1 Intro

These notes follow very closely the exposition in Chapter II.3 of Serre's *A Course in Arithmetic*. The text is quite condensed: we tried to explicit the hidden ideas.

Write M_k for the set of modular forms of weight k. Recall that $M_k = \emptyset$ for k an odd number. Also,

Proposition 1.1. *Each $\mathcal{M}_k$ forms a complex vector space. Further, $\mathcal{M}$ is a graded ring.*

Proof. Let $f \in \mathcal{M}_k$, $g \in \mathcal{M}_{k'}$, and $\lambda \in \mathbb{C}$.
Suppose $k = k'$. Then, $(f + \lambda g)[\gamma]_k = f[\gamma]_k + \lambda g[\gamma]_k = f + \lambda g$, and the holomorphy conditions are respected. This shows $\mathcal{M}_k$ is a vector space over $\mathbb{C}$.
Now, we show that fg is also modular, but of weight $k + k'$. This gives the (graded) ring structure to $\mathcal{M}$.

$$(fg)[\gamma]_{k+k'}(\tau) = j(\gamma, \tau)^{k+k'} f(\tau)g(\tau) = j(\gamma, \tau)^k f(\tau) j(\gamma, \tau)^{k'} g(\tau) = f(\tau)g(\tau)$$

□

We develop a formula which will enable us to decribe explicitely the vector spaces M_k. The result can be derived from the Riemann-Roch theorem, and should be derived in this way for modular forms over general congruence subgroups. In our case, the evaluation of an integral suffices.

2 The Integral

The main result is a formula relating the zeros and poles of a modular form to its degree. We get it by computing an integral in two diffent ways. First, a definition

Definition 2.1. Let f be a meromorphic function on $\mathcal{H}$, and $p \in \mathcal{H}$. Then, define the order of f at p, written $\nu_p(f)$, to be the least number n so that $\frac{f(z)}{(z-p)^n}$ is holomorhic at p.

This allows us to discriminate poles and zeroes, since $\nu_1(z-1) \neq \nu_1\left((z-1)^2\right)$. Also, note that the order of a zero is a positive number and the order of a pole is a negative number. The reason for this convention will become apparent. Recall that we consider "$f(\infty)$" to be the value of $\hat{f}$ at $q = 0$, where $\hat{f}$ is the fourier transform (a function of $e^{1\pi i z}$). In this logic, define $\nu_\infty(f) = \nu_0(\hat{f})$.

Theorem 2.1. *Let f be a nonzero weakly modular form of weight $2k$. Then*

$$\nu_\infty(f) + \frac{1}{2}\nu_i(f) + \frac{1}{3}\nu_\rho(f) + \sum_{\rho \in Y(G)}^{\sim} \nu_\rho(f) = \frac{2k}{12}$$

The proof will be in two parts. We integrate $\frac{df}{f}$ on the domain U shown in the image in two different ways. The following lemma justifies the chocie of domain.

Lemma 2.1. *Our function f can only have finitely many zeroes and poles in U.*

Proof. More generaly, meromorphic functions can only have finitely many zeros or poles on compact sets. This is because every infinite subset of a compact set has an accumulation point, but the set of zeroes/poles can't accumulate. □

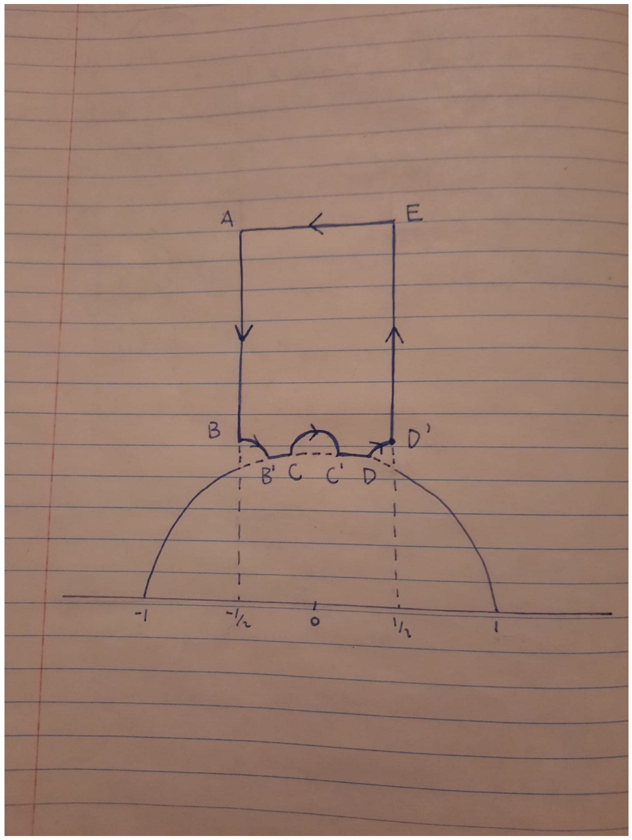

Figure 1: The contour U.

Hence by taking the limit as $[A, E]$ goes to infinity, we recover all the fundamental domain. We add an extra assumption before we continue: that f has no zeros or poles on the vertical segments.

Now, we need a way to extract the order of poles and zeros. To this end, we use a general principle of complex analysis.

Lemma 2.2. *Let f be holomorphic on U, and suppose f vanishes at only finitely many points. Then, $\frac{1}{2\pi i}\oint_U \frac{f'(z)}{f(z)}dz = \sum_{p\in U} \nu_p f$. The sum converges since there are only finitely many nonzero terms.*

Proof. Say f has a zero or pole at $p_1, ..., p_n$ in U. Let $\nu_i = \nu_{p_i}(f) > 0$. Then, we investigate the Laurent Series of f'/f at p_i.

$$(z-p_i)^{-\nu} f(z) = a_0 + a_1(z-p_i)^1 + \Omega((z-p_i)^2) \implies f(z) = (z-p_i)^{\nu}\left(a_0 + a_1(z-p_i)^1 + \Omega((z-p_i)^2)\right)$$

$$f'(z) = \nu a_0(z-p_i)^{\nu-1} + a_1(\nu+1)(z-p_i)^{\nu} + ...$$

$$\frac{f'(z)}{f(z)} = \frac{(z-p_i)^{\nu-1}(-\nu a_0 + \Omega(z-p_i)}{(z-p_i)^{\nu-1}(a_0(z-p_i) + \Omega\left((z-p_i)^2\right)} = \frac{\nu}{z-p_i} + \Omega(1)$$

Hence, the residue is by definition $\mathrm{Res}_{p_i}(\frac{f'}{f}) = \nu$. Hence, the residue theorem yields that $\frac{1}{2\pi i}\oint_U \frac{f'(z)}{f(z)}dz = \sum_{p_i} \nu_{p_i} f = \sum_{p\in U} \nu_p f$. $\square$

Lemma 2.3. *The integral of f on the domain U is*

$$\frac{1}{2\pi i}\oint_U \frac{df}{f} = -\nu_\infty(f) - \frac{1}{3}\nu_\rho(f) - \frac{1}{2}\nu_i(f) + \frac{2k}{12}$$

Proof. This expression follows from a trick-less computation of the contour integral: we integrate line by line.

First, note that for $z \in [A, B]$, $z+1 \in [D', E]$ and modularity yields $f(z+1) = f(z)$. Because of the direction we integrate in, $\int_A^B \frac{df}{f} + \int_{D'}^E \frac{df}{f} = 0$.

A similar phenomenon appears when looking at $[B', C]$ and $[D, D']$, since then $f(Sz) = z^k f(z)$. After applying the differential, $df(Sz) = d(z^k)f(z) + z^k df = k\frac{dz f(Sz)}{z} + \frac{df(z)f(Sz)}{f(z)}$, so

$$\frac{df(Sz)}{f(Sz)} = k\frac{dz}{z} + \frac{df(z)}{f(z)}$$

We can now compute the integrals

$$\int_{B'}^C \frac{df}{f} + \int_{C'}^D \frac{df}{f} = \int_{B'}^C \frac{df(z)}{f(z)} - \frac{df(Sz)}{f(Sz)} = -k\int_{B'}^C \frac{dz}{z}$$

As the radii of the arcs go to 0, the arc $[B', C]$ is $1/12$ of the unit circle. Hence, by switching to polar coordinates, one evaluates the integral to be $-1/12$. Thus $\int_{B'}^C + \int_{C'}^D \frac{df}{f} = \frac{k}{12}$.

Now, we look at $[E, A]$. Note that when applying the change of variables $q = e^{2\pi i z}$, this segment becomes a circle $\mathcal{C}$ of some radius < 1 centered at 0. Note also that the zeroes and poles of f are all contained in U, so the circle $\mathcal{C}$ does not contain any zero or pole other than at $q = 0$. Thus, by the residue theorem and being careful about orientation,

$$\frac{1}{2\pi i}\int_E^A \frac{df}{f} = \frac{1}{2\pi i}\oint_{\mathcal{C}} \frac{df}{f} = -\nu_0(\hat{f}) = -\nu_\infty(f)$$

Finally, we look at the arcs. Recall that we made the circle contaning $[B, B']$ big enough for it to contain ρ, but small enough so that it was the only zero or pole inside. Hence, $\frac{1}{2\pi i}\oint_{\mathcal{C}} \frac{df}{f} = -\nu_\rho(f)$. Now however, notice that the angle $B\rho B'$ is $\frac{\pi}{3}$ by a geometry exercise, so $\frac{1}{2\pi i}\int_B^{B'} \frac{df}{f} = -\frac{1}{6}\nu_\rho(f)$. Similarly, $\frac{1}{2\pi i}\int_C^{C'} \frac{df}{f} = -\frac{1}{2}\nu_i(f)$, and $\frac{1}{2\pi i}\int_D^{D'} \frac{df}{f} = -\frac{1}{6}\nu_\rho(f)$.

Combining, we have

$$\frac{1}{2\pi i}\oint_U \frac{df}{f} = -\nu_\infty(f) - \frac{2}{6}\nu_\rho(f) - \frac{1}{2}\nu_i(f) + \frac{2k}{12}$$

□

The theorem now follows:

proof of Theorem. The theorem follows from the previous lemmas. Indeed,

$$\frac{1}{2\pi i}\oint_U \frac{df}{f} = \frac{1}{2\pi i}\oint_U \frac{f'(z)}{f(z)}dz = \sum_{\rho\in X(G)}^{*} \nu_\rho(f)$$

and also

$$\frac{1}{2\pi i}\oint_U \frac{df}{f} = -\nu_\infty(f) - \frac{1}{3}\nu_\rho(f) - \frac{1}{2}\nu_i(f) + \frac{2k}{12}$$

Rearranging the term gives the desired result. □

There is one hickup before we continue. It might have been the case that f has a pole or zero on the walls on the fundamental domain. In this case, make circles around the point of interest. Since T will map one to the other and they have opposite orientation, they cancel each other's contribution.

3 The Spaces

We are now equipped to prove our main results. First, we recall some basic objects.

Definition 3.1. A cusp form is a modular form with $f(\infty)$. The set is written S_k, which by definition is the kernel of the functional $f \to f(\infty)$.

By the first isomorphism theorem, one sees that the quotient M_k/S_k is either 0 or $\mathbb{C}$, depending if that functional was surjective or not. Note that it is either the 0 map, or surjective, since $wf \to wf(\infty)$ for $w \in \mathbb{C}$. Hence, $dim(M_k/S_k) = 0$ or 1.

Recall from earlier talks that for $2k \geq 2$, one can define the Eisenstein series, a modular form of weight $2k$. Also, $E_k(\infty) \neq 0$. So in fact, when Eisenstein series exist, we have $M_k = S_k \oplus \mathbb{C}E_k$.

For an example of a cusp form, the discriminant $\Delta = g_2^3 - g_3^2$ is a cusp form of weight 12 (where $g_2 = 60G_4$, and $g_3 = 140G_6$).

We now have every character ready to act in the proof of our main theorem. The tightness we wish to use to describe the sets comes from the formula proven earlier.

Theorem 3.1. *For $k < 0$ and $k = 2$, $M_k = 0$.*

Proof. Recall the formula describing the order of poles and zeroes of a form of weight $2k$.

$$\nu_\infty(f) + \frac{1}{2}\nu_i(f) + \frac{1}{3}\nu_\rho(f) + \tilde{\sum_{\rho \in X(G)}} \nu_\rho(f) = \frac{2k}{12}$$

Now, f is holormophic on the upper half plane and at infinity, so $\nu_p(f) \geq 0$. Hence the LHS must be positive, so $k \geq 0$. This shows there are no forms of weight $k < 0$. If $k = 2$, we must find a positive triple that solves $a + \frac{b}{2} + \frac{c}{3} = \frac{1}{6}$. It is easy to see that there are none. This shows the first bullet. □

Theorem 3.2. *If $f \in M_{2k-12}$, $\Delta f \in S_{2k}$. The map is an isomorphism.*

Proof. We can apply this formula in the other direction: to get information about a specific form. For example, there is a unique solution to $a + \frac{b}{2} + \frac{c}{3} = \frac{1}{3}$. Hence, modular forms of weight 4 satisfy $\nu_\infty(f) = 0 = \nu_i(f)$ and $\nu_\rho(f) = 1$. This is the case for G_4. We get similar results for G_6. Combining, we see that Δ has $\nu_\infty(\Delta) = 1$ and $\nu_p(\Delta) = 0$ for all other p. Hence the discrimant function is not vanishing on the upper half plane.

Now, let we show division by Δ is the inverse to the map we wish to show is an isomorphism. It is well defined since Δ is non vanishing. If $g \in S_k$ and $f = g/\Delta$, then f has weight $k - 12$. We have to show f is holomorphic on the plane and at infinity. This again is done using the formula, since $\nu_p(g) = \nu_p(f) - \nu_p(\Delta)$. So $\nu_p(g) = \nu_p(f)$ for $p \neq \infty$, and $\nu_p(f) - 1$ at $p = \infty$. But g was a cusp form, so $\nu_\infty(g) > 0$. Hence f is a modular form of weight $k - 12$, and multiplication by Δ is an isomorphism. □

Theorem 3.3. *For $k \geq 0$, there are 2 cases. If $k \equiv 2 \pmod{12}$, then $dim(M_{2k}) = \lfloor 2k/12 \rfloor$. Otherwise, $dim(M_{2k}) = \lfloor 2k/12 \rfloor + 1$.*

Proof. Let k be one of $0, 2, 3, 4, 5$. Then, $S_{2k} \cong M_{2k-12} = 0$. Thus the M_{2k} have dimension 1. We know already forms in these: the Eisenstein series. Hence $M_{2k} = \mathbb{C}E_{2k}$.

To go to higher k, notice that $S_{2k} \cong M_{2k-12}$ and $S_{2k-12} = M_{2k-12} \oplus \mathbb{C}$ pair to give $dim(M_{2k}) = dim(M_{2k-12}) + 1$. The result is proven. □