GENERALISED HEEGNER CYCLES AND p-ADIC RANKIN L-SERIES

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Contents

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Introduction

This article studies a distinguished collection of algebraic cycles on varieties which are fibered over modular curves. The cycles in question generalise the Heegner cycles on Kuga-Sato varieties that are studied in [Sch], [Ne2] and [Zh], and will henceforth be referred to as *generalised Heegner cycles*. The main result (Theorem 5.13 of Chapter 5.3) is a p-adic analogue of the Gross-Zagier formula which relates the images of generalised Heegner cycles under a p-adic Abel-Jacobi map to the special values of certain p-adic Rankin L-series at critical points that lie outside the range of p-adic interpolation. Even in the 0-dimensional limit case where generalised Heegner cycles are nothing but Heegner divisors on modular curves, this analogue differs from the p-adic Gross-Zagier formula proved in [PR1], and provides a concrete instance of the p-adic Beilinson conjectures of [PR2], [PR3]. It can also be viewed as the direct analogue of Leopoldt's evaluation at $s = 1$ of the classical p-adic L-function attached to an even Dirichlet character in terms of p-adic logarithms of cyclotomic units. In this analogy, the Kubota-Leopoldt p-adic L-function is replaced by the p-adic Rankin L-function attached to a cusp form and a theta series of an imaginary quadratic field, and the cyclotomic units are replaced by (generalised) Heegner cycles.

Recall that the Kuga-Sato variety W_r is a smooth compactification of the r-fold product of the universal generalised elliptic curve over a modular curve $C = C_\Gamma$ attached to $\Gamma = \Gamma_1(N)$. It is naturally fibered over C, with generic fiber isomorphic to an r-fold product of elliptic curves. The variety W_{2r} is equipped with a supply of so-called *Heegner cycles* (in the Chow group with rational coefficients) of dimension r, which were introduced in [GZ], §V.4. (See also [Ne2], §II.3.6, where a more precise definition is given.) These cycles are supported on fibers above CM points of C and are defined over abelian extensions of imaginary quadratic fields. The main theorem of [Zh] relates their heights to the central critical derivatives of Rankin convolution L-series of cusp forms of weight $2r + 2$ with weight one binary theta series attached to *finite order* Hecke characters of an imaginary quadratic field. In the case $r = 0$, where the Heegner cycles are Heegner points on the modular curve $C = W_0$, this is the theorem of Gross and Zagier [GZ]. A p-adic analogue of these formulae has also been established (in [PR1] for $r = 0$ and in [Ne2] for general r) in which the Arakelov height pairing is replaced by a p-adic height pairing and the complex L-series by a suitable two-variable p-adic L-function.

The present work replaces the Kuga-Sato variety W_{2r} by the $(2r+1)$-dimensional variety

$$X_r := W_r \times A^r,$$

where A is a fixed elliptic curve with complex multiplication by the ring of integers of an imaginary quadratic field K, defined, say, over the Hilbert class field H of K. Like W_{2r}, the variety X_r is fibered over the modular curve C and is also equipped with an infinite collection of special cycles defined over abelian extensions of K. These *generalised Heegner cycles* are naturally indexed by isogenies $\varphi : A \longrightarrow A'$. The cycle attached to φ, denoted Δ_φ, is supported on the fiber $(A')^r \times A^r$ above a point of C attached to A', and is essentially equal to the r-fold self-product of the graph of φ.

Section 2.3 of Chapter 2 defines the cycles Δ_φ precisely and establishes some of their basic properties. In particular, it shows that generalised Heegner cycles are homologically trivial. One can therefore consider their images under various (étale, p-adic, and also complex) Abel-Jacobi maps defined on homologically trivial cycles modulo rational equivalence. Moreover, it is observed in Section 2.4 that the classical Heegner cycles on W_{2r} attached to the imaginary quadratic field K can be obtained as the images of generalised Heegner cycles on X_{2r} under a suitable algebraic correspondence. It follows that generalised Heegner cycles carry at least as much arithmetic information as Heegner cycles on Kuga-Sato varieties. One expects that they carry substantially more: namely, that their heights should encode the central critical derivatives of Rankin L-series attached to the convolution of cusp forms of weight $k := r + 2$ on Γ with theta series of weight $\leq k - 1$ attached to certain Hecke characters of K (and not just with those arising from finite order characters).

Chapter 3 describes the images of generalised Heegner cycles under the p-adic Abel-Jacobi map for a prime p not dividing N. More precisely, Section 3.1 introduces the étale Abel-Jacobi map

$$\mathrm{AJ}_F^{\mathrm{et}} : \mathrm{CH}^{r+1}(X_r)_{0,\mathbf{Q}}(F) \longrightarrow H^1(F, H^{2r+1}_{\mathrm{et}}(\bar{X}_r, \mathbf{Q}_p)(r+1)) \tag{0.0.1}$$

attached to any field F containing H, where $H^1(F, M)$ denotes the (continuous) group cohomology of $G_F := \mathrm{Gal}(\bar{F}/F)$ with values in a G_F–module M. (Here and elsewhere, the subscript 0 stands for homologically trivial and the subscript $\mathbf{Q}$ denotes the Chow group with rational coefficients.) As shown in

Appendix A, the variety X_r admits a proper smooth model over $\operatorname{Spec}\mathbf{Z}[\frac{1}{N}]$ and hence the image of $\mathrm{AJ}_F^{\mathrm{et}}$ (for F a finite extension of $\mathbf{Q}_p$) is contained in the Bloch-Kato subspace H^1_f. The comparison theorems between p-adic étale cohomology and de Rham cohomology then allow us to view (0.0.1) as a map AJ_F (called the p-adic Abel-Jacobi map)

$$\mathrm{AJ}_F : \mathrm{CH}^{r+1}(X_r)_{0,\mathbf{Q}}(F) \longrightarrow \mathrm{Fil}^{r+1} H^{2r+1}_{\mathrm{dR}}(X_r/F)^\vee. \tag{0.0.2}$$

Chapter 3 explains how this map can be computed analytically via Coleman's theory of p-adic integration of differential forms attached to certain classes in the de Rham cohomology $H^{2r+1}_{\mathrm{dR}}(X_r/F)$.

We now describe briefly the *anticyclotomic p-adic L-function* that is constructed in Chapters 4 and 5. Let $S_k(\Gamma_0(N),\varepsilon)$ denote the space of cusp forms of weight k, level N and character ε. The quadratic imaginary field K is said to satisfy the *Heegner hypothesis* (relative to N) if $\mathcal{O}_K$ possesses a cyclic ideal $\mathfrak{N}$ of norm N, i.e., an ideal for which

$$\mathcal{O}_K/\mathfrak{N} = \mathbf{Z}/N\mathbf{Z}. \tag{0.0.3}$$

Assume that this hypothesis is satisfied, and fix a normalised newform $f \in S_k(\Gamma_0(N),\varepsilon_f)$. Let χ be a Hecke character of K of infinity type (j_1,j_2) with $j_1+j_2=k$ and satisfying

$$\chi|_{\mathbb{A}_{\mathbf{Q}}^\times} = \varepsilon_f \cdot \mathbf{N}^k \tag{0.0.4}$$

where $\mathbf{N}$ is the usual norm character. This condition implies that the Rankin L-series $L(f,\chi^{-1},s)$ is self-dual and its functional equation relates its values at s to those at $-s$, so that 0 is the point of symmetry. Such χ will be called *central critical* for f.

At the cost of possibly interchanging j_1 and j_2, we will assume that $j_1 \geq 0$. Let $\Sigma_{\mathrm{cc}}(\mathfrak{N})$ denote the set of central critical characters of conductor dividing $\mathfrak{N}$ and satisfying (0.0.4), as well as the following auxiliary condition: for all finite primes q, the epsilon factor $\varepsilon_q(f,\chi^{-1}) = +1$. Given our other hypotheses, this auxiliary condition is automatic except at those primes q ramified in K, that divide N but do not divide the conductor of ε_f. (In the text, we allow more generally the conductor of χ to divide $c\mathfrak{N}$ where c is an auxiliary odd rational integer prime to Nd_K, where $-d_K$ is the discriminant of K.) The set $\Sigma_{\mathrm{cc}}(\mathfrak{N})$ can be written as the disjoint union of two subsets:

$$\Sigma_{\mathrm{cc}}(\mathfrak{N}) = \Sigma^{(1)}_{\mathrm{cc}}(\mathfrak{N}) \cup \Sigma^{(2)}_{\mathrm{cc}}(\mathfrak{N}),$$

where $\Sigma^{(1)}_{\mathrm{cc}}(\mathfrak{N})$ consists of the characters of infinity type $(k-1-j, 1+j)$ with $0 \leq j \leq r$, and $\Sigma^{(2)}_{\mathrm{cc}}(\mathfrak{N})$ consists of those of infinity type $(k+j,-j)$ with $j \geq 0$. When $\chi \in \Sigma^{(1)}_{\mathrm{cc}}(\mathfrak{N})$, the sign $\varepsilon_\infty(f,\chi^{-1})$ equals -1, hence the sign in the functional equation for $L(f,\chi^{-1},s)$ is also -1, and therefore the function $\chi \mapsto L(f,\chi^{-1},0)$ vanishes identically on $\Sigma^{(1)}_{\mathrm{cc}}(\mathfrak{N})$. On the other hand, for $\chi \in \Sigma^{(2)}_{\mathrm{cc}}(\mathfrak{N})$, the sign $\varepsilon_\infty(f,\chi^{-1})$ equals $+1$ whence the sign in the functional equation for $L(f,\chi^{-1},s)$ is $+1$ as well, and so one expects that the associated central critical values should be non-zero most of the time.

Chapter 4 is devoted to proving an explicit version of Waldspurger's formula relating the central L-values $L(f,\chi^{-1},0)$, for $\chi \in \Sigma^{(2)}_{\mathrm{cc}}(\mathfrak{N})$, to period integrals on tori. Such explicit formulae have been studied by several authors recently, for example [Xue], [MW] and more recently [Hi3]. However our approach is somewhat different in that we always insist that our torus embeddings come from Heegner points and that the test vectors are of minimal level. The relevant period integrals then reduce to finite sums of values of (certain non-holomorphic derivatives of) the form f at all conjugates of a CM point, twisted by the character χ^{-1}, which is a key to providing a link to the p-adic Abel-Jacobi images of generalized Heegner cycles supported on the same set of conjugate CM points.

Section 5.1 recalls the algebraicity properties of these special values: for all $\chi \in \Sigma^{(2)}_{\mathrm{cc}}(\mathfrak{N})$,

$$L_{\mathrm{alg}}(f,\chi^{-1}) := \tilde{C}(f,\chi) \times \frac{L(f,\chi^{-1},0)}{\Omega^{2(k+2j)}} \tag{0.0.5}$$

is an algebraic number. Here $\tilde{C}(f,\chi)$ is an explicit, elementary constant and Ω is a CM period attached to K whose value depends on the choice of a regular differential ω_A on A/H. After fixing an embedding

$$\iota : \bar{\mathbf{Q}} \longrightarrow \bar{\mathbf{Q}}_p,$$

the values $L_{\mathrm{alg}}(f,\chi^{-1})$ attached to $\chi \in \Sigma^{(2)}_{\mathrm{cc}}(\mathfrak{N})$ can be viewed as p-adic numbers. Section 5.2 takes up the question of their p-adic interpolation. As explained in that section, the set $\Sigma^{(2)}_{\mathrm{cc}}(\mathfrak{N})$ is endowed with a natural p-adic topology, and can be viewed as a dense subset of its completion $\hat{\Sigma}_{\mathrm{cc}}(\mathfrak{N})$. Assume that the rational prime p is split in $K/\mathbf{Q}$, so that $\iota(K) \subset \mathbf{Q}_p$. Under this assumption, there is a unique prime $\mathfrak{p}$ of K above p for which $\chi(\mathfrak{p})$ is a p-adic unit. The main result of Section 5.2 is that, after setting

$$L_p(f,\chi) = \Omega_p^{2(k+2j)}(1-\chi^{-1}(\bar{\mathfrak{p}})a_p + \varepsilon_f(p)\chi(\bar{\mathfrak{p}})^{-2}p^{k-1})^2 L_{\mathrm{alg}}(f,\chi^{-1})$$

for an appropriate p-adic period Ω_p (which also depends on the choice of ω_A), the assignment $\chi \mapsto L_p(f,\chi^{-1})$ extends to a (necessarily unique) continuous function on $\hat{\Sigma}_{\mathrm{cc}}(\mathfrak{N})$, which we refer to as the *anticyclotomic p-adic L-function* attached to f and K.

Now, let χ be a character in $\Sigma_{\mathrm{cc}}^{(1)}(\mathfrak{N})$, having infinity type $(k-1-j, 1+j)$ for some $0 \le j \le r$. While the classical L-value $L(f,\chi^{-1},0)$ vanishes, the character χ can be viewed as an element of $\hat{\Sigma}_{\mathrm{cc}}(\mathfrak{N})$ (lying outside the range of classical interpolation defining the anticyclotomic p-adic L-function $L_p(f,\chi)$), and the special value $L_p(f,\chi)$—which may be thought of as a p-adic avatar of $L'(f,\chi^{-1},0)$—is not forced to vanish a priori. Our main result relates $L_p(f,\chi)$ to the Abel-Jacobi images of generalised Heegner cycles. For the sake of illustration, we state the main result under the following simplifying assumptions, postponing the more general statement to Theorem 5.13 of Chapter 5.3:

(1) The quadratic imaginary field K has class number one and odd discriminant $-d_K < -3$. Let $\varepsilon_K : (\mathbf{Z}/d_K\mathbf{Z})^\times \longrightarrow \{\pm 1\}$ be the associated odd Dirichlet character, and denote by the same symbol the quadratic character of $(\mathcal{O}_K/\sqrt{-d_K}\mathcal{O}_K)^\times$ induced from the identification of $\mathcal{O}_K/\sqrt{-d_K}\mathcal{O}_K$ with $\mathbf{Z}/d_K\mathbf{Z}$.
(2) The newform f belongs to $S_k(\Gamma_0(N), \varepsilon_K^k)$. (Note that it is necessary that d_K divides N when k is odd.)
(3) The grossencharacter $\chi \in \Sigma_{\mathrm{cc}}^{(1)}(\mathfrak{N})$ is of the form
$$\chi((\alpha)) = \varepsilon_K^k(\alpha)\alpha^{k-1-j}\bar{\alpha}^{1+j},$$
for some integer $0 \le j \le r$.

In this special setting, our main result is:

Main Theorem. *Let $\Delta = \Delta_1$ be the generalised Heegner cycle attached to $1 : A \longrightarrow A$, viewed as an element of $\mathrm{CH}^{r+1}(X_r)_0(\mathbf{Q}_p)_{\mathbf{Q}}$ via ι. Then*

$$\frac{L_p(f,\chi)}{\Omega_p^{2(r-2j)}} = \left(1-\chi^{-1}(\bar{\mathfrak{p}})a_p + \chi^{-2}(\bar{\mathfrak{p}})p^{k-1}\right)^2 \cdot \left(\frac{1}{j!}\,\mathrm{AJ}_{\mathbf{Q}_p}(\Delta)(\omega_f \wedge \omega_A^j \eta_A^{r-j})\right)^2,$$

where $\mathrm{AJ}_{\mathbf{Q}_p}$ is the p-adic Abel-Jacobi map of (0.0.2), ω_f is the class in $\Omega^{r+1}(W_r)$ attached to f in Corollary 2.3 of Section 2.1, and $\omega_A^j\eta_A^{r-j}$ is the class in $H^r(A^r)$ defined in (1.4.6) of Section 1.4.

Note that it is a special value and not a derivative of the p-adic L-series that occurs on the analytic side of this formula, while the algebraic side involves the Abel-Jacobi images of generalised Heegner cycles rather than their (p-adic) heights. Note also that if ω_A is replaced by a nonzero multiple $\lambda\omega_A$, then both sides of the equation above are multiplied by $\lambda^{2(2j-r)}$.

Those approaching this paper for the first time may find it pedagogically helpful to focus on the simplest case $r = j = 0$, where f is a newform of weight 2 and $\chi \in \Sigma_{\mathrm{cc}}^{(1)}(\mathfrak{N})$ is a grossencharacter of infinity type $(1,1)$. In this case, the Main Theorem above involves the formal group logarithms of points in the Jacobians of modular curves arising from certain divisors supported on Heegner points. It relates these p-adic logarithms to the *values* of the p-adic L-function $L_p(f,\chi)$ at characters of finite order (shifted by the norm). One thus obtains a new p-adic variant of the Gross-Zagier formula in the "traditional" setting of Heegner points on modular curves. As a first guide to the somewhat lengthy arguments required to deal with forms and Hecke characters of general weights and levels, here is a brief outline of the proof of the Main Theorem in this simplest non-trivial setting, assuming further that K has class number one and a unit group of order 2, and that $\chi := \chi_0$ is the trivial character of weight $(1,1)$ sending the (principal) ideal (α) to its norm $\alpha\bar{\alpha}$. This norm character is the specialisation at $j = 0$ of the sequence $\chi_j \in \Sigma_{\mathrm{cc}}^{(2)}(\mathfrak{N})$ of grossencharacters of infinity type $(1+j, 1-j)$ defined by

$$\chi_j((\alpha)) := \alpha^{1+j}\bar{\alpha}^{1-j}.$$

Let δ_2^{j-1} denote the $(j-1)$st iterate of the Shimura-Maass differential operator as defined in Sec. 1.2; this sends weight 2 real analytic modular forms to those of weight $2j$. For all $j \geq 1$, Theorem 5.5 identifies the quantity $L_{\mathrm{alg}}(f, \chi_j^{-1})$ of equation (0.0.5) with $(\delta_2^{j-1} f)(P_A)^2$, where P_A denotes the triple (A, ω_A, t_A) attached to the elliptic curve A with CM by the maximal order of K, the differential ω_A and a suitable $\Gamma_1(N)$-level structure t_A on A. (Here modular forms are viewed as functions on triples as explained in Section 1.1.) Using the well-known fact that the unit root splitting of the Hodge filtration agrees with the Hodge decomposition for ordinary CM elliptic curves, Proposition 1.12 identifies $(\delta_2^{j-1} f)(P_A)$ with $(\theta^{j-1} f)(P_A)$, where $\theta = q\frac{d}{dq}$ is the Atkin-Serre theta operator on p-adic modular forms defined in (1.3.2). This key identification leads to the p-adic interpolation of the special values $L_{\mathrm{alg}}(f, \chi_j^{-1})$ described in Section 5.2, and hence, to the Rankin p-adic L-function $L_p(f, \chi_j)$ which arises in the Main Theorem above. This p-adic L-function satisfies the equality

$$L_p(f, \chi_j) = (\theta^{j-1} f^\flat)(P_A^{(p)})^2, \qquad \forall j \geq 0,$$

where $f^\flat$ is the "p-depleted" modular form associated to f defined in (3.8.4), and $P_A^{(p)} = (A, \Omega_p^{-1}\omega_A, t_A)$. Taking a p-adic limit when $j \to 0$ shows that

$$L_p(f, \chi) = (\theta^{-1} f^\flat)(P_A^{(p)})^2.$$

One can see (either directly, or by specializing the calculations of Section 3 to the case where $r = 0$) that the function $\theta^{-1} f^\flat$ – a p-adic, and in fact overconvergent, modular form of weight 0 – is the unique rigid analytic primitive of the exact rigid differential $\omega_{f^\flat}$ which vanishes at the cusp ∞, and its value at the triple $P_A^{(p)}$ is an explicit multiple of the formal group logarithm, relative to the differential ω_f, of the degree zero divisor $\Delta_1 = (A, t_A) - (\infty)$ on the modular curve C.

We close this introduction by listing a few of the arithmetic applications of Theorem 5.13.

Rubin's formula. The article [BDP-cm] exploits Theorem 5.13 in the special case where f is itself a weight two binary theta series attached to the quadratic imaginary field K to give a new proof of the main result of [Ru], which relates the values of the Katz p-adic L-function attached to K to the p-adic logarithms of global points on elliptic curves with complex multiplication by K.

Chow-Heegner points. Because it involves Abel-Jacobi images rather than p-adic heights, Theorem 5.13 is used in [BDP-ch] to study the algebraicity of the certain points on CM elliptic curves arising from *higher dimensional* cycles in the Chow groups of certain algebraic varieties whose cohomology realises the ℓ-adic repreentations attached to theta series of higher (possibly odd) weight. This construction, provides a basic illustration of the phenomenon of "Chow-Heegner points" arising from the image of algebraic cycles under Abel-Jacobi maps (both complex and p-adic). The relevance of Theorem 5.13 to the notion of Chow-Heegner points was in fact the original motivation for the present article, although Theorem 5.13 is considerably more general than the special case exploited in [BDP-ch].

Coniveau and the Bloch-Beilinson conjecture. The article [BDP-co] illustrates how Theorem 5.13 may be used to prove part of the Bloch-Beilinson conjecture for the Rankin-Selberg motives that are studied in this article. In particular, by verifying that specific values of the p-adic L-function $L_p(f, \chi)$ are not zero, one can often show that generalised Heegner cycles are not just nonzero in the Chow group but also nonzero in a certain graded piece for the coniveau filtration on the Chow group, as predicted by a refined version ([Bl-1], [Bl-2]) of the Bloch-Beilinson conjecture.

Euler systems. Let F be any global field over which A is defined. For each cuspidal newform f on C of weight $r+2$ and each character χ as in the previous statement, there is a G_F–equivariant projection

$$\pi_{f,\chi} : H^{2r+1}_{\mathrm{et}}(\bar{X}_r, \mathbf{Q}_p)(r{+}1) \longrightarrow (V_f \otimes \chi)(r{+}1) =: V_{f,\chi},$$

where V_f is the Deligne representation attached to f and χ is viewed as a one-dimensional p-adic representation of G_F in the usual way. Each generalised Heegner cycle Δ_φ, defined over an appropriate extension $F_\varphi \supset H$, gives rise to a global cohomology class:

$$\kappa_\varphi := \pi_{f,\chi}(\mathrm{AJ}^{\mathrm{et}}_{F_\varphi}(\Delta_\varphi)) \in H^1(F_\varphi, V_{f,\chi}),$$

which belongs to a generalised Selmer group $H^1_{\mathrm{Sel}}(F_\varphi, V_{f,\chi})$ attached to the p-adic Galois representation $V_{f,\chi}$. If $\mathfrak{p}$ is a prime of F_φ above p and p does not divide the level of Γ, the discriminant of K, or the degree of φ, then the natural image $\mathrm{res}_{\mathfrak{p}}(\kappa_\varphi)$ of κ_φ in the local cohomology group $H^1(F_{\varphi,\mathfrak{p}}, V_{f,\chi})$ belongs to the subgroup $H^1_f(F_{\varphi,\mathfrak{p}}, V_{f,\chi})$ corresponding to cristalline extensions of $V_{f,\chi}$ by $\mathbf{Q}_p$. The Main Theorem above relates $\mathrm{res}_{\mathfrak{p}}(\kappa_\varphi)$ to the values of the p-adic L-function $L_p(f,\chi)$ at points lying outside the range of classical interpolation. This suggests that the collection $\{\kappa_\varphi\}$ of global cohomology classes, as φ ranges over the isogenies $A \longrightarrow A'$, should give rise to an *Euler system* attached to the compatible system $V_{f,\chi}$ of p-adic representations of G_F. See Section 2.4 for a discussion of the relation between these cycles and classical L-series, and [Cas1], [Cas2] where the connection between the results of this paper and the theory of Euler systems obtained by interpolating generalised Heegner cycles in p-adic families is described in more detail.

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1. Preliminaries

1.1. **Algebraic modular forms.** Let $N \geq 1$ be an integer and let $\Gamma = \Gamma_1(N)$ be the standard congruence subgroup of level N:

$$\Gamma_1(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbf{SL}_2(\mathbf{Z}) \quad \text{such that } a-1,\ d-1,\ c \equiv 0 \pmod{N} \right\}.$$

We shall begin by recalling the geometric definition of modular forms over a field F that is given in [Ka2] and [Hi4].

If R is a ring in which N is invertible and E is an elliptic curve over R, we observe that a closed immersion $t : \mathbf{Z}/N\mathbf{Z} \hookrightarrow E$ of group schemes over $\mathrm{Spec}\, R$ gives rise to a section $s : \mathrm{Spec}(R) \longrightarrow E$ of order N by restriction to the section 1 of $\mathbf{Z}/N\mathbf{Z}$.

Definition 1.1. An *elliptic curve with* Γ*-level structure* over a ring R is a pair (E, t) consisting of

1. an elliptic curve E over $\mathrm{Spec}(R)$;
2. a closed immersion $t : \mathbf{Z}/N\mathbf{Z} \hookrightarrow E$ of group schemes over $\mathrm{Spec}\, R$.

A triple (E, t, ω) where (E, t) is an elliptic curve with Γ-level structure and $\omega \in \Omega^1_{E/R}$ is a global section of Ω^1_E over $\mathrm{Spec}(R)$ is called a *marked elliptic curve with* Γ*-level structure.*

The notion of R-isomorphisms between elliptic curves or marked elliptic curves with Γ-level structure is defined in the obvious way. Denote by $\mathrm{Ell}(\Gamma, R)$ the set of isomorphism classes of elliptic curves with Γ-level structure over R, and by $\widetilde{\mathrm{Ell}}(\Gamma, R)$ the set of isomorphism classes of marked elliptic curves with Γ-level structure.

Definition 1.2. A *weakly holomorphic algebraic modular form of weight* k *on* Γ *defined over a field* F is a rule which to every isomorphism class of triples $(E, t, \omega) \in \widetilde{\mathrm{Ell}}(\Gamma, R)$ defined over an F-algebra R associates an element $f(E, t, \omega) \in R$ satisfying:

1. (Compatibility with base change). For all F-algebra homomorphisms $j : R \longrightarrow R'$,
$$f((E, t, \omega) \otimes_j R') = j(f(E, t, \omega)).$$
2. (Weight k condition). For all $\lambda \in R^\times$,
$$f(E, t, \lambda\omega) = \lambda^{-k} f(E, t, \omega).$$

Let $(\mathrm{Tate}(q), t, \omega_{\mathrm{can}})_{/F((q^{1/d}))}$ be the Tate elliptic curve $\mathbf{G}_m/q^{\mathbf{Z}}$, equipped with some level N structure t defined over $F((q^{1/d}))$ (for some $d|N$) and the canonical differential $\omega_{\mathrm{can}} := \frac{du}{u}$ over $F((q))$, where u is the usual parameter on $\mathbf{G}_m$.

Definition 1.3. An *algebraic modular form on* Γ *over* F is a weakly holomorphic modular form satisfying

$$f\,(\mathrm{Tate}(q), t, \omega_{\mathrm{can}}) \text{ belongs to } F[[q^{1/d}]], \quad \text{for all } t.$$

If these values belong to $q^{1/d}F[[q^{1/d}]]$, then f is called a *cusp form.*

We denote by

$$S_k(\Gamma, F) \subset M_k(\Gamma, F) \subset M_k^\dagger(\Gamma, F)$$

the F-vector spaces of cusp forms, algebraic modular forms, and weakly holomorphic modular forms respectively on Γ over F.

Write

$$C^0 = Y_1(N), \qquad C = X_1(N) = Y_1(N) \cup Z_N,$$

for the usual modular curves over $\mathbf{Q}$ associated to Γ. The cuspidal subscheme Z_N is finite over $\mathbf{Q}$. If $N \geq 3$, the group $\Gamma_1(N)$ is torsion-free and the curve C^0 is a fine moduli scheme having a canonical smooth proper model over $\mathrm{Spec}(\mathbf{Z}[1/N])$. It represents the functor on $\mathbf{Z}[1/N]$-algebras which to R associates the set $\mathrm{Ell}(\Gamma, R)$ of Definition 1.1. We will not make use of the integral model for now, and will view the curves C^0 and C as defined over some base field F (of characteristic 0) for the rest of this chapter.

Let $\pi : \mathcal{E} \longrightarrow C^0$ be the universal elliptic curve with level N structure over C^0, and let $\underline{\omega} := \pi_*\Omega^1_{\mathcal{E}/C^0}$ be the line bundle of relative differentials on $\mathcal{E}/C^0$. A weakly holomorphic modular form $f \in M_k^\dagger(\Gamma, F)$ can be viewed as a global section of the sheaf $\underline{\omega}^k$ over C^0, by setting

$$f(E, t) = f(E, t, \omega)\omega^k, \tag{1.1.1}$$

where (E, t) is viewed as a point of $C^0(R)$ and ω is an arbitrarily chosen generator (locally on $\mathrm{Spec}\, R$) of $\Omega^1_{E/R}$. Note that the expression on the right of (1.1.1) does not depend on the choice of ω.

Consider the relative de Rham cohomology sheaf on C^0:

$$\mathcal{L}_1 := \mathbb{R}^1\pi_*(0 \to \mathcal{O}_{\mathcal{E}} \to \Omega^1_{\mathcal{E}/C^0} \to 0).$$

It is a rank 2 algebraic vector bundle over C^0 whose fibre at any geometric point $x : \mathrm{Spec}\, L \longrightarrow C^0$ is given by

$$(\mathcal{L}_1)_x := H^1_{\mathrm{dR}}(\mathcal{E}_x),$$

with $\mathcal{E}_x := \mathcal{E} \times_x \mathrm{Spec}\, L$. There is a non-degenerate (Poincaré) pairing

$$\langle\ ,\ \rangle : \mathcal{L}_1 \times \mathcal{L}_1 \longrightarrow \mathcal{O}_{C^0},$$

and the Hodge filtration on the fibres corresponds to an exact sequence of coherent sheaves over C^0:

$$0 \longrightarrow \underline{\omega} \longrightarrow \mathcal{L}_1 \longrightarrow \underline{\omega}^{-1} \longrightarrow 0. \tag{1.1.2}$$

The vector bundle $\mathcal{L}_1$ is also equipped with the canonical integrable Gauss-Manin connection

$$\nabla : \mathcal{L}_1 \longrightarrow \mathcal{L}_1 \otimes \Omega^1_{C^0}. \tag{1.1.3}$$

The Kodaira-Spencer map KS is defined to be the composite

$$\mathrm{KS} : \underline{\omega} \longrightarrow \mathcal{L}_1 \xrightarrow{\nabla} \mathcal{L}_1 \otimes \Omega^1_{C^0} \longrightarrow \underline{\omega}^{-1} \otimes \Omega^1_{C^0},$$

in which the first and last arrows arise from (1.1.2). This map is an isomorphism of sheaves over C^0, and therefore gives rise to an identification

$$\sigma : \underline{\omega}^2 \xrightarrow{\sim} \Omega^1_{C^0}, \qquad \sigma(\omega_1 \otimes \omega_2) := \langle \omega_1, \nabla\omega_2 \rangle. \tag{1.1.4}$$

In addition to the geometric interpretation (1.1.1), it will also be convenient to view modular forms $f \in M^\dagger_{r+2}(\Gamma, F)$ as global sections of the sheaf $\underline{\omega}^r \otimes \Omega^1_{C^0}$, by the rule

$$\omega_f(E, t) := f(E, t, \omega) \cdot \omega^r \otimes \sigma(\omega^2). \tag{1.1.5}$$

Assume for simplicity that all the cusps of $X_1(N)$ are *regular* in the sense of [DS, §3.2]. (This condition is satisfied as soon as $N > 4$.) The line bundles $\underline{\omega}$ and $\mathcal{L}_1$ and their attendant structures extend naturally to the complete curve C as follows:

- The line bundle $\underline{\omega}$ admits an extension to C (denoted again by $\underline{\omega}$) which is characterised by the property
$$H^0(C,\underline{\omega}^k) = M_k(\Gamma, F).$$
By Definition 1.3, the local sections of $\underline{\omega}$ in the neighborhood $\operatorname{Spec} F(\zeta_N)[\![q^{1/d}]\!]$ of the cusp attached to the pair $(\mathrm{Tate}(q), q^{1/d}\zeta_N)$ are expressions of the form $h\omega_{\mathrm{can}}$ with $h \in F(\zeta_N)[\![q^{1/d}]\!]$, where we recall that ω_{can} is the canonical differential on the Tate curve.
- The exact sequence (1.1.2), together with the given extensions of $\underline{\omega}$ and $\underline{\omega}^{-1}$ to C, determine an extension of $\mathcal{L}_1$ to C, in such a way that (1.1.2) becomes an exact sequence of sheaves over this base. The local sections of $\mathcal{L}_1$ in a neighborhood of the cusp $(\mathrm{Tate}(q), q^{1/d}\zeta_N)$ are $F(\zeta_N)[\![q^{1/d}]\!]$-linear combinations of ω_{can} and the local section ξ_{can} defined by

(1.1.6)
$$\nabla\omega_{\mathrm{can}} =: \xi_{\mathrm{can}} \otimes \frac{dq}{q}.$$
(The sheaf $\mathcal{L}_1$ is described in Sec. 2.4. of [Schol1], where it is denoted $\mathcal{E}$.)
- The Gauss-Manin connection ∇ of (1.1.3) extends to a connection with log poles

(1.1.7)
$$\nabla : \mathcal{L}_1 \longrightarrow \mathcal{L}_1 \otimes \Omega^1_C(\log Z_N),$$
where $\Omega^1_C(\log Z_N)$ denotes the sheaf of differentials on C with logarithmic singularities on the cuspidal subscheme Z_N. Over $\operatorname{Spec} F(\zeta_N)[\![q^{1/d}]\!]$, it is described by the equation

(1.1.8)
$$\nabla\omega_{\mathrm{can}} = \xi_{\mathrm{can}} \otimes \frac{dq}{q}, \qquad \nabla\xi_{\mathrm{can}} = 0.$$
- Finally, the Kodaira-Spencer isomorphism σ gives an identification

(1.1.9)
$$\sigma : \underline{\omega}^2 \xrightarrow{\sim} \Omega^1_C(\log Z_N)$$
of sheaves over C. Over $\operatorname{Spec} F(\zeta_N)[\![q^{1/d}]\!]$, it is determined by

(1.1.10)
$$\sigma(\omega^2_{\mathrm{can}}) = \frac{dq}{q}.$$
- With these definitions, the rules (1.1.1) and (1.1.5) give identifications

$$\begin{aligned} &(1.1.11) & M_{r+2}(\Gamma, F) &= H^0(C, \underline{\omega}^{r+2}) = H^0(C, \underline{\omega}^r \otimes \Omega^1_C(\log Z_N)), \\ &(1.1.12) & S_{r+2}(\Gamma, F) &= H^0(C, \underline{\omega}^r \otimes \Omega^1_C). \end{aligned}$$

For any $r \geq 1$, let
$$\mathcal{L}_r := \operatorname{Sym}^r \mathcal{L}_1.$$
The sheaf $\mathcal{L}_r$ inherits from (1.1.2) a canonical Hodge filtration by sheaves of $\mathcal{O}_C$-modules:
$$\mathcal{L}_r \supset \mathcal{L}_{r-1} \otimes \underline{\omega} \supset \cdots \supset \underline{\omega}^r,$$
and the relative Poincaré duality

(1.1.13)
$$\langle\ ,\ \rangle : \mathcal{L}_r \times \mathcal{L}_r \longrightarrow \mathcal{O}_C$$
whose reduction to the geometric fibers is given by the rule

(1.1.14)
$$\langle \alpha_1 \cdots \alpha_r, \beta_1 \cdots \beta_r \rangle = \frac{1}{r!} \sum_{\sigma \in S_r} \langle \alpha_1, \beta_{\sigma 1} \rangle \cdots \langle \alpha_r, \beta_{\sigma r} \rangle,$$
where S_r denotes the symmetric group on r letters. The connection ∇ on $\mathcal{L}_1$ gives rise to a connection (which will also be denoted ∇)
$$\nabla : \mathcal{L}_r \longrightarrow \mathcal{L}_r \otimes \Omega^1_C(\log Z_N).$$
Let $\tilde{\nabla}$ denote the composite

(1.1.15)
$$\tilde{\nabla} : \mathcal{L}_r \xrightarrow{\nabla} \mathcal{L}_r \otimes \Omega^1_C(\log Z_N) \xrightarrow{\mathrm{id} \otimes \sigma^{-1}} \mathcal{L}_r \otimes \underline{\omega}^2 \longrightarrow \mathcal{L}_r \otimes \mathcal{L}_2 \longrightarrow \mathcal{L}_{r+2},$$
where the penultimate arrow is induced from (1.1.2) and the last arises from the natural projection
$$\operatorname{Sym}^r \otimes \operatorname{Sym}^2 \longrightarrow \operatorname{Sym}^{r+2}.$$

The map $\tilde{\nabla}$ (which, like ∇, is a homomorphism of abelian sheaves but *not* of $\mathcal{O}_C$-modules) gives rise to differential operators on modular forms. More precisely, let

$$\Psi : \mathcal{L}_1 \longrightarrow \underline{\omega} \tag{1.1.16}$$

be a splitting of the Hodge filtration (1.1.2), and let $\Psi^{(k)}$ denote the corresponding homomorphism $\mathcal{L}_k \longrightarrow \underline{\omega}^k$. The splitting Ψ determines a differential operator

$$\Theta_\Psi : M_r(\Gamma, F) \longrightarrow M_{r+2}(\Gamma, F), \qquad (\Theta_\Psi f)(E, t) := \Psi^{(r+2)}(\tilde{\nabla} f)(E, t). \tag{1.1.17}$$

Example 1.4. We can construct a splitting Ψ as in (1.1.16) as follows. The datum of a pair $(E, \omega)_{/R}$ determines (locally on $\operatorname{Spec} R$) canonical elements $x \in H^0(E, \mathcal{O}_E(2O_E))$ and $y \in H^0(E, \mathcal{O}_E(3O_E))$ satisfying

$$y^2 = 4x^3 + g_2 x + g_3, \quad \text{for some } g_2, g_3 \in R, \qquad \text{and} \quad \frac{dx}{y} = \omega.$$

The decomposition

$$H^1_{\mathrm{dR}}(E/R) = R\left[\frac{dx}{y}\right] \oplus R\left[\frac{x\,dx}{y}\right]$$

determines a canonical algebraic (but not functorial) splitting Ψ_{alg} of the Hodge filtration on $\mathcal{L}_1$. The resulting differential operator Θ_{alg} on $M_r(\Gamma, F)$ is given in terms of q-expansions by the formula

$$\Theta_{\mathrm{alg}}(f) = \theta f - \frac{r}{12} P f, \qquad \theta = q\frac{d}{dq},$$

where

$$P = 1 - 24 \sum_{n \geq 1} \sigma_1(n) q^n, \qquad (\text{with } \sigma_1(n) = \sum_{d|n} d)$$

arises from the Eisenstein series of weight 2. (Cf. §A1.4 of [Ka2].)

1.2. **Modular forms over C.** Assume now that $F = \mathbf{C}$. The set $C(\mathbf{C})$ of complex points of C is a compact Riemann surface, and the analytic map

$$\mathrm{pr} : \mathcal{H} \longrightarrow C^0(\mathbf{C}), \qquad \mathrm{pr}(\tau) := \left(\mathbf{C}/\langle 1, \tau\rangle, \frac{1}{N}\right)$$

identifies $C^0(\mathbf{C})$ with the quotient $\Gamma \backslash \mathcal{H}$, where we recall that $\Gamma = \Gamma_1(N)$. The coherent sheaf $\mathcal{L}_r$ gives rise to an analytic sheaf $\mathcal{L}_r^{\mathrm{an}}$ on the Riemann surface $C(\mathbf{C})$; let $\tilde{\mathcal{L}}_r^{\mathrm{an}} := \mathrm{pr}^* \mathcal{L}_r^{\mathrm{an}}$ denote its pullback to $\mathcal{H}$.

Recall the elliptic fibration $\pi : \mathcal{E} \longrightarrow C^0$, and let

$$\mathbb{L}_1^B := R^1 \pi_* \mathbf{Z}, \qquad \mathbb{L}_r^B := \operatorname{Sym}^r \mathbb{L}_1^B,$$

be the locally constant sheaves of $\mathbf{Z}$-modules whose fibers at $x \in C^0(\mathbf{C})$ are identified with the Betti cohomology $H^1_B(\mathcal{E}_x, \mathbf{Z})$ and $\operatorname{Sym}^r H^1_B(\mathcal{E}_x, \mathbf{Z})$ respectively. The local system

$$\mathbb{L}_r := \mathbb{L}_r^B \otimes_{\mathbf{Z}} \mathbf{C} \tag{1.2.1}$$

is identified with the sheaf of horizontal sections of $(\mathcal{L}_r^{\mathrm{an}}, \nabla)$ over $C^0(\mathbf{C})$. (Cf. [De1], thm. 2.17.)

A modular form $f \in M_k^\dagger(\Gamma, \mathbf{C})$ gives rise to a holomorphic section of $\underline{\omega}^k$ viewed as an analytic sheaf over $C^0(\mathbf{C})$. It also gives rise to a holomorphic function on $\mathcal{H}$ by the rule

$$f(\tau) := f\left(\mathbf{C}/\langle 1, \tau\rangle, \frac{1}{N}, 2\pi i\, dw\right), \tag{1.2.2}$$

where w is the standard coordinate on $\mathbf{C}/\langle 1, \tau\rangle$. This function obeys the familiar transformation rule

$$f\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^k f(\tau), \qquad \text{for all } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_1(N), \tag{1.2.3}$$

and the modular form f is completely determined by the associated function $f(\tau)$.

The Hodge filtration on $H^1_{\mathrm{dR}}(\mathbf{C}/\langle 1, \tau\rangle)$ admits a canonical, functorial (but *non holomorphic*) splitting

$$H^1_{\mathrm{dR}}(\mathbf{C}/\langle 1, \tau\rangle) := \mathbf{C}\, dw \oplus \mathbf{C}\, d\bar{w}, \tag{1.2.4}$$

called the *Hodge decomposition*. In terms of the local coordinates τ, $\bar{\tau}$, dw, and $d\bar{w}$, the Gauss-Manin connection and Kodaira-Spencer map are described by

$$\nabla dw = \left(\frac{dw - d\bar{w}}{\tau - \bar{\tau}}\right) \otimes d\tau, \qquad \sigma((2\pi i dw)^2) = 2\pi i d\tau. \tag{1.2.5}$$

The global sections of $\underline{\omega}^{r+2}$ and $\underline{\omega}^r \otimes \Omega^1_C$ attached to f in (1.1.1) and (1.1.5) are therefore given by the complex formulae

$$f\left(\mathbf{C}/\langle 1, \tau\rangle, \frac{1}{N}\right) = f(\tau)(2\pi i dw)^{r+2} \qquad \omega_f\left(\mathbf{C}/\langle 1, \tau\rangle, \frac{1}{N}\right) = f(\tau)(2\pi i dw)^r \otimes (2\pi i d\tau). \tag{1.2.6}$$

Let $\mathcal{L}_r^{\mathrm{ra}}$ denote the real analytic sheaf on C^0 associated to $\mathcal{L}_r^{\mathrm{an}}$ by forgetting the complex structure on C and retaining only its associated real analytic structure, and denote by $\underline{\omega}^r_{\mathrm{ra}}$ the subsheaf of $\mathcal{L}_r^{\mathrm{ra}}$ for the real analytic topology associated to $\underline{\omega}^r$. The global sections of $\underline{\omega}^r_{\mathrm{ra}}$ over C^0 are called *real analytic modular forms* of weight r on Γ. They are identified, via (1.2.2), with real analytic functions on $\mathcal{H}$ satisfying the transformation property (1.2.3).

Following [Ka4], (1.8.3), we recall the Hodge decomposition of real analytic sheaves

$$\mathcal{L}_1^{\mathrm{ra}} = \underline{\omega}_{\mathrm{ra}} \oplus \bar{\underline{\omega}}_{\mathrm{ra}}, \tag{1.2.7}$$

which induces (1.2.4) over the points of $C^0(\mathbf{C})$. It gives rise to real analytic splittings

$$\Psi_{\mathrm{Hodge}} : \mathcal{L}_1^{\mathrm{ra}} \longrightarrow \underline{\omega}_{\mathrm{ra}}, \qquad \Psi^{(r)}_{\mathrm{Hodge}} : \mathcal{L}_r^{\mathrm{ra}} \longrightarrow \underline{\omega}^r_{\mathrm{ra}}. \tag{1.2.8}$$

A section f of $\underline{\omega}^r_{\mathrm{ra}}$ which is of the form $\Psi^{(r)}_{\mathrm{Hodge}}(s)$ for some holomorphic section s of $\mathcal{L}_r$ over C is called a *nearly holomorphic modular form* on Γ. The holomorphic section s of $\mathcal{L}_r$ associated to a given nearly holomorphic modular form f is unique (cf. equation (5a) in §10.1 of [Hi2]). Following a common abuse of notation, a nearly holomorphic modular form is treated interchangeably as as a real analytic section $f(\tau)(2\pi i dw)^r$ of $\underline{\omega}^r_{\mathrm{ra}}$ and as a real analytic function $f(\tau)$ on $\mathcal{H}$ transforming under Γ like a modular form of weight r.

Let Θ_{Hodge} be the differential operator on nearly holomorphic modular forms associated to the splitting (1.2.8) as in (1.1.17), i.e., satisfying

$$\Theta_{\mathrm{Hodge}}(f) = \Psi^{(r+2)}_{\mathrm{Hodge}}(\tilde{\nabla}(s)), \qquad \text{for all } f = \Psi^{(r)}_{\mathrm{Hodge}}(s) \text{ with } s \in H^0(C, \mathcal{L}_r).$$

The following lemma relates Θ_{Hodge} to the classical Shimura-Maass differential operator δ_r defined by

$$\delta_r f(\tau) := \frac{1}{2\pi i}\left(\frac{\partial}{\partial \tau} + \frac{r}{\tau - \bar{\tau}}\right) f(\tau), \tag{1.2.9}$$

which maps real analytic modular forms of weight r to real analytic modular forms of weight $r+2$.

Lemma 1.5. *Let f be any nearly holomorphic modular form of weight r on Γ. Then*

$$\Theta_{\mathrm{Hodge}} f = \delta_r f. \tag{1.2.10}$$

Proof. Write $f = \Psi^{(r)}_{\mathrm{Hodge}}(s)$, where s is the holomorphic section of $\mathcal{L}_r$ giving rise to f, and expand s in terms of the local coordinates τ and w as

$$s = s_0(\tau) d\bar{w}^r + s_1(\tau) d\bar{w}^{r-1} dw + \cdots + s_{r-1}(\tau) d\bar{w} dw^{r-1} + f(\tau)(2\pi i dw)^r.$$

Since s is a holomorphic section, its periods vary holomorphically, and therefore $\nabla s = \nabla^{1,0} s$, where $\nabla^{1,0}$ is the component of the Gauss-Manin connection on $\mathcal{L}_r^{\mathrm{ra}}$ obtained by differentiating periods of real analytic sections in the holomorphic direction. Since the periods attached to the local section $d\bar{w}$ are antiholomorphic, it follows that $\nabla^{1,0}(d\bar{w}) = 0$, and therefore, by (1.2.5), which continues to hold when ∇ is replaced by $\nabla^{1,0}$,

$$\begin{aligned} \nabla s = \nabla^{1,0} s &\equiv \nabla^{1,0}(f(\tau)(2\pi i dw)^r) \pmod{d\bar{w} H^0(C^0, \mathcal{L}_{r-1} \otimes \Omega^1_C)} \\ &\equiv (2\pi i)^r \cdot \left(f_\tau(\tau) dw^r + r f(\tau) dw^{r-1}\left(\frac{dw - d\bar{w}}{\tau - \bar{\tau}}\right)\right) \otimes d\tau, \end{aligned}$$

where $f_\tau := \frac{\partial f}{\partial \tau}$ is the derivative of f with respect to the holomorphic variable τ. It follows from the last identity in (1.2.5) and the definition of $\tilde{\nabla}$ that

$$\begin{aligned}\Psi^{(r+2)}_{\mathrm{Hodge}}(\tilde{\nabla}(s)) &= (2\pi i)^{r+1}\cdot \Psi^{(r+2)}_{\mathrm{Hodge}}\left(f_\tau(\tau)dw^{r+2} + rf(\tau)dw^{r+1}\left(\frac{dw-d\bar{w}}{\tau-\bar{\tau}}\right)\right)\\ &= \delta_r f(\tau)(2\pi i dw)^{r+2}.\end{aligned}$$

The lemma follows. □

More generally, letting

$$\Theta^j_{\mathrm{Hodge}} : f \mapsto \Psi^{(r+2j)}_{\mathrm{Hodge}}(\tilde{\nabla}^j(s)),$$

one obtains $\Theta^j_{\mathrm{Hodge}}(f) = \delta^j_r f$, where $\delta^j_r := \delta_{r+2j-2}\circ\cdots\circ\delta_r$ is the j-th iterate of the Shimura-Maass derivative, sending nearly holomorphic modular forms of weight r to nearly holomorphic modular forms of weight $r+2j$.

1.3. **p-adic modular forms.** A ring is called a p-adic ring if the natural homomorphism to its pro-p completion is an isomorphism. If R is a p-adic ring, then a triple $(E,t,\omega)_{/R}$ as in Definition 1.2 is said to be *ordinary* if the mod p reduction of E (viewed as an elliptic curve over R/pR) has invertible Hasse invariant. We briefly recall Katz's definition of p-adic modular forms, which is modelled on Definition 1.2. In this definition we continue to assume that k is an integer ≥ 2.

Definition 1.6. A *p-adic modular form of weight k on* Γ *defined over a p-adic ring Z* is a rule which to every isomorphism class of ordinary triples $(E,t,\omega)\in \widetilde{\mathrm{Ell}}(\Gamma,R)$ defined over a p-adic Z-algebra R associates an element $f(E,t,\omega)\in R$ satisfying

(1) (Compatibility with base change). For all Z-algebra homomorphisms $j: R\longrightarrow R'$,
$$f((E,t,\omega)\otimes_j R') = j(f(E,t,\omega)).$$
(2) (Weight k condition). For all $\lambda\in R^\times$,
$$f(E,t,\lambda\omega) = \lambda^{-k} f(E,t,\omega).$$
(3) (Behavior at the cusps). Let $(\mathrm{Tate}(q),t,\omega_{\mathrm{can}})$ be the Tate elliptic curve $\mathbf{G}_m/q^{\mathbf{Z}}$ equipped with any level N structure t defined over the p-adic completion of $Z[\zeta_N]((q^{1/d}))$, and the canonical differential ω_{can} over $Z((q))$. Then
$$f\left(\mathrm{Tate}(q),t,\omega_{\mathrm{can}}\right) \text{ belongs to } Z[\zeta_N][[q^{1/d}]],$$
and $f\left(\mathrm{Tate}(q),t^\sigma,\omega_{\mathrm{can}}\right) = f\left(\mathrm{Tate}(q),t,\omega_{\mathrm{can}}\right)^\sigma$ for all $\sigma\in \mathrm{Aut}(Z(\zeta_N)/Z)$.

We will now recall the geometric interpretation of p-adic modular forms as sections of suitable rigid analytic line bundles. Assume that the prime p does not divide N, so that C extends to a canonical smooth proper model $\mathcal{C}$ over $\mathrm{Spec}\,\mathbf{Z}_p$. Write $C_{\mathbf{F}_p} := \mathcal{C}\times_{\mathbf{Z}_p}\mathbf{F}_p$, and let

$$\mathrm{red}_p : C(\mathbf{C}_p)\longrightarrow C_{\mathbf{F}_p}(\bar{\mathbf{F}}_p)$$

denote the natural reduction map.

Let $\{P_1,\ldots,P_t\}$ be the finite subset of $C_{\mathbf{F}_p}(\bar{\mathbf{F}}_p)$ consisting of the supersingular points. The *residue disc* attached to P_j, denoted $D(P_j)$, is the set of points of $C(\mathbf{C}_p)$ which have the same image as P_j under red_p. Let

$$C^{\mathrm{ord}} = C(\mathbf{C}_p) - D(P_1) - \cdots - D(P_t).$$

Since the P_j are smooth points of $C_{\mathbf{F}_p}(\bar{\mathbf{F}}_p)$, the residue discs $D(P_j)$ are conformal to the open unit disc $U\subset \mathbf{C}_p$ consisting of $z\in\mathbf{C}_p$ with $|z|<1$. The set C^{ord} is an example of an *affinoid* subset of $C(\mathbf{C}_p)$ with good reduction. (These concepts are discussed in somewhat more detail in Section 3.5. For general definitions and a more systematic discussion, see also, for example, Sections II and III of [Col2].)

The algebraic vector bundle $\mathcal{L}_r$ on C gives rise to a rigid analytic coherent sheaf $\mathcal{L}_r^{\mathrm{rig}}$ on C^{ord}, equipped with the Gauss-Manin connection

$$\nabla : \mathcal{L}_r^{\mathrm{rig}}\longrightarrow \mathcal{L}_r^{\mathrm{rig}}\otimes\Omega^1(\log Z_N),$$

and a subsheaf $\underline{\omega}^r$ for the rigid analytic topology on C^{ord}. A p-adic modular form f of weight r for Γ corresponds, via (1.1.1), to a rigid analytic section of $\underline{\omega}^r$ over C^{ord}.

Following [Ka4], Theorem (1.11.27), there is a unique decomposition of rigid analytic sheaves

$$\mathcal{L}_1^{\mathrm{rig}} = \underline{\omega} \oplus \mathcal{L}_1^{\mathrm{Frob}} \tag{1.3.1}$$

such that the Frobenius endomorphism preserves (and acts invertibly) on $\mathcal{L}_1^{\mathrm{Frob}}$. In the p-adic theory, this "unit root" decomposition plays a role analogous to that of the Hodge decomposition in the complex setting. Most importantly, (1.3.1) gives rise to a rigid analytic splitting over C^{ord}

$$\Psi_{\mathrm{Frob}} : \mathcal{L}_1^{\mathrm{rig}} \longrightarrow \underline{\omega}.$$

Let Θ_{Frob} be the differential operator associated to this splitting as in (1.1.17). It maps p-adic modular forms of weight r to p-adic modular forms of weight $r+2$. The following lemma relates Θ_{Frob} to the classical Atkin-Serre theta operator whose effect on q-expansions $f(\mathrm{Tate}(q), \zeta_N, \omega_{\mathrm{can}}) = \sum a_n q^n$ is given by

$$\theta f(\mathrm{Tate}(q), \zeta_N, \omega_{\mathrm{can}}) = q\frac{d}{dq}\sum_{n=1}^{\infty} a_n q^n = \sum_{n=1}^{\infty} n a_n q^n. \tag{1.3.2}$$

Lemma 1.7. *For all p-adic modular forms f of weight r,*

$$\Theta_{\mathrm{Frob}} f = \theta f. \tag{1.3.3}$$

Proof. Since a p-adic modular form is determined by its q-expansion, it is enough to check the identity on the Tate curve. By (1.1.8),

$$\begin{aligned} \nabla f\,(\mathrm{Tate}(q), \zeta_N) &= \nabla(f(q)\omega_{\mathrm{can}}^r) \\ &= \left(q\frac{d}{dq} f(q)\omega_{\mathrm{can}}^r + r f(q)\omega_{\mathrm{can}}^{r-1}\xi_{\mathrm{can}}\right)\frac{dq}{q}. \end{aligned}$$

Therefore, by (1.1.10),

$$\tilde{\nabla} f\,(\mathrm{Tate}(q), \zeta_N) = q\frac{d}{dq} f(q)\omega_{\mathrm{can}}^{r+2} + r f(q)\omega_{\mathrm{can}}^{r+1}\xi_{\mathrm{can}}. \tag{1.3.4}$$

Since the Frobenius endomorphism respects the Gauss-Manin connection, it preserves the line spanned by the unique horizontal section ξ_{can} of $\mathcal{L}_1$ over $Z'[\![q]\!]$, and therefore ξ_{can} is stable under Frobenius. (Cf. Sec. A2.2 of [Ka2].) It follows that the unit root subspace of the Tate curve $\mathrm{Tate}(q)$ over the p-adic completion R of $Z'(\!(q)\!)$ is equal to

$$H^1_{\mathrm{dR}}(\mathrm{Tate}(q))^{\mathrm{Frob}} = R\xi_{\mathrm{can}}.$$

Hence $\Psi_{\mathrm{Frob}}(\xi_{\mathrm{can}}) = 0$. Applying $\Psi_{\mathrm{Frob}}^{(r+2)}$ to equation (1.3.4) shows that

$$\Theta_{\mathrm{Frob}} f(\mathrm{Tate}(q), \zeta_N, \omega_{\mathrm{can}}) = \theta f(\mathrm{Tate}(q), \zeta_N, \omega_{\mathrm{can}}).$$

□

1.4. **Elliptic curves with complex multiplication.** Let K be an imaginary quadratic field of discriminant $-d_K$, let $\mathcal{O}_K$ be its ring of integers, and let H denote the Hilbert class field of K. Let A be a fixed elliptic curve defined over H satisfying

$$\mathrm{End}_H(A) \simeq \mathcal{O}_K.$$

The identification $\mathcal{O}_K = \mathrm{End}_H(A)$ is normalised so that the endomorphism $[\alpha]$ induces multiplication by α on $\Omega^1_{A/H}$.

Cohomology. The Hodge filtration on the de Rham cohomology $H^1_{\mathrm{dR}}(A/F)$ (over any field F which contains H) admits a canonical, functorial *algebraic* splitting

$$H^1_{\mathrm{dR}}(A/F) = H^{1,0}_{\mathrm{dR}}(A/F) \oplus H^{0,1}_{\mathrm{dR}}(A/F), \tag{1.4.1}$$

which agrees with the Hodge decomposition of $H^1_{\mathrm{dR}}(A/\mathbf{C})$ when $F = \mathbf{C}$ and with the unit root decomposition over a p-adic ring when A is ordinary. This decomposition is characterised by the conditions

$$H^{1,0}_{\mathrm{dR}}(A/F) = \Omega^1_{A/F}, \qquad \lambda^*\eta = \lambda^\rho\eta, \quad \forall\ \lambda \in \mathcal{O}_K, \quad \eta \in H^{0,1}_{\mathrm{dR}}(A/F),$$

where $\lambda \mapsto \lambda^\rho$ is the non-trivial automorphism of K. The choice of a non-zero differential $\omega_A \in \Omega^1_{A/F} = H^{1,0}_{\mathrm{dR}}(A/F)$ thus determines a generator η_A of $H^{0,1}_{\mathrm{dR}}(A/F)$ satisfying

$$\langle \omega_A, \eta_A \rangle = 1, \tag{1.4.2}$$

where $\langle\ ,\ \rangle$ denotes the algebraic cup product pairing on de Rham cohomology.

Let S_r denote the symmetric group on r letters. Multiplication by -1 on A, combined with the natural permutation action of S_r on A^r, gives rise to an action of the wreath product

$$\Xi_r := (\mu_2)^r \rtimes S_r \tag{1.4.3}$$

on A^r. Let $j : \Xi_r \longrightarrow \mu_2$ be the homomorphism which is the identity on μ_2 and the sign character on S_r, and let

$$\epsilon_A := \frac{1}{2^r r!} \sum_{\sigma \in \Xi_r} j(\sigma)\sigma \in \mathbf{Q}[\mathrm{Aut}(A^r)] \tag{1.4.4}$$

denote the associated idempotent in the rational group ring of $\mathrm{Aut}(A^r)$. By functoriality, it induces an idempotent on $H^*_{\mathrm{dR}}(A^r/F)$. Recall the Künneth decomposition

$$H^*_{\mathrm{dR}}(A^r/F) = \oplus_{(i_1,\ldots,i_r)} H^{i_1}_{\mathrm{dR}}(A/F) \otimes \cdots \otimes H^{i_r}_{\mathrm{dR}}(A/F), \tag{1.4.5}$$

where the direct sum is taken over all r-tuples $(i_1, \ldots, i_r)$ with $0 \le i_j \le 2$. The natural action of S_r on $H^1_{\mathrm{dR}}(A/F)^{\otimes r}$ gives rise to a subspace $\mathrm{Sym}^r H^1_{\mathrm{dR}}(A/F)$ consisting of classes which are fixed by this action.

Lemma 1.8. *The image of the projector ϵ_A acting on $H^*_{\mathrm{dR}}(A^r/F)$ is equal to* $\mathrm{Sym}^r H^1_{\mathrm{dR}}(A/F)$. *More precisely,*

$$\epsilon_A H^j_{\mathrm{dR}}(A^r/F) = \begin{cases} 0 & \text{if } j \neq r; \\ \mathrm{Sym}^r H^1_{\mathrm{dR}}(A/F) & \text{if } j = r. \end{cases}$$

Proof. Since multiplication by (-1) acts as -1 on $H^1_{\mathrm{dR}}(A/F)$ and as 1 on $H^0_{\mathrm{dR}}(A/F)$ and $H^2_{\mathrm{dR}}(A/F)$, it follows that ϵ_A annihilates all the terms in the Künneth decomposition (1.4.5) except $H^1_{\mathrm{dR}}(A/F)^{\otimes r}$. The natural action of S_r on this term corresponds to the geometric permutation action of S_r on A^r, twisted by the sign character. It follows that the restriction of ϵ_A to $H^1_{\mathrm{dR}}(A/F)^{\otimes r}$ induces the natural projection onto the space $\mathrm{Sym}^r H^1_{\mathrm{dR}}(A/F)$ of symmetric tensors. □

For any j such that $0 \le j \le r$, we define $\omega_A^j \eta_A^{r-j}$ by

$$\begin{aligned} \omega_A^j \eta_A^{r-j} &:= \epsilon_A^*(p_1^*\omega_A \wedge \cdots p_j^*\omega_A \wedge p_{j+1}^*\eta_A \wedge \cdots \wedge p_r^*\eta_A) \\ &= \frac{j!(r-j)!}{r!} \sum_{I \subset \{1,\ldots,r\}} p_1^*\varpi_{1,I} \wedge \ldots \wedge p_r^*\varpi_{r,I}, \end{aligned} \tag{1.4.6}$$

where $\varpi_{i,I} := \omega_A$ or η_A according as $i \in I$ or $i \notin I$.

Note that the classes $\omega_A^j \eta_A^{r-j}$ form a basis of the vector space

$$\epsilon_A H^r_{\mathrm{dR}}(A^r/F) = \mathrm{Sym}^r H^1_{\mathrm{dR}}(A/F).$$

Isogenies. It will always be assumed that A satisfies the following "Heegner hypothesis" relative to a fixed positive integer N that is mentioned in (0.0.3) of the Introduction.

Assumption 1.9. *There is an ideal $\mathfrak{N}$ of $\mathcal{O}_K$ of norm N such that $\mathcal{O}_K/\mathfrak{N} = \mathbf{Z}/N\mathbf{Z}$. (Such an ideal is called a* cyclic ideal of norm N *in $\mathcal{O}_K$.)*

Since both A and its endomorphisms are defined over the Hilbert class field H, the group scheme $A[\mathfrak{N}]$ of $\mathfrak{N}$-torsion in A is a cyclic subgroup scheme of A of order N defined over this field. The absolute Galois group G_H acts naturally on its set of geometric points. Let $\tilde{H}$ be the smallest extension of H over which this Galois representation becomes trivial. The choice of a section $t_A : \mathrm{Spec}(\tilde{H}) \longrightarrow A[\mathfrak{N}]$ of order N gives rise to a Γ-level structure on A defined over any field F that contains $\tilde{H}$. Fix such a t_A once and for all.

Consider the set of pairs (φ, A'), where A' is an elliptic curve and $\varphi : A \longrightarrow A'$ is an isogeny (defined over $\bar{K}$). Two pairs (φ_1, A_1') and (φ_2, A_2') are said to be *isomorphic* if there is a $\bar{K}$-isomorphism $\iota : A_1' \longrightarrow A_2'$ satisfying $\iota\varphi_1 = \varphi_2$. Let

$$\mathrm{Isog}(A) := \{\text{Isomorphism class of pairs } (\varphi, A')\}.$$

The absolute Galois group $G_K = \mathrm{Gal}(\bar{K}/K)$ acts naturally on $\mathrm{Isog}(A)$ and a pair (φ, A') admits a representative defined over a field $F \subset \bar{K}$ if it is fixed by the group $G_F \subset G_K$. Fix $(\varphi, A') \in \mathrm{Isog}(A)$. Since A has complex multiplication by $\mathcal{O}_K$, the endomorphism ring of A' is an order in $\mathcal{O}_K$. Such an order is completely determined by its conductor, and therefore there is a unique integer $c \geq 1$ such that $\mathrm{End}_F(A') = \mathcal{O}_c := \mathbf{Z} + c\mathcal{O}_K$. A pair (φ, A') is said to be *of conductor* c if $\mathrm{End}_F(A') = \mathcal{O}_c$. Clearly this notion is well defined on isomorphism classes, and hence we may set

$$\mathrm{Isog}_c(A) := \{\text{Isomorphism classes of pairs } (\varphi, A') \text{ of conductor } c\}.$$

More generally, let $\mathrm{Isog}^{\mathfrak{N}}(A)$ be the subset of $\mathrm{Isog}(A)$ consisting of pairs (φ, A') where φ is an isogeny whose kernel intersects $A[\mathfrak{N}]$ trivially, and set $\mathrm{Isog}_c^{\mathfrak{N}}(A) := \mathrm{Isog}_c(A) \cap \mathrm{Isog}^{\mathfrak{N}}(A)$.

Let $P_K(\mathcal{O}_c)$ denote the group of projective rank one $\mathcal{O}_c$-submodules of K and let $P(\mathcal{O}_c)$ denote the subsemigroup of modules that are contained in $\mathcal{O}_c$ and are relatively prime to $\mathfrak{N}_c := \mathfrak{N} \cap \mathcal{O}_c$. The semigroup $P(\mathcal{O}_c)$ acts naturally on $\mathrm{Isog}_c(A)$ and $\mathrm{Isog}_c^{\mathfrak{N}}(A)$ by the rule $\mathfrak{a} * (\varphi, A') = (\varphi_{\mathfrak{a}}\varphi, A'_{\mathfrak{a}})$, where

$$\varphi_{\mathfrak{a}} : A' \longrightarrow A'_{\mathfrak{a}} := A'/A'[\mathfrak{a}] \tag{1.4.7}$$

is the natural isogeny. Note that, if $\mathfrak{a} = \mathcal{O}_c \cdot a$ is free, then $\mathfrak{a} * (\varphi, A') = (a\varphi, A')$.

Let (A_1, t_1, ω_1) and (A_2, t_2, ω_2) be two marked elliptic curves with Γ-level structure. The following notion of an isogeny

$$\varphi : (A_1, t_1, \omega_1) \longrightarrow (A_2, t_2, \omega_2)$$

will be convenient from the notational viewpoint.

Definition 1.10. An isogeny from (A_1, t_1, ω_1) to (A_2, t_2, ω_2) is an isogeny $\varphi : A_1 \longrightarrow A_2$ on the underlying elliptic curves satisfying

$$\varphi(t_1) = t_2, \qquad \varphi^*(\omega_2) = \omega_1.$$

The action of $P(\mathcal{O}_c)$ on $\mathrm{Isog}_c^{\mathfrak{N}}(A)$ that was just defined gives rise to an action of $P(\mathcal{O}_c)$ on the set of isomorphism classes of triples (A', t', ω') with $\mathrm{End}(A') = \mathcal{O}_c$ and $t' \in A'[\mathfrak{N}_c]$, by the rule

$$\mathfrak{a} * (A', t', \omega') = (A'_{\mathfrak{a}}, \varphi_{\mathfrak{a}}(t'), \omega'_{\mathfrak{a}}), \qquad \text{where } \varphi_{\mathfrak{a}}^*(\omega'_{\mathfrak{a}}) = \omega'. \tag{1.4.8}$$

Remark 1.11. Let $\mathbb{A}_{K,f}$ denote the ring of finite adèles of K and let $\hat{\mathcal{O}}_c$ denote $(\mathcal{O}_c \otimes \hat{\mathbf{Z}})$, viewed as a subring of $\mathbb{A}_{K,f}$. The group $P_K(\mathcal{O}_c)$ is naturally identified with $\mathbb{A}_{K,f}^\times / \hat{\mathcal{O}}_c^\times$, by associating to $\mathfrak{a}$ a generator $(a_v) \in \mathbb{A}_{K,f}^\times$ of $\mathfrak{a} \otimes_{\mathcal{O}_c} \hat{\mathcal{O}}_c$.

1.5. **Values of modular forms at CM points.** Following the notations of Section 1.4, we continue to let (A, t_A, ω_A) be a marked elliptic curve with Γ-level structure and complex multiplication by $\mathcal{O}_K$, defined over a field F, and let $\varphi : (A, t_A, \omega_A) \longrightarrow (A', t', \omega')$ be an isogeny of marked elliptic curves over F.

Fix complex and p-adic embeddings $\iota_\infty : F \longrightarrow \mathbf{C}$ and $\iota_p : F \longrightarrow \mathbf{C}_p$, and use these to view A and A' as curves over $\mathbf{C}$ and $\mathcal{O}_{\mathbf{C}_p}$ (by fixing a good integral model) respectively. If f belongs to the space $M_k^\dagger(\Gamma, F)$ of modular forms over F, then by definition $f(A', t', \omega')$ belongs to F as well. Note that f can be viewed as a p-adic modular form, after possibly rescaling it. The following algebraicity theorem asserts that a similar conclusion holds for $\Theta_{\mathrm{Hodge}}(f)$ and $\Theta_{\mathrm{Frob}}(f)$, evaluated on $\iota_\infty(A', t', \omega')$ and $\iota_p(A', t', \omega')$ respectively.

Proposition 1.12. *Let $(A', t', \omega')_{/F}$ be a marked elliptic curve with complex multiplication by an order in K. Assume that A', viewed as an elliptic curve over $\mathcal{O}_{\mathbf{C}_p}$, is ordinary. Then:*

1. *The complex number $\Theta_{\mathrm{Hodge}} f(A', t', \omega')$ belongs to $\iota_\infty(F)$.*
2. *The p-adic number $\Theta_{\mathrm{Frob}} f(A', t', \omega')$ belongs to $\iota_p(F)$.*
3. *Viewing these two quantities as elements of F, we have*

$$\Theta_{\mathrm{Hodge}} f(A', t', \omega') = \Theta_{\mathrm{Frob}} f(A', t', \omega').$$

Proof. Part (1) is due to Shimura [Shim1] and parts (2) and (3) are due to Katz [Ka4]. Our proof below follows Katz's approach. (See also the article of Hida [Hi4].) The key point is that any endomorphism $\alpha \in \mathcal{O}_K$ of A' respects the algebraic splitting of the Hodge filtration on $H^1_{\mathrm{dR}}(A'/F)$ defined in equation (1.4.1) of Section 1.4, and acts on $H^{0,1}_{\mathrm{dR}}(A'/F)$ via multiplication by $\bar{\alpha}$. It follows that $H^1_{\mathrm{dR}}(A'/F) =$

$\Omega^1(A'/F) \oplus H^{0,1}_{\mathrm{dR}}(A'/F)$ agrees with the Hodge decomposition of $H^1_{\mathrm{dR}}(A' \otimes_{\iota_\infty} \mathbf{C})$ and with the unit root decomposition of $H^1_{\mathrm{dR}}(A' \otimes_{\iota_p} \mathbf{C}_p)$, which both share this property. More precisely,

$$\begin{aligned} H^{0,1}_{\mathrm{dR}}(A'/F) \otimes_{\iota_\infty} \mathbf{C} &= H^{0,1}_{\mathrm{dR}}(A' \otimes_{\iota_\infty} \mathbf{C}), \\ H^{0,1}_{\mathrm{dR}}(A'/F) \otimes_{\iota_p} \mathbf{C}_p &= H^{1}_{\mathrm{dR}}(A' \otimes_{\iota_p} \mathbf{C}_p)^{\mathrm{Frob}}. \end{aligned}$$

Therefore, $\Psi^{(r+2)}_{\mathrm{Hodge}}\tilde{\nabla} f(A',t')$ and $\Psi^{(r+2)}_{\mathrm{Frob}}\tilde{\nabla} f(A',t')$ both belong to $\mathrm{Sym}^{r+2}\,\Omega^1(A'/F)$, and are equal. The proposition follows. □

2. Generalised Heegner cycles

2.1. **Kuga-Sato varieties.** Let $\pi : \mathcal{E} \longrightarrow C$ be the universal generalised elliptic curve with $\Gamma_1(N)$ level structure, extending the universal elliptic curve over C^0 introduced in Section 1.1, which exists because our running assumption that $N > 4$. The variety $W_1 := \mathcal{E}$ is smooth and proper, and the geometric fibers of π above a closed point $x \in C$ are singular precisely when x is a cusp. The geometric fiber $\pi^{-1}(x)$ is then isomorphic to a chain of projective lines intersecting at ordinary double points whose dual graph is an m-gon for a suitable $m|N$, depending on x. Let $W_1^* \subset W_1$ denote the relative identity component of the Néron model of $\mathcal{E}$ over $X_1(N)$, whose geometric fibers above the cusps are isomorphic to the multiplicative group $\mathbf{G}_m$.

Fix an integer $r \geq 0$, and let

$$W_r^* := W_1^* \times_C W_1^* \times_C \cdots \times_C W_1^* \quad \subset \quad W_r^\sharp := \mathcal{E} \times_C \mathcal{E} \times_C \cdots \times_C \mathcal{E},$$

denote the r-fold fiber products of W_1^* and $\mathcal{E}$ respectively over C.

Write W_r for the canonical desingularisation of $W_r^\sharp$, as described in [De2], Lemmas 5.4 and 5.5, and [Schol2] 1.0.3, for example. In these references, these constructions are performed for the universal elliptic curve over the modular curve $X(N)$ with full level N structure, but can be adapted to deal with the case of $X_1(N)$; see the Appendix for further details on this more general construction, even over $\mathrm{Spec}\,\mathbf{Z}[\frac{1}{N}]$.

Denote by

$$W_r^0 := W_r \times_C C^0 = W_r^\sharp \times_C C^0 = W_r^* \times_C C^0$$

the complement in W_r of the geometric fibers above the cusps, and let $W_r^{\mathrm{reg}} \in W_r^\sharp$ be the locus where the natural projection $W_r^\sharp \longrightarrow C$ is smooth. As in 1.3.2. of [Schol2] there is a non-canonical isomorphism

$$W_r^{\mathrm{reg}} \times_C Z_\infty = \coprod_{d|N} \left(Z_\infty(d) \times (\mathbf{G}_m \times \mathbf{Z}/d\mathbf{Z})^r\right), \tag{2.1.1}$$

where $Z_\infty \subset C$ denotes the cuspidal subscheme and $Z_\infty(d) \subset Z_\infty$ is the (possibly empty) subscheme of cusps with ramification degree d over the modular curve of level one. The varieties $\mathcal{E}$, C, $W_r^\sharp$, W_r^*, W_r and W_r^0 are all defined over $\mathbf{Q}$, and can therefore be viewed as defined over any field F of characteristic 0. It will be convenient to fix such an F at the outset.

Translation by the sections of order N gives rise to an action of $(\mathbf{Z}/N\mathbf{Z})^r$ on $W_r^\sharp$, which extends to W_r by the canonical nature of the desingularisation. The group $(\mathbf{Z}/N\mathbf{Z})^r$ also acts transitively (but not freely, in general) on the set of components of $W_r^\sharp$ above any cusp of C arising in (2.1.1). Let σ_a denote the automorphism of W_r associated to $a \in (\mathbf{Z}/N\mathbf{Z})^r$, and let

$$\epsilon_W^{(1)} = \frac{1}{N^r} \sum_{a \in (\mathbf{Z}/N\mathbf{Z})^r} \sigma_a$$

denote the corresponding idempotent in the rational group ring of $(\mathbf{Z}/N\mathbf{Z})^r$. Similarly, the group Ξ_r of (1.4.3) can be viewed as a subgroup of $\mathrm{Aut}(W_r/C)$ acting on the fibers of the natural projection from W_r to C. Let $\epsilon_W^{(2)}$ be the idempotent in the group ring $\mathbf{Z}[1/2r!][\mathrm{Aut}(W_r/C)]$ which is defined by the same formula as in (1.4.4) with A^r replaced by W_r/C. The idempotents $\epsilon_W^{(1)}$ and $\epsilon_W^{(2)}$ commute, and therefore the composition

$$\epsilon_W = \epsilon_W^{(2)} \epsilon_W^{(1)} \tag{2.1.2}$$

defines a projector in the ring of rational correspondences on W_r.

Let

$$\Omega^0(\mathcal{L}_r) := \mathcal{L}_r, \qquad \Omega^1(\mathcal{L}_r) := \mathcal{L}_r \otimes \Omega^1_C + \nabla(\mathcal{L}_r).$$

The complex

$$0 \longrightarrow \Omega^0(\mathcal{L}_r) \xrightarrow{\nabla} \Omega^1(\mathcal{L}_r) \longrightarrow 0 \tag{2.1.3}$$

of sheaves over C is the smallest subcomplex of

$$0 \longrightarrow \mathcal{L}_r \xrightarrow{\nabla} \mathcal{L}_r \otimes \Omega^1_C(\log Z_N) \longrightarrow 0 \tag{2.1.4}$$

which contains $\mathcal{L}_r$ and $\mathcal{L}_r \otimes \Omega^1_C$ in degrees 0 and 1 respectively. The de Rham cohomology of C attached to $\mathcal{L}_r$, denoted $H^i_{\mathrm{dR}}(C/F, \mathcal{L}_r, \nabla)$, is defined to be the ith hypercohomology of the complex (2.1.4):

$$H^i_{\mathrm{dR}}(C/F, \mathcal{L}_r, \nabla) := \mathbb{H}^i(C/F, \mathcal{L}_r \otimes \Omega^\bullet(\log Z_N)).$$

The *parabolic de Rham cohomology* of C attached to $\mathcal{L}_r$ is defined, following Section 2.6 of [Schol1] as the hypercohomology of the subcomplex (2.1.3):

$$H^i_{\mathrm{par}}(C/F, \mathcal{L}_r, \nabla) := \mathbb{H}^i(C/F, \Omega^\bullet(\mathcal{L}_r)).$$

In degree 0, we have

$$H^0_{\mathrm{par}}(C/F, \mathcal{L}_r, \nabla) = H^0_{\mathrm{dR}}(C/F, \mathcal{L}_r, \nabla).$$

As explained in the proof of Theorem 2.7 (i) of [Schol1], the parabolic cohomology $H^1_{\mathrm{par}}(C/F, \mathcal{L}_r, \nabla)$ in degree 1 is equipped with a natural filtration

$$0 \longrightarrow H^0(C/F, \underline{\omega}^r \otimes \Omega^1_C) \longrightarrow H^1_{\mathrm{par}}(C/F, \mathcal{L}_r, \nabla) \longrightarrow H^1(C/F, \underline{\omega}^{-r}) \longrightarrow 0. \tag{2.1.5}$$

The de Rham cohomology groups $H^i_{\mathrm{dR}}(X/F)$ (attached to any variety X over F) and $H^i_{\mathrm{dR}}(C/F, \mathcal{L}_r, \nabla)$ will sometimes be abbreviated to $H^i_{\mathrm{dR}}(X)$ and $H^i_{\mathrm{dR}}(C, \mathcal{L}_r, \nabla)$, and likewise for the parabolic cohomology groups, when no confusion results from suppressing the field of definition F in the notation.

Lemma 2.1. *If $r = 0$, then $H^0_{\mathrm{dR}}(C, \mathcal{L}_r, \nabla) = F$, and $H^0_{\mathrm{dR}}(C, \mathcal{L}_r, \nabla) = 0$ otherwise.*

Proof. Fix an embedding of F into $\mathbf{C}$ and consider $H^0_{\mathrm{dR}}(C/\mathbf{C}, \mathcal{L}_r, \nabla) = H^0_{\mathrm{dR}}(C/F, \mathcal{L}_r, \nabla) \otimes_F \mathbf{C}$. By the GAGA principle,

$$H^0_{\mathrm{dR}}(C/\mathbf{C}, \mathcal{L}_r, \nabla) = H^0_{\mathrm{dR}}(C, \mathcal{L}_r^{\mathrm{an}}, \nabla).$$

The restriction map

$$H^0_{\mathrm{dR}}(C, \mathcal{L}_r^{\mathrm{an}}, \nabla) \longrightarrow H^0_{\mathrm{dR}}(C^0, \mathcal{L}_r^{\mathrm{an}}, \nabla)$$

is injective, and

$$H^0_{\mathrm{dR}}(C^0, \mathcal{L}_r^{\mathrm{an}}, \nabla) = H^0(C^0, \mathbb{L}_r),$$

where $\mathbb{L}_r$ is the local system introduced in (1.2.1). This local system corresponds to the r-th symmetric power of the standard two-dimensional representation $\mathbf{C}^2$ of $\Gamma \subset \mathbf{SL}_2(\mathbf{Z}) \subset \mathbf{SL}_2(\mathbf{C})$, and therefore

$$H^0(C^0, \mathbb{L}_r) = H^0(\Gamma, \mathrm{Sym}^r(\mathbf{C}^2)) = \begin{cases} \mathbf{C} & \text{if } r = 0, \\ 0 & \text{otherwise.} \end{cases}$$

The lemma follows. □

We wish to describe the image of ϵ_W on the (middle) cohomology of W_r, and relate this image to $H^1_{\mathrm{par}}(C, \mathcal{L}_r, \nabla)$.

Lemma 2.2. *Assume $r \geq 1$.*

1. *The image of $\epsilon_W^{(2)}$ (and of ϵ_W) acting on $H^*_{\mathrm{dR}}(W_r^0/F)$ is canonically isomorphic to $H^1_{\mathrm{dR}}(C, \mathcal{L}_r, \nabla)$.*
2. *The image of ϵ_W acting on $H^*_{\mathrm{dR}}(W_r/F)$ is canonically isomorphic to $H^1_{\mathrm{par}}(C, \mathcal{L}_r, \nabla)$.*
3. *Furthermore, the Hodge filtration on $\epsilon_W H^*_{\mathrm{dR}}(W_r/F) = \epsilon_W H^{r+1}_{\mathrm{dR}}(W_r/F)$ is given by (2.1.5), i.e.,*

$$\begin{aligned} \mathrm{Fil}^0 = {} & H^1_{\mathrm{par}}(C, \mathcal{L}_r, \nabla), \\ \mathrm{Fil}^1 = \mathrm{Fil}^2 = \cdots = \mathrm{Fil}^{r+1} = {} & H^0(C, \underline{\omega}^r \otimes \Omega^1_C), \\ \mathrm{Fil}^{r+2} = \cdots = {} & 0, \end{aligned}$$

where Fil^j denotes the j-th step in the Hodge filtration on $\epsilon_W H^{r+1}_{\mathrm{dR}}(W_r)$.

Proof. The arguments below are mild adaptations of those in [Schol2], [Schol3].

(1) By [De1] Cor. 3.15, the natural map

$$H^i_{\mathrm{dR}}(C, \mathcal{L}_r, \nabla) \to H^i_{\mathrm{dR}}(C^0, \mathcal{L}_r|_{C^0}, \nabla) := \mathbb{H}^i(C^0, \Omega^{\cdot}(\mathcal{L}_r)|_{C^0})$$

is an isomorphism. Consider the Leray spectral sequence for de Rham cohomology ([Ka1], Remark 3.3) applied to the map $W_r^0 \to C^0$: i.e.,

$$E_2^{pq} = H^p_{\mathrm{dR}}(C^0, H^q_{\mathrm{dR}}(W_r^0/C^0), \nabla) \Rightarrow H^{p+q}_{\mathrm{dR}}(W_r^0).$$

By the same argument as in [De2] Lemma 5.3, this spectral sequence degenerates at E_2 and identifies the space $H^p_{\mathrm{dR}}(C^0, H^q_{\mathrm{dR}}(W_r^0/C^0), \nabla)$ with the subspace of $H^{p+q}_{\mathrm{dR}}(W_r^0)$ on which $[m]$ acts as m^q. (Here $[m]$ denotes multiplication by m on the fibers of W_r^0/C^0.) Applying the projector $\epsilon_W^{(2)}$, we find

$$\epsilon_W^{(2)} H^*_{\mathrm{dR}}(W_r^0/C^0) = \epsilon_W^{(2)} H^r_{\mathrm{dR}}(W_r^0/C^0) = \mathcal{L}_r|_{C^0}$$

and

$$H^1_{\mathrm{dR}}(C^0, \mathcal{L}_r|_{C^0}, \nabla) \simeq \epsilon_W^{(2)} H^{r+1}_{\mathrm{dR}}(W_r^0) = \epsilon_W^{(2)} H^*_{\mathrm{dR}}(W_r^0).$$

A similar statement holds with $\epsilon_W^{(2)}$ replaced by ϵ_W, since translation by $W_r^0(C^0)$ on $H^1_{\mathrm{dR}}(W_r^0/C^0)$ is trivial (as $W_r^0 \to C^0$ is an abelian scheme).

(2) We use the following fact due to Scholl: there is a canonical isomorphism

$$\epsilon_W H^i_{\cdot}(W_r) \simeq \epsilon_W^{(2)} H^i_{\cdot}(W_r^*),$$

for $\cdot = B$, et or dR. This is proved in [Schol2], Thm. 3.1.0 for the case of full level structure, and the modifications needed to extend this to $X_1(N)$ are described in [Schol3], §2, especially §2.9-2.12. Now consider the Gysin sequence for the inclusion $W_r^0 \hookrightarrow W^*$, writing $Z := W_r^* \setminus W_r^0$:

$$\to H^i(W^*) \to H^i(W_r^0) \to H^{i-1}(Z)(-1) \to H^{i+1}(W^*) \to$$

Since (by [Schol2], Lemma 1.3.1 (i) and [Schol3] §2.9)

$$\epsilon_W^{(2)} H^i(Z) = \begin{cases} 0, \text{ if } i \neq r; \\ H^0(Z_\infty)(-r), \text{ if } i = r, \end{cases}$$

we see from part (1) of the lemma that $\epsilon_W^{(2)} H^i(W^*) = 0$ for $i \neq r+1, r+2$. Further there is an exact sequence (in any cohomology theory)

$$\begin{tikzcd} 0 \arrow[r] & \epsilon_W^{(2)} H^{r+1}(W_r^*) \arrow[r] \arrow[d, equal] & \epsilon_W^{(2)} H^{r+1}(W_r^0) \arrow[r, "\rho"] & H^0(Z_\infty)(-r-1) \arrow[r, "\sigma"] & \epsilon_W^{(2)} H^{r+2}(W_r^*) \arrow[r] \arrow[d, equal] & 0 \\ & \epsilon_W H^{r+1}(W_r) & & & \epsilon_W H^{r+2}(W_r) & \end{tikzcd}$$

The map σ vanishes since its source and image are pure of weight $2r+2$ and $r+2$ respectively, and $r \neq 0$, hence $\epsilon_W H^{r+2}(W_r) = 0$. In the de Rham realization, we have from part (1) that $\epsilon_W^{(2)} H^{r+1}(W_r^0) = H^1_{\mathrm{dR}}(C, \mathcal{L}_r, \nabla) = H^1_{\mathrm{dR}}(C^0, \mathcal{L}_r|_{C^0}, \nabla)$ and hence $\epsilon_W^{(2)} H^{r+1}_{\mathrm{dR}}(W_r^*)$ is identified naturally with the kernel of the map

$$H^1_{\mathrm{dR}}(C^0, \mathcal{L}_r|_{C^0}, \nabla) \xrightarrow{\rho_{\mathrm{dR}}} H^0_{\mathrm{dR}}(Z_\infty, -r-1),$$

which is just $H^1_{\mathrm{par}}(C, \mathcal{L}_r, \nabla)$.
(3) See Thm. 2.7 (i) and Remark 2.8 of [Schol1] . ☐

Corollary 2.3. *The assignment*

$$f \mapsto \omega_f = f(E, t, \omega)\omega^r \otimes \sigma(\omega^2)$$

induces an identification

$$S_{r+2}(\Gamma, F) \xrightarrow{\sim} \mathrm{Fil}^{r+1} \epsilon_W H^{r+1}_{\mathrm{dR}}(W_r/F).$$

Proof. This follows from Part 2 of Lemma 2.2 combined with (1.1.12) (the case $r = 0$ being well known). ☐

2.2. **The variety X_r and its cohomology.** Recall that A is the elliptic curve with complex multiplication by $\mathcal{O}_K$ that was fixed in Section 1.4, defined over the Hilbert class field H of K. Fix a field $F \supset H$, and, for each $r \geq 0$, consider the $(2r+1)$-dimensional variety over F given by

$$X_r := W_r \times A^r.$$

Like the Kuga-Sato variety W_{2r}, the variety X_r is equipped with a proper morphism

$$\pi_r : X_r \longrightarrow C$$

with $2r$-dimensional fibers. The fibers above points of C^0 are products of elliptic curves of the form $E^r \times A^r$, where E varies and A is fixed.

The projectors ϵ_A and ϵ_W defined in (1.4.4) and (2.1.2) respectively give rise to commuting idempotents in the ring of correspondences on X_r which preserve the fibers of the projection $\pi_r : X_r \longrightarrow C$. We set

$$\epsilon_X := \epsilon_W \epsilon_A. \tag{2.2.1}$$

By functoriality, the idempotent ϵ_X acts as a projector on the various cohomology groups associated to the variety X_r.

We define a coherent sheaf of $\mathcal{O}_C$-modules by setting

$$\mathcal{L}_{r,r} = \mathcal{L}_r \otimes \mathrm{Sym}^r H^1_{\mathrm{dR}}(A). \tag{2.2.2}$$

Note that $\mathcal{L}_{r,r}$ is equipped with the self-duality

$$\langle\ ,\ \rangle : \mathcal{L}_{r,r} \times \mathcal{L}_{r,r} \longrightarrow \mathcal{O}_C \tag{2.2.3}$$

arising from Poincaré duality on the fibers. It is described explicitly in terms of equation (1.1.14) and its analogue for $\mathrm{Sym}^r H^1_{\mathrm{dR}}(A)$. Let

$$\mathbb{L}_{r,r} := \mathbb{L}_r \otimes \mathrm{Sym}^r H^1_{\mathrm{dR}}(A/\mathbf{C}) \tag{2.2.4}$$

denote the corresponding locally constant sheaf (for the complex topology on $C^0(\mathbf{C})$). The sheaf $\mathbb{L}_{r,r}$ is the sheaf of horizontal sections of $\mathcal{L}^{\mathrm{an}}_{r,r}$ relative to the Gauss-Manin connection

$$\nabla : \mathcal{L}_{r,r} \longrightarrow \mathcal{L}_{r,r} \otimes \Omega^1_C(\log Z_N).$$

This connection is induced by the Gauss-Manin connection on $\mathcal{L}_r$ combined with the trivial connection on $H^1_{\mathrm{dR}}(A)$. The de Rham cohomology attached to $(\mathcal{L}_{r,r}, \nabla)$ is defined in the same way as for $(\mathcal{L}_r, \nabla)$, and one has

$$\begin{aligned} H^1_{\mathrm{dR}}(C, \mathcal{L}_{r,r}, \nabla) &= H^1_{\mathrm{dR}}(C, \mathcal{L}_r, \nabla) \otimes \mathrm{Sym}^r H^1_{\mathrm{dR}}(A), \\ H^1_{\mathrm{par}}(C, \mathcal{L}_{r,r}, \nabla) &= H^1_{\mathrm{par}}(C, \mathcal{L}_r, \nabla) \otimes \mathrm{Sym}^r H^1_{\mathrm{dR}}(A). \end{aligned}$$

Proposition 2.4. *Assume that r is ≥ 1. The image of the projector ϵ_X acting on $H^*_{\mathrm{dR}}(X_r)$ is given by*

$$\epsilon_X H^*_{\mathrm{dR}}(X_r) = H^1_{\mathrm{par}}(C, \mathcal{L}_{r,r}, \nabla) = H^1_{\mathrm{par}}(C, \mathcal{L}_r, \nabla) \otimes \mathrm{Sym}^r H^1_{\mathrm{dR}}(A).$$

In particular,

$$\epsilon_X H^j_{\mathrm{dR}}(X_r) = \begin{cases} 0 & \text{if } j \neq 2r+1; \\ H^1_{\mathrm{par}}(C, \mathcal{L}_{r,r}, \nabla) & \text{if } j = 2r+1. \end{cases}$$

Furthermore, if Fil^j denotes the j-th step in the Hodge filtration on $\epsilon_X H^{2r+1}_{\mathrm{dR}}(X_r)$, then

$$\mathrm{Fil}^{r+1} = H^0(C, \underline{\omega}^r \otimes \Omega^1_C) \otimes \mathrm{Sym}^r H^1_{\mathrm{dR}}(A). \tag{2.2.5}$$

Proof. This follows directly from Lemmas 1.8 and 2.2 in light of the Künneth decomposition for the cohomology of $X_r = W_r \times A^r$. $\square$

Proposition 2.5. *The assignment $f \otimes \alpha \mapsto \omega_f \wedge \alpha$ induces an identification*

$$S_{r+2}(\Gamma, F) \otimes \mathrm{Sym}^r H^1_{\mathrm{dR}}(A/F) = \mathrm{Fil}^{r+1} \epsilon_X H^{2r+1}_{\mathrm{dR}}(X_r/F).$$

Proof. This follows directly from Corollary 2.3, combined with Proposition 2.4 when $r \geq 1$. $\square$

Given any integer $0 \leq j \leq r$, note in particular that the class

$$\omega_f \wedge \omega_A^j \eta_A^{r-j},$$

where $\omega_A^j \eta_A^{r-j}$ is the class introduced in (1.4.6), belongs to $H^0(C, \underline{\omega}^r \otimes \Omega_C^1) \otimes \mathrm{Sym}^r H^1_{\mathrm{dR}}(A)$ and can thus be viewed, via Proposition 2.5, as an element of the middle step Fil^{r+1} in the Hodge filtration of $\epsilon_X H^{2r+1}_{\mathrm{dR}}(X_r/F)$.

2.3. **Definition of the cycles.** In this section we will assume the Heegner hypothesis 1.9 that was discussed in Section 1.4. As in Section 1.4, fix once and for all a Γ-level structure t_A on A, in such a way that t_A belongs to $A[\mathfrak{N}]$.

The datum (A, t_A) determines a point P_A on C, and a canonical embedding ι_A of A^r into the fiber in W_r above P_A. More generally, any pair $(\varphi, A') \in \mathrm{Isog}^{\mathfrak{N}}(A)$ determines a point $P_{A'}$ on C attached to the pair $(A', \varphi(t_A))$, and an embedding

$$\iota_{A'} : (A')^r \longrightarrow W_r$$

defined over F.

We associate to any $(\varphi, A') \in \mathrm{Isog}^{\mathfrak{N}}(A)$ a codimension $r+1$ cycle Υ_φ on X_r by letting $\mathrm{Graph}(\varphi) \subset A \times A'$ denote the graph of φ, and setting

$$\Upsilon_\varphi \quad := \quad \mathrm{Graph}(\varphi)^r \subset (A \times A')^r \xrightarrow{\simeq} (A')^r \times A^r \subset W_r \times A^r,$$

where the last inclusion is induced from the pair $(\iota_{A'}, \mathrm{id}_A^r)$. We then set

$$\Delta_\varphi := \epsilon_X \Upsilon_\varphi,$$

where ϵ_X is the idempotent given in equation (2.2.1), viewed as an element of the ring of algebraic correspondences from X_r to itself. Note that Δ_φ is supported on the fiber $\pi_r^{-1}(P_{A'})$ of π_r above $P_{A'}$ and gives an element in $\mathrm{CH}^{r+1}(X_r)_{\mathbf{Q}}$, the Chow group of codimension $r+1$ cycles with rational coefficients.

Remark 2.6. The generalised Heegner cycles Δ_φ are all defined over abelian extensions of K. More precisely, if (φ, A') belongs to $\mathrm{Isog}_c^{\mathfrak{N}}(A)$, then the associated cycles can be defined over the compositum of the abelian extension $\tilde{H}/K$ over which the isomorphism class of (A, t_A) is defined with the ring class field H_c of conductor c.

When $r = 0$, the generalised Heegner cycle Δ_φ is a CM point on the modular curve C. In this case, we replace Δ_φ by $\Delta_\varphi - \infty$, where ∞ is any cusp, in order to make Δ_φ homologically trivial. The same is true when $r \geq 1$, by Proposition 2.4 which implies that $\epsilon_X H^{2r+2}(X, \mathbf{Q}) = 0$. Thus we record the following:

Proposition 2.7. *The cycle Δ_φ is homologically trivial on X_r.*

Remark 2.8. Another approach to proving the homological triviality of Δ_φ, by deforming these cycles to the fibers supported above the cusps of the modular curve, is described in [Sch]. The approach we have given adapts more readily to the setting of Shimura curves attached to arithmetic subgroups of $\mathbf{SL}_2(\mathbf{R})$ with compact quotient.

2.4. **Relation with Heegner cycles and L-series.** This motivational section discusses the relation between generalised Heegner cycles and the more classical Heegner cycles on Kuga-Sato varieties that are studied in [Ne2] and [Zh], as well as the expected relation with derivatives of L-series.

Keeping the same notations as in the previous section, the "traditional" Heegner cycles are codimension $r+1$ cycles on the Kuga-Sato variety W_{2r} which are supported on fibers for the natural projection to the modular curve C. These cycles are indexed by elliptic curves with Γ-level structure having endomorphisms by an order in an imaginary quadratic field. More precisely, if A' is an elliptic curve with endomorphism by the order $\mathcal{O}_c = \mathbf{Z}[(d + \sqrt{-d})/2]$ of conductor c of the imaginary quadratic field K, then we set

$$\Upsilon_{A'}^{\mathrm{heeg}} := \mathrm{graph}(\sqrt{-d})^r \subset (A' \times A')^r,$$

$$\Delta_{A'}^{\mathrm{heeg}} := \epsilon_W(\Upsilon_{A'}^{\mathrm{heeg}}).$$

We will now construct an explicit correspondence from the $(4r+1)$-dimensional variety X_{2r} to the $(2r+1)$-dimensional variety W_{2r} which maps generalised Heegner cycles to Heegner cycles.

Let $\Pi = W_{2r} \times A^r$, viewed as a subvariety of $W_{2r} \times X_{2r} = W_{2r} \times W_{2r} \times (A^2)^r$ via the map

$$(\mathrm{id}_{W_{2r}}, \mathrm{id}_{W_{2r}}, (\mathrm{id}_A, \sqrt{-d_K})^r).$$

This subvariety induces a correspondence from X_{2r} to W_{2r}, yielding a map on Chow groups:

$$\Phi_\Pi : \mathrm{CH}^{2r+1}(X_{2r})_{\mathbf{Q}} \longrightarrow \mathrm{CH}^{r+1}(W_{2r})_{\mathbf{Q}}.$$

If $\varphi : A \longrightarrow A'$ is an isogeny of elliptic curves with Γ-level structure, a direct calculation (which will not be used in the sequel, and is therefore left to the reader) shows that the cycles $\Phi_\Pi(\Delta_\varphi)$ and $\Delta_{A'}^{\mathrm{heeg}}$ generate the same $\mathbf{Q}$-subspace of $\mathrm{CH}^{r+1}(W_{2r})_Q$.

This relation shows that the generalised Heegner cycles carry at least as much information as the classical Heegner cycles on Kuga-Sato varieties studied in [Ne2] and [Zh]. One expects them to enjoy similar relationships with central critical derivatives of Rankin L-series. More precisely, we expect that the Arakelov heights of the generalized Heegner cycles Δ_φ should encode the derivatives $L'(f,\chi^{-1},0)$ where χ are Hecke characters of infinity type $(k-1-j, 1+j)$ with $0 \le j \le r$. The case $r=0$ corresponds to the classical Gross-Zagier formula, and the case where r is even and $j = r/2$ corresponds to the setting that is treated in [Zh]. We expect that there should also be a generalisation of the p-adic result of [Ne2] expressing the p-adic height of generalised Heegner cycles in terms of the derivative in the cyclotomic direction of a two variable p-adic L-function attached to f and χ, at a point which corresponds to the special value $L(f,\chi^{-1},0)$ and lies in the range of classical interpolation defining this p-adic L-function.

The present article avoids height calculations altogether by focusing instead on the images of generalised Heegner cycles under Abel-Jacobi maps, both complex and p-adic. In the p-adic setting, we will relate these images to the special values of an anticyclotomic p-adic L-function attached to f and K at a point lying *outside* its range of classical interpolation.

3. p-adic Abel-Jacobi maps

The goal of this section is to compute the images of the generalised Heegner cycles Δ_φ under the p-adic Abel-Jacobi map. The resulting formulae of Sections 3.7 and 3.8 are a key ingredient in the proof of our p-adic Gross-Zagier formula. Some of the techniques used in this chapter, particularly those of Sections 3.1–3.4, are drawn from [IS], which treats the case of Heegner cycles on the r-fold product of the universal "fake" elliptic curve over a Shimura curve attached to a quaternion algebra which is ramified at p. This Shimura curve admits an explicit description as a rigid analytic quotient of the p-adic upper half-plane, via the Cerednik-Drinfeld theory of p-adic uniformisation of Shimura curves. The present chapter treats classical modular curves at primes p of good reduction, for which no p-adic uniformisation à la Cerednik-Drinfeld is available. The techniques employed in Section 3.5 onwards therefore differ markedly from those of [IS].

3.1. **The étale Abel-Jacobi map.** Recall the generalised Heegner cycle Δ_φ associated to the pair $(\varphi, A') \in \mathrm{Isog}_c(A)$ where $\varphi : (A,t) \to (A',t')$ is an isogeny of elliptic curves with Γ-level structure. Let $P = P_{A'}$ be the point of C associated to the pair (A',t') and let

$$X_P := \pi_r^{-1}P, \qquad X_r^\flat := X_r - X_P.$$

Fix any field F over which the pair (X_r, Δ_φ) is defined, and a rational prime p. Consider the following Gysin sequence in p-adic étale cohomology (cf. Corollary 16.2 of [Mi]). After setting $X = \bar{X}_r$, $Z = \bar{X}_P$, $U = \bar{X}_r^\flat$ and $\mathcal{F} = \mathbf{Q}_p(r+1)$ in the statement of that corollary (with r replaced by $2r$), we obtain the following exact sequence in the category Rep_F of continuous p-adic representations of $G_F = \mathrm{Gal}(\bar{F}/F)$:

$$H^{2r-1}_{\mathrm{et}}(\bar{X}_P, \mathbf{Q}_p)(r) \longrightarrow H^{2r+1}_{\mathrm{et}}(\bar{X}_r, \mathbf{Q}_p)(r{+}1) \longrightarrow H^{2r+1}_{\mathrm{et}}(\bar{X}_r^\flat, \mathbf{Q}_p)(r{+}1) \longrightarrow H^{2r}_{\mathrm{et}}(\bar{X}_P, \mathbf{Q}_p)(r)_0 \longrightarrow 0, \tag{3.1.1}$$

where

$$H^{2r}_{\mathrm{et}}(\bar{X}_P, \mathbf{Q}_p)(r)_0 := \ker\left(H^{2r}_{\mathrm{et}}(\bar{X}_P, \mathbf{Q}_p)(r) \longrightarrow H^{2r+2}_{\mathrm{et}}(\bar{X}_r, \mathbf{Q}_p)(r{+}1)\right).$$

By applying the projector ϵ_X to (3.1.1), we obtain

$$0 \longrightarrow \epsilon_X H^{2r+1}_{\mathrm{et}}(\bar{X}_r, \mathbf{Q}_p)(r{+}1) \longrightarrow \epsilon_X H^{2r+1}_{\mathrm{et}}(\bar{X}_r^\flat, \mathbf{Q}_p)(r{+}1) \longrightarrow \epsilon_X H^{2r}_{\mathrm{et}}(\bar{X}_P, \mathbf{Q}_p)(r) \longrightarrow 0, \tag{3.1.2}$$

where we have used the fact that, when $r > 0$,

$$\epsilon_X H^{2r-1}_{\mathrm{et}}(X_P)(r) = 0, \qquad \epsilon_X H^{2r}_{\mathrm{et}}(X_P)(r)_0 = \epsilon_X H^{2r}_{\mathrm{et}}(X_P)(r).$$

Since $\Delta = \Delta_\varphi$ is equal to $\epsilon_X \Delta_\varphi$ by definition, its image under the étale cycle class map

$$\mathrm{cl}_P : \mathrm{CH}^r(X_P)_{\mathbf{Q}}(F) \longrightarrow H^{2r}_{\mathrm{et}}(\bar{X}_P, \mathbf{Q}_p)(r)$$

belongs to $\epsilon_X H^{2r}_{\rm et}(\bar{X}_P, \mathbf{Q}_p)(r)$. Let

$$\mathrm{cl}_\Delta : \mathbf{Q}_p \longrightarrow \epsilon_X H^{2r}_{\rm et}(\bar{X}_P, \mathbf{Q}_p)(r)$$

be the map of p-adic representations of G_F defined by $\mathrm{cl}_\Delta(1) = \mathrm{cl}_P(\Delta)$, and consider the extension V_Δ of $\mathbf{Q}_p$ by $\epsilon_X H^{2r+1}_{\rm et}(\bar{X}_r, \mathbf{Q}_p)(r{+}1)$ arising from pullback in the following commutative diagram with exact rows in which the right-most square is cartesian:

(3.1.3)

$$\begin{tikzcd}
0 \arrow[r] & \epsilon_X H^{2r+1}_{\rm et}(\bar{X}_r, \mathbf{Q}_p)(r{+}1) \arrow[r] \arrow[d, equal] & V_\Delta \arrow[r] \arrow[d] & \mathbf{Q}_p \arrow[r] \arrow[d, "\mathrm{cl}_\Delta"] & 0 \\
0 \arrow[r] & \epsilon_X H^{2r+1}_{\rm et}(\bar{X}_r, \mathbf{Q}_p)(r{+}1) \arrow[r] & \epsilon_X H^{2r+1}_{\rm et}(\bar{X}_r^\flat, \mathbf{Q}_p)(r{+}1) \arrow[r] & \epsilon_X H^{2r}_{\rm et}(\bar{X}_P, \mathbf{Q}_p)(r) \arrow[r] & 0.
\end{tikzcd}$$

Given two objects V'', V' in the category Rep_F, write

$$\mathrm{Ext}_F(V'', V') := H^1(F, \hom(V'', V'))$$

for the set of isomorphism classes of extensions

$$0 \longrightarrow V' \longrightarrow E \longrightarrow V'' \longrightarrow 0.$$

(Here $H^1(F, -)$ denotes continuous Galois cohomology and $\hom(V'', V')$ is the object of Rep_F equipped with the natural action of G_F.)

Definition 3.1. The étale Abel-Jacobi map

$$\mathrm{AJ}^{\rm et}_F : \mathrm{CH}^{r+1}(X_r)_{0,\mathbf{Q}}(F) \longrightarrow H^1(F, \epsilon_X H^{2r+1}_{\rm et}(\bar{X}_r, \mathbf{Q}_p(r{+}1)))$$

sends the class of the null-homologous codimension-$(r{+}1)$ cycle Δ to the isomorphism class of the extension V_Δ of (3.1.3) in

$$\mathrm{Ext}_F(\mathbf{Q}_p, \epsilon_X H^{2r+1}_{\rm et}(\bar{X}_r, \mathbf{Q}_p)(r{+}1)) = H^1(F, \epsilon_X H^{2r+1}_{\rm et}(\bar{X}_r, \mathbf{Q}_p)(r{+}1)).$$

Remark 3.2. Definition 3.1 applies directly to cycle classes in $\mathrm{CH}^{r+1}(X_r)_{0,\mathbf{Q}}(F)$ which are represented by a cycle supported on X_P. Usually, the map $\mathrm{AJ}^{\rm et}_F$ is defined on a general cycle Δ by replacing in the diagrams above X_P by Δ and $X^\flat$ by $X - \Delta$, respectively. In this case, one obtains an analogue of the commutative diagram (3.1.3) without the need of applying ϵ_X. It can be checked, following the argument that is explained in [Ne2, Prop. II.2.4] that this more general definition, once composed with ϵ_X, is compatible with Definition 3.1, which is adapted to our subsequent calculations.

3.2. **The comparison isomorphism.** The p-adic Abel-Jacobi map arises from the map $\mathrm{AJ}^{\rm et}_F$ by considering the case where F is a finite extension of $\mathbf{Q}_p$. Let $\mathcal{O}_F$ denote the ring of integers of F and let k be its residue field. We will make the following assumptions on F, which are satisfied in our application:

(1) The extension F is a finite unramified extension of $\mathbf{Q}_p$.
(2) The varieties C and X_r over F extend to smooth proper models $\mathcal{C}$ and $\mathcal{X}_r$ over $\mathcal{O}_F$.

If φ belongs to $\mathrm{Isog}^{\mathfrak{N}}_c(A)$, and p does not divide $Nc\,\mathrm{Disc}(K)$, then the field F can be taken to be the p-adic completion of the compositum of $\tilde{H}$, the extension of the Hilbert class field of K over which $A[\mathfrak{N}]$ is defined, with H_c, the Hilbert class field of conductor c. By abuse of notation, we will use the same letter σ to denote the p-power Frobenius automorphism of k and its canonical lift to F.

The de Rham cohomology groups $H^j_{\rm dR}(X_r/F)$, equipped with their σ-semilinear Frobenius endomorphisms and Hodge filtrations, are examples of *filtered Frobenius modules.* (See [B], [Fo], [I] or [FI] for details concerning the category of these objects.)

The fundamental *comparison theorem* between p-adic étale cohomology and de Rham cohomology of varieties over p-adic fields relates the p-adic representation $H^j_{\rm et}(\bar{X}_r, \mathbf{Q}_p)$ of G_F to the filtered Frobenius module $H^j_{\rm dR}(X_r/F)$. To any continuous p-adic representation V of G_F we may associate the F-vector space

$$D_{\rm cris}(V) := (V \otimes_{\mathbf{Q}_p} B_{\rm cris})^{G_F},$$

where $B_{\rm cris}$ is Fontaine's ring of cristalline periods over F, which is called the *cristalline Dieudonné module* attached to V. Recall that a p-adic representation V of G_F is said to be *cristalline* if

$$\dim_F D_{\rm cris}(V) = \dim_{\mathbf{Q}_p}(V).$$

The category of cristalline representations of G_F is an abelian tensor subcategory of Rep_F. Given objects V_1 and V_2 of this category, denote by $\mathrm{Ext}_{\mathrm{cris}}(V_1, V_2)$ the group of extensions of V_2 by V_1 which are cristalline. The Dieudonné module attached to a cristalline representation V inherits from B_{cris} the structure of a filtered Frobenius module. The following deep theorem will be used to make the p-adic Abel-Jacobi map amenable to computation.

Theorem 3.3 (Faltings). *The p-adic representation $H^{2r+1}_{\mathrm{et}}(\bar{X}_r, \mathbf{Q}_p)(r+1)$ is cristalline, and there is a canonical, functorial isomorphism of filtered Frobenius modules:*

$$D_{\mathrm{cris}}(H^{2r+1}_{\mathrm{et}}(\bar{X}_r, \mathbf{Q}_p)(r+1)) = H^{2r+1}_{\mathrm{dR}}(X_r/F)(r+1).$$

Proof. See [Fa], Theorem 5.6, or [T]. □

The comparison theorem will be applied via the following corollary:

Corollary 3.4. *The assignment $V \mapsto D_{\mathrm{cris}}(V)$ induces an isomorphism*

$$\mathrm{comp} : \mathrm{Ext}_{\mathrm{cris}}(\mathbf{Q}_p, H^{2r+1}_{\mathrm{et}}(\bar{X}_r, \mathbf{Q}_p)(r+1)) \xrightarrow{\sim} \mathrm{Ext}_{\mathrm{ffm}}(F, H^{2r+1}_{\mathrm{dR}}(X_r/F)(r+1)). \tag{3.2.1}$$

Proof. The injectivity follows from from the comparison theorem and the fact that the functor D_{cris} is fully faithful, while the surjectivity follows from a comparison with the Bloch-Kato exponential, as in Prop. 1.21 and Cor. 1.22 of [Ne1]. □

3.3. **Extensions of filtered Frobenius modules.** We will now give a general abstract description of the group of extensions in the category of filtered Frobenius modules.

Let H be a filtered Frobenius module of strictly negative weight, and consider an extension

$$0 \longrightarrow H \xrightarrow{i} E \xrightarrow{\rho} F \longrightarrow 0 \tag{3.3.1}$$

of filtered Frobenius modules. Let η_E^{hol} and η_E^{frob} be elements of $\mathrm{Fil}^0 E$ and $E^{\phi^n=1}$ respectively, satisfying

$$\rho(\eta_E^{\mathrm{hol}}) = 1, \qquad \rho(\eta_E^{\mathrm{frob}}) = 1. \tag{3.3.2}$$

The element

$$\eta_E := \eta_E^{\mathrm{hol}} - \eta_E^{\mathrm{frob}}$$

is in the kernel of ρ and hence can be viewed as an element of H. The lifts η_E^{hol} and η_E^{frob} are well-defined up to $\mathrm{Fil}^0 H$ and $H^{\phi^n=1}$ respectively. By the assumption on the weight of H, we have $H^{\phi^n=1} = 0$, and the class of η_E in $H/\mathrm{Fil}^0 H$ does not depend on the choices that were made in (3.3.2). The reader should compare the following proposition with Lemma 2.1 of Section 2 of [IS], which treats the more complicated situation of extensions of filtered Frobenius monodromy modules arising from semistable (and not necessarily cristalline) p-adic representations of G_F.

Proposition 3.5. *The assignment $E \mapsto \eta_E$ yields an isomorphism*

$$\mathrm{Ext}_{\mathrm{ffm}}(F, H) = H/\mathrm{Fil}^0 H.$$

Sketch of proof. The isomorphism $E^{\phi^n=1} \longrightarrow F$ induced by ρ determines a canonical vector space splitting of (3.3.1) which preserves the ϕ-module structure of the extension, but need not respect with the filtrations. In other words, the extension (3.3.1) is trivial when viewed as an extension of ϕ-modules. Fix the resulting identification

$$E = H \oplus F, \tag{3.3.3}$$

so that η_E^{frob} is identified with the element $(0, 1)$ of $H \oplus F$. We are left with the problem of classifying the filtrations which may arise on the splitting of ϕ-modules (3.3.3). This splitting is compatible with filtrations if and only if $\eta_E^{\mathrm{hol}} = (h, 1)$ is such that h belongs to $\mathrm{Fil}^0 H$ (since in this case $\mathrm{Fil}^0 E = \mathrm{Fil}^0 H \oplus F$ and this equality determines the filtration on E in all degrees). In general, the datum $\eta_E^{\mathrm{hol}} = (h, 1)$ completely determines the filtration on E in terms of the filtration on H (since $\mathrm{Fil}^0 E = \mathrm{span}(\mathrm{Fil}^0 H, \eta_E^{\mathrm{hol}})$), and $(h, 1)$ and $(h', 1)$ give rise to the same filtration if and only if $h - h'$ belongs to $\mathrm{Fil}^0 H$. □

3.4. **The p-adic Abel-Jacobi map.** We can now define the p-adic Abel-Jacobi map attached to the p-adic field F introduced in Section 3.2. By Theorem 3.1.1. of [Ne3] (see also [Ni]), the image of $\mathrm{CH}^{r+1}(X_r)_{0,\mathbf{Q}}(F)$ by the étale Abel-Jacobi map $\mathrm{AJ}_F^{\mathrm{et}}$ is contained in the subgroup

$$H^1_f(F, \epsilon_X H^{2r+1}_{\mathrm{et}}(\bar{X}_r, \mathbf{Q}_p)(r{+}1)) := \mathrm{Ext}_{\mathrm{cris}}(\mathbf{Q}_p, \epsilon_X H^{2r+1}_{\mathrm{et}}(\bar{X}_r, \mathbf{Q}_p)(r{+}1))$$

of $H^1(F, \epsilon_X H^{2r+1}_{\mathrm{et}}(\bar{X}_r, \mathbf{Q}_p)(r{+}1))$ whose elements correspond to cristalline extensions. By Corollary 3.4, this group is identified with $\mathrm{Ext}_{\mathrm{ffm}}(F, \epsilon_X H^{2r+1}_{\mathrm{dR}}(X_r/F)(r+1))$. Applying Proposition 3.5 to the filtered Frobenius module $H = \epsilon_X H^{2r+1}_{\mathrm{dR}}(X_r/F)(r{+}1)$ which is of weight -1, we find an isomorphism

$$(3.4.1)\quad J : \mathrm{Ext}_{\mathrm{ffm}}(F, \epsilon_X H^{2r+1}_{\mathrm{dR}}(X_r/F)(r{+}1)) \longrightarrow \frac{\epsilon_X H^{2r+1}_{\mathrm{dR}}(X_r/F)(r{+}1)}{\mathrm{Fil}^0\, \epsilon_X H^{2r+1}_{\mathrm{dR}}(X_r/F)(r{+}1)} = \mathrm{Fil}^1\, \epsilon_X H^{2r+1}_{\mathrm{dR}}(X_r/F)(r)^\vee,$$

where the last identification arises from the Poincaré duality

$$\epsilon_X H^{2r+1}_{\mathrm{dR}}(X_r/F)(r) \times \epsilon_X H^{2r+1}_{\mathrm{dR}}(X_r/F)(r{+}1) \longrightarrow F,$$

in which the spaces $\mathrm{Fil}^1\, \epsilon_X H^{2r+1}_{\mathrm{dR}}(X_r/F)(r)$ and $\mathrm{Fil}^0\, \epsilon_X H^{2r+1}_{\mathrm{dR}}(X_r/F)(r{+}1)$ are exact annihilators of each other.

The p-adic Abel-Jacobi map, denoted AJ_F, is the diagonal map in the diagram

$$\begin{tikzcd}
\mathrm{CH}^{r+1}(X_r)_{0,\mathbf{Q}}(F) \arrow[r, "\mathrm{AJ}_F^{\mathrm{et}}"] \arrow[dddr, bend right, "\mathrm{AJ}_F"'] & H^1_f(F, \epsilon_X H^{2r+1}_{\mathrm{et}}(\bar{X}_r, \mathbf{Q}_p)(r{+}1)) \arrow[d, equal] \\
& \mathrm{Ext}_{\mathrm{cris}}(\mathbf{Q}_p, \epsilon_X H^{2r+1}_{\mathrm{et}}(\bar{X}_r, \mathbf{Q}_p)(r{+}1)) \arrow[d, "\mathrm{comp}"] \\
& \mathrm{Ext}_{\mathrm{ffm}}(F, \epsilon_X H^{2r+1}_{\mathrm{dR}}(X_r)(r{+}1)) \arrow[d, "J"] \\
& (\mathrm{Fil}^1\, \epsilon_X H^{2r+1}_{\mathrm{dR}}(X_r/F)(r))^\vee,
\end{tikzcd}$$

where the second vertical isomorphism is given in (3.2.1).

After invoking Proposition 2.5, we can view AJ_F as a map

$$(3.4.2)\qquad \mathrm{AJ}_F : \mathrm{CH}^{r+1}(X_r)(F)_{0,\mathbf{Q}} \longrightarrow (S_{r+2}(\Gamma, F) \otimes \mathrm{Sym}^r\, H^1_{\mathrm{dR}}(A/F))^\vee.$$

Further, applying the comparison isomorphisms to the diagram (3.1.3) gives a corresponding diagram of filtered Frobenius modules:

$$(3.4.3)\quad \begin{tikzcd}
0 \arrow[r] & \epsilon_X H^{2r+1}_{\mathrm{dR}}(X_r/F)(r{+}1) \arrow[r] \arrow[d, equal] & D_\Delta \arrow[r] \arrow[d] & F \arrow[r] \arrow[d, "\mathrm{cl}_\Delta"] & 0 \\
0 \arrow[r] & \epsilon_X H^{2r+1}_{\mathrm{dR}}(X_r/F)(r{+}1) \arrow[r] & \epsilon_X H^{2r+1}_{\mathrm{dR}}(X_r^\flat/F)(r{+}1) \arrow[r] & \epsilon_X H^{2r}_{\mathrm{dR}}(X_P/F)(r) \arrow[r] & 0.
\end{tikzcd}$$

By Proposition 2.4 (and an analogue with C replaced by $C - \{P\}$), this diagram can be rewritten as

$$(3.4.4)\quad \begin{tikzcd}
0 \arrow[r] & H^1_{\mathrm{par}}(C, \mathcal{L}_{r,r}, \nabla)(r{+}1) \arrow[r] \arrow[d, equal] & D_\Delta \arrow[r] \arrow[d] & F \arrow[r] \arrow[d, "\mathrm{cl}_\Delta"] & 0 \\
0 \arrow[r] & H^1_{\mathrm{par}}(C, \mathcal{L}_{r,r}, \nabla)(r{+}1) \arrow[r] & H^1_{\mathrm{par}}(C - \{P\}, \mathcal{L}_{r,r}, \nabla)(r{+}1) \arrow[r] & \mathcal{L}_{r,r}(P)(r) \arrow[r] & 0.
\end{tikzcd}$$

The image of the cycle class Δ under the p-adic Abel-Jacobi map is thus described by the class of the extension D_Δ in the category of filtered Frobenius modules.

3.5. **de Rham cohomology over p-adic fields.** In this section we give an explicit description of the action of the Frobenius operator on

$$\epsilon_X H^{2r+1}_{\mathrm{dR}}(X_r/F) = H^1_{\mathrm{par}}(C, \mathcal{L}_{r,r}, \nabla)$$

in terms of $\mathcal{L}_{r,r}$-valued rigid analytic differentials on appropriate subsets of the curve C. The reader is referred to [Col1] and [Col2] for more details on the concepts and definitions discussed below.

Viewing C as a rigid analytic space over F, let $\mathcal{O}_C^{\mathrm{rig}}$ denote the sheaf of rigid analytic functions on C and let $\mathcal{L}_{r,r}^{\mathrm{rig}}$ denote the rigid analytic coherent sheaf on C associated to $\mathcal{L}_{r,r}$.

We will now define certain basic affinoid subsets of C for the rigid analytic topology. For this, recall from Section 3.2 that $\mathcal{C}$ is a smooth proper model of C over $\mathrm{Spec}(\mathcal{O}_F)$. Write $C_k := \mathcal{C} \times_{\mathcal{O}_F} k$, and let

$$\mathrm{red}_p : C(\mathbf{C}_p) \longrightarrow C_k(\bar{k})$$

denote the natural reduction map.

Let $P_1, \ldots, P_t$ be any collection of points on $C(F)$ which map to distinct points of $C_k(k)$ under red_p and contain all the cusps of C. Recall that the residue disc attached to P_j, denoted $D(P_j)$, is the set of points of $C(\mathbf{C}_p)$ which have the same image as P_j under red_p. Let

$$\mathcal{A} = C(\mathbf{C}_p) - D(P_1) - \cdots - D(P_t).$$

Because the P_j reduce to smooth points of $C_k(k)$, the residue discs $D(P_j)$ are conformal to the open unit disc $U \subset \mathbf{C}_p$ consisting of $z \in \mathbf{C}_p$ with $|z| < 1$. For each $j = 1, \ldots, t$, fix an isomorphism $h_j : D(P_j) \longrightarrow U$ sending P_j to 0. Given a rational number $r_j < 1$, we then let

$$D[P_j, r_j] = \{z \in D(P_j) \text{ such that } |h_j(z)| \leq r_j\}$$

denote the "closed disc of radius r_j in $D(P_j)$". Finally, fixing a collection of rational numbers $r_1, \ldots, r_t$ with $0 < r_j < 1$, we write

$$\begin{aligned} \mathcal{W} &= C(\mathbf{C}_p) - D[P_1, r_1] - \cdots - D[P_t, r_t] \\ &= \mathcal{A} \cup \mathcal{V}_1 \cup \cdots \cup \mathcal{V}_t, \end{aligned}$$

where

$$\mathcal{V}_j := \mathcal{V}(P_j, r_j, 1) := \{z \in D(P_j) \text{ such that } r_j < |h_j(z)| < 1\}.$$

Define the positive *orientation* of the *annulus* $\mathcal{V}_j$ by choosing the subset $\{z \in D(P_j) \text{ such that } |h_j(z)| \leq r_j\}$ of its complement.

The set $\mathcal{A}$ is an example of an *affinoid* subset of $C(\mathbf{C}_p)$ with good reduction, while the set $\mathcal{W}$ is an example of a *wide open neighborhood* of the affinoid $\mathcal{A}$. The set $\mathcal{V}_j$ is called a *wide open annulus* around the point P_j. The wide open space $\mathcal{W}$ is thus obtained by adjoining to $\mathcal{A}$ a finite union of open annuli about the boundaries of the deleted residue discs. For general definitions and a more systematic discussion of these concepts, see for example Sections II and III of [Col2].

Because $\mathcal{W}$ is contained in $C^0(\mathbf{C}_p)$, the Gauss-Manin connection (1.1.3) gives rise to a rigid analytic connection

$$\nabla : \mathcal{L}_{r,r}^{\mathrm{rig}} \longrightarrow \mathcal{L}_{r,r}^{\mathrm{rig}} \otimes \Omega^1_{\mathcal{W}}.$$

The *de Rham cohomology* $H^1_{\mathrm{dR}}(\mathcal{W}, \mathcal{L}_{r,r}^{\mathrm{rig}}, \nabla)$ is defined to be the quotient

$$H^1_{\mathrm{dR}}(\mathcal{W}, \mathcal{L}_{r,r}^{\mathrm{rig}}, \nabla) := \frac{\mathcal{L}_{r,r}^{\mathrm{rig}}(\mathcal{W}) \otimes \Omega^1_{\mathcal{W}}}{\nabla \mathcal{L}_{r,r}^{\mathrm{rig}}(\mathcal{W})}.$$

A meromorphic $\mathcal{L}_{r,r}$-valued differential on C which is regular on $C - \{P_1, \ldots, P_t\}$ can be viewed as a rigid section of $\mathcal{L}_{r,r} \otimes \Omega^1_C$ over $\mathcal{W}$. In this way one obtains by restriction a natural map from the algebraic de Rham cohomology over $\mathbf{C}_p$ to the rigid de Rham cohomology.

Theorem 3.6. *The natural restriction map*

$$H^1_{\mathrm{dR}}(C - \{P_1, \ldots, P_t\}, \mathcal{L}_{r,r}, \nabla) \longrightarrow H^1_{\mathrm{dR}}(\mathcal{W}, \mathcal{L}_{r,r}^{\mathrm{rig}}, \nabla)$$

is an isomorphism.

Proof. In the case $r = 0$, this is Theorem 4.2 of [Col2]. The proof in the general case follows from a similar argument, as explained in the proof of Proposition 10.3 of [Col3]. ☐

A set $\mathcal{W}'$ of the form

$$\mathcal{W}' = C(\mathbf{C}_p) - D[P_1, r_1'] - \cdots - D[P_t, r_t'], \qquad \text{with } r_j < r_j' < 1$$

is called a *wide open neighborhood* of the affinoid $\mathcal{A}$ in $\mathcal{W}$. The following is an immediate corollary of Theorem 3.6.

Corollary 3.7. *Let $\mathcal{W}'$ be any wide open neighborhood of $\mathcal{A}$ in $\mathcal{W}$. The natural map*

$$res_{\mathcal{W},\mathcal{W}'} : H^1_{\mathrm{dR}}(\mathcal{W}, \mathcal{L}^{\mathrm{rig}}_{r,r}, \nabla) \longrightarrow H^1_{\mathrm{dR}}(\mathcal{W}', \mathcal{L}^{\mathrm{rig}}_{r,r}, \nabla)$$

induced by restriction is an isomorphism.

We want to describe the image of $H^1_{\mathrm{dR}}(C, \mathcal{L}_{r,r}, \nabla)$ in $H^1_{\mathrm{dR}}(\mathcal{W}, \mathcal{L}^{\mathrm{rig}}_{r,r}, \nabla)$. For this, we recall the notion of the p-adic annular residue

$$\mathrm{res}_{\mathcal{V}_j}(\omega) \in (H^0(\mathcal{V}_j, \mathcal{L}^{\mathrm{rig}}_{r,r})^{\nabla=0})^\vee$$

of a $\mathcal{L}^{\mathrm{rig}}_{r,r}$-valued one-differential form ω on $\mathcal{W}$. It is defined by the formula

$$\mathrm{res}_{\mathcal{V}_j}(\omega)(\alpha) = \mathrm{res}_{\mathcal{V}_j}\langle \alpha, \omega\rangle, \qquad \text{for all } \alpha \in H^0(\mathcal{V}_j, \mathcal{L}^{\mathrm{rig}}_{r,r})^{\nabla=0},$$

where the residue on the right hand side is the usual p-adic annular residue of the rigid analytic one-form $\langle \alpha, \omega\rangle$ on the oriented annulus $\mathcal{V}_j$, as it is defined in Section II of [Col2] for example.

By Proposition 3.1.2 of [Ka3], the sheaf $\mathcal{L}_{r,r}$ admits a basis of horizontal sections on each non-cuspidal residue disc $D(P_j)$, so that the target of the residue map on the corresponding annulus is identified with

$$(H^0(\mathcal{V}_j, \mathcal{L}^{\mathrm{rig}}_{r,r})^{\nabla=0})^\vee = (H^0(D(P_j), \mathcal{L}_{r,r})^{\nabla=0})^\vee = \mathcal{L}_{r,r}(P_j)^\vee = \mathcal{L}_{r,r}(P_j),$$

where the last identification arises from the self-duality on $\mathcal{L}_{r,r}(P_j)$. We will always view the residue map on a non-cuspidal residue disc as taking values on $\mathcal{L}_{r,r}(P_j)$, so that for all $\alpha \in \mathcal{L}_{r,r}(P_j)$ one has

$$\langle \alpha, \mathrm{res}_{\mathcal{V}_j}(\omega)\rangle = \mathrm{res}_{\mathcal{V}_j}\langle \alpha^\nabla, \omega\rangle,$$

where α^∇ is the unique horizontal section on $D(P_j)$ satisfying $\alpha^\nabla(P_j) = \alpha$.

On the cuspidal residue disc of the cusp P attached to the pair $(\mathrm{Tate}(q), t)$, the space of horizontal sections of $\mathcal{L}_r$ is one-dimensional and generated by the local section ξ^r_{can}. One therefore has

$$\mathrm{res}_{\mathcal{V}_j}\left(\left(\sum_{j=0}^{r} a_j(q)\omega^j_{\mathrm{can}}\xi^{r-j}_{\mathrm{can}}\right)\frac{dq}{q}\right)(b\xi^r_{\mathrm{can}}) = \mathrm{res}_{q=0}\left(ba_r(q)\frac{dq}{q}\right) = ba_r(0).$$

Note that if ω is any global section of $\mathcal{L}_{r,r} \otimes \Omega^1_C$ over $C - \{P_1, \ldots, P_t\}$, it can also be viewed as a rigid section over $\mathcal{W}$, and

(3.5.1) $$\mathrm{res}_{\mathcal{V}_j}\,\omega = \mathrm{res}_{P_j}\,\omega.$$

If P_j is not a cusp, the residue $\mathrm{res}_{P_j}\,\omega$ that appears on the right of this formula satisfies

$$\langle G(P_j), \mathrm{res}_{P_j}\,\omega\rangle = \mathrm{res}_{P_j}\langle G, \omega\rangle.$$

In this formula, G can be taken to be any regular (not necessarily horizontal) section of $\mathcal{L}_{r,r}$ over $D(P_j)$, and the residue on the right is the residue at P_j of the differential $\langle G, \omega\rangle$ on $D(P_j) - \{P_j\}$.

The following rigid-analytic analogue of the classical residue theorem for meromorphic differentials on curves (cf. for example [Col2]) will play an important role in the calculations of the next section.

Theorem 3.8. *If $\omega \in \Omega^1_{\mathcal{W}}$ is a rigid analytic one-form on $\mathcal{W}$, then*

$$\sum_{j=1}^{t} \mathrm{res}_{\mathcal{V}_j}\,\omega = 0.$$

Proposition 3.9. *A class $\kappa \in H^1_{\mathrm{dR}}(\mathcal{W}, \mathcal{L}^{\mathrm{rig}}_{r,r}, \nabla)$ represented by an $\mathcal{L}^{\mathrm{rig}}_{r,r}$-valued differential form ω belongs to the natural image of $H^1_{\mathrm{par}}(C, \mathcal{L}_{r,r}, \nabla)$ under restriction if and only if*

$$\mathrm{res}_{\mathcal{V}_j}(\omega) = 0, \qquad \textit{for } j = 1, \ldots, t.$$

Proof. The Gysin exact sequence applied to the cohomology of the pair of rigid spaces $\mathcal{W} \subset C^0$ shows that

$$H^1_{\mathrm{dR}}(C, \mathcal{L}_{r,r}, \nabla) = \left\{\omega \text{ s.t. } \mathrm{res}_{\mathcal{V}_j}(\omega) = 0 \quad \text{for all non-cuspidal annuli } \mathcal{V}_j\right\}.$$

On the other hand, the definition of $H^1_{\mathrm{par}}(C, \mathcal{L}_{r,r}, \nabla)$ shows that this space is identified with the space of classes in $H^1_{\mathrm{dR}}(C, \mathcal{L}_{r,r}, \nabla)$ represented by $\mathcal{L}_{r,r}$-valued differentials ω satisfying

$$\mathrm{res}_{\mathcal{V}_j}(\omega) = 0, \qquad \text{for all cuspidal annuli } \mathcal{V}_j.$$

The result follows. □

Let κ_1, κ_2 be classes in $H^1_{\mathrm{par}}(C, \mathcal{L}_{r,r}, \nabla)$ and let ω_1, ω_2 be rigid analytic sections of $\mathcal{L}^{\mathrm{rig}}_{r,r} \otimes \Omega^1_C$ over $\mathcal{W}$ representing them. The fact that $\mathrm{res}_{\mathcal{V}_j}(\omega_1) = 0$ on all the annuli $\mathcal{V}_j \subset \mathcal{W}$ allows us to find an analytic solution $F_{\omega_1,j}$ on $\mathcal{V}_j$ to the equation

$$\nabla F_{\omega_1,j} = \omega_1,$$

which is well-defined up to horizontal sections of $\mathcal{L}^{\mathrm{rig}}_{r,r}$ over $\mathcal{V}_j$. Such an $F_{\omega_1,j}$ is called a *local primitive* of ω_1 on $\mathcal{V}_j$. Note that the expression $\mathrm{res}_{\mathcal{V}_j}\langle F_{\omega_1,j}, \omega_2\rangle$ does not depend on the choice of the local primitive $F_{\omega_1,j}$, since ω_2 is of the second kind.

The following proposition expresses the Poincaré duality on $H^1_{\mathrm{par}}(C, \mathcal{L}_{r,r}, \nabla)$ in terms of the residues of rigid $\mathcal{L}_{r,r}$-valued forms on $\mathcal{W}$.

Proposition 3.10. *For all $\kappa_1, \kappa_2 \in H^1_{\mathrm{par}}(C, \mathcal{L}_{r,r}, \nabla)$,*

$$\langle \kappa_1, \kappa_2\rangle = \sum_{j=1}^{t} \mathrm{res}_{\mathcal{V}_j}\langle F_{\omega_1,j}, \omega_2\rangle,$$

where $\omega_1, \omega_2 \in H^1_{\mathrm{dR}}(\mathcal{W}, \mathcal{L}^{\mathrm{rig}}_{r,r}, \nabla)$ are representatives for κ_1 and κ_2 and $F_{\omega_1,j}$ is any local primitive for ω_1 on $\mathcal{V}_j$.

Proof. This follows from Lemma 7.1 of [Col3] combined with equation (3.5.1) comparing the rigid analytic and algebraic residue maps. □

Theorem 3.6 will now be used to give an explicit description of the action of the Frobenius operator on the algebraic de Rham cohomology. Since the points $P_1, \ldots, P_t$ are defined over F, the points $\tilde{P}_j := \mathrm{red}_p(P_j)$ are defined over k and the curve $U_k := C_k - \{\tilde{P}_1, \ldots, \tilde{P}_t\}$ is a smooth affine open subset of C_k. As before, let σ denote the Frobenius automorphism of k which sends x to x^p, and let $U_k^\sigma = U_k \times_\sigma k$. There is a canonical morphism $\phi : U_k \longrightarrow U_k^\sigma$ characterised by

$$\phi^* f^\sigma = f^p, \qquad \text{for all } f \in \mathcal{O}_{C_k}(U_k).$$

Definition 3.11. A morphism

$$\phi_{\mathcal{A}} : \mathcal{A} \longrightarrow \mathcal{A}^\sigma$$

which lifts the canonical Frobenius morphism $U_k \longrightarrow U_k^\sigma$ to characteristic 0 is called a *lifting of Frobenius* for the affinoid $\mathcal{A}$.

A Frobenius lifting always exists under our hypotheses (cf. Corollary 1.1a of [Col1]). Assume from now on that the set $\{\tilde{P}_1, \ldots, \tilde{P}_t\}$ is stable under ϕ, so that $\mathcal{A}^\sigma = \mathcal{A}$.

Definition 3.12. A *Frobenius neighbourhood* of $\mathcal{A}$ in $\mathcal{W}$ is a pair $(\mathcal{W}', \phi)$, where $\mathcal{A} \subset \mathcal{W}' \subset \mathcal{W}$ is a wide open neighborhood of $\mathcal{A}$ in $\mathcal{W}$ and $\phi : \mathcal{W}' \longrightarrow \mathcal{W}$ is a morphism whose restriction to $\mathcal{A}$ is a lifting of Frobenius in the sense of Definition 3.11.

Definition 3.13. An *overconvergent Frobenius isocrystal* on $\mathcal{W}$ is a triple $(\mathcal{L}, \phi, \mathrm{Fr})$, where

(1) $\mathcal{L}$ is a rigid analytic coherent sheaf on $\mathcal{W}$ equipped with a rigid analytic integrable connection

$$\nabla : \mathcal{L} \longrightarrow \mathcal{L} \otimes \Omega^1_{\mathcal{W}};$$

(2) $(\mathcal{W}', \phi)$ is a Frobenius neighborhood of $\mathcal{A}$ in $\mathcal{W}$;
(3) Fr is a horizontal morphism

$$\mathrm{Fr} : \phi^* \mathcal{L} \longrightarrow \mathcal{L}|_{\mathcal{W}'}.$$

The condition that Fr be horizontal amounts to requiring that the diagram

$$\begin{tikzcd} \phi^*\mathcal{L} \arrow[r, "\nabla"] \arrow[d, "\mathrm{Fr}"'] & \phi^*\mathcal{L}\otimes\Omega^1_{\mathcal{W}'} \arrow[d, "\mathrm{Fr}\otimes\mathrm{id}"] \\ \mathcal{L} \arrow[r, "\nabla"] & \mathcal{L}\otimes\Omega^1_{\mathcal{W}'} \end{tikzcd}$$

be commutative.

Given a Frobenius neighborhood $(\mathcal{W}',\phi)$ of $\mathcal{A}$ in $\mathcal{W}$, the canonical functorial action of a lifting of Frobenius on the relative de Rham cohomology $H^{2r}_{\mathrm{dR}}(X_r/C)$ is compatible with the Gauss-Manin connection and gives rise to a horizontal morphism $\mathrm{Fr}:\phi^*\mathcal{L}^{\mathrm{rig}}_{r,r}\longrightarrow\mathcal{L}^{\mathrm{rig}}_{r,r}|_{\mathcal{W}'}$. In this way, the triple $(\mathcal{L}^{\mathrm{rig}}_{r,r|\mathcal{W}},\phi,\mathrm{Fr})$ is equipped with the structure of an overconvergent Frobenius isocrystal.

The action of the p-power Frobenius operator (denoted by the letter Φ_0, to distinguish it from the lifting ϕ of Frobenius on the curve C) on $H^1_{\mathrm{dR}}(\mathcal{W},\mathcal{L}^{\mathrm{rig}}_{r,r},\nabla)$ is then given by the sequence of maps:

$$H^1_{\mathrm{dR}}(\mathcal{W},\mathcal{L}^{\mathrm{rig}}_{r,r},\nabla)\xrightarrow{\phi^*}H^1_{\mathrm{dR}}(\mathcal{W}',\phi^*\mathcal{L}^{\mathrm{rig}}_{r,r},\nabla)\xrightarrow{\mathrm{Fr}}H^1_{\mathrm{dR}}(\mathcal{W}',\mathcal{L}^{\mathrm{rig}}_{r,r},\nabla)\xrightarrow{\sim}H^1_{\mathrm{dR}}(\mathcal{W},\mathcal{L}^{\mathrm{rig}}_{r,r},\nabla),$$

where the last map is the inverse of the restriction $\mathrm{res}_{\mathcal{W},\mathcal{W}'}$ which is an isomorphism by Corollary 3.7. (Cf. the discussion preceding Th. 10.1 of [Col3], or the more detailed discussion in [CI].)

Notice that the operator Φ_0 acting on the group $H^1_{\mathrm{dR}}(\mathcal{W},\mathcal{L}^{\mathrm{rig}}_{r,r},\nabla)$ preserves the natural images of $H^1_{\mathrm{dR}}(C,\mathcal{L}_{r,r},\nabla)$ and of $H^1_{\mathrm{par}}(C,\mathcal{L}_{r,r},\nabla)$. (This follows from Proposition 3.9 for example.) The map Φ_0 on $H^1_{\mathrm{par}}(C,\mathcal{L}_{r,r},\nabla)$ agrees with the Frobenius endomorphism on $\epsilon_X H^{2r+1}_{\mathrm{dR}}(X_r/F)$ via the identification $H^1_{\mathrm{par}}(C,\mathcal{L}_{r,r},\nabla)=\epsilon_X H^{2r+1}_{\mathrm{dR}}(X_r/F)$. It is σ-semilinear. In order to work with an F-linear endomorphism, we set

$$\Phi=\Phi_0^n,\qquad\text{where } n=[F:\mathbf{Q}_p].$$

By abuse of notation, we will also denote by Φ the Frobenius endomorphism acting on the space $H^0_{\mathrm{la}}(C,\mathcal{L}_r)^\nabla$ of locally analytic horizontal sections of $\mathcal{L}_r$ over C, as it is described in the paragraph preceding Thm. 10.1 of [Col3].

A similar discussion applies of course when $\mathcal{L}_{r,r}$ is replaced by $\mathcal{L}_r$, and the symbol Φ will also be used to denote the F-linear Frobenius endomorphism acting on $H^1_{\mathrm{par}}(C,\mathcal{L}_r,\nabla)$ and $H^0_{\mathrm{la}}(C,\mathcal{L}_r)^\nabla$.

3.6. **The Coleman primitive.**

Lemma 3.14. *Let ω be a global (rigid) section of the sheaf $\underline{\omega}^r\otimes\Omega^1_C$ over C, and let $[\omega]\in H^1_{\mathrm{par}}(C,\mathcal{L}_r,\nabla)$ be its associated cohomology class. There exists a polynomial $P\in F[x]$ satisfying*

(1) *$P(\Phi)([\omega])=0$.*
(2) *The map $P(\Phi)$ is an isomorphism on $H^0_{\mathrm{la}}(C,\mathcal{L}_r)^\nabla$, and $P(1)\neq 0$.*

Proof. This follows from the ideas explained in Section 11 of [Col3]. (Cf. in particular the argument following Lemma 11.1 of loc.cit.) One can use the fact that the eigenvalues of Φ acting on $H^1_{\mathrm{dR}}(C,\mathcal{L}_r,\nabla)$ and on any (finite-dimensional) Φ-stable subspace of $H^0_{\mathrm{la}}(C,\mathcal{L}_r)^\nabla$ differ, since they have complex absolute values $p^{\frac{r+1}{2}}$ and $p^{\frac{r}{2}}$ respectively. □

Theorem 3.15 (Coleman)**.** *Let ω be a global section of the sheaf $\underline{\omega}^r\otimes\Omega^1_C$ over C. Choose a polynomial P satisfying the properties of Lemma 3.14, and let d be its degree. There exists a locally analytic section F_ω of $\mathcal{L}_r$ over C satisfying the following conditions:*

(1) *$\nabla F_\omega=\omega$;*
(2) *$P(\Phi)(F_\omega)$ is a rigid analytic section of $\mathcal{L}_r$ on some wide open neighborhood $\mathcal{W}'$ of $\mathcal{A}$ in $\mathcal{W}$ satisfying $\phi^n(\mathcal{W}')\subset\mathcal{W}$, for all $n\leq d$.*

The locally analytic section F_ω is called the Coleman primitive *of ω on C.*

Proof. See Theorem 10.1 of [Col3]. Note that our setting, where p is assumed to not divide the level of the modular curve C, differs from the semistable reduction case considered in [Col3]. In fact it is simpler, and the assumptions that are required for Theorem 10.1 of loc.cit., such as the "regular singular annuli" assumption on the cuspidal annuli, are satisfied a fortiori in the setting of Theorem 3.15. Note also that Theorem 10.1 as stated produces a locally analytic primitive on each wide open $\mathcal{W}$, but expressing C as a finite union of wide opens and gluing the different primitives (which, by their uniqueness, agree on the overlaps) leads to a locally analytic primitive on all of C. The uniqueness clause in the definition of the Coleman primitive also implies that F_ω is defined over the field F over which ω is defined. □

Remark 3.16. The definition of F_ω depends a priori on several choices: the choice of an affinoid $\mathcal{A}$ in C, a lifting of Frobenius to $\mathcal{A}$, a Frobenius neighborhood $\mathcal{W}'$ of $\mathcal{A}$ in $\mathcal{W}$ and the polynomial P. It can be shown that the Coleman primitive does not depend on these choices, and therefore the Coleman primitives on a covering of C by affinoid regions can be pieced together to give a locally analytic section of $\mathcal{L}_r$ over C which is well-defined up to global rigid analytic horizontal sections of $\mathcal{L}_r$ over C. This latter space is trivial when $r > 0$ and is the space of constant functions on C when $r = 0$. (Cf. Proposition 5.1 of [Col3].)

Remark 3.17. It can be shown that the Coleman primitive F_ω is in fact analytic on each residue disc $D(P)$ associated to any point P of $C(\mathbf{Q}_p^{\mathrm{unram}})$.

3.7. **p-adic integration and the p-adic Abel-Jacobi map.** The following is one of the main results of this chapter.

Proposition 3.18. *Let Δ_φ be a generalised Heegner cycle attached to an isogeny of ordinary pairs $\varphi : (A,t)\longrightarrow(A',t')$, and let $P_{A'}$ be the point of C attached to (A',t'). Then*

$$\mathrm{AJ}_F(\Delta_\varphi)(\omega_f\wedge\alpha) = \langle F_f(P_{A'})\wedge\alpha, \mathrm{cl}_{P_{A'}}(\Delta_\varphi)\rangle,$$

where the pairing on the right is the natural one on $\mathcal{L}_{r,r}(P_{A'})$, and F_f is the Coleman primitive of $\omega_f \in H^0(C, \underline{\omega}^r\otimes\Omega^1_C)$.

Proof. In order to ease notations, we drop the index φ in this proof, by setting $\Delta = \Delta_\varphi$, and write $P = P_A'$ and $U = C - \{P\}$. By definition of the p-adic Abel-Jacobi map, we have

$$\mathrm{AJ}_F(\Delta)(\omega_f\wedge\alpha) = \langle\omega_f\wedge\alpha, \eta_\Delta\rangle,$$

where the class η_Δ represents the extension D_Δ of (3.4.4) following the recipe given in Section 3.3. We may write

$$\eta_\Delta = \eta_\Delta^{\mathrm{hol}} - \eta_\Delta^{\mathrm{frob}},$$

where

(1) The cohomology class $\eta_\Delta^{\mathrm{hol}}$ is represented by a section of $\mathcal{L}_{r,r}\otimes\Omega^1_C(\log Z_N)$ over U having residue 0 at the cusps and a simple pole at P with residue equal to $\mathrm{cl}_P(\Delta)$. By abuse of notation, we will use the same symbol $\eta_\Delta^{\mathrm{hol}}$ to denote the associated $\mathcal{L}_{r,r}$-valued differential on C. If $P_1,\ldots,P_t$ were chosen in such a way that $P_1 = P$, and G_j is any rigid analytic section of $\mathcal{L}_{r,r}^{\mathrm{rig}}$ over $D(P_j)$, then by (3.5.1), for all non-cuspidal annuli $\mathcal{V}_j$,

$$\mathrm{res}_{\mathcal{V}_1}\langle G_1,\eta_\Delta^{\mathrm{hol}}\rangle = \langle G_1(P),\mathrm{cl}_P(\Delta)\rangle,\qquad \mathrm{res}_{\mathcal{V}_j}\langle G_j,\eta_\Delta^{\mathrm{hol}}\rangle = 0\quad\text{for } j\geq 2. \tag{3.7.1}$$

If $\mathcal{V}_j$ is a cuspidal annulus, then we at least have

$$\mathrm{res}_{\mathcal{V}_j}\langle F_{f,j}\wedge\alpha,\eta_\Delta^{\mathrm{hol}}\rangle = 0, \tag{3.7.2}$$

where $F_{f,j}$ is a local primitive of ω_f on $\mathcal{V}_j$. To see this, use the fact that $\eta_\Delta^{\mathrm{hol}}$ has residue 0 along $\mathcal{V}_j$ to write $\eta_\Delta^{\mathrm{hol}} = \nabla H_\Delta$ for some section of $\mathcal{L}_{r,r}^{\mathrm{rig}}$ over $\mathcal{V}_j$, and observe that

$$0 = \mathrm{res}_{\mathcal{V}_j} d\langle F_{f,j}\wedge\alpha, H_\Delta\rangle = \mathrm{res}_{\mathcal{V}_j}(\langle\omega_f, H_\Delta\rangle + \langle F_{f,j}\wedge\alpha,\eta_\Delta^{\mathrm{hol}}\rangle) = \mathrm{res}_{\mathcal{V}_j}\langle F_{f,j}\wedge\alpha,\eta_\Delta^{\mathrm{hol}}\rangle.$$

(2) The differential $\eta_\Delta^{\mathrm{frob}}$ is a section of $\mathcal{L}_{r,r}^{\mathrm{rig}}\otimes\Omega^1_C$ over $\mathcal{W}$, chosen so that it satisfies

$$\Phi\eta_\Delta^{\mathrm{frob}} = \eta_\Delta^{\mathrm{frob}} + \nabla G, \tag{3.7.3}$$

for some rigid section G of $\mathcal{L}_{r,r}^{\mathrm{rig}}$ over $\mathcal{W}'$, and of course

$$\mathrm{res}_{\mathcal{V}_1}\langle G_1,\eta_\Delta^{\mathrm{Frob}}\rangle = \langle G_1(P),\mathrm{cl}_P(\Delta)\rangle. \tag{3.7.4}$$

By Proposition 3.10, the Poincaré pairing between $H^1_{\mathrm{dR}}(C, \mathcal{L}_{r,r}(r), \nabla)$ and $H^1_{\mathrm{dR}}(C, \mathcal{L}_{r,r}(r+1), \nabla)$ is given by the formula

$$\langle \omega_f \wedge \alpha, \eta_\Delta \rangle = \sum_{j=1}^{t} \mathrm{res}_{\mathcal{V}_j} \langle F_{f,j} \wedge \alpha, \eta_\Delta \rangle \tag{3.7.5}$$

$$= \left(\sum_{j=1}^{t} \mathrm{res}_{\mathcal{V}_j} \langle F_{f,j} \wedge \alpha, \eta_\Delta^{\mathrm{hol}} \rangle \right) - \left(\sum_{j=1}^{t} \mathrm{res}_{\mathcal{V}_j} \langle F_{f,j} \wedge \alpha, \eta_\Delta^{\mathrm{frob}} \rangle \right), \tag{3.7.6}$$

where the sum is taken over the t annuli $\mathcal{V}_j$ in $\mathcal{W} - \mathcal{A}$, and $F_{f,j}$ is an analytic primitive of ω_f on the residue disc $D(P_j)$. Note that if ω^∇ is any horizontal section of $\mathcal{L}_{r,r}$ on $D(P_j)$, the residue of the differential $\langle \omega^\nabla, \eta_\Delta \rangle$ on the annulus $\mathcal{V}_j$ is zero, and therefore the expression on the right of (3.7.5) is independent of the choice of local primitives on each residue disc. The same is not true for either of the sums that appear on the right of (3.7.6), since the differentials $\eta_\Delta^{\mathrm{hol}}$ and $\eta_\Delta^{\mathrm{Frob}}$ each have non-zero residue along the annulus $\mathcal{V}_1$.

In order to compute each of the terms appearing in (3.7.6) individually, we need to make a "coherent" choice of local primitives. This is done by fixing a Coleman primitive F_f of ω_f. Once this choice is made, the two terms appearing in (3.7.6) are controlled in the following two lemmas.

Lemma 3.19. *If $F_{f,j}$ is any choice of local primitives of ω_f on each residue disc $D(P_j)$, then*

$$\sum_{j=1}^{t} \mathrm{res}_{\mathcal{V}_j} \langle F_{f,j} \wedge \alpha, \eta_\Delta^{\mathrm{hol}} \rangle = \langle F_{f,1}(P_{A'}) \wedge \alpha, \mathrm{cl}_{P_{A'}}(\Delta) \rangle.$$

Proof. Since the local primitive $F_{f,j} \wedge \alpha$ is analytic on the residue disc $D(P_j)$, and since $\tilde{\eta}_\Delta^{\mathrm{hol}}$ has 0 residue on $\mathcal{V}_j$ when $j \geq 2$, it follows from (3.7.1) and (3.7.2) that

$$\sum_{j=1}^{t} \mathrm{res}_{\mathcal{V}_j} \langle F_{f,j} \wedge \alpha, \eta_\Delta^{\mathrm{hol}} \rangle = \mathrm{res}_{\mathcal{V}_1} \langle F_{f,1} \wedge \alpha, \eta_\Delta^{\mathrm{hol}} \rangle = \langle F_f(P_{A'}) \wedge \alpha, \mathrm{cl}_{P_{A'}}(\Delta) \rangle.$$

The lemma follows. □

Lemma 3.20. *Let F_f be the Coleman primitive of ω_f on C. Then*

$$\sum_{j=1}^{t} \mathrm{res}_{\mathcal{V}_j} \langle F_f \wedge \alpha, \eta_\Delta^{\mathrm{frob}} \rangle = 0. \tag{3.7.7}$$

Proof. We begin by noting that for each $j = 1, \ldots, t$,

$$\begin{aligned} \mathrm{res}_{\mathcal{V}_j} \langle F_f \wedge \alpha, \eta_\Delta^{\mathrm{Frob}} \rangle &= \mathrm{res}_{\mathcal{V}_j} \langle \Phi F_f \wedge \alpha, \Phi \eta_\Delta^{\mathrm{Frob}} \rangle \\ &= \mathrm{res}_{\mathcal{V}_j} \langle \Phi F_f \wedge \alpha, \eta_\Delta^{\mathrm{Frob}} \rangle + \mathrm{res}_{\mathcal{V}_j} \langle \Phi F_f \wedge \alpha, \nabla G \rangle, \end{aligned} \tag{3.7.8}$$

where G is the rigid analytic section of $\mathcal{L}_{r,r}$ over $\mathcal{W}'$ given by (3.7.3). The fact that Φ is horizontal for the Gauss-Manin connection (combined with the Leibniz rule) shows that

$$d\langle \Phi F_f \wedge \alpha, G \rangle = \langle \Phi F_f \wedge \alpha, \nabla G \rangle + \langle \Phi \omega_f \wedge \alpha, G \rangle.$$

In particular, the expression appearing on the right is exact on each annulus $\mathcal{V}_j$, and therefore

$$\begin{aligned} \sum_{j=1}^{t} \mathrm{res}_{\mathcal{V}_j} \langle \Phi F_f \wedge \alpha, \nabla G \rangle &= -\sum_{j=1}^{t} \mathrm{res}_{\mathcal{V}_j} \langle \Phi \omega_f \wedge \alpha, G \rangle \\ &= 0, \end{aligned}$$

where the last vanishing follows from the rigid analytic residue theorem (Theorem 3.8), in light of the fact that $\langle \Phi \omega_f \wedge \alpha, G \rangle$ belongs to $\Omega^1_{\mathcal{W}'}$. Hence by summing equation (3.7.8) over $j = 1, \ldots, t$, we get

$$\sum_{j=1}^{t} \mathrm{res}_{\mathcal{V}_j} \langle F_f \wedge \alpha, \eta_\Delta^{\mathrm{Frob}} \rangle = \sum_{j=1}^{t} \mathrm{res}_{\mathcal{V}_j} \langle \Phi F_f \wedge \alpha, \eta_\Delta^{\mathrm{Frob}} \rangle.$$

More generally, if L is any polynomial in $F[x]$, we get

$$L(1)\sum_{j=1}^{t}\mathrm{res}_{\mathcal{V}_j}\langle F_f\wedge\alpha,\eta_\Delta^{\mathrm{Frob}}\rangle=\sum_{j=1}^{t}\mathrm{res}_{\mathcal{V}_j}\langle L(\Phi)F_f\wedge\alpha,\eta_\Delta^{\mathrm{Frob}}\rangle.$$

Now, choosing the polynomial $L(x)=P(x)$ as in Lemma 3.14 preceding the definition of the Coleman primitive, we get

$$L(1)\sum_{j=1}^{t}\mathrm{res}_{\mathcal{V}_j}\langle F_f\wedge\alpha,\eta_\Delta^{\mathrm{Frob}}\rangle=\sum_{j=1}^{t}\mathrm{res}_{\mathcal{V}_j}\langle L(\Phi)F_f\wedge\alpha,\eta_\Delta^{\mathrm{Frob}}\rangle=0,$$

where the vanishing follows by noting that $L(\Phi)F_f\wedge\alpha$ is a rigid analytic section of $\mathcal{L}_{r,r}$ over $\mathcal{W}'$ and applying Theorem 3.8 once again. Lemma 3.20 now follows from the fact that $L(1)\neq 0$. □

The proof of Proposition 3.18 now follows from (3.7.6) combined with Lemmas 3.19 and 3.20, which show that

$$\mathrm{AJ}_F(\Delta_\varphi)(\omega_f\wedge\alpha)=\langle\omega_f\wedge\alpha,\eta_\Delta\rangle=\langle F_f(P_{A'})\wedge\alpha,\mathrm{cl}_{P_{A'}}(\Delta)\rangle$$

when F_f is a Coleman primitive for ω_f. □

Proposition 3.21. *With the same notations as in Proposition 3.18,*

$$\mathrm{AJ}_F(\Delta_\varphi)(\omega_f\wedge\alpha)=\langle\varphi^*F_f(P_{A'}),\alpha\rangle_A,$$

where the pairing $\langle\ ,\ \rangle_A$ *on the right is the natural one on* $\mathrm{Sym}^r H^1_{\mathrm{dR}}(A/F)$.

Proof. Let

$$\varrho:=(\varphi^r,\mathrm{id}^r):A^r\longrightarrow\Upsilon_\varphi\subset(A')^r\times A^r.$$

Note that

$$\varrho^*(F_f(P_{A'})\wedge\alpha)=\varphi^*(F_f(P_{A'}))\wedge\alpha,\qquad\qquad \varrho([A^r])=\mathrm{cl}_{P_{A'}}(\Upsilon_\varphi),$$

where $[A^r]\in H^0_{\mathrm{dR}}(A^r/F)$ is the fundamental class associated to the variety A^r. Let

$$\langle\ ,\ \rangle_{A,j}:H^{2r-j}_{\mathrm{dR}}(A^r/F)\times H^j_{\mathrm{dR}}(A^r/F)\longrightarrow H^{2r}(A^r/F)=F$$

denote the Poincaré pairing, so that the restriction of $\langle\ ,\ \rangle_{A,r}$ to $\mathrm{Sym}^r H^1_{\mathrm{dR}}(A/F)\subset H^r_{\mathrm{dR}}(A/F)$ agrees with $\langle\ ,\ \rangle_A$. Observe that

$$\langle F_f(P_{A'})\wedge\alpha,\mathrm{cl}_{P_{A'}}(\Delta_\varphi)\rangle=\langle F_f(P_{A'})\wedge\alpha,\mathrm{cl}_{P_{A'}}(\Upsilon_\varphi)\rangle=\langle F_f(P_{A'})\wedge\alpha,\varrho([A^r])\rangle. \tag{3.7.9}$$

The functoriality properties of the Poincaré pairing imply that

$$\begin{aligned}\langle F_f(P_{A'})\wedge\alpha,\varrho([A^r])\rangle &= \langle\varrho^*(F_f(P_{A'})\wedge\alpha),[A^r]\rangle_{A,0}\\ &= \langle\varphi^*(F_f(P_{A'}))\wedge\alpha,[A^r]\rangle_{A,0}=\langle\varphi^*(F_f(P_{A'})),\alpha\rangle_A.\end{aligned} \tag{3.7.10}$$

Proposition 3.21 follows by combining Proposition 3.18 with (3.7.9) and (3.7.10). □

Let $\{\bar{P}_1,\ldots,\bar{P}_t\}$ be the set of supersingular points of C_k, and let $P_j\in C(F)$ be an arbitrary lift of $\bar{P}_j$ under the reduction map. The residue discs $D(P_j)$ are called the *supersingular discs* of C and the complement $\mathcal{A}:=C^{\mathrm{ord}}$ is called the *ordinary locus* of C. A *locally analytic p-adic modular form* of weight k is a locally analytic section of ω^k over C^{ord}. Following equation (1.1.1), a modular form G of this type can also be viewed as a function on ordinary triples of *generalised* elliptic curves $(E,t,\omega)_{/R}$, where R is a p-adic ring of finite type over $\mathbf{Z}_p$, satisfying

$$G(E,t,\lambda\omega)=\lambda^{-k}G(E,t,\omega),\qquad\qquad\text{for all }\lambda\in R^\times.$$

Following Chapter VII of [DR], in particular Corollaire 2.2, the formal completion along a cusp of a suitable cuspidal p-adic neighborhood $D\simeq\mathrm{Spec}(R)$ in C^{ord} can be identified with $\mathrm{Spf}(Z[\![q^{1/d}]\!])$, for Z finite unramified over $\mathbf{Z}_p$ and $d\mid N$, in such a way that the universal object over D pulls back to $\mathrm{Tate}(q)$, equipped with a suitable level structure. By an abuse of notation, we will denote by $G(\mathrm{Tate}(q),t,\omega_{\mathrm{can}})$ the q-expansion obtained by evaluating G at a generalised marked elliptic curve corresponding to $(\mathrm{Tate}(q),t,\omega_{\mathrm{can}})$ via the above identifications.

For $0\leq j\leq r$, let G_j denote the "j-th component" of the Coleman primitive F_f, defined (as a function on ordinary triples) by the rule

$$G_j(E,t,\omega):=\langle F(E,t),\omega^j\eta^{r-j}\rangle,$$

where η is the generator of the unit root subspace of $H^1_{\mathrm{dR}}(E/R)$, normalised so that $\langle\omega,\eta\rangle = 1$. The rule G_j thus defined satisfies

$$G_j(E,t,\lambda\omega) = \lambda^{2j-r}G_j(E,t,\omega), \qquad \text{for all } \lambda \in R^\times,$$

and therefore defines a locally analytic p-adic modular form of weight $r-2j$.

The next lemma expresses the Abel-Jacobi images of the cycles Δ_φ in terms of the modular forms G_j.

Lemma 3.22. *Let*

$$\varphi : (A,t,\omega) \longrightarrow (A',t',\omega')$$

be an isogeny of ordinary marked elliptic curves of degree $d_\varphi = \deg(\varphi)$, *and let* Δ_φ *be the associated generalised Heegner cycle on* X_r. *Then*

$$\mathrm{AJ}_F(\Delta_\varphi)(\omega_f \wedge \omega^j\eta^{r-j}) = d_\varphi^j G_j(A',t',\omega').$$

Proof. By Proposition 3.21,

$$\mathrm{AJ}_F(\Delta_\varphi)(\omega_f \wedge \omega^j\eta^{r-j}) = \langle \varphi^* F_f(A',t'), \omega^j\eta^{r-j}\rangle_A. \tag{3.7.11}$$

Since $\langle \varphi^*\omega', \varphi^*\eta'\rangle = d_\varphi$, we have

$$\varphi^*(\eta') = d_\varphi \eta. \tag{3.7.12}$$

Hence

$$\begin{aligned}
\langle \varphi^* F_f(A',t'), \omega^j\eta^{r-j}\rangle_A &= d_\varphi^{j-r}\langle \varphi^* F_f(A',t'), \varphi^*((\omega')^j(\eta')^{r-j})\rangle_A \\
&= d_\varphi^j \langle F_f(A',t'), (\omega')^j(\eta')^{r-j}\rangle_{A'} \\
&= d_\varphi^j G_j(A',t',\omega').
\end{aligned}$$

□

3.8. **Calculation of the Coleman primitive.** We now turn to the explicit calculation of the Coleman primitive F_f of the regular $\mathcal{L}_r^{\mathrm{rig}}$-valued differential ω_f, or rather, of its components G_j. In order to do this, we begin by introducing an operator $VU-UV$ on locally analytic p-adic modular forms which plays the role of the operator $P(\Phi)$ in Theorem 3.15 defining the Coleman primitive, in the sense that it maps the locally analytic forms G_j to genuine p-adic modular forms in the sense of Section 1.3. As a consequence of the use of this operator, it will be possible to resort to q-expansions in our calculation of Coleman primitive (cf. the proof of Proposition 3.24).

We recall the definition of the operators U and V (as they are described in [Se] for example). Given an ordinary triple (E,t,ω), let

$$\varphi_j^{(p)} : (E,t,\omega) \longrightarrow (E_j,t_j,\omega_j), \qquad j = 0,1,\dots,p$$

denote the distinct p-isogenies on E, ordered in such a way that $\varphi_0^{(p)}$ is the distinguished p-isogeny whose kernel is the canonical subgroup of E. For instance, when $(E,t,\omega) = (\mathrm{Tate}(q),\zeta_N,\omega_{\mathrm{can}})$, the canonical subgroup is μ_p and we can take

$$(E_0,t_0,\omega_0) = \left(\mathrm{Tate}(q^p),\zeta_N^p,\frac{1}{p}\omega_{\mathrm{can}}\right), \qquad (E_j,t_j,\omega_j) = (\mathrm{Tate}(q^{1/p}\zeta_p^j),\zeta_N,\omega_{\mathrm{can}}). \tag{3.8.1}$$

The Hecke operators U and V are defined by setting

$$(G|U)(E,t,\omega) := G(U(E,t,\omega)), \qquad (G|V)(E,t,\omega) := G(V(E,t,\omega)),$$

where

$$U(E,t,\omega) := \frac{1}{p}\sum_{j=1}^{p}(E_j,t_j,\omega_j), \qquad V(E,t,\omega) := (E_0,\frac{1}{p}t_0,p\omega_0).$$

These operators are related to the usual Hecke operator T_p by the rule

$$T_p = U + \frac{1}{p}[p]V,$$

where $[p]$ denotes the isogeny given by multiplication by p. In particular,

$$VU - UV = 1 - T_pV + \frac{1}{p}[p]V^2. \tag{3.8.2}$$

The diamond operator $\langle a\rangle$ attached to $a\in(\mathbf{Z}/N\mathbf{Z})^\times$ is defined on locally analytic p-adic modular forms by the rule

$$G|\langle a\rangle(E,t,\omega)=G(E,a^{-1}t,\omega).$$

Given a locally analytic p-adic modular form G, we set

$$G^\flat:=G|(VU-UV).$$

In terms of the q-expansion

$$G\left(\mathrm{Tate}(q),\zeta_N,\omega_{\mathrm{can}}\right)=\sum_{n=1}^\infty b_nq^n$$

of G, the operators U and V satisfy

$$(G|U)\left(\mathrm{Tate}(q),\zeta_N,\omega_{\mathrm{can}}\right)=\sum_{n=1}^\infty b_{np}q^n,\qquad (G|V)\left(\mathrm{Tate}(q),\zeta_N,\omega_{\mathrm{can}}\right)=\sum_{n=1}^\infty b_nq^{np}, \tag{3.8.3}$$

so that the q-expansion of $G^\flat$ is given by

$$G^\flat(\mathrm{Tate}(q),\zeta_N,\omega_{\mathrm{can}})=\sum_{(p,n)=1}b_nq^n. \tag{3.8.4}$$

Lemma 3.23. *Let K be a quadratic imaginary field in which the prime $(p)=\mathfrak{p}\bar{\mathfrak{p}}$ splits, and let (A',t') be a point in C^{ord} corresponding to an elliptic curve A' with complex multiplication by (an order in) K. Let G be a locally analytic p-adic modular form of weight k satisfying*

$$T_pG=b_pG,\qquad \langle p\rangle G=\epsilon_G(p)G.$$

Then

$$G^\flat(A',t',\omega')=G(A',t',\omega')-\frac{\epsilon_G(p)b_p}{p^k}G(\mathfrak{p}*(A',t',\omega'))+\frac{\epsilon_G(p)}{p^{k+1}}G(\mathfrak{p}^2*(A',t',\omega')),$$

where the action of ideals on CM triples is the one given in (1.4.8).

Proof. Because A' has complex multiplication, its canonical subgroup is identified with the kernel $A'[\mathfrak{p}]$ of multiplication by $\mathfrak{p}$, and therefore,

$$V(A',t',\omega')=\mathfrak{p}*(A',p^{-1}t',p\omega'),\qquad [p]V^2(A',t',\omega')=\mathfrak{p}^2*(A',p^{-1}t',p\omega').$$

Therefore,

$$\begin{aligned}G^\flat(A',t',\omega') &= G((1-T_pV+\frac{1}{p}[p]V^2)(A',t',\omega'))\\ &= G(A',t',\omega')-b_pG(\mathfrak{p}*(A',p^{-1}t',p\omega'))+\frac{1}{p}G(\mathfrak{p}^2*(A',p^{-1}t',p\omega'))\\ &= G(A',t',\omega')-\frac{\epsilon_G(p)b_p}{p^k}G(\mathfrak{p}*(A',t',\omega'))+\frac{\epsilon_G(p)}{p^{k+1}}G(\mathfrak{p}^2*(A',t',\omega')).\end{aligned}$$

The result follows. □

Proposition (3.24) below gives an explicit formula for $G_j^\flat$ in terms of the Atkin-Serre operator θ defined in equation (1.3.2) acting on the modular form f. Note that, for any $j\geq 0$, the expression

$$\theta^{-1-j}f^\flat:=\lim_{t\to-1-j}\theta^tf^\flat$$

is a p-adic modular form of weight $r-2j$. (Cf. Théorème 5 (b) of [Se].)

Proposition 3.24. *For all $(E,t)\in C^{\mathrm{ord}}$,*

$$G_j^\flat(E,t,\omega)=j!\theta^{-1-j}f^\flat(E,t,\omega). \tag{3.8.5}$$

(In particular the Coleman primitive $F_f^\flat$ of $\omega_{f^\flat}$ is a rigid analytic section of $\mathcal{L}_r^{\mathrm{rig}}$ over C^{ord}.)

Proof. For $0 \le j \le r$, set $F^\flat := F^\flat_f = F_f|(VU - UV)$. Then

$$G^\flat_j(E,t,\omega) = \langle F^\flat(E,t), \omega^j\eta^{r-j}\rangle.$$

Equation (3.8.5) amounts to the statement that

$$\theta G^\flat_0 = f^\flat, \qquad \theta G^\flat_j = jG^\flat_{j-1}, \quad \text{for } 1 \le j \le r. \tag{3.8.6}$$

We verify that this holds on q-expansions, working with the basis $(\omega_{\rm can}, \xi_{\rm can})$ for the de Rham cohomology of the Tate curve which is described in equation (1.1.6) of Section 1.1. To check (3.8.6), note that

$$\begin{aligned}
\nabla G^\flat_0(\mathrm{Tate}(q),\zeta_N) &= \nabla\left(G^\flat_0(\mathrm{Tate}(q),\zeta_N,\omega_{\rm can})\omega^r_{\rm can}\right)\\
&= \nabla\left(\langle F^\flat(\mathrm{Tate}(q),\zeta_N),\xi^r_{\rm can}\rangle\omega^r_{\rm can}\right)\\
&= \langle\omega_{f^\flat}(\mathrm{Tate}(q),\zeta_N),\xi^r_{\rm can}\rangle\omega^r_{\rm can} + r\langle F^\flat(\mathrm{Tate}(q),\zeta_N),\xi^r_{\rm can}\rangle\omega^{r-1}_{\rm can}\xi_{\rm can}\frac{dq}{q}\\
&= f^\flat(\mathrm{Tate}(q),\zeta_N,\omega_{\rm can})\omega^r_{\rm can}\frac{dq}{q} + r\langle F^\flat(\mathrm{Tate}(q),\zeta_N),\xi^r_{\rm can}\rangle\omega^{r-1}_{\rm can}\xi_{\rm can}\frac{dq}{q},
\end{aligned}$$

where the last equality follows from (1.1.10).

After applying the inverse of the Kodaira-Spencer isomorphism and using (1.1.10) again, we find

$$\tilde\nabla G^\flat_0(\mathrm{Tate}(q),\zeta_N) = f^\flat(\mathrm{Tate}(q),\zeta_N,\omega_{\rm can})\omega^{r+2}_{\rm can} + r\langle F^\flat(\mathrm{Tate}(q),\zeta_N),\xi^r_{\rm can}\rangle\omega^{r+1}_{\rm can}\xi_{\rm can}.$$

Applying the unit root splitting $\Psi_{\rm Frob}$ to this identity then gives

$$\Theta_{\rm Frob}G^\flat_0(\mathrm{Tate}(q),\zeta_N) = f^\flat(\mathrm{Tate}(q),\zeta_N).$$

This proves (3.8.6) for $j = 0$, in light of Lemma 1.7. For the case $j \ge 1$, we note that, because $\langle\omega_{f^\flat},\omega^j_{\rm can}\xi^{r-j}_{\rm can}\rangle = 0$,

$$\begin{aligned}
\nabla G^\flat_j(\mathrm{Tate}(q),\zeta_N) &= \nabla\left(G^\flat_j(\mathrm{Tate}(q),\zeta_N,\omega_{\rm can})\omega^{r-2j}_{\rm can}\right)\\
&= \nabla(\langle F^\flat(\mathrm{Tate}(q),\zeta_N),\omega^j_{\rm can}\xi^{r-j}_{\rm can}\rangle\omega^{r-2j}_{\rm can})\\
&= j\langle F^\flat(\mathrm{Tate}(q),\zeta_N)),\omega^{j-1}_{\rm can}\xi^{r-j+1}_{\rm can}\rangle\omega^{r-2j}_{\rm can}\frac{dq}{q}\\
&\quad +(r-2j)\langle F^\flat(\mathrm{Tate}(q),\zeta_N)),\omega^j_{\rm can}\xi^{r-j}_{\rm can}\rangle\omega^{r-2j-1}_{\rm can}\xi_{\rm can}\frac{dq}{q}\\
&= jG^\flat_{j-1}(\mathrm{Tate}(q),\zeta_N,\omega_{\rm can})\omega^{r-2j}_{\rm can}\frac{dq}{q}\\
&\quad +(r-2j)G^\flat_j(\mathrm{Tate}(q),\zeta_N,\omega_{\rm can})\omega^{r-2j-1}_{\rm can}\xi_{\rm can}\frac{dq}{q}.
\end{aligned}$$

Applying σ^{-1} followed by the unit root splitting to this identity gives

$$\Psi_{\rm Frob}\tilde\nabla G^\flat_j(\mathrm{Tate}(q),\zeta_N) = jG^\flat_{j-1}(\mathrm{Tate}(q),\zeta_N,\omega_{\rm can})\omega^{r+2-2j}_{\rm can}.$$

Therefore,

$$\Theta_{\rm Frob}G^\flat_j(\mathrm{Tate}(q),\zeta_N,\omega_{\rm can}) = jG^\flat_{j-1}(\mathrm{Tate}(q),\zeta_N,\omega_{\rm can}),$$

and (3.8.6) follows from Lemma 1.7 for all $1 \le j \le r$. This completes the proof of Proposition 3.24. (See also Lemma 9.2 of [Col3], where a similar result is proved.) □

4. Period integrals and central values of Rankin-Selberg L-functions

4.1. **Rankin L-series and their special values.** Let $f = \sum a_n e^{2\pi i n z} \in S_k(\Gamma_0(N),\varepsilon_f)$ be a normalised newform. Write

$$L(f,s) = \sum_{n\ge1} a_n n^{-s} = \prod_q (1-\alpha_q q^{-s})^{-1}(1-\beta_q q^{-s})^{-1}$$

for its Hecke L-series, where the product on the right, taken over all the rational primes, should be taken as the definition of the parameters $\{\alpha_q,\beta_q\}$. In particular, $\alpha_q\beta_q = q^{k-1}\varepsilon_f(q)$ if q does not divide N, and $\alpha_q\beta_q = 0$ otherwise. Let N_{ε_f} denote the conductor of ε_f.

In this section, it will also be convenient to view f as a function on pairs (L, t), where L is a lattice in $\mathbf{C}$ and t is an element of exact order N in $\mathbf{C}/L$. The lattice function f is determined by the rules

$$f(\langle 1, \tau\rangle, 1/N) = f(\tau), \qquad \text{for all } \tau \in \mathcal{H}, \tag{4.1.1}$$

$$f(\lambda L, \lambda t) = \lambda^{-k} f(L, t), \qquad \text{for all } \lambda \in \mathbf{C}^\times, \tag{4.1.2}$$

$$f(L, at) = \varepsilon_f(a) f(L, t), \qquad \text{for all } a \in (\mathbf{Z}/N\mathbf{Z})^\times. \tag{4.1.3}$$

Let $w_f \in \mathbf{C}^\times$ be the scalar of norm one defined by the rule

$$w_N(f) = w_f f_\rho, \tag{4.1.4}$$

where $f_\rho \in S_k(\Gamma_0(N), \bar{\varepsilon}_f)$ is the modular form obtained by applying complex conjugation to the coefficients of f and w_N is the Atkin-Lehner involution (which is described precisely in Lemma 5.2 and the discussion preceding it). We note that the Hecke L-series $L(f, s)$ satisfies the functional equation

$$\Lambda(f, s) = w_f \Lambda(f_\rho, k - s),$$

where $\Lambda(f, s) = (2\pi)^{-s}\Gamma(s)N^{s/2}L(f, s)$.

Let K be an imaginary quadratic field with discriminant $-d_K$, equipped with a fixed complex embedding. Recall that for any pair of integers (ℓ_1, ℓ_2), a Hecke character of K of infinity type (ℓ_1, ℓ_2) is a continuous homomorphism

$$\chi : \mathbb{A}_K^\times \longrightarrow \mathbf{C}^\times$$

satisfying

$$\chi(\alpha \cdot x \cdot z_\infty) = \chi(x) \cdot z_\infty^{-\ell_1} \bar{z}_\infty^{-\ell_2}, \qquad \text{for all } \alpha \in K^\times, \quad z_\infty \in K_\infty^\times.$$

For each prime $\mathfrak{q}$ of K, let $\chi_{\mathfrak{q}} : K_{\mathfrak{q}}^\times \longrightarrow \mathbf{C}^\times$ denote the local character associated to χ. The *conductor* of χ is the largest integral ideal $\mathfrak{f}_\chi$ of K such that $\chi_{\mathfrak{q}}(u) = 1$ for all $u \in (1 + \mathfrak{f}_\chi \mathcal{O}_{K,\mathfrak{q}})^\times \hookrightarrow K_{\mathfrak{q}}^\times$. In the usual way, we can identify χ with a character on $\mathcal{O}_K$-ideals prime to $\mathfrak{f}_\chi$ by defining

$$\chi(\mathfrak{a}) = \prod_{\mathfrak{q}|\mathfrak{a}} \chi_{\mathfrak{q}}(\pi_{\mathfrak{q}})^{v_{\mathfrak{q}}(\mathfrak{a})}, \tag{4.1.5}$$

where $\pi_{\mathfrak{q}}$ is any uniformizer at $\mathfrak{q}$, this assignment being independent of the choice of $\pi_{\mathfrak{q}}$. As a function on ideals, χ satisfies $\chi((\alpha)) = \alpha^{\ell_1}\bar{\alpha}^{\ell_2}$ for all principal ideals (α) with $\alpha \equiv 1 \mod \mathfrak{f}_\chi$.

The focus of this section is on the special values of the Rankin-Selberg L-function $L(f \times \theta_\chi, s)$ where θ_χ denotes the theta function associated to χ. For simplicity we will denote this L-function by $L(f, \chi, s)$. If we set $\alpha_{p^j} := \alpha_p^j$ and $\beta_{p^j} := \beta_p^j$, then it can be defined as an Euler product of terms $L_{\mathfrak{p}}(f, \chi, s)$ where for good $\mathfrak{p}$, i.e. for $\mathfrak{p} \nmid \mathfrak{f}_\chi N$,

$$L_{\mathfrak{p}}(f, \chi, s) = (1 - \chi(\mathfrak{p})\alpha_{\mathrm{N}\mathfrak{p}}(\mathrm{N}\mathfrak{p})^{-s})^{-1}(1 - \chi(\mathfrak{p})\beta_{\mathrm{N}\mathfrak{p}}(\mathrm{N}\mathfrak{p})^{-s})^{-1}.$$

The local factors at ramified places are described in [Jac] §15. Indeed, up to a shift $L(f, \chi, s)$ is identified with the Rankin-Selberg L-function $L(\pi_f \times \pi_\chi, s)$, where π_f and π_χ are the automorphic representations of $\mathbf{GL}_2(\mathbb{A}_{\mathbf{Q}})$ associated to f and θ_χ respectively. More precisely, after normalizing π_f and π_χ to be unitary, we have

$$L(f, \chi, s) = L\left(\pi_f \times \pi_\chi, s - \frac{k - 1 + \ell_1 + \ell_2}{2}\right).$$

Set $\ell := |\ell_1 - \ell_2|$ and $\ell_0 := \min(\ell_1, \ell_2)$. Define

$$L_\infty(f, \chi, s) = \Gamma_{\mathbf{C}}(s - \ell_0)\Gamma_{\mathbf{C}}(s - \min(k - 1, \ell) - \ell_0),$$

where $\Gamma_{\mathbf{C}}(s) = 2 \cdot (2\pi)^{-s}\Gamma(s)$, and set

$$\Lambda(f, \chi, s) := L_\infty(f, \chi, s) \cdot L(f, \chi, s).$$

The function $\Lambda(f, \chi, s)$ (defined a priori in some right half plane) extends to a meromorphic function on $\mathbf{C}$ and satisfies a functional equation of the form

$$\Lambda(f, \chi, s) = \epsilon(f, \chi, s)\Lambda(f_\rho, \bar{\chi}, k + \ell_1 + \ell_2 - s),$$

where f_ρ is as in (4.1.4) and $\epsilon(f, \chi, s)$ is an epsilon factor again described in [Jac] §15. In the case of interest to us below, $\pi_f \times \pi_\chi$ will be self-dual and the value of $\epsilon(f, \chi, s)$ at the center of the critical strip, denoted $\epsilon(f,\chi)$, is equal to ± 1. If ε_K is the quadratic character associated to K and ε_χ is the Dirichlet character attached to χ by

$$\varepsilon_\chi := \chi|_{\mathbb{A}_{\mathbf{Q}}^\times} \cdot \mathbf{N}^{-(\ell_1+\ell_2)},$$

then the function $\Lambda(f,\chi,s)$ is known to be holomorphic when $\varepsilon_f\varepsilon_\chi\varepsilon_K$ is non-trivial. (For more details on the above, see [Jac], § 19.)

An integer n is said to be *critical* (in the sense of Deligne) for $L(f,\chi,s)$ if none of the Gamma factors that occur on either side of the functional equation for $L(f,\chi,s)$ have a pole at $s=n$. The corresponding values of $L(f,\chi,s)$ will be called *critical values*. Deligne has made precise conjectures (proved by Shimura [Shim2]) that predict the rationality of these critical L-values over specific number fields, after dividing them by appropriate (ostensibly transcendental) periods. It turns out that the nature of the period depends qualitatively on the infinity type of χ. Indeed, assuming for the moment that χ is of type $(0,\ell)$ with $\ell \geq 0$, the form of the gamma factor $L_\infty(f,\chi,s)$ shows that the following two cases arise naturally:

Case 1: $\ell \leq k-2$. In this case the critical integers j for $L(f,\chi,s)$ are those in the closed segment $[\ell+1,k-1]$. The transcendental part of $L(f,\chi,j)$ depends only on f and not on χ, and is expressible in terms of the Petersson inner product $\langle f,f\rangle$.

Case 2: $\ell \geq k$. In this case the critical integers j for $L(f,\chi,s)$ are those in the closed segment $[k,\ell]$. The transcendental part of $L(f,\chi,j)$ depends only on K and not on f, and is expressible as a power of a CM period attached to K. (This period will be defined precisely in Section 5.1.)

We now return to considering characters χ of more general infinity type (ℓ_1,ℓ_2). It will be convenient in what follows to work with the L-function $L(f,\chi^{-1},s)$. Note that the critical values of $L(f,\chi^{-1},s)$ (as χ and s both vary) are completely captured by the critical values of $L(f,\chi^{-1},0)$ (as only χ is made to vary). This motivates the following definition.

Definition 4.1. A Hecke character χ of infinity type (ℓ_1,ℓ_2) is said to be *critical* if $s=0$ is a critical point for $L(f,\chi^{-1},s)$.

Let us define χ_0 by $\chi_0 := \chi^{-1}\cdot\mathbf{N}^{\ell_1}$ so that the infinity type of χ_0 is $(0,\ell_1-\ell_2)$. Then

$$L(f,\chi^{-1},s) = L(f,\chi_0\mathbf{N}^{-\ell_1},s) = L(f,\chi_0,s+\ell_1).$$

By the previous discussion applied to χ_0 (and to χ_0^ρ: see remark below), the character χ of weight (ℓ_1,ℓ_2) is then critical if one of the following hypotheses is satisfied:

Case 1: $1 \leq \ell_1,\ell_2 \leq k-1$. This implies that $\ell \leq k-2$.

Case 2: $\ell_1 \geq k$ and $\ell_2 \leq 0$, and Case $2'$: $\ell_1 \leq 0$ and $\ell_2 \geq k$. In both these cases, $\ell \geq k$.

Let $\Sigma^{(1)}$, $\Sigma^{(2)}$ and $\Sigma^{(2')}$ denote the set of Hecke characters satisfying the conditions in Case 1, Case 2 and Case $2'$ respectively, so that the set Σ of all critical characters is the disjoint union

$$\Sigma = \Sigma^{(1)} \sqcup \Sigma^{(2)} \sqcup \Sigma^{(2')}.$$

Remark 4.2. The weights of characters in $\Sigma^{(1)}$ are the integer lattice points in the lightly shaded square in Figure 1, and those attached to characters in $\Sigma^{(2)}$ are the lattice points in the darker lower right quadrant of this figure. The region $\Sigma^{(2')}$ is the reflection of $\Sigma^{(2)}$ around the principal diagonal, and the map $\chi \mapsto \chi^\rho$ (where χ^ρ is the composition of χ with complex conjugation on $\mathbb{A}_K^\times$) interchanges these two regions.

A character $\chi \in \Sigma$ is said to be *central critical* if

$$\ell_1+\ell_2 = k, \quad \varepsilon_\chi = \varepsilon_f.$$

The terminology is justified by the fact that in this case $\pi_f \times \pi_{\chi^{-1}}$ is self-dual and 0 is the central (critical) point for $L(f,\chi^{-1},s)$. Let Σ_{cc} denote the set of central critical characters, and write (for $i=1,2,2'$)

$$\Sigma_{\mathrm{cc}}^{(i)} := \Sigma_{\mathrm{cc}} \cap \Sigma^{(i)}.$$

The weights of central critical characters are the lattice points on the central critical line which is depicted in Figure 1.

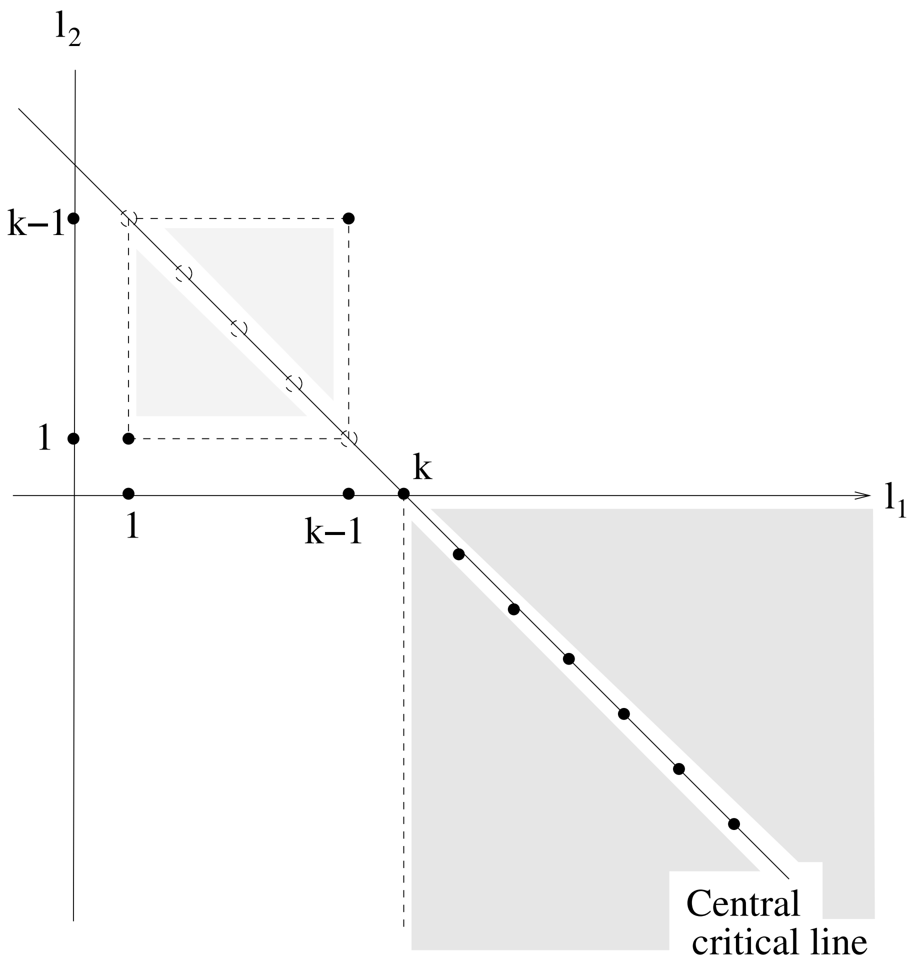

FIGURE 1. Critical and central critical weights for $\chi \mapsto L(f, \chi^{-1}, 0)$

Remark 4.3. This article is concerned with the p-adic L-function obtained by interpolating the L-values $L(f, \chi^{-1}, 0)$ for χ in $\Sigma^{(2)}$ or $\Sigma^{(2')}$. Since this L-value is unchanged if χ is replaced by χ^{ρ}, we may assume that $\ell_1 \geq 0$ and work simply with the region $\Sigma^{(2)}$. The main result of this paper (Theorem 5.13) relates the special values of this p-adic L-function at characters χ in $\Sigma^{(1)}_{\mathrm{cc}}$ (which is outside the range of interpolation) to the p-adic Abel-Jacobi images of generalised Heegner cycles. It would also be very interesting to study the values of this p-adic L-function at χ in $\Sigma^{(2')}_{\mathrm{cc}}$. We do not address this issue here. However, one could speculate that a study of the triple product L-function analogous to the one for the Rankin-Selberg L-function in this article may shed light on this issue. This intuition is suggested by the way in which the results of the present article are used in [BDP-ch] to yield information about the Katz p-adic L-function at critical characters that are outside the range of p-adic interpolation.

We assume henceforth that K satisfies the Heegner hypothesis for f i.e., that all the primes $q \mid N$ are either split or ramified in K, and further that if $q^2 \mid N$, then q is split in K. This implies that there exists a cyclic $\mathcal{O}_K$-ideal $\mathfrak{N}$ of norm N. We fix once and for all such a choice of $\mathfrak{N}$. We also fix an integer c prime to Nd_K, and set (as in Sec. 1.4) $\mathfrak{N}_c := \mathfrak{N} \cap \mathcal{O}_c$. Thus $\mathfrak{N}_c$ is a proper $\mathcal{O}_c$-ideal and $\mathcal{O}_c/\mathfrak{N}_c \simeq \mathcal{O}_K/\mathfrak{N} \simeq \mathbf{Z}/N\mathbf{Z}$. Let $U_c = \hat{\mathcal{O}}_c^{\times}$ denote the corresponding compact open subgroup of $\mathbb{A}_{K,f}^{\times}$, so that $U_c = \prod_q U_{c,q}$ with $U_{c,q} := (\mathcal{O}_c \otimes \mathbf{Z}_q)^{\times}$. For ε any character of conductor $N_\varepsilon | N$, we define $\mathfrak{N}_\varepsilon$ to be the unique ideal in $\mathcal{O}_K$ that divides $\mathfrak{N}$ and has norm equal to N_ε. Let ψ_ε be the composite homomorphism

$$U_c = \hat{\mathcal{O}}_c^{\times} \hookrightarrow \hat{\mathcal{O}}_K^{\times} \to \prod_{\mathfrak{q} | \mathfrak{N}_\varepsilon} (\mathcal{O}_{K,\mathfrak{q}}/\mathfrak{N}_\varepsilon \mathcal{O}_{K,\mathfrak{q}})^{\times} \simeq \prod_{q | N_\varepsilon} (\mathbf{Z}_q/N_\varepsilon \mathbf{Z}_q)^{\times} \xrightarrow{\prod \varepsilon_q} \mathbf{C}^{\times}. \tag{4.1.6}$$

Equivalently, if we set $\mathfrak{N}_{c,\varepsilon} := \mathfrak{N}_\varepsilon \cap \mathcal{O}_c$, then ψ_ε is the composite

$$U_c = \hat{\mathcal{O}}_c^\times \to (\hat{\mathcal{O}}_c/\mathfrak{N}_{c,\varepsilon}\hat{\mathcal{O}}_c)^\times \simeq (\mathcal{O}_K/\mathfrak{N}_\varepsilon\mathcal{O}_K)^\times \simeq (\mathbf{Z}/N_\varepsilon\mathbf{Z})^\times \xrightarrow{\varepsilon^{-1}} \mathbf{C}^\times.$$

The following definition will be key in what follows.

Definition 4.4. A Hecke character χ of K is said to be of finite type $(c, \mathfrak{N}, \varepsilon)$ if c divides $\mathfrak{f}_\chi$ and

$$\chi|_{U_c} = \psi_\varepsilon.$$

Note that a character χ of finite type $(c, \mathfrak{N}, \varepsilon)$ is necessarily unramified outside $c\mathfrak{N}_\varepsilon$. Further, we may think of χ as a character on proper $\mathcal{O}_c$-ideals prime to $\mathfrak{N}_{c,\varepsilon}$. Indeed, any such ideal $\mathfrak{a}$ is locally principal, i.e. $\mathfrak{a} = x\mathcal{O}_c$ for some $x = (x_\mathfrak{q}) \in \mathbb{A}_{K,\mathrm{fin}}^\times$, and we set

$$\chi(\mathfrak{a}) := \prod_{\mathfrak{q}\nmid\mathfrak{N}_\varepsilon} \chi_\mathfrak{q}(x_\mathfrak{q}). \tag{4.1.7}$$

This is independent of the choice of x since $\chi|_{\mathcal{O}_{c,q}^\times} = \psi_\varepsilon|_{\mathcal{O}_{c,q}^\times} = 1$ for $q \nmid N$, and χ is unramified at the primes of K dividing N but not dividing $\mathfrak{N}_\varepsilon$. Viewed in this manner, χ satisfies

$$\chi((\alpha)) = \alpha^{\ell_1}\bar{\alpha}^{\ell_2}\varepsilon(\alpha \bmod \mathfrak{N}_\varepsilon) \tag{4.1.8}$$

for any $\alpha \in K^\times$ that is a unit at all the primes dividing $\mathfrak{N}_\varepsilon$.

Let $\Sigma_{\mathrm{cc}}(\mathfrak{N})$ denote the set of those characters in $\Sigma_{\mathrm{cc}}^{(1)} \sqcup \Sigma_{\mathrm{cc}}^{(2)}$ that are of finite type $(c, \mathfrak{N}, \varepsilon_f)$ and that satisfy the following auxiliary condition: the local sign $\varepsilon_q(f, \chi^{-1}) = +1$ for all finite primes q. In view of our other hypotheses, this condition is automatic except possibly at those primes q lying in the set

$$S(f) := \{q : \quad q \mid (N, d_K),\ q \nmid N_{\varepsilon_f}\}.$$

For $i = 1, 2$, we define $\Sigma_{\mathrm{cc}}^{(i)}(\mathfrak{N})$ by

$$\Sigma_{\mathrm{cc}}^{(i)}(\mathfrak{N}) := \Sigma_{\mathrm{cc}}^{(i)} \cap \Sigma_{\mathrm{cc}}(\mathfrak{N}),$$

so that $\Sigma_{\mathrm{cc}}(\mathfrak{N})$ is the disjoint union:

$$\Sigma_{\mathrm{cc}}(\mathfrak{N}) = \Sigma_{\mathrm{cc}}^{(1)}(\mathfrak{N}) \sqcup \Sigma_{\mathrm{cc}}^{(2)}(\mathfrak{N}).$$

For $\chi \in \Sigma_{\mathrm{cc}}(\mathfrak{N})$, writing $(k+j, -j)$ for the weight of χ, we see that $\chi \in \Sigma_{\mathrm{cc}}^{(2)}(\mathfrak{N})$ or $\Sigma_{\mathrm{cc}}^{(1)}(\mathfrak{N})$ according as $j \geq 0$ or $j \in [-(k-1), -1]$. Let $\chi \in \Sigma_{\mathrm{cc}}^{(2)}(\mathfrak{N})$ be a Hecke character of infinity type $(k+j, -j)$. Recall the Shimura-Maass operator δ_k of equation (1.2.9) and let

$$\delta_k^j = \delta_{k+2j-2} \cdots \delta_{k+2}\delta_k$$

be the differential operator sending holomorphic modular forms of weight k to nearly holomorphic modular forms of weight $k+2j$. The modular form $\delta_k^j f$ can also be viewed as a function on pairs (L, t) consisting of a lattice L in $\mathbf{C}$ and an element t of order N in $\mathbf{C}/L$, satisfying the homogeneity properties of (4.1.3) with k replaced by $k+2j$.

In what follows, we will also fix a generator t of $\mathfrak{N}_c^{-1}/\mathcal{O}_c \simeq \mathbf{Z}/N\mathbf{Z}$. Let $\mathfrak{a}$ be a proper $\mathcal{O}_c$-ideal prime to $\mathfrak{N}_c$ and choose $\alpha \in K^\times$ such that $\mathfrak{b} := \alpha\mathfrak{a} \subset \mathcal{O}_c$ and $\alpha \equiv 1 \bmod \mathfrak{N}$. Then the image of t under the composite map

$$\mathfrak{N}_c^{-1}/\mathcal{O}_c \to \mathfrak{N}_c^{-1}\mathfrak{b}^{-1}/\mathfrak{b}^{-1} \xrightarrow{\cdot\alpha} \mathfrak{N}_c^{-1}\mathfrak{a}^{-1}/\mathfrak{a}^{-1}$$

is independent of the choice of α, and will be denoted $t_\mathfrak{a}$. Thus the choice of t gives rise to a generator $t_\mathfrak{a}$ of $\mathfrak{N}_c^{-1}\mathfrak{a}^{-1}/\mathfrak{a}^{-1}$ for every proper $\mathcal{O}_c$-ideal $\mathfrak{a}$ prime to $\mathfrak{N}_c$.

Lemma 4.5. *Let $\mathfrak{a}$ be any proper $\mathcal{O}_c$-ideal prime to $\mathfrak{N}_c$ and suppose χ is a Hecke character in $\Sigma_{\mathrm{cc}}^{(2)}(\mathfrak{N})$ of infinity type $(k+j, -j)$. With t fixed, the expression*

$$\chi^{-1}(\mathfrak{a})\mathrm{N}\mathfrak{a}^{-j} \cdot \delta_k^j f(\mathfrak{a}^{-1}, t_\mathfrak{a}) \tag{4.1.9}$$

depends only on the class of $\mathfrak{a}$ in $\mathrm{Pic}(\mathcal{O}_c)$.

Proof. Note that since $\mathfrak{a}$ is prime to $\mathfrak{N}_c$, it is certainly prime to $\mathfrak{N}_{c,\varepsilon}$ as well and so the expression $\chi^{-1}(\mathfrak{a})$ is well defined. The lemma then follows immediately from the equations (4.1.2) (with f replaced by $\delta_k^j(f)$ and k by $k+2j$), (4.1.3) and (4.1.8). $\square$

Theorem 4.6. *Let f be a normalised eigenform in $S_k(\Gamma_0(N), \varepsilon_f)$ and let $\chi \in \Sigma_{cc}^{(2)}(\mathfrak{N})$ be a Hecke character of K of infinity type $(k+j, -j)$. Suppose also that c and d_K are odd, and let w_K denote the number of roots of unity in K. Then*

$$C(f,\chi,c) \cdot L(f,\chi^{-1},0) = \left| \sum_{[\mathfrak{a}] \in \mathrm{Pic}(\mathcal{O}_c)} \chi^{-1}(\mathfrak{a}) \mathrm{N}\mathfrak{a}^{-j} \cdot (\delta_k^j f)(\mathfrak{a}^{-1}, t_{\mathfrak{a}}) \right|^2,$$

where the representatives $\mathfrak{a}$ of the ideal classes in $\mathrm{Pic}(\mathcal{O}_c)$ are chosen to be prime to $\mathfrak{N}_c$ and the constant $C(f,\chi,c)$ is given by

$$C(f,\chi,c) = \frac{1}{4}\pi^{k+2j-1}\Gamma(j+1)\Gamma(k+j) w_K |d_K|^{1/2} \cdot c \,\mathrm{vol}(\mathcal{O}_c)^{-\ell} \cdot 2^{\# S_f} \cdot \prod_{q|c} \frac{(q - \varepsilon_K(q))}{q-1}.$$

Remark 4.7. The restriction that c and d_K are odd is made for convenience to simplify the local calculations in Section 4.6 at primes dividing cd_K.

The rest of this chapter will be devoted to proving Theorem 4.6 using Waldspurger's results relating period integrals to L-values. The reader whose main interest is in p-adic methods can take this result on faith and continue reading from Sec. 5.1 onwards.

4.2. **Differential operators.** We recall some general facts about the Shimura-Maass operators that were introduced in Sec. 1.2 and appear in the statement of the theorem above. Let Γ be a congruence subgroup of $\mathbf{SL}_2(\mathbf{Z})$ and denote by $C_k^\infty(\Gamma)$ the space of C^∞-modular forms of weight k on Γ. We also denote by $\tilde{C}_k^\infty(\Gamma)$ the space of C^∞-functions on $\mathcal{H}$ such that

$$f(\gamma z) = (c'z+d')^k |c'z+d'|^{-k} f(z)$$

for all $\gamma = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \in \Gamma$. (For the moment we will use the symbol f for an arbitrary modular form in $C_k^\infty(\Gamma)$ or $\tilde{C}_k^\infty(\Gamma)$.) Recall that the weight k Shimura-Maass raising operator $\delta_k : C_k^\infty(\Gamma) \to C_{k+2}^\infty(\Gamma)$ is defined by

$$\delta_k(f) = \frac{1}{2\pi i}\left(\frac{\partial}{\partial z} + \frac{k}{z - \bar{z}}\right) f. \tag{4.2.1}$$

Via the isomorphism

$$C_k^\infty(\Gamma) \simeq \tilde{C}_k^\infty(\Gamma), \qquad f(z) \mapsto \tilde{f}(z) := f(z) y^{k/2}, \tag{4.2.2}$$

we see that δ_k is identified with $-\frac{1}{4\pi} R_k$ where

$$R_k : \tilde{C}_k^\infty(\Gamma) \to \tilde{C}_{k+2}^\infty(\Gamma), \qquad R_k(f) = \left((z-\bar{z})\frac{\partial}{\partial z} + \frac{k}{2}\right) f. \tag{4.2.3}$$

Let us define (following the discussion in [Bump] §2.1)

$$L_k : \tilde{C}_k^\infty(\Gamma) \to \tilde{C}_{k-2}^\infty(\Gamma), \qquad L_k(f) = -\left((z-\bar{z})\frac{\partial}{\partial \bar{z}} + \frac{k}{2}\right) f, \tag{4.2.4}$$

and

$$\Delta_k : \tilde{C}_k^\infty(\Gamma) \to \tilde{C}_k^\infty(\Gamma), \qquad \Delta_k(f) = -y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) + iky\frac{\partial}{\partial x}. \tag{4.2.5}$$

These operators satisfy

$$\Delta_k = -L_{k+2}R_k - \frac{k}{2}\left(1 + \frac{k}{2}\right) = -R_{k-2}L_k + \frac{k}{2}\left(1 - \frac{k}{2}\right). \tag{4.2.6}$$

Note that via the isomorphism (4.2.2), the lowering operator L_k corresponds to $f \mapsto 2i\frac{\partial}{\partial \bar{z}} f$ on $C_k^\infty(\Gamma)$. Thus if f is holomorphic, then $L_k(\tilde{f}) = 0$.

Definition 4.8. Let j be a nonnegative integer and $f \in \tilde{C}_k^\infty(\Gamma)$. Then $R^j f$ is defined by

$$R^j f = (R_{k+2j-2} \circ R_{k+2j-4} \circ \cdots \circ R_{k+2} \circ R_k) f.$$

Lemma 4.9. *Suppose that $f \in C_k^\infty(\Gamma)$ is holomorphic. Then for $j \geq 0$, the form $R^j \tilde{f}$ is an eigenfunction of Δ_{k+2j} with eigenvalue $\mu_j + \lambda_j$ where $\mu_j := j(k+j-1)$ and $\lambda_j := \frac{k+2j}{2}\left(1 - \frac{k+2j}{2}\right)$.*

Proof. Since f is holomorphic, we have $L_k(\tilde{f}) = 0$. Hence $\Delta_k \tilde{f} = \frac{k}{2}\left(1 - \frac{k}{2}\right)$ by (4.2.6) and the result holds for $j = 0$. We now work inductively, assuming the result holds for $j-1$. By (4.2.6) again,

$$\begin{aligned}
\Delta_{k+2j} R^j \tilde{f} &= (-R_{k+2j-2} L_{k+2j} + \lambda_j) R^j \tilde{f} \\
&= -R_{k+2j-2} L_{k+2j} R_{k+2j-2} R^{j-1} \tilde{f} + \lambda_j R^j \tilde{f} \\
&= R_{k+2j-2}\left(\Delta_{k+2j-2} + \frac{k+2j-2}{2}\left(1 + \frac{k+2j-2}{2}\right)\right) R^{j-1}\tilde{f} + \lambda_j R^j \tilde{f} \\
&= R_{k+2j-2}\left(\mu_{j-1} + \lambda_{j-1} + \frac{k+2j-2}{2}\left(1 + \frac{k+2j-2}{2}\right)\right) R^{j-1}\tilde{f} + \lambda_j R^j \tilde{f} \\
&= R_{k+2j-2}\left(\mu_{j-1} + k + 2j - 2\right) R^{j-1}\tilde{f} + \lambda_j R^j \tilde{f} \\
&= \left(\mu_{j-1} + k + 2j - 2 + \lambda_j\right) R^j \tilde{f} = (\mu_j + \lambda_j) R^j \tilde{f}. \quad \square
\end{aligned}$$

Definition 4.10. Let $f, g \in C_k^\infty(\Gamma)$ and suppose at least one of f or g is a cusp form. Then set

$$\langle f, g \rangle = \frac{1}{[\mathbf{SL}_2(\mathbf{Z}) : \Gamma]} \int_{\Gamma\backslash\mathcal{H}} f(z)\overline{g(z)} y^k \frac{dxdy}{y^2}.$$

Likewise, for $f, g \in \tilde{C}_k^\infty(\Gamma)$ with at least one being cuspidal, we set

$$\langle f, g \rangle = \frac{1}{[\mathbf{SL}_2(\mathbf{Z}) : \Gamma]} \int_{\Gamma\backslash\mathcal{H}} f(z)\overline{g(z)} \frac{dxdy}{y^2}.$$

Clearly, for $f, g \in C_k^\infty(\Gamma)$, we have $\langle f, g \rangle = \langle \tilde{f}, \tilde{g} \rangle$.

Lemma 4.11. *Suppose that $f, g \in C_k^\infty(\Gamma)$ are holomorphic. Then*

$$\langle R^j \tilde{f}, R^j \tilde{g} \rangle = \frac{\Gamma(j+1)\Gamma(k+j)}{\Gamma(k)} \langle \tilde{f}, \tilde{g} \rangle, \tag{4.2.7}$$

and

$$\langle \delta_k^j f, \delta_k^j g \rangle = \frac{1}{(4\pi)^{2j}} \frac{\Gamma(j+1)\Gamma(k+j)}{\Gamma(k)} \langle f, g \rangle. \tag{4.2.8}$$

Proof. Clearly (4.2.7) and (4.2.8) are equivalent. We will prove (4.2.7) inductively. Invoking [Bump] Prop. 2.1.3, equation (4.2.6) and Lemma 4.9 in turn, we find

$$\begin{aligned}
\langle R^j \tilde{f}, R^j \tilde{g} \rangle &= \langle R^{j-1}\tilde{f}, -L_{k+2j} R_{k+2j-2} R^{j-1} \tilde{g} \rangle \\
&= \langle R^{j-1}\tilde{f}, \left(\Delta_{k+2j-2} + \frac{k+2j-2}{2}\left(1 + \frac{k+2j-2}{2}\right)\right) R^{j-1}\tilde{g} \rangle \\
&= \langle R^{j-1}\tilde{f}, \left(\mu_{j-1} + \lambda_{j-1} + \frac{k+2j-2}{2}\left(1 + \frac{k+2j-2}{2}\right)\right) R^{j-1}\tilde{g} \rangle = \mu_j \langle R^{j-1}\tilde{f}, R^{j-1}\tilde{g} \rangle.
\end{aligned}$$

Hence

$$\langle R^j \tilde{f}, R^j \tilde{g} \rangle = \langle \tilde{f}, \tilde{g} \rangle \cdot \prod_{1 \leq t \leq j} \mu_t = \frac{\Gamma(j+1)\Gamma(k+j)}{\Gamma(k)} \langle \tilde{f}, \tilde{g} \rangle. \quad \square$$

4.3. **Period integrals and values at CM points.** Let $A_0 := \mathbf{C}/\mathcal{O}_c$ and t_0 be the $\mathfrak{N}$-torsion point on A_0 corresponding to our choice of $t \in \mathfrak{N}_c^{-1}/\mathcal{O}_c$. The pair (A_0, t_0) determines a point P_{A_0} on the modular curve $X_1(N)$. Let $\tau \in \mathcal{H}$ be any any point lying over P_{A_0}. Thus there is a unique isomorphism

$$A_\tau := \mathbf{C}/\mathbf{Z}\tau + \mathbf{Z} \overset{\cdot \Lambda_\tau}{\simeq} \mathbf{C}/\mathcal{O}_c$$

sending $[1/N]$ to t_0, which on tangent spaces is given by multiplication by a scalar $\Lambda_\tau \in K^\times$. Hence $\mathcal{O}_c = \Lambda_\tau(\mathbf{Z}\tau + \mathbf{Z})$ and

$$\frac{\Lambda_\tau}{N} \equiv t \mod \mathcal{O}_c.$$

Thus

$$\Lambda_\tau \in \overline{\mathfrak{N}_c}, \qquad \text{and} \qquad (\Lambda_\tau, \mathfrak{N}_c) = 1. \tag{4.3.1}$$

Let $\boldsymbol{\xi} : K \hookrightarrow M_2(\mathbf{Q})$ be the embedding that describes the action of K on $H_1(A_\tau(\mathbf{C}), \mathbf{Q})$ with respect to the basis $(\tau, 1)$ i.e. given by

$$\alpha \cdot \left[\begin{array}{c} \tau \\ 1 \end{array}\right] = \boldsymbol{\xi}(\alpha) \left[\begin{array}{c} \tau \\ 1 \end{array}\right].$$

Explicitly, for $a, b \in \mathbf{Q}$,

$$\boldsymbol{\xi}(a + b\tau) = \begin{pmatrix} a + b\,\mathrm{Tr}(\tau) & -b\mathrm{N}\tau \\ b & a \end{pmatrix}. \tag{4.3.2}$$

Let $\mathrm{M}_0(N)$ be the order defined by

$$\mathrm{M}_0(N) := \left\{ \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \in M_2(\mathbf{Z}) : c' \equiv 0 \mod N \right\}.$$

Then, via the embedding $\boldsymbol{\xi}$,

$$K \cap \mathrm{M}_0(N) = \mathrm{End}(A_\tau, \langle [1/N] \rangle) = \mathrm{End}(\mathbf{C}/\mathcal{O}_c, \langle t \rangle) = \mathcal{O}_c,$$

so that $\boldsymbol{\xi}$ is a *Heegner* embedding of conductor c. A different choice of τ will give an embedding $\boldsymbol{\xi}'$ that is conjugate to $\boldsymbol{\xi}$ by an element of $\Gamma_0(N)$. Note that $\boldsymbol{\xi}$ gives rise to a map of algebraic groups

$$\boldsymbol{\xi} : \mathrm{Res}_{K/\mathbf{Q}} \mathbb{G}_m \hookrightarrow \mathbf{GL}_{2,\mathbf{Q}}$$

and hence a map on adelic points $\boldsymbol{\xi}_{\mathbb{A}} : \mathbb{A}_K^\times \hookrightarrow \mathbf{GL}_2(\mathbb{A}_\mathbf{Q})$. We consider $\mathbb{A}_K^\times$ as a subgroup of $\mathbf{GL}_2(\mathbb{A}_\mathbf{Q})$ via this embedding.

As in the previous section, let $\delta_k^j f$ denote the nearly holomorphic modular form of weight $\ell := k + 2j$ obtained by applying the Shimura-Maass differential operator j times to f. We use the embedding $\boldsymbol{\xi}$ to associate to the classical modular form $\delta_k^j f$ an automorphic form F^j on $\mathbf{GL}_2(\mathbb{A}_\mathbf{Q})$ as follows. First, let

$$U_q' := (\mathrm{M}_0(N) \otimes \mathbf{Z}_q)^\times, \quad U' := \widehat{\mathrm{M}_0(N)}^\times = \prod_q U_q' \subset \mathbf{GL}_2(\mathbb{A}_f)$$

and define a character $\omega_f = \prod_q \omega_{f,q}$ of U' by setting

$$\omega_{f,q} \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} = \varepsilon_{f,q}(d')$$

for $\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \in U_q'$. Now, for $g \in \mathbf{GL}_2(\mathbb{A}_\mathbf{Q})$, write

$$g = \gamma \cdot (u\gamma_\infty), \qquad \text{with} \quad \gamma \in \mathbf{GL}_2(\mathbf{Q}), \quad u \in U', \quad \gamma_\infty \in \mathbf{GL}_2(\mathbf{R})^+.$$

Then set

$$F^j(g) = \delta_k^j(f)(\gamma_\infty(\tau)) j(\gamma_\infty, \tau)^{-\ell} \omega_f(u),$$

where we define

$$J(\gamma', z) := c'z + d' \qquad \text{and} \qquad j(\gamma', z) := \det(\gamma')^{-1/2}(c'z + d'),$$

for any $\gamma' = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \in \mathbf{GL}_2(\mathbf{R})$. One checks easily that this definition is independent of the choice of decomposition of g. Further, for any $\alpha \in K_\infty^\times$,

$$F^j(g\alpha) = F^j(g) j(\alpha, \tau)^{-\ell} = \alpha^{-\ell} \mathbf{N}_K(\alpha)^{\ell/2} F^j(g).$$

Here $\mathbf{N}_K = \mathbf{N} \circ N_{K/\mathbf{Q}}$ is the usual norm character on K, $\mathbf{N}$ being the norm character on $\mathbf{Q}$.

Lemma 4.12. *The restriction of the character ω_f of U' to U_c (via the embedding $\boldsymbol{\xi}_{\mathbb{A}}$) is ψ_{ε_f}.*

Proof. For $q \nmid N$, the restrictions of ω_f to U_q' and of ψ_{ε_f} to $U_{c,q}$ are both trivial. Suppose therefore that q divides N. Let $a + b\tau \in \mathcal{O}_c \cap U_{c,q}$. By (4.3.2), we have $a \in \mathbf{Z}$ and $b \in N\mathbf{Z}$. Since N/Λ_τ lies in $\mathfrak{N}_c \otimes \mathbf{Z}_q$ and $\Lambda_\tau \tau \in \mathcal{O}_c$, the element $N\tau = (N/\Lambda_\tau) \cdot \Lambda_\tau \tau$ also lies in $\mathfrak{N}_c \otimes \mathbf{Z}_q$, so that

$$\psi_{\varepsilon_f, q}(a + b\tau) = \varepsilon_{f,q}(a) = \omega_{f,q}(\boldsymbol{\xi}_q(a + b\tau)).$$

Since $\mathcal{O}_c \cap U_{c,q}$ is dense in $U_{c,q}$, it follows that $\psi_{\varepsilon_f}(u) = \omega_f(\boldsymbol{\xi}_{\mathbb{A}}(u))$ for all $u \in U_{c,q} \subseteq U_c$. $\square$

Proposition 4.13. *Suppose $\chi \in \Sigma_{cc}^{(2)}(\mathfrak{N})$ is of infinity type $(k+j,-j)$. Let η and η' be Grossencharacters defined by*

$$\eta := \chi^{-1}\mathbf{N}_K^{-j}, \qquad \eta' := \eta\mathbf{N}_K^{\ell/2},$$

so that η' is unitary. Then

$$\frac{1}{h_c}\sum_{[\mathfrak{a}]\in \mathrm{Pic}(\mathcal{O}_c)} \chi^{-1}(\mathfrak{a})\mathrm{N}\mathfrak{a}^{-j}\cdot(\delta_k^j f)(\mathfrak{a}^{-1}, t_{\mathfrak{a}}) = (2\pi i)^{\ell}\Lambda_\tau^{-\ell}\int_{K^\times K_\infty^\times\setminus \mathbb{A}_K^\times} F^j(\boldsymbol{\xi}_{\mathbb{A}}(x))\cdot \eta'(x)d^\times x,$$

where $h_c := \#\,\mathrm{Pic}(\mathcal{O}_c)$ and the measure $d^\times x$ on $K^\times K_\infty^\times \setminus \mathbb{A}_K^\times$ is chosen to have total volume 1.

Proof. Let us pick elements $y_i \in \hat{\mathcal{O}}_c$ such that $\mathbb{A}_K^\times = \sqcup_{i=1}^h K^\times\cdot U_c\cdot K_\infty^\times\cdot y_i$. We may assume that we have picked y_i to satisfy

$$y_{i,\mathfrak{q}} \equiv 1 \mod \mathfrak{N}\mathcal{O}_{K,\mathfrak{q}} \text{ for } \mathfrak{q}\mid\mathfrak{N}. \tag{4.3.3}$$

Let $\mathfrak{a}_i := y_i\mathcal{O}_c$ be the associated proper $\mathcal{O}_c$-ideal, so that

$$\eta(y_i) = \eta(\mathfrak{a}_i) = \chi^{-1}(\mathfrak{a}_i)\mathrm{N}\mathfrak{a}_i^{-j}. \tag{4.3.4}$$

Let $U'' := \prod_q U_q''$ be the subgroup of U' defined by $U_q'' := U_q'$ if $q\nmid N$ and

$$U_q'' := \left\{ \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \in U_q' : d' \equiv 1 \mod N \right\}.$$

By strong approximation for $\mathbf{GL}_2$, we may write

$$\xi_{\mathbb{A}}(y_i) = g_i(g_{U,i}\cdot\gamma_i) \qquad \text{with} \quad g_i \in \mathbf{GL}_{2,\mathbf{Q}}, \quad g_{U,i}\in U'', \quad \gamma_i \in \mathbf{GL}_2(\mathbf{R})^+.$$

Since $g_i\gamma_i = 1$, we have $\gamma_i^{-1} = g_i \in \mathbf{GL}_2(\mathbf{Q})^+$. Further, since ξ is a Heegner embedding, we have $g_i g_{U,i} \in \widehat{\mathrm{M}_0(N)}$ and consequently $\gamma_i^{-1} \in \widehat{\mathrm{M}_0(N)} \cap \mathbf{GL}_2(\mathbf{Q})^+$. i.e.

$$\gamma_i^{-1} = \begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix} \in M_2(\mathbf{Z})\cap \mathbf{GL}_2(\mathbf{Q})^+, \quad c_i \in N\mathbf{Z}. \tag{4.3.5}$$

In fact, on account of (4.3.3) and the fact that $N\tau \in \mathfrak{N}_c\otimes \mathbf{Z}_q$ for $q\mid N$ (see the proof of Lemma 4.12 above), we also have $d_i \equiv 1 \mod N$. Now, for $u\in U_c$,

$$F^j(\boldsymbol{\xi}_{\mathbb{A}}(xu)) = F^j(\boldsymbol{\xi}_{\mathbb{A}}(x))\omega_f(\boldsymbol{\xi}_{\mathbb{A}}(u)) = F^j(\boldsymbol{\xi}_{\mathbb{A}}(x))\varepsilon_f(u).$$

Hence

$$\begin{aligned}\int_{K^\times K_\infty^\times\setminus \mathbb{A}_K^\times} F^j(\boldsymbol{\xi}_{\mathbb{A}}(x))\cdot\eta'(x)d^\times x &= \frac{1}{h_c}\sum_{i=1}^{h_c}\delta_k^j(f)(\gamma_i\tau)j(\gamma_i,\tau)^{-\ell}\omega_f(g_{U,i})\eta'(y_i)\\ &= \frac{1}{h_c}\sum_{i=1}^{h_c}\delta_k^j(f)(\gamma_i\tau)J(\gamma_i,\tau)^{-\ell}\eta(y_i),\end{aligned}$$

since $\omega_f(g_{U,i}) = 1$. Taking into account (4.3.4), it will suffice to show that

$$(2\pi i)^\ell\Lambda_\tau^{-\ell}\delta_k^j(f)(\gamma_i\tau)J(\gamma_i,\tau)^{-\ell} = (\delta_k^j f)(\mathfrak{a}_i^{-1}, t_{\mathfrak{a}_i}).$$

From the choice of γ_i, we see that the class of $\gamma_i\tau$ in $X_1(N)$ corresponds to the pair $(\mathbf{C}/\mathfrak{a}_i^{-1}, t_{\mathfrak{a}_i})$, and there is a unique isomorphism

$$\mathbf{C}/(\mathbf{Z}\gamma_i\tau+\mathbf{Z}) \overset{\cdot\lambda_i}{\simeq} \mathbf{C}/\mathfrak{a}_i^{-1},$$

sending $[1/N]$ to $t_{\mathfrak{a}_i}$, with a scalar $\lambda_i\in K^\times$. Note that

$$J(\gamma_i,\tau)^{-1} = J(\gamma_i^{-1},\gamma_i\tau) = c'(\gamma_i\tau)+d'.$$

The scalar λ_i may then be identified from the fact that there is a commutative diagram:

$$\begin{array}{ccc} \mathbf{C}/(\mathbf{Z}\tau+\mathbf{Z}) & \xrightarrow{J(\gamma_i,\tau)^{-1}} & \mathbf{C}/(\mathbf{Z}\gamma_i\tau+\mathbf{Z}) \\ \big\downarrow{\scriptstyle \Lambda_\tau} & & \big\downarrow{\scriptstyle \lambda_i} \\ \mathbf{C}/\mathcal{O}_c & \longrightarrow & \mathbf{C}/\mathfrak{a}_i^{-1}. \end{array}$$

Thus $\lambda_i = \Lambda_\tau \cdot J(\gamma_i, \tau)$, and

$$\begin{aligned} \delta_k^j(f)(\mathfrak{a}_i^{-1}, t_{\mathfrak{a}_i}) &= \delta_k^j(f)(\mathbf{C}/(\mathbf{Z}\gamma_i\tau + \mathbf{Z}), \lambda_i^{-1}dz, [1/N]) \\ &= \Lambda_\tau^{-\ell}(2\pi i)^\ell \delta_k^j(f)(\gamma_i\tau)J(\gamma_i,\tau)^{-\ell}. \quad \square \end{aligned}$$

In the next few sections we will study the period integral

$$L_{\eta',\boldsymbol{\xi}}(F^j) := \int_{K^\times K_\infty^\times\backslash\mathbb{A}_K^\times} F^j(\boldsymbol{\xi}_{\mathbb{A}}(x)) \cdot \eta'(x) d^\times x \tag{4.3.6}$$

using the method of Waldspurger.

4.4. **Explicit theta lifts.** Let ψ denote the additive character of $\mathbb{A}/\mathbf{Q}$ given by $\psi((x_v)_v) = \prod_v \psi_v(x_v)$, where

$$\psi_\infty(x) = e^{2\pi i x}, \qquad \psi_q(x) = e^{-2\pi i x} \qquad \text{for} \quad x \in \mathbf{Z}\left[\frac{1}{q}\right] \subset \mathbf{Q}_q.$$

Let $(V, \langle , \rangle)$ be an even dimensional orthogonal space over $\mathbf{Q}$, and denote by $\mathbf{O}(V)$ (resp. $\mathbf{GO}(V)$) its isometry group (resp. orthogonal similitude group). Recall the Weil representation $r_\psi = \prod_v r_{\psi,v}$ of the group $\mathbf{SL}_2(\mathbb{A}) \times \mathbf{O}(V)(\mathbb{A})$ on the Schwartz space $\mathcal{S}(V(\mathbb{A}))$. On the orthogonal group, $r_{\psi,v}$ is given by

$$r_{\psi,v}(g)\varphi(x) = \varphi(g^{-1}\cdot x) \qquad \text{for } g \in \mathbf{O}(V)(\mathbf{Q}_v), \varphi \in \mathcal{S}(V(\mathbf{Q}_v)).$$

On $\mathbf{SL}_2(\mathbf{Q}_v)$, the representation $r_{\psi,v}$ is described by its action on the matrices

$$U(a) := \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}, \quad D(a) := \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}, \quad W := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

by the equations

$$\begin{aligned} r_{\psi,v}(U(a))\varphi(x) &= \psi_v(\frac{1}{2}\langle ax, x\rangle)\varphi(x), \\ r_{\psi,v}(D(a))\varphi(x) &= \chi_{V,v}(a)|a|_v^{\dim(V)/2}\varphi(ax), \\ r_{\psi,v}(W)\varphi(x) &= \gamma_{V,v}\hat{\varphi}(x), \end{aligned}$$

where $\chi_{V,v}$ is a quadratic character and $\gamma_{V,v}$ is an eighth root of unity, that can be read off from [JL] §1. In the cases of interest to us, they can also be found listed in the table in [P], §3.4. The Fourier transform $\hat{\varphi}$ is defined by

$$\hat{\varphi}(x) = \int_{V(\mathbf{Q}_v)} \varphi(y)\psi_v(\langle y, x\rangle)dy,$$

the measure dy on $V(\mathbf{Q}_v)$ being chosen such that $\hat{\hat{\varphi}}(x) = \varphi(-x)$.

We will need to extend the Weil representation to similitude groups, following Harris-Kudla [HK1]. Let $\mathcal{R}$ be the group defined by:

$$\mathcal{R} := \{(g,h) \in \mathbf{GL}_2 \times \mathbf{GO}(V) : \det(g) = \nu(h)\}$$

where ν denotes the similitude character of $\mathbf{GO}(V)$. Then r_ψ can be extended to $\mathcal{R}(\mathbb{A})$ by

$$r_\psi(g,h)\varphi = r_\psi\left(g \cdot \begin{pmatrix} 1 & 0 \\ 0 & \det g^{-1} \end{pmatrix}\right) L(h)\varphi,$$

where

$$L(h)\varphi(x) = |\nu(h)|^{-\dim(V)/4}\varphi(h^{-1}x).$$

Let $\mathbf{GO}(V)^0$ denote the algebraic connected component of $\mathbf{GO}(V)$. If F is an automorphic form on $\mathbf{GL}_2(\mathbb{A})$ and $\varphi \in \mathcal{S}(V(\mathbb{A}))$, we define for $h \in \mathbf{GO}(V)(\mathbb{A})$,

$$\theta_\varphi(F)(h) := \int_{\mathbf{SL}_2(\mathbf{Q})\backslash\mathbf{SL}_2(\mathbb{A})} \sum_{x \in V(\mathbf{Q})} r_\psi(gg', h)\varphi(x)F(gg')d^{(1)}g,$$

where g' is chosen such that $\det(g') = \nu(h)$. Likewise, in the opposite direction, if F' is an automorphic form on $\mathbf{GO}(V)^0(\mathbb{A})$, and $g \in \mathbf{GL}_2(\mathbb{A})$ is such that $\det(g) \in \nu(\mathbf{GO}(V)(\mathbb{A}))$, we set

$$\theta_\varphi^t(F')(g) := \int_{\mathbf{O}(V)(\mathbf{Q})\backslash\mathbf{O}(V)(\mathbb{A})} \sum_{x \in V(\mathbf{Q})} r_\psi(g, hh')\varphi(x)F'(hh')dh,$$

where $h' \in \mathbf{GO}(V)^0(\mathbb{A})$ is chosen such that $\det(g) = \nu(h')$. (We refer the reader to [P], §1, for the choices of measures in the above and in what follows.) If π (resp. Π) is an automorphic representation of $\mathbf{GL}_2(\mathbb{A})$ (resp. of $\mathbf{GO}(V)^0(\mathbb{A})$), we define

$$\theta(\pi) := \{\theta_\varphi(F) : F \in \pi, \varphi \in \mathcal{S}(V(\mathbb{A}))\};$$
$$\theta^t(\Pi) := \{\theta^t_\varphi(F') : F' \in \Pi, \varphi \in \mathcal{S}(V(\mathbb{A}))\}.$$

Now set $V := M_2(\mathbf{Q})$ and consider V as an orthogonal space over $\mathbf{Q}$ with bilinear form

$$\langle x, y\rangle = \frac{1}{2}(xy^\iota + yx^\iota), \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix}^\iota = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

The associated quadratic form is just $x \mapsto xx^\iota = \det(x)$. The group $\mathbf{GO}(V)^0$ is identified with the quotient $\mathbf{Q}^\times \setminus \mathbf{GL}_2 \times \mathbf{GL}_2$ via the map $(\alpha, \beta) \mapsto \delta(\alpha, \beta)$ where $\delta(\alpha, \beta)(x) = \alpha x \beta^{-1}$. Thus an automorphic representation of $\mathbf{GO}(V)^0(\mathbb{A})$ is identified with a pair (π_1, π_2) of representations of $\mathbf{GL}_2(\mathbb{A})$, such that the product of the central characters of π_1 and π_2 is trivial. To ease notation, we will often just write (α, β) to denote the element $\delta(\alpha, \beta)$.

Let π denote the (unitary) automorphic representation of $\mathbf{GL}_2(\mathbb{A})$ associated to f. The following theorem is the classical Jacquet-Langlands correspondence realized using theta functions, and is essentially due to Shimizu [SH]. (See also [Wa] §3.2.)

Theorem 4.14. (1) *$\theta(\bar\pi) = \bar\pi \times \pi$, where $\bar\pi = \pi^\vee = \pi \otimes \varepsilon_f^{-1}$.*
(2) *$\theta^t(\pi \times \bar\pi) = \pi$.*

We will need a statement involving specific forms in π and $\bar\pi$ and explicit theta functions i.e. explicit choices of Schwartz functions. For any finite prime q, let q^{n_q} be the exact power of q dividing N and for any set A, let $\mathbf{I}_A$ denote the characteristic function of A. For q a prime dividing N, we will set

$$\varphi_q^1 \begin{pmatrix} a & b \\ c & d \end{pmatrix} := \begin{cases} \mathbf{I}_{\mathbf{Z}_q}(a)\mathbf{I}_{\mathbf{Z}_q}(b)\mathbf{I}_{q^{n_q}\mathbf{Z}_q}(c)\mathbf{I}_{\mathbf{Z}_q}(d), & \text{if } q \nmid N_{\varepsilon_f}, \\ \varepsilon_{f,q}(d)\mathbf{I}_{\mathbf{Z}_q}(a)\mathbf{I}_{\mathbf{Z}_q}(b)\mathbf{I}_{q^{n_q}\mathbf{Z}_q}(c)\mathbf{I}_{\mathbf{Z}_q^\times}(d), & \text{if } q \mid N_{\varepsilon_f}; \end{cases} \tag{4.4.1}$$

$$\varphi_q^2 \begin{pmatrix} a & b \\ c & d \end{pmatrix} := \begin{cases} \frac{1}{q}\mathbf{I}_{\mathbf{Z}_q}(a)\mathbf{I}_{\mathbf{Z}_q}(b)\mathbf{I}_{q^{n-1}\mathbf{Z}_q}(c)\mathbf{I}_{\mathbf{Z}_q}(d), & \text{if } q \nmid N_{\varepsilon_f}, \\ \frac{1}{q}\varepsilon_{f,q}(d)\mathbf{I}_{\mathbf{Z}_q}(a)\mathbf{I}_{\mathbf{Z}_q}(b)\mathbf{I}_{q^{n_q-1}\mathbf{Z}_q}(c)\mathbf{I}_{\mathbf{Z}_q^\times}(d), & \text{if } q \mid N_{\varepsilon_f}. \end{cases} \tag{4.4.2}$$

Let Σ denote the set of primes dividing N. For now we will fix a subset Ξ of Σ and consider the following Schwartz function: $\varphi^\Xi := \otimes_q \varphi_q^\Xi$ where

(i) For $q \nmid N$, $\varphi_q^\Xi = \mathbf{I}_{\mathrm{M}_0(N)\otimes\mathbf{Z}_q} = \mathbf{I}_{M_2(\mathbf{Z}_q)}$;
(ii) For $q \mid N$, $\varphi_q^\Xi = \varphi_q^1$ or φ_q^2 according as $q \notin \Xi$ or $q \in \Xi$;
(iii) For $q = \infty$, we identify $M_2(\mathbf{R}) = (K \otimes \mathbf{R}) + (K \otimes \mathbf{R})^\perp = \mathbf{C} + \mathbf{C}^\perp$ and set $\varphi_\infty^\Xi = \varphi_\infty$, with

$$\varphi_\infty(\mathbf{u} + \mathbf{v}) = \bar{\mathbf{u}}^\ell p_j(4\pi\langle \mathbf{v}, \mathbf{v}\rangle)e^{-2\pi(|\langle \mathbf{u},\mathbf{u}\rangle| + |\langle \mathbf{v},\mathbf{v}\rangle|)}, \tag{4.4.3}$$

for $\mathbf{u} \in \mathbf{C}, \mathbf{v} \in \mathbf{C}^\perp$, where p_j denotes the jth Laguerre polynomial

$$p_j(X) = \sum_{s=0}^{j} \binom{j}{s} \frac{(-X)^s}{s!}.$$

Lemma 4.15. *Suppose $\kappa_\theta := \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \in \mathbf{SO}_2(\mathbf{R})$ and $\kappa_1, \kappa_2 \in (K \otimes \mathbf{R})^{(1)} \subset \mathbf{GL}_2(\mathbf{R})$. Then*

$$r_\psi(\kappa_\theta, (\kappa_1, \kappa_2))\varphi_\infty = e^{ik\theta} \cdot \kappa_1^\ell \cdot \kappa_2^{-\ell}\varphi_\infty.$$

Proof. This is [Xue], Prop. 2.2.5. □

For $q \mid N$, let us set $U_q^1 := U_q'$ (recall that U_q' was defined to be $(\mathrm{M}_0(N) \otimes \mathbf{Z}_q)^\times$) and

$$U_q^2 := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbf{GL}_2(\mathbf{Z}_q) : \quad a, d \in \mathbf{Z}_q^\times, \quad b \in q\mathbf{Z}_q, \quad c \in q^{n_q-1}\mathbf{Z}_q \right\}.$$

We also set U_q^Ξ equal to U_q' if $q \nmid N$ and equal to U_q^1 or U_q^2 according as $q \notin \Xi$ or $q \in \Xi$, if $q \mid N$.

Lemma 4.16. *Let q be a finite prime and suppose $\alpha, \beta \in U'_q$, $\gamma \in U_q^{\Xi}$ are such that*

$$\det(\alpha) = \det(\beta) \cdot \det(\gamma)^{-1},$$

so that $(\alpha, (\beta, \gamma))$ may be viewed as an element of $\mathcal{R}(\mathbf{Q}_q)$.

(1) *Suppose $q \nmid N_{\varepsilon_f}$. Then*

$$r_\psi(\alpha, (\beta, \gamma))\varphi_q^{\Xi} = \varphi_q^{\Xi}.$$

(2) *Suppose $q \mid N_{\varepsilon_f}$. Then*

$$r_\psi(\alpha, (\beta, \gamma))\varphi_q^{\Xi} = \varepsilon_{f,q}(\mathbf{a}(\alpha))\varepsilon_{f,q}(\mathbf{d}(\beta)^{-1}\mathbf{d}(\gamma))\varphi_q^{\Xi},$$

where for any matrix α in $\mathbf{GL}_2$, we define $\mathbf{a}(\alpha)$ and $\mathbf{d}(\alpha)$ to be the upper left and lower right entries of α respectively.

Proof. Let us write φ_q instead of φ_q^{Ξ} for simplicity. Clearly we may assume that

$$\det(\alpha) = \det(\beta)\det(\gamma)^{-1} = 1.$$

Then

$$r_\psi(\alpha, (\beta, \gamma))\varphi_q(x) = r_\psi(\alpha)L(\beta, \gamma)\varphi_q(x) = r_\psi(\alpha)\varphi_q(\beta^{-1}x\gamma).$$

In case (1), we have $\varphi_q(\beta^{-1}x\gamma) = \varphi_q(x)$, while in case (2), $\varphi_q(\beta^{-1}x\gamma) = \varepsilon_{f,q}(\mathbf{d}(\beta)^{-1}\mathbf{d}(\gamma))\varphi_q(x)$. So it suffices to consider the action of $r_\psi(\alpha)$ on φ_q. Let us first check case (1). If further $q \nmid N$, then α is in the subgroup generated by matrices of the form $D(a)$, $U(y)$ and W with $a \in \mathbf{Z}_q^\times$ and $y \in \mathbf{Z}_q$. Thus we may assume that α is in fact one of these three possibilities. Since $\varphi_q = \mathbf{I}_{M_2(\mathbf{Z}_q)}$ in this case, one checks easily that

$$r_\psi(D(a))\varphi_q(x) = \varphi_q(ax) = \varphi_q(x); \tag{4.4.4}$$

$$r_\psi(U(y))\varphi_q(x) = \psi_q(y\det(x))\varphi_q(x) = \varphi_q(x); \tag{4.4.5}$$

$$r_\psi(W)\varphi_q(x) = \hat{\varphi}_q(x) = \varphi_q(x). \tag{4.4.6}$$

Next let us suppose that we are still in case (1) but $q \mid N$ and $q^n \mid\mid N$, so that

$$\varphi_q\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{cases} \mathbf{I}_{\mathbf{Z}_q}(a)\mathbf{I}_{\mathbf{Z}_q}(b)\mathbf{I}_{q^n\mathbf{Z}_q}(c)\mathbf{I}_{\mathbf{Z}_q}(d), & \text{if } q \notin \Xi; \\ \frac{1}{q}\mathbf{I}_{\mathbf{Z}_q}(a)\mathbf{I}_{\mathbf{Z}_q}(b)\mathbf{I}_{q^{n-1}\mathbf{Z}_q}(c)\mathbf{I}_{\mathbf{Z}_q}(d), & \text{if } q \in \Xi. \end{cases}$$

Note that

$$\hat{\varphi}_q\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{cases} \frac{1}{q^n}\mathbf{I}_{\mathbf{Z}_q}(a)\mathbf{I}_{q^{-n}\mathbf{Z}_q}(b)\mathbf{I}_{\mathbf{Z}_q}(c)\mathbf{I}_{\mathbf{Z}_q}(d), & \text{if } q \notin \Xi; \\ \frac{1}{q^n}\mathbf{I}_{\mathbf{Z}_q}(a)\mathbf{I}_{q^{-(n-1)}\mathbf{Z}_q}(b)\mathbf{I}_{\mathbf{Z}_q}(c)\mathbf{I}_{\mathbf{Z}_q}(d), & \text{if } q \in \Xi. \end{cases}$$

Set

$$V(z) := \begin{pmatrix} 1 & 0 \\ z & 1 \end{pmatrix}.$$

Then α is in the subgroup generated by matrices of the form $D(a)$, $U(y)$ and $V(z)$ with $a \in \mathbf{Z}_q^\times$, $y \in \mathbf{Z}_q$ and $z \in q^n\mathbf{Z}_q$. Now one checks immediately that the relations (4.4.4) and (4.4.5) continue to hold for such q. As for $V(z)$, note that $V(z) = D(-1)WU(z)W$. Further, for $z \in q^n\mathbf{Z}_q$,

$$r_\psi(U(z))\hat{\varphi}_q = \varphi_q.$$

Hence for such z,

$$r_\psi(V(z))\varphi_q = r_\psi(D(-1)WU(z)W)\varphi_q = r_\psi(D(-1)WU(z))\hat{\varphi}_q = r_\psi(D(-1)W)\hat{\varphi}_q = r_\psi(D(-1))\hat{\hat{\varphi}}_q = \varphi_q.$$

Thus case (1) is entirely verified. We now deal with case (2). In this case,

$$\varphi_q\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{cases} \varepsilon_{f,q}(d)\mathbf{I}_{\mathbf{Z}_q}(a)\mathbf{I}_{\mathbf{Z}_q}(b)\mathbf{I}_{q^n\mathbf{Z}_q}(c)\mathbf{I}_{\mathbf{Z}_q^\times}(d), & \text{if } q \notin \Xi; \\ \frac{1}{q}\varepsilon_{f,q}(d)\mathbf{I}_{\mathbf{Z}_q}(a)\mathbf{I}_{\mathbf{Z}_q}(b)\mathbf{I}_{q^{n-1}\mathbf{Z}_q}(c)\mathbf{I}_{\mathbf{Z}_q^\times}(d), & \text{if } q \in \Xi. \end{cases}$$

Thus

$$r_\psi(D(a))\varphi_q(x) = \varphi_q(ax) = \varepsilon_{f,q}(a)\varphi_q(x)$$

for $a \in \mathbf{Z}_q^\times$ and $r_\psi(U(y))\varphi_q(x) = \psi_q(y\det(x))\varphi_q(x) = \varphi_q(x)$ for $y \in \mathbf{Z}_q$.

It remains to consider the action of $r_\psi(V(z))$ on φ_q for $z \in q^n\mathbf{Z}_q$. For this we need as before to compute the Fourier transform of φ_q. Suppose that $\mathrm{cond}(\varepsilon_{f,q}) = q^m\mathbf{Z}_q$, so that $m \le n$. Then

$$\hat{\varphi}_q\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{cases} \frac{1}{q^{m+n}}\varepsilon_{f,q}^{-1}(q^m a)\mathbf{I}_{q^{-m}\mathbf{Z}_q^\times}(a)\mathbf{I}_{q^{-n}\mathbf{Z}_q^\times}(b)\mathbf{I}_{\mathbf{Z}_q}(c)\mathbf{I}_{\mathbf{Z}_q}(d), & \text{if } q \notin \Xi; \\ \frac{1}{q^{m+n}}\varepsilon_{f,q}^{-1}(q^m a)\mathbf{I}_{q^{-m}\mathbf{Z}_q^\times}(a)\mathbf{I}_{q^{-(n-1)}\mathbf{Z}_q^\times}(b)\mathbf{I}_{\mathbf{Z}_q}(c)\mathbf{I}_{\mathbf{Z}_q}(d), & \text{if } q \in \Xi. \end{cases}$$

Thus $r_\psi(V(z))\hat{\varphi}_q = \hat{\varphi}_q$ in this case as well, and we see as above that $r_\psi(V(z))\varphi_q = \varphi_q$. $\square$

We need the following lemma in order to study explicit theta lifts in both directions. For any $q \in \Sigma$, and for $\beta \in \mathbf{GL}_2(\mathbb{A})$, we define

$$\Phi_q(\beta) := \int_{\mathbf{SL}_2(\mathbf{Q}_q)} \varphi_q^2(\alpha_q^{-1})F^j(\beta\alpha_q)d^{(1)}\alpha_q.$$

Lemma 4.17. *Let Σ' denote the subset of Σ consisting of those primes q such that $\pi_{f,q} \simeq \pi(\mu_1,\mu_2)$ is a ramified principal series representation with μ_1 unramified and μ_2 ramified of conductor exactly q^{n_q}, where $q^{n_q}||N$. Then for $q \in \Sigma$, the function $\Phi_q(\beta)$ is identically zero unless $q \in \Sigma'$. If $q \in \Sigma'$, then*

$$\Phi_q(\beta) = q^{-1/2}\mu_1(q)^{-1}F^j(\beta\gamma_q),$$

where γ_q is the element of $\mathbf{GL}_2(\mathbb{A})$ that is $\begin{pmatrix} q & 0 \\ 0 & 1 \end{pmatrix}$ at q and 1 at all other places.

Proof. Let us write n instead of n_q for ease of notation. We suppose first that $q \in \Sigma \setminus \Sigma'$. In this case, $\pi_{f,q}$ is either supercuspidal or a ramified special representation or a ramified principal series $\simeq \pi(\mu_1,\mu_2)$ where μ_1 and μ_2 both have conductor dividing q^{n-1}. In any case, the central character $\varepsilon_{f,q}$ has conductor dividing q^{n-1}. (See [Tu1] Prop. 3.4.) We claim then that

$$\Phi_q(\beta u) = \varepsilon_{f,q}(d)\Phi_q(\beta), \tag{4.4.7}$$

for $u = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_q(n-1)$, where for any integer $m \ge 1$, we define

$$\Gamma_q(m) := \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbf{GL}_2(\mathbf{Z}_q) : c \equiv 0 \mod q^m\right\}.$$

It suffices to verify (4.4.7) for γ a matrix in one of the three forms:

$$D(a,b) := \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}, \quad a,b, \in \mathbf{Z}_q^\times; \qquad U(y), \quad y \in \mathbf{Z}_q; \qquad \text{and} \qquad V(z), \quad z \in q^{n-1}\mathbf{Z}_q.$$

This follows from the following set of computations. First, let $a,b \in \mathbf{Z}_q^\times$. Then

$$\begin{aligned} \Phi_q(\beta \cdot D(a,b)) &= \int_{\mathbf{SL}_2(\mathbf{Q}_q)} \varphi_q^2(\alpha_q^{-1})F^j(\beta \cdot D(a,b) \cdot \alpha_q \cdot D(a,b)^{-1} \cdot D(a,b))d^{(1)}\alpha_q \\ &= \varepsilon_{f,q}(b)\int_{\mathbf{SL}_2(\mathbf{Q}_q)} \varphi_q^2(D(a,b) \cdot \alpha_q^{-1} \cdot D(a,b)^{-1})F^j(\beta \cdot \alpha_q)d^{(1)}\alpha_q \\ &= \varepsilon_{f,q}(b)\int_{\mathbf{SL}_2(\mathbf{Q}_q)} \varphi_q^2(\alpha_q^{-1})F^j(\beta \cdot \alpha_q)d^{(1)}\alpha_q = \varepsilon_{f,q}(b)\Phi_q(\beta). \end{aligned}$$

Next, let $y \in \mathbf{Z}_q$. Then

$$\begin{aligned} \Phi_q(\beta \cdot U(y)) &= \int_{\mathbf{SL}_2(\mathbf{Q}_q)} \varphi_q^2(\alpha_q^{-1})F^j(\beta \cdot U(y) \cdot \alpha_q)d^{(1)}\alpha_q \\ &= \int_{\mathbf{SL}_2(\mathbf{Q}_q)} \varphi_q^2(\alpha_q^{-1} \cdot U(y))F^j(\beta\alpha_q)d^{(1)}\alpha_q. \end{aligned}$$

Suppose $\alpha_q^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then

$$\alpha_q^{-1} \cdot U(y) = \begin{pmatrix} a & ay+b \\ c & cy+d \end{pmatrix}.$$

If $\varphi_q^2(\alpha_q^{-1}) \neq 0$, then $a, b, d \in \mathbf{Z}_q$ and $c \in q^{n-1}\mathbf{Z}_q$. Hence $cy + d \equiv d \mod q^{n-1}$. Since the conductor of $\varepsilon_{f,q}$ divides q^{n-1}, it follows that $\varphi_q^2(\alpha_q^{-1}U(y)) = \varphi_q^2(\alpha_q^{-1})$ for all α_q, and consequently $\Phi_q(\beta \cdot U(y)) = \Phi_q(\beta)$. Finally, let $z \in q^{n-1}\mathbf{Z}_q$. Then

$$\begin{aligned} \Phi_q(\beta \cdot V(z)) &= \int_{\mathbf{SL}_2(\mathbf{Q}_q)} \varphi_q^2(\alpha_q^{-1}) F^j(\beta \cdot V(z) \cdot \alpha_q) d^{(1)}\alpha_q \\ &= \int_{\mathbf{SL}_2(\mathbf{Q}_q)} \varphi_q^2(\alpha_q^{-1} \cdot V(z)) F^j(\beta\alpha_q) d^{(1)}\alpha_q. \end{aligned}$$

But

$$\alpha_q^{-1}V(z) = \begin{pmatrix} a+bz & b \\ c+dz & d \end{pmatrix}.$$

Since $z \in q^{n-1}\mathbf{Z}_q$, one finds that $\varphi_q^2(\alpha_q^{-1}V(z)) = \varphi_q^2(\alpha_q^{-1})$ for all α_q. This proves (4.4.7). But now by Casselman's theorem, we see that $\Phi_q(\beta)$ must be identically zero for such q.

We now turn to $q \in \Sigma'$. In this case, one cannot argue as above since $\varepsilon_{f,q}$ has conductor q^n. However the argument above shows that Φ_q is right invariant by $V(z)$ for $z \in q^{n-1}\mathbf{Z}_q$, and by $U(y)$ for $y \in q\mathbf{Z}_q$, and transforms by $\varepsilon_{f,q}(b)$ under the right action of $D(a,b)$. We conclude that if u lies in the subgroup

$$\left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbf{GL}_2(\mathbf{Z}_q) : \quad a, d \in \mathbf{Z}_q^\times, \quad b \in q\mathbf{Z}_q, \quad c \in q^{n-1}\mathbf{Z}_q \right\},$$

then $\Phi_q(\beta \cdot u) = \varepsilon_{f,q}(\mathbf{d}(u))\Phi_q(\beta)$. By Casselman's theorem, we see that

$$\Phi_q(\beta\gamma_q^{-1}) = \tilde{c} \cdot F^j(\beta)$$

for some scalar $\tilde{c}$. We now compute the value of $\tilde{c}$. Letting $\Gamma_q^{(1)}(m) := \Gamma_q(m) \cap \mathbf{SL}_2(\mathbf{Q}_q)$, note that

$$\Phi_q(\beta) = \frac{1}{q} \int_{\Gamma_q^{(1)}(n-1)} \varepsilon_{f,q}(\mathbf{d}(\alpha_q^{-1})) F^j(\beta\alpha_q) d^{(1)}\alpha_q.$$

Let us first suppose that $n \geq 2$. Then the collection

$$V(x) = \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix}, \qquad x \in q^{n-1}\mathbf{Z}_q/q^n\mathbf{Z}_q.$$

is a set of coset representatives for $\Gamma_q^{(1)}(n-1)/\Gamma_q^{(1)}(n)$. Hence

$$\begin{aligned} \tilde{c} \cdot F^j(\beta\gamma_q) = \Phi_q(\beta) &= \frac{1}{q} \sum_{x \in q^{n-1}\mathbf{Z}_q/q^n\mathbf{Z}_q} \int_{\Gamma_q^{(1)}(n)} \varepsilon_{f,q}(\mathbf{d}(\alpha_q^{-1}V(x))) F^j(\beta V(x)\alpha_q) d^{(1)}\alpha_q \\ &= \frac{1}{q} \sum_{x \in q^{n-1}\mathbf{Z}_q/q^n\mathbf{Z}_q} \int_{\Gamma_q^{(1)}(n)} \varepsilon_{f,q}(\mathbf{d}(\alpha_q^{-1})) F^j(\beta V(x)) \varepsilon_{f,q}(\mathbf{d}(\alpha_q)) d^{(1)}\alpha_q \\ &= \frac{1}{q} \operatorname{vol}(U_q'^{(1)}) \sum_{x \in q^{n-1}\mathbf{Z}_q/q^n\mathbf{Z}_q} F^j(\beta V(x)). \end{aligned} \tag{4.4.8}$$

To find the value of $\tilde{c}$ we may substitute $\beta = 1$ and compute in a convenient model for the local representation $\pi_{f,q} \simeq \pi(\mu_1, \mu_2)$. We use the standard model of the induced representation $V(\mu_1, \mu_2)$, and denote by f_q a new vector in this representation, normalized so that $f_q(\mathbf{1}) = 1$. Then (see [SR], Prop. 2.1.2)

$$f_q(\gamma_q) = \mu_1(q)^{1-n}|q|_q^{1/2}$$

while

$$f_q \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix} = \begin{cases} \mu_1(q)^{-n}, & \text{if } v_q(x) \geq n; \\ 0, & \text{if } v_q(x) < n. \end{cases}$$

It follows that

$$\tilde{c} = \frac{1}{q}\mu_1(q)^{-1}|q|_q^{-1/2} \operatorname{vol}(U_q'^{(1)}) = q^{-1/2}\mu_1(q)^{-1} \operatorname{vol}(U_q'^{(1)}).$$

If on the other hand $n = 1$, then the matrices $V(x)$ with $x \in \mathbf{Z}_q/q\mathbf{Z}_q$ along with W form a set of coset representatives for $\Gamma_q^{(1)}/\Gamma_q^{(1)}(1)$. Again we can use the standard model of the induced representation to compute the value of $\tilde{c}$. However, since $f_q(W) = 0$ (see [SR], Prop. 2.1.2), the expression for $\tilde{c}$ remains the same in this case too.

□

Definition 4.18. For $\Xi \subset \Sigma$, we set $F^j_\Xi(g) = F^j(g \cdot \prod_{q \in \Xi} \gamma_q)$, where γ_q is as in Lemma 4.17 above.

Proposition 4.19.

$$\theta^t_{\varphi^\Xi}(F^j \times \overline{F^j_\Xi}) = C_1^\Xi \cdot F^{0,\sharp},$$

where

$$C_1^\Xi := \begin{cases} 0, \text{ if } \Xi \not\subset \Sigma'; \\ (4\pi)^{-(j-1)} \frac{\Gamma(k+j)}{\Gamma(k)} \operatorname{vol}(U'^{(1)}) \cdot \langle F^j, F^j \rangle \cdot \prod_{q \in \Xi} (q^{-1/2} \mu_1(q)) \text{ if } \Xi \subseteq \Sigma', \end{cases} \tag{4.4.9}$$

and $F^{0,\sharp}$ is the unique form in π characterized by

(i) *If $q \nmid N$, then $F^{0,\sharp}(gu) = F^{0,\sharp}(g)$ for $u \in \mathbf{GL}_2(\mathbf{Z}_q)$.*

(ii) *If $q \mid N$, then $F^{0,\sharp}(gu) = \varepsilon_{f,q}(a) F^{0,\sharp}(g)$ for $u = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_q(n_q)$.*

(iii) *Let $a \in \mathbf{R}^\times$, $a_\infty := d(a) \in \mathbf{GL}_2(\mathbf{R})$, $\kappa_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \in \mathbf{SO}_2(\mathbf{R})$. Let $(1, a_\infty \kappa_\theta)$ denote the element of $\mathbf{GL}_2(\mathbb{A})$ which is 1 at all finite places and $a_\infty \kappa_\theta$ at the infinite place. Then*

$$W_{F^{0,\sharp},\psi}(1, a_\infty \kappa_\theta) = a^{k/2} e^{-2\pi a} e^{ik\theta} \mathbf{I}_{\mathbf{R}^+}(a).$$

Here $W_{\cdot,\psi}$ denotes as usual the ψ-Whittaker coefficient and $\langle F^j, F^j \rangle$ denotes the Petersson inner product:

$$\langle F^j, F^j \rangle = \frac{1}{2} \int_{\mathbf{PGL}_2(\mathbf{Q}) \backslash \mathbf{PGL}_2(\mathbb{A})} F^j(\beta) \overline{F^j(\beta)} d^\times \beta.$$

Proof. Let $F' := \theta^t_{\varphi^\Xi}(F^j \times \overline{F^j_\Xi})$. We first show that $F' = C_1^\Xi \cdot F^{0,\sharp}$ for some constant C_1^Ξ. Note that for $u \in U'$ and $\kappa_\theta \in \mathbf{SO}_2(\mathbf{R})$, by Lemmas 4.15 and 4.16,

$$\begin{aligned} F'(gu\kappa_\theta) &= \int_{\mathbf{O}(V)(\mathbf{Q}) \backslash \mathbf{O}(V)(\mathbb{A})} \sum_{x \in V(\mathbf{Q})} r_\psi(gu\kappa_\theta, h \cdot (u,1)) \varphi^\Xi(x) (F^j \times \overline{F^j_\Xi})(h \cdot (u,1)) dh \\ &= e^{ik\theta} \prod_{q \mid N_{\varepsilon_f}} \varepsilon_{f,q}(\mathbf{a}(u_q)) \varepsilon_{f,q}(\mathbf{d}(u_q)^{-1}) \cdot \varepsilon_{f,q}(\mathbf{d}(u_q)) F'(g) \\ &= e^{ik\theta} \prod_{q \mid N_{\varepsilon_f}} \varepsilon_{f,q}(\mathbf{a}(u_q)) F'(g). \end{aligned} \tag{4.4.10}$$

Since $\theta^t_\psi(\pi \otimes \bar{\pi}) = \pi$, it follows by Casselman's theorem that $F' = C_1^\Xi \cdot F^{0,\sharp}$ for some scalar C_1^Ξ. Clearly, C_1^Ξ is just the first Fourier coefficient of F'. To evaluate C_1^Ξ, we compute the Whittaker coefficients of F'. As in [Wa] Sec. 3.2.1,

$$W_{F',\psi}(g) = \frac{1}{2} \int_{\mathbf{PGL}_2(\mathbf{Q}) \backslash \mathbf{PGL}_2(\mathbb{A})} \Psi(g, \beta) \overline{F^j_\Xi}(\beta) d^\times \beta,$$

where

$$\Psi(g, \beta) = \int_{\mathbf{GL}_2(\mathbb{A})^{\det(g)}} r_\psi(g, (\alpha, 1)) \varphi^\Xi(1) F^j(\beta\alpha) d^{(1)} \alpha.$$

Note that

$$\begin{aligned} \Psi(\mathbf{1}, \beta) &= \int_{\mathbf{SL}_2(\mathbb{A})} r_\psi(\mathbf{1}, (\alpha, 1)) \varphi^\Xi(1) F^j(\beta\alpha) d^{(1)} \alpha \\ &= \int_{\mathbf{SL}_2(\mathbb{A})} \varphi^\Xi(\alpha^{-1}) F^j(\beta\alpha) d^{(1)} \alpha. \end{aligned}$$

This integral can be computed one place at a time since both F^j and φ^Ξ are pure tensors. We first consider finite primes q such that $q \notin \Xi$. In this case, if $\varphi_q(\alpha_q^{-1}) \neq 0$, then $\alpha_q^{-1} \in U'_q$. Hence $\alpha_q \in U'_q$

as well. If further $q \nmid N_{\varepsilon_f}$, then $\varphi_q(\alpha_q^{-1}) = 1$ and $F^j(\beta\alpha_q) = F^j(\beta)$. On the other hand, if $q \mid N_{\varepsilon_f}$, then $\varphi_q(\alpha_q^{-1}) = \varepsilon_{f,q}(\mathbf{d}(\alpha_q)^{-1})$ and $F^j(\beta\alpha_q) = \varepsilon_{f,q}(\mathbf{d}(\alpha_q))F^j(\beta)$, so that in any case, for $q \notin \Xi$, we have

$$\int_{\mathbf{SL}_2(\mathbf{Q}_q)} \varphi_q^{\Xi}(\alpha_q^{-1})F^j(\beta\alpha_q)d^{(1)}\alpha_q = \mathrm{vol}({U'_q}^{(1)}) \cdot F^j(\beta).$$

For $q \in \Xi$, it follows from Lemma 4.17 that

$$\int_{\mathbf{SL}_2(\mathbf{Q}_q)} \varphi_q^{\Xi}(\alpha_q^{-1})F^j(\beta\alpha_q)d^{(1)}\alpha_q = \begin{cases} 0, \text{ if } q \notin \Sigma'; \\ \mathrm{vol}({U'_q}^{(1)}) \cdot q^{-1/2}\mu_1^{-1}(q)F^j(\beta\gamma_q), \text{ if } q \in \Sigma'. \end{cases}$$

Finally, the computation of the local integral at the infinite place can be found in [Xue], Prop. 4.3.4. Accounting for our different choice of measures, this contribution equals $e^{-2\pi}(4\pi)^{-(j-1)}\Gamma(k+j)/\Gamma(k)$. Puting together the local computations, we find

$$\Psi(\mathbf{1},\beta) = \begin{cases} 0, \text{ if } \Xi \not\subset \Sigma'; \\ e^{-2\pi} \cdot (4\pi)^{-(j-1)}\frac{\Gamma(k+j)}{\Gamma(k)} \cdot \mathrm{vol}({U'}^{(1)}) \cdot \prod_{q\in\Xi}(q^{-1/2}\mu_1^{-1}(q)) \cdot F_{\Xi}^j(\beta), \text{ if } \Xi \subset \Sigma'. \end{cases}.$$

Thus $C_1^{\Xi} = 0$ unless $\Xi \subseteq \Sigma'$ and in that case,

$$\begin{aligned} C_1^{\Xi} &= e^{2\pi}W_{F',\psi}(\mathbf{1}) = (4\pi)^{-(j-1)}\,\mathrm{vol}({U'}^{(1)})\frac{\Gamma(k+j)}{\Gamma(k)} \cdot \prod_{q\in\Xi}(q^{-1/2}\mu_1^{-1}(q))\langle F_{\Xi}^j, F_{\Xi}^j\rangle \\ &= (4\pi)^{-(j-1)}\frac{\Gamma(k+j)}{\Gamma(k)} \cdot \mathrm{vol}({U'}^{(1)})\langle F^j, F^j\rangle \cdot \prod_{q\in\Xi}(q^{-1/2}\mu_1^{-1}(q)). \quad \square \end{aligned}$$

Proposition 4.20.

$$\theta_\varphi(\overline{F^{0,\sharp}}) = C_2^{\Xi} \cdot (\overline{F^j} \times F_{\Xi}^j),$$

where

$$C_2^{\Xi} = \begin{cases} 0, \text{ if } \Xi \not\subset \Sigma'; \\ \frac{(4\pi)^{j+1}}{\Gamma(j+1)}\Im(\tau)^{\ell}\,\mathrm{vol}({U'}^{(1)})\prod_{q\in\Sigma'}(q^{-1/2}\mu_1^{-1}(q)) \text{ if } \Xi \subseteq \Sigma'. \end{cases} \tag{4.4.11}$$

(Recall that Σ' was defined in Lemma 4.17.)

Proof. By a calculation as in (4.4.10) above and another application of Casselman's theorem, we have $\theta_\varphi(\overline{F^{0,\sharp}}) = C_2^{\Xi} \cdot (\overline{F^j} \times F_{\Xi}^j)$ for some constant C_2^{Ξ}. To compute C_2^{Ξ}, one studies the theta lift in the opposite direction and uses the seesaw principle. Indeed, the seesaw principle and Proposition 4.19 imply that

$$C_2^{\Xi}\langle F^j, F^j\rangle^2 = \langle\theta_\varphi(\overline{F^{0,\sharp}}), \overline{F^j} \times F_{\Xi}^j\rangle = \langle\overline{F^{0,\sharp}}, \overline{\theta_\varphi^t(F^j \times \overline{F_{\Xi}^j})}\rangle = C_1^{\Xi}\langle F^{0,\sharp}, F^{0,\sharp}\rangle.$$

i.e., $C_2^{\Xi} = C_1^{\Xi}\langle F^{0,\sharp}, F^{0,\sharp}\rangle/\langle F^j, F^j\rangle^2$. But (see Lemma 4.11),

$$\langle F^j, F^j\rangle/\langle F^{0,\sharp}, F^{0,\sharp}\rangle = \Im(\tau)^{-\ell}(4\pi)^{-2j}\Gamma(j+1)\Gamma(k+j)/\Gamma(k).$$

(The term $\Im(\tau)^{-\ell}$ appears since F^0 and $F^{0,\sharp}$ are normalized differently: the former is the adelic form associated to f and the base point τ, while the latter uses the base point i. To translate from one to other involves picking an element $\gamma \in \mathbf{SL}_2(\mathbf{R})$ such that $\gamma i = \tau$ and one checks that $\langle F^0, F^0\rangle/\langle F^{0,\sharp}, F^{0,\sharp}\rangle = j(\gamma, i)^{2\ell} = \Im(\tau)^{-\ell}$.) The proposition now follows by using the value of C_1^{Ξ} from Prop. 4.19. $\square$

We now make the following key definition, namely that of the Schwartz function in the explicit theta correspondence.

Definition 4.21. The explicit Schwartz function φ is defined by $\varphi := \otimes_q\varphi_q$, where φ_∞ is as in (4.4.3) and for finite primes q, the φ_q are as below:

(i) If $q \nmid N$, then $\varphi_q = \mathbf{I}_{\mathrm{M}_0(N)\otimes\mathbf{Z}_q} = \mathbf{I}_{M_2(\mathbf{Z}_q)}$.

(ii) If $q \mid N$, then $\varphi_q = \varphi_q^1$ for $q \notin \Sigma'$ and $\varphi_q := \varphi_q^1 - \varphi_q^2$ for $q \in \Sigma'$. Recall that φ_q^1 and φ_q^2 were defined previously in (4.4.1) and (4.4.2) respectively and Σ' was defined in Lemma 4.17.

The following lemma which will be used in the next section is an easy consequence of the fact that η is of type $(c, \mathfrak{N}, \varepsilon_f^{-1})$.

Lemma 4.22. *For $q \in \Sigma'$, fix an isomorphism $K_q \simeq \mathbf{Q}_q \times \mathbf{Q}_q$ such that via this identification the embedding $\xi_q : K_q \hookrightarrow M_2(\mathbf{Q}_q)$ is conjugate by an element of U'_q to the embedding $(a,b) \mapsto \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$. Let $\overline{\eta'_q} = (\eta_1, \eta_2)$ via this identification. Then*

(1) *η_1 is unramified and η_2 is ramified.*
(ii) *$\eta_2 \mu_2^{-1}$ is unramified.*

4.5. **Seesaw duality and the Siegel-Weil formula.** Let $V_1 = K$ (viewed as a subspace of V via $\boldsymbol{\xi}$) and let $V_2 = V_1^{\perp}$. Then

$$\mathbf{GO}(V_1)^0 \simeq \mathbf{GO}(V_2)^0 \simeq K^\times,$$
$$\mathbf{H} := \mathbf{G}(\mathbf{O}(V_1) \times \mathbf{O}(V_2))^0 = \mathbf{G}(K^\times \times K^\times),$$

and via this identification the map $\delta : K^\times \times K^\times \to \mathbf{H}$ is

$$\delta(\alpha, \beta) = (\alpha\beta^{-1}, \alpha(\beta^\rho)^{-1}).$$

Since $\overline{\eta'}(\alpha)\eta'(\beta) = \overline{\eta'}(\alpha\beta^{-1})$, the character $(\overline{\eta'}, \eta')$ of $K^\times \times K^\times$ is the pullback via δ of the character $\boldsymbol{\eta} := (\overline{\eta'}, 1)$ on $\mathbf{H}$. Suppose that

$$\varphi_q = \sum_{i_q \in I_q} \varphi_1^{i_q} \otimes \varphi_2^{i_q} \in \mathcal{S}(V_1 \otimes \mathbf{Q}_q) \otimes \mathcal{S}(V_2 \otimes \mathbf{Q}_q).$$

Then by an application of seesaw duality for the seesaw pair

$$\begin{tikzcd}[column sep=huge, row sep=huge]
\mathbf{GL}_2 \times \mathbf{GL}_2 \arrow[d, dash] \arrow[dr, dash] & \mathbf{GO}(V) \arrow[d, dash] \arrow[dl, dash] \\
\mathbf{GL}_2 & \mathbf{G}(\mathbf{O}(V_1) \times \mathbf{O}(V_2))
\end{tikzcd}$$

we have (as in [HK2] (14.5)),

$$\begin{aligned}
&\int_{\mathbf{H}(\mathbf{Q})\backslash\mathbf{H}(\mathbb{A})} \theta_{\psi,\varphi}(\overline{F^{0,\sharp}})|_{\mathbf{H}(\mathbb{A})}(h)\boldsymbol{\eta}(h)d^\times h \\
&\qquad = \int_{\mathbf{GL}_2(\mathbf{Q})\mathbb{A}^\times\backslash\mathbf{GL}_2(\mathbb{A})} \overline{F^{0,\sharp}}(g) \cdot \theta^t_\varphi(\boldsymbol{\eta})|_{\mathbf{GL}_2(\mathbb{A})}(g)dg \\
&\qquad = \int_{\mathbf{GL}_2(\mathbf{Q})\mathbb{A}^\times\backslash\mathbf{GL}_2(\mathbb{A})} \overline{F^{0,\sharp}}(g) \cdot \sum_{i=(i_q)\in I=\prod_q I_q} \theta^t_{\otimes_q \varphi_1^{i_q}}(\overline{\eta'})(g)\theta^t_{\otimes_q\varphi_2^{i_q}}(1)(g)dg. \qquad (4.5.1)
\end{aligned}$$

Here $\theta^t(\overline{\eta'})$ and $\theta^t(1)$ are defined as follows. Set

$$\mathbf{GL}_2(\mathbb{A})^K := \left\{ g \in \mathbf{GL}_2(\mathbb{A}) : \det(g) \in \mathbf{N}_K(\mathbb{A}_K^\times) \right\}.$$

For $g \in \mathbf{GL}_2(\mathbb{A})^K$, $\varsigma \in \mathcal{S}(V_1(\mathbb{A}))$ and $h \in \mathbb{A}_K^\times$ such that $\det(g) = \mathbf{N}_K(h)$,

$$\theta^t_\varsigma(\overline{\eta'})(g) := \int_{K^{(1)}\backslash K^{(1)}_{\mathbb{A}}} \sum_{x \in V_1} r_\psi(g, hh_1)\varsigma(x)\overline{\eta'}(hh_1)d^{(1)}h_1.$$

One then extends the definition to the index 2 subgroup $\mathbf{GL}_2(\mathbf{Q}) \cdot \mathbf{GL}_2(\mathbb{A})^K$ of $\mathbf{GL}_2(\mathbb{A})$ by requiring it to be left invariant by $\mathbf{GL}_2(\mathbf{Q})$. Finally, one extends it by zero outside this index two subgroup. The theta lift $\theta^t(1)$ is defined similarly with $\overline{\eta'}$ replaced by the trivial character and V_1 replaced by V_2. Here the measure $d^{(1)}h_1$ is chosen such that it lifts to a Haar measure on $K^{(1)}_{\mathbb{A}}$ and $\mathrm{vol}(K^{(1)} \setminus K^{(1)}_{\mathbb{A}}) = 1$.

Now, by the Siegel-Weil formula, the theta lift $\theta^t(1)$ is an Eisenstein series. Unfolding this Eisenstein series by the standard Rankin-Selberg method, one finds that the integral in (4.5.1) above is equal to the expression $I(\varphi, \boldsymbol{\xi})$, where (defining Φ^s as in [P] Prop. 3.1.),

$$I(\varphi, \boldsymbol{\xi}) := \zeta(2)^{-1} \int_{\mathbb{A}_{\mathbf{Q}}^\times} \int_{K_0} W_{\bar\psi}(\overline{F^{0,\sharp}})(d(a)k) \sum_{i=(i_q)\in I=\prod_q I_q} W_\psi(\theta^t_{\otimes_q\varphi_1^{i_q}}(\overline{\eta'}))(d(a)k)\Phi^s_{\otimes_q\varphi_2^{i_q}}(d(a)k)(1)|a|^{-1}d^\times a dk,$$

Here $K_0 = \prod_q \mathbf{GL}_2(\mathbf{Z}_q) \times \mathbf{SO}_2(\mathbf{R}))$, the measure dk is a product of local Haar measures such that $\mathrm{vol}(\mathbf{GL}_2(\mathbf{Z}_q)) = 1$ and $\mathrm{vol}(\mathbf{SO}_2(\mathbf{R})) = 2\pi$, and the factor $\zeta(2)^{-1}$ accounts for the change in measure normalization. We now state two propositions that will be useful in computing the integral above.

We note first that $W_{\bar\psi}(\overline{F^{0,\sharp}}) = \overline{W_\psi(F^{0,\sharp})}$ and $W_\psi(F^{0,\sharp}) = \prod_v W_{\psi,v}(F^{0,\sharp})$ where $W_{\psi,v}(F^{0,\sharp})$ is normalized to take value 1 on the identity matrix in $\mathbf{GL}_2(\mathbf{Z}_q)$ for finite q and $W_{\psi,\infty}(F^{0,\sharp})(d(a)) = e^{-2\pi a} a^{\frac{k}{2}} \mathbf{I}_{\mathbf{R}^+}(a)$. The proposition below (which is simply copied from [SR] Sec. 2.4 taking into account that $F^{0,\sharp}$ transforms by the central character of the upper left entry at ramified places as opposed to the lower right entry as in loc. cit.) lists the values of $W_{\psi,q}(F^{0,\sharp})$ on matrices of the form $d(a) := \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}$.

Proposition 4.23. *Let $a \in \mathbf{Q}_q^\times$. Then $W_{\psi,q}(F^{0,\sharp})(d(a))$ is equal to*

(i) $|a|^{1/2} \left(\sum_{r+s=v_q(a)} \mu_1(q)^r \mu_2(q)^s\right) \mathbf{I}_{\mathbf{Z}_q}(a)$ *if $\pi_{f,q} \simeq \pi(\mu_1,\mu_2)$ is an unramified principal series representation.*

(ii) $|a|\mu(a)\mathbf{I}_{\mathbf{Z}_q}(a)$, *if $\pi_{f,q} \simeq \mathrm{St}(\mu)$ is a special representation with μ unramified.*

(iii) $\mathbf{I}_{\mathbf{Z}_q^\times}(a)$, *if $\pi_{f,q} \simeq \mathrm{St}(\mu)$ is a special representation with μ ramified.*

(iv) $|a|^{1/2}\mu_2(a)\mathbf{I}_{\mathbf{Z}_q}(a)$, *if $\pi_{f,q} \simeq \pi(\mu_1,\mu_2)$ is a ramified principal series representation with μ_1 unramified and μ_2 ramified.*

(v) $\varepsilon_{f,q}(a)\mathbf{I}_{\mathbf{Z}_q^\times}(a)$, *if $\pi_{f,q} \simeq \pi(\mu_1,\mu_2)$ is a ramified principal series representation with both μ_1 and μ_2 ramified, or if $\pi_{f,q}$ is supercuspidal.*

For simplicity, in our local calculations below, we will simply write W_F for $W_{\bar\psi,q}(\overline{F^{0,\sharp}})$. The following proposition follows from the discussion in [P] §3.3.

Proposition 4.24. *The Whittaker function $W_\psi(\theta^t_{\otimes_q \vartheta_q}(\overline{\eta'}))$ factors as*

$$W_\psi(\theta^t_{\otimes_q \vartheta_q}(\overline{\eta'})) = \frac{1}{h_K} \prod_q W_{\Theta,\vartheta_q},$$

where for any prime q, either finite or infinite,

$$\begin{aligned} W_{\Theta,\vartheta_q}(d(a)) &= \int_{K_q^{(1)}} \vartheta_q(a(hh')^{-1}) \overline{\eta'_q}(hh')dh \\ &= |a|_q^{1/2} \int_{K_q^1} \vartheta_q((hh')^\rho) \overline{\eta'_q}(hh')dh, \end{aligned} \tag{4.5.2}$$

for any h' such that $\mathrm{N}(h') = a$. (Here the Haar measure dh on $K_v^{(1)}$ is chosen such that $\mathrm{vol}(K_\infty^{(1)}) = 1$ and for finite primes q, $\mathrm{vol}(\mathcal{O}_K \otimes \mathbf{Z}_q)^{(1)} = 1$.) Also,

$$\Phi^s_{\otimes \varsigma_q}(d(a)) = |a|^s \prod_q \varsigma_q(0).$$

More generally, suppose $j_q : K_q \to V_q$ is an embedding of quadratic spaces, where $K_q = K \otimes \mathbf{Q}_q$ and $V_q = V(\mathbf{Q}_q)$. For $\varsigma \in \mathcal{S}(V_q) = \mathcal{S}(K_q)) \otimes \mathcal{S}(K_q^\perp)$, write $\varsigma = \sum_i \varsigma_{1,i} \otimes \varsigma_{2,i}$ and define

$$I(\varsigma, j_q) = \sum_i \int_{\mathbf{Q}_q^\times} \int_{K_{0,q}} W_F(d(a)k) W_{\Theta,\varsigma_{1,i}}(d(a)k) \Phi^s_{\varsigma_{2,i}}(d(a)k)|a|^{-1} d^\times a dk \tag{4.5.3}$$

Since $W_{\Theta,\varsigma_{1,i}} \cdot \Phi^s_{2,i}(\cdot)$ is bilinear in $(\varsigma_{1,i}, \varsigma_{2,i})$, the expression on the right in (4.5.3) is independent of the decomposition $\varsigma = \sum_i \varsigma_{1,i} \otimes \varsigma_{2,i}$. In this notation, we have

$$I(\varphi, \boldsymbol{\xi}) = \frac{\zeta(2)^{-1}}{h_K} \prod_{q<\infty} I(\varphi_q, \boldsymbol{\xi}_q) \cdot I(\varphi_\infty, \boldsymbol{\xi}_\infty). \tag{4.5.4}$$

Thus to compute $I(\varphi, \boldsymbol{\xi})$ it suffices to compute $I(\varphi_q, \boldsymbol{\xi}_q)$ for all q. However, for finite primes q, it is easier to compute $I(\varphi_q, \xi'_q)$ for a modified embedding ξ'_q which is defined by

$$\xi'_q(x) = u_q^{-1} \xi_q(x) u_q$$

for some suitable choice of $u_q \in U'_q$. If φ'_q is the Schwartz function defined by

$$\varphi'_q(x) = \varphi_q(u_q^{-1} x u_q),$$

then it is immediate that

$$I(\varphi'_q, \boldsymbol{\xi}_q) = I(\varphi_q, \xi'_q).$$

Define φ' by $\varphi' = (\otimes_q \varphi'_q) \otimes \varphi_\infty$.

Proposition 4.25. *Suppose that the $u_q \in U'_q$ have been chosen such that for all $q \in \Sigma'$, ξ'_q is given on $K_q = \mathbf{Q}_q \times \mathbf{Q}_q$ by*

$$\xi'_q(a, b) = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}.$$

Then

$$\int_{\mathbf{H}(\mathbf{Q})\backslash\mathbf{H}(\mathbb{A})} \theta_{\varphi'}(\overline{F^{0,\sharp}})|_{\mathbf{H}(\mathbb{A})}(h)\boldsymbol{\eta}(h)d^\times h = \frac{(4\pi)^{j+1}\Im(\tau)^\ell}{\Gamma(j+1)} \operatorname{vol}(U'^{(1)}) \cdot \prod_{q\in\Sigma'} (1 - \mu_1^{-1}(q)\eta_1(q)q^{-1/2}) \cdot |L_{\eta',\boldsymbol{\xi}}(F^j)^2|.$$

Proof. Let $u \in \mathbf{GL}_2(\mathbb{A}_f)$ be the element whose coordinate at q is u_q. Observe that $\varphi' = r_\psi(1, (u, u))\varphi$. Hence $\theta_{\varphi'}(\overline{F^{0,\sharp}})(h) = \theta_\varphi(\overline{F^{0,\sharp}})(h \cdot (u, u))$, and

$$\begin{aligned}
&\int_{\mathbf{H}(\mathbf{Q})\backslash\mathbf{H}(\mathbb{A})} \theta_{\varphi'}(\overline{F^{0,\sharp}})|_{\mathbf{H}(\mathbb{A})}(h)\boldsymbol{\eta}(h)d^\times h \\
&\quad= \theta_\varphi(\overline{F^{0,\sharp}})|_{\mathbf{H}(\mathbb{A})}(h\cdot(u,u))\boldsymbol{\eta}(h)d^\times h \\
&\quad= \sum_{\Xi\subset\Sigma'} (-1)^{|\Xi|}\theta_{\varphi^\Xi}(\overline{F^{0,\sharp}})|_{\mathbf{H}(\mathbb{A})}(h\cdot(u,u))\boldsymbol{\eta}(h)d^\times h \\
&\quad= \sum_{\Xi\subset\Sigma'} (-1)^{|\Xi|}C_2^\Xi \int_{K^\times\times K^\times\backslash\mathbb{A}_K^\times\times\mathbb{A}_K^\times} ((\overline{F^j}\times F_\Xi^j)(\alpha u, \beta u)\cdot(\overline{\eta'}\times\eta')(\alpha,\beta)d^\times\alpha d^\times\beta \\
&\quad= \sum_{\Xi\subset\Sigma'} (-1)^{|\Xi|}C_2^\Xi \cdot \overline{L_{\eta',\boldsymbol{\xi}}(F^j(\cdot u))}\cdot L_{\eta',\boldsymbol{\xi}}(F_\Xi^j(\cdot u)).
\end{aligned}$$

But setting $\alpha_q := (q^{-1}, 1) \in K_q^\times$, $\alpha_\Xi := \prod_{q\in\Xi}\alpha_q$ and $\gamma_\Xi := \prod_{q\in\Xi}\gamma_q$, we have $\boldsymbol{\xi}_\mathbb{A}(\alpha_\Xi)\cdot u\gamma_\Xi u^{-1} = 1$ and

$$\begin{aligned}
L_{\eta',\boldsymbol{\xi}}(F_\Xi^j(\cdot u)) &= \int_{K^\times\backslash\mathbb{A}_K^\times} F^j(\boldsymbol{\xi}_\mathbb{A}(x)u\gamma_\Xi)\eta'(x)d^\times x = \int_{K^\times\backslash\mathbb{A}_K^\times} F^j(\boldsymbol{\xi}_\mathbb{A}(x\alpha_\Xi)u\gamma_\Xi)\eta'(x\alpha_\Xi)d^\times x \\
&= \eta'(\alpha_\Xi)L_{\eta',\boldsymbol{\xi}}(F^j(\cdot u)) = \left(\prod_{q\in\Xi}\eta_1(q)\right)\cdot L_{\eta',\boldsymbol{\xi}}(F^j(\cdot u)).
\end{aligned}$$

Since $F^j(\cdot u) = F^j(\cdot)\omega_f(u)$, the proposition follows by using the value of C_2^Ξ from (4.4.11). $\square$

We record the following corollary, which follows from the above proposition and the preceding discussion.

Corollary 4.26.

$$I(\varphi', \boldsymbol{\xi}) = \frac{(4\pi)^{j+1}\Im(\tau)^\ell}{\Gamma(j+1)} \operatorname{vol}(U'^{(1)}) \cdot \prod_{q\in\Sigma'} (1 - \mu_1^{-1}(q)\eta_1(q)q^{-1/2}) \cdot |L_{\eta',\boldsymbol{\xi}}(F^j)^2|.$$

Applying (4.5.4) (with φ replaced by φ'), we see that to compute $|L_{\eta',\boldsymbol{\xi}}(F^j)|^2$, it suffices to compute $I(\varphi'_q, \boldsymbol{\xi}_q) = I(\varphi_q, \xi'_q)$ for convenient choices of ξ'_q satisfying the hypotheses of the lemma above. This is the content of the next section.

4.6. **Local zeta integrals.** To handle the local computations, it will be useful to set up the following notation. Define

$$J(\varsigma, \vartheta) := \int_{\mathbf{Q}_q^\times} W_F(d(a))W_{\Theta,\varsigma}(d(a))\Phi_\vartheta^s(d(a))|a|^{-1}d^\times a,$$

and for $\alpha \in \mathbf{GL}_2(\mathbf{Q}_q)$,

$$J(\varsigma, \vartheta, \alpha) := \int_{\mathbf{Q}_q^\times} W_F(d(a)\alpha)W_{\Theta,\varsigma}(d(a))\Phi_\vartheta^s(d(a))|a|^{-1}d^\times a.$$

We first dispose the simple case $q=\infty$.

Proposition 4.27. *For $q=\infty$, we have*

$$I(\varphi_\infty,\xi_\infty)=(2\pi)\cdot(4\pi)^{-(k+j)}\Gamma(k+j).$$

Proof. One sees easily that $I(\varphi_\infty, j_\infty)=J(\varsigma,\vartheta)$, where

$$\varsigma(\mathbf{u})=\overline{\mathbf{u}}^l e^{-2\pi\langle\mathbf{u},\mathbf{u}\rangle}$$

and

$$\vartheta(\mathbf{v})=p_j(4\pi\langle\mathbf{v},\mathbf{v}\rangle)e^{-2\pi\langle\mathbf{v},\mathbf{v}\rangle}.$$

Thus $\Phi^s_{\varsigma_1}(d(a))=|a|^s\vartheta(0)$. Taking $h'=a^{1/2}$ in (4.5.2), we find

$$\begin{aligned} W_{\Theta,\varsigma}(d(a)) &= \mathbf{I}_{\mathbf{R}^+}(a)|a|^{1/2}\int_{K_\infty^{(1)}}\varsigma(a^{1/2}h^{-1})h^{-\ell}dh \\ &= a^{\frac{\ell+1}{2}}e^{-2\pi a}\mathbf{I}_{\mathbf{R}^+}(a)=a^{\frac{\ell+1}{2}}e^{-2\pi a}\mathbf{I}_{\mathbf{R}^+}(a). \end{aligned}$$

Thus $I(\varphi_\infty,\xi_\infty)=2\pi\cdot\int_{\mathbf{R}^\times}a^{\frac{k}{2}}e^{-2\pi a}\cdot a^{\frac{\ell+1}{2}}e^{-2\pi a}\cdot|a|^{s-1}\mathbf{I}_{\mathbf{R}^+}(a)d^\times a$ and

$$I(\varphi_\infty,\xi_\infty)|_{s=1/2} \;=\; 2\pi\cdot\int_0^\infty a^{\frac{k+\ell}{2}}e^{-4\pi a}d^\times a=(2\pi)\cdot(4\pi)^{-(k+j)}\Gamma(k+j).$$

□

Next let q be a finite prime, and denote by $\mathfrak{o}_q$ and $\mathfrak{r}_q$ the maximal orders in K_q and $\mathbf{Q}_q$ respectively. We split the calculations into several cases:

I : $q\nmid cNd_K$.
II : $q\mid c$.
III : $q^{n_q}||N$, with $n_q\geq 2$.
IV : $q||N$, $q\nmid d_K$.
V : $q||N$, $q\mid d_K$.
VI : $q\mid d_K$, $q\nmid N$.

For the rest of this section, we simply write I for $I(\varphi',\boldsymbol{\xi}_q)=I(\varphi,\xi'_q)$.

4.6.1. *Case I: $q\nmid cNd_K$.* In this case all the data is unramified and we have by a standard computation:

$$I=L_q(\bar\pi_f,\pi_{\bar\eta},s)L_q(2s,\varepsilon_K)^{-1}.$$

4.6.2. *Case II: $q\mid c$.* Write $\mathfrak{o}_q=\mathbf{Z}_q+\mathbf{Z}_q\varpi$, where $\mathrm{tr}(\varpi)=0$. Let $\varpi^2=u$. We may assume that $\xi'_q(\varpi)=\begin{pmatrix}0 & 1/q^r\\ uq^r & 0\end{pmatrix}$, where $q^r||c$. Set $\mathrm{j}_q:=\begin{pmatrix}1&0\\0&-1\end{pmatrix}$. For $0\leq i,j\leq q^r-1$, set

$$\varsigma_{i,j}=\mathbf{I}_{\mathbf{Z}_q+(q^r\mathbf{Z}_q+i+\frac{j}{q^r})\varpi}\qquad \vartheta_{i,j}=\mathbf{I}_{(\mathbf{Z}_q+(q^r\mathbf{Z}_q+i+\frac{j}{q^r})\varpi)\mathrm{j}_q}.$$

Then

$$\varphi_q=\sum_{i,j}\varsigma_{i,j}\otimes\vartheta_{i,j}.$$

Since W_F and φ_q are invariant under $\mathbf{GL}_2(\mathbf{Z}_q)$, it follows that

$$I=\sum_{i,j}J(\varsigma_{i,j},\vartheta_{i,j})=J(\varsigma_{0,0},\vartheta_{0,0}).$$

Now,

$$W_{\Theta,\varsigma_{0,0}}(d(a))=\int_{\mathbf{Q}_q^\times}\varsigma_{0,0}(t,at^{-1})\eta_1(at^{-1})\eta_2(t)d^\times t=\int_{\substack{0\leq v_q(t)\leq v_q(a)\\ v_q(t-at^{-1})\geq r}}\eta_1(at^{-1})\eta_2(t)d^\times t.$$

Suppose $v_q(a)\geq 2r-1$. Then either $v_q(t)\geq r$ or $v_q(at^{-1})\geq r$. In this case, $v_q(t-at^{-1})\geq r\iff$ both $v_q(t)\geq r$ and $v_q(at^{-1})\geq r$. For such a then, the region of integration in the last integral above is unchanged if a is replaced by ua for any $u\in\mathbf{Z}_q^\times$. Thus $W_{\Theta,\varsigma_{0,0}}(d(au))=\eta_1(u)W_{\Theta,\varsigma_{0,0}}(a)$. Since $W_F(d(au))=W_F(d(a))$, by picking u such that $\eta_1(u)\neq 1$, we see that

$$\int_{v_q(a)\geq 2r-1}W_F(d(a))W_{\Theta,\varsigma_{0,0}}(d(a))|a|^{s-1}d^\times a=0.$$

So we may restrict attention to a such that $0 \le v_q(a) \le 2r-2$, and let t be in the region of integration above. Since either $v_q(t) \le r-1$ or $v_q(at^{-1}) \le r-1$, we see that $v_q(t-at^{-1}) \ge r$ is only possible if $v_q(t) = v_q(at^{-1})$. This implies that $v_q(a)$ must be even. Suppose that $v_q(a) = 2m \le 2r-2$ so that $m \le r-1$, and $v_q(t) = m$. Write $a = q^{2m}u$, $t = q^m v$ with $u, v \in \mathbf{Z}_q^\times$. The condition $v_q(t-at^{-1}) \ge r$ then translates to $v_q(v^2-u) \ge r-m$, and $\eta_1(at^{-1})\eta_2(t) = \eta_1(q^m uv^{-1})\eta_2(q^m v) = \varepsilon_{f,q}(q)^m \eta_1(uv^{-2})$ since $\eta_1\eta_2 = \varepsilon_{f,q}$ is unramified. Then for m fixed,

$$\int_{v_q(a)=m} W_F(d(a)) W_{\Theta,\varsigma_{0,0}}(d(a)|a|^{s-1} d^\times a = \text{constant} \cdot \int\int_{\substack{u,v\in\mathbf{Z}_q^\times \\ u\equiv v^2 \mod q^{r-m}}} \eta_1(uv^{-2}) d^\times v d^\times u.$$

Suppose $m > 0$. Since the conductor of η_1 is q^r, there exists $\alpha \in \mathbf{Z}_q^\times$, $\alpha \equiv 1 \mod q^{r-m}$ such that $\eta_1(\alpha) \neq 1$. Then for v fixed the integral over u is seen to be zero by making a change of variables $u \mapsto \alpha u$. Thus we are reduced to considering only the case $m = 0$, and

$$I = \mathrm{vol}((u,v) \in \mathbf{Z}_q^\times \times \mathbf{Z}_q^\times, u \equiv v^2 \mod q^r) = \frac{1}{q^{r-1}(q-1)} = \frac{1}{q^r}\zeta_{K,q}(1) \cdot L_q(\bar{\pi}_f, \pi_{\bar{\eta}'}, s) L_q(2s, \varepsilon_K)^{-1}|_{s=1/2}.$$

Here $\zeta_{K,q}(1) = (1-\frac{1}{q})^{-2}$ if q is split in K and equal to $(1-\frac{1}{q^2})$ if q is inert in K.

4.6.3. *Case III: $q^n || N$ with $n \ge 2$.* In this case, q is split in K i.e. $q = \mathfrak{q}\bar{\mathfrak{q}}$ and $K \otimes \mathbf{Q}_q \simeq \mathbf{Q}_q \times \mathbf{Q}_q$ corresponding to the completions at $\mathfrak{q}$ and $\bar{\mathfrak{q}}$ respectively. We suppose that $\mathfrak{q}$ and $\bar{\mathfrak{q}}$ are chosen such that $\mathfrak{N} \otimes \mathbf{Z}_q = \bar{\mathfrak{q}}^n$. We may assume

$$\xi'_q(a,b) = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}.$$

Then $\overline{\eta'_q} = (\eta_1, \eta_2)$ where η_1 and $\eta_2\varepsilon_{f,q}^{-1}$ are both unramified. Set $\mathfrak{j}_q := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Then

$$\vartheta((a,b)\mathfrak{j}_q) = \mathbf{I}_{\mathbf{Z}_q}(a)(\mathbf{I}_{q^n\mathbf{Z}_q} - \frac{1}{q}\mathbf{I}_{q^{n-1}\mathbf{Z}_q})(b),$$

and

$$\varsigma(a,b) = \begin{cases} \mathbf{I}_{\mathbf{Z}_q}(a)\mathbf{I}_{\mathbf{Z}_q}(b), & \text{if } q \nmid N_{\varepsilon_f}; \\ \mathbf{I}_{\mathbf{Z}_q}(a)\mathbf{I}_{\mathbf{Z}_q^\times}(b)\varepsilon_{f,q}(b), & \text{if } q \mid N_{\varepsilon_f}. \end{cases}$$

Now

$$\mathbf{GL}_2(\mathbf{Z}_q) = \Gamma_q(1) \bigsqcup \bigsqcup_{z=0}^{q-1} U(z) w \Gamma_q(1),$$

and

$$\Gamma_q(1) = \bigsqcup_{y \in q\mathbf{Z}_q/q^n\mathbf{Z}_q} V(y)\Gamma_q(n).$$

so that

$$\mathbf{GL}_2(\mathbf{Z}_q) = \bigsqcup_{y\in q\mathbf{Z}_q/q^n\mathbf{Z}_q} V(y)\Gamma_q(n) \bigsqcup \bigsqcup_{\substack{y\in q\mathbf{Z}_q/q^n\mathbf{Z}_q \\ z\in\mathbf{Z}_q/q\mathbf{Z}_q}} U(z)wV(y)\Gamma_q(n).$$

Now $V(y) = -wU(-y)w$ and $wV(y) = U(-y)w$. Thus

$$\begin{aligned} r_\psi(w,1)\vartheta((a,b)\mathfrak{j}_q) &= \hat{\vartheta}((a,b)\mathfrak{j}_q) = \frac{1}{q^n}\mathbf{I}_{q^{-n}\mathbf{Z}_q^\times}(a)\mathbf{I}_{\mathbf{Z}_q}(b), \\ r_\psi(U(-y),1)\hat{\vartheta}((a,b)\mathfrak{j}_q) &= \frac{1}{q^n}\psi_q(yab)\mathbf{I}_{q^{-n}\mathbf{Z}_q^\times}(a)\mathbf{I}_{\mathbf{Z}_q}(b), \\ r_\psi(wV(y),1)\vartheta(0) &= r_\psi(U(-y)w,1)\vartheta(0) = r_\psi(U(-y),1)\hat{\vartheta}(0) = \hat{\vartheta}(0) = 0, \end{aligned}$$

and

$$\begin{aligned} r_\psi(V(y),1)\vartheta(0) &= r_\psi(-wU(-y)w,1)\vartheta(0) = \int \frac{1}{q^n}\psi_q(yab)\mathbf{I}_{q^{-n}\mathbf{Z}_q^\times}(a)\mathbf{I}_{\mathbf{Z}_q}(b)\,da\,db \\ &= \frac{1}{q^n}\int I_{\mathbf{Z}_q}(ya)\mathbf{I}_{q^{-n}\mathbf{Z}_q^\times}(a)\,da \\ &= \begin{cases} 0, & \text{if } y \notin q^n\mathbf{Z}_q; \\ 1-\frac{1}{q}, & \text{if } y \in q^n\mathbf{Z}_q. \end{cases} \end{aligned}$$

Thus

$$I = (1-\frac{1}{q})\operatorname{vol}(\Gamma_q(n))\int W_F(d(a))W_{\Theta,\varsigma}(d(a))|a|^{s-1}d^\times a.$$

Now suppose first that $q \nmid N_{\varepsilon_f}$. Then η_1 and η_2 are both unramified and

$$W_{\Theta,\varsigma}(d(a)) = |a|^{1/2}\frac{\eta_1(aq)-\eta_2(aq)}{\eta_1(q)-\eta_2(q)}\mathbf{I}_{\mathbf{Z}_q}(a).$$

In this case $\pi_{f,q}$ is either supercuspidal or ramified principal series $\simeq \pi(\mu_1,\mu_2)$ with both μ_1 and μ_2 ramified. In any case, $W_F(d(a)) = \mathbf{I}_{\mathbf{Z}_q^\times}(a)$ and

$$I = (1-\frac{1}{q})\operatorname{vol}(\Gamma_q(n)) = \frac{1}{q^{n-1}(q+1)}\cdot L_q(\bar\pi_f,\pi_{\bar\eta'},s)L_q(2s,\varepsilon_K)^{-1}|_{s=1/2}.$$

Next suppose that $q \mid N_{\varepsilon_f}$. Then

$$W_{\Theta,\varsigma}(d(a)) = |a|^{1/2}\eta_2(a)\mathbf{I}_{\mathbf{Z}_q}(a).$$

As for W_F, we have

$$W_F(d(a)) = \begin{cases} \varepsilon_{f,q}^{-1}(a)\mathbf{I}_{\mathbf{Z}_q^\times}(a), & \text{if } q\notin\Sigma'; \\ \mu_2^{-1}(a)|a|^{1/2}\mathbf{I}_{\mathbf{Z}_q}(a), & \text{if } q\in\Sigma' \text{ and } \pi_{f,q}\simeq\pi(\mu_1,\mu_2) \text{ with } \mu_2 \text{ ramified.}\end{cases}$$

From this we find

$$I = \begin{cases} (1-\frac{1}{q})\operatorname{vol}(\Gamma_q(n)) = \frac{1}{q^{n-1}(q+1)}\cdot L_q(\bar\pi_f,\pi_{\bar\eta'},s)L_q(s,\varepsilon_K)^{-1}|_{s=1/2}, & \text{if } q\in\Sigma\setminus\Sigma' \\ (1-\frac{1}{q})\operatorname{vol}(\Gamma_q(n))(1-\mu_2^{-1}\eta_2(q)q^{-s}) = \frac{1}{q^{n-1}(q+1)}\cdot\frac{L_q(\bar\pi_f,\pi_{\bar\eta'},s)L_q(s,\varepsilon_K)^{-1}}{1-\mu_1^{-1}\eta_1(q)q^{-s}}|_{s=1/2}, & \text{if } q\in\Sigma'.\end{cases}$$

4.6.4. *Case IV: $q||N$, $q\nmid d_K$.* In this case, q is split in K i.e. $q=\mathfrak{q}\bar{\mathfrak{q}}$ and $K\otimes\mathbf{Q}_q\simeq\mathbf{Q}_q\times\mathbf{Q}_q$ corresponding to the completions at $\mathfrak{q}$ and $\bar{\mathfrak{q}}$ respectively. We suppose that $\mathfrak{q}$ and $\bar{\mathfrak{q}}$ are chosen such that $\mathfrak{N}\otimes\mathbf{Z}_q=\bar{\mathfrak{q}}$. We may assume

$$\xi_q'(a,b) = \begin{pmatrix} a & 0 \\ 0 & b\end{pmatrix}.$$

The character $\bar\eta_q'$ is identified with (η_1,η_2). Set $\mathfrak{j}_q := \begin{pmatrix} 0 & 1 \\ 1 & 0\end{pmatrix}$. Then

$$\vartheta((a,b)\mathfrak{j}_q) = \mathbf{I}_{\mathbf{Z}_q}(a)(\mathbf{I}_{q\mathbf{Z}_q}-\frac{1}{q}\mathbf{I}_{\mathbf{Z}_q})(b),$$

and

$$\varsigma(a,b) = \begin{cases} \mathbf{I}_{\mathbf{Z}_q}(a)\mathbf{I}_{\mathbf{Z}_q}(b), & \text{if } q\nmid N_{\varepsilon_f}; \\ \mathbf{I}_{\mathbf{Z}_q}(a)\mathbf{I}_{\mathbf{Z}_q^\times}(b)\varepsilon_{f,q}(b), & \text{if } q\mid N_{\varepsilon_f}.\end{cases}.$$

$$I = \frac{1}{q+1}\left(J(\varsigma,\vartheta)+qJ(w,\hat\varsigma,\hat\vartheta)\right).$$

But $\vartheta(0) = 1-\frac{1}{q}$ and $\hat\vartheta(0)=0$. Hence

$$I = \frac{1}{q+1}J(\varsigma,\vartheta) = \frac{1}{q+1}\cdot(1-\frac{1}{q})\cdot\int W_F(d(a))W_{\Theta,\varsigma}(d(a))|a|^{s-1}d^\times a,$$

where

$$W_{\Theta,\varsigma}(d(a)) = |a|^{1/2}\int_{\mathbf{Q}_q^\times}\varsigma(t,at^{-1})\eta_1(at^{-1})\eta_2(t)d^\times t.$$

Suppose $q\nmid N_{\varepsilon_f}$. Then η_1 and η_2 are unramified and

$$W_{\Theta,\varsigma}(d(a)) = |a|^{1/2}\left(\sum_{r+s=v_q(a)}\eta_1(q)^r\eta_2(q)^s\right)\mathbf{I}_{\mathbf{Z}_q}(a).$$

In this case, $\pi_{f,q}$ is a special representation $\mathrm{St}(\mu)$ with μ unramified and $W_F(d(a)) = |a|\mu^{-1}(a)\mathbf{I}_{\mathbf{Z}_q}(a)$. Hence

$$I = \frac{1}{(q+1)}\frac{1-1/q}{(1-q^{-1/2}\mu^{-1}(q)\eta_1(q)q^{-s})(1-q^{-1/2}\mu^{-1}(q)\eta_2(q)q^{-s})} = \frac{1}{q+1}L_q(\bar\pi_f,\pi_{\bar\eta'},s)L_q(2s,\varepsilon_K)^{-1}|_{s=1/2}.$$

Next suppose $q \mid N_{\varepsilon_f}$, so that η_1 is unramified and η_2 is ramified but $\eta_2\varepsilon_{f,q}^{-1}$ is unramified. Then

$$W_{\Theta,\varsigma}(d(a)) = |a|^{1/2}\eta_2(a).$$

In this case, $\pi_{f,q}$ is a ramified principal series representation $\pi(\mu_1,\mu_2)$ with say μ_1 unramified and μ_2 ramified. Since $W_F(d(a)) = |a|^{1/2}\mu_2^{-1}(a)\mathbf{I}_{\mathbf{Z}_q}(a)$, we get

$$I = \frac{1}{(q+1)}\frac{1-1/q}{1-\mu_2^{-1}(q)\eta_2(q)q^{-s}} = \frac{1}{q+1}L_q(\bar{\pi}_f, \pi_{\bar{\eta}'}, s)L_q(2s,\varepsilon_K)^{-1}|_{s=1/2}.$$

4.6.5. *Case V: $q||N$, and $q \mid d_K$.* Then $n=1$. Recall that we have assumed q odd in this case. Let $\varpi_q \in K_q := K\otimes \mathbf{Q}_q$ be such that $\Pi_q := \varpi_q^2$ is a uniformizer in $\mathbf{Z}_q$. We may assume

$$\xi_q'(\varpi_q) = \begin{pmatrix} 0 & 1 \\ \Pi_q & 0 \end{pmatrix}.$$

Set $\mathfrak{j}_q := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. First we suppose we are in

Subcase Va: $q \nmid N_{\varepsilon_f}$, i.e., $q \in S(f)$. Then $\varphi_q = \varsigma\otimes\vartheta$ where

$$\varsigma(a+b\varpi_q) = \mathbf{I}_{\mathbf{Z}_q}(a)\mathbf{I}_{\mathbf{Z}_q}(b), \qquad \vartheta((c+d\varpi_q)\mathfrak{j}_q) = \mathbf{I}_{\mathbf{Z}_q}(c)\mathbf{I}_{\mathbf{Z}_q}(d),$$

so that

$$\hat{\varsigma}(a+b\varpi_q) = q^{-1/2}\mathbf{I}_{\mathbf{Z}_q}(a)\mathbf{I}_{\frac{1}{q}\mathbf{Z}_q}(b), \qquad \hat{\vartheta}((c+d\varpi_q)\mathfrak{j}_q) = q^{-1/2}\mathbf{I}_{\mathbf{Z}_q}(c)\mathbf{I}_{\frac{1}{q}\mathbf{Z}_q}(d),$$
$$I = \frac{1}{q+1}\left(J(\varsigma,\vartheta) + qJ(w,\hat{\varsigma},\hat{\vartheta})\right).$$

Let β_q denote the matrix $\begin{pmatrix} 1 & 0 \\ 0 & \Pi_q^{-1} \end{pmatrix}$. Then

$$r_\psi\left(\beta_q, \varpi_q^{-1}\right)\varsigma(a+b\varpi_q) = \mathbf{I}_{\mathbf{Z}_q}(a)\mathbf{I}_{\frac{1}{q}\mathbf{Z}_q}(b) = q^{1/2}\hat{\varsigma}(a+b\varpi_q)$$

and likewise $r_\psi\left(\beta_q,\varpi_q^{-1}\right)\vartheta = q^{1/2}\hat{\vartheta}$. Thus

$$I = \frac{1}{q+1}\left(J(\varsigma,\vartheta) + \overline{\eta_q'}(\varpi_q)J(\beta_q^{-1}w,\varsigma,\vartheta)\right).$$

But $\pi_{f,q}$ is special, say $\simeq \mathrm{St}(\mu)$, hence $W_F(g\beta_q^{-1}w) = \mu(\Pi_q)W_F(g)$. Hence

$$I = \frac{(1-\mu(\Pi_q)\overline{\eta_q'}(\varpi_q))}{q+1}J(\varsigma,\vartheta) = \frac{2}{q+1}J(\varsigma,\vartheta),$$

on account of our assumption that $\varepsilon_q(f,\chi^{-1}) = +1$ and [Tu2], Prop. 1.7. Since $\overline{\eta_q'}$ is unramified in this case, we can write $\eta_q' = \eta_1\circ \mathrm{N}_{K_q/\mathbf{Q}_q} = \eta_2\circ\mathrm{N}_{K_q/\mathbf{Q}_q}$ where η_1 is an unramified character of $\mathbf{Q}_q^\times$ and $\eta_2 = \eta_1\cdot\varepsilon_{K,q}$. Then $W_{\Theta,\varsigma}(d(a)) = |a|^{1/2}(\eta_1(a)+\eta_2(a))\mathbf{I}_{\mathbf{Z}_q}(a)$. Since $W_F(d(a)) = |a|\mu^{-1}(a)\mathbf{I}_{\mathbf{Z}_q}(a)$, we find

$$I = \frac{2}{q+1}\cdot\frac{1}{(1-q^{-1/2}\mu^{-1}(q)\eta_1(q)q^{-s})} = \frac{2}{q+1}\cdot L_q(\bar{\pi}_f,\pi_{\bar{\eta}'},s)L_q(2s,\varepsilon_K)^{-1}|_{s=1/2}.$$

Subcase Vb: $q \mid N_{\varepsilon_f}$. Then

$$\varphi_q = \sum_{\substack{i,j\in\mathbf{Z}_q/q\mathbf{Z}_q\\ i\neq j}} \varepsilon_{f,q}(i-j)\cdot\varsigma_i\otimes\vartheta_j$$

where

$$\varsigma_i(a+b\varpi_q) = \mathbf{I}_{q\mathbf{Z}_q+i}(a)\mathbf{I}_{\mathbf{Z}_q}(b), \qquad \vartheta_j((c+d\varpi_q)\mathfrak{j}_q) = \mathbf{I}_{q\mathbf{Z}_q+j}(c)\mathbf{I}_{\mathbf{Z}_q}(d),$$
$$I(\varphi_q) = \frac{1}{q+1}\cdot\sum_{\substack{i,j\in\mathbf{Z}_q/q\mathbf{Z}_q\\ i\neq j}}\varepsilon_{f,q}(i-j)\left(J(\varsigma_i,\vartheta_j) + qJ(\hat{\varsigma}_i,\hat{\vartheta}_j)\right).$$

Note that $\hat{\vartheta}_j$ is independent of j. Thus, for any fixed i, the sum $\sum_{j\neq i}\varepsilon_{f,q}(i-j)J(\hat{\varsigma}_i,\hat{\vartheta}_j) = 0$. Also $\vartheta_j(0) = \delta_{j0}$. Consequently,

$$I(\varphi_q) = \frac{1}{q+1}\sum_{i\neq 0}\varepsilon_{f,q}(i)J(\varsigma_i,\vartheta_0) = \frac{1}{q+1}J(\varsigma,\vartheta_0),$$

where $\varsigma := \sum_{i\neq 0} \varepsilon_{f,q}(i)\varsigma_i$. Now $W_{\Theta,\varsigma}(d(a)) = \varepsilon_{f,q}(a)(1+\varepsilon_{K,q}(a))\mathbf{I}_{\mathbf{Z}_q^\times}(a)$. Since $\pi_{f,q}$ is ramified principal series of the form $\pi(\mu_1,\mu_2)$ with μ_1 unramified and μ_2 ramified, we have $W_F(d(a)) = |a|^{1/2}\mu_2^{-1}(a)\mathbf{I}_{\mathbf{Z}_q}(a)$ and

$$I = \frac{1}{q+1} = \frac{1}{q+1}\cdot L_q(\bar{\pi}_f, \pi_{\bar{\eta}'}, s)L_q(2s,\varepsilon_K)^{-1}|_{s=1/2}.$$

4.6.6. *Case VI: $q \mid d_K$, $q \nmid N$.* Again we may assume

$$\xi_q'(\varpi_q) = \begin{pmatrix} 0 & 1 \\ \pi_q & 0 \end{pmatrix}.$$

Set $\mathfrak{j}_q := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Then

$$\varphi_q = \sum_{i=0}^{q-1} \varsigma_i \otimes \vartheta_i,$$

where

$$\varsigma_i(a+b\varpi_q) = \mathbf{I}_{\mathbf{Z}_q}(a)\mathbf{I}_{\frac{i}{q}+\mathbf{Z}_q}(b), \qquad \vartheta_i((a+b\varpi_q)\mathfrak{j}_q) = \mathbf{I}_{\mathbf{Z}_q}(a)\mathbf{I}_{\frac{i}{q}+\mathbf{Z}_q}(b).$$

Since $q \nmid N$, we have $I = \sum_i J(\varsigma_i,\vartheta_i) = J(\varsigma_0,\vartheta_0)$. Since $\overline{\eta_q'}$ is unramified in this case, we can write $\eta_q' = \eta_1\circ \mathrm{N}_{K_q/\mathbf{Q}_q} = \eta_2\circ\mathrm{N}_{K_q/\mathbf{Q}_q}$ where η_1 is an unramified character of $\mathbf{Q}_q^\times$ and $\eta_2 = \eta_1\cdot\varepsilon_{K,q}$. Then $W_{\Theta,\varsigma_0}(d(a)) = |a|^{1/2}(\eta_1(a)+\eta_2(a))\mathbf{I}_{\mathbf{Z}_q}(a)$. If $\pi_{f,q}\simeq\pi(\mu_1,\mu_2)$, then $W_F(d(a)) = |a|^{1/2}\frac{\mu^{-1}(aq)-\mu_2^{-1}(aq)}{\mu_1^{-1}(q)-\mu_2^{-1}(q)}\mathbf{I}_{\mathbf{Z}_q}(a)$ and

$$I = \frac{1}{(1-\mu_1^{-1}(q)\eta_1(q)q^{-s})(1-\mu_2^{-1}(q)\eta_1(q)q^{-s})} = L_q(\bar{\pi}_f,\pi_{\bar{\eta}'},s)L_q(2s,\varepsilon_K)^{-1}.$$

4.7. **The explicit form of Waldspurger's formula.** We can now state the main result on the absolute value squared of the period integral $L_{\eta,\boldsymbol{\xi}}(F^j)$ defined in equation (4.3.6). We will need the class number formula $L(1,\varepsilon_K) = 2\pi h_K/w_K\sqrt{|d_K|}$ and the volume of $U'^{(1)}$:

$$\mathrm{vol}(U'^{(1)}) = \zeta(2)^{-1}\cdot\prod_{q^{n_q}||N}\frac{1}{q^{n_q-1}(q+1)}.$$

Combining these with Cor. 4.26, equation (4.5.4) (with φ replaced by φ') and the computations of the previous section, we obtain:

Theorem 4.28. *Suppose cd_K is odd and η is a character of K of infinity type $(-\ell,0)$ ($\ell = k+2j$) and finite type $(c,\mathfrak{N},\varepsilon_f^{-1})$. Then*

$$|L_{\eta',\boldsymbol{\xi}}(F^j)|^2 = C\cdot L(\frac{1}{2},\overline{\pi}_f\times\pi_{\bar{\eta}'}) = C\cdot L(\frac{1}{2},\pi_f\times\pi_{\eta'}),$$

with

$$C = \frac{\Gamma(j+1)\Gamma(k+j)w_K\sqrt{|d_K|}\Im(\tau)^{-\ell}}{(4\pi)^{k+2j+1}\cdot h_K^2\cdot c}\cdot 2^{\#S(f)}\cdot\prod_{q|c}\zeta_{K,q}(1). \tag{4.7.1}$$

Since $L(\frac{1}{2},\pi_f\times\pi_{\eta'}) = L(f,\chi^{-1},0)$, $|\Lambda_\tau|^2\Im(\tau) = \mathrm{vol}(\mathcal{O}_c)$ and $h_c/h_K = c\prod_{q|c}(1-\varepsilon_K(q)/q)$, we obtain Theorem 4.6 by combining Theorem 4.28 and Proposition 4.13.

5. Anticyclotomic p-adic L-functions

5.1. **Periods and algebraicity.** We will now use Theorem 4.6 of Section 4.1 to deduce algebraicity properties of the central critical values $L(f,\chi^{-1},0)$ attached to characters $\chi\in\Sigma_{cc}^{(2)}(\mathfrak{N})$. In order to do this, recall the dictionary between pairs (L,t) as in Section 4.1 and triples (E,t,ω) consisting of an elliptic curve over $\mathbf{C}$, a point t on E of order N, and a differential $\omega\in\Omega^1_{E/\mathbf{C}}$. Under this correspondence, the pair (L,t) corresponds to the triple $(\mathbf{C}/L,t,2\pi i dw)$, where the differential $2\pi i dw$ arises from the standard coordinate w on $\mathbf{C}$; in the other direction, the triple (E,t,ω) corresponds to the pair (Λ_ω,t) where $2\pi i\Lambda_\omega$

is the period lattice attached to the differential ω. Viewing a nearly holomorphic modular form of weight $k+2j$ as a function on triples, we can rewrite the expression $\delta_k^j f(\mathfrak{a}^{-1},t)$ that appears in Theorem 4.6 as

$$\delta_k^j f(\mathfrak{a}^{-1},t) = \delta_k^j f(\mathbf{C}/\mathfrak{a}^{-1}, t, 2\pi i dw) = \delta_k^j f(\mathfrak{a} * (A_0, t, 2\pi i dw)),$$

where $A_0 := \mathbf{C}/\mathcal{O}_c$, and we recall that the action of $\mathcal{O}_c$-ideals of norm prime to N on marked elliptic curves with Γ-level structure of the form (A_0,t_0,ω_0) is the one described in equation (1.4.8) of Section 1.4.

Recall the triple (A, t_A, ω_A) with $\mathrm{End}_F(A) = \mathcal{O}_K$ that was fixed until now. The curve A_0 is the image of A by an isogeny $\varphi_0 : A \longrightarrow A_0$ of degree c. Let (A_0,t_0,ω_0) be the marked elliptic curve induced from (A,t_A,ω_A) via φ_0, i.e., the unique triple for which

$$\varphi_0 : (A, t_A, \omega_A) \longrightarrow (A_0, t_0, \omega_0) \tag{5.1.1}$$

is an isogeny of marked elliptic curves with Γ-level structure in the sense of Definition 1.10.

Given a Hecke character $\chi \in \Sigma_{cc}^{(2)}(\mathfrak{N})$ of infinity type $(k+j,-j)$, it will be convenient to set

$$\chi_j := \chi \mathrm{N}^j$$

for the associated Hecke character of infinity type $(k+2j,0)$. Following the usual conventions, we will view χ_j as a multiplicative function on the fractional $\mathcal{O}_c$-ideals that are prime to $\mathfrak{N}c$. This character satisfies

$$\chi_j(x\mathfrak{a}) = x^{k+2j}\varepsilon_f(x \bmod \mathfrak{N})\chi_j(\mathfrak{a}), \tag{5.1.2}$$

for all $x \in K^\times$ that are prime to $\mathfrak{N}c$. After fixing the triple $(A_0,t,2\pi i dw)$, with t an (arbitrarily chosen, but fixed from now on) generator of $A_0[\mathfrak{N}]$, the expression

$$\chi_j^{-1}(\mathfrak{a})\delta_k^j f(\mathfrak{a} * (A_0, t, 2\pi i dw))$$

depends only on the class of $\mathfrak{a}$ in $\mathrm{Pic}(\mathcal{O}_c)$. (Cf. Lemma 4.5.) We can now restate Theorem 4.6 of Section 4.1 as follows:

Theorem 5.1. *Let f be a normalised eigenform in $S_k(\Gamma_0(N), \varepsilon_f)$ and let $\chi \in \Sigma_{cc}^{(2)}(\mathfrak{N})$ be a Hecke character of K of infinity type $(k+j,-j)$. Then*

$$C(f,\chi,c)L(f,\chi^{-1},0) = \left| \sum_{[\mathfrak{a}]\in \mathrm{Pic}(\mathcal{O}_c)} \chi_j^{-1}(\mathfrak{a}) \cdot \delta_k^j f(\mathfrak{a} * (A_0, t, 2\pi i dw)) \right|^2, \tag{5.1.3}$$

where the sum is taken over a system of representatives of the elements of $\mathrm{Pic}(\mathcal{O}_c)$ *that are prime to* $\mathfrak{N}c$*, and the constant* $C(f,\chi,c)$ *is given in Theorem 4.6.*

Note that the sum appearing in the right-hand side of (5.1.3) does depend on the choice of generator t of $A_0[\mathfrak{N}]$, but only up to multiplication by an N-th root of unity; in particular, its absolute value is independent of the choice of t that was made.

For the purposes of algebraicity statements, p-adic interpolation, and the applications that are given in [BDP-cm] and [BDP-ch], it will be useful to have a formula in which the absolute value signs that occur in Theorem 5.1 are replaced by squares. In order to do this, we will need to examine the behavior of

$$J(f,\chi) := \sum_{[\mathfrak{a}]\in \mathrm{Pic}(\mathcal{O}_c)} \chi_j^{-1}(\mathfrak{a}) \cdot \delta_k^j f(\mathfrak{a} * (A_0, t, 2\pi i dw)) \tag{5.1.4}$$

under complex conjugation.

The choice of a primitive N-th root of unity ζ and of a square root of $-N$ determines an Atkin-Lehner involution w_N acting on triples (E,t,ω) by the rule

$$w_N(E,t,\omega) = (E/\langle t\rangle, t', \sqrt{-N}\omega'),$$

where t' is the image in $E/\langle t\rangle$ of any element $t'' \in E[N]$ satisfying

$$\langle t, t''\rangle = \zeta$$

for the Weil pairing $\langle\ ,\ \rangle$, and ω' is the differential on $E' = E/\langle t\rangle$ which pulls back to ω under the natural projection. It is straightforward to verify that the function w_N is an involution on triples, and that it satisfies the commutation relation

$$\mathfrak{a} * w_N(A_0, t, 2\pi i dw) = w_N \mathfrak{a} * (A_0, N\mathfrak{a}^{-1}t, 2\pi i dw). \tag{5.1.5}$$

Recall the decomposition $N = \mathfrak{N}\bar{\mathfrak{N}}$ of N as a product of two cyclic ideals of $\mathcal{O}_c$ of norm N. Choose an integral $\mathcal{O}_c$-ideal $\mathfrak{b}$ and a nonzero element $b_N \in \mathcal{O}_c$ satisfying

$$(\mathfrak{b}, Nc) = 1, \qquad \mathfrak{b}\mathfrak{N} = (b_N). \tag{5.1.6}$$

The multiplication by b_N map identifies the quotient $A_0[N]/A_0[\mathfrak{N}]$ with the submodule $A_0[\bar{\mathfrak{N}}]$ of $A_0[N]$. Furthermore, the elliptic curve A_0 and its differential dw are defined over $\mathbf{R}$. Hence complex conjugation preserves them, but interchanges $A_0[\mathfrak{N}]$ and $A_0[\bar{\mathfrak{N}}]$. The pair $(\mathfrak{b}, b_N)$ therefore determines an element t'' of $A_0[N]$ satisfying

$$A_0[N] = (\mathbf{Z}/N\mathbf{Z})t + (\mathbf{Z}/N\mathbf{Z})t'', \qquad b_N t'' = \bar{t}. \tag{5.1.7}$$

This element is uniquely determined by b_N up to addition of a multiple of t. Therefore the primitive Nth root of unity

$$\zeta := \langle t, t'' \rangle \tag{5.1.8}$$

depends only on b_N and not on the choice of t'' satisfying (5.1.7). Let w_N denote the Atkin-Lehner involution associated to the root of unity ζ. If f is a modular form in $S_k(\Gamma_0(N), \varepsilon_f)$, recall that f_ρ is the form in $S_k(\Gamma_0(N), \bar{\varepsilon}_f)$ whose fourier coefficients are the complex conjugates of those of f. If f is a normalised eigenform and a_n denotes the eigenvalue of the Hecke operator T_n acting on f, then we have the relation

$$\bar{a}_n = \varepsilon_f^{-1}(n) a_n, \tag{5.1.9}$$

for all n which are relatively prime to N. In particular, the form f_ρ is also a normalised eigenform and corresponds to the twist of f by the character ε_f^{-1}. The following lemma is well-known.

Lemma 5.2. *Suppose that $f \in S_k(\Gamma_0(N), \varepsilon_f)$ is a newform. Then there exists a complex scalar w_f of norm one satisfying (for all triples (E, t, ω))*

$$f_\rho(w_N(E, t, \omega)) = w_f f(E, t, \omega).$$

Proof. The operator w_N satisfies the following commutation relation relative to the Hecke operators:

$$T_n w_N = \langle n \rangle w_N T_n, \qquad \langle n \rangle w_N = w_N \langle n^{-1} \rangle. \tag{5.1.10}$$

Equations (5.1.9) and (5.1.10) imply that the eigenvalue of T_n acting on $w_N f_\rho$ is equal to a_n. By multiplicity one, it follows that $w_N f_\rho$ is a non-zero scalar multiple of f, i.e., $w_N f_\rho = w_f f$, for some $w_f \in \mathbf{C}^\times$. The fact that w_N is defined over $\mathbf{R}$, and hence commutes with the action of complex conjugation, implies also that $w_N f = \bar{w}_f f_\rho$, and therefore $|w_f|^2 = 1$ since $w_N^2 = 1$. □

It should be noted that the scalar w_f is not entirely intrinsic to f, but depends on the choice of N-th root of unity ζ that was made in (5.1.8) prior to defining the Atkin-Lehner involution w_N. Over $\mathbf{C}$, it is customary to take $\zeta = e^{\frac{2\pi i}{N}}$ but our choice of ζ may differ.

After these preliminaries, we define a complex scalar of norm one by the rule:

$$w(f, \chi) := w_f \cdot \varepsilon_f(\mathrm{N}\mathfrak{b})^{-1} \chi_j(\mathfrak{b})(-N)^{k/2+j} b_N^{-k-2j}. \tag{5.1.11}$$

Ostensibly, this scalar depends on the choice of $(\mathfrak{b}, b_N)$ satisfying (5.1.6), but in fact we have:

Lemma 5.3. *The scalar $w(f, \chi)$ satisfies the following properties:*

1. *It depends only on f and χ and not on the choice of pair $(\mathfrak{b}, b_N)$ satisfying (5.1.6);*
2. *It belongs to the finite extension L of K generated by K_f, K_χ, and $\sqrt{-N}$;*
3. *For all $\sigma \in \mathrm{Gal}(L/K)$,*

$$w(f^\sigma, \chi^\sigma) = w(f, \chi)^\sigma.$$

Proof. Properties (2) and (3) follow directly from the definition of $w(f, \chi)$. The truth of (1) follows from Theorem 5.4 below (since none of the terms other than $w(f, \chi)$ that appear in (5.1.12) depend on $(\mathfrak{b}, b_N)$) but it may be helpful to supply an independent, self-contained argument. If the pair $(\mathfrak{b}, b_N)$ is replaced by the pair $(\mathfrak{b}', b'_N)$, then

$$\mathfrak{b}' = \mathfrak{b}(a), \qquad b'_N = b_N a,$$

where a is an element of $K^\times$ which is prime to $\mathfrak{N}c$. The conditions (5.1.7) and (5.1.8) that are required to be satisfied by b_N and b'_N imply that $a \equiv 1 \pmod{\bar{\mathfrak{N}}}$. The constants $w(f,\chi)$ attached to the choices $(\mathfrak{b}, b_N)$ and $(\mathfrak{b}', b'_N)$ therefore differ by a factor of

$$\varepsilon_f(a\bar{a})^{-1}\chi_j(a)a^{-k-2j} = \varepsilon_f(a \bmod \mathfrak{N})^{-1}\chi_j(a)a^{-k-2j}.$$

But this factor is equal to 1, by (5.1.2). □

Theorem 5.4. *Let f be a normalised eigenform in $S_k(\Gamma_0(N),\varepsilon_f)$ and let $\chi \in \Sigma_{cc}^{(2)}(\mathfrak{N})$ be a Hecke character of K of infinity type $(k+j,-j)$. Then*

$$C(f,\chi,c)L(f,\chi^{-1},0) = w(f,\chi)\left(\sum_{[\mathfrak{a}]\in\mathrm{Pic}(\mathcal{O}_c)} \chi_j^{-1}(\mathfrak{a})\cdot\delta_k^j f(\mathfrak{a}*(A_0,t,2\pi i dw))\right)^2, \tag{5.1.12}$$

where the constants $C(f,\chi,c)$ and $w(f,\chi)$ are described in Theorem 4.6 and in equation (5.1.11) respectively.

Proof. Theorem 5.4 is proved by computing the effect of complex conjugation on the quantity $J(f,\chi)$ of equation (5.1.4). Observe that

(1) Since $\overline{(A_0, 2\pi i dw)} = (A_0, 2\pi i dw)$ and b_N satisfies (5.1.7) and (5.1.8), the action of complex conjugation on $(A_0,t,2\pi i dw)$ is given by
$$\overline{(A_0,t,2\pi i dw)} = (A_0,\bar{t},2\pi i dw) = \mathfrak{b} * w_N(A_0,t,b_N\sqrt{-N}^{-1}2\pi i dw).$$
(2) The action of complex conjugation on $\chi_j^{-1}(\mathfrak{a})$ is given by
$$\overline{\chi_j^{-1}(\mathfrak{a})} = \varepsilon_f(\mathrm{N}\mathfrak{a})\chi_j^{-1}(\bar{\mathfrak{a}}).$$

Hence we have

$$\begin{aligned}\overline{\chi_j^{-1}(\mathfrak{a})\delta_k^j f(\mathfrak{a}*(A_0,t,2\pi i dw))} &= \varepsilon_f(\mathrm{N}\mathfrak{a})\chi_j^{-1}(\bar{\mathfrak{a}})\delta_k^j f_\rho(\bar{\mathfrak{a}}*(A_0,\bar{t},2\pi i dw)) \\ &= \varepsilon_f(\mathrm{N}\mathfrak{a})\chi_j^{-1}(\bar{\mathfrak{a}})\delta_k^j f_\rho(\bar{\mathfrak{a}}\mathfrak{b}*w_N(A_0,t,b_N\sqrt{-N}^{-1}2\pi i dw)) \\ &= (-N)^{k/2+j}b_N^{-k-2j}\varepsilon_f(\mathrm{N}\mathfrak{a})\chi_j^{-1}(\bar{\mathfrak{a}})\cdot\delta_k^j f_\rho(\bar{\mathfrak{a}}\mathfrak{b}*w_N(A_0,t,2\pi i dw)).\end{aligned} \tag{5.1.13}$$

But now, by (5.1.5), we have:

$$\begin{aligned}\delta_k^j f_\rho(\bar{\mathfrak{a}}\mathfrak{b}*w_N(A_0,t,2\pi i dw)) &= \delta_k^j f_\rho(w_N\bar{\mathfrak{a}}\mathfrak{b}*(A_0,(N\bar{\mathfrak{a}}\mathfrak{b})^{-1}t,2\pi i dw)) \\ &= w_f\varepsilon_f(N\bar{\mathfrak{a}}\mathfrak{b})^{-1}\cdot\delta_k^j f(\bar{\mathfrak{a}}\mathfrak{b}*(A_0,t,2\pi i dw)).\end{aligned} \tag{5.1.14}$$

Combining equations (5.1.13) and (5.1.14), we obtain

$$\begin{aligned}&\overline{\chi_j^{-1}(\mathfrak{a})\delta_k^j f(\mathfrak{a}*(A_0,t,2\pi i dw))} \\ &\quad = w_f\cdot(-N)^{k/2+j}b_N^{-k-2j}\chi_j(\mathfrak{b})\varepsilon_f(\mathrm{N}\mathfrak{b})^{-1}\chi_j(\bar{\mathfrak{a}}\mathfrak{b})^{-1}\delta_k^j f(\bar{\mathfrak{a}}\mathfrak{b}*(A_0,t,2\pi i dw)).\end{aligned}$$

Summing this relation over all classes $\mathfrak{a} \in \mathrm{Pic}\,\mathcal{O}_c$, we obtain

$$\overline{J(f,\chi)} = w(f,\chi)J(f,\chi),$$

and Theorem 5.4 follows. □

We now turn to the algebraicity properties of $L(f,\chi^{-1},0)$. We begin by defining a complex period attached to K. For this, we observe that the complex elliptic curve A_0 has endomorphism ring equal to the order $\mathcal{O}_c$ of conductor c, and therefore is defined over a subfield H_c of $\mathbf{C}$ which is isomorphic to the ring class field of K of conductor c. The choice of the differential $\omega_0 \in \Omega^1(A_0/H_c)$ determined by (5.1.1) determines a complex period Ω, defined as the non-zero complex scalar satisfying

$$\omega_0 = \Omega\cdot 2\pi i dw, \tag{5.1.15}$$

where w is the standard complex coordinate on $A_0(\mathbf{C}) = \mathbf{C}/\mathcal{O}_c$.

Theorem 5.5 below asserts that the ratios $w^{-1}(f,\chi)C(f,\chi,c)L(f,\chi^{-1},0)/\Omega^{2(k+2j)}$ are algebraic numbers. In order to make a more precise claim about the fields of definition, we remark that the point t_0 belongs (by assumption) to the $\mathfrak{N}$-torsion subgroup of A_0, which is defined over H_c. Let H'_c be the abelian extension of H_c over which the individual $\mathfrak{N}$-torsion points of A_0 are defined, so that in particular the pair (A_0, t_0) is defined over H_c'. The Galois group of $\mathrm{Gal}(H_c'/H_c)$ is canonically identified with a subgroup of $(\mathbf{Z}/N\mathbf{Z})^\times$ via its faithful action on $A_0[\mathfrak{N}]$. Let $\tilde{H}_c \subset H_c'$ be the subfield which is fixed by $\ker(\varepsilon_f)$. Let $F \subset \mathbf{C}$ be the finite extension of K generated by $\tilde{H}_c$, by the values of the Hecke character χ on $\mathbb{A}_{K,f}^\times$, and by the Fourier coefficients of f. We can now state Shimura's algebraicity theorem on the special values $L(f, \chi^{-1}, 0)$ in a precise form.

Theorem 5.5. *For all $\chi \in \Sigma_{\mathrm{cc}}^{(2)}(\mathfrak{N})$ of infinity type $(k+j, -j)$, the quantity*

$$L_{\mathrm{alg}}(f, \chi^{-1}, 0) := w(f,\chi)^{-1} C(f, \chi, c) \cdot L(f, \chi^{-1}, 0)/\Omega^{2(k+2j)}$$

belongs to F.

Proof. By Theorem 5.4,

$$\begin{aligned} w(f,\chi)^{-1}C(f,\chi,c)L(f,\chi^{-1},0) &= \left(\sum_{[\mathfrak{a}] \in \mathrm{Pic}(\mathcal{O}_c)} \chi_j^{-1}(\mathfrak{a}) \cdot \delta_k^j f(\mathfrak{a} * (A_0, t_0, 2\pi i dw)) \right)^2 \\ &= \left(\sum_{[\mathfrak{a}] \in \mathrm{Pic}(\mathcal{O}_c)} \chi_j^{-1}(\mathfrak{a}) \cdot \delta_k^j f(\mathfrak{a} * (A_0, t_0, \Omega^{-1}\omega_0)) \right)^2 \\ &= \Omega^{2(k+2j)} \left(\sum_{[\mathfrak{a}] \in \mathrm{Pic}(\mathcal{O}_c)} \chi_j^{-1}(\mathfrak{a}) \cdot \delta_k^j f(\mathfrak{a} * (A_0, t_0, \omega_0)) \right)^2 . \end{aligned}$$

It follows from Lemma 1.5 that

$$L_{\mathrm{alg}}(f, \chi^{-1}, 0) = \left(\sum_{[\mathfrak{a}] \in \mathrm{Pic}(\mathcal{O}_c)} \chi_j^{-1}(\mathfrak{a}) \cdot \Theta_{\mathrm{Hodge}}^j f(\mathfrak{a} * (A_0, t_0, \omega_0)) \right)^2 . \tag{5.1.16}$$

Part 1 of Proposition 1.12 implies that the terms $\Theta_{\mathrm{Hodge}}^j f(\mathfrak{a} * (A_0, t_0, \omega_0))$ belong to F. Theorem 5.5 follows. ☐

Remark 5.6. The datum of $\mathcal{O}_c$ determines the elliptic curve A_0/H_c together with the embedding of H_c into $\mathbf{C}$. Both sides of (5.1.16) depend on the further choice of a regular differential ω_0 on A_0/H_c which was determined by our choice of ω_A. Note that a change in ω_A (or ω_0) affects both sides of (5.1.16) in the same way.

5.2. **p-adic interpolation.** Let p be a rational prime which splits in $K/\mathbf{Q}$, and fix a prime $\mathfrak{p}$ of K above p. Extending the associated embedding of K into $\mathbf{Q}_p$ to an embedding $\iota_p : F \longrightarrow \mathbf{C}_p$. The special values $L_{\mathrm{alg}}(f, \chi^{-1}, 0)$ can be viewed, through the embedding ι_p, as p-adic numbers. The following theorem gives a p-adic formula for these special values, in terms of the Atkin-Serre operator θ on p-adic modular forms.

Theorem 5.7. *For all $\chi \in \Sigma_{\mathrm{cc}}^{(2)}(\mathfrak{N})$ of infinity type $(k+j, -j)$,*

$$L_{\mathrm{alg}}(f, \chi^{-1}, 0) = \left(\sum_{\mathfrak{a} \in \mathrm{Pic}(\mathcal{O}_c)} \chi_j^{-1}(\mathfrak{a}) (\theta^j f)(\mathfrak{a} * (A_0, t_0, \omega_0)) \right)^2 .$$

Proof. The fact that p is split in K implies that the elliptic curve $\iota_p(A_0)$ has good ordinary reduction. By part 3 of Proposition 1.12, combined with (5.1.16), we have:

$$L_{\mathrm{alg}}(f, \chi^{-1}, 0) = \left(\sum_{[\mathfrak{a}] \in \mathrm{Pic}(\mathcal{O}_c)} \chi_j^{-1}(\mathfrak{a}) \cdot \Theta_{\mathrm{Frob}}^j f(\mathfrak{a} * (A_0, t_0, \omega_0)) \right)^2 . \tag{5.2.1}$$

Theorem 5.7 now follows from Lemma 1.7. ☐

Although the set $\Sigma_{\rm cc}^{(2)}(\mathfrak{N})$ is infinite, its elements take values in a finite extension of K. By possibly enlarging the finite extension F of K that appears in the statement of Theorem 5.5, we will assume that it contains the values $\chi(\mathfrak{a})$ as χ ranges over all characters in $\Sigma_{\rm cc}^{(2)}(\mathfrak{N})$ and $\mathfrak{a}$ ranges over $\mathbb{A}_{K,f}^{\times}$.

Let $\mathbb{A}'_{K,f}$ denote the subgroup of $\mathbb{A}_{K,f}^{\times}$ of idèles which are prime to p, and choose any prime $\mathfrak{p}_F$ of F above $\mathfrak{p}$. We observe that the values $\chi(\mathfrak{a})$ as $\mathfrak{a}$ ranges over $\mathbb{A}'_{K,f}$ are integral at $\mathfrak{p}_F$, i.e., they belong to the ring of integers $\mathcal{O}_{F,\mathfrak{p}_F}$ of the completion $F_{\mathfrak{p}_F}$. It follows that $\Sigma_{\rm cc}^{(2)}(\mathfrak{N})$ is naturally embedded in the space $\mathcal{F}(\mathbb{A}'_{K,f},\mathcal{O}_{F,\mathfrak{p}_F})$ of $\mathcal{O}_{F,\mathfrak{p}_F}$-valued functions on $\mathbb{A}'_{K,f}$. We equip $\Sigma_{\rm cc}^{(2)}(\mathfrak{N})$ with the topology induced by the compact open topology on this function space, i.e., the topology of uniform convergence on $\mathbb{A}'_{K,f}$ relative to the p-adic topology on $\mathcal{O}_{F,\mathfrak{p}}$. Let $\hat{\Sigma}_{\rm cc}(\mathfrak{N})$ be the completion of $\Sigma_{\rm cc}^{(2)}(\mathfrak{N})$ relative to this topology.

To p-adically interpolate the values $L_{\rm alg}(f,\chi^{-1},0)$ we need to modify them by dropping a suitable Euler factor at p, and multiplying by a suitable p-adic period. We begin by attaching to A_0 a p-adic period Ω_p as follows. Let $\mathcal{A}_0$ be a good integral model of A_0 over $\mathcal{O}_{\mathbf{C}_p}$. The formal completion $\hat{\mathcal{A}}_0$ of $\mathcal{A}_0$ along its identity section is (non-canonically) isomorphic to $\hat{\mathbb{G}}_m$ over $\mathcal{O}_{\mathbf{C}_p}$; fix such an isomorphism $\iota:\hat{\mathcal{A}}_0\longrightarrow\hat{\mathbb{G}}_m$. (This amounts to fixing an isomorphism between the p-divisible groups μ_{p^∞} and $\mathcal{A}_0[\mathfrak{p}^\infty]$, which is determined up to a scalar in $\mathbf{Z}_p^{\times}$.) Fixing the isomorphism ι once and for all, we define $\Omega_p\in\mathbf{C}_p^{\times}$ by the rule, analogous to (5.1.15)

$$\omega_0=\Omega_p\cdot\omega_{\rm can},\qquad\text{where}\qquad \omega_{\rm can}:=\iota^*\frac{du}{u},\tag{5.2.2}$$

and u denotes the standard coordinate on $\hat{\mathbb{G}}_m$.

For all $\chi\in\Sigma_{\rm cc}^{(2)}(\mathfrak{N})$ of infinity type $(k+j,-j)$, we set

$$L_p(f,\chi):\ =\ \Omega_p^{2(k+2j)}(1-\chi^{-1}(\bar{\mathfrak{p}})a_p+\chi^{-2}(\bar{\mathfrak{p}})\varepsilon_f(p)p^{k-1})^2L_{\rm alg}(f,\chi^{-1},0)\tag{5.2.3}$$

$$=\ \Omega_p^{2(k+2j)}(1-\alpha_p\chi^{-1}(\bar{\mathfrak{p}}))^2(1-\beta_p\chi^{-1}(\bar{\mathfrak{p}}))^2L_{\rm alg}(f,\chi^{-1},0),\tag{5.2.4}$$

where α_p,β_p denote the parameters of f at p described at the beginning of Section 4.1.

Remark 5.8. Note that both $L_{\rm alg}(f,\chi)$ and Ω_p depend on the choice of the differential ω_A on A, but that the ratio $L_{\rm alg}(f,\chi)/\Omega_p^{2(k+2j)}$ does not depend on this choice, once an isomorphism ι between $\hat{\mathcal{A}}_0$ and $\hat{\mathbf{G}}_m$ has been chosen. Replacing ι by a $\mathbf{Z}_p^{\times}$–multiple $a\iota$ has the effect of multiplying $L_p(f,\chi)$ by $a^{2(k+2j)}$.

Recall the form $f^{\flat}=f|_{(VU-UV)}$ that was introduced in equation (3.8.4).

Theorem 5.9. *Assume that p is split in $K/\mathbf{Q}$. For all $\chi\in\Sigma_{\rm cc}^{(2)}(\mathfrak{N})$ of infinity type $(k+j,-j)$ (with $j\geq 0$), we have*

$$L_p(f,\chi)=\left(\sum_{[\mathfrak{a}]\in{\rm Pic}(\mathcal{O}_c)}\chi_j^{-1}(\mathfrak{a})\cdot\theta^jf^{\flat}(\mathfrak{a}*(A_0,t,\omega_{\rm can}))\right)^2.$$

Proof. Set

$$S_\chi:=\sum_{[\mathfrak{a}]}\chi_j^{-1}(\mathfrak{a})\cdot\theta^jf(\mathfrak{a}*(A_0,t_0,\omega_0)),$$

and

$$S_\chi^{\flat}:=\sum_{[\mathfrak{a}]}\chi_j^{-1}(\mathfrak{a})\cdot\theta^jf^{\flat}(\mathfrak{a}*(A_0,t_0,\omega_0)).$$

Now $p^ja_p\cdot\theta^jf=\theta^jf|T_p=\theta^jf|(U+\varepsilon_f(p)p^{k+2j-1}V)$ and

$$(\theta^jf|V)(\mathfrak{a}*(A_0,t_0,\omega_0))=(\theta^jf)(\bar{\mathfrak{p}}^{-1}\mathfrak{a}*(A_0,t_0,\omega_0)).$$

Thus

$$\begin{aligned}\theta^jf^{\flat}(\mathfrak{a}*(A_0,t_0,\omega_0)) &= \{\theta^jf|(VU-UV)\}(\mathfrak{a}*(A_0,t_0,\omega_0))\\ &= \{\theta^jf|(1-T_pV+\varepsilon_f(p)p^{k+2j-1}V^2)\}(\mathfrak{a}*(A_0,t_0,\omega_0))\\ &= \theta^jf(\mathfrak{a}*(A_0,t_0,\omega_0))-p^ja_p\cdot\theta^jf(\bar{\mathfrak{p}}^{-1}\mathfrak{a}*(A_0,t_0,\omega_0))+\\ &\qquad\qquad \varepsilon_f(p)p^{k+2j-1}\theta^jf(\bar{\mathfrak{p}}^{-2}\mathfrak{a}*(A_0,t_0,\omega_0)).\end{aligned}$$

Multiplying this equation by $\chi_j^{-1}(\mathfrak{a})$ and summing over all the classes $[\mathfrak{a}] \in \mathrm{Pic}(\mathcal{O}_c)$ gives the identity

$$S_\chi^\flat = \left\{1 - a_p\chi^{-1}(\bar{\mathfrak{p}}) + \varepsilon_f(p)p^{k-1}\chi^{-1}(\bar{\mathfrak{p}}^2)\right\} S_\chi.$$

The result now follows from Theorem 5.7 combined with the homogeneity properties of the p-adic modular form $\theta^j f^\flat$ of weight $k+2j$. □

Proposition 5.10. *The function $\chi \mapsto L_p(f,\chi)$ extends to a continuous function on $\hat{\Sigma}_{\mathrm{cc}}(\mathfrak{N})$.*

Proof. Let $\chi_1, \chi_2 \in \Sigma_{\mathrm{cc}}^{(2)}(\mathfrak{N})$ be two elements (of infinity type $(k+j_1, -j_1)$ and $(k+j_2,-j_2)$ respectively) satisfying

$$\chi_1(\mathfrak{a}) \equiv \chi_2(\mathfrak{a}) \pmod{\mathfrak{p}^M}, \qquad \text{for all } \mathfrak{a} \in \mathbb{A}'_{K,f}.$$

By evaluating at idèles in $\mathbb{A}'_{K,f}$ that are congruent to 1 modulo $\mathfrak{N}$, we see that necessarily

$$j_1 \equiv j_2 \pmod{(p-1)p^{M-1}}.$$

Now we observe that, since

$$\theta^j f^\flat(\mathrm{Tate}(q), t, \omega_{\mathrm{can}}) = \sum_{(p,n)=1} n^j a_n q^n,$$

the q-expansions of $\theta^{j_1} f$ and $\theta^{j_2} f$ are congruent modulo p^M, and therefore agree modulo $\mathfrak{p}^M$. If E is any ordinary elliptic curve over $\mathcal{O}_{F\mathfrak{p}}$, and ω_{can} is any canonical differential on it as in (5.2.2), it follows that

$$\theta^{j_1} f^\flat(E, t, \omega_{\mathrm{can}}) \equiv \theta^{j_2} f^\flat(E,t,\omega_{\mathrm{can}}) \pmod{\mathfrak{p}^M}.$$

(Cf. for example Sec.I.3.5 of [Gou].) It follows from the formula for $L_p(f,\chi)$ given in Theorem 5.9 that

$$L_p(f,\chi_1) \equiv L_p(f,\chi_2) \pmod{\mathfrak{p}^M}.$$

The proposition follows. □

The function $L_p(f,\cdot)$ on $\hat{\Sigma}_{\mathrm{cc}}(\mathfrak{N})$ is a type of anticyclotomic p-adic L-function attached to f and K (and the triple $(c, \mathfrak{N}, \varepsilon_f)$).

Remark 5.11. The p-adic L-functions attached to Rankin convolutions of p-adic families of modular forms have been constructed in great generality by Hida [Hi1]. In fact, our p-adic L-function $L_p(f,\cdot)$ is the restriction of a more general "two-variable p-adic L-function" defined over $\hat{\Sigma}(\mathfrak{N})$, the existence of which can be deduced from the main result of [Hi1].

Note that one obtains from Hida's work two different p-adic L-functions by interpolating the L-values corresponding to critical characters in $\Sigma^{(1)}(\mathfrak{N})$ and $\Sigma^{(2)}(\mathfrak{N})$ respectively. The p-adic L-function obtained by interpolating $L(f,\chi^{-1},0)$ with $\chi \in \Sigma^{(1)}(\mathfrak{N})$ has received much attention in the literature; for instance, it is studied in the article [PR1] of Perrin-Riou (for $k=2$) and in [Ne2] (for k even and ≥ 2.) Our focus in this article has been instead on the p-adic L-function obtained by p-adic interpolation of the special values corresponding to (central critical characters) $\chi \in \Sigma^{(2)}(\mathfrak{N})$.

5.3. **The main theorem.** For the convenience of the reader, we collect the notations and the running assumptions that were made in the previous sections and are in force in the statement of Theorem 5.13 below.

Assumption 5.12. (1) *The form f is a normalised cuspidal eigenform in $S_k(\Gamma_0(N), \varepsilon_f)$.*

(2) *c is an odd rational integer prime to Nd_K.*

(3) *The quadratic imaginary field K has odd discriminant and satisfies the Heegner hypothesis stated in Assumption 1.9, so that the order $\mathcal{O}_c$ of K of conductor c admits a cyclic ideal $\mathfrak{N}$ of norm N.*

(4) *The sets $\Sigma_{\mathrm{cc}}^{(1)}(\mathfrak{N})$ and $\Sigma_{\mathrm{cc}}^{(2)}(\mathfrak{N})$ consist of characters χ of finite type $(c, \mathfrak{N}, \varepsilon_f)$ and satisfying $\varepsilon_q(f,\chi^{-1}) = +1$ for all finite primes q, as described in Defn. 4.4 and the subsequent paragraph.*

(5) *The rational prime $(p) = \mathfrak{p}\bar{\mathfrak{p}}$ is split in $K/\mathbf{Q}$ and prime to Nc.*

A character $\chi \in \Sigma_{\mathrm{cc}}^{(1)}(\mathfrak{N})$ can be approximated by elements of $\Sigma_{\mathrm{cc}}^{(2)}(\mathfrak{N})$ (relative to the topology on $\Sigma_{\mathrm{cc}}(\mathfrak{N})$ discussed in the previous section) as follows. Let h denote the class number of K, and let ψ_t be the Hecke character of K of infinity type $(th, -th)$ and trivial central character defined by

$$\psi_t(\mathfrak{a}) = a^t/\bar{a}^t, \qquad \text{where } (a) = \mathfrak{a}^h.$$

If t is a sufficiently large positive integer, then the Hecke character $\chi\psi_t$ belongs to $\Sigma^{(2)}_{cc}(\mathfrak{N})$, and it converges to χ as t converges to 0 in $\mathbf{Z}/(p-1)\mathbf{Z}\times\mathbf{Z}_p$. This fact allows us to view $\Sigma^{(1)}_{cc}(\mathfrak{N})$ as a subset of $\hat{\Sigma}_{cc}(\mathfrak{N})$.

The following Theorem, which relates the value of $L_p(f,\chi)$ at $\chi\in\Sigma^{(1)}_{cc}(\mathfrak{N})$ (which lies outside the range of interpolation for the p-adic L-function) to Abel-Jacobi images of generalized Heegner cycles, is the main result of this paper.

Theorem 5.13. *Suppose that $\chi\in\Sigma^{(1)}_{cc}(\mathfrak{N})$ is a character of infinity type $(k-1-j,1+j)$, with $0\le j\le r$. Then*

$$\begin{aligned}\frac{L_p(f,\chi)}{\Omega_p^{2(r-2j)}} &= \left(1-\chi^{-1}(\bar{\mathfrak{p}})a_p+\chi^{-2}(\bar{\mathfrak{p}})\varepsilon_f(p)p^{k-1}\right)^2\times\\ &\quad\left(\frac{c^{-j}}{j!}\sum_{[\mathfrak{a}]\in\mathrm{Pic}(\mathcal{O}_c)}\chi^{-1}(\mathfrak{a})\mathrm{N}(\mathfrak{a})\cdot\mathrm{AJ}_F(\Delta_{\varphi_\mathfrak{a}\varphi_0})(\omega_f\wedge\omega_A^j\eta_A^{r-j})\right)^2.\end{aligned}$$

Proof. The proof of Proposition 5.10 shows that the formula in Theorem 5.9 for $L_p(f,\chi)$ at $\chi\in\Sigma^{(2)}_{cc}(\mathfrak{N})$ extends to $\chi\in\Sigma^{(1)}_{cc}(\mathfrak{N})$ in the obvious way, and gives

$$L_p(f,\chi)=\left(\sum_{[\mathfrak{a}]\in\mathrm{Pic}(\mathcal{O}_c)}\chi^{-1}_{-1-j}(\mathfrak{a})\cdot\theta^{-1-j}f^\flat(\mathfrak{a}*(A_0,t_0,\omega_{\mathrm{can}}))\right)^2.$$

Therefore, by (5.2.2) and the fact that $\theta^{-1-j}f^\flat$ is a p-adic modular form of weight $r-2j$, we have

$$\frac{L_p(f,\chi)}{\Omega_p^{2(r-2j)}}=\left(\sum_{[\mathfrak{a}]\in\mathrm{Pic}(\mathcal{O}_c)}\chi^{-1}_{-1-j}(\mathfrak{a})\cdot\theta^{-1-j}f^\flat(\mathfrak{a}*(A_0,t_0,\omega_0))\right)^2.$$

By Proposition 3.24,

$$\frac{L_p(f,\chi)}{\Omega_p^{2(r-2j)}}=\left(\frac{1}{j!}\sum_{[\mathfrak{a}]\in\mathrm{Pic}(\mathcal{O}_c)}\chi^{-1}_{-1-j}(\mathfrak{a})\cdot G_j^\flat(\mathfrak{a}*(A_0,t_0,\omega_0))\right)^2. \tag{5.3.1}$$

In view of Proposition 3.24 and of the relation $\theta^jf|T_p=p^ja_p\cdot\theta^jf$, for $j\ge0$, one sees by p-adic approximation that

$$T_pG_j=p^{-1-j}a_pG_j.$$

Then, by Lemma 3.23,

$$G_j^\flat(\mathfrak{a}*(A_0,t_0,\omega_0))=G_j(\mathfrak{a}*(A_0,t_0,\omega_0))-\frac{\epsilon_f(p)a_p}{p^{r-j+1}}G_j(\mathfrak{p}\mathfrak{a}*(A_0,t_0,\omega_0))+\frac{\epsilon_f(p)}{p^{r-2j+1}}G(\mathfrak{p}^2\mathfrak{a}*(A_0,t_0,\omega_0)).$$

Substituting this expression for $G_j^\flat(\mathfrak{a}*(A_0,t_0,\omega_0)$ into (5.3.1) and rewriting the second and the third summands by substituting $\mathfrak{a}$ for $\mathfrak{a}\mathfrak{p}$ and $\mathfrak{a}\mathfrak{p}^2$ respectively, we obtain

$$\begin{aligned}\frac{L_p(f,\chi)}{\Omega_p^{2(r-2j)}} &= \left(1-\frac{\chi_{-1-j}(\mathfrak{p})a_p\varepsilon_f(p)}{p^{r-j+1}}+\frac{\chi^2_{-1-j}(\mathfrak{p})\varepsilon_f(p)}{p^{r-2j+1}}\right)^2\times\\ &\quad\left(\frac{1}{j!}\sum_{[\mathfrak{a}]\in\mathrm{Pic}(\mathcal{O}_c)}\chi^{-1}_{-1-j}(\mathfrak{a})\cdot G_j(\mathfrak{a}*(A_0,t_0,\omega_0))\right)^2.\end{aligned} \tag{5.3.2}$$

Using the fact that

$$\chi_{-1-j}(\mathfrak{p})=\chi(\mathfrak{p})p^{-1-j}=\varepsilon_f(p)^{-1}p^{r+1-j}\chi(\bar{\mathfrak{p}})^{-1},$$

the Euler factor that appears in (5.3.2) can be rewritten as

$$\mathcal{E}_p(f,\chi):=\left(1-\chi^{-1}(\bar{\mathfrak{p}})a_p+\chi^{-2}(\bar{\mathfrak{p}})\varepsilon_f(p)p^{k-1}\right)^2.$$

Now, applying Lemma 3.22 to the isogeny

$$\varphi_\mathfrak{a}\varphi_0:(A,t_A,\omega_A)\longrightarrow\mathfrak{a}*(A_0,t_0,\omega_0)$$

of degree $c\mathrm{N}(\mathfrak{a})$, and using the fact that $\chi^{-1}_{-1-j}(\mathfrak{a}) = \chi^{-1}(\mathfrak{a})\mathrm{N}(\mathfrak{a})^{1+j}$, we find

$$\frac{L_p(f,\chi)}{\Omega_p^{2(r-2j)}} = \mathcal{E}_p(f,\chi)\left(\frac{c^{-j}}{j!}\sum_{[\mathfrak{a}]\in\mathrm{Pic}(\mathcal{O}_c)} \chi^{-1}(\mathfrak{a})\mathrm{N}(\mathfrak{a})\cdot \mathrm{AJ}_F(\Delta_{\varphi_{\mathfrak{a}}\varphi_0})(\omega_f\wedge\omega_A^j\eta_A^{r-j})\right)^2,$$

as was to be shown. □

Appendix A. Kuga-Sato schemes by Brian Conrad

The aim of this appendix is to explain the relative version of Deligne's method for constructing a smooth projective compactification of the fiber powers E^k of the universal elliptic curve E with "enough level-N structure" over an open modular curve Y over $\mathbf{Z}[1/N]$ (for applications in this paper with $Y = Y_1(N)$). This was originally developed in 1968 for applications over finite fields F of characteristic not dividing N (see [De2, Lemma 5.5]), and later found uses for $X(N)$ over $\mathbf{Z}[1/N]$ (see [Schol2, 4.2.1]). For applications over such fields F (e.g., $\mathbf{Q}$ or finite fields) one can compactify $E_F \to Y_F$ over the associated smooth complete modular curve X_F by using the technique of minimal regular proper models of relative smooth proper curves over a Dedekind base (such as $E_F \to Y_F$ relative to the Dedekind base X_F), together with their relation to Néron models of elliptic curves, and then try to explicitly resolve singularities of fiber powers over X_F of that minimal regular proper model. Thus, when working over such a field F there is no need for the concept of a generalized elliptic curve (which was introduced only in 1972 in the work of Deligne and Rapoport [DR], building on Artin's theory of algebraic spaces).

The viewpoint of minimal regular proper models is insufficient in the relative situation over $\mathbf{Z}[1/N]$ since now X is 2-dimensional rather than Dedekind. In such settings we use the proper flat universal generalized elliptic curve $\overline{E} \to X$ over $\mathbf{Z}[1/N]$ (for a modular curve X classifying rigid fiberwise ample level-N structures on generalized elliptic curves over $\mathbf{Z}[1/N]$-schemes) as a compactification of E over $\mathbf{Z}[1/N]$. Such $\overline{E}$ are smooth over $\mathbf{Z}[1/N]$ (see Lemma A.2) but not smooth over X, so for $k \geq 2$ the compactification $\overline{E}^k$ of E^k is not smooth over $\mathbf{Z}[1/N]$ (as we will see explicitly below). In Scholl's work with $X(N)$ over $\mathbf{Z}[1/N]$ in [Schol2, 4.2.1], for each $k \geq 2$ he used Deligne's method to construct a smooth projective $\mathbf{Z}[1/N]$-scheme equipped with a proper birational map onto the fiber power $\overline{E}^k$ over X such that the map is an isomorphism over E^k and can be described étale-locally near the fibers over the cuspidal locus on X. The method is a series of successive blow-ups, organized in terms of the number of coordinates of a geometric point $\xi = (\xi_1,\ldots,\xi_k) \in \overline{E}^k$ for which ξ_i is singular in its geometric fiber for $\overline{E} \to X$.

The hard part is to give an *intrinsic* description of what to blow-up at each step; once we have defined an intrinsic algorithm, we can carry out computations étale-locally to see that we reach a smooth $\mathbf{Z}[1/N]$-scheme. These étale-local computations are sketched over $\mathbf{Q}$ in Scholl's work (see [Schol2, 2.0.1–2.1.1]) but the details on how to carry it out over $\mathbf{Z}[1/N]$ are omitted there (and the intrinsic definitions of what the pieces correspond to in terms of $\overline{E}^k$ is not given). Thus, at the request of the referee, in this appendix we explain the procedure in more detail over $\mathbf{Z}[1/N]$.

We shall axiomatize the calculation so that it applies to "all" modular curves (with enough étale level structure). The intrinsic nature of the method also makes it applicable to cases in which the modular curve only exists as a Deligne–Mumford stack (such as $X_0(N)$ over $\mathbf{Z}[1/N]$ for any $N \geq 1$), but we leave that generalization to the interested reader. The "étale" nature of the level structure (i.e., using N-torsion level structures over $\mathbf{Z}[1/N]$-schemes) is essential to the method because only in such cases can certain deformation-theoretic problems with generalized elliptic curves be reduced to the case of a Tate curve with geometrically irreducible fibers; see [DR, III, 1.4.2; VII, 2.1].

Fix an integer $N \geq 1$, and let X be a modular curve over $\mathbf{Z}[1/N]$ classifying a rigid fiberwise ample level-N structure on generalized elliptic curves over $\mathbf{Z}[1/N]$-schemes (e.g., $\Gamma_1(N)$-structures with $N \geq 5$, or full level-N structures with $N \geq 3$). Here, by "rigid" we mean that generalized elliptic curves equipped with such a level structure admit no nontrivial automorphisms. The work of Deligne and Rapoport provides such modular curves X as smooth proper $\mathbf{Z}[1/N]$-schemes with fibers of pure dimension 1, equipped with a universal generalized elliptic curve $\overline{E} \to X$. (Even though such an X is initially built only as an algebraic space, it is a scheme. This can be seen in a couple of ways, perhaps the most concrete being that the

j-map from X to $\mathbf{P}^1_{\mathbf{Z}[1/N]}$ is quasi-finite, and any algebraic space that is separated and quasi-finite over a noetherian scheme is a scheme [K, II, 6.16].)

Remark A.1. For the reader who is interested in schemes being projective rather than just proper, we make some side remarks now (not to be used in what follows). The fiberwise ample level structure on $\overline{E}$ over X defines a closed subgroup scheme of the open X-smooth locus $\overline{E}^{\mathrm{sm}}$ that is finite étale over X and so is closed in $\overline{E}$ with ideal sheaf in $\mathcal{O}_{\overline{E}}$ that is a line bundle on $\overline{E}$ whose inverse is fiberwise ample over X. But a fiberwise ample line bundle on a proper finitely presented scheme over a base S is relatively ample over S [EGA, IV$_3$, 9.6.4], so the projectivity and flatness of X over $\mathbf{Z}[1/N]$ implies that $\overline{E}$ is projective and flat over $\mathbf{Z}[1/N]$. Likewise, the fiber powers $\overline{E}^k$ over X are projective and flat over $\mathbf{Z}[1/N]$ for all $k \geq 1$. In particular, any scheme obtained from $\overline{E}^k$ by a composition of successive blow-ups is projective over $\mathbf{Z}[1/N]$. This ensures that the $\mathbf{Z}[1/N]$-smooth compactification of $\overline{E}^k$ built below is projective over $\mathbf{Z}[1/N]$.

We now recall that for any generalized elliptic curve $f : \mathcal{E} \to S$ over a scheme, Deligne and Rapoport introduced canonical closed subscheme structures $S_\infty \subset S$ and $\mathcal{E}^{\mathrm{sing}} \subset \mathcal{E}$ respectively supported at the set of $s \in S$ such that $\mathcal{E}_s$ is not $k(s)$-smooth and at the set of $\xi \in \mathcal{E}$ at which the proper fppf map $\mathcal{E} \to S$ is not smooth. Explicitly, $\mathcal{E}^{\mathrm{sing}}$ is defined by the annihilator ideal of $\Omega^2_{\mathcal{E}/S}$ (the first Fitting ideal of $\Omega^1_{\mathcal{E}/S}$), and S_∞ is defined to be the scheme-theoretic image of $\mathcal{E}^{\mathrm{sing}}$ in S. The formation of both of these commutes with any base change on S (though this has some hidden subtleties for S_∞; see [Con, 2.1.11, 2.1.12]). We call these closed subschemes the "loci of non-smoothness" in S and $\mathcal{E}$ for f. Their compatibility with base change on S enables us to compute completions along these loci via deformation theory.

Let $X_\infty \subset X$ be the locus of non-smoothness for the universal generalized elliptic curve $\overline{E} \to X$. Computations with the deformation theory of generalized elliptic curves equipped with ample level-N structure over $\mathbf{Z}[1/N]$ show that X_∞ is (finite) étale over $\mathbf{Z}[1/N]$ (see [DR, III, 1.2(iv); IV, 3.4(ii)]). The structure of $\overline{E}$ around $\overline{E}^{\mathrm{sing}}$ can also be understood via deformation theory, leading to:

Lemma A.2. *The scheme $\overline{E}$ is smooth over $\mathbf{Z}[1/N]$.*

Proof. The problem is to prove smoothness at non-smooth points ξ in fibers over points $x \in X_\infty$, and since $\overline{E}$ is fppf over $\mathbf{Z}[1/N]$ it suffices to work on geometric fibers over $\mathrm{Spec}(\mathbf{Z}[1/N])$. In other words, for an algebraically closed field F of characteristic not dividing N and the universal generalized elliptic curve $\overline{E}_F \to X_F$, we want to prove that the surface $\overline{E}_F$ is smooth at points $\xi \in \overline{E}(F)$ that are non-smooth in the fiber over $x \in X_\infty(F)$. It is equivalent to prove the formal smoothness of $\mathcal{O}^\wedge_{\overline{E}_F,\xi}$ over F. But $\mathcal{O}^\wedge_{\overline{E}_F,\xi}$ coincides with the completed local ring at ξ on the formal completion of $\overline{E}_F \to X_F$ along x. This latter formal completion is the universal deformation of $(\overline{E}_F)_x$ equipped with its ample level-N structure, and $\mathcal{O}^\wedge_{X_F,x}$ is its universal deformation ring. Since $\mathrm{char}(F) \nmid N$, by [DR, III, 1.2(iv); VII, (1.1.1), 1.11, 2.1] there is an F-isomorphism between the universal deformation ring $\mathcal{O}^\wedge_{X_F,x}$ and $F[\![q]\!]$ such that the completed local ring at ξ is $F[\![q]\!]$-isomorphic to $F[\![q,u,v]\!]/(uv-q) = F[\![u,v]\!]$. □

Now we shall prove a general resolution result for generalized elliptic curves over a family of smooth curves:

Theorem A.3. *Let S be a scheme, $X \to S$ a smooth map with all fibers of pure dimension 1, and $f : \overline{E} \to X$ a generalized elliptic curve such that:*

1. *the locus of non-smoothness $X_\infty \subset X$ for f is étale over S,*
2. *the scheme $\overline{E}$ is S-smooth.*

For each $k \geq 1$, let $\overline{E}^k$ denote the kth fiber power over X.

There exists a smooth S-scheme Z_k and a proper birational map $Z_k \to \overline{E}^k$ that is an isomorphism over E^k. The map $Z_k \to \overline{E}^k$ is a composition of finitely many blow-ups.

We emphasize that although $\overline{E}$ is assumed to be S-smooth, in practice it is not X-smooth, so the closed subscheme $\overline{E}^{\mathrm{sing}}$ (which encodes non-smoothness over X) is generally not empty. The proof of the theorem consists of giving an explicit definition of the blow-up process. If $k = 1$ then we may take $Z_1 = \overline{E}$ by hypothesis (2), so we now assume $k \geq 2$.

By hypothesis (1), the pair (X, X_∞) looks étale-locally like $(\mathbf{A}^1_R, 0)$. Thus, the étale-local structure of relative semi-stable curves [FK, III, 2.7] and the "homogeneity" of $\overline{E}$ around $\overline{E}^{\mathrm{sing}}$ (via translation by $\overline{E}^{\mathrm{sm}}$) implies that Zariski-locally over an affine open $\operatorname{Spec} R$ in S the pair $(\overline{E}, \overline{E}^{\mathrm{sing}})$ has a common étale neighborhood with

$$(\operatorname{Spec}(R[q,u,v]/(uv-q)), \{q = u = v = 0\})$$

(see the proof of [DR, II, 1.16]). Up to permutation of coordinates, a geometric point $\xi = (\xi_1, \ldots, \xi_k) \in \overline{E}^k$ that is non-smooth over S has $\xi_1, \ldots, \xi_r$ non-smooth in $\overline{E}$ over X and $\xi_{r+1}, \ldots, \xi_k$ smooth in $\overline{E}$ over X for some $r \geq 2$ (the case $r = 1$ being ruled out by the hypothesis that $\overline{E}$ is S-smooth). Thus, $(\overline{E}^k, \xi)$ has a common étale neighborhood with the spectrum of

$$R[q, X_1, Y_1, \ldots, X_r, Y_r, T_{r+1}, \ldots, T_k]/\big(X_1Y_1 = \cdots = X_rY_r = q\big) \simeq R[X_1, Y_1, \ldots, X_r, Y_r, T_{r+1}, \ldots, T_k]/\big(X_1Y_1 = \cdots = X_rY_r\big). \tag{A.0.3}$$

Of course, we have an analogous ring for any permutation of the ξ_i's.

Let $\overline{F}^k$ denote the k-fold fiber product of $\overline{E}$ over X_∞. We define a stratification of $\overline{F}^k \hookrightarrow \overline{E}^k$ by closed subschemes

$$\overline{F}^k = F^k_k \supsetneq F^k_{k-1} \supsetneq \cdots \supsetneq F^k_0 \supseteq F^k_{-1} = \emptyset$$

where, for $0 \leq r \leq k$, $F^k_r \subseteq \overline{F}^k$ is the scheme-theoretic union of the closed subschemes defined by requiring at least $k - r$ factors to lie in $\overline{E}^{\mathrm{sing}}$. For example, working étale locally over $\overline{E}$, we see that F^k_{k-2} is supported at precisely the closed non-smooth locus for the fppf map $\overline{E}^k \to S$.

Define $E^k\langle 0\rangle = \overline{E}^k$ and $F^k_i\langle 0\rangle = F^k_i$ for $0 \leq i \leq k$. For $1 \leq r \leq k-1$, we recursively define $E^k\langle r\rangle = \mathrm{Bl}_{F^k_{r-1}\langle r-1\rangle}\big(E^k\langle r-1\rangle\big)$, and we let $F^k_i\langle r\rangle$ be the proper transform in $E^k\langle r\rangle$ of $F^k_i\langle r-1\rangle$ for $r \leq i \leq k-1$. (Equivalently, $F^k_i\langle r\rangle$ is the blow-up of $F^k_i\langle r-1\rangle$ along $F^k_{r-1}\langle r-1\rangle$.)

We claim several properties:

(i) $E^k\langle r\rangle$ and all $F^k_i\langle r\rangle$ are S-flat,

(ii) $F^k_r\langle r\rangle$ is contained in the closed locus where the S-flat $E^k\langle r\rangle$ is non-smooth over S for all $0 \leq r \leq k-2$ (so the map $E^k\langle k-1\rangle \to E^k\langle 0\rangle = \overline{E}^k$ is an isomorphism over the S-smooth locus of $\overline{E}^k$, which contains $\overline{E}^k$),

(iii) $E^k\langle k-1\rangle$ is S-smooth,

(iv) the formation of these blow-ups and strict transforms commutes with any base change on S (via the evident base change morphisms).

To verify these claims we may work étale-locally over a non-smooth point of $\overline{E}^k$ over affine open $\operatorname{Spec} R \subset S$, which amounts to replacing $\overline{E}^k$ with

$$\widetilde{E}^m\langle 0\rangle = R[X_1, Y_1, \ldots, X_m, Y_m, T_{m+1}, \ldots, T_k]/\big(X_1Y_1 = \cdots = X_mY_m\big),$$

where $2 \leq m \leq k$.

We define $\widetilde{F}^m_i\langle 0\rangle$ to be the R-flat locus in $\widetilde{E}^m\langle 0\rangle$ where $m - i$ pairs (X_j, Y_j) vanish. Using inductive definitions analogous to those above, we define $\widetilde{E}^m\langle r\rangle$ and $\widetilde{F}^m_i\langle r\rangle$ (with $r \leq i \leq m-1$) for $0 \leq r \leq m-1$. We can replace the above claims with analogues in this new setting, so we aim to prove:

- $\widetilde{F}^m_i\langle r\rangle$ and $\widetilde{E}^m\langle r\rangle$ are R-flat and their formation commutes with base change on R;
- $\widetilde{F}^m_r\langle r\rangle$ is contained in the closed non-smooth locus for $\widetilde{E}^m\langle r\rangle$ over R for all $0 \leq r \leq m-2$ (so the blow-up steps are always isomorphisms over the smooth locus of the previous stage);
- $\widetilde{E}^m\langle m-1\rangle$ is R-smooth.

This will clearly finish the proof. The $T_{m+1}, \ldots, T_k$ just get "carried along", so they can (and will) now be dropped.

It is easy to see that $\widetilde{E}^m\langle 1\rangle$ has an open cover by $2m$-copies U_j of $\mathbf{A}^1 \times \widetilde{E}^{m-1}\langle 0\rangle$ such that $U_j \cap \widetilde{F}^m_i\langle 1\rangle = \mathbf{A}^1 \times \widetilde{F}^{m-1}_{i-1}\langle 0\rangle$ for $1 \leq i \leq m-1$. Here, we define $\widetilde{E}^1\langle 0\rangle = \operatorname{Spec} R[X_1, Y_1]/(X_1Y_1)$ and $\widetilde{F}^1_0 = (0,0)$.

By induction on r for each m (with the case $r = 0$ always trivial and the case $r = 1$ just settled for all m) we see that for $0 \leq r \leq m-2$ there exists an open cover of $\widetilde{E}^m\langle r\rangle$ by copies V_j of $\mathbf{A}^r \times \widetilde{E}^{m-r}\langle 0\rangle$ with $V_j \cap \widetilde{F}^m_i\langle r\rangle = \mathbf{A}^r \times \widetilde{F}^{m-r}_{i-r}\langle 0\rangle$ for all $r \leq i \leq m-1$. Thus, $\widetilde{F}^m_r\langle r\rangle$ is contained in $\widetilde{F}^{m-r}_0$, which in turn is contained in the closed locus of non-smooth points in $\widetilde{E}^{m-r}\langle 0\rangle$ over R since $m-r \geq 2$. These Zariski-local descriptions yield the desired R-flatness and compatibility with base change on R.

Taking $r = m-2$ at the end of the induction, $\widetilde{E}^m\langle m-2\rangle$ is covered by open subschemes R-isomorphic to $\mathbf{A}^{m-2} \times \widetilde{E}^2\langle 0\rangle$. Since

$$\widetilde{E}^2\langle 0\rangle = \operatorname{Spec} R[X_1, Y_1, X_2, Y_2]/(X_1Y_1 - X_2Y_2)$$

with $\widetilde{F}_0^2\langle 0\rangle$ equal to the origin over R, so it remains to note that the R-scheme $\mathrm{Bl}_{(0)}(\widetilde{E}^2\langle 0\rangle)$ is covered by copies of $\mathbf{A}^3$.

References

[B] Berger, Laurent, *An introduction to the theory of p-adic representations.* Geometric aspects of Dwork theory. Vol. I, II, 255–292, Walter de Gruyter GmbH & Co. KG, Berlin, 2004.

[Bl-1] Bloch, Spencer, *Algebraic cycles and values of L-functions.* J. Reine Angew. Math. 350 (1984), 94-108.

[Bl-2] Bloch, Spencer, *Algebraic cycles and values of L-functions. II.* Duke Math. J. 52 (1985), no. 2, 379-397.

[BDP-cm] Massimo Bertolini, Henri Darmon and Kartik Prasanna, *p-adic Rankin L-series and rational points on CM elliptic curves*, submitted.

[BDP-ch] Massimo Bertolini, Henri Darmon and Kartik Prasanna, *Chow-Heegner points on CM elliptic curves and values of p-adic L-series*, submitted.

[BDP-co] Massimo Bertolini, Henri Darmon and Kartik Prasanna, *p-adic L-functions and the coniveau filtration on Chow groups*, preprint.

[Bump] Bump, Daniel, *Automorphic forms and representations.* Cambridge Studies in Advanced Mathematics, 55. Cambridge University Press, Cambridge, 1997.

[Cas1] Castella, Francesc, *Heegner cycles and higher weight specializations of big Heegner points*, submittted.

[Cas2] Castella, Francesc, McGill PhD thesis, in progress.

[Col1] Coleman, Robert F., *Torsion points on curves and p-adic abelian integrals*, Ann. of Math. (2) 121 (1985), no. 1, 111–168.

[Col2] Coleman, Robert F., *Reciprocity laws on curves*, Compositio Math. **72** (1989) 205-235.

[Col3] Coleman, Robert F., *A p-adic Shimura isomorphism and p-adic periods of modular forms*, p-adic monodromy and the Birch and Swinnerton-Dyer conjecture (Boston, MA, 1991), 21–51, Contemp. Math., 165, Amer. Math. Soc., Providence, RI, 1994.

[CI] Coleman, Robert F. and Iovita, Adrian, *Hidden structures on semistable curves*, Astérisque No. 331 (2010), 179–254.

[Con] Conrad, Brian, *Arithmetic moduli of generalized elliptic curves*, J. Inst. Math. Jussieu **6** (2007), 209–278.

[De1] Deligne, Pierre. Equations différentielles à points singuliers réguliers. Lecture Notes in Mathematics, Vol. 163. Springer-Verlag, Berlin-New York, 1970.

[De2] Deligne, Pierre, *Formes modulaires et représentations ℓ-adiques*, Sém. N. Bourbaki, 1968-1969, exp. 355, 139–172.

[DR] Deligne, Pierre, Rapoport, Michael, *Les schémas de modules de courbes elliptiques.* Modular functions of one variable, II (Proc. Internat. Summer School, Univ. Antwerp, Antwerp, 1972), pp. 143–316. Lecture Notes in Math., Vol. 349, Springer, Berlin, 1973.

[DS] Diamond, Fred and Shurman, Jerry, *A first course in modular forms*, Graduate Texts in Mathematics **228**, Springer-Verlag, 2005.

[Fa] Faltings, Gerd, *Crystalline cohomology and p-adic Galois-representations*, Algebraic analysis, geometry, and number theory (Baltimore, MD, 1988), 25–80, Johns Hopkins Univ. Press, Baltimore, MD, 1989.

[FI] Fontaine, Jean-Marc; Illusie, Luc, *p-adic periods: a survey*, Proceedings of the Indo-French Conference on Geometry (Bombay, 1989), 57–93, Hindustan Book Agency, Delhi, 1993.

[Fo] Fontaine, Jean-Marc, *Modules galoisiens, modules filtrés, et anneaux de Barsotti-Tate*, Astérisque **65** (1979) 3–80.

[FK] Freitag, Eberhard and Kiehl, Reinhardt, *Étale cohomology and the Weil conjectures*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3), Springer-Verlag, Berlin, 1988.

[Gou] Gouvêa, Fernando Q., *Arithmetic of p-adic modular forms*, Lecture Notes in Mathematics, **1304**. Springer-Verlag, Berlin, 1988.

[GZ] Gross, Benedict H. and Zagier, Don B., *Heegner points and derivatives of L-series*, Invent. Math. 84 (1986), no. 2, 225–320.

[EGA] Grothendieck, Alexander, *Eléments de Géométrie Algébrique*, Publ. Math. IHES **4, 8, 11, 17, 20, 24, 28, 32**, 1960–7.

[HK1] Harris, Michael; Kudla, Stephen S. *Arithmetic automorphic forms for the nonholomorphic discrete series of GSp(2).* Duke Math. J. 66 (1992), no. 1, 59121.

[HK2] Harris, Michael; Kudla, Stephen S. *The central critical value of a triple product L-function.* Ann. of Math. (2) 133 (1991), no. 3, 605672.

[Hi1] Hida, Haruzo, *A p-adic measure attached to the zeta functions associated with two elliptic modular forms II*, Ann. Inst. Fourier (Grenoble) 38 (1988), no. 3, 1–83.

[Hi2] Hida, Haruzo. Elementary theory of L-functions and Eisenstein series. London Mathematical Society Student Texts, 26. Cambridge University Press, Cambridge, 1993.

[Hi3] Hida, Haruzo, *Central critical values of modular Hecke L-function*, Kyoto Journal of Mathematics, 50 (2010), 777–826

[Hi4] Hida, Haruzo, *CM periods, L-values and the CM main conjecture*, to appear.

[I] Illusie, Luc., *Cohomologie de de Rham et cohomologie étale p-adique (d'après G. Faltings, J.-M. Fontaine et al)*, Séminaire Bourbaki, Vol. 1989/90, Astérisque No. 189-190 (1990), Exp. No. 726, 325–374.

[IS] Iovita, Adrian; Spiess, Michael, *Derivatives of p-adic L-functions, Heegner cycles and monodromy modules attached to modular forms*, Invent. Math. **154** (2003), no. 2, 333–384.

[Jac] Jacquet, Hervé *Automorphic forms on* GL(2). *Part II*, Lecture Notes in Mathematics, Vol. 278. Springer-Verlag, Berlin-New York, 1972.

[JL] Jacquet, H.; Langlands, R. P. *Automorphic forms on GL(2).* Lecture Notes in Mathematics, Vol. 114. Springer-Verlag, Berlin-New York, 1970.

[Ka1] Katz, Nicholas M., *Nilpotent connections and the monodromy theorem: Applications of a result of Turrittin.* Inst. Hautes Études Sci. Publ. Math. No. 39 (1970), 175–232.

[Ka2] Katz, Nicholas M., *p-adic properties of modular schemes and modular forms*, Modular functions of one variable, III (Proc. Internat. Summer School, Univ. Antwerp, Antwerp, 1972), pp. 69–190. Lecture Notes in Mathematics, Vol. 350, Springer, Berlin, 1973.

[Ka3] Katz, Nicholas M., *Travaux de Dwork*, Séminaire Bourbaki, 24me anne (1971/1972), Exp. No. 409, pp. 167–200. Lecture Notes in Math., Vol. 317, Springer, Berlin, 1973.

[Ka4] Katz, Nicholas M., *p-adic L-functions for CM fields*, Invent. Math. 49 (1978), no. 3, 199–297.

[K] Knutson, Don, *Algebraic spaces*, Lecture Notes in Mathematics, Vol. 203, Springer-Verlag, Berlin, 1971.

[Mi] Milne, Jim, *Lectures on Etale Cohomology.* Available on-line at `http://www.jmilne.org/math/CourseNotes/LEC.pdf`.

[MW] Martin, Kimball; Whitehouse, David, *Central L-values and toric periods for* GL(2), Int. Math. Res. Not. IMRN 2009, no. 1, 141–191.

[Ne1] Nekovář, Jan, *On p-adic height pairings*, Séminaire de Théorie des Nombres, Paris, 1990–91, 127–202, Progr. Math., 108, Birkhäuser Boston, Boston, MA, 1993.

[Ne2] Nekovář, Jan, *On the p-adic height of Heegner cycles*, Math. Ann. 302 (1995), no. 4, 609–686.

[Ne3] Nekovář, Jan, *p-adic Abel-Jacobi maps and p-adic heights*, The arithmetic and geometry of algebraic cycles (Banff, AB, 1998), 367–379, CRM Proc. Lecture Notes, 24, Amer. Math. Soc., Providence, RI, 2000.

[Ni] Niziol, Wieslawa, *On the image of p-adic regulators.* Invent. Math. 127 (1997), no. 2, 375–400.

[PR1] Perrin-Riou, Bernadette, *Points de Heegner et dérivées de fonctions L p-adiques*, Invent. Math. 89 (1987), no. 3, 455–510.

[PR2] Perrin-Riou, Bernadette, *Fonctions L p-adiques d'une courbe elliptique et points rationnels*, Ann. Inst. Fourier (Grenoble) 43 (1993), no. 4, 945–995.

[PR3] Perrin-Riou, Bernadette, *p-adic L-functions and p-adic representations*, Translated from the 1995 French original by Leila Schneps and revised by the author. SMF/AMS Texts and Monographs, 3. American Mathematical Society, Providence, RI; Société Mathématique de France, Paris, 2000.

[P] Prasanna, Kartik, *Integrality of a ratio of Petersson norms and level-lowering congruences*, Ann. of Math. (2), 163 (2006), no. 3, 901-967.

[Ru] Rubin, Karl, *p-adic L-functions and rational points on elliptic curves with complex multiplication.* Invent. Math. **107** (1992) 323-350.

[SR] Schmidt, Ralf, *Some remarks on local newforms for* GL(2). J. Ramanujan Math. Soc. 17 (2002), no. 2, 115–147.

[Sch] Schoen, Chad, *Complex multiplication cycles on elliptic modular threefolds*, Duke Math. J. 53 (1986), no. 3, 771–794.

[Schol1] Scholl, Anthony J., *Modular forms and de Rham cohomology; Atkin–Swinnerton-Dyer congruences*, Invent. Math. 79 (1985), no. 1, 49–77.

[Schol2] Scholl, Anthony J., *Motives for modular forms*, Invent. Math. 100 (1990), no. 2, 419–430.

[Schol3] Scholl, Anthony J., *Vanishing cycles and non-classical parabolic cohomology.* Invent. Math. 124 (1996), no. 1–3, 503–524.

[Se] Serre, Jean-Pierre, *Formes modulaires et fonctions zêta p-adiques*, in Modular functions of one variable, III (Proc. Internat. Summer School, Univ. Antwerp, 1972), pp. 191–268. Lecture Notes in Math., Vol. 350, Springer, Berlin, 1973.

[SH] Shimizu, Hideo, *Theta series and automorphic forms on* GL_2, J. Math. Soc. Japan 24 (1972), 638–683.

[Shim1] Shimura, Goro, *On some arithmetic properties of modular forms of one and several variables*, Ann. of Math. (2) 102 (1975), no. 3, 491–515.

[Shim2] Shimura, Goro, *The special values of the zeta functions associated with cusp forms*, Comm. Pure Appl. Math. 29 (1976), no. 6, 783–804.

[T] Tsuji, Takeshi, *p-adic étale cohomology and crystalline cohomology in the semi-stable reduction case.* Invent. Math. 137 (1999), no. 2, 233–411.

[Tu1] Tunnell, Jerrold B., *On the local Langlands conjecture for $GL(2)$.* Invent. Math. 46 (1978), no. 2, 179–200.

[Tu2] Tunnell, Jerrold B., *Local ε-factors and characters of* $\mathbf{GL}(2)$. Amer. J. Math. 105 (1983), no. 6, 1277–1307.

[Wa] Watson, Thomas C. *Rankin triple products and quantum chaos*, to appear in Ann. of Math., Ph.D. dissertation, Princeton University, Princeton, 2002.

[Xue] Xue, Hui, *Central values of L-functions over CM fields*, J. Number Theory 122 (2007), no. 2, 342–378.

[Zh] Zhang, Shouwu, *Heights of Heegner cycles and derivatives of L-series*, Invent. Math. 130 (1997), no. 1, 99–152.