Math 457B: Honors Algebra 4

Final Exam

Thursday, April 22, 2:00-5:00 PM

The 7 questions in this exam are each worth 15 points. The final grade will be out of 100.

1. Give examples of extensions E/F of fields satisfying the following properties:

(a) E/F is separable but not normal.
(b) E/F is normal but not separable.
(c) E/F is neither normal nor seperable.

2. Let E be the splitting field of the polynomial $x^4 - 2$ over the field $\mathbf{Q}$ of rational numbers. Compute the Galois group G of E, and give a complete list of the subfields of E along with the subgroups of G to which they correspond under the Galois correspondence.

3. Write down the character table of the permutation group S_4 on 4 elements.

4. Let F be a field, let $f(x) \in F[x]$ be a seperable irreducible polynomial of degree 4, and let r_1, r_2, r_3, and r_4 be the roots of f in a splitting field E of f over F.. Show that the element $r_1 r_2 + r_3 r_4$ is the root of a cubic polynomial in $F[x]$.

5. Let G be a finite group, let $\varrho : G \to GL_n(\mathbf{C})$ be an irreducible representation of G, let χ be its associated character, and let $f : G \to \mathbf{C}$ be a class function on G. Show that

$$\frac{1}{\#G}\sum_{f\in G}\overline{f(g)}\varrho(g) = \langle f, \chi\rangle \frac{I_n}{n},$$

where I_n is the identity matrix, and

$$\langle f_1, f_2\rangle = \frac{1}{\#G}\sum_{g\in G}\overline{f_1(g)}f_2(g),$$

for any two class functions f_1 and f_2 on G.

6. Let $G = \mathbf{GL}_3(\mathbf{F}_2)$ be our favorite simple group of order 168.

(a) Construct two transitive G-sets X_1 and X_2 of cardinality 7.

(b) Show that the linear permutation representations $\mathbf{C}[X_1]$ and $\mathbf{C}[X_2]$ of G are isomorphic as representations of G.

(c) Let E/F be a Galois extension with Galois group G. Show that E contains two extensions of F of degreee 7 over F which are not isomorphic to each other.

7. Let F be a field of prime characteristic p. A polynomial in $F[x]$ is said to be *linear* if it is of the form

$$f(x) = a_n x^{p^n} + a_{n-1}x^{p^{n-1}} + \cdots + a_1 x^p + a_0 x,$$

with $a_j \in F$.

(a) Give a necessary and sufficient condition for $f(x)$ to be seperable.

(b) When $f(x)$ is seperable, show that the Galois group of $f(x)$ is a subgroupof $\mathbf{GL}_n(\mathbf{F}_p)$, where $\mathbf{F}_p$ is the finite field with p elements.

(c) Suppose that $F = \mathbf{F}_p$, and that $f(x)/x$ is irreducible. What is the Galois group of $f(x)$?