189-570A: Higher Algebra
Assignment 3

Due: Monday, November 20

1. Let k be a field and G be a finite group whose order is not divisible by the characteristic of k. Show that the group ring $k[G]$ is isomorphic to a direct sum of rings

$$k[G] \simeq M_{n_1}(D_1) \oplus \cdots \oplus M_{n_r}(D_r), \tag{1}$$

where each D_i is a (not necessarily commutative) field containing k in its center, and $M_n(D)$ denotes the $n \times n$ matrix ring with entries in k.

2. For each of the following values of k and G, find the invariants r, $(n_1, \ldots, n_r)$ and $(D_1, \ldots, D_r)$ for which the isomorphism (1) holds. (For this question, you will find it useful to consult the character tables you have already constructed for various groups.)

a) $G = \mathbf{Z}/3\mathbf{Z}$, $k = \mathbf{R}$.
b) $G = \mathbf{Z}/5\mathbf{Z}$, $k = \mathbf{Q}$.
c) $G = \mathbf{Z}/5\mathbf{Z}$, $k = \mathbf{R}$.
d) $G = D_8$, $k = \mathbf{R}$.
e) $G = Q$, $k = \mathbf{R}$.
f) $G = S_5$, $k = \mathbf{Q}$.
g) $G = A_5$, $k = \mathbf{Q}$.

3. Let H_1 and H_2 be subgroups of a finite group G. Show that the following two conditions are equivalent.

a) For all conjugacy classes C of G, the intersections $C \cap H_1$ and $C \cap H_2$ have the same cardinalty.

b) For all representations V of G, the dimensions of V^{H_1} and V^{H_2} are the same. (Here V^H denotes the subspace of vectors that are fixed by all elements of H.)

(Hint: You may find it useful to take another look at problem 10 of assignment 1.)

4. Let χ be a $\mathbf{C}$-valued character of a finite group G. Show that $|\chi(g)| \leq \chi(1)$, and that this inequality is strict if G is simple.

5. Let g be an element of order 3 in a finite group G. Show that $\chi(g)$ is rational for all characters χ, if and only if g is conjugate to g^2. What can you say if g has order 5?

6. Let A and B be Noetherian local rings with maximal ideals m_A and m_B respectively. Let $f : A \longrightarrow B$ be a ring homomorphism satisfying $f^{-1}(m_B) = m_A$, and assume that

a) The induced homomorphism $A/m_A \longrightarrow B/m_B$ is an isomorphism;
b) The induced homomorphism $m_A \longrightarrow m_B/m_B^2$ is surjective;
c) B is finitely generated as an A-module, via f.
Show that f is surjective.

7. Let A be a Noetherian local ring without nilpotent elements, and let E be a finitely generated A-module. Assume that for each prime ideal p of E, the dimension of E_p/pE_p over A_p/pA_p is independent of p. Show that E is a free A-module.

8. In the case where $A = \mathbf{Z}_p[[T]]$ and $E \subset A \times A$ is the set of (f, g) with $f(0) = g(0)$, show that E is not free by exhibiting two prime ideals p_1 and p_2 of A for which the dimensions of $E_{p_j}/p_j E_{p_j}$ are different.

9. Show that if R is a Noetherian ring and S is a multiplicative subset of R then $S^{-1}R$ is Noetherian. (Hence, in particular, the class of Noetherian rings is preserved by localisation.)