§2. Metaplectic Groups and Representations.

In [21] Kubota constructs a non-trivial two-fold covering group of $GL_2(\mathbb{A})$ over a totally imaginary number field. In this Section I shall describe the basic properties of such a group over an arbitrary number field and complete some details of Kubota's construction at the same time. I shall also recall Weil's construction in a form suitable for our purposes.

I start by discussing the local theory and first collect some elementary facts about topological group extensions.

Let G denote a group and T a subgroup of the torus regarded as a trivial G-space. A two-cocycle (or multiplier, or factor set) on G is a map from $G \times G$ to T satisfying

$$(2.1) \qquad \alpha(g_1 g_2, g_3)\alpha(g_1, g_2) = \alpha(g_1, g_2 g_3)\alpha(g_2, g_3)$$

and

$$(2.2) \qquad \alpha(g, e) = \alpha(e, g) = 1$$

for all g, g_i in G. In addition, if G is locally compact, α will be called Borel if it is Borel measurable.

Following Moore [28], let $Z^2(G,T)$ denote the group of Borel 2-cocycles on G and let $B^2(G,T)$ denote its subgroup of "trivial" cocycles (cocycles of the form $s(g_1)s(g_2)s(g_1g_2)^{-1}$ with s a map from G to T). Then the quotient group $H^2(G,T)$ (the two-dimensional cohomology group of G with coefficients in T) represents equivalence classes of topological coverings groups of G by T which are central as group extensions.

To see this, let α be a representative of the cohomology class $\bar{a}$ in $H^2(G,T)$. Form the Borel space $G \times T$ and define multiplication in $G \times T$ by

$$(2.3) \qquad \{g_1, \zeta_1\}\{g_2, \zeta_2\} = \{g_1 g_2, \alpha(g_1, g_2)\zeta_1\zeta_2\} \quad .$$

One can check that $G \times T$ is a standard Borel group and that the product of Haar measures on G and T is an invariant measure for $G \times T$. Thus by [25] $G \times T = \overline{G}$ admits a unique locally compact topology compatible with the given Borel structure.

Note that the natural maps from T to $\overline{G}$ and $\overline{G}$ to G are continuous. (They are homomorphisms, and obviously Borel, hence they are automatically continuous.) The latter map, moreover, induces a homeomorphism of $\overline{G}/T$ with G. Thus we have an exact sequence of locally compact groups

$$1 \to T \to \overline{G} \to G \to 1 .$$

This sequence is central as a group extension <u>and depends only on the equivalence class of α</u>. Its natural Borel cross-section is $\iota \colon g \to (g, 1)$.

2.1. Local Theory.

Let F denote a local field of zero characteristic. If F is archimedean, F is $\mathbb{R}$ or $\mathbb{C}$; if F is non-archimedean, F is a finite algebraic extension of the p-adic field $\mathbb{Q}_p$.

If F is non-archimedean, let O denote the ring of integers of F, U its group of units, P its maximal prime ideal, and ϖ a generator of P. Let $q = |\varpi|^{-1}$ denote the residual characteristic of F.

The local metaplectic group is defined by a two-cocycle on $GL_2(F)$ which involves the Hilbert or quadratic norm residue symbol of F.

The Hilbert symbol $(\cdot,\cdot)$ is a symmetric bilinear map from $F^x \times F^x$ to Z_2 which takes (x,y) to 1 iff x in F^x is a norm from $F(\sqrt{y})$. In particular, (x,y) is identically 1 if y is a square. Thus $(\cdot,\cdot)$ is trivial on $(F^x)^2 \times (F^x)^2$ for <u>every</u> F and trivial on $F^x \times F^x$ itself if $F = \mathbb{C}$.

Some properties of the Hilbert symbol which we shall repeatedly use throughout this paper are collected below.

<u>Proposition 2.1.</u> (i) For each F, $(\cdot,\cdot)$ is continuous,

$$(a,b) = (a,-ab) = (a,(1-a)b) \, , \tag{2.4}$$

and

$$(a,b) = (-ab,a+b) \; ; \tag{2.5}$$

(ii) If q is odd, (u,v) is identically 1 on $U \times U$;

(iii) If q is even, and v in U is such that $v \equiv 1(4)$, then (u,v) is identically 1 on U.

The proof of this Proposition can be gleaned from Section 63 of O'meara [31] and Chapter 12 of Artin-Tate [1].

Now suppose $s = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in SL_2(F)$ and set $x(s)$ equal to c or d according if c is non-zero or not.

<u>Theorem 2.2</u>. The map α: $SL_2(F) \times SL_2(F) \to Z_2$, defined by

$$\alpha(s_1,s_2) = (x(s_1),x(s_2))(-x(s_1)x(s_2),x(s_1s_2)) \ , \tag{2.6}$$

is a Borel two-cocycle on $SL_2(F)$. Moreover, this cocycle is cohomologically trivial if and only if $F = \mathbb{C}$.

This Theorem is the main result of [20]. According to our preliminary remarks it determines an exact sequence of topological groups

$$1 \to Z_2 \to \overline{SL_2}(F) \to SL_2(F) \to 1$$

where $\overline{SL_2}(F)$ is realized as the group of pairs $\{s,\zeta\}$ with multiplication given by (2.3). The topology for $\overline{SL_2}(F)$, however, is <u>not</u> the product topology of $SL_2(F)$ and Z_2 <u>unless</u> $F = \mathbb{C}$. Indeed, suppose N is a neighborhood basis for the identity in $SL_2(F)$. Then a neighborhood basis for $\overline{SL_2}(F)$ is provided by the sets $(U,1)$ where $U \in N$ <u>and</u> $\alpha(U,U)$ is identically one.

<u>Proposition 2.3</u>. If $F \neq \mathbb{C}$, each non-trivial topological extension of $SL_2(F)$ by Z_2 is isomorphic to the group $\overline{SL_2}(F)$ just constructed.

<u>Proof</u>. If $F = \mathbb{R}$, each such extension is automatically a connected Lie group, hence isomorphic to "the" two-sheeted cover of $SL_2(F)$ obtained by factoring its universal cover by $2\mathbb{Z}$. Suppose, on the other hand, that F is non-archimedean. What has to be shown is that $H^2(SL_2(F),Z_2) = Z_2$. For this we appeal to a result of C. Moore's.

Let E_F denote the (finite cyclic) group of roots of unity in F . Consider the short exact sequence

$$1 \to Z_2 \to E_F \to E_F/Z_2 \to 1 \quad .$$

The corresponding long exact sequence of cohomology groups is

$$\cdots H^1(SL_2(F), E_F/Z_2) \to H^2(SL_2(F), Z_2) \to H^2(SL_2(F), E_F) \to$$
$$H^2(SL_2(F), E_F/Z_2) \to H^3(SL_2(F), Z_2) \to \cdots .$$

Recall that $H^1(SL_2(F), T) = \mathrm{Hom}(SL_2(F), T)$. Moreover $SL_2(F)$ equals its commutator subgroup. Therefore $H^1(SL_2(F), E_F/Z_2) = \{1\}$ and $H^2(SL_2(F), Z_2)$ imbeds as a subgroup of $H^2(SL_2(F), E_F)$. But from Theorem 10.3 of [29] it follows that $H^2(SL_2(F), E_F) = E_F$. Thus the desired conclusion follows from the fact that $H^2(SL_2(F), Z_2)$ is non-trivial and each of its elements obviously has order at most two. $\square$

Remark 2.4. As already remarked, a non-trivial two-fold cover of $SL_2(F)$ for F a non-archimedean field seems first to have been constructed by Weil in [47]. His construction, which we shall recall in Subsection 2.3, is really an existence proof. His general theory leads first to an extension

$$1 \to T \to M_p(2) \to SL_2(F) \to 1 ,$$

where T is the torus and $M_p(2)$ is a group of unitary operators on $L^2(F)$. Then it is shown that $M_p(2)$ determines a non-trivial element of order two in $H^2(SL_2(F), T)$.

Remark 2.5. In [20] Kubota constructs n-fold covers of $SL_2(F)$. His idea is to replace Hilbert's symbol in (2.6) by the n-th power norm residue symbol in F (assuming F contains the n-th roots of unity). In [29] Moore treats similar questions for a wider range of classical p-adic linear groups.

Now we must extend α to

$$G = GL_2(F) .$$

In fact we shall describe a two-fold cover of G which is a trivial non-central extension of $SL_2(F)$ by F^x, i.e. a semi-direct product of these groups.

If $g = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ belongs to G, write $g = \begin{bmatrix} 1 & 0 \\ 0 & \det(g) \end{bmatrix} p(g)$ where

$$p(g) = \begin{bmatrix} a & b \\ \frac{c}{\det(g)} & \frac{d}{\det(g)} \end{bmatrix} \in SL_2(F) \quad . \tag{2.7}$$

For g_1, g_2 in G, define

$$\alpha^*(g_1,g_2) = \alpha(p(g_1)^{\det(g_2)}, p(g_2))\, v(\det(g_2), p(g_1)) \tag{2.8}$$

where

$$s^y = \begin{bmatrix} 1 & 0 \\ 0 & y \end{bmatrix}^{-1} s \begin{bmatrix} 1 & 0 \\ 0 & y \end{bmatrix} \tag{2.9}$$

and

$$v(y,s) = \begin{cases} 1 & \text{if } c \neq 0 \\ (y,d) & \text{otherwise} \end{cases} \tag{2.10}$$

if $s = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Note that the restriction of α^* to $SL_2(F) \times SL_2(F)$ coincides with α .

Proposition 2.6. If $y \in F^x$, and $\bar{s} = \{s, \zeta\} \in \overline{SL_2}(F)$, put $\bar{s}^y$ equal to $\{s^y, \zeta v(y,s)\}$. Then $\bar{s} \to \bar{s}^y$ is an automorphism of $\overline{SL_2}(F)$, and the semi-direct product of $\overline{SL_2}(F)$ and F^x it determines is isomorphic to the covering group $\overline{G}$ of G determined by (2.10).

Proof. It suffices to prove that

$$\alpha(s_1,s_2) = \alpha(s_1^{\,y}, s_2^{\,y})\, v(y,s_1)\, v(y,s_2)\, v(y,s_1 s_2)$$

and this is verified in Kubota [21] by direct computation.

Remark 2.7. We shall refer to $\overline{G}$ as "the" metaplectic group even though there are several (cohomologically distinct) ways to extend α to G . We shall also realize it as the set of pairs $\{g, \zeta\}$ with $g \in G$, $\zeta \in Z_2$, and multiplication described by $\{g_1, \zeta_1\}\{g_2, \zeta_2\} = \{g_1 g_2, \alpha^*(g_1,g_2)\zeta_1\zeta_2\}$.

Now let $B, A, N,$ and K denote the usual subgroups of G . Thus

$$A = \left\{ \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix}, \ a_i \in F^x \right\}, \qquad N = \left\{ \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix}, \ b \in F \right\}$$

$$B = NA = \left\{ \begin{bmatrix} a_1 & b \\ 0 & a_2 \end{bmatrix} \right\}$$

and K is the standard maximal compact subgroup of G ($U(2)$ if $F = \mathbb{C}$, $O(2)$ if $F = \mathbb{R}$, and $GL(2,O)$ otherwise).

If H is <u>any</u> subgroup of G, $\overline{H}$ will denote its complete inverse image in $\overline{G}$. Moreover, if $\overline{G}$ splits over H, then $\overline{H}$ is the direct product of Z_2 and some subgroup H' of $\overline{G}$ isomorphic to H. We shall denote H' by H even though H' need not be uniquely determined by H.

In general, it is important to know whether or not $\overline{G}$ splits over the subgroups listed above. In particular, the Proposition below is useful in constructing the global metaplectic group. (Recall that if x belongs to a non-archimedean field its <u>order</u> is defined by the equation $x = \omega^{\mathrm{ord}(x)} u$, $u \in U$.)

<u>Proposition 2.8</u>. Suppose $F \neq \mathbb{C}$ or $\mathbb{R}$ and N (as usual) is a positive integer divisible by 4. Then $\overline{G}$ splits over the compact group

$$K^N = \{ [\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}] \in K\colon \ a \equiv 1, \ c \equiv 0 (\mathrm{mod}\ N) \} .$$

More precisely, for $g = [\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}] \in G$, set

$$(2.11) \qquad s([\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}]) = \begin{cases} (c, d\ \det(g)) & \text{if } cd \neq 0 \text{ and } \mathrm{ord}(c) \text{ is odd} \\ 1 & \text{otherwise} \end{cases} .$$

Then <u>for all $g_1, g_2 \in K^N$</u>,

$$(2.12) \qquad \alpha^*(g_1, g_2) = s(g_1) s(g_2) s(g_1 g_2)^{-1} .$$

<u>Proof</u>. Theorem 2 of [21] asserts that (2.12) is valid for all g_1, g_2 in $\{ [\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}] \in K\colon \ [\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}] \equiv 1_2 (\mathrm{mod}\ N) \}$. But careful inspection

of Kubota's proof reveals that the conditions $b \equiv 0 \pmod N$ and $d \equiv 1 \pmod N$ are superfluous. Indeed Kubota's proof is computational and the crucial observation which makes it work is the Lemma below. We include its proof since Kubota does not.

<u>Lemma 2.9</u>. Suppose $k = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in K^N$. Then

$$s(k) = \begin{cases} (c, d(\det k)) & \text{if } c \neq 0 \text{ and } c \notin U \\ 1 & \text{otherwise.} \end{cases} \tag{2.13}$$

<u>Proof</u>. Throughout this proof assume $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in K^N$. In particular, $\det(k) = ad-bc$ belongs to U.

Suppose first that $c \neq 0$ and $c \notin U$. Then clearly $cd \neq 0$. Indeed $d = 0$ implies that $\det(k) = -bc$, a contradiction since $c \notin U$ implies $-bc \notin U$. Thus <u>if</u> order (c) is odd,

$$s(k) = (c, d(\det(k))$$

by definition.

On the other hand, if $\operatorname{ord}(c) = 2n$ (where $n \neq 0$ since $c \notin U$) it remains to prove that

$$(c, d(\det k)) = 1 .$$

But $d(\det k) = ad^2 - dbc$, so $k \in K^N$ implies that $d(\det k) \equiv 1 \pmod 4$. Thus

$$(c, d(\det(k))) = (\omega^{2n}u, u') = (u, u') = 1$$

by (iii) of Proposition 2.1.

To complete the proof it suffices to verify that $c \neq 0$ and $c \notin U$ if $cd \neq 0$ and $\operatorname{ord}(c)$ is odd. This is obvious, however, since if $c \in U$, then $\operatorname{ord}(c) = 0$, a contradiction, since $\operatorname{ord}(c)$ must be odd. $\square$

Note that when the residual characteristic of F is odd, $K^N = K = GL(2, O_F)$.

Definition 2.10. If $F = \mathbb{R}$ or $\mathbb{C}$, let $s(g)$ denote the function on G which is identically one. If F is non-archimedean, let $s(g)$ be as in (2.11), and in general, let $\beta(g_1, g_2)$ denote the factor set $\alpha^*(g_1, g_2)\ s(g_1)s(g_2)s(g_1g_2)$.

Obviously β determines a covering group of G isomorphic to $\overline{G}$. But according to Proposition 2.8, its restriction to $K^N \times K^N$ is identically one. Thus $\overline{K}^N$ is isomorphic to $K^N \times Z_2$, and K^N lifts as a subgroup of $\overline{G}$ via the map $k \to \{k, 1\}$. For this reason we shall henceforth deal exclusively with β.

Lemma 2.11. Suppose

$$g_i = \begin{bmatrix} \mu_i & x_i \\ 0 & \lambda_i \end{bmatrix} \in B, \quad i = 1,2 .$$

Then $\beta(g_1, g_2) = (\mu_1, \lambda_2)$.

Proof. Since

$$\begin{bmatrix} \mu_i & x_i \\ 0 & \lambda_i \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & \mu_i\lambda_i \end{bmatrix} \begin{bmatrix} \mu_i & x_i \\ 0 & \mu_i^{-1} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & \det(g_i) \end{bmatrix} p(g_i) ,$$

it follows that

$$\alpha^*(g_1, g_2) = \alpha\left(\begin{bmatrix} \mu_1 & x_1\mu_2\lambda_2 \\ 0 & \mu_1^{-1} \end{bmatrix}, \begin{bmatrix} \mu_2 & x_2 \\ 0 & \mu_2^{-1} \end{bmatrix}\right) v(\mu_2\lambda_2, \begin{bmatrix} \mu_1 & x_1 \\ 0 & \mu_1^{-1} \end{bmatrix})$$

$$= (\mu_1^{-1}, \mu_2^{-1})(-\mu_1^{-1}\mu_2^{-1}, \mu_1^{-1}\mu_2^{-1})(\mu_2\lambda_2, \mu_1^{-1}) .$$

But using (2.4) together with the symmetry and bilinearity of Hilbert's symbol, this last expression is easily seen to equal (μ_1, λ_2). Thus the Lemma follows from the identity

(2.14) $$s(\left[\begin{smallmatrix}\mu & x\\ 0 & \lambda\end{smallmatrix}\right]) = 1$$

valid for all $\left[\begin{smallmatrix}\mu & x\\ 0 & \lambda\end{smallmatrix}\right] \in B$. $\square$

<u>Corollary 2.12</u>. Suppose $\gamma = \left[\begin{smallmatrix}\mu & 0\\ 0 & \lambda\end{smallmatrix}\right]$. Then $\overline{\gamma} = \{\gamma,\zeta\}$ is central in $\overline{A}$ iff γ is a square in A.

<u>Proof</u>. Suppose $\overline{\gamma}' = \{\gamma',\zeta'\} = \{\left[\begin{smallmatrix}\mu' & 0\\ 0 & \lambda'\end{smallmatrix}\right],\zeta'\}$ is arbitrary in $\overline{A}$. Then $\overline{\gamma}\,\overline{\gamma}' = \overline{\gamma}'\,\overline{\gamma}$ iff $\{\gamma\gamma',\beta(\gamma,\gamma')\zeta\zeta\} = \{\gamma'\gamma,\beta(\gamma',\gamma)\zeta\zeta'\}$ iff $\beta(\gamma,\gamma') = \beta(\gamma',\gamma)$ iff

(2.15) $$(\mu,\lambda') = (\mu',\lambda) .$$

So suppose first that γ is a square in A. Then both μ and λ are squares in F^x and (2.15) obtains by default (both sides equal one for <u>all</u> μ',λ').

On the other hand, if μ, say, is not a square in F^x, then by the non-triviality of Hilbert's symbol,

$$(\mu,\lambda') = -1 \quad \text{for } \underline{\text{some}} \quad \lambda' \in F^x .$$

This means that (2.15) fails for $\mu' = 1$, say. Thus $\overline{\gamma}$ will not commute with $\{\left[\begin{smallmatrix}1 & 0\\ 0 & \lambda'\end{smallmatrix}\right],1\}$ and the proof is complete. $\square$

<u>Corollary 2.12'</u>. The subgroup N of G lifts as a subgroup of $\overline{G}$. So does A^2 with

$$A^2 = \{\gamma \in A\colon \gamma = \delta^2,\ \delta \in A\} .$$

<u>Proof</u>. Obvious.

<u>Corollary 2.13</u>. Fix $F \neq \mathbb{C}$. Then:

(a) The center of $\overline{G}$ is

$$Z(\overline{G}) = \{\{\left[\begin{smallmatrix}z & 0\\ 0 & z\end{smallmatrix}\right],\zeta\}\colon z \in (F^x)^2\};$$

(b) Suppose

$$\overline{G}^2 = \{\{g,\zeta\} \in \overline{G}\colon \det(g) \in (F^x)^2\} .$$

Then the center of $\overline{G}^2$ is $\overline{Z} = \{\{\left[\begin{smallmatrix}z & 0\\ 0 & z\end{smallmatrix}\right],\zeta\}\colon z \in F^x\}$.

Proof. Since $\overline{g} = \{g,\zeta\} \in Z(\overline{G})$ only if $g \in Z(G)$, i.e. only if $g = \begin{bmatrix} z & 0 \\ 0 & z \end{bmatrix}$ with $z \in F^x$, it suffices to prove that $\{\begin{bmatrix} z & 0 \\ 0 & z \end{bmatrix},\zeta\}$ commutes with every $\overline{g}' = (\begin{bmatrix} a & b \\ c & d \end{bmatrix},\zeta') \in \overline{G}$ iff z is a square in F^x.

Clearly $\{\begin{bmatrix} z & 0 \\ 0 & z \end{bmatrix},\zeta\}$ commutes with $\overline{g}'$ iff $\beta(\begin{bmatrix} z & 0 \\ 0 & z \end{bmatrix},g') = \beta(g',\begin{bmatrix} z & 0 \\ 0 & z \end{bmatrix})$. But a simple computation shows that

$$a*(\begin{bmatrix} z & 0 \\ 0 & z \end{bmatrix},g') = (z,c)$$

and

$$\alpha*(g',\begin{bmatrix} z & 0 \\ 0 & z \end{bmatrix}) = (z,c)(z, \det(g')) .$$

So since

$$s(g')^{-1}s(\begin{bmatrix} z & 0 \\ 0 & z \end{bmatrix})^{-1}s(g'\begin{bmatrix} z & 0 \\ 0 & z \end{bmatrix}) = s(\begin{bmatrix} z & 0 \\ 0 & z \end{bmatrix})^{-1}s(g')^{-1}s(\begin{bmatrix} z & 0 \\ 0 & z \end{bmatrix}g'),$$

$$\{\begin{bmatrix} z & 0 \\ 0 & z \end{bmatrix},\zeta\} \in Z(\overline{G}) \quad \text{if and only if}$$

(2.16) $$(z, \det(g')) = 1$$

for all $g' \in G$, or, since Hilbert's symbol is non-trivial, $\{\begin{bmatrix} z & 0 \\ 0 & z \end{bmatrix},\zeta\} \in Z(\overline{G})$ iff z is a square. On the other hand, if $\det(g') \in (F^x)^2$, (2.16) holds for all z, and hence (b) too is established. □

2.2. Global Theory.

In this paragraph, F will denote an arbitrary number field, v a place of F, F_v the completion of F at v, and $\mathbb{A}$ the adeles of F. Thus the G of Section 2.1 becomes $G_v = GL_2(F_v)$, its maximal compact subgroup is K_v, A is A_v, etc. Our interest henceforth is in the global group

$$G_{\mathbb{A}} = GL_2(\mathbb{A})$$

and its two fold cover $\overline{G}_{\mathbb{A}}$.

If g and g' are arbitrary in $G_{\mathbb{A}}$, put $\beta_v(g,g') = \beta(g_v,g_v')$, if $g = (g_v)$ and $g' = (g_v')$. According to Proposition 2.8, $\beta_v(g,g') = 1$ for almost every v. Thus it makes sense to define $\beta_{\mathbb{A}}$ on $G_{\mathbb{A}} \times G_{\mathbb{A}}$ by

$$\beta_{\mathbb{A}}(g,g') = \prod_v \beta_v(g,g') \tag{2.17}$$

the product extending over all the primes of F. Since $\beta_{\mathbb{A}}$ is obviously a Borel factor set on $G_{\mathbb{A}}$ it determines an extension of $G_{\mathbb{A}}$ which we shall denote by $\overline{G}_{\mathbb{A}}$ and realize as the set of pairs $\{g,\zeta\}$, $g \in G_{\mathbb{A}}$, $\zeta \in Z_2$, with group multiplication given by

$$\{g_1,\zeta_1\}\{g_2,\zeta_2\} = \{g_1g_2,\beta_{\mathbb{A}}(g_1,g_2)\zeta_1\zeta_2\}.$$

Important subgroups of $G_{\mathbb{A}}$ include

$$K_0^N = \prod_{v<\infty} K_v^N$$

and

$$G_F = GL_2(F).$$

Proposition 2.14. The subgroup K_0^N of $G_{\mathbb{A}}$ lifts to a subgroup of $\overline{G}_{\mathbb{A}}$ via the map $k_0 \to \{k_0,1\}$.

Proof. Proposition 2.8.

Proposition 2.15. For each $\gamma \in G_F$, let

$$s_{\mathbb{A}}(\gamma) = \prod_v s_v(\gamma)$$

the product extending over all (finite) primes v of F. Then the map

$$\gamma \to \{\gamma, s_{\mathbb{A}}(\gamma)\}$$

provides an isomorphism between G_F and a subgroup of $\overline{G}_{\mathbb{A}}$.

<u>Proof</u>. Note first that if $\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in G_F$, the v-order of c is 0 for almost every v. Thus $s_v(\gamma) = 1$ for almost every v and the product appearing in (2.18) is finite, i.e. $s_{\mathbb{A}}(\gamma)$ is well-defined. To prove the Proposition it will suffice to prove that

$$s_{\mathbb{A}}(\gamma) s_{\mathbb{A}}(\gamma') \beta_{\mathbb{A}}(\gamma,\gamma') = s_{\mathbb{A}}(\gamma\gamma') \tag{2.18}$$

for all γ, γ' in G_F.

So fix γ and γ' in G_F. For almost all v, <u>all</u> the entries of γ <u>and</u> γ' will be units. Thus $\alpha_v(\gamma,\gamma') = 1$ for almost all v. (By Proposition 2.1, Hilbert's symbol is trivial on units if v is finite and the residual characteristic of F_v is odd.) On the other hand, for <u>all</u> v,

$$\alpha_v(\gamma,\gamma') = (r_1,r_2)_v (r_3,r_4)_v$$

with $r_1,r_2,r_3,r_4 \in F$. Consequently, by the product formula for Hilbert's symbol (quadratic reciprocity for F),

$$\prod_v \alpha_v(\gamma,\gamma') = 1 .$$

That is,

$$\beta_{\mathbb{A}}(\gamma,\gamma') = s_{\mathbb{A}}(\gamma) s_{\mathbb{A}}(\gamma') s_{\mathbb{A}}(\gamma\gamma')$$

as was to be shown. $\square$

Because of this Proposition we can make sense now out of the homogeneous space

$$\overline{X} = Z_\infty^0 G_F \backslash \overline{G}_{\mathbb{A}} \tag{2.19}$$

where Z_∞^0 denotes the subgroup of the center of G_∞ consisting of positive real matrices $\begin{bmatrix} z & 0 \\ 0 & z \end{bmatrix}$. (Cf. Corollaries 2.12' and 2.13.) This space will be the focus of our attention in Section 3.

The Proposition below makes it possible to relate functions on $\overline{X}$ with classical forms defined in the upper half-plane. Before stating it we collect some facts concerning the quadratic power residue symbol.

Suppose $a, b \in F^{x}$ with b relatively prime to a and 2. Let S_∞ denote the set of archimedean places of F. Suppose that

$$(b) = \prod_{v} v^{\mathrm{ord}_v(b)}$$

where (b) denotes the F-ideal generated by b and the product extends over all prime ideals of O_F (the ring of integers of F). Only finitely many v will be such that $\mathrm{ord}_v(b) \neq 0$ and each such v will be relatively prime to 2 and a.

The quadratic power residue symbol $(\frac{a}{v})$ is then 1 if $x^2 = a$ has a solution in O_v and -1 otherwise. The <u>quadratic power residue symbol</u> $(\frac{a}{b})$ is

$$(\tfrac{a}{b}) = \prod_{\substack{v \notin S_\infty \\ v \nmid 2}} (\tfrac{a}{v})^{\mathrm{ord}_v(b)}$$

Now consider the congruence subgroup

(2.20) $$\Gamma_1(N) = \{\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL(2,O):\ a \equiv d \equiv 1,\ c \equiv 0(N)\}.$$

Clearly

(2.21) $$\Gamma_1(N) = G_F \cap G_\infty^0\, K_0^N$$

where $G_\infty^0 = \prod_{v \in S_\infty} G_v^0$, the product of the connected components of the archimedean completions of G. Note that for <u>any</u> $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ in $SL(2,O)$, d is relatively prime to c and 2.

<u>Proposition 2.16</u>. If d is relatively prime to c and 2, let

(2.22) $$(\tfrac{c}{d})_s = (\tfrac{c}{d}) \prod_{v \in S_\infty} (c,d)_v$$

If $\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ belongs to $\Gamma_1(N)$, put

(2.23) $$\chi(\gamma) = \begin{cases} (\frac{c}{d})_s & \text{if } c \neq 0 \\ 1 & \text{otherwise.} \end{cases}$$

Then

$$(2.24) \qquad s_{\mathbb{A}}(\gamma) = \chi(\gamma)$$

for $\gamma \in \Gamma_1(N)$.

Proof. From the definition of everything in sight it follows that

$$s_{\mathbb{A}}(\gamma) = \prod_v s_v(\gamma) = \prod_{v \text{ finite}} s_v(\gamma)$$

Thus the Proposition is obvious if $c = 0$ (since both sides of (2.24) are then one).

Assume now that $c \neq 0$ and recall that c and d are relatively prime. By Lemma 2.9,

$$s_v(\gamma) = \begin{cases} (c,d,\det(\gamma))_v & \text{if } c \neq 0 \text{ and } c \notin U_v \\ 1 & \text{otherwise} \end{cases}$$

Therefore

$$s_v(\gamma) = \begin{cases} (c,d(\det \gamma)_v & \text{if } v \mid c \\ 1 & \text{otherwise} \end{cases}$$

and consequently

$$s_{\mathbb{A}}(\gamma) = \prod_{v \mid c} (c,d(\det \gamma))_v .$$

By assumption, $\det(\gamma)$ is a unit for each v. Moreover, $\det(\gamma) \equiv 1(4)$. Thus by Proposition 2.1, and the product fromula for Hilbert's symbol

$$\prod_{v \mid c} (c,\det \gamma)_v = \prod_{v \nmid c} (c,\det(\gamma)) \prod_{v \in S_\infty} (c,\det \gamma) = 1.$$

Consequently $\prod_{v \mid c} (c,d(\det \gamma) = \prod_{v \mid c} (c,d)_v$, which means it suffices to prove

$$(2.25) \qquad \prod_{v \mid c} (c,d)_v = \left(\frac{c}{d}\right)_S .$$

To simplify matters, assume F has class number one. (The Proposition is undoubtedly true without this assumption.) To prove (2.25) recall the following basic formulas ([14], Chapter 13): for each prime integer w in F dividing both c and 2,

$$(2.26) \qquad \left(\frac{w}{d}\right) = \prod_{\substack{v \in S_\infty \\ v|2}} (w,d)_v$$

(Supplementary Reciprocity Law); for each prime w which divides c but not 2,

$$(2.27) \qquad \left(\frac{w}{d}\right) = (w,d)_w \prod_{\substack{v \in S_\infty \\ v|2}} (w,d)_v$$

(Power reciprocity formula). Note that if d is a v-unit for odd v dividing c, $(w,d)_v = 1$. Consequently (2.26) can be rewritten as

$$(2.28) \qquad \left(\frac{w}{d}\right) = \prod_{\substack{v|c \\ v\nmid 2}} (w,d)_v \prod_{\substack{v \in S_\infty \\ v|2}} (w,d)_v .$$

Now suppose $c = \prod_{w|c} w^{\mathrm{ord}_w(c)}$. By the multiplicativity of the power residue symbol, (2.28) and (2.27) can be multiplied for each $w|c$ to obtain

$$\left(\frac{c}{d}\right) = \left(\prod_{v|c} (c,d)_v\right) \prod_{v \in S_\infty} (c,d)_v .$$

Thus the proof is complete. (Units play no role since $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma_1(N)$ implies $(u,d)_v = 1$.) $\square$

Two special cases of Proposition 2.16 will be of particular interest in Section 3. The first assume F is a <u>totally imaginary</u> number field, in which case

$$(2.29) \qquad \chi(\gamma) = \begin{cases} \left(\frac{c}{d}\right) & \text{if } c \neq 0 \\ 0 & \text{otherwise} \end{cases}$$

since $(c,d)_\infty$ is identically one.

Corollary 2.17. (Cf. Kubota [19], cf. Theorem 6.1 of Bass-Milnor-Serre [2]). If F is totally imaginary, $\chi(\gamma)$ defines a character of $\Gamma_1(N)$, i.e. $\chi(\gamma_1\gamma_2) = \chi(\gamma_1)\chi(\gamma_2)$ for all $\gamma_1, \gamma_2 \in \Gamma_1(N)$.

Proof. Since F is totally imaginary, $\beta_{\mathbb{A}}$ is identically one on $G_\infty^0 K_0^N$ in $G_{\mathbb{A}}$. So suppose γ_1 and γ_2 are arbitrary in $\Gamma_1(N)$. Since γ_1 and γ_2 belong to G_F,

$$s_{\mathbb{A}}(\gamma_1\gamma_2) = \beta_{\mathbb{A}}(\gamma_1,\gamma_2)s_{\mathbb{A}}(\gamma_1)s_{\mathbb{A}}(\gamma_2)$$

(by (2.18)). But $\beta_{\mathbb{A}}(\gamma_1,\gamma_2) = 1$. Thus the Corollary is immediate from Proposition 2.16. $\square$

The existence of such a character (with the assumption that F is totally imaginary) provides a starting point for Kubota's recent investiations ([21] through [23]). Its relation to the congruence subgroup problem is discussed in [2].

The second example I have in mind assumes $F = Q$. In this case,

$$\chi(\gamma) = \begin{cases} \left(\frac{d}{|d|}\right) & \text{if } c \neq 0,\ c > 0 \text{ or } d > 0 \\ -\left(\frac{c}{|d|}\right) & \text{if } c \neq 0,\ c < 0 \text{ and } d < 0 \\ 1 & \text{if } c = 0. \end{cases}$$

Now χ no longer defines a character of $\Gamma_1(N)$ ($\overline{G}_\infty$ is not a trivial extension of G_∞). However $\chi(\gamma)$ is still a "multiplier system" for $\Gamma_1(N)$. More precisely, if $\gamma = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$, let $J^*(\gamma,z) = (cz+d)^{1/2}$. Here $w^{1/2}$ is chosen so that $-\pi/2 < \arg(w^{1/2}) \leq \pi/2$.

Then $|\chi(\gamma)| = 1$ and

$$\frac{\chi(\gamma_1\gamma_2)}{\chi(\gamma_1)\chi(\gamma_2)} = \frac{J^*(\gamma_1,\gamma_2 z)J^*(\gamma_2,z)}{J^*(\gamma_1\gamma_2,z)} \tag{2.31}$$

for all $\gamma \in \Gamma_1(N)$. I.e., $\chi(\gamma)$ is a multiplier system for $\Gamma_1(N)$ of dimension $1/2$.

2.3. Weil's Metaplectic Representation.

In this Section I want to sketch Weil's general theory of the metaplectic representation and reformulate parts of it in a form suitable for the construction of automorphic forms on the metaplectic group.

Roughly speaking, to each abstract symplectic group, one associates a projective representation (Weil's metaplectic representation). This representation operates in L^2 of the group and its associated multiplier is of order two. Thus Weil's representation reproduces a two fold covering group which repreduces $\overline{SL}_2(F)$ when the underlying group is $SL(2)$.

We begin by recalling some basic properties of projective representations and the group extensions they determine.

If H is a Hilbert space, the torus imbeds in the obvious way as a central subgroup of $U(H)$ equipped with the strong operator topology. A projective (unitary) representation of G in H is then a continuous homomorphism π^* from G to $U(H)/T$.

Our interest will be in the cocycle (or multiplier) representations of G canonically associated to such projective representations π^*. These are obtained by choosing a Borel cross-section f of $U(H)/T$ in $U(H)$ and introducing the map $\pi = f\circ\pi^*$. If $f(T) = 1$, this map from G to $U(H)$ must automatically be Borel and satisfy

$$\pi(e) = I \tag{2.32}$$

and

$$\pi(g_1)\pi(g_2) = \alpha(g_1,g_2)\pi(g_1,g_2) \quad . \tag{2.33}$$

for all $g_1,g_2 \in G$. Here α is a Borel function from $G \times G$ to T which (by the associative law in G) must be a two-cocycle on G with values in T.

Note that α above obviously depends on the choice of cross-section f. However, if f' is another such cross-section, the cocycle it determines will be cohomologous to α. Thus each projective representation uniquely determines an element of $H^2(G,T)$.

Any Borel map from G to $U(H)$ satisfying (2.32) and (2.33) is called a multiplier representation with multiplier α (or, simply, an α-representation) and all such representation arise from projective representations as above. More precisely, if π is any α-representation, set $\pi^* = p \circ \pi$, where p is the natural projection from $U(H)$ to $U(H)/T$. Then $\pi = f \circ \pi^*$, and α lies in the cohomology class determined by π^*. In this sense, each projective representation is essentially a family of multiplier representations with cohomologous multipliers.

Throughout this paper we shall want to distinguish between representations of the metapletic group which factor through Z_2 and those that do not. Moreover, we shall often want to confuse those that don't with cocycle representations of $GL(2)$.

In general, if $\overline{G}$ is an extension of the locally compact group G by some subgroup T of the torus we shall call π genuine if $\pi(t) = t \circ I$ for all $t \in T \subset G$. For the sake of exposition we assume below that multiplication in $\overline{G}$ is determined by some fixed cocycle α.

Proposition 2.18. Let $\iota : G \to \overline{G}$ denote the (Borel) cross-section $\iota(g) = (g,1)$ and suppose π is a genuine representation of $\overline{G}$. Then:

(a) The map $\pi \circ \iota$ from G to $U(H)$ is an α-representation of G;

(b) The correspondence $\pi \to \pi \circ \iota$ is a bijection between the collection of genuine representations of $\overline{G}$ and α-representations of G;

(c) This correspondence preserves unitary equivalence and direct sums.

Proof. Part (a) follows immediately from the fact that

$\ell(g_1g_2) = \alpha(g_1,g_2)\ell(g_1)\ell(g_2)$. To prove (b), suppose π' is an α-representation of G in H and define

$$\pi(t\ell(g)) = t\pi'(g).$$

Then π is a Borel homomorphism of $\overline{G}$ into $U(H)$, whence continuous, and obviously "genuine". The rest of the proposition follows from the fact that an operator A intertwines the α-representations π_1' and π_2' if and only if it intertwines the corresponding ordinary representations π_1 and π_2. $\square$

We shall now describe Weil's construction in earnest.

The symplectic groups of Weil's general theory are attached to locally compact abelian groups G so the notation $\langle x,x^*\rangle$ will denote the value of the character x^* in G^* (the dual group to G) at $x \in G$. Since Weil's theory is essentially empty unless G is <u>isomorphic</u> to G^* we shall assume throughout that this is the case.

<u>Example 2.19</u>. Let denote a local field of characteristic zero and V a finite-dimensional vector space defined over F. Fix q to be a non-degenerate quadratic form on V and τ the canonical non-trivial additive character of F described in [46]. The identity

$$\langle X,Y\rangle = \tau(q(X,Y)), \quad (X,Y \in V)$$

where

$$q(X,Y) = q(X+Y)-q(X)-q(Y),$$

establishes a self-duality for the additive group of V equipped with its obvious topology. Thus Weil's theory will be applicable in particular to $G = V$ together with (q,τ). This example in fact will suffice for the applications we have in mind.

For each $w = (u,u^*)$ in $G \times G^*$ let $U'(w)$ denote the unitary operator in $L^2(G)$ defined by

$$U'(w)\phi(x) = \langle x,u^*\rangle\phi(x+u).$$

Then $U'(w_1)U'(w_2) = F(w_1,w_2)U'(w_1+w_2)$ where

$$F(w_1,w_2) = \langle u_1, u_2^*\rangle$$

if $w_i = (u_i, u_i^*)$. That is, U' is a multiplier representation of $G \times G^*$ with multiplier $1/F$ and the family of operators

$$U(w,t) = tU'(w) \quad (w \in G \times G,\ t \in T)$$

comprises a group with composition law given by

$$(2.34) \qquad (w_1,t_1)(w_2,t_2) = (w_1+w_2, F(w_1,w_2)t_1t_2).$$

In particular, the multiplier representation U' determines an extension of $G \times G^*$ by T which is called the <u>Heisenberg group</u> of G and denoted by $A(G)$. In case $G = \mathbb{R}$, $A(G)$ is (the exponentiation of) the familiar Heisenberg group

$$\left\{ \begin{bmatrix} 1 & x & t \\ 0 & 1 & x^* \\ 0 & 0 & 1 \end{bmatrix} \right\}.$$

In general, T is the center of $A(G)$; $A(G)$ may be viewed <u>either</u> as $G \times G^* \times T$ equipped with the group law (2.34) <u>or</u> as the group of operators $\{U(w,t)\}$ (in which case it is denoted by $\overline{A}(G)$). The crucial fact is that $U(w,t)$ is an <u>irreducible</u> representation; in fact, <u>$U(w,t)$ is the unique irreducible representation of $A(G)$ which leaves T pointwise fixed</u>. Therefore at least the first part of the result below is plausible.

<u>Theorem 2.20 (Segal)</u>. Let $B_0(G)$ denote the group of automorphisms of $A(G)$ leaving T pointwise fixed, $\overline{B_0(G)}$ the normalizer of $\overline{A(G)}$ in $L^2(G)$, and let

$$p_0 : \overline{B_0(G)} \to B_0(G)$$

be the natural projection. Then p_0 is <u>onto</u> with kernel T.

In other words, there exists a multiplier representation ·

of $B_0(G)$ in $L^2(G)$ whose range (choosing the constant in all possible ways) coincides with $\overline{B_0(G)}$. Indeed if $s \in B_0(G)$, the formula $U^s(w,t) = U((w,t))s)$ defines an irreducible unitary representation of $A(G)$ which fixes $A(G)$ pointwise and hence is equivalent to U. Therefore $U((w,t)s) = r^{-1}(s)U(w,t)r(s)$ for some unitary operator $r(s)$ in $L^2(G)$ (uniquely determined up to a scalar in T) and

$$s \to r(s)$$

determines a multiplier representation of $B_0(G)$ (the Weil representation) satisfying $p_0(r(s)) = s$.

This "abstract" Weil representation is relevant to $SL(2)$ since $B_0(G)$ is the semi-direct product of $(G \times G^*)^*$ and an abstract symplectic group which reduces to $SL_2(F)$ when $G = F$.

More precisely, since s in $B_0(G)$ fixes $(1,t)$, it is completely determined by its restriction to $G \times G^* \times \{1\}$ where it is of the form $(w,1) \overset{s}{\to} (w\sigma, f(w))$. Thus on $G \times G^* \times T$ it is of the form (σ, f) where

$$(w,t)s = (w,t)(\sigma,f) = (w\sigma, f(w)t),$$

σ is an automorphism of $G \times G^*$, $f: G \times G^* \to T$ is continuous, and

$$\text{(2.35)} \qquad \frac{F(w_1\sigma, w_2\sigma)}{F(w_1,w_2)} = \frac{f(w_1+w_2)}{f(w_1)f(w_2)} .$$

Conversely, each pair (σ,f) satisfying (2.35) defines an automorphism of $A(G)$ fixing T pointwise, so $B_0(G)=\{(\sigma,f)\}$, with group law

$$(\sigma,f)(\sigma',f') = (\sigma\sigma',f'')$$

if $f''(w) = f(w)f'(w\sigma)$.

Now let $Sp(G)$ denote the abstract symplectic group of automorphisms of $G \times G^*$ which leave invariant the bicharacter

$$\frac{F(w_1,w_2)}{F(w_2,w_1)} = \frac{\langle x_1,x_2^*\rangle}{\langle x_2,x_1^*\rangle} \qquad (w_i = (x_i,x_i^*))$$

Using (2.35) one checks that if $(\sigma,f) = s \in B_0(G)$ then $\sigma \in Sp(G)$. Our claim was that the exact sequence

$$(2.36) \qquad 1 \to (G\times G^*)^* \to B_0(G) \to Sp(G) \to 1$$

actually splits.

To check this it is convenient to describe $Sp(G)$ in matrix form. Since each $\sigma \in Aut(G\times G^*)$ is of the form

$$(x,x^*) \to (x,x^*)\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}$$

where $\alpha: G \to G$, $\beta: G \to G^*$, $\gamma: G^* \to G^*$, we shall, following Weil, write

$$\sigma = \begin{bmatrix}\alpha & \beta\\ \gamma & \delta\end{bmatrix},$$

and define

$$\sigma^I = \begin{bmatrix}\delta^* & -\beta^*\\ -\gamma^* & \alpha^*\end{bmatrix},$$

where α^* denotes the appropriate map dual to α. Then $\sigma \in Aut(G\times G^*)$ is symplectic iff $\sigma\sigma^I = I$.

Given $\begin{bmatrix}a & b\\ c & d\end{bmatrix} = \sigma \in Sp(G)$, define f_σ on $G\times G^*$ by

$$f_\sigma(u,u^*) = \langle u, 2^{-1}u\alpha\beta^*\rangle\langle 2^{-1}u^*\gamma\delta^*, u^*\rangle\langle u^*\gamma, u\beta\rangle .$$

Here we are assuming, as we do throughout, that $x \mapsto 2x$ is an automorphism of G. Then f_σ and σ satisfy (2.35) and the map

$$\sigma \to (\sigma,f)$$

is a monomorphism of $sp(G)$ into $B_0(G)$ which splits (2.36).

In short, Theorem 2.20 produces an exact sequence

$$(2.37) \qquad 1 \to T \to \overline{B_0(G)} \to B_0(G) \to 1$$

and $B_0(G)$ contains a copy of $Sp(G)$.

Example 2.21. Suppose F is local and $G = (V,q,\tau)$ is as in Example 2.19. Then $SL_2(F)$ can be imbedded homomorphically in $Sp(G)$ by allowing each element of F to act on V via scalar multiplication. (Since $\alpha^* = a$ for each $\alpha \in F$, $\sigma = [\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}] \in SL(2,F)$ obviously satisfies $\sigma\sigma^I = I$; note that $Sp(G) = SL_2(F)$ if $G = V$ but in all other cases properly contains it.) From this it follows that one can associate to each quadratic form (q,V) a natural projective representation of $SL(F)$ in $L^2(V)$ which we shall call the Weil representation attached to (q,V) and denote by r_q.

Before further analyzing this representation we need to adapt Weil's general theory to the special context of Examples 2.19 and 2.21. Recall that V is a vector space over F and τ is a non-trivial additive character of F (fixed once and for all). If X^* belongs to the linear dual V^* we shall denote its value at X in V by $[X,X^*]$. The natural isomorphism between V and V^* is then $[X,Y] = q(X,Y)$.

Using $[\cdot,\cdot]$ in place of $\langle\cdot,\cdot\rangle$ one can now "linearize" the theory just sketched, introducing:

(a) the bilinear form $B(z_1,z_2) = [X_1,X_2^*]$ in place of the bicharacter $F(W_1,W_2)$;

(b) the Heisenberg group $A(V)$ in place of $A(G)$;

(c) the symplectic group $Sp(V)$ in place of $Sp(G)$; and

(d) the pseudo-symplectic group $Ps(V)$ in place of $B_0(G)$.

A typical element of $Ps(V)$ is of the form (σ,f) where $\sigma \in \mathrm{Aut}(V \times V^*)$ belongs to $Sp(V)$ and $f: V \times V^* \to F$ satisfies

$$f(z_1+z_2)-f(z_1)-f(z_2) = B(z_1\sigma,z_2\sigma)-B(z_1,z_2). \tag{2.38}$$

This relation, of course, is the linearlization of (2.35): taking τ of both sides of it yields (2.35). Note that $Ps(V) \neq B_0(G)$. In fact $Ps(V)$ is the subgroup of $B_0(G)$ consisting of pairs (σ,f) with f quadratic; cf. (2.39) below.

Note now that (2.38) associates to each σ in $Sp(V)$ one and only one quadratic form on $V \times V^*$. (We are assuming F

has characteristic zero, in particular, characteristic not equal to two.) Therefore the homomorphism

$$(\sigma,f) \to f$$

is an <u>isomorphism</u> between $Ps(V)$ and $Sp(V)$.

Since the same map from $B_0(G)$ to $Sp(G)$ has kernel isomorphic to $(G\times G^*)^*$ the analogy between the exponentiated and unexponentiated theories breaks down here. However, the map

$$(2.39) \qquad (\sigma,f) \xrightarrow{\mu} (\sigma,\tau\circ f)$$

does <u>imbed</u> $Ps(V)$ homomorphically in $B_0(G)$ so we can still obtain an extension of $Ps(V)$ by pulling back (2.37) through μ:

$$\begin{tikzcd} 1 \arrow[r] & T \arrow[r] & M_p(V) \arrow[r, "p"] & Sp(V) \arrow[r] \arrow[d, "\mu"] & 1 \\ 1 \arrow[r] & T \arrow[r] & \overline{B_0(G)} \arrow[r, "p_0"] & B_0(G) \arrow[r] & 1 \end{tikzcd}$$

The group

$$Mp(V) = \{(s,\overline{s}) \in P_s(V) \times \overline{B_0(G)} : \; p_0(\overline{s}) = \mu(s)\}$$

is Weil's general metaplectic group. It is a central extension of $Sp(V)$ by T .

Weil's results concerning the non-triviality of $Mp(V)$ include the following:

(a) $Mp(V)$ always reduces to an extension of $Sp(V)$ by Z_2, i.e. the cohomology class it determines in $H^2(G,T)$ is of order 2 ;

(b) the extension $Mp(V)$ is in general non-trivial; in particular, if $F \neq \mathbb{C}$, and V is one-dimensional, $Mp(V)$ is always a non-trivial cover of $Sp(V)$ by Z_2 .

These results yield a (topological) central extension

$$(2.40) \qquad 1 \to Z_2 \to \overline{Sp}(V) \to Sp(V) \to 1$$

which by Proposition 2.3 must coincide with

$$1 \to Z_2 \to \overline{SL}_2(F) \to SL_2(F) \to 1$$

when $V = F$. I.e., Weil's metaplectic group generalizes the metaplectic cover of $SL_2(F)$ constructed in 2.2.

Although Weil's construction does not immediately yield an explicit factor set for $\overline{SL}_2(F)$ it does a priori realize this group as a group of operators. In fact, it provides a host of representations for this group. These correspond to quadratic forms over F and in explicit terms are realized as follows.

Fix a Haar measure $d_\tau y$ on F such that the Fourier transform

$$\hat{f}(x) = \int f(y)\tau(2xy)\,d_\tau y$$

satisfies $(\hat{f})^\wedge(x) = f(-x)$. Recall that τ is the canonical character of F whose conductor is O_F. Given (q,V), fix an orthogonal basis $X_1,\ldots,X_n$ of V such that if $X = \Sigma x_i X_i$, then $q(X) = q(x_1,\ldots,x_n) = \Sigma \ell_i x_i^2$, with $\ell_i = \frac{1}{2}\,q(X_i,X_i)$. Let τ_i denote the character $\tau_i(y) = \tau(\ell_i y)$ of F, $i = 1,\ldots,n$, and let $d_i y$ denote the corresponding Haar measure on F, normalized as above.

If F is non-archimedean, the limit

$$\gamma(\tau_i) = \lim_{m\to\infty} \int_{P^m} \tau_i(y^2)\,d_i y$$

is known to exist (by Weil [47] it is an eighth root of unity). Consequently the invariant

$$\gamma(q,\tau) = \prod_{i=1}^{n} \gamma(\tau_i)$$

is well-defined. If $F = \mathbb{R}$, and $\tau_i(x) = \exp(\pi\sqrt{-1}\,\ell_i x)$, set $\gamma(\tau_i)$ equal to $\exp(\frac{\pi}{4}\sqrt{-1}\,\mathrm{sgn}(\ell_i))$ and define $\gamma(q,\tau)$ as above. Finally, if $F = \mathbb{C}$, set $\gamma(q,\tau)$ identically equal to 1. The Fourier transform on $L^2(V)$ is defined to be

$$\hat{\Phi}(X) = \int_V \Phi(Y)\tau(q(X,Y))dY$$

where $dY = \prod_{i=1}^{n} d_i y$.

Theorem 2.22. For each $t \in F^x$ let τ_t denote the character of F defined by $\tau_t(x) = \tau(tx)$ and denote $\gamma(q,\tau_t)$ by $\gamma(q,t)$. Suppose we use (q,V,τ) to imbed $SL_2(F)$ in $Sp(V)$ (and $B_0(V)$). Then a cross-section of $\overline{Sp}(V)$ over $SL_2(F)$ (c.f. (2.40)) is provided by the maps

$$(2.41) \qquad \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} \to r(\begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix})\Phi(X) = \tau_t(bq(X))\Phi(X)$$

$$(2.42) \qquad \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \to r(\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix})\Phi(X) = \gamma(q,t)^{-1}\hat{\Phi}(-X)$$

$(\Phi \in L^2(V))$. More precisely, for each $t \in F^x$, these maps extend to a multiplier representation of $SL_2(F)$ in $L^2(V)$ whose associated cocycle is of order two.

Remark 2.23. In (2.42) the Fourier transform is taken with respect to τ_t and Haar measure $d_tY = |t|^{n/2}dY$. Thus

$$\hat{\Phi}(X) = \int_V \Phi(Y)\tau_t(q(X,Y))|t|^{n/2}dY .$$

Proof of Theorem 2.22. Without loss of generality we assume $t=1$. Since $dY = \prod_{t=1}^{n} d_i Y$ it is easy to check that the operators in question are tensor products of the operators in $L^2(F)$ corresponding to τ_i, namely

$$r(\begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix})f(x) = \tau_i(bx^2)f(x)$$

and

$$r(\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix})f(x) = \gamma(\tau_i)^{-1}\hat{f}(-x) ,$$

$i = 1,\dots,n$. Thus the Theorem is reduced to the case $V = F$, $q(x) = x^2$, precisely Theorem 1.4.1 of Shalika [35].

We note that $SL_2(F)$ is generated by elements of the form

$\left[\begin{smallmatrix}1 & b\\ 0 & 1\end{smallmatrix}\right]$ and $\left[\begin{smallmatrix}0 & -1\\ 1 & 0\end{smallmatrix}\right]$ subject to certain relations. Thus one could follow Shalika directly and prove Theorem 2.22 by checking that these relations are preserved by the operators $r(\left[\begin{smallmatrix}1 & b\\ 0 & 1\end{smallmatrix}\right])$ and $r(\left[\begin{smallmatrix}0 & -1\\ 1 & 0\end{smallmatrix}\right])$. In any case, the resulting multiplier representation of $SL_2(F)$ will be denoted r_q and called "the" Weil representation attached to the quadratic form q. (This is an explicit realization of the representation r_q of Example 2.21).

Corollary 2.24. r_q is ordinary if and only if V is even dimensional.

We conclude this Section with some remarks. Although the group of operators $\overline{B}_0(G)$ is irreducible (it contains $\overline{A}(G)$), the representations r_q are never irreducible. In fact, their decomposition is now known to be of great importance for the theory of automorphic forms and group representations.

For the case when q is the norm form of a quadratic field, see [35], [45], [36], and [18]; for the case when q is the norm form of a division algebra in four variables see [37] and [18]. As already remarked in the Section 1, no such complete results have yet been obtained when q is a quadratic form in an odd number of variables. This is because until recently no one seriously attacked the representation theory of the two-fold covering groups of $SL_2(F)$.

In these Notes we shall describe how a Weil representation in one and three variables decomposes and how its decomposition relates automorphic forms on $GL(1)$ and $GL(2)$ to automorphic forms on the metaplectic group. The general philosophy is explained in Subsection 2.4.

2.4. <u>A philosophy for Weil's representation.</u>

The purpose of this Subsection is to describe a simple principle which (though unproven) underlies most of the results of these Notes. Roughly speaking, the idea is that quadratic forms index 1-1 correspondences between automorphic forms on the metaplectic group and automorphic forms on the orthogonal group.

Let F denote a local or global field of characteristic zero, (q,V) a quadratic space over F, and r_q the corresponding representation of $SL_2(F)$. Let H denote the orthogonal group of q. This group acts on $L^2(V)$ through its natural action on V. The resulting representation of H in $L^2(V)$ may be assumed to be unitary and we denote it by $A(h)$.

Now fix F to be local. From Theorem 2.22 it follows that for $h \in H$ and $\overline{s} \in \overline{SL}_2(F)$ the operator $A(h)$ commutes with $r_q(\overline{s})$. In fact it seems plausible that r_q and A generate each others commuting algebras. Suppose for the moment that this is so. Then the primary constituents of r_q correspond 1-1 to primary constituents of A. In particular, the commuting diagram

$$\begin{tikzcd} & L^2(V) & \\ \overline{SL}_2 \arrow[ur, "r_q"] & & H \arrow[ul, "A"'] \end{tikzcd}$$

leads to a correspondence between irreducible representations of H which occur in A and irreducible representations of $\overline{SL}_2$ which occur in r_q. Following R. Howe, we call this correspondence D for "duality". Globally, D should pair together automorphic forms on H which occur in A with automorphic forms on $\overline{SL}_2$ which occur in r_q. This is the correspondence alluded to above.

Since the existence of this correspondence D seems to rest on the hypothesis that r_q and A generate each others commuting algebras, some further remarks are in order. The precise formulation of this hypothesis, in the greater generality of "dual reductive pairs", was communicated to me by Howe; as such, it is but one facet of his inspiring theory of "oscillator representations" for the metaplectic group (manuscript in preparation). In the context of SL_2, at least over the reals, this fact had also been suspected by S. Rallis and G. Schiffmann (cf. [52]). On the other hand, important special examples of D had already appeared in the literature. In [36] Shalika and Tanaka discovered that the Weil representation attached to the norm form of a quadratic extension of F yielded a correspondence between forms on SL(2) and SO(2). This seems to have been the first published example of D. (Actually the correspondence of Shalika-Tanaka was not 1-1 since they dealt with SO(2) instead of O(2).) For quadratic forms given by the norm forms of quaternion algebras over F the resulting corresponding between automorphic forms on the quaternion algebra and GL(2) was developed by Jacquet-Langlands ([18], Chapter III) following earlier work of Shimizu. In both cases, the fact that r_q and A generate each others commuting algebras was not established apriori; rather it appeared as a consequence of the complete decomposition of r_q.

One purpose of these Notes is to describe the duality correspondences belonging to two forms in an odd number of variables, namely $g_1(x) = x^2$, and $q_3(x_1,x_2,x_3) = x_1^2-x_1^2-x_3^2$. The inspiration for our discussion derives directly from general ideas of Howe's, an initial suggestion of Langland's, and earlier works of Kubota [ ], Shintani [ ], and Niwa [ ]. Our results, specifally Cor.4.18, Cor.4.20, and Theorem 6.3, lend further evidence to the general principle asserted above. However, as in earlier works, the fact that r_q and A generate each others commuting algebras, appears as a consequence ofthe existence of D.

2.5 Extending Weil's representation to GL_2

Weil's construction of the metaplectic group produces a multiplier representation r_q of SL_2. There are several reasons now why it is advantageous to extend this construction to GL_2. One is that Weil's representation for SL_2 often depends not only on q but also on the choice of additive character τ. By contrast, the natural analogue of r_q for GL_2 depends only on q.

In this Subsection I want to explain how Weil's original representation depends on τ. More precisely, I want to define an analogue of r_q for GL_2 which is independent of τ. The case when q is the norm form of a quadratic or quaternionic space is discussed in [18]. I shall treat the cases

$$q_1(x) = x^2$$

and

$$q_3(x_1,x_2,x_3) = x_1^2 - x_2^2 - x_3^2$$

following a suggestion of Cartier's.

Throughout this paragraph the following conventions will be in force:

1) F will be a local field of characteristic zero;
2) $n=1$ or 3 according as $q=q_1$ or q_3;
3) r_n will denote r_{q_1} or r_{q_3};
4) $\{[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}],1\}$ will be abbreviated by $[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}]$.

Note that the formulas for r_n given in Theorem 2.22 depend on τ_t. Therefore, for emphasis, I shall denote r_n by $r_n(\tau_t)$; the notation r_n will be reserved for the representation eventually introduced for GL_2.

Proposition 2.27. (Cf. [18] Lemma 1.4.). If $a \in F^x$, and $\overline{s} = \{s,\zeta\} \in \overline{SL}_2(F)$, define $\overline{s}^a$ to be $\{s^a, \zeta\, v(a,s)\}$ (cf. Proposition 2.6). Then

$$r_n(\tau_a)(\bar{s}) = r_n(\tau)(\bar{s}^a)$$

for all $a \in F^x$ and $\bar{s} \in SL_2(F)$.

Proof. We may assume without loss of generality that $n = 1$ and $\bar{s}$ is a generator of $\overline{SL}_2(F)$. Suppose first that $\bar{s} = \bar{w} = \{[\begin{smallmatrix}0 & 1\\ -1 & 0\end{smallmatrix}], 1\}$. Then

$$\bar{s}^a = \bar{w}^a = [\begin{smallmatrix}0 & a\\ -a^{-1} & 0\end{smallmatrix}] = [\begin{smallmatrix}a & 0\\ 0 & a^{-1}\end{smallmatrix}]\, \bar{w}\, \{1,(a,a)\}$$

and

$$r_n(\tau)(\bar{w}^a)\Phi(X) = (a,a)\ \gamma(q_n,\tau)(r_n(\tau)([\begin{smallmatrix}a & 0\\ 0 & a^{-1}\end{smallmatrix}])\hat{\Phi})(X) \ .$$

Some tedious computations with Hilbert symbols also show that

$$[\begin{smallmatrix}a & 0\\ 0 & a^{-1}\end{smallmatrix}] = \bar{w}[\begin{smallmatrix}1 & a^{-1}\\ 0 & 1\end{smallmatrix}]\ \bar{w}\ [\begin{smallmatrix}1 & a\\ 0 & 1\end{smallmatrix}]\ \bar{w}\ [\begin{smallmatrix}1 & a^{-1}\\ 0 & 1\end{smallmatrix}]\{1,(a,a)\} \ .$$

Consequently

$$(2.43) \qquad r_n(\tau)([\begin{smallmatrix}a & 0\\ 0 & a^{-1}\end{smallmatrix}])\Phi(X) = (a,a)\ |a|^{n/2}\ \frac{\gamma(q_n,\tau_a)}{\gamma(q_n,\tau)}\ \Phi(aX)$$

and

$$r_n(\tau)(\bar{w}^a)\Phi(X) = |a|^{n/2}\ \gamma(q_n,\tau_a)\hat{\Phi}(aX) \ .$$

But $|a|^{n/2}\gamma(q_n,\tau_a)\hat{\Phi}(aX) = r_n(\tau_a)(\bar{w})\Phi(X)$. Therefore $r_n(\tau)(\bar{w}^a) = r_n(\tau_a)(\bar{w})$, as was to be shown. The analogous identity for $\bar{s} = \{[\begin{smallmatrix}1 & b\\ 0 & 1\end{smallmatrix}], 1\}$ is completely straightforward. □

Corollary 2.28. The representation $r_n(\tau_a)$ is independent of $a \in F^x$ iff $r_n(\tau)$ extends to a representation of $\overline{GL}_2(F)$.

Proof. Suppose $r_n(\tau_a)$ is equivalent to $r_n(\tau)$ for all $a \in F^x$. Then for each $a \in F^x$ there is a unitary operator R_a in $L^2(F^n)$ such that

$$r_n(\tau_a)(\bar{s}) = R_a\ r_n(\tau)R_a^{-1}$$

for all $\bar{s} \in \overline{SL}_2(F)$. Equivalently, by Proposition 2.27,

$$r_n(\tau)(\bar{s}^a) = R_a\ r_n(\tau_a)R_a^{-1} \quad .$$

But $\overline{GL}_2(F)$ is the semi-direct product of $\overline{SL}_2(F)$ and $F^x=\{[\begin{smallmatrix}1 & 0\\ 0 & a\end{smallmatrix}]\}$. Thus $r_n(\tau)$ can be extended to a representation of $\overline{GL}_2(F)$ by defining $r_n(\tau)([\begin{smallmatrix}1 & 0\\ 0 & a\end{smallmatrix}],1)$ to be R_a .

Conversely, if r_n extends to $\overline{GL}_2(F)$, $r_n(\tau)([\begin{smallmatrix}1 & 0\\ 0 & a\end{smallmatrix}],1)$ will intertwine $r_n(\tau)$ with $r_n(\tau_a)$. $\square$

<u>Example 2.29</u>. If $a \in (F^x)^2$, say $a = \alpha^2$, then $R_a\Phi(x)=|\alpha|^{1/2}h(\alpha x)$ intertwines $r_1(\tau_a)$ and $r_1(\tau)$; in particular, $r_1(\tau)$ extends to a representation of

$$\overline{GL}_2{}^2(F) = \{\{g,\zeta\} \in \overline{GL}_2(F) : \det(g) \in (F^x)^2\} \quad .$$

In general, $r_q(\tau_a)$ will <u>not</u> be equivalent to $r_q(\tau)$. Indeed $r_q(\tau)$ will extend only to a representation of the subgroup

$$\{\{g,\zeta\} \in \overline{GL}_2(F) : \det(g) \ \text{ fixes } \ r_q(\tau)\} \quad .$$

To get around this problem we "fatten up" $r_q(\tau)$ <u>before</u> attempting to extend it. This way there is more room in the representation space for intertwining operators to act.

<u>Definition 2.30</u>. Let dt denote the restriction of additive Haar measure on F to F^x . Let r_q denote the direct integral of the representations $r_q(\tau_t)$ with respect to dt .

Note that the space of r_q is isomorphic to $L^2(F^n \times F^x)$. Moreover, the methods of Corollary 2.28 imply that r_q extends to a representation of $\overline{GL}_2(F)$ satisfying

$$r_q([\begin{smallmatrix}1 & 0\\ 0 & a\end{smallmatrix}])\Phi_t(X) = |a|^{-1/2}\Phi_{ta^{-1}}(X) \quad . \tag{2.44}$$

<u>Proposition 2.31</u>. The action of r_q in $L^2(F^n \times F^x)$ is given by the formulas

$$r_q([\begin{smallmatrix}1 & b\\ 0 & 1\end{smallmatrix}])\Phi(X,t) = \tau_t(bq(X))\Phi(X,t) \tag{2.45}$$

$$(2.46)\qquad r_q(\overline{w})\Phi(X,t) = \gamma(q,t)\hat{\Phi}(X,t) = \gamma(q,t)\int_{F^n} \Phi(Y,t)\tau_t(q(X,Y)d_tY$$

$$(2.47)\qquad r_q([\begin{smallmatrix} a & 0 \\ 0 & a^{-1}\end{smallmatrix}])\Phi(X,t) = (a,a)|a|^{n/2}\frac{\gamma(q,\tau_{at})}{\gamma(q,\tau_t)}\Phi(aX,t)$$

and

$$(2.48)\qquad r_q([\begin{smallmatrix} 1 & 0 \\ 0 & a\end{smallmatrix}])\Phi(X,t) = |a|^{-1/2}\Phi(X,ta^{-1}) .$$

Proof. Apply the definition of direct integral. □

Proposition 2.31 implies we could have defined r_q directly for GL_2 using formulas (2.45), (2.46), and (2.48). In fact this construction of r_q was communicated to me by Cartier awhile ago and my Definition 2.30 simply reformulates his ideas.

Note that

$$(2.49)\qquad r_1([\begin{smallmatrix} a & 0 \\ 0 & a\end{smallmatrix}])\Phi(X,t) = \frac{\gamma(q_1,ta)}{\gamma(q_1,t)}|a|^{-1/2}\Phi(aX,ta^{-2}) .$$

Thus one easily checks that $r_1([\begin{smallmatrix} a & 0 \\ 0 & a\end{smallmatrix}])$ commutes with $r_1(\overline{g})$ for all $\overline{g}\in\overline{GL}_2(F)$ iff $\det(g)\in(F^x)^2$. This is consistent with Corollary 2.13(a).

Now it is natural to ask what shape the philosophy of Subsection 2.4 takes in the present context? Roughly speaking, the answer is that the orthogonal group of q is simply replaced by the group of similitudes of q.

Suppose first that $q=q_1$. The group of similitudes of q_1 is $F^x = GL_1(F)$ and its action in $L^2(F\times F^x)$ is given by

$$(A(y)\Phi)(x,t) = |a|^{-1/2}\Phi(ax,ta^{-2}) .$$

Note that $A(y)$ clearly commutes with r_1.

Now suppose $q=q_3$. If F^3 is realized as the space of 2×2 symmetric matrices with coefficients in F then $q(X)=\det(X)$.

The group of similitudes of q_3 is essentially $GL_2(F)$. More precisely, define

$$g\circ X = {}^t g\, X\, g$$

when $g \in GL_2(F)$ and $X \in F^3$. Then $q(g\circ X) = (\det g)^2 q(X)$. The action of $GL_2(F)$ in $L^2(F^3 \times F^\times)$ is given by

$$(A(g)\Phi)(X,t) = |\det g|^2 \Phi(g\circ X, t(\det g)^{-2})$$

and A once again commutes with r_3.

The result now is that r_1 (resp. r_3) should decompose into irreducible representations of $\widetilde{G}$ indexed by irreducible representations of $GL_1(F)$ (resp. irreducible representations of $GL_2(F)$ which occur in A). Globally this means automorphic forms on the metaplectic group which occur in r_1 (resp. r_3) should be indexed by automorphic forms on GL_1 (resp. automorphic forms on GL_2 which occur in A). The global results that can be obtained or expected are described in Subsections 6.1 and 6.2. The local analysis is carried out in Subsections 4.3, 4.4, and 5.5.

2.6. Theta-functions

Classically, automorphic forms are constructed from theta-series attached to quadratic forms. This is the procedure reformulated in representation theoretic terms by Weil in [47]. For SL_2 the idea is this.

Suppose F is a number field and q is an F-rational quadratic form in n variables. Define a distributuon θ on F^n by

$$\theta(\Phi) = \sum_{\xi \in F^n} \Phi(\xi).$$

Piecing together local Weil representations a representation r_q of $\overline{SL}_2(\mathbb{A})$ is defined with the property that $\theta(r_q(\overline{g})\Phi)$ is $SL_2(F)$-invariant. In particular, $\theta(r_q(s)\Phi = \theta(\Phi)$ for all $s \in SL_2(F)$. This is the theta function attached to q. For GL_2, the point of departure is Proposition 2.32 below.

For v a non-archimedean place of F let O_v denote the ring of integers of F_v and U_v its group of units. Let $r_v(q)$ denote the corresponding Weil representation of $GL_2(F_v)$ in $L^2(F^n \times F^x)$. As in Subsection 2.1, K_v^N is the subgroup of $GL_2(O_v)$ consisting of matrices $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ with $a \equiv 1$ and $c \equiv 0 \pmod N$. Here N is a positive integer divisible by 4. Thus $K_v^N = GL_2(O_v)$ if F_v has odd residual characteristic.

<u>Proposition 2.32</u>. Suppose $q = q_1$ or q_3. Then for almost every v, $r_v(q)$ is class 1, i.e., the restriction of $r_v(q)$ to $GL_2(O_v)$ has at least one fixed vector. More precisely, for each place v, define $\Phi_v^o \in L^2(F^n \times F^x)$ by

$$(2.50) \qquad \Phi_v^o(x_1,\ldots,x_n,t) = \left(\prod_{i=1}^{n} 1_{O_v}\right) \otimes 1_{U_v^N}.$$

Here 1_{O_v} denotes the characteristic function of O_F and $1_{U_v^N}$ denotes the characteristic function of $U_v^n = \{y \in U_v : y \equiv 1 \pmod N\}$.

<u>For all odd v,</u>

(2.51) $$r_v(q)(k)\Phi_o = \Phi_o$$

for $k \in K_v^N$.

<u>Proof</u>. The group $GL_2(O_v)$ is generated by the matrices

$$\begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} \ (b \in O_v)$$

$$w$$

and

$$\begin{bmatrix} 1 & 0 \\ 0 & a \end{bmatrix} \ (a \in U_v).$$

Thus it suffices to check (2.51) for these generators.

Recall that our canonical additive character τ_v has conductor O_v. In particular, $\tau_v(tbq(X)) = 1$ if $q(X) \in O_v$, $b \in O_v$, and $t \in U_v$. Therefore $r_v(q)\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}\Phi_v^o(X,t) = \Phi_v^o(X,t)$. Note also that the Fourier transform of 1_{O_v} is 1_{O_v}. Thus $r_v(q)(w)\Phi_v^o(X,t) = \Phi_v^o(X,t)$. The fact that $r_v(q)(\begin{bmatrix} 1 & 0 \\ 0 & a \end{bmatrix})\Phi_v^o(X,t) = |a|^{-1/2}\Phi_v^o(X,ta^{-1}) = \Phi_v^o(X,t)$ is obvious since $a \in U_v$. $\square$

To define r_q globally, and to introduce theta-functions on $\overline{G_{\mathbb{A}}}$, we need first to define an appropriate space of Schwartz-functions on $F_{\mathbb{A}}^n \times F_{\mathbb{A}}^x$. We shall say that Φ on $F_{\mathbb{A}}^n \times F_{\mathbb{A}}^x$ is "Schwartz-Bruhat" if $\Phi(X,t) = \prod_v \Phi_v(X_v,t_v)$ and:

(i) $\Phi_v(X_v,t_v)$ is infinitely differentiable and rapidly decreasing on $F_v^n \times F_v^x$ for each archimedean v;

(ii) for each finite v, Φ_v is the restriction to $F_v^n \times F_v^x$ of a locally constant compactly supported function on F_v^{n+1};

(iii) <u>for almost every finite v</u>, $\Phi_v = \Phi_v^o$ (the function defined by (2.50)).

Denote this space of functions by $\mathcal{S}(F_{\mathbb{A}}^n \times F_{\mathbb{A}}^x)$. Since $\mathcal{S}(F_{\mathbb{A}}^n \times F_{\mathbb{A}}^x)$ is dense in $L^2(F_{\mathbb{A}}^n \times F_{\mathbb{A}}^x)$, we can define a representation r_q of $\overline{G_{\mathbb{A}}}$ in $L^2(F_{\mathbb{A}}^n \times F_{\mathbb{A}}^x)$ through the formula

$$r_q(\overline{g})\Phi = \otimes_v r_v(q)(\overline{g}_v)\Phi_v.$$

By virtue of Proposition 2.32 this definition is meaningful for at least all $\Phi \in \mathcal{S}(F_{\mathbb{A}}^n \times F_{\mathbb{A}}^{x})$. Indeed for almost all v, $\overline{g}_v \in GL_2(O_v)$ and $\Phi_v = \Phi_v^o$. Thus $r_v(q)(\overline{g}_v)\Phi_v = \Phi_v$ and $r_q(\overline{g})\Phi$ again belongs to $\mathcal{S}(F_{\mathbb{A}}^n \times F_{\mathbb{A}}^{x})$. By continuity, $r_q(\overline{g})$ operates (unitarily) in L^2. The role of τ_v in the local theory is now played by a non-trivial character τ of $F\backslash\mathbb{A}$.

The theta-functions on $\overline{G_{\mathbb{A}}}$ corresponding to q (or r_q) are defined as follows. For each $\Phi \in \mathcal{S}(F_{\mathbb{A}}^n \times F_{\mathbb{A}}^{x})$ define $\theta(\Phi,\overline{g})$ on $\overline{G_{\mathbb{A}}}$ by

$$\theta(\Phi,\overline{g}) = \sum_{(\xi,\eta)\in F^n \times F^x} (r_q(\overline{g}))\Phi(\xi,\eta) .$$

<u>Proposition 2.33</u>. For each $\gamma \in GL_2(F)$,

$$\theta(\Phi,\gamma\overline{g}) = \theta(\Phi,\overline{g}); \tag{2.52}$$

i.e., $\theta(\Phi,\overline{g})$ is $GL_2(F)$ invariant.

<u>Proof</u>. We may assume without loss of generality that $\overline{g} = \{1,1\}$ and γ is a generator of $GL_2(F)$. Suppose first that $\gamma = \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix}$ with $b \in F$. Then

$$r_q(\gamma)\Phi(X,t) = \tau(btq(X)\Phi(X,t)$$

with $\tau = \Pi\tau_v$ trivial on F. But q is F-rational. Thus $r_q(\gamma)\Phi(\xi,\eta) = \Phi(\xi,\eta)$ for $(\xi,\eta) \in F^n \times F^x$ and (2.52) is immediate.

Now suppose $\gamma = w$. From Weil's theory (cf. Theorem 5 of [47]) it follows that $\gamma(q,\tau_t) \equiv 1$. Moreover, $|t| = 1$ if $t \in F^x$. Therefore,

$$\Sigma\, r_q(w)\Phi(\xi,\eta) = \Sigma|\eta|^{n/2}\, \gamma(q,\tau_\eta)\hat{\Phi}(\eta\xi,\eta)$$

$$= \sum_{(\xi,\eta)} \hat{\Phi}(\eta\xi,\eta)$$

$$= \Sigma\ \hat{\Phi}(\xi,\eta)$$

$$= \Sigma\ \Phi(\xi,\eta).$$

The last step above follows from Poisson's summation formula.

Finally suppose $\gamma = \begin{bmatrix} 1 & 0 \\ 0 & a \end{bmatrix}$ with $a \in F^x$. By formula (2.48), $r_q(\gamma)\Phi(\xi,\eta) = |a|^{-1/2}\ \Phi(\xi,\eta a^{-1})$. But $|a|^{-1/2} = 1$. Therefore $\Sigma\ r_q(\gamma)\Phi(\xi,\eta) = \Sigma\ \Phi(\xi,\eta)$ as desired. $\square$

Observation 2.34. The function $\theta(\Phi,\overline{g})$ always defines an automorphic form on $\overline{G_{\mathbb{A}}}$, i.e., $\theta(\Phi,\overline{g})$ is always G_F invariant. Nevertheless, it may often be the case that $\theta(\overline{g})$ is identically zero. Suppose, for example, that $\Phi(-X,t) = -\Phi(X,t)$, i.e. Φ is an odd function in X. Then $r_q(\overline{g})\Phi$ is also odd for all $\overline{g} \in \overline{G_{\mathbb{A}}}$. For simplicity, suppose also that $F = \mathbb{Q}$. From formula (2.47) it follows that $r_q(\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix})\Phi(X,t) = -\Phi(X,t)$, i.e.

$$\theta(\Phi,\ \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\overline{g}) = -\ \theta(\Phi,\overline{g}).$$

But by the G_F-invariance of θ,

$$\theta(\Phi,\overline{g}) = \theta(\Phi,\ \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\overline{g}).$$

Therefore $\theta(\Phi,\overline{g}) \equiv 0$.

We close this paragraph by demonstrating how $\theta(\Phi,\overline{g})$ generalizes the basic theta function

$$\theta(z) = \sum_{n=-\infty}^{\infty} e^{2\pi i n^2 z}.$$

Proposition 2.35. Fix $n = 1$, $F = \mathbb{Q}$, and $r_q = r_1$. Suppose $\Phi = \Pi\ \Phi_p$ where $\Phi_p = \Phi_p^0$ for all finite p and $\Phi_\infty(x,t) = |t|^{-1/4}\ e^{-|t|2\pi x^2}$. Then if $\overline{g} = \{\begin{bmatrix} y^{1/2} & xy^{-1/2} \\ 0 & y^{-1/2} \end{bmatrix}, 1\}$,

$$\theta(\Phi,\overline{g}) = y^{1/4}\theta(x+iy).$$

Proof. Since $\begin{bmatrix} y^{1/2} & xy^{-1/2} \\ 0 & y^{-1/2} \end{bmatrix} = \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix}\begin{bmatrix} y^{1/2} & 0 \\ 0 & y^{-1/2} \end{bmatrix}$,

$$\Sigma\, r_q(\overline{g})\Phi(\xi,\eta) = \sum_{(\xi,\eta)} |y|^{1/4}\, e^{-2\pi y\xi^2}\, e^{2\pi i\xi^2 x\eta}.$$

Here we choose $\tau = \Pi\, \tau_p$ with $\tau_\infty(x) = e^{2\pi ix}$ and τ_p trivial on O_p for $p < \infty$. Thus (ξ,η) contributes to the summation above only if $\xi \in \mathbb{Z}$ and $\eta = 1$ (recall $\Phi_2 = 1_{O_2} \cdot 1_{U_2^N}$). I.e.,

$$\theta(\Phi,\overline{g}) = y^{1/4} \sum_{n=-\infty}^{\infty} e^{2\pi in^2(x+iy)},$$

as was to be shown. $\Box$

Note that for $a \in (0,\infty)$, and Φ as above,

$$r(\begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix})\Phi(\xi,\eta) = |a|^{-1/2}\, \Phi(ax, ta^{-2})$$

$$= |a|^{-1/2}\, |ta^{-2}|^{-1/4}\, e^{-|t|2\pi x^2}$$

$$= |t|^{-1/4}\, e^{-|t|2\pi x^2} = \Phi(\xi,\eta).$$

Thus $\theta(\Phi,\overline{g})$ is actually defined on

$$\overline{X} = Z_\infty^0\, G_F \backslash \overline{G_{\mathbb{A}}}\ .$$