# 189-235A: Basic Algebra I

# Practice Final Exam

*This exam has ten questions, worth 10 points each. The bonus question is worth 20 points. The final grade will be out of* 100*, even though the maximum possible grade is* 120.

1. Show that 64 divides $9^n - 8n - 1$ for every $n \geq 0$, by induction on $n$.

2. Solve each of the following congruence equations.
   a) $6x = 3 \pmod{30}$,
   b) $x^2 + 4x + 5 = 0 \pmod{13}$.
   c) $x^7 = 1 \pmod{101}$.

3. The *exponent* of a finite group is the smallest positive integer $n$ such that $a^n = 1$ for all $a \in G$. Let $G$ be an abelian group.
   a) Show that $G$ is cyclic if and only if its exponent is equal to its cardinality.
   b) Use part a) to show that a finite subgroup of the multiplicative group of a field is necessarily cyclic.

4. Prove that the rings $\mathbf{R}[x]/(x^2 + 1)$ and $\mathbf{R}[x]/(x^2 - 1)$ are *not* isomorphic.

5. Give a non-commutative group $G$ of order 125 in which $a^5 = 1$ for all $a \in G$.

6. Are the groups $D_4 \times \mathbf{Z}_3$ and $S_3 \times \mathbf{Z}_4$ isomorphic? Prove or disprove.

7. Give the definitions of: *maximal ideal* and *prime ideal* in a commutative ring $R$. Show that if $I$ is a prime ideal then $R/I$ is an integral domain.

8. State the isomorphism theorem for rings. Show that the ring $R_1 = \mathbf{Z}[x]/(3x-2)$ is isomorphic to the subring $R_2$ of $\mathbf{Q}$ consisting of all rational numbers whose denominator is a power of 3. (Hint: construct a surjective homomorphism from $\mathbf{Z}[x]$ to $R_2$ whose kernel is the ideal $(3x-2)\mathbf{Z}[x]$.)

9. For each of the following rings $R$, state whether or not there exists a homomorphism from $R$ to $\mathbf{Z}_3$, and, if so, how many there are.

9a) $R = \mathbf{Z}_7$;
9b) $R = \mathbf{Z}_3[x]$;
9c) $R = \mathbf{Z}_3[x]/(x^2+2)$;
9d) $R = \mathbf{Z}_3[x]/(x^2-2)$.

10. Show that the symmetric group $S_5$ contains elements of order 6, and that they are all conjugate. How many such elements are there?

**Extra Credit Problem**

11. Let $G = \mathbf{GL}_3(\mathbf{Z}_2)$ be the group of invertible $3 \times 3$ matrices with entries in $\mathbf{Z}_2$. What is the cardinality of this group? Write down representatives for each of the distinct conjugacy classes, and give their orders, ie., compute the class equation for $G$. Use this to show that $G$ has no non-trivial normal subgroups.