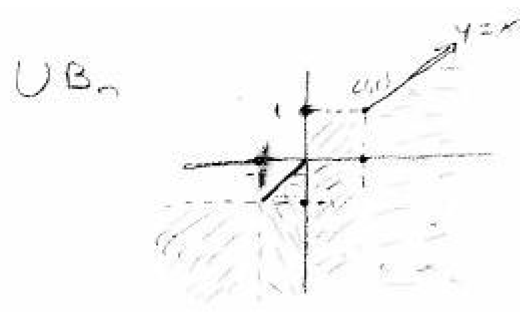
MATH 235: Assignment 1 solutions

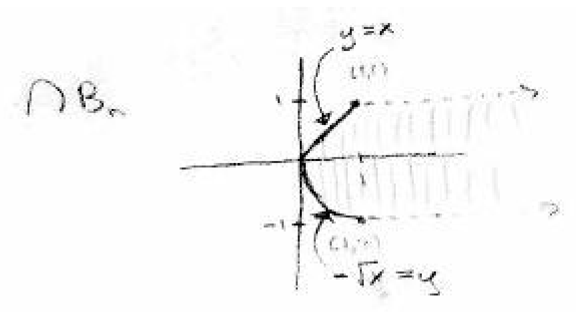
2.*a*) $\bigcup_{n=1}^{N}[-n,n]=[-N,N]$, $\bigcap_{n=1}^{N}[-n,n]=[-1,1]$.
b) $\bigcup_{n=1}^{\infty}[n,n+1]=[1,\infty)$, $\bigcup_{n=1}^{\infty}(n,n+2)=(1,\infty)$.
c) $\bigcup_{n=1}^{\infty}(n,n+1)=(1,\infty)\backslash\mathbb{N}$, $\bigcup_{n=1}^{\infty}(1/n,1]=(0,1]$
d) $\bigcap_{n=1}^{\infty}A_n=\{0,1\}$ (Note that $0\in\mathbb{N}$ and hence in this intersection, many people forgot this. Also the definition of A_n runs through ALL $x\in\mathbb{N}$, not just 1 element as some of you were confused about. For example $A_2=\{0,1,4,9,\ldots\}$ contains all perfect squares in $\mathbb{N}$.

e) The best way to solve this problem is to draw the sets for a few small n (both even and odd!) and try to observe limiting and intersecting points of the curves $y=x^n$, then prove these limits hold. For example, every curve contains $\{(0,0),(1,1)\}$. When $x\geq 1$, the curves $y=x^n$ all lie "underneath" the curve $y=x$ and as n gets large, they approach the segment $(1,\infty)\times\{1\}$ from above (and also $(1,\infty)\times\{-1\}$ from below for even n). This tells you all you need to know about the intersection / union for the parts where $x\geq 1$. When $0<x<1$, all the curves lie "above" $y=x$ (or below $y^2=x$ in the even case and $y<0$) and the points on the line segment $(0,1)\times\{1\}$ (also $(0,1)\times\{-1\}$ in even case) are never attained but are limits of the A_n (in the even case). Again this tells you what you need for $0<x<1$. Clearly when $x=0$, the only point in the intersection is $(0,0)$ while $\{0\}\times(-\infty,0]$ lies in the union. For $x<0$, there will be no points in the intersection because A_n is disjoint from this set when n even. For the union however, again observe that the curves are bounded from above by $y=x$ for $-1\leq x\leq 0$ and approach the limiting ray $(-\infty,-1)\times\{-1\}$ from below while never actually attaining it. We conclude:

$$\begin{aligned}\bigcup B_n &= (-\infty,-1)\times(-\infty,-1)\cup\{(x,y):-1\leq x\leq 0\ \&\ y\leq x\}\\ &\quad\cup(0,1)\times(-\infty,1)\cup\{(x,y):1\leq x\ \&\ y\leq x\}\\ \bigcap B_n &= \{(x,y):0\leq x\leq 1\ \&\ -\sqrt{x}\leq y\leq x\}\cup(1,\infty)\times(-1,1)\end{aligned}$$

Below is a simple sketch of the sets. It is a good check of your understanding to make sure you can identify the diagram with what I wrote above.





3. Proof by induction: Since it is clear that $0^3 = 0 = 0^2$, the base case is obvious. Assume the claim holds for all $0 \leq n < N$. Then

$$\begin{aligned}
\sum_{i=0}^{N} i^3 &= \sum_{i=0}^{N-1} i^3 + N^3 \\
&= (1 + \ldots + (N-1))^2 + N^3 \\
&= \frac{N^2(N-1)^2}{2^2} + N^3 \qquad \text{By Euler's formula} \\
&= \frac{N^2(N^2 - 2N + 1 + 4N)}{4} \\
&= \frac{N^2(N+1)^2}{2^2} \\
&= (1 + \ldots + (N-1) + N)^2 \qquad \text{again by Euler's formula.}
\end{aligned}$$

Thus it holds for N. Hence by induction, the formula holds for all $n \in \mathbb{N}$

4. Solution 1: As was suggested in the hint, we first compute $g(n) := f(0, n)$. It is not hard to see that $g(n) - g(n-1) = n$ because there are n points on the diagonal starting at $(0, n-1)$. Therefore, $g(n) = 0 + 1 + \ldots + n$ which is given by Euler's formula $g(m) = \frac{n(n+1)}{2}$. Now observe that $f(m, n) = f(m-1, n+1) + 1$ so long as $m > 0$. Applying this recursively, we deduce

$$f(m, n) = f(0, m+n) + m.$$

Therefore $f(m, n) = g(m+n) + m = \frac{(m+n)(m+n+1)}{2} + m$.

Solution 2: Consider the diagonals of points $D_k := \{(m, n) : m+n = k\}, k \in \mathbb{N}$. If $m + n = k$, then obviously the point (m, n) lies on the k-th diagonal. Each diagonal contains $k + 1$ points, thus there are exactly

$$1 + 2 + \ldots + k = \frac{k(k+1)}{2} = \frac{(m+n)(m+n+1)}{2}$$

points counted before the k-th diagonal (since all points on diagonals less than k are counted before the points on D_k). (m, n) is the $(m+1)$-th point counted on D_k, and $f(m, n)$ is equal to the number of points counted up to (m, n) minus 1, because we start at 0. Therefore

$$f(m, n) = \frac{(m+n)(m+n+1)}{2} + (m+1) - 1 = \frac{(m+n)(m+n+1)}{2} + m.$$

Note: Solution 1 and solution 2 are virtually identical in their logic if not their wordings. I included both because they were the two most common approaches used in the solutions handed in. However the statement "$g(n) - g(n-1) = n$ because there are n points on the diagonal starting at $(0, n-1)$" is something that was missing from virtually everyone's proof that was similar to solution 1. It is good to recognize patterns, but you need to explain why they are occurring so that you can be sure they will continue.

5.a) $y = x^3 + 3x + 1$, so using variables a, b, c, d for the coefficients, we have $a = 1, b = 0, c = 3, d = 1$. Thus $\Delta = -4(1)(3)^3 - 27(1)^2(1)^2 < 0$, so there is 1 real root. Via Cardano's method for the depressed cubic we have

$$\begin{aligned} t_1 &= u + v \\ &= \sqrt[3]{-\frac{1}{2} + \sqrt{\frac{1}{4} + \frac{27}{27}}} + \sqrt[3]{-\frac{1}{2} - \sqrt{\frac{1}{4} + \frac{27}{27}}} \\ &= \sqrt[3]{\frac{-1 + \sqrt{5}}{2}} + \sqrt[3]{\frac{-1 - \sqrt{5}}{2}} \end{aligned}$$

b) $y = x^3 - 3x + 1$, so this time $\Delta = -4(1)(-3)^3 - 27(1)^2(1)^2 > 0$. Hence there are 3 real roots. Thus we also need to solve for variables u, v.

$$\begin{aligned} t_1 &= u + v \\ &= \sqrt[3]{-\frac{1}{2} + \sqrt{\frac{1}{4} - \frac{27}{27}}} + \sqrt[3]{-\frac{1}{2} - \sqrt{\frac{1}{4} - \frac{27}{27}}} \\ &= \sqrt[3]{\frac{-1 + i\sqrt{3}}{2}} + \sqrt[3]{\frac{-1 - i\sqrt{3}}{2}} \\ &= \sqrt[3]{e^{2\pi i/3}} + \sqrt[3]{e^{-2\pi i/3}}. \end{aligned}$$

There are obviously different choices of u and v that we can take (depending on choice of cube root) but we must have $uv = -(-3)/3 = 1$. Thus the 3 possible choices are

$$(u, v) = (e^{2\pi i/9}, e^{-2\pi i/9}), (e^{8\pi i/9}, e^{-8\pi i/9}), (e^{-4\pi i/9}, e^{4\pi i/9}).$$

Therefore the 3 real roots are

$$\begin{gathered} 2\cos(2\pi/9) \\ 2\cos(4\pi/9) = 2(2\cos(2\pi/9)^2 - 1) \\ 2\cos(8\pi/9) = 2(2(2\cos(2\pi/9)^2 - 1)^2 - 1. \end{gathered}$$

6. a) There are numerous options to pick, just remember that surjective means every element of B needs to be hit. An example solution is

$$\begin{gathered} f_1 = \{(1, a), (2, b), (3, c), (4, c)\} \\ f_2 = \{(1, a), (2, b), (3, c), (4, b)\} \\ f_3 = \{(1, a), (2, b), (3, c), (4, a)\} \\ f_4 = \{(1, c), (2, b), (3, c), (4, a)\} \end{gathered}$$

(Notation: Here the map f_1 represented as a set of pairs means $f_1(1) = a, f_1(2) = b$, etc.)

b) Again, here is an example

$$\begin{aligned} f_1 &= \{(a,1),(b,2),(c,3)\} \\ f_2 &= \{(a,1),(b,2),(c,4)\} \\ f_3 &= \{(a,4),(b,2),(c,3)\} \\ f_4 &= \{(a,1),(b,4),(c,3)\} \end{aligned}$$

c) A function is determined by where it sends each element. There are 4 elements of A, and each can go to any of the 3 elements of B. Hence there are $3^4 = 81$ total functions from A to B.
d) A surjective function from A to B is determined by the following choices: (1) Which element of B gets hit twice? (2) Which 2 elements of A hit the element chosen in (1)? (3) Where do the remaining two elements of A go?
There are 3 choices for (1), 6 choices for (2) and 2 choices for (3) (Note that to be surjective, the 2 remaining elements must go to 2 distinct points). Thus the total number of surjective functions is $6 \cdot 3 \cdot 2 = 36$.
e) Because the cardinality of A is larger than that of B, there are no injective functions from A to B.
f) Same reasoning as (*c*), there are $4^3 = 64$ functions from B to A.
g) There are no surjective functions from B to A due to cardinality of A larger than that of B.
h) As in *d*, this comes down to counting our choices. An injective function from B to A is determined by: (1) which element of A is not hit? (2) Which of the remaining 3 elements does a hit? (3) Which of the remaining 2 elements does b hit? There are 4 choices for (1), 3 choices for (2), and 2 choices for (3). Hence there are 24 injective functions.

7. *a*) Let $J : X \to X$ be the function $x \to f(g(x))$ and let $K : X \to X$ be the function $x \to g(h(x))$. Then

$$\begin{aligned} (f \cdot J)(x) &= f(J(x)) \\ &= f(g(h(x)) \\ &= K(h(x)) \\ &= (K \cdot h)(x) \end{aligned}$$

Therefore $f \cdot (g \cdot h) = (f \cdot g) \cdot h$.

b) There are many easy choices that work here. The important thing is to make sure that you define X and your two functions actually map X into itself (many of you did not do this). For example:

$$X = \mathbb{R}, f(x) = x^2, g(x) = x + 1.$$

Then $f(g(1)) = 4$ and $g(f(1)) = 2$. Note that I showed these two functions are not equal by evaluating them at a point. This is better than simply writing out the formula and stating "they aren't equal", because the same function can be written in many different ways. Note that for example if $X = \mathbb{R}$, the following choices of functions would be bad

$$f = \log(x), f = \sqrt{x}, f = \frac{1}{x}$$

because these functions are not defined on all of $\mathbb{R}$. If $X = \mathbb{Z}$, then the function $f = \frac{1}{2}x$ would be bad because it would not map X into itself.

8. Observe that $(1+i)^2 = 1 + 2i - 1 = 2i$. Hence

$$\begin{aligned} (1+i)^{83} &= (1+i)((1+i)^2)^{41} \\ &= (1+i)(2i)^{41} \\ &= i(1+i)(2^{41}) \qquad \text{Note } i^4 = 1 \\ &= -2^{41} + i \cdot 2^{41} \end{aligned}$$

9.

$$\begin{aligned} 321456 &= 123654 * 2 + 74148 \\ 123654 &= 74148 * 1 + 49506 \\ 74148 &= 49506 * 1 + 24642 \\ 49506 &= 24642 * 2 + 222 \\ 24642 &= 222 * 111 + 0 \end{aligned}$$

Hence $\mathrm{GCD}(321456, 123654) = 222$.

10. Let $X = \{x \in \mathbb{N} : (a+b) + x = a + (b+x), \quad \forall a, b \in \mathbb{N}\}$. We wish to show $X = \mathbb{N}$, which we will do by induction. Base case: $x = 0$. Then for all a, b we have

$$\begin{aligned} (a+b) + 0 &= a + b \qquad \text{by Peano's first axiom} \\ &= a + (b+0) \qquad \text{again by the first axiom.} \end{aligned}$$

Thus $0 \in X$. Now suppose all natural numbers up to an including N lie in X. Then

$$\begin{aligned} (a+b) + S(N) &= S((a+b)+N) \qquad \text{by Peano's second axiom} \\ &= S(a + (b+N)) \qquad \text{by induction hypothesis} \\ &= a + S(b+N) \qquad \text{second axiom} \\ &= a + (b + S(N)). \end{aligned}$$

Hence it holds for the successor of N (namely $N+1$). Thus by induction $X = \mathbb{N}$.

11. There is a very obvious injection from any set A into its power set, namely send $x \in A$ to the subset $\{x\} \in 2^A$. The existence of an injection shows $|A| \leq |2^A|$. Thus it suffices to show the sets do not have equal cardinality, ie. there does not exist a bijection between them. To prove this we suppose otherwise and derive a contradiction.
Hence suppose $\phi : A \to 2^A$ is bijective. Let $U \subset A$ be defined as the set

$$U := \{x \in A : x \notin \phi(x)\}.$$

Since ϕ is bijective, there must be some element $y \in A$ such that $\phi(y) = U$. Is y in U? If $y \in U = \phi(y)$ then by definition of U, $y \notin U$. On the other hand if $y \notin U = \phi(y)$, then by definition of U, $y \in U$. So if we assume either case, we conclude y must be both an element of U and also not an element of U. Obviously this is a contradiction and hence our assumption that such a ϕ existed in the first place cannot hold.