Some Comments on Section 7

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1 Alternative proof of 7.3 (i)

Let $k_{\mathfrak{p}}$ be the completion of k at $\mathfrak{p}$. As usual, we denote by $U_{\mathfrak{p}}$, π, and $k(\mathfrak{p})$ the group of units, the uniformizer, and the residue field, respectively. Define $\nu_{\mathfrak{p}}$ analogously to ν:

$$\text{classes in } k_{\mathfrak{p}}^*/(k_{\mathfrak{p}}^*)^l \text{ whose evaluation is 0 mod } l$$

Theorem 1.1. *We have maps:*

(I) $\nu \to \nu_{\mathfrak{p}}$*. Moreover, the map*

$$\nu \hookrightarrow \prod_{\mathfrak{p} \notin T} \nu_{\mathfrak{p}} \tag{1.1}$$

is injective as long as the set T *of excluded primes is finite.*

(II) $\nu_{\mathfrak{p}} \cong k^*(\mathfrak{p})/l \cong \mathbb{F}_l$ *for all* $\mathfrak{p} \nmid l$

Proof of (II) Recall that $k_{\mathfrak{p}}^* \cong \pi^{\mathbb{Z}} \times U_{\mathfrak{p}}$. Hence,

$$\text{elements of } k_{\mathfrak{p}}^* \text{ whose evaluation is 0 mod } l \cong (\pi^l)^{\mathbb{Z}} \times U_{\mathfrak{p}}$$

Therefore,

$$\nu_{\mathfrak{p}} \cong U_{\mathfrak{p}}/l$$

$(U_{\mathfrak{p}})^l$ contains all elements that are congruent to $1 \mod \mathfrak{p}$, by Hensel's lemma. Hence,

$$U_{\mathfrak{p}}/l \cong k(\mathfrak{p})^*/l \cong \mathbb{F}_l$$

As in (7.3), we get the last isomorphism because $k(\mathfrak{p})^*$ is cyclic and contains ζ_l.

Proof of (I) The map is well defined. Injectivity follows from the following lemma:

Lemma 1.2. *If* x *is an* l^{th} *power at all but finitely many primes, it is an* l^{th} *power globally.*

Proof. The hypothesis of the lemma implies that all but finitely many primes of k split in $k(x^{1/l})$. We apply (5) on page 5 to conclude that $x^{1/l} \in k$. $\square$

To get our set of auxiliary primes, we use the following lemma from linear algebra:

Lemma 1.3. *If an $n \times \infty$ matrix has rank n, some $n \times n$ submatrix has rank n.*

We apply this to (1.1). It follows from Lemma 1.3 that there exists a set of primes P in the complement of T such that

$$\nu \cong \prod_{\mathfrak{p} \in P} \nu_{\mathfrak{p}}$$

2 Alternative proof of 7.2 (b)

We give a proof of 7.2 (b), that is analogous to the proof of 7.2 (a) and, similarly, relies on the two identities below:

$$x^l - 1 = \prod_i (x - \zeta^i) \Rightarrow \sum_i val(x - \zeta^j) = val(x^l - 1) \tag{2.1}$$

$$val\left((x - \zeta^i) - (x - \zeta^j)\right) = val(\zeta^i - \zeta^j) = e_0 \quad \textbf{for } i \neq j \tag{2.2}$$

Lemma 2.1. *We have the following explicit description of l-torsion in $(\mathfrak{o}/\lambda^n)^*$, as n varies:*

Case 1: If $\frac{n}{l} \leq e_0$,

$$val(x^l - 1) \geq n \Leftrightarrow val(x - \zeta^i) \geq \left\lceil \frac{n}{l} \right\rceil \forall\, i \Leftrightarrow x \in 1 + \lambda^{\lceil \frac{n}{l} \rceil}$$

Case 2: If $\frac{n}{l} > e_0$,

$$val(x^l - 1) \geq n \Leftrightarrow \begin{cases} val(x - \zeta^{i_0}) = n - (l-1)e_0 & \textit{for some (unique) } i_0, \\ \\ val(x - \zeta^i) = e_0 & \textit{for all } i \neq i_0 \end{cases}$$

Proof. We combine (2.1) and (2.2) just as we did for the proof of 7.2 (a). $\square$

Lemma 2.2.

$$\frac{\#\left\{ x \in \mathfrak{o}/\lambda^{e_0 l + 1} \,\middle|\, x^l = 1 \right\}}{\#\left\{ x \in \mathfrak{o}/\lambda^{e_0 l} \,\middle|\, x^l = 1 \right\}} = l \tag{2.3}$$

Proof. Lemma 2.1 allows us to explicitly compute: $\#\left\{ x \in \mathfrak{o}/\lambda^n \;\middle|\; x^l = 1 \right\}$ for all n.[1] ☐

Corollary.

$$\frac{\#\left\{ \left(\mathfrak{o}\middle/\lambda^{e_0 l+1}\right)^* \middle/ l \right\}}{\#\left\{ \left(\mathfrak{o}\middle/\lambda^{e_0 l}\right)^* \middle/ l \right\}} = l \tag{2.4}$$

Proof of 7.2 (b): Now we have the exact sequence:

$$0 \longrightarrow 1 + \lambda^{e_0 l} \longrightarrow \left(\mathfrak{o}\middle/\lambda^{e_0 l+1}\right)^* \longrightarrow \left(\mathfrak{o}\middle/\lambda^{e_0 l}\right)^* \longrightarrow 0 \tag{2.5}$$

(2.4) implies that the kernel of

$$\left(\mathfrak{o}\middle/\lambda^{e_0 l+1}\right)^* \middle/ l \longrightarrow \left(\mathfrak{o}\middle/\lambda^{e_0 l}\right)^* \middle/ l$$

has cardinality l. By (2.5), this kernel is the image of

$$1 + \lambda^{e_0 l} \longrightarrow \left(\mathfrak{o}\middle/\lambda^{e_0 l+1}\right)^* \middle/ l.$$

Therefore, this image likewise has cardinality l, as desired.

[1] Intuitively, the denominator of (2.3) counts the elements in a coset, while the numerator counts the elements in l *disjoint* cosets. In this case, the relevant cosets have the same cardinality.