3)\rangle_{\mathcal{E}^r,\mathrm{fp}} = \langle \varphi^*_{A,\mathrm{dR}}(\eta), P(\Phi)^{-1}\varphi^*_{BC}(\rho(P, \omega_2, \omega_3))\rangle_{\mathcal{E}^r,\mathrm{dR}}. \tag{87}$$

Pulling-back under the natural immersion $\mathcal{E}^{r_1} \hookrightarrow \mathcal{E}^r$ induced by A, we obtain that the right-hand side of (87) equals

$$\langle \iota_A^*\eta, P(\Phi)^{-1}\xi(P, \omega_2, \omega_3)\rangle_{\mathcal{E}^{r_1},\mathrm{dR}} = \langle \iota_A^*\eta, \xi(\omega_2, \omega_3)\rangle_{\mathcal{E}^{r_1},\mathrm{dR}}.$$

Theorem 3.8 follows.

3.4. **A formula for $\mathrm{AJ}_p(\Delta_{k,\ell,m})$ in terms of p-adic modular forms.** Let g and h denote modular forms of weights $\ell = r_2 + 2$ and $m = r_3 + 2$, respectively, on $\Gamma_1(N)$. For the calculations of this section, it is only necessary to assume that g and h are eigenvectors for the Hecke operator T_p; in particular it will not be assumed that they are new of level N. Let α_g and β_g be the roots of the Hasse polynomial $x^2 - a_p(g)x + \chi_g(p)p^{\ell-1}$ for g at p, ordered in such a way that $\mathrm{ord}_p(\alpha_g) \leq \mathrm{ord}_p(\beta_g)$, and let

$$g_\alpha = g - \beta_g V g, \qquad g_\beta = g - \alpha_g V g \tag{88}$$

denote the respective p-stabilisations, on which the U_p operator acts with eigenvalue α_g and β_g respectively. Similar notational conventions are adopted for h. Note that with these conventions, when g is ordinary then g_α is the ordinary p-stabilisation of g that was previously denoted $g^{(p)}$, but that no ordinariness assumption on g or h are made in this section.

Let f denote a modular form of weight $k = r_1 + 2$ on $\Gamma_1(N)$ which is an eigenvector for all the good Hecke operators, and is ordinary at p (but is not necessarily new of level N) and let $\eta_f^{\text{u-r}} \in H^1_{\mathrm{par}}(X_{\mathbb{C}_p}, \mathcal{L}_{r_1})^{\text{u-r}}$ be the unique lift to the unit root subspace of the cohomology class in $H^1(X_{K_f}, \underline{\omega}^{-r_1})$ attached to f as in Corollary 2.13. Recall that e_{ord} denotes Hida's ordinary projector. Let e_{f^*} denote the commuting idempotent in the Hecke algebra giving the projection onto the f^*-isotypic part, where f^* is the modular form which is dual to f, obtained by applying complex conjugation to the Fourier coefficients of f, and set

$$e_{f^*,\mathrm{ord}} = e_{f^*}e_{\mathrm{ord}}.$$

The goal of the next two theorems is to give an explicit expression for the class $\xi(\omega_g, \omega_h) \in H^1_{\mathrm{dR}}(X, \mathcal{L}_{r_1})$ that appears in Theorem 3.8, or rather, its image under $e_{f^*,\mathrm{ord}}$.

Theorem 3.12 below treats the setting where $k = \ell = m = 2$, in which the cycle $\Delta_{2,2,2}$ is simply the Gross-Kudla-Schoen modified diagonal on $X_1(N)^3$. This setting is notationally simpler and therefore easier to follow, but already brings out the key ideas needed to handle the general case, which will then be treated in Theorem 3.13.

Theorem 3.12. *Suppose that $k = \ell = m = 2$, so that $\xi(\omega_g, \omega_h)$ can be viewed as an overconvergent modular form of weight 2. The ordinary projection $e_{f^*,\mathrm{ord}}(\xi(\omega_g, \omega_h))$ is equal to the classical modular form*

$$e_{f^*,\mathrm{ord}}(\xi(\omega_g, \omega_h)) = -\frac{\mathcal{E}_1(f)}{\mathcal{E}(f,g,h)} e_{f^*,\mathrm{ord}}(d^{-1}g^{[p]} \times h),$$

where $\mathcal{E}_1(f)$ and $\mathcal{E}(f,g,h)$ are defined as in Theorem 1.3 of the introduction.

Proof. Let

$$\begin{aligned} P_g(x) = 1 - a_p(g)p^{-1}x + \chi_g(p)p^{-1}x^2 &= (1-\alpha_g p^{-1}x)(1-\beta_g p^{-1}x), \\ P_h(x) = 1 - a_p(h)p^{-1}x + \chi_h(p)p^{-1}x^2 &= (1-\alpha_h p^{-1}x)(1-\beta_h p^{-1}x), \end{aligned}$$

denote the Hasse polynomials attached to the forms g and h, and set

$$\begin{aligned} &P_{gh}(x) = (1-\alpha_g\alpha_h p^{-2}x)(1-\alpha_g\beta_h p^{-2}x)(1-\beta_g\alpha_h p^{-2}x)(1-\beta_g\beta_h p^{-2}x), \\ &P_{\alpha\alpha}(x) = (1-\alpha_g\alpha_h p^{-2}x)^{-1}P_{gh}(x), \qquad P_{\alpha\beta}(x) = (1-\alpha_g\beta_h p^{-2}x)^{-1}P_{gh}(x), \\ &P_{\beta\alpha}(x) = (1-\beta_g\alpha_h p^{-2}x)^{-1}P_{gh}(x), \qquad P_{\beta\beta}(x) = (1-\beta_g\beta_h p^{-2}x)^{-1}P_{gh}(x). \end{aligned}$$

Recall that Φ_2 and Φ_3 denote the canonical lift Φ of frobenius operating on the Hasse domains, viewed as contained in the second and third factors respectively of the three-fold of $X_1(N)^3$. The operator Φ is related to the V operator on (nearly) overconvergent p-adic modular forms of weight two by the rule $\Phi = pV$. After writing $\Phi_2 = pV_2$ and $\Phi_3 = pV_3$, the operator $\Phi := \Phi_2\Phi_3 = p^2V_2V_3$ corresponds to the the canonical lift of Frobenius acting on $\Omega^2_{\mathrm{rig}}(\mathcal{W}_\epsilon \times \mathcal{W}_\epsilon)$.

The operators $P_g(\Phi_2)$ and $P_h(\Phi_3)$ annihilate the classes of ω_g and ω_h in $H^1_{\mathrm{rig}}(\mathcal{W}_\epsilon)$. More precisely, as in (19),

$$\begin{aligned} P_g(\Phi_2)\omega_g &= (1-a_p(g)V_2 + \chi_g(p)pV_2^2)g(q) = g^{[p]}(q) = dG^{[p]}, \\ P_h(\Phi_3)\omega_h &= (1-a_p(h)V_3 + \chi_h(p)pV_3^2)h(q) = h^{[p]}(q) = dH^{[p]}, \end{aligned} \tag{89}$$

where $G^{[p]}$ and $H^{[p]} \in \mathcal{O}_{\mathcal{W}_\epsilon}$ are given by

$$G^{[p]}(q) = d^{-1}g^{[p]} = \sum_{p\nmid n} \frac{a_n(g)}{n} q^n, \qquad H^{[p]}(q) = d^{-1}h^{[p]} = \sum_{p\nmid n} \frac{a_n(h)}{n} q^n.$$

These primitives are overconvergent modular forms of level N and weight 0. Note that they have been normalized so that they vanish at the cusp ∞.

Equation (89) indicates that $P_{gh}(\Phi_2\Phi_3)$ annihilates the class of the rigid differential $\omega_g\omega_h$ in $H^2_{\mathrm{dR}}(X_1(N)^2)$. Hence, there is a rigid one-form $\rho(P_{gh}, \omega_g, \omega_h)$ satisfying

$$d\rho(P_{gh}, \omega_g, \omega_h) = P_{gh}(\Phi_2\Phi_3)\omega_g\omega_h. \tag{90}$$

To describe $\rho(P_{gh}, \omega_g, \omega_h)$ more explicitly, we exploit the identities

$$g = (\alpha_g - \beta_g)^{-1}(\alpha_g g_\alpha - \beta_g g_\beta), \qquad h = (\alpha_h - \beta_h)^{-1}(\alpha_h h_\alpha - \beta_h h_\beta)$$

expressing g and h as linear combinations of the stabilisations appearing in (88). This leads to an expression for the right-hand side of (90) as a sum of four contributions:

$$P_{gh}(\Phi_2\Phi_3)\omega_g\omega_h = \theta_{\alpha\alpha} - \theta_{\alpha\beta} - \theta_{\beta\alpha} + \theta_{\beta\beta},$$

where $\theta_{\alpha\alpha}$ is the rigid two-form on $\mathcal{W}_\epsilon \times \mathcal{W}_\epsilon$ (for a suitable $\epsilon > 0$ given by

$$\begin{aligned} \theta_{\alpha\alpha} &= \frac{\alpha_g\alpha_h P_{gh}(\Phi_2\Phi_3)(g_\alpha \otimes h_\alpha)}{(\alpha_g-\beta_g)(\alpha_h-\beta_h)} \\ &= \frac{\alpha_g\alpha_h P_{\alpha\alpha}(\Phi_2\Phi_3)(1-\alpha_g\alpha_h V_2V_3)(g_\alpha\otimes h_\alpha)}{(\alpha_g-\beta_g)(\alpha_h-\beta_h)}, \end{aligned} \tag{91}$$

and $\theta_{\alpha\beta}$ (resp. $\theta_{\beta\alpha}$ and $\theta_{\beta\beta}$) are defined in the identical way, after replacing (α_g, α_h) by (α_g, β_h) (resp. (β_g, α_h), (β_g, β_h).) We next observe that

$$\begin{aligned} (1-\alpha_g\alpha_h V_2V_3)(g_\alpha\otimes h_\alpha) &= \frac{1}{2}\Big((1-\alpha_gV_2)(1+\alpha_hV_3) \;+\; (1+\alpha_gV_2)(1-\alpha_hV_3)\Big)\, g_\alpha\otimes h_\alpha \\ &= \frac{1}{2}\left(g^{[p]}\otimes(1+\alpha_hV_3)h_\alpha \;+\; (1+\alpha_gV_2)g_\alpha\otimes h^{[p]}\right). \end{aligned} \tag{92}$$

It follows from (91) and (92), together with the fact that $d\Phi = \Phi d$, that

$$\theta_{\alpha\alpha} = d\rho_{\alpha\alpha}, \tag{93}$$

where

$$\rho_{\alpha\alpha} = -\frac{\alpha_g\alpha_h P_{\alpha\alpha}(\Phi_2\Phi_3)\left(G^{[p]}\otimes(1+\alpha_hV_3)h_\alpha - (1+\alpha_gV_2)g_\alpha\otimes H^{[p]}\right)}{2(\alpha_g-\beta_g)(\alpha_h-\beta_h)}. \tag{94}$$

The rigid one-forms $\rho_{\alpha\beta}$, $\rho_{\beta\alpha}$, and $\rho_{\beta\beta}$ are defined similarly, after replacing (α_g, α_h) as before by (α_g, β_h), (β_g, α_h), and (β_g, β_h) respectively, and the differential one-form $\rho(P_{gh}, \omega_g, \omega_h)$ of (90) can be chosen to be

$$\rho(P_{gh}, \omega_g, \omega_h) = \rho_{\alpha\alpha} - \rho_{\alpha\beta} - \rho_{\beta\alpha} + \rho_{\beta\beta}.$$

Let $\xi_{\alpha\alpha}$, $\xi_{\alpha\beta}$, $\xi_{\beta\alpha}$, and $\xi_{\beta\beta}$ denote the pullbacks to the modified diagonal of the one-forms $\rho_{\alpha\alpha}$, $\rho_{\alpha\beta}$, $\rho_{\beta\alpha}$, and $\rho_{\beta\beta}$, so that for example

$$\xi_{\alpha\alpha} = -\frac{\alpha_g\alpha_h P_{\alpha\alpha}(\Phi)\left(G^{[p]}\times(1+\alpha_hV)h_\alpha - (1+\alpha_gV)g_\alpha\times H^{[p]}\right)}{2(\alpha_g-\beta_g)(\alpha_h-\beta_h)}. \tag{95}$$

The frobenius operator Φ acts on $e_{f^*,\mathrm{ord}}(H^1_{\mathrm{rig}}(\mathcal{W}_\epsilon))$ with eigenvalue $p\alpha_{f^*}^{-1} = \beta_f$, and hence, by Lemma 2.17,

$$\begin{aligned} e_{f^*,\mathrm{ord}}\left(P_{\alpha\alpha}(\Phi)(G^{[p]}(1+\alpha_hV)h_\alpha)\right) &= P_{\alpha\alpha}(\beta_f)e_{f^*,\mathrm{ord}}(G^{[p]}h), \\ e_{f^*,\mathrm{ord}}\left(P_{\alpha\alpha}(\Phi)((1+\alpha_gV)g_\alpha H^{[p]})\right) &= P_{\alpha\alpha}(\beta_f)e_{f^*,\mathrm{ord}}(gH^{[p]}). \end{aligned}$$

Invoking Lemma 2.17 once again, we find

$$e_{f^*,\mathrm{ord}}(gH^{[p]}) = e_{f^*,\mathrm{ord}}(g^{[p]}H^{[p]}) = -e_{f^*,\mathrm{ord}}(G^{[p]}h^{[p]}) = -e_{f^*,\mathrm{ord}}(G^{[p]}h),$$

where the second equality follows by noting that $g^{[p]}H^{[p]} + G^{[p]}h^{[p]} = d(G^{[p]}H^{[p]})$ is exact and invoking the fact that exact rigid differentials are in the kernel of the ordinary projection. Hence applying the projector $e_{f^*,\mathrm{ord}}$ to (95) gives

$$e_{f^*,\mathrm{ord}}(\xi_{\alpha\alpha}) = -\frac{\alpha_g\alpha_h P_{\alpha\alpha}(\beta_f)}{(\alpha_g-\beta_g)(\alpha_h-\beta_h)}e_{f^*,\mathrm{ord}}(G^{[p]}h). \tag{96}$$

The definition of $\xi(\omega_g, \omega_h)$ given in (72) therefore implies that

$$e_{f^*,\mathrm{ord}}(\xi(\omega_g,\omega_h)) = \xi'_{\alpha\alpha} - \xi'_{\alpha\beta} - \xi'_{\beta\alpha} + \xi'_{\beta\beta}, \tag{97}$$

where

$$\xi'_{\alpha\alpha} = -\frac{\alpha_g\alpha_h}{(\alpha_g-\beta_g)(\alpha_h-\beta_h)(1-\alpha_g\alpha_h\beta_f p^{-2})}e_{f^*,\mathrm{ord}}(G^{[p]}h),$$

and likewise for the other three contributions to $e_{f^*,\mathrm{ord}}(\xi(\omega_g,\omega_h))$. A direct calculation reveals that

$$\begin{aligned}
\xi'_{\alpha\alpha}-\xi'_{\alpha\beta} &= -\frac{\alpha_g}{(\alpha_g-\beta_g)(1-\alpha_g\alpha_h\beta_f p^{-2})(1-\alpha_g\beta_h\beta_f p^{-2})}e_{f^*,\mathrm{ord}}(G^{[p]}h),\\
\xi'_{\beta\alpha}-\xi'_{\beta\beta} &= -\frac{\beta_g}{(\alpha_g-\beta_g)(1-\beta_g\alpha_h\beta_f p^{-2})(1-\beta_g\beta_h\beta_f p^{-2})}e_{f^*,\mathrm{ord}}(G^{[p]}h).
\end{aligned}$$

Subtracting these two equations and invoking (97) gives

$$e_{f^*,\mathrm{ord}}(\xi(\omega_g,\omega_h)) = -\frac{(1-\chi_f(p)^{-1}\beta_f^2p^{-2})}{\mathcal{E}(f,g,h)}e_{f^*,\mathrm{ord}}(d^{-1}g^{[p]}\times h),$$

as was to be shown. $\square$

We now turn to the case of general weights (k,ℓ,m). Before stating the result, recall that the triple of weights (k,ℓ,m) is still assumed to be balanced, and that, following notations similar to those in the proof of Proposition 2.9, we have set

$$k=\ell+m-2-2t, \quad \text{with } t\geq 0, \qquad c=(k+\ell+m-2)/2,$$

so that $\xi(\omega_g,\omega_h)$ corresponds to a class in $H^1_{\mathrm{dR}}(X,\mathcal{L}_{r_1}(-t))$.

Theorem 3.13. *The projection $e_{f^*,\mathrm{ord}}(\xi(\omega_g,\omega_h))$ is represented by the classical modular form*

$$e_{f^*,\mathrm{ord}}(\xi(\omega_g,\omega_h)) = -\frac{(-1)^t\cdot t!\cdot\mathcal{E}_1(f)}{\mathcal{E}(f,g,h)}e_{f^*,\mathrm{ord}}(d^{-1-t}g^{[p]}\times h),$$

where $\mathcal{E}_1(f)$ and $\mathcal{E}(f,g,h)$ are defined as in Theorem 1.3 of the introduction.

Proof. Let ω_g and ω_h be the global sections of $\underline{\omega}^{r_2}\otimes\Omega^1_X$ and of $\underline{\omega}^{r_3}\otimes\Omega^1_X$ over $X=X_1(N)$ attached to g and h respectively. Since these sections are algebraic, they can also be viewed as $\mathcal{L}_{r_2}$ and $\mathcal{L}_{r_3}$-valued rigid differentials on $\mathcal{W}_\epsilon/\mathbb{C}_p$ for any $\epsilon>0$. Modifying slightly the definitions of the weight two setting, we define

$$P_g(x)=(1-\alpha_g p^{-r_2-1}x)(1-\beta_g p^{-r_2-1}x), \qquad P_h(x)=(1-\alpha_h p^{-r_3-1}x)(1-\beta_h p^{-r_3-1}x).$$

The operators $P_g(\Phi_2)$ and $P_h(\Phi_3)$ annihilate the classes of ω_g and ω_h in $H^1_{\mathrm{rig}}(\mathcal{W}_\epsilon,\mathcal{L}_{r_2})$ and $H^1_{\mathrm{rig}}(\mathcal{W}_\epsilon,\mathcal{L}_{r_3})$ respectively, and in terms of q-expansions,

$$\begin{aligned}
P_g(\Phi_2)(\omega_g) &= (1-\alpha_g V_2)(1-\beta_g V_2)g = g^{[p]},\\
P_h(\Phi_3)(\omega_h) &= (1-\alpha_h V_3)(1-\beta_h V_3)g = h^{[p]}.
\end{aligned}$$

Let $G^{[p]}$ and $H^{[p]}$ denote the overconvergent sections of $\mathcal{L}_{r_2}$ and $\mathcal{L}_{r_3}$ satisfying

$$\nabla G^{[p]}=\omega_{g^{[p]}}, \qquad \nabla H^{[p]}=\omega_{h^{[p]}}.$$

The q-expansions of these sections are given in equation (47) of Section 2.4, i.e.,

$$G^{[p]}(q) = \sum_{j=0}^{r_2}(-1)^j j!\binom{r_2}{j}d^{-1-j}g^{[p]}(q)\omega_{\mathrm{can}}^{r_2-j}\eta_{\mathrm{can}}^j, \tag{98}$$

$$H^{[p]}(q) = \sum_{j=0}^{r_3}(-1)^j j!\binom{r_3}{j}d^{-1-j}h^{[p]}(q)\omega_{\mathrm{can}}^{r_3-j}\eta_{\mathrm{can}}^j. \tag{99}$$

As in the weight two setting, we let

$$\begin{aligned}
P_{gh}(x) = {} & (1-\alpha_g\alpha_h p^{-r_2-r_3-2}x)(1-\alpha_g\beta_h p^{-r_2-r_3-2}x)(1-\beta_g\alpha_h p^{-r_2-r_3-2}x)\\
& \times(1-\beta_g\beta_h p^{-r_2-r_3-2}x).
\end{aligned}$$

Just as before, the operator $P_{gh}(\Phi_2\Phi_3)$ annihilates the class of $\omega_g\otimes\omega_h$ in the hypercohomology group $H^2_{\mathrm{dR}}(X_1(N)^2, \mathcal{L}_{r_2}\otimes\mathcal{L}_{r_3})$, so there is a rigid $\mathcal{L}_{r_2}\otimes\mathcal{L}_{r_3}$-valued one-form $\rho(P_{gh},\omega_g,\omega_h)$ on $\mathcal{W}_\epsilon\times\mathcal{W}_\epsilon$ satisfying

$$\nabla\rho(P_{gh},\omega_g,\omega_h) = P_{gh}(\Phi_2\Phi_3)\omega_g\omega_h. \tag{100}$$

The same manipulations as in the weight two case allow us to express $\rho(P_{gh},\omega_g,\omega_h)$ as a sum of four contributions:

$$\rho(P_{gh},\omega_g,\omega_h) = \rho_{\alpha\alpha} - \rho_{\alpha\beta} - \rho_{\beta\alpha} + \rho_{\beta\beta},$$

where for instance

$$\rho_{\alpha\alpha} = -\frac{\alpha_g\alpha_h P_{\alpha\alpha}(\Phi_2\Phi_3)\left(G^{[p]}\otimes(1+\alpha_h V_3)\omega_{h_\alpha} - (1+\alpha_g V_2)\omega_{g_\alpha}\otimes H^{[p]}\right)}{2(\alpha_g-\beta_g)(\alpha_h-\beta_h)},$$

just as in equations (93) and (94). Let $\xi_{\alpha\alpha}$, $\xi_{\alpha\beta}$, $\xi_{\beta\alpha}$, and $\xi_{\beta\beta}$ denote the pullbacks to the generalised diagonal cycle $\mathcal{E}^r$ of the one-forms $\rho_{\alpha\alpha}$, $\rho_{\alpha\beta}$, $\rho_{\beta\alpha}$, and $\rho_{\beta\beta}$ via the map φ_{BC}, viewed as overconvergent sections of $\mathcal{L}_A(-t)\otimes\Omega^1$ on $\mathcal{W}_\epsilon$. Write

$$\xi^{\text{n-oc}}_{\alpha\alpha} = \mathrm{Spl}_{\text{u-r}}(\xi_{\alpha\alpha}), \qquad \xi^{\text{oc}}_{\alpha\alpha} = \Pi^{\text{oc}}_N(\xi^{\text{n-oc}}_{\alpha\alpha}),$$

for the nearly overconvergent and overconvergent modular forms attached to $\xi_{\alpha\alpha}$. By equation (44) defining Π^{oc}_N, the elements $\xi_{\alpha\alpha}$ and $\xi^{\text{oc}}_{\alpha\alpha}$ have the same image in cohomology, and therefore the following equality holds in $H^1_{\mathrm{dR}}(X,\mathcal{L}_{r_1}(-t))$:

$$\xi(P_{gh},\omega_g,\omega_h) = \xi^{\text{oc}}_{\alpha\alpha} - \xi^{\text{oc}}_{\alpha\beta} - \xi^{\text{oc}}_{\beta\alpha} + \xi^{\text{oc}}_{\beta\beta}.$$

Applying $e_{f^*,\mathrm{ord}}$ to this equation and invoking Lemma 2.7 gives

$$e_{f^*,\mathrm{ord}}(\xi(P_{gh},\omega_g,\omega_h)) = e_{f^*,\mathrm{ord}}(\xi^{\text{n-oc}}_{\alpha\alpha} - \xi^{\text{n-oc}}_{\alpha\beta} - \xi^{\text{n-oc}}_{\beta\alpha} + \xi^{\text{n-oc}}_{\beta\beta}),$$

and the same argument as in the proof of Proposition 2.9 (cf. in particular equation (49)) shows that

$$\xi^{\text{n-oc}}_{\alpha\alpha} = -\frac{\alpha_g\alpha_h P_{\alpha\alpha}(\Phi)\left(d^{-1-t}g^{[p]}\times(1+\alpha_h V)h_\alpha - (1+\alpha_g V)g_\alpha\times d^{-1-t}h^{[p]}\right)}{2(\alpha_g-\beta_g)(\alpha_h-\beta_h)}. \tag{101}$$

By exactly the same reasoning used to obtain (96) in the weight two case, and using the fact that Φ acts on $e_{f^*,\mathrm{ord}}(H^1_{\mathrm{dR}}(X,\mathcal{L}_{r_1}(-t)))$ with eigenvalue $p^{r_1+1+t}\alpha_{f^*}^{-1} = \beta_f p^t$, we find after applying the projector $e_{f^*,\mathrm{ord}}$ to (101) that

$$e_{f^*,\mathrm{ord}}(\xi^{\text{n-oc}}_{\alpha\alpha}) = -\frac{(-1)^t\cdot t!\cdot\alpha_g\alpha_h P_{\alpha\alpha}(\beta_f p^t)}{(\alpha_g-\beta_g)(\alpha_h-\beta_h)} e_{f^*,\mathrm{ord}}(d^{-1-t}g^{[p]}\times h). \tag{102}$$

It now follows from (72) that

$$e_{f^*,\mathrm{ord}}(\xi(\omega_g,\omega_h)) = \xi'_{\alpha\alpha} - \xi'_{\alpha\beta} - \xi'_{\beta\alpha} + \xi'_{\beta\beta}, \tag{103}$$

where

$$\xi'_{\alpha\alpha} = -\frac{(-1)^t t!\alpha_g\alpha_h}{(\alpha_g-\beta_g)(\alpha_h-\beta_h)(1-\alpha_g\alpha_h\beta_f p^{-c})} e_{f^*,\mathrm{ord}}(d^{-1-t}g^{[p]}\times h),$$

and likewise for the other three contributions to $e_{f^*,\mathrm{ord}}(\xi(\omega_g,\omega_h))$. The result now follows from a simple direct calculation, as in the conclusion of the proof of Theorem 3.12. □

We can now conclude this section with a (partial) formula for the image of the generalised Gross-Kudla Schoen cycle $\Delta_{k,\ell,m}$ under the p-adic Abel-Jacobi map.

Theorem 3.14. *With notations as in the statement of Theorems 3.12 and 3.13,*

$$\mathrm{AJ}_p(\Delta_{k,\ell,m})(\eta^{\text{u-r}}_f\omega_g\omega_h) = (-1)^{t+1}\frac{t!\cdot\mathcal{E}_1(f)}{\mathcal{E}(f,g,h)}\langle\eta^{\text{u-r}}_f, d^{-1-t}g^{[p]}\times h\rangle,$$

for all $\eta^{\text{u-r}}_f\in H^1_{\mathrm{dR}}(X,\mathcal{L}_{r_1})^{\text{u-r}}[\pi_{f^*}]$.

Proof. Assume first that $(k, \ell, m) = (2, 2, 2)$. By Proposition 2.3, the class $\eta_f^{\text{u-r}}$ is orthogonal to the kernel of $e_{f^*,\text{ord}}$, and hence

$$\langle \eta_f^{\text{u-r}}, \xi(\omega_g, \omega_h) \rangle = \langle \eta_f^{\text{u-r}}, e_{f^*,\text{ord}} \xi(\omega_g, \omega_h) \rangle.$$

Theorem 3.14 therefore follows from Theorem 3.12. The case of general weights (k, ℓ, m) follows from an identical argument using Theorem 3.13 instead of Theorem 3.12. $\square$

4. Garrett-Rankin triple product L-functions

4.1. **Classical L-functions.** As in the introduction, let

$$f \in S_k(N_f, \chi_f), \quad g \in S_\ell(N_g, \chi_g), \quad h \in S_m(N_h, \chi_h)$$

be a triplet of normalized primitive cuspidal eigenforms such that $\chi_f \cdot \chi_g \cdot \chi_h = 1$, and let

$$\mathbb{Q}_{f,g,h} = \mathbb{Q}_f \cdot \mathbb{Q}_g \cdot \mathbb{Q}_h = \mathbb{Q}(\{a_n(f), a_n(g), a_n(h)\}_{n \geq 1})$$

denote the field generated by the fourier coefficients of f, g and h. Write also $N = \operatorname{lcm}(N_f, N_g, N_h)$.

The Garrett-Rankin triple product L-function $L(f, g, h; s)$ is defined by an Euler product

$$L(f, g, h; s) = \prod_p L^{(p)}(f, g, h; p^{-s})^{-1},$$

where, for $p \nmid N$ the local factor $L^{(p)}(f, g, h; T)$ is the degree 8 polynomial

$$\begin{aligned} L^{(p)}(f, g, h; T) \quad = \quad & (1 - \alpha_{f,p}\alpha_{g,p}\alpha_{h,p}T) \times (1 - \alpha_{f,p}\alpha_{g,p}\beta_{h,p}T) \\ & \times (1 - \alpha_{f,p}\beta_{g,p}\alpha_{h,p}T) \times (1 - \alpha_{f,p}\beta_{g,p}\beta_{h,p}T) \\ & \times (1 - \beta_{f,p}\alpha_{g,p}\alpha_{h,p}T) \times (1 - \beta_{f,p}\alpha_{g,p}\beta_{h,p}T) \\ & \times (1 - \beta_{f,p}\beta_{g,p}\alpha_{h,p}T) \times (1 - \beta_{f,p}\beta_{g,p}\beta_{h,p}T). \end{aligned} \tag{104}$$

Piatetski-Shapiro and Rallis have given a precise recipe [PSR] for the local Euler factors $L^{(p)}(f, g, h; s)$ at the bad primes $p \mid N$ and have shown that there exists an archimedean factor $L^{(\infty)}(f, g, h; s)$ for which the completed L-function

$$\Lambda(f, g, h; s) = L(f, g, h; s) \cdot L^{(\infty)}(f, g, h; s)$$

satisfies the functional equation

$$\Lambda(f, g, h; s) = \varepsilon(f, g, h)\Lambda(f, g, h; k + \ell + m - 2 - s), \tag{105}$$

where $\varepsilon(f, g, h) \in \{\pm 1\}$. The sign in this functional equation determines the parity of the order of vanishing of $L(f, g, h, s)$ at the center of symmetry

$$c = c_{f,g,h} = \frac{k + \ell + m - 2}{2}, \tag{106}$$

at which there is no pole (cf. [PSR, Theorem 5.2]). The sign $\varepsilon(f, g, h)$ can be expressed as a product

$$\varepsilon(f, g, h) = \prod_q \varepsilon_q(f, g, h)$$

of local root numbers $\varepsilon_q(f, g, h) \in \{\pm 1\}$ indexed by the places $q \leq \infty$ of $\mathbb{Q}$. As already anticipated in the introduction, we assume throughout that $\varepsilon_q(f, g, h) = +1$ for all finite primes. According to e. g. [Pr90] and the references therein, we then have

$$\varepsilon(f, g, h) = \varepsilon_\infty(f, g, h) = \begin{cases} -1 & \text{if } (k, \ell, m) \text{ are balanced.} \\ +1 & \text{if } (k, \ell, m) \text{ are unbalanced.} \end{cases} \tag{107}$$

We now recall a fundamental result which relates the central critical value of $L(f, g, h, s)$ to certain *trilinear period integrals*. For this, assume that (k, ℓ, m) is unbalanced, namely (without loss of generality, by eventually permuting f, g and h), that $k = \ell + m + 2t$ with

$t \geq 0$. Recall also the Shimura-Maass operators δ_ℓ defined in §2.3 and the space $S_k(N)[\pi_f]$ introduced in §2.5.

For any eigenform $\phi \in S_k(N)$ with character χ (not necessarily new at N), we write ϕ^* for the dual form obtained by twisting ϕ by the character χ^{-1}. The eigenform ϕ^* has nebentype character χ^{-1}, and its Hecke eigenvalues are complex conjugate to the Hecke eigenvalues attached to ϕ.

Definition 4.1. Assume that $k = \ell + m + 2t$ with $t \geq 0$. The *trilinear period* attached to

$$(\breve{f}, \breve{g}, \breve{h}) \in S_k(N)[\pi_f] \times S_\ell(N)[\pi_g] \times S_m(N)[\pi_h]$$

is the expression

$$I(\breve{f}, \breve{g}, \breve{h}) := (\breve{f}^*, \delta_\ell^t \breve{g} \times \breve{h})_N.$$

One of the main results of M. Harris and S. Kudla [HaKu], conjectured by H. Jacquet and refined recently by A. Ichino [I] and T. C. Watson [Wa] is:

Theorem 4.2. *Let (f, g, h) be a triple of modular forms of unbalanced weights (k, ℓ, m) with $k = \ell + m + 2t$ and $t \geq 0$. Then there exist*

- *holomorphic modular forms $\breve{f} \in S_k(N)[\pi_f]$, $\breve{g} \in S_\ell(N)[\pi_g]$, and $\breve{h} \in S_m(N)[\pi_h]$,*
- *for each prime $q \mid N\infty$, a constant $C_q \in \mathbb{Q}_{f,g,h}$ which depends only on the local components at q of the vectors $\breve{f}$, $\breve{g}$ and $\breve{h}$ in the admissible representations of $\mathrm{GL}_2(\mathbb{Q}_q)$ associated to π_f, π_g and π_h;*

such that

$$\frac{\prod_{q|N\infty} C_q}{\pi^{2k}} \cdot L(f, g, h, c) = |I(\breve{f}, \breve{g}, \breve{h})|^2. \tag{108}$$

Furthermore, there is a choice of $(\breve{f}, \breve{g}, \breve{h})$ for which the constants C_q are all non-zero.

Remark 4.3. If $N_f = N_g = N_h = N$, then the space $S_k(N)[\pi_f]$ is one-dimensional, and likewise for g and h. In that case there is only one choice (up to scaling) for the vectors $\breve{f}$, $\breve{g}$ and $\breve{h}$. Otherwise, the triplet $(\breve{f}, \breve{g}, \breve{h})$ has to be chosen more carefully: see [GrPr] and [DiNy] for explicit recipes when at least one of π_f, π_g or π_h is not supercuspidal at the primes $q \mid \gcd(N_f, N_g, N_h)$. It could be tempting to choose $\breve{f}$, $\breve{g}$ and $\breve{h}$ to be the new vectors in $S_k(N_f)$, $S_\ell(N_g)$ and $S_m(N_h)$ attached to f, g and h respectively, but such a choice may not always be suitable. For example, if there is a prime q dividing N_f but not N_g or N_h, then $C_q = 0$ for this choice of test vectors.

4.2. p-adic L-functions.

Let $\bar{\mathbb{Q}}$ denote the algebraic closure of $\mathbb{Q}$ in $\mathbb{C}$ and fix throughout an odd prime $p \nmid N$, an algebraic closure $\bar{\mathbb{Q}}_p$ of $\mathbb{Q}_p$ and an embedding $\iota_p : \bar{\mathbb{Q}} \hookrightarrow \bar{\mathbb{Q}}_p$. Let $\mathrm{ord}_p : \bar{\mathbb{Q}}_p^\times \to \mathbb{Q}$ be the valuation on $\bar{\mathbb{Q}}_p$ normalized so that $\mathrm{ord}_p(p) = 1$.

Fix a triple (f, g, h) of newforms and keep the notations of the previous section. As before we assume that $\chi_f\chi_g\chi_h = 1$, $\varepsilon_q(f, g, h) = +1$ for all finite primes $q \mid N$, and the triple (k, ℓ, m) of weights is unbalanced.

Let $\breve{f} \in S_k(N; K_f)[\pi_f]$, $\breve{g} \in S_\ell(N; K_g)[\pi_g]$ and $\breve{h} \in S_m(N; K_h)[\pi_h]$ be test vectors for which the local constants C_q of Theorem 4.2 are non-zero and the central critical value of $L(\pi_f, \pi_g, \pi_h, s)$ is (up to elementary non-zero fudge factors) equal to the square of $I(\breve{f}, \breve{g}, \breve{h})$.

Assume in addition that f, g and h are ordinary with respect to ι_p and let

$$\breve{\mathbf{f}} = (\Lambda_f, \Omega_f, \Omega_{f,\mathrm{cl}}, \breve{\mathbf{f}}), \quad \breve{\mathbf{g}} = (\Lambda_g, \Omega_g, \Omega_{g,\mathrm{cl}}, \breve{\mathbf{g}}), \quad \breve{\mathbf{h}} = (\Lambda_h, \Omega_h, \Omega_{h,\mathrm{cl}}, \breve{\mathbf{h}})$$

be the Hida families of (not necessarily new) forms on $\Gamma_1(N)$ interpolating $\breve{f}$, $\breve{g}$ and $\breve{h}$ in weights k, ℓ and m respectively, as described in (58). Adapting the notations of the introduction, we write

$$\Sigma := \{(x, y, z) \in \Omega_{f,\mathrm{cl}} \times \Omega_{g,\mathrm{cl}} \times \Omega_{h,\mathrm{cl}}\},$$

and set

$$\begin{aligned}\Sigma_f &= \{(x,y,z)\in\Sigma, \quad \text{such that } 2t := \kappa(x)-\kappa(y)-\kappa(z)\geq 0\},\\ \Sigma_{\rm bal} &= \{(x,y,z)\in\Sigma, \quad \text{such that } (\kappa(x),\kappa(y),\kappa(z)) \text{ is balanced.}\}.\end{aligned}$$

For each $(x,y,z)\in\Sigma_f$, we may consider the algebraic number

$$J(\breve{f}_x,\breve{g}_y,\breve{h}_z) := \frac{I(\breve{f}_x,\breve{g}_y,\breve{h}_z)}{(f_x^*,f_x^*)_N}\in\bar{\mathbb{Q}},$$

which we view in $\mathbb{C}_p$ via ι_p. In light of the discussion in the previous section, our approach to defining the p-adic L-function attached to $\pi_f\otimes\pi_g\otimes\pi_h$ will be to interpolate (suitable multiples of) the ratios $J(\breve{f}_x,\breve{g}_y,\breve{h}_z)$ as (x,y,z) ranges over Σ_f. In order to do this, we let $\boldsymbol{\phi} := e_{\rm ord}(d^\bullet\breve{\mathbf{g}}^{[p]}\times\breve{\mathbf{h}})$ be the ordinary family of modular forms attached to $\breve{\mathbf{g}}$ and $\breve{\mathbf{h}}$ following the recipe in Proposition 2.18.

Furthermore, define the Hida family $\breve{\mathbf{f}}^* = \breve{\mathbf{f}}\otimes\chi_f^{-1}$; note that $(\breve{f}^*)_x = (\breve{f}_x)^*$ for all $x\in\Omega_{f,\rm cl}$. Recall that according to the notations settled in §2.6, a classical point corresponds to a character of the form $\gamma\mapsto\gamma^k$ for some $k\in\mathbb{Z}_{\geq 2}$. We warn the reader that the equality $(\breve{f}^*)_x = (\breve{f}_x)^*$ does not hold true for arithmetic points corresponding to characters $\gamma\mapsto\psi(\gamma)\gamma^k$ for some Dirichlet character ψ of non-trivial p-power conductor.

Definition 4.4. The *Garrett-Rankin triple product p-adic L-function* attached to the triple $(\breve{\mathbf{f}},\breve{\mathbf{g}},\breve{\mathbf{h}})$ of Λ-adic modular forms is the element

$$\mathscr{L}_p{}^f(\breve{\mathbf{f}},\breve{\mathbf{g}},\breve{\mathbf{h}}) := J(\breve{\mathbf{f}}^*, e_{\rm ord}(d^\bullet\breve{\mathbf{g}}^{[p]}\times\breve{\mathbf{h}}))\in\Lambda_f'\otimes_\Lambda(\Lambda_g\otimes\Lambda_h\otimes\Lambda)$$

attached to the families $\breve{\mathbf{f}}$ and $\boldsymbol{\phi} = e_{\rm ord}(d^\bullet\breve{\mathbf{g}}^{[p]}\times\breve{\mathbf{h}})$ following the notations of Lemma 2.19.

Remark 4.5. The definition of $\mathscr{L}_p{}^f(\breve{\mathbf{f}},\breve{\mathbf{g}},\breve{\mathbf{h}})$ depends not just on the p-adic families of automorphic representations interpolating π_f, π_g, and π_h, but also on the choice of a triple $(\breve{f},\breve{g},\breve{h})$ of test vectors in these automorphic representations. Note also that while

$$\mathscr{L}_p{}^f(\breve{\mathbf{f}},\breve{\mathbf{g}},\breve{\mathbf{h}}) = \mathscr{L}_p{}^f(\breve{\mathbf{f}},\breve{\mathbf{h}},\breve{\mathbf{g}}),$$

the Λ-adic family $\breve{\mathbf{f}}$ plays an essentially different role from the other two in the definition of $\mathscr{L}_p{}^f(\breve{\mathbf{f}},\breve{\mathbf{g}},\breve{\mathbf{h}})$. There are thus three *a priori distinct* p-adic L-functions attached to $(\breve{\mathbf{f}},\breve{\mathbf{g}},\breve{\mathbf{h}})$, namely, $\mathscr{L}_p{}^f(\breve{\mathbf{f}},\breve{\mathbf{g}},\breve{\mathbf{h}})$, $\mathscr{L}_p{}^g(\breve{\mathbf{f}},\breve{\mathbf{g}},\breve{\mathbf{h}}) := \mathscr{L}_p{}^g(\breve{\mathbf{g}},\breve{\mathbf{f}},\breve{\mathbf{h}})$, and $\mathscr{L}_p{}^h(\breve{\mathbf{f}},\breve{\mathbf{g}},\breve{\mathbf{h}})$.

Any element $\mathscr{L}\in\Lambda_f'\otimes_\Lambda(\Lambda_g\otimes\Lambda_h\otimes\Lambda)$ as in Definition 4.4 has poles at (x,y,z) for only finitely many $x\in\Omega_f$, and hence is completely determined by its values at the points of $\Omega_{f,\rm cl}\times_\Omega(\Omega_{g,\rm cl}\times\Omega_{h,\rm cl}\times\Omega)$ where it is defined. Furthermore, it is always defined at (x,y,z) when x belongs to $\Omega_{f,\rm cl}$. By definition, for all $(x,y,z)\in\Sigma_f$, after setting $\kappa(x)=\kappa(y)+\kappa(z)+2t$,

$$\mathscr{L}_p{}^f(\breve{\mathbf{f}},\breve{\mathbf{g}},\breve{\mathbf{h}})(x,y,z) = \frac{(\breve{f}_x^{*(p)}, e_{\rm ord}(d^t\breve{g}_y^{[p]}\times\breve{h}_z))_{N,p}}{(f_x^{*(p)},f_x^{*(p)})_{N,p}}. \tag{109}$$

In particular, the special value $\mathscr{L}_p{}^f(\breve{\mathbf{f}},\breve{\mathbf{g}},\breve{\mathbf{h}})(x,y,z)$ is algebraic and belongs to the field K generated by α_{f_x} and by the fourier coefficients of $\breve{f}_x$, $\breve{g}_y$ and $\breve{h}_z$.

The following proposition supplements (109) with a formula for the p-adic special value $\mathscr{L}_p{}^f(\breve{\mathbf{f}},\breve{\mathbf{g}},\breve{\mathbf{h}})$ at a point $(x,y,z)\in\Sigma_{\rm bal}$.

Proposition 4.6. *For all $(x,y,z)\in\Sigma_{\rm bal}$, let $(f,g,h) := (f_x,g_y,h_z)$ and define (k,ℓ,m,t) by setting*

$$(k,\ell,m) = (\kappa(x),\kappa(y),\kappa(z)), \qquad k=\ell+m-2-2t.$$

Then

$$\mathscr{L}_p{}^f(\breve{\mathbf{f}},\breve{\mathbf{g}},\breve{\mathbf{h}})(x,y,z) = \mathcal{E}_0(f)^{-1}\langle\eta_{\breve{f}_x}^{\rm u\text{-}r}, e_{\rm ord}(d^{-1-t}\breve{g}_y^{[p]}\times\breve{h}_z)\rangle,$$

where

- $\mathcal{E}_0(f)$ *is the local factor given in* (7)*;*
- $\eta^{\text{u-r}}_{\breve{f}_x} \in H^1_{\text{par}}(X_{\mathbb{C}_p}, \mathcal{L}_r)^{\text{u-r}}$ *is the unique lift to the unit root subspace of the cohomology class in* $H^1(X_{K_{f_x}}, \underline{\omega}^{-r})$ *attached to* $\breve{f}^*_x$ *as in Corollary 2.13.*
- *the classical modular form* $e_{\text{ord}}(d^t \breve{g}^{[p]}_y \times \breve{h}_z)$ *on* $\Gamma_1(N) \cap \Gamma_0(p)$ *is viewed as a class in* $H^0(\mathcal{W}_\epsilon, \mathcal{L}_r \otimes \Omega^1_X) \subset H^1_{\text{par}}(X_{\mathbb{C}_p}, \mathcal{L}_r)$*;*
- $\langle\ ,\ \rangle_X$ *denotes the Poincaré duality on* $H^1_{\text{par}}(X_{\mathbb{C}_p}, \mathcal{L}_r)$*.*

Proof. Let $e_{f^*_x}$ denote the idempotent attached to the eigenform f^*_x, set

$$\phi^{(p)}_{xyz} := e_{f^*_x, \text{ord}}(d^t g^{[p]}_y \times h_z),$$

and let $\phi_{xyz} \in S_k(N; \mathbb{C}_p)[\pi_{f^*_x}]$ be the classical modular form whose p-ordinary stabilisation is equal to $\phi^{(p)}_{xyz}$. The definition of $\mathscr{L}_p{}^f(\breve{\mathbf{f}}, \breve{\mathbf{g}}, \breve{\mathbf{h}})$, in view of the last equality in equation (61) of Lemma 2.19, implies that, for all $(x,y,z) \in \Sigma_f \cup \Sigma_{\text{bal}}$,

$$\mathscr{L}_p{}^f(\breve{\mathbf{f}}, \breve{\mathbf{g}}, \breve{\mathbf{h}})(x,y,z) = \langle \eta_{\breve{f}_x}, \phi_{xyz} \rangle. \tag{110}$$

Since ϕ_{xyz} belongs to $\text{Fil}^{r+1} H^1_{\text{par}}(X_{\mathbb{C}_p}, \mathcal{L}_r)$, we can also express this identity in terms of the Poincaré pairing on $H^1_{\text{par}}(X_{\mathbb{C}_p}, \mathcal{L}_r)$ by replacing $\eta_{\breve{f}_x}$ by its lift to $H^1_{\text{par}}(X_{\mathbb{C}_p}, \mathcal{L}_r)$ under the unit root splitting:

$$\mathscr{L}_p{}^f(\breve{\mathbf{f}}, \breve{\mathbf{g}}, \breve{\mathbf{h}})(x,y,z) = \langle \eta^{\text{u-r}}_{\breve{f}_x}, \phi_{xyz} \rangle. \tag{111}$$

By Proposition 2.11, we then have

$$\mathscr{L}_p{}^f(\breve{\mathbf{f}}, \breve{\mathbf{g}}, \breve{\mathbf{h}})(x,y,z) = \langle \eta^{\text{u-r}}_{\breve{f}_x}, e_{\text{ord}} \phi_{xyz} \rangle. \tag{112}$$

On the other hand, applying the ordinary projector to the identity $\phi^{(p)}_{xyz} = \phi_{xyz} - \beta_{f^*_x} V \phi_{xyz}$ shows that

$$\phi^{(p)}_{xyz} = (1 - \frac{\beta_{f^*_x}}{\alpha_{f^*_x}}) e_{\text{ord}} \phi_{xyz} = (1 - \frac{\beta_{f_x}}{\alpha_{f_x}}) e_{\text{ord}} \phi_{xyz} = \mathcal{E}_0(f_x) e_{\text{ord}} \phi_{xyz}, \tag{113}$$

and therefore

$$\begin{aligned} \mathscr{L}_p{}^f(\breve{\mathbf{f}}, \breve{\mathbf{g}}, \breve{\mathbf{h}})(x,y,z) &= \mathcal{E}_0(f_x)^{-1} \langle \eta^{\text{u-r}}_{\breve{f}_x}, \phi^{(p)}_{xyz} \rangle = \mathcal{E}_0(f_x)^{-1} \langle \eta^{\text{u-r}}_{\breve{f}_x}, e_{f^*_x, \text{ord}}(d^t \breve{g}^{[p]}_y \times \breve{h}_z) \rangle \\ &= \mathcal{E}_0(f_x)^{-1} \langle \eta^{\text{u-r}}_{\breve{f}_x}, e_{\text{ord}}(d^t \breve{g}^{[p]}_y \times \breve{h}_z) \rangle. \end{aligned}$$

The proposition follows. $\square$

The next theorem evaluates $\mathscr{L}_p{}^f(\breve{\mathbf{f}}, \breve{\mathbf{g}}, \breve{\mathbf{h}})$ at points $(x,y,z) \in \Sigma_f$ and relates these values to the complex periods $I(\breve{f}_x, \breve{g}_y, \breve{h}_z)$.

Theorem 4.7. *Let* (x,y,z) *be a point of* Σ_f *and set*

$$(f,g,h) := (f_x, g_y, h_z), \qquad (\breve{f}, \breve{g}, \breve{h}) := (\breve{f}_x, \breve{g}_y, \breve{h}_z), \qquad (k, \ell, m) = (\kappa(x), \kappa(y), \kappa(z)).$$

Then

$$\mathscr{L}_p{}^f(\breve{\mathbf{f}}, \breve{\mathbf{g}}, \breve{\mathbf{h}})(x,y,z) = \frac{\mathcal{E}(f,g,h)}{\mathcal{E}_0(f)\mathcal{E}_1(f)} \times \frac{I(\breve{f}, \breve{g}, \breve{h})}{(f^*, f^*)_N}, \tag{114}$$

where $\mathcal{E}(f,g,h)$*,* $\mathcal{E}_0(f)$ *and* $\mathcal{E}_1(f)$ *are given in equations* (6), (7) *and* (8) *of Theorem 1.3.*

Remark 4.8. Theorems 4.2 and 4.7 above justify the designation of $\mathscr{L}_p{}^f(\breve{\mathbf{f}}, \breve{\mathbf{g}}, \breve{\mathbf{h}})$ as a p-adic Garrett-Rankin triple product L-function in that it interpolates the square-roots of the central critical values of the complex L-series $L(f_x, g_y, h_z; s)$ for $(x, y, z) \in \Sigma_f$. Indeed, assume that

$$L(f, g, h; c) \neq 0,$$

so that Theorems 4.2 and 4.7 ensure that $\mathscr{L}_p{}^f(\breve{\mathbf{f}}, \breve{\mathbf{g}}, \breve{\mathbf{h}})$ is non-zero in $\mathcal{L}_f \otimes \Lambda_g \otimes \Lambda_h$. By the Weierstrass preparation theorem, $\mathscr{L}_p{}^f(\breve{\mathbf{f}}, \breve{\mathbf{g}}, \breve{\mathbf{h}})$ vanishes at a finite number of points and therefore $\langle \breve{f}_x, \breve{g}_y \cdot \delta^n(\breve{h}_z)\rangle_N^2 \neq 0$ for all but finitely many $(x, y, z) \in \Sigma_f$. In addition, Jacquet's conjecture as formulated and proved in [HaKu], imply that the hypothesis $\varepsilon_q(f_x, g_y, h_z) = +1$ for all finite primes q still holds true for all such (x, y, z). Hence Theorem 4.2 can again be applied for the triple (f_x, g_y, h_z), which implies that (114) expresses $\mathscr{L}_p{}^f(\breve{\mathbf{f}}, \breve{\mathbf{g}}, \breve{\mathbf{h}})(x, y, z)$ as a non-zero multiple of the square-root of $L(f_x, g_x, h_x; c_{f_x, g_y, h_z})$. $\square$

We now turn to the proof of Theorem 4.7. As before, p is a prime which does not divide N, and $S_k^{\text{n-oc}}(N; \mathbb{C}_p)$ denotes the space of nearly overconvergent p-adic modular forms of level N, as described in Section 2.4. Given a classical newform $f = \sum_n a_n q^n$ of some level $N_f|N$, recall that

$$e_{f^*} : S_k^{\text{ord}}(N; \mathbb{C}_p) \longrightarrow S_k^{\text{ord}}(N; \mathbb{C}_p)[\pi_{f^*}]$$

$$e_{f^*,\text{ord}} = e_{f^*} e_{\text{ord}} : S_k^{\text{n-oc}}(N; \mathbb{C}_p) \longrightarrow S_k^{\text{ord}}(N; \mathbb{C}_p)[\pi_{f^*}]$$

denote the associated projectors to the f^*-isotypic component of the ordinary subspace in the space of overconvergent p-adic modular forms. The operators U and V whose action on q-expansions is given in (13) also act on the spaces of nearly overconvergent modular forms of a given level.

Lemma 4.9. *Let $g \in S_\ell^{\text{n-oc}}(N; \mathbb{C})$ and $h \in S_m^{\text{n-oc}}(N; \mathbb{C})$ be nearly overconvergent modular forms of weights $\ell, m \geq 2$ which are eigenvectors for all the good Hecke operators T_q with $q \nmid N$. (They are not assumed to be new or to be eigenvectors for the operators T_q with $q|N$.) Let*

$$\phi_{fgh} := e_{f^*,\text{ord}}(gh) \in S_k^{ord}(N; \mathbb{C})[\pi_{f^*}], \qquad (\textit{with } k = \ell + m).$$

For each $r \geq 0$, the modular form

$$e_{f^*,\text{ord}}((V^r g) \times h) \in S_k^{\text{ord}}(N; \mathbb{C})[\pi_{f^*}]$$

is a multiple of ϕ_{fgh}.

Proof. By Lemma 2.17,

$$0 = e_{\text{ord}}(g^{[p]} \times Vh) = e_{\text{ord}}(g \times Vh) - a_p(g) e_{\text{ord}}(Vg \times Vh) + \chi_g(p) p^{\ell-1} e_{\text{ord}}(V^2 g \times Vh). \tag{115}$$

But since

$$e_{\text{ord}} V = U^{-1} e_{\text{ord}}, \qquad e_{f^*,\text{ord}} V = \alpha_{f^*}^{-1} e_{f^*,\text{ord}},$$

it follows after applying the projector e_{f^*} to (115) that

$$e_{f^*,\text{ord}}(g \times Vh) + \chi_g(p) p^{\ell-1} \alpha_{f^*}^{-1} e_{f^*,\text{ord}}(Vg \times h) = a_p(g) \alpha_{f^*}^{-1} \phi_{fgh}. \tag{116}$$

The same argument with the roles of g and h interchanged shows that

$$\chi_h(p) p^{m-1} \alpha_{f^*}^{-1} e_{f^*,\text{ord}}(g \times Vh) + e_{f^*,\text{ord}}(Vg \times h) = a_p(h) \alpha_{f^*}^{-1} \phi_{fgh}. \tag{117}$$

The case $r = 1$ of the proposition is obtained by simultaneously solving equations (116) and (117) above. The case of general $r > 1$ can then be treated by induction on r using the identity

$$e_{f^*,\text{ord}}(V^r g \times h) = \alpha_{f^*}^{-1} a_p(g) e_{f^*,\text{ord}}(V^{r-1} g \times h) - \alpha_{f^*}^{-2} \chi_h(p) p^{m-1} e_{f^*,\text{ord}}(V^{r-1} g \times h), \tag{118}$$

which follows from Lemma 2.17, more precisely from the fact that $e_{\text{ord}}(V^r g \times h^{[p]}) = 0$. $\square$

Lemma 4.10. *The following identities, as well as their counterparts with* (α_g, α_h) *replaced by* (α_g, β_h)*,* $(\beta_g, \alpha_h$*, and* (β_g, β_h)*, are satisfied:*

(1) $e_{f^*,\mathrm{ord}}((Vg_\alpha) \times (Vh_\alpha)) = \alpha_{f^*}^{-1} e_{f^*,\mathrm{ord}}(g_\alpha h_\alpha)$;
(2) $e_{f^*,\mathrm{ord}}(g_\alpha \times Vh_\alpha) = \alpha_g \alpha_{f^*}^{-1} e_{f^*,\mathrm{ord}}(g_\alpha h_\alpha)$;
(3) $e_{f^*,\mathrm{ord}}(Vg_\alpha \times h_\alpha) = \alpha_h \alpha_{f^*}^{-1} e_{f^*,\mathrm{ord}}(g_\alpha h_\alpha)$;
(4) $e_{f^*,\mathrm{ord}}(g^{[p]} \times h) = e_{f^*,\mathrm{ord}}(g \times h^{[p]}) = (1 - \alpha_g \alpha_h \alpha_{f^*}^{-1}) e_{f^*,\mathrm{ord}}(g_\alpha h_\alpha)$.

Proof. To prove the first assertion, note that

$$e_{\mathrm{ord}}((Vg_\alpha) \times (Vh_\alpha)) = e_{\mathrm{ord}} \circ V(g_\alpha h_\alpha),$$

and that

$$e_{\mathrm{ord}} \circ V = \lim U_p^{n!} \circ V = \lim U_p^{n!-1} = U_p^{-1} e_{\mathrm{ord}}.$$

The first assertion now follows from the fact that U_p operates via multiplication by α_{f^*} on $S_k^{\mathrm{ord}}(N; \mathbb{C}_p)[\pi_f^*]$. For the second assertion, we invoke Lemma 2.17 to assert that

$$(g^{[p]} \times (Vh_\alpha)) = (g - a_p(g)Vg) \times (Vh_\alpha)$$

is in the kernel of the ordinary projection to conclude that

$$e_{\mathrm{ord}}(g \times Vh_\alpha) = a_p(g) e_{f^*,\mathrm{ord}}((Vg) \times (Vh)),$$

and the second assertion now follows directly from the first. The third assertion follows from the identical argument with the roles of g and h interchanged. The last identity then follows from the previous ones by a direct calculation using the fact that $g^{[p]} = g_\alpha - \alpha_g V g_\alpha$. □

Proposition 4.11. *Let* $f \in S_k(N)$ *be a holomorphic form of weight* k *and let* $g \in S_\ell^{\mathrm{n\text{-}oc}}(N)$ *and* $h \in S_m^{\mathrm{n\text{-}oc}}(N)$ *be nearly holomorphic modular forms, with* $k = \ell + m$*. Assume that* f*,* g *and* h *are eigenvectors for the good Hecke operators of level* N *(but are not necessarily new of this level.) Then for all primes* $p \nmid N$*,*

$$e_{f^*,\mathrm{ord}}(g^{[p]} h) = \frac{\mathcal{E}(f,g,h)}{\mathcal{E}_1(f)} e_{f^*,\mathrm{ord}}(gh), \tag{119}$$

where $\mathcal{E}(f,g,h)$ *is as in equation* (6) *of Theorem 1.3 (with* $c = k-1$*).*

Proof. The last identity of Lemma 4.10 shows, after setting $\phi_{fg^{[p]}h} := e_{f^*,\mathrm{ord}}(g^{p]}h)$, that

$$e_{f^*,\mathrm{ord}}(g_\alpha h_\alpha) = (1 - \alpha_g \alpha_h \alpha_{f^*}^{-1})^{-1} \phi_{fg^{[p]}h}; \tag{120}$$

$$e_{f^*,\mathrm{ord}}(g_\alpha h_\beta) = (1 - \alpha_g \beta_h \alpha_{f^*}^{-1})^{-1} \phi_{fg^{[p]}h}; \tag{121}$$

$$e_{f^*,\mathrm{ord}}(g_\beta h_\alpha) = (1 - \beta_g \alpha_h \alpha_{f^*}^{-1})^{-1} \phi_{fg^{[p]}h}; \tag{122}$$

$$e_{f^*,\mathrm{ord}}(g_\beta h_\beta) = (1 - \beta_g \beta_h \alpha_{f^*}^{-1})^{-1} \phi_{fg^{[p]}h}. \tag{123}$$

Combining (120) and (121) with the identity

$$h = (\alpha_h - \beta_h)^{-1} (\alpha_h h_\alpha - \beta_h h_\beta),$$

and similarly with (122) and (123), we obtain

$$e_{f^*,\mathrm{ord}}(g_\alpha h) = ((1 - \alpha_g \alpha_h \alpha_{f^*}^{-1})(1 - \alpha_g \beta_h \alpha_{f^*}^{-1}))^{-1} \phi_{fg^{[p]}h}; \tag{124}$$

$$e_{f^*,\mathrm{ord}}(g_\beta h) = ((1 - \beta_g \alpha_h \alpha_f^{-1})(1 - \beta_g \beta_h \alpha_{f^*}^{-1}))^{-1} \phi_{fg^{[p]}h}. \tag{125}$$

The result now follows from these two equations by a direct calculation using the fact that

$$g = (\alpha_g - \beta_g)^{-1} (\alpha_g g_\alpha - \beta_g g_\beta).$$

□

Remark 4.12. A formula similar to Proposition 4.11 can be found in Proposition 2.2.2. of [HaTi]. The approach followed there is conceptually sound but more complicated than the one followed above; possibly for this reason, an error seems to have crept into the roughly five pages of elementary but tedious calculations in the proof of Prop. 2.2.2. of loc.cit, and the factor denoted there by $E_P(f,g,h)$ does not agree with the factor $\mathcal{E}(f,g,h)$ of Proposition 4.11. We note that the factor $\mathcal{E}(f,g,h)$ occuring in Proposition 4.11 is consistent with the conjectural recipe for the "correction term at p" arising in the theory of p-adic L-functions, as described in e. g. in [Pan, p. 285].

Corollary 4.13. *Let $f \in S_k(N)$, $g \in S_\ell(N)$ and $h \in S_m(N)$ be holomorphic forms, and assume that $k = \ell + m + 2t$ with $t \in \mathbb{Z}^{\geq 0}$. Assume that f, g and h are eigenvectors for the good Hecke operators in level N (but are not necessarily new in this level.) Then for all primes $p \nmid N$,*

$$e_{f^*,\mathrm{ord}}(d^t g^{[p]} \times h) = \frac{\mathcal{E}(f,g,h)}{\mathcal{E}_1(f)} e_{f^*,\mathrm{ord}}(d^t g \times h).$$

Proof. This follows from Proposition 4.11 with (f,g,h) replaced by

$$(f, d^t g, h) \in S_k(N) \times S^{\text{n-oc}}_{\ell+2t}(N) \times S_m(N),$$

in light of the fact that $d^t(g^{[p]}) = (d^t g)^{[p]}$, $\alpha_{d^t g} = p^t \alpha_g$ and $\beta_{d^t g} = p^t \beta_g$. $\square$

Proof of Theorem 4.7. By (109), the value of $\mathscr{L}_p{}^f(\breve{\mathbf{f}}, \breve{\mathbf{g}}, \breve{\mathbf{h}})$ at $(x,y,z) \in \Sigma_f$ is

$$\mathscr{L}_p{}^f(\breve{\mathbf{f}}, \breve{\mathbf{g}}, \breve{\mathbf{h}})(x,y,z) = \frac{(\breve{f}_x^{*(p)}, e_{f_x^*,\mathrm{ord}}(d^t \breve{g}_y^{[p]} \times \breve{h}_z))_{N,p}}{(f_x^{*(p)}, f_x^{*(p)})_{N,p}} \tag{126}$$

$$= \frac{\mathcal{E}(f_x, g_y, h_z)}{\mathcal{E}_1(f_x)} \frac{(\breve{f}_x^{*(p)}, e_{f_x^*,\mathrm{ord}}(d^t \breve{g}_y \times \breve{h}_z))_{N,p}}{(f_x^{*(p)}, f_x^{*(p)})_{N,p}}, \tag{127}$$

where the second equality follows from Corollary 4.13. But then by Proposition 2.8,

$$\frac{(\breve{f}_x^{*(p)}, e_{f_x^*,\mathrm{ord}}(d^t \breve{g}_y \times \breve{h}_z))_{N,p}}{(f_x^{*(p)}, f_x^{*(p)})_{N,p}} = \frac{(\breve{f}_x^{*(p)}, e_{f_x^*,\mathrm{ord}}\Pi_N^{\mathrm{hol}}(\delta^t \breve{g}_y \times \breve{h}_z))_{N,p}}{(f_x^{*(p)}, f_x^{*(p)})_{N,p}}. \tag{128}$$

On the other hand, by equation (113),

$$\frac{(\breve{f}_x^{*(p)}, e_{f_x^*,\mathrm{ord}}\Pi_N^{\mathrm{hol}}(\delta_\ell^t \breve{g}_y \times \breve{h}_z))_{N,p}}{(f_x^{*(p)}, f_x^{*(p)})_{N,p}} = \frac{(\breve{f}_x^{*(p)}, (e_{f_x^*}\Pi_N^{\mathrm{hol}}(\delta_\ell^t \breve{g}_y \times \breve{h}_z))^{(p)})_{N,p}}{\mathcal{E}_0(f_x)(f_x^{*(p)}, f_x^{*(p)})_{N,p}}$$

$$= \frac{(\breve{f}_x^*, e_{f_x^*}\Pi_N^{\mathrm{hol}}(\delta_\ell^t \breve{g}_y \times \breve{h}_z))_N}{\mathcal{E}_0(f_x)(f_x^*, f_x^*)_N} \tag{129}$$

$$= \frac{(\breve{f}_x^*, \delta_\ell^t \breve{g}_y \times \breve{h}_z)_N}{\mathcal{E}_0(f_x)(f_x^*, f_x^*)_N} \tag{130}$$

$$= \frac{I(\breve{f}_x, \breve{g}_y, \breve{h}_z)}{\mathcal{E}_0(f_x)(f_x^*, f_x^*)_N}. \tag{131}$$

Theorem 4.7 follows.

5. The p-adic Gross-Zagier formula

Recall the class $\eta_{\breve{f}} \in H^1(X_{K_f}, \underline{\omega}^{-r})$ attached to the test vector $\breve{f} \in S_k(N; K_f)[\pi_f]$ following the discussion after (55). Thanks to the chosen embedding of K_f into $\mathbb{C}_p$, the class $\eta_{\breve{f}}$ can be viewed as belonging to $H^1(X_{\mathbb{C}_p}, \underline{\omega}^{-r})[\pi_f]$. Let

$$\eta_{\breve{f}}^{\text{u-r}} \in H^1_{\mathrm{dR}}(X_{\mathbb{C}_p}, \mathcal{L}_r)$$

denote its unique lift to the unit root subspace. We can now state and prove the p-adic Gross-Zagier formula of this paper, relating the image of the generalised diagonal cycles under the p-adic Abel-Jacobi map to the value of the Garrett-Rankin triple product p-adic L-function at points corresponding to *balanced* triples of classical modular forms.

Theorem 5.1. *Given* $(x, y, z) \in \Sigma_{\mathrm{bal}}$*, set*

$$(f, g, h) = (f_x, g_y, h_z), \qquad (\breve{f}, \breve{g}, \breve{h}) = (\breve{f}_x, \breve{g}_y, \breve{h}_z), \qquad (k, \ell, m) = (\kappa(x), \kappa(y), \kappa(z)),$$

and let $\Delta := \Delta_{k,\ell,m} \subset W$ *be the generalised diagonal cycle attached to the weights* (k, ℓ, m)*. Then after writing* $k = \ell + m - 2 - 2t$ *with* $t \geq 0$*,*

$$\mathscr{L}_p{}^f(\breve{\mathbf{f}}, \breve{\mathbf{g}}, \breve{\mathbf{h}})(x, y, z) = (-1)^t \frac{\mathcal{E}(f, g, h)}{t! \cdot \mathcal{E}_0(f)\mathcal{E}_1(f)} \times \mathrm{AJ}_p(\Delta)(\eta^{\text{u-r}}_{\breve{f}} \otimes \omega_{\breve{g}} \otimes \omega_{\breve{h}}).$$

Proof. This follows directly from Theorem 3.14 and Proposition 4.6. □

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