189-457B: Algebra 4
Assignment 4
Due: Wednesday, March 22

1. Does every extension K of degree 4 over a field F contain a sub-extension of degree 2 over F? If yes, prove your statement, and if no, give a counterexample.

The following series of exercises is meant to get you to prove that every finite group occurs as the Galois group of some finite extension K/F, where F is itsef a finite extension of $\mathbf{Q}$.

2. Let p be a prime number and let S_p denote the group of permutations on p elements. Show that any subgroup $G \subset S_p$ that contains a transposition (i.e., a permutation that interchanges two elements and leaves all others fixed) and a permutation of order p is necessarily equal to S_p. Show that the same conclusion holds if G contains a transposition and acts *transitively* on $\{1, 2, \ldots, p\}$. Finally, show that this conclusion is *false* if the prime p is replaced by an even integer > 2.

3. Using the result in part 2, show that any irreducible polynomial in $\mathbf{Q}[x]$ of degree p having exactly $p-2$ real roots has Galois group S_p over $\mathbf{Q}$ (i.e., the automorphism group of its splitting field acts on the roots of this polynomial as the full permutation group on p elements.)

4. If $f(x) \in \mathbf{R}[x]$ is any polynomial having exactly k *distinct* real roots, show that there exists $\epsilon > 0$ for which $f(x) + a$ has exactly k real roots, for all $a \in \mathbf{R}$ with $|a| < \epsilon$. Give an example to show that the assumption that the roots of $f(x)$ are distinct is essential for the conclusion to hold.

5. For any n, show that there is an irreducible polynomial in $\mathbf{Q}[x]$ of degree n having exactly $n-2$ real roots. (Hint: starting with any polynomial $f(x) \in \mathbf{Q}[x]$ with exactly $n-2$ distinct real roots, exercise 4 shows that $f(x)+a$ has the same property for infinitely many $a \in \mathbf{Q}$. Now, make a judicious choice of $f(x) \in \mathbf{Z}[x]$ and $a \in \mathbf{Q}$ for which the Eisenstein irreducibility criterion can be applied.)

6. Show that every finite groupcan be realised as a subgroup of S_p for a large enough prime p. Conclude from this and from what you have done in exercises 2-5 that every finite group is the Galois group of an extension K/F, where F is itself a finite extension of $\mathbf{Q}$.

Remark: It is natural to ask whether any finite group can occur as the Galois group of a finite extension of $\mathbf{Q}$. This is widely believed to be the case, but a proof is not known. The problem of realising any finite group as a Galois group of an extension of $\mathbf{Q}$ is known as the "inverse problem of Galois theory", and is one of the most famous open problems in the subject.

7. Show that any sequence of elements $x_i \in \bar{\mathbf{Q}}$ (or in any field, for that matter) that satisfies the recursion

$$x_{i-1}x_{i+1} = x_i + 1, \qquad \text{for all } i \geq 1$$

is necessarily periodic of period 5. If $(x_1, x_2, x_3, x_4, x_5)$ is the basic period in such a sequence, show that the 5 elementary symmetric functions of $x_1, \ldots, x_5$ can be expressed in terms of the quantities

$$s := x_1 + \cdots + x_5, \qquad t = x_1x_2 + x_2x_3 + x_3x_4 + x_4x_5 + x_5x_1$$

by the formulae

$$x_1 + \cdots x_2 = s; \qquad x_1x_2 + x_1x_3 + \cdots x_4x_5 = s + t + 5;$$

$$x_1x_2x_3 + \cdots x_1x_2x_4 + \cdots + x_3x_4x_5 = s^2 + s - 2t - 5;$$

$$x_1x_2x_3x_4 + \cdots + x_2x_3x_4x_5 = 2s + t + 5; \qquad x_1x_2x_3x_4x_5 = s + 3.$$

8. Using exercise 7, show that the degree 5 polynomial

$$x^5 - sx^4 + (s + t + 5)x^3 - (s^2 + s - 2t - 5)x^2 + (2s + t + 5)x - (s + 3)$$

with coefficients in the field $\mathbf{Q}(s, t)$ of rational functions in the two indeterminates s and t has Galois group contained in the dihedral group D_{10} of cardinality 10.

9. *Experimental exercise.* This exercise assumes some familiarity with a computer algebra package like Pari/GP, which is available for free on the internet. Enter the three-variable polynomial

$$\mathtt{f} = \mathtt{x}^5 - \mathtt{s} * \mathtt{x}^4 + (\mathtt{s} + \mathtt{t} + 5) * \mathtt{x}^3 - (\mathtt{s}^2 + \mathtt{s} - 2 * \mathtt{t} - 5) * \mathtt{x}^2$$
$$+ (2 * \mathtt{s} + \mathtt{t} + 5) * \mathtt{x} - (\mathtt{s} + 3)$$

in the Pari command line, and ask Pari to calculate the Galois group of the specialised polynomial as the parameters s and t range over all integer values between 1 and 10, by typing something like

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for(s = 1, 10, for(t = 1, 10, print(polgalois(eval(f)))))
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What do you observe?