McGill University
Faculty of Science Practice Final examination

Abstract Algebra 4

Math 457-B

INSTRUCTIONS.

This is a practice exam to help in preparing you for the final exam. It will not be graded but you are encouraged to come to my office hours for feedback on your solutions.

1. Please write your answers clearly in the exam booklets provided.
2. You may quote any result/theorem seen in the lectures or in the assignments without proving it (unless, of course, it is what the question asks you to prove).
3. This is a closed book exam.
4. Translation dictionary is not permitted.
5. Calculators are not permitted.

This exam comprises the cover page and two pages of questions, numbered 1 to 10. Each question is worth 10 points.

1. Let R be a ring and let M be a module over R.
a) State the definition of what it means for a module to be free over R.
b) Give an example to show that an R-submodule of a free module need not be free.
c) Suppose that M has a free submodule M', and that the quotient M/M' is also free over R. Show that M is a free R-module.

2. Recall that a module is simple if it contains no non-zero proper R submodules, and that it is semisimple if it is a direct sum of simple modules. Let F be a field and let $R = F[x]$ be the polynomial ring over F.
a) Show that every module over F is semisimple.
b) Give an example of an R-module which, when viewed as a vector space over $F \subset R$, is two-dimensional over F, and is not semisimple over R.

3. Compute the Smith Normal Form of the matrix

$$M = \begin{pmatrix} 26 & 44 & 18 \\ 8 & 14 & 6 \\ 12 & 12 & 12 \end{pmatrix}.$$

What is the quotient $(\mathbf{Z}^3/M\mathbf{Z}^3)$ isomorphic to as a $\mathbf{Z}$-module (i.e., an abelian group?)

4. Let $F \subset L \subset K$ be field extensions. Show that

$$[K : F] = [K : L][L : F].$$

5. Let $F \subset L \subset K$ be field extensions. If L is Galois over F and K is Galois over L, is it necessarily true that K is Galois over F? If yes, prove this assertion, and if no, give a example in which K is not Galois over F.

6. Let F be a field of characteristic p and let $f(x)$ be a polynomial in $F[x]$ of the form

$$f(x) = a_0 x + a_1 x^p + a_2 x^{p^2} = 0, \qquad \text{with } a_0 \neq 0.$$

a) Show that $f(x)$ is seperable and conclude that the splitting field K of $f(x)$ over F is a Galois extension of F.
b) Show that the set W of roots of $f(x)$ in K is closed under addition and mutiplication by elements of the finite field with p elements, $\mathbf{F}_p := \mathbf{Z}/p\mathbf{Z} \subset F$. Conclude that W has the structure of a vector space over the finite field $\mathbf{F}_p$. What is its dimension?
c) Show that the Galois group of $f(x)$ is isomorphic to a subgroup of the group $\mathbf{GL}_2(\mathbf{F}_p)$ of 2×2 matrices with entries in $\mathbf{F}_p$.

7. Consider the polynomial $f(x) = x^5 - 3$ in $\mathbf{Q}[x]$.
a) Show that $f(x)$ is irreducible, and write down its splitting field K over $\mathbf{Q}$.
b) Show that K contains the field $L = Q(\zeta_5)$ generated by the fifth roots of unity.
c) Show that $\mathrm{Gal}(K/L)$ is a normal subgroup of $G := \mathrm{Gal}(K/\mathbf{Q})$, of cardinality 5.

8. Show that there are no field extensions of the field $\mathbf{R}$ of real numbers which are of odd degree over $\mathbf{R}$.

9. Let k be a field and let $K = k(x_1, dots, x_n)$ be the field of *rational functions* in the n indeterminates $x_1, \ldots, x_n$ over k. Let $F = k(s_1, \ldots, s_n)$ be the subfield generated by the n elementary symmetric polynomials

$$s_1 := x_1 + \cdots + x_n, \quad s_2 = x_1x_2 + x_1x_3 + \cdots + x_{n-1}x_n,$$

$$s_n = x_1 \cdots x_n$$

in the indeterminates $x_1, \ldots, x_n$. Show tat K is a Galois extension of F. What is the Galois grop of K/F?

10. Let $f(x)$ and $g(x)$ be polynomials with rational coefficients, of degrees 4 and 3 respectively, and let $h = f(g(x))$ be the degree 12 polynomial obtained by composing f with g.
a) Show that the splitting field of h over $\mathbf{Q}$ has degree at most $31104 = 4! \cdot (3!)^4$.
b) Show that the equation $f(x) = 0$ is solvable by radicals.