189-457B: Algebra 4

Assignment 5

Due: Wednesday, April 5

1. List all the subfields of the field $\mathbf{Q}(\zeta)$ generated by a primitive 16th root of unity ζ.

2. Show that the symmetric group S_{12} contains subgroups of cardinalities 31104 and 82994. (Hint: $31104 = (3!)^4 \cdot 4!$ and $82994 = (4!)^3 \cdot 3!$.) Explain how you might try to go about constructing degree 12 polynomials with those Galois groups.

Each of the following questions depends on the previous ones. The goal of the series is to guide you towards the proof of the following beautiful theorem of Galois: *"Pour qu'une équation de degré premier soit résoluble par radicaux, il faut et il suffit que deux quelconques de ses racines étant connues, les autres s'en déduisent rationnellement."* (Evariste Galois, Bulletin de M. Férussac, XIII (avril 1830), p. 271).

3. Let G be a transitive subgroup of the symmetric group S_n on n letters, and let H be a normal subgroup of G. Show that the action of G on the set $X := \{1, \ldots, n\}$ induces a natural action of G on the set

$$X_H := \{Hx, \quad x \in X\}$$

of subsets of X consisting of the orbits for H in X. Use this to conclude that all the H-orbits in X have the same cardinality. Give an example to illustrate the failure of this conclusion when H is not assumed to be normal in G.

4. Let p be a prime number. Show that any non-trivial normal subgroup of a transitive subgroup of S_p also acts transitively on $\{1, \dots, p\}$.

5. Show that any transitive subgroup of S_p contains a non-trivial Sylow p subgroup, of cardinality p.

6. Let G be a transitive subgroup of S_p and let H be a non-trivial normal subgroup of G. Show that any Sylow p-subgroup of G is also contained in H. (Hint: remember your Sylow theorems!)

7. Show that any transitive solvable subgroup of S_p contains a *unique* Sylow p subgroup, and hence is contained in the normaliser of its Sylow p-subgroup.

8. After identifying $X := \{1, \dots, p\}$ with $\mathbf{Z}/p\mathbf{Z}$, show that the normaliser of the Sylow p-subgroup generated by the cyclic permutation $T : x \mapsto x + 1$ is the group of affine linear transformations of the form $x \mapsto ax + b$ with $a \in (\mathbf{Z}/p\mathbf{Z})^\times$ and $b \in \mathbf{Z}/p\mathbf{Z}$. (Hint: show that any element σ in this normaliser satisfies the functional equation

$$\sigma(x + 1) = \sigma(x) + a, \quad \text{for all } x \in X,$$

for some a depending on σ. Now set $b := \sigma(0)$ and derive a closed expression for σ by induction on x.)

9. Show that any transitive solvable subgroup of S_p is conjugate to a subgroup of the group of affine linear transformations of cardinality $p(p-1)$ described in exercise 8.

10. Prove the theorem of Galois quoted above: an irreducible polynomial f of prime degree p is solvable by radicals if and only if the splitting field of f is generated by any two roots of f.