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# NOTE ON A POLYNOMIAL OF EMMA LEHMER

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ABSTRACT. Recently, Emma Lehmer constructed a parametric family of units in real quintic fields of prime conductor $p = t^4 + 5t^3 + 15t^2 + 25t + 25$ as translates of Gaussian periods. Later, Schoof and Washington showed that these units were fundamental units. In this note, we observe that Lehmer's family comes from the covering of modular curves $X_1(25) \to X_0(25)$. This gives a conceptual explanation for the existence of Lehmer's units: they are modular units (which have been studied extensively). By relating Lehmer's construction with ours, one finds expressions for certain Gauss sums as values of modular units on $X_1(25)$.

## 1. LEHMER'S POLYNOMIAL

Throughout the discussion, we fix a choice $\{\zeta_n\}$ of primitive $n$th roots of unity for each $n$, say by $\zeta_n = e^{2\pi i/n}$.

Let

$$\begin{aligned} P_5(Y, T) = Y^5 + T^2Y^4 - 2(T^3 + 3T^2 + 5T + 5)Y^3 \\ + (T^4 + 5T^3 + 11T^2 + 15T + 5)Y^2 \\ + (T^3 + 4T^2 + 10T + 10)Y + 1 \end{aligned} \tag{1}$$

be the quintic polynomial constructed in [5]. The discriminant of $P_5(Y, T)$, viewed as a polynomial in $Y$ with coefficients in $\mathbf{Q}(T)$, is

$$D(T) = (T^3 + 5T^2 + 10T + 7)^2(T^4 + 5T^3 + 15T^2 + 25T + 25)^4.$$

The projective curve $C$ in $\mathbf{P}_2$ defined by the affine equation (1) has three nodal singularities whose $T$-coordinates are the roots of the first factor of $D(T)$. The points $(y, t)$, where $t$ is a root of the second factor, are branch points for the covering of $C$ onto the $T$-line.

As shown in [5], the polynomial $P_5(Y, T)$ defines a regular Galois extension of $\mathbf{Q}(T)$ with Galois group $\mathbf{Z}/5\mathbf{Z}$. By the analysis above, it is ramified at the four conjugate points $T = -\sqrt{5}\zeta_5, \sqrt{5}\zeta_5^2, -\sqrt{5}\zeta_5^{-1}, \sqrt{5}\zeta_5^{-2}$, the zeros of the

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minimal polynomial

$$T^4 + 5T^3 + 15T^2 + 25T + 25.$$

(Here $\sqrt{5}$ denotes the positive square root.) If $t \in \mathbf{Z}$ is chosen so that

$$p = t^4 + 5t^3 + 15t^2 + 25t + 25$$

is prime (hence, in particular, $p \equiv 1 \mod 5$), then the roots $r_1, \dots, r_5$ of $P_5(Y, t)$ are translates of Gaussian periods:

$$r_i = (t/5)\eta_i + [(t/5) - t^2]/5,$$

where $\eta_j = \sum_{x \in \Gamma_j} \zeta_p^x$ and $\Gamma_j$ denotes the $j$th coset of $(\mathbf{Z}/p\mathbf{Z})^{*5}$ in $(\mathbf{Z}/p\mathbf{Z})^*$.

Since $C$ admits a five-to-one map to $\mathbf{P}_1$ which is totally ramified at four points, the geometric genus of $C$ is 4 by the Riemann-Hurwitz theorem. On the other hand, $C$ is realized as a plane curve of degree $d = 6$, and its arithmetic genus is $(d-1)(d-2)/2 = 10$. Let $C'$ denote the normalization of $C$; it is a smooth projective curve of genus 4. The covering $C' \to \mathbf{P}_1$ defines a Galois covering of $\mathbf{P}_1$ with Galois group $\mathbf{Z}/5\mathbf{Z}$, and has the following properties:

1. It is ramified only over the four closed points in $R = \{-\sqrt{5}\zeta_5, \sqrt{5}\zeta_5^2, -\sqrt{5}\zeta_5^{-1}, \sqrt{5}\zeta_5^{-2}\}$.
2. The closed points of the fiber above $\infty \in \mathbf{P}_1$ are rational.

**Proposition 1.1.** *Properties* 1 *and* 2 *determine the covering* $C'$ *uniquely up to* $\mathbf{Q}$-*isomorphism.*

*Proof.* Let $(\mathbf{P}_1 - R)$ be the projective line with the points of $R$ removed, viewed as a curve over $\mathbf{Q}$. The space $V = H^1_{et}(\mathbf{P}_1 - R, \mathbf{Z}/5\mathbf{Z})$ is a vector space of dimension 3 over $\mathbf{F}_5$, and is endowed with a natural action of $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$. In fact, one has

$$V = H^1_{et}(\mathbf{P}_1 - R, \mu_5) \otimes \mu_5^{-1},$$

where $\mu_5$ denotes the group scheme of 5th roots of unity. By Kummer theory, $H^1_{et}(\mathbf{P}_1 - R, \mu_5)$ is identified with the subspace of $\overline{\mathbf{Q}}(T)^*/\overline{\mathbf{Q}}(T)^{*5}$ spanned by the elements

$$(T + \zeta_5\sqrt{5})/(T - \zeta_5^2\sqrt{5}), \qquad (T - \zeta_5^2\sqrt{5})/(T + \zeta_5^{-1}\sqrt{5}),$$
$$(T + \zeta_5^{-1}\sqrt{5})/(T - \zeta_5^{-2}\sqrt{5}), \qquad (T - \zeta_5^{-2}\sqrt{5})/(T + \zeta_5\sqrt{5}),$$

whose product is 1. Hence the action of $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ on $H^1_{et}(\mathbf{P}_1 - R, \mu_5)$ factors through $\mathrm{Gal}(\mathbf{Q}(\zeta_5)/\mathbf{Q})$, and is isomorphic to the regular representation of $\mathrm{Gal}(\mathbf{Q}(\zeta_5)/\mathbf{Q})$ minus the trivial representation. It follows that $V$ decomposes as a direct sum of three irreducible one-dimensional Galois representations,

$$V = V_0 \oplus V^{\omega} \oplus V^{\omega^2},$$

where $V_0$ is the trivial representation, and $V^{\omega}$, $V^{\omega^2}$ denote one-dimensional spaces on which $\mathrm{Gal}(\mathbf{Q}(\zeta_5)/\mathbf{Q})$ acts via the Teichmüller character $\omega$ and the square of the Teichmüller character $\omega^2$, respectively. In particular, $V_0$ is the unique one-dimensional subspace of $V$ which is fixed by $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$. But the cyclic quintic coverings of $\mathbf{P}_1$ which are Galois over $\mathbf{Q}$ and unramified outside $R$ correspond exactly to such subspaces. Hence, property 1 determines $C'$ uniquely as a curve over $\overline{\mathbf{Q}}$. (Alternatively, one could use the "rigidity criterion" of Matzat, cf. [6, p. 368].) It is not hard to see that there is a unique rational form of the covering $C'$ such that the closed points above $\infty \in \mathbf{P}_1$ are all rational (twisting this rational form by a cocycle $c$ in $H^1(\mathbf{Q}, \mathrm{Aut}(C'/\mathbf{P}_1))$ will cause these points to be defined over the larger extension "cut out" by $c$). Thus, property 2 determines $C' \to \mathbf{P}_1$ up to $\mathbf{Q}$-isomorphism. □

## 2. A MODULAR COVERING INTERPRETATION OF LEHMER'S QUINTIC

We assume in this section some basic facts about modular forms and the geometry of modular curves. A good reference for this material is [7].

Let $X_0(25)$ and $X_1(25)$ denote the modular curves of level 25, compactified by adjoining a finite set of cusps. The curve $X_0(25)$ is of genus 0 and is isomorphic to $\mathbf{P}_1$ over $\mathbf{Q}$. The covering $X_1(25) \to X_0(25)$ is Galois with Galois group canonically isomorphic to $G = (\mathbf{Z}/25\mathbf{Z})^*/\langle \pm 1 \rangle$. The quotient $X$ of $X_1(25)$ by the involution $7 \in G$ gives a cyclic covering of $X_0(25)$ of degree 5.

Let $T_5 = \eta(z)/\eta(25z)$ and $F_5 = (\eta(z)/\eta(5z))^6$ be Hauptmoduls for $X_0(25)$ and $X_0(5)$, respectively. One has

$$F_5 = T_5^5/(T_5^4 + 5T_5^3 + 15T_5^2 + 25T_5 + 25).$$

The curve $X_0(5)$ has two cusps $C_1$ and $C_2$ corresponding to the values $F_5 = 0$ and $F_5 = \infty$, respectively. Hence, $X_0(25)$ has six cusps: a unique one lying above $C_1$, corresponding to $T_5 = 0$; and five cusps above $C_2$, given by $T_5 = \infty$, $-\sqrt{5}\zeta_5$, $\sqrt{5}\zeta_5^2$, $-\sqrt{5}\zeta_5^{-1}$, $\sqrt{5}\zeta_5^{-2}$ (cf. [1]). The covering $X \to X_0(25)$ is ramified at the four nonrational cusps, and the fiber above the cusp $T_5 = \infty$ is composed of rational points (cf. [1, p. 226]). By Proposition 1.1, $X$ can be described by Lehmer's quintic; the zeros $r_1, \dots, r_5$ of $P_5(Y, T_5)$ are modular functions on $X_1(25)$ (in fact, on $X$) with divisor supported at the $P_i$, where $P_1, \dots, P_5$ are the closed points of $X$ which lie above the cusp $T_5 = \infty$ of $X_0(25)$. By using Hensel's lemma to solve explicitly the equation $P_5(Y, T_5) = 0$, one obtains the following $q$-expansions for the $r_i$:

$$\begin{aligned}
r_1 &= -q^3 + q^4 + q^{10} - q^{11} - q^{12} + q^{13} - q^{15} + q^{17} + \cdots, \\
r_2 &= q^{-1} + 1 + q^6 + q^7 - q^{10} - q^{11} + \cdots, \\
r_3 &= -q - q^3 + q^4 + q^6 - q^{12} - q^{14} + q^{18} + q^{20} \cdots, \\
r_4 &= -q^{-2} - q - q^2 - q^5 + q^{15} + q^{17} + q^{18} \cdots, \\
r_5 &= q^{-1} + q^5 + q^7 - q^8 - q^{12} + q^{13} - q^{14} + \cdots.
\end{aligned} \tag{2}$$

By [8, p. 548], the transformation

$$r \mapsto \frac{(T_5+2)+T_5 r - r^2}{1+(T_5+2)r}$$

permutes the roots of $P(Y, T_5)$ cyclically; one can thus label the $r_i$ in such a way that a generator of $\mathrm{Gal}(X/X_0(25)) \simeq \mathbf{Z}/5\mathbf{Z}$ sends $r_i$ to $r_{i+1}$, where the subscripts are taken modulo 5. The five cusps of $X$ lying above the cusp $T_5 = \infty$ are permuted cyclically by the Galois group of $X$ over $X_0(25)$. By considering the $q$-expansions above, we may fix a labelling of the cusps $P_1, \dots, P_5$ so that a generator of $\mathrm{Gal}(X/X_0(25))$ sends $P_i$ to $P_{i+1}$ and such that

$$\mathrm{Divisor}(r_1) = 3P_1 - P_2 + P_3 - 2P_4 - P_5.$$

Now, let $a$ belong to $\mathbf{Z}/25\mathbf{Z}$, and define

$$\wp_a(\tau) = \wp(a/25; \tau),$$

where

$$\wp(z;\tau) = \frac{1}{z^2} + \sum_{(m,n)\in\mathbf{Z}^2-0}\left(\frac{1}{(z-n-m\tau)^2} - \frac{1}{(n+m\tau)^2}\right)$$

is the Weierstrass $\wp$-function. It is well known that the functions

$$\wp_{a,b}(\tau) = \wp_a(\tau) - \wp_b(\tau)$$

are modular units on $X_1(25)$. The divisors of these functions are computed in [1]. In particular, we find that

$$\mathrm{Divisor}\left(\frac{\wp_{7,9}\wp_{6,3}\wp_{1,12}\wp_{8,4}}{\wp_{1,3}\wp_{7,4}\wp_{6,7}\wp_{8,1}}\right) = 3P_1 - P_2 + P_3 - 2P_4 - P_5,$$

where the $P_i$ denote the cusps on $X$ which are above the cusp $\infty$ of $X_0(25)$. By expressing the function on the left in terms of so-called Klein forms $t_{(a_1,a_2)}$ (cf. [2]), the above simplifies to give

$$\mathrm{Divisor}\left(\frac{t_{(0,1)}t_{(0,7)}}{t_{(0,9)}t_{(0,12)}}\right) = 3P_1 - P_2 + P_3 - 2P_4 - P_5.$$

Let us abbreviate $t_{(0,a)}$ to $t_a$. By comparing divisors and $q$-expansions, one finds the following infinite product expressions for the $r_i$:

$$r_1 = \frac{t_1 t_7}{t_9 t_{12}}(25\tau) = -q^3 \prod_{n\equiv\pm1,\pm7(25)}(1-q^n)\Big/\prod_{n\equiv\pm9,\pm12(25)}(1-q^n),$$

$$r_2 = \frac{t_2 t_{11}}{t_1 t_7}(25\tau) = q^{-1} \prod_{n\equiv\pm2,\pm11(25)}(1-q^n)\Big/\prod_{n\equiv\pm1,\pm7(25)}(1-q^n),$$

$$r_3 = \frac{t_4 t_3}{t_{11} t_2}(25\tau) = -q \prod_{n\equiv\pm4,\pm3(25)} (1-q^n) / \prod_{n\equiv\pm11,\pm2(25)} (1-q^n),$$

$$r_4 = \frac{t_8 t_6}{t_3 t_4}(25\tau) = -q^{-2} \prod_{n\equiv\pm8,\pm6(25)} (1-q^n) / \prod_{n\equiv\pm3,\pm4(25)} (1-q^n),$$

$$r_5 = \frac{t_9 t_{12}}{t_6 t_8}(25\tau) = q^{-1} \prod_{n\equiv\pm9,\pm12(25)} (1-q^n) / \prod_{n\equiv\pm6,\pm8(25)} (1-q^n).$$

The Galois group $\mathrm{Gal}(X_1(25)/X_0(25)) = (\mathbf{Z}/25\mathbf{Z})^*/\langle\pm1\rangle$ acts on the $t_a$ by multiplying the subscripts (which are viewed as belonging to $(\mathbf{Z}/25\mathbf{Z})^*/\langle\pm1\rangle$). Hence, to go from $r_i$ to $r_{i+1}$, one applies the Galois automorphism $2 \in \mathrm{Gal}(X/X_0(25)) = (\mathbf{Z}/25\mathbf{Z})^*/\langle\pm1, \pm7\rangle$.

## 3. Gauss sums

Given a prime $p \equiv 1 \pmod 5$, let $\Psi_p : \mathbf{F}_p \to \mathbf{C}^*$ be the additive character sending 1 to $\zeta_p$. We consider the Gauss sum

$$g(p) = \sum_{x\in\mathbf{F}_p} \chi(x)\Psi_p(x),$$

where $\chi$ is a character of $\mathbf{F}_p^*$ of order 5. The value of $g(p)$ is independent of $\chi$, up to the action of $\mathrm{Gal}(\mathbf{Q}(\zeta_5)/\mathbf{Q})$.

By combining Lehmer's explicit determination of the roots of her polynomial as Gaussian periods, and our identification of these roots with certain modular forms of level 25, we obtain:

**Theorem 3.1.** *If $\eta(\tau)/\eta(25\tau) = n \in \mathbf{Z}$, and $\eta(5\tau)^6/(\eta(\tau)\eta(25\tau)^5) = p$ is prime, then*

$$\prod_{i=1}^{4} (\eta(\tau)/\eta(25\tau) - \sigma_i^{-1}(\zeta_5\sqrt{5}))^{i/5} = (n/5)g(p),$$

*where $\sigma_i \in \mathrm{Gal}(\mathbf{Q}(\zeta_5)/\mathbf{Q})$ sends $\zeta_5$ to $\zeta_5^i$.*

There is some ambiguity in the formula, since the value of $g(p)$ depends on the choice of a multiplicative character $\chi$, and the left-hand side is really only defined up to a fifth root of 1. We are asserting that there is a way of making these choices so that the formula holds.

Observe that the left-hand side is a modular unit (i.e., a unit for the covering $X_1(25) \to X_0(1)$). Thus the above expresses Gauss sums as values of certain modular units on $X_1(25)$. It seems that the other coverings of lower degree studied by Lehmer yield similar results. It would be interesting to obtain such formulas a priori: this might provide a justification for the fact that translates of Gaussian period polynomials yield cyclic units for extensions of small degree.

*Note.* The idea of studying families of units in cyclic extensions of $\mathbf{Q}$ arising from the modular covering $X_1(N) \to X_0(N)$ has been explored by Odile

Lecacheux (see, for example, the paper [3], which studies units in sextic extensions which arise from the modular covering $X_1(13) \to X_0(13)$). Independently of the author, Lecacheux has also observed the connection between Lehmer's quintic and the modular curve $X_1(25)$ [4].

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## Bibliography

1. Daniel S. Kubert, *Universal bounds on the torsion of elliptic curves*, Proc. London Math. Soc. (3) **33** (1976), 193–237.
2. Daniel S. Kubert and Serge Lang, *Modular units*, Springer-Verlag, New York, 1981.
3. Odile Lecacheux, *Unités d'une famille de corps cycliques réels de degré* 6 *liés à la courbe modulaire* $X_1(13)$, J. Number Theory **31** (1989), 54–63.
4. ——, private communication.
5. Emma Lehmer, *Connection between Gaussian periods and cyclic units*, Math. Comp. **50** (1988), 535–541.
6. B. H. Matzat, *Rationality criteria for Galois extensions*, Galois Groups Over $\mathbf{Q}$, Proc. Workshop held March 23–27, 1987 (Y. Ihara, K. Ribet, and J.-P. Serre, eds.), MSRI Publications, 1989, pp. 361–384.
7. A. Ogg, *Survey of modular functions of one variable*, Modular Functions of One Variable (Proc. Antwerp 1972), Lecture Notes in Math., vol. 320, Springer, 1973.
8. René Schoof and Lawrence C. Washington, *Quintic polynomials and real cyclotomic fields with large class numbers*, Math. Comp. **50** (1988), 543–556.

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