Euler systems and the Birch and Swinnerton-Dyer conjecture

Henri Darmon
(joint work with Massimo Bertolini, Victor Rotger)

The Birch and Swinnerton-Dyer conjecture for an elliptic curve $E/\mathbb{Q}$ asserts that

$$\mathrm{ord}_{s=1} L(E,s) = \mathrm{rank}(E(\mathbb{Q})), \tag{1}$$

where $L(E,s)$ is the Hasse-Weil L-function attached to E. The scope of the conjecture can be broadened somewhat by introducing an Artin representation

$$\varrho : G_{\mathbb{Q}} \longrightarrow \mathrm{Aut}(V_\varrho) \simeq \mathbf{GL}_n(\mathbb{C}), \tag{2}$$

and studying the Hasse-Weil-Artin L-function $L(E,\varrho,s)$, namely, the L-function attached to $H^1_{\mathrm{et}}(E_{\bar{\mathbb{Q}}}, \mathbb{Q}_p) \otimes V_\varrho$, viewed as a (compatible system of) p-adic representations. The "equivariant Birch and Swinnerton-Dyer conjecture" states that

$$\mathrm{ord}_{s=1} L(E,\varrho,s) = \dim_{\mathbb{C}} \hom_{G_{\mathbb{Q}}}(V_\varrho, E(H)\otimes\mathbb{C}), \tag{3}$$

where H is a finite extension of $\mathbb{Q}$ through which ϱ factors. Denote by $\mathrm{BSD}_r(E,\varrho)$ the assertion that the right-hand side of (3) is equal to r when the same is true of the left-hand side. Virtually nothing is known about $\mathrm{BSD}_r(E,\varrho)$ when $r > 1$. For $r \leq 1$, there are the following somewhat fragmentary results, listed in roughly chronological order:

Theorem (Gross-Zagier 1984, Kolyvagin 1989) *If ϱ is induced from a ring class character of an imaginary quadratic field, and $r \leq 1$, then $BSD_r(E,\varrho)$ holds.*

Theorem A (Kato, 1990) *If ϱ is abelian (i.e., corresponds to a Dirichlet character), then $BSD_0(E,\varrho)$ holds.*

Theorem B (Bertolini-Darmon-Rotger, 2011) *If ϱ is an odd, irreducible, two-dimensional representation whose conductor is relatively prime to the conductor of E, then $BSD_0(E,\varrho)$ holds.*

Theorem C (Darmon-Rotger, 2012) *If $\varrho = \varrho_1 \otimes \varrho_2$, where ϱ_1 and ϱ_2 are odd, irreducible, two-dimensional representations of $G_{\mathbb{Q}}$ satisfying:*

(1) $\det(\varrho_1) = \det(\varrho_2)^{-1}$, *so that ϱ is isomorphic to its contragredient representation;*
(2) *ϱ is regular, i.e., there is a $\sigma \in G_{\mathbb{Q}}$ for which $\varrho(\sigma)$ has distinct eigenvalues;*
(3) *the conductor of ϱ is prime to that of E;*

then $BSD_0(E,\varrho)$ holds.

This lecture endeavoured to explain the proofs of Theorems A, B, and C, emphasising the fundamental unity of ideas underlying all three.

The key ingredients are certain global cohomology classes

$$\kappa(f,g,h) \in H^1(\mathbb{Q}, V_f \otimes V_g \otimes V_h(c))$$

attached to triples (f,g,h) of modular forms of respective weights (k,ℓ,m); here V_f, V_h and V_g denote the Serre-Deligne representations attached to f, g and h, and it is assumed that the triple tensor product of Galois representations admits a Kummer-self-dual Tate twist, denoted $V_f \otimes V_g \otimes V_h(c)$. (This is true when the product of nebentype characters associated to f, g and h is trivial.)

When f, g and h are all of weight two and level dividing N, and f is cuspidal, associated to an elliptic curve E, say, the class $\kappa(f, g, h)$ admits a geometric construction via p-adic étale regulators/Abel-Jacobi images of

(1) Beilinson-Kato elements in the higher Chow group $\mathrm{CH}^2(X_1(N), 2)$ of the modular curve $X_1(N)$, when g and h are Eisenstein series of weight two arising as logarithmic derivatives of suitable Siegel units;
(2) Beilinson-Flach elements in the higher Chow group $\mathrm{CH}^2(X_1(N)^2, 1)$ when g is cuspidal and h is an Eisenstein series;
(3) Gross-Kudla-Schoen diagonal cycles in the Chow group $\mathrm{CH}^2(X_1(N)^3)$, when all forms are cuspidal.

When g and h are of weight one rather than two, and hence, are associated to certain (possibly reducible) odd two-dimensional Artin representations, the construction of $\kappa(f, g, h)$ via K-theory and algebraic cycles ceases to be available. The class $\kappa(f, g, h)$ is obtained instead by a process of p-adic analytic continuation, interpolating the geometric constructions at all classical weight two points of Hida families passing through g and h in weight one, and then specialising to this weight. The resulting $\kappa(f, g, h)$ is called the *generalised Kato class* atttached to the triple (f, g, h) of modular forms of weights $(2, 1, 1)$.

The generalised Kato classes arising from (p-adic limits of) Beilinson-Kato elements, Beilinson-Flach elements, and Gross-Kudla-Schoen cycles are germane to the proofs of Theorems A, B and C respectively. The key point in all three proofs is an *explicit reciprocity law* which asserts that the global class $\kappa(f, g, h)$ is *non-cristalline at p* pecisely when the classical central critical value $L(f \otimes g \otimes h, 1) = L(E, \varrho, 1)$ is non-zero. The non-cristalline classes attached to (f, g, h) (of which there are actually four, attached to various choices of ordinary p-stabilisations of g and h) can then be used (by a standard argument involving local and global Tate duality) to conclude that the natural inclusion of $E(H)$ into $E(H \otimes \mathbb{Q}_p)$ becomes zero when restricted to $\varrho_g \otimes \varrho_h$-isotypic components, and hence, that $\hom_{G_{\mathbb{Q}}}(V_\varrho, E(H) \otimes \mathbb{C})$ is trivial when $L(E, \varrho, 1) \neq 0$.

The lecture strived to set the stage for the two that immediately followed, which were both devoted to further developments arising from these ideas:

(1) Victor Rotger's lecture studied the generalised Kato classes $\kappa(f, g, h)$ when $L(f, g, h, 1) = 0$. In that case, they belong to the Selmer group of E/H, and can be viewed as p-adic avatars of $L''(E, \varrho, 1)$;
(2) Sarah Zerbes' lecture reported on [LLZ1], [LLZ2], [KLZ] in which the study of Beilinson-Flach elements undertaken in [BDR] is generalised, extended and refined. By making more systematic use of the Euler system properties of Beilinson-Flach elements, notably the possibility of "tame deformations" at primes $\ell \neq p$, the article [KLZ] is also able to establish strong finiteness results for the relevant ϱ-isotypic parts of the Shafarevich-Tate group of E over H, in the setting of Theorem B.

References

[BDR] M. Bertolini, H. Darmon, and V. Rotger, *Beilinson-Flach elements and Euler systems II: the Birch and Swinnerton-Dyer conjecture for Hasse-Weil-Artin L-series*, submitted.

[DR] H. Darmon and V. Rotger, *Diagonal cycles and Euler systems II: the Birch and Swinnerton-Dyer conjecture for Hasse-Weil-Artin L-series*, submitted.

[Ka] K. Kato. *p-adic Hodge theory and values of zeta functions of modular forms*, Cohomologies p-adiques et applications arithmétiques III, Astérisque no. 295 (2004).

[KLZ] G. Kings, D. Loeffler and S. Zerbes, *Rankin-Selberg Euler systems and p-adic interpolation*, submitted.

[LLZ1] A. Lei, D. Loeffler, and S. Zerbes, *Euler systems for Rankin-Selberg convolutions*, Annals of Mathematics, to appear.

[LLZ2] A. Lei, D. Loeffler, and S. Zerbes, *Euler systems for modular forms over imaginary quadratic fields*, submitted.