Work of Kudla, Funke-Millson and application to construction of cocycles : Part Two

1 Goal

Our goal for today is to give topological interpretations for the integrals we saw last time and restate the results. We also would like to go back to our proposed construction and highlight the importance of results of Kudla and Funke-Millson.

2 Recall

Last time we proposed a construction of modular symbols. Let $X_0 \in V_{\mathbb{R}}$ then the following mapping

$$(r,s) \mapsto \sum_{\substack{X \in \Gamma X_0 \\ (X,X)=m}} a_X(r,s)\Delta_X$$

gives a modular symbol given that $a_X(r,s)$ satisfy certain properties. We ended last time by saying that we can define

$$a_X(r,s) := \int_{C(r,s)} \widetilde{\eta}(\tau, X)$$

where $\widetilde{\eta}(\tau, X)$ is a slight modification of the 1-form $\eta(\tau, X)$ considered previously (the difference is of a factor of $\sqrt{2}$ and $q^{(X,X)}$). Our goal is to provide a clearer statement and also present a topological interpretation.

We recall that $\eta(\tau, X)$ is a 1-form on $\mathcal{H}^{(3)}$ and $C(r,s)$ is a 1-cycle "joining r and s". We also saw the following correspondence between lines in vector space and points on $\mathcal{H}^{(3)}$ (and "boundary"). Under the correspondence we can associate an isotropic line ℓ_r to any $r \in \mathbb{P}^1(K)$.

Vector space	Manifold
Negative length lines in $V_{\mathbb{R}}$	Point in $\mathcal{H}^{(3)}$
Zero length lines in $V_{\mathbb{R}}$	$\mathbb{C} \cup \{\infty\}$

The final theorem was a statement of modularity which we state in the remaining part of this section. Let $\Lambda \subset V_{\mathbb{Q}}$ and assume that $V_{\mathbb{Q}}$ has isotropic vectors (see Remark (1)). For $h \in \Lambda^\vee/\Lambda$ and $r,s \in \mathbb{P}^1(K)$, let $H(r,s)$ be the 2-dimensional space generated by ℓ_r and ℓ_s. Define the following sets

$$S^0_{(r,s)}(h+\Lambda) := \{X \in h+\Lambda : \textit{Projection of } X \textit{ along } H(r,s) \textit{ is of length zero}\}$$
$$S^{>0}_{(r,s)}(h+\Lambda) := \{X \in h+\Lambda : \textit{Projection of } X \textit{ along } H(r,s) \textit{ is of positive length }\}.$$

We are now ready to state the result.

Theorem 1. *The series* $\sum_{h \in \Lambda^\vee/\Lambda} g_h(r,s)\mathbf{e}_h$ *is a vector valued modular form of weight* 2 *and type* ρ_Λ *where*

$$g_h(r,s)(q) := \sum_{m \in \mathbb{Q}_{\geq 0}} \left(\int_r^s \sum_{\substack{X \in S^0_{(r,s)}(h+\Lambda) \\ (X,X)=m}} \widetilde{\eta}(\tau, X) \right) q^m + \sum_{m \in \mathbb{Q}_{>0}} \left(\sum_{\substack{X \in S^{>0}_{(r,s)}(h+\Lambda) \\ (X,X)=m}} \int_r^s \widetilde{\eta}(\tau, X) \right) q^m.$$

Remark 1. *We note that the assumption that* $V_{\mathbb{Q}}$ *has isotropic vectors might not be completely precise. However I am confident that there is a assumption (very similar) which makes the above theorem a valid one. In a general result when there is no restriction on presence of isotropic vectors, the result still holds that there is a modular form whose coefficients are certain integrals. However the precise integrals look a bit different and at the moment I am somewhat unclear on this yoga.*

3 Topological Interpretation

There is a nice topological interpretation of these integrals as intersection numbers of cycles. We won't go into mathematical details of defining "intersection numbers" but rather content ourselves with interpreting them as "physical intersections".

Special Cycles associated to a vector For each vector $X \in V_{\mathbb{R}}$ we look at $X^\perp$ which is a three dimensional space. Based on the general recipe of considering negative length lines we consider

$$B(X^\perp) := \{Z \in X^\perp : (Z,Z) = -1\}.$$

We now consider three different cases

1. $(X,X) > 0$: $X^\perp$ is of signature $(2,1)$ and thus $B(X^\perp)$ is a real manifold of dimension two. In fact we can identify it with Poincaré upper half plane.
2. $(X,X) = 0$: $X^\perp$ is a positive semidefinite space and can be decomposed as $U \widehat{\oplus} X$ where U is a space of signature $(2,0)$. Therefore $B(X^\perp)$ is empty.
3. $(X,X) < 0$: $X^\perp$ is of signature $(3,0)$ and thus there is no negative length line in $X^\perp$ and as a consequence $B(X^\perp)$ is empty.

From this point of view the only interesting ones to us are actually those of positive length. Note that we have the inclusion of quadratic spaces $X^\perp \subset V_{\mathbb{R}}$ which gives us an embedding of manifolds as $B(X^\perp) \hookrightarrow B(V_{\mathbb{R}})$. Infact if one chooses appropriate coordinates and identify $B(V_{\mathbb{R}})$ with $\mathcal{H}^{(3)}$, the embedded manifold $B(X^\perp)$ will be a geodesic hemisphere or a plane in $\mathcal{H}^{(3)}$.

Remark 2. *The plane would occur if X is orthogonal to ℓ_∞.*

We now define "intersection numbers". For $(X,X) > 0$,

$$B(X^\perp) \cap C(r,s) := \frac{1}{\sqrt{2}} \frac{1}{q^{(X,X)}} \int_C \eta(\tau, X) \in \left\{ +1, +\frac{1}{2}, , 0, -\frac{1}{2}, -1 \right\}.$$

The $\pm\frac{1}{2}$ *appear precisely when X has an isotropic projection along $H(r,s)$.* Therefore we can now interpret the integrals as physical intersections of oriented manifolds $B(X^\perp)$ and $C(r,s)$ in $\mathcal{H}^{(3)}$.

3.1 Modular symbols valued in Integers

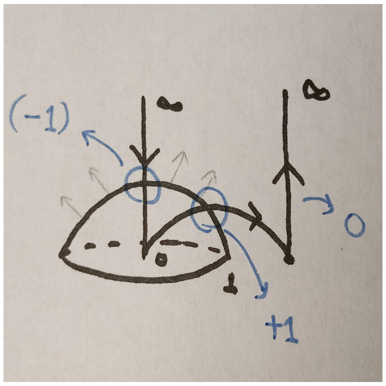
This is a very short section but is a culmination of all the hard work seen so far. We present the following crucial theorem.

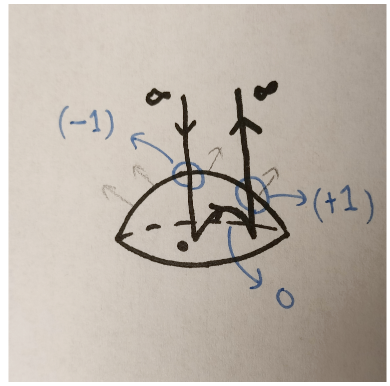
Theorem 2. *Let $r, s \in \mathbb{P}^1(K)$ and $X \in V_\mathbb{R}$. The "intersection numbers" defined previously satisfy the following properties:*

1. $B(X^\perp) \cap C(r,s) + B(X^\perp) \cap C(s,r) = 0.$
2. $B(X^\perp) \cap C(r,s) + B(X^\perp) \cap C(s,t) + B(X^\perp) \cap C(t,r) = 0.$
3. $B((\gamma X)^\perp) \cap (\gamma r, \gamma s) = B(X^\perp) \cap C(r,s)$ *for all* $\gamma \in \Gamma$.

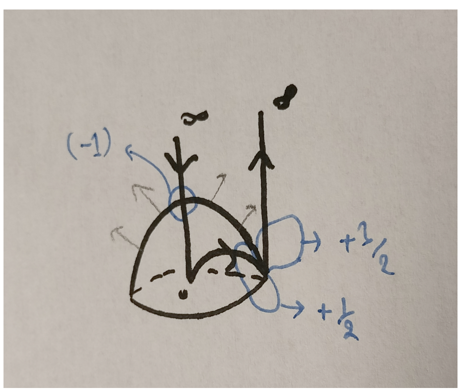
We can quite clearly see that these $B(X^\perp) \cap C(r,s)$ are candidates for $a_X(r,s)$. In fact if we can have a supply of vectors X such that $a_X(r,s) = 0$ for almost all vectors in its Γ-orbit, we will have a modular symbol valued in integers. The good news is that there exist infinitely many such Γ-orbits in our setup! In fact we can restrict our attention to vectors X whose projection along any $H(r,s)$ is always of non-zero length and this restricted set gives many examples of divisor valued modular symbols.

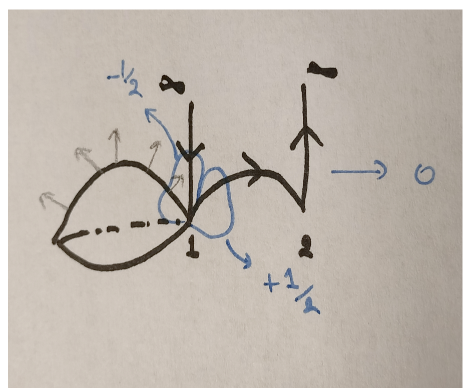
An argument by picture. We won't give a proof of the aforementioned theorem but rather try to give a few pleasant pictures to generate confidence in the statement.





The above two pictures describe intersection numbers when considering intersections of $B(X^{\perp})$ with three oriented 1-cycles : $C(r,s), C(s,t)$, and $C(t,r)$ when intersections are transversal. The hemisphere $B(X^{\perp})$ is oriented by choosing an outward normal and then we can assign ± 1 depending on the orientation of 1-cycle. If there is no intersection we assign zero.





The above two pictures describe intersection numbers when considering intersections of $B(X^{\perp})$ with three oriented 1-cycles : $C(r,s), C(s,t)$, and $C(t,r)$ when the intersections are at "boundary". In the left picture, the "boundary" intersection is $+1/2$ for both 1-cycles as the orientations align with outward normal. However for the right picture, there is a $+1/2$ (resp. $-1/2$) as the cycle from $1 \to 2$ is oriented along the outward normal (resp. as the cycle from $\infty \to 1$ is oriented opposite to the outward normal).

4 Lifting to Cocycles valued in functions

We start with the following lemma.

Lemma 3. *If $\sum_{X\in S} a_X = 0$ for $a_X \in \mathbb{Z}$ and some set $S \subset V_{\mathbb{Q}}$, then there is a function $f \in \mathbb{C}(z_1, z_2)^\times$ such that*

$$\operatorname{div}(f) = \sum_{X\in S} a_X \Delta_X.$$

From previous lecture. *We recall that for each $X \in V_{\mathbb{Q}}$ we associated a divisor Δ_X defined as*

$$\Delta_X := \left\{ ([X_1 : Y_1], [X_2 : Y_2]) : \begin{pmatrix} X_1 \\ Y_1 \end{pmatrix}^T SX \begin{pmatrix} X_2 \\ Y_2 \end{pmatrix} = 0 \right\}.$$

Proof. It suffices to prove that $\Delta_{v_1} - \Delta_{v_2}$ is a divisor of some function. This fact is easier to esatblish by just explicitly writing out the defining equations for Δ_{v_1} and Δ_{v_2}. Say we have

$$\begin{aligned} \Delta_{v_1} &= \{([X_1 : Y_1], [X_2 : Y_2]) : \mathcal{F}_1(X_1, Y_1, X_2, Y_2) = 0\} \\ \Delta_{v_2} &= \{([X_1 : Y_1], [X_2 : Y_2]) : \mathcal{F}_2(X_1, Y_1, X_2, Y_2) = 0\} \end{aligned}$$

Then we define our $f(z_1, z_2)$ by

$$f(z_1, z_2) = \frac{\mathcal{F}_1(z_1, 1, z_2, 1)}{\mathcal{F}_2(z_1, 1, z_2, 1)}$$

where $\mathcal{F}_1(z_1, z_2)$ is a dehomogonization of $\mathcal{F}_1(X_1, Y_1, X_2, Y_2)$. $\square$

Let $\operatorname{Div}^\dagger$ be a divisor module generated by $\Delta_X' s$ and $\mathcal{F}^\dagger$ denote all the rational functions whose divisors are in $\operatorname{Div}^\dagger$. We then have the following exact sequence

$$0 \to \mathcal{F}^\dagger/\mathbb{C}^\times \to \operatorname{Div}^\dagger \to \mathbb{Z} \to 0.$$

The exactness in the middle follows from Lemma (3). This induces a long exact sequence in cohomology

$$\cdots \to H^1(\Gamma, \mathcal{F}^\dagger/\mathbb{C}^\times) \to H^1(\Gamma, \operatorname{Div}^\dagger) \to H^1(\Gamma, \mathbb{Z}) \to \cdots .$$

Our construction from the previous section gives us the following cocycle valued in divisors

$$\begin{aligned} J_{X_0} : \Gamma &\to \operatorname{Div}^\dagger \\ \gamma &\mapsto= \sum_{\substack{X\in\Gamma X_0 \\ (X,X)=m}} a_X(\infty, \gamma\infty)\Delta_X. \end{aligned}$$

Applying the degree map gives us

$$\deg(J_{X_0}(\infty, \gamma\infty)) = \sum_{\substack{X\in\Gamma X_0 \\ (X,X)=m}} a_X(\infty, \gamma\infty) \in \mathbb{Z}.$$

We saw that we were not able to give a suitable definition of $a_X(r,s)$ for all vectors X, but for the ones we could do we defined it as

$$a_X(\infty, \gamma\infty) = \int_{\infty}^{\gamma\infty} \eta(\tau, X),$$

and we have already seen that

$$\sum_{\substack{X \in \Gamma h + \Lambda \\ (X,X)=m}} \int_{\infty}^{\gamma\infty} \eta(\tau, X)$$

is m-th coefficient of a vector valued modular form of weight 2 and type ρ_Λ (or at least a part of). The key point for us is that spaces of modular forms finite dimensional and thus there are many linear relations between their coefficients. Hence we can make use of such linear relations to have a combination of $J'_{X_0}s$ such that their image under the connecting map vanishes and thus the divisor valued cocycle lifts to a cocycle valued in function modular scalars. We summarize this discussion in the following theorem.

Theorem 4. *Let $\{c(m_i, h_i)\}_{i=1}^r$ be integers such that for each $X \in h_i + \Lambda$ of length m_i the projection of X along all $H(r,s) \subset V_{\mathbb{Q}}$ is anisotropic.*

$$\sum_{i=1}^{r} c(h_i, m_i) c_{m_i,h_i}(g) = 0$$

for all $g \in M_{2,\mathbb{D}_\Lambda}$ where $c_{m_i,h_i}(g)$ denotes $(h_i, m_i)-$th coefficient of vector valued modular forms. Then there exists a rational cocycle $J \in H^1(\Gamma, \mathcal{F}^\dagger/\mathbb{C}^\times)$ such that

$$\operatorname{div}_*(J(\gamma)) = \sum_{i=1}^{r} c(h_i, m_i) \left(\sum_{\substack{X \in h_i + \Lambda \\ (X,X)=m_i}} a_X(\infty, \gamma\infty) \Delta_X \right).$$