189-457B: Abstract Algebra
Practice Midterm Exam

The first four questions are worth 25 *points, and the last is worth* 10 *points of extra credit.*
The final grade will be out of 100.

1. Let $G = \{1, r, r^2, r^3, V, H, D_1, D_2\}$ be the dihedral group of order 8, where 1 is the identity transformation, r is the counterclockwise rotation by an angle of 90 degrees, and V, H, D_1 and D_2 are the reflections about the vertical, horizontal and two diagonal axes of symetry of the square, respectively.

Write down the character table of G.

2. Let G be a finite group in which every element is conjugate to its inverse.

(a) Give an example of a group G satisfying this condition.

(b) Show that the character of any complex representation of a group G satisfying this condition is real-valued, i.e., that all the entries in the character table for G are real.

3. Let G be a finite group and let $\varrho : G \longrightarrow \mathrm{GL}_n(\mathbf{R})$ be a homomorphism from G to the group of invertible $n \times n$ matrices with real entries. For any integer $t \geq 1$, show that the matrix

$$M = \sum_{\mathrm{order}(g)=t} \varrho(g),$$

where the sum is taken over all the elements of G of order t, is diagonalisable. (Hint: consider the adjoint of M relative to a suitable inner product.)

4. Let χ be the character of a two-dimensional representation of a finite group G, and assume that g is an element of G of order 4 for which $\chi(g) = 0$. Show that $\chi(g^2)$ is equal to 2 or -2.

The following question will count as extra credit (10 points) for those who do it successfully.

5. Let D_8 be the dihedral group of order 8, and let Q be the quaternion group of order 8. Show that the complex groups rings $\mathbf{C}[D_8]$ and $\mathbf{C}[Q]$ are isomorphic, but that the group rings $\mathbf{R}[D_8]$ and $\mathbf{R}[Q]$ are not isomorphic.