189-235A: Basic Algebra I

Practice Final Exam

This exam has ten questions, worth 10 points each. The bonus question is worth 20 points. The final grade will be out of 100, *even though the maximum possible grade is* 120.

1. Show that 64 divides $9^n - 8n - 1$ for every $n \geq 0$, by induction on n.

2. Solve each of the following congruence equations.
a) $6x = 3 \pmod{30}$,
b) $x^2 + 4x + 5 = 0 \pmod{13}$.
c) $x^7 = 1 \pmod{101}$.

3. The *exponent* of a finite group is the smallest positive integer n such that $a^n = 1$ for all $a \in G$. Let G be an abelian group.
a) Show that G is cyclic if and only if its exponent is equal to its cardinality.
b) Use part a) to show that a finite subgroup of the multiplicative group of a field is necessarily cyclic.

4. Prove that the rings $\mathbf{R}[x]/(x^2 + 1)$ and $\mathbf{R}[x]/(x^2 - 1)$ are *not* isomorphic.

5. Give a non-commutative group G of order 125 in which $a^5 = 1$ for all $a \in G$.

6. Are the groups $D_4 \times \mathbf{Z}_3$ and $S_3 \times \mathbf{Z}_4$ isomorphic? Prove or disprove.

7. Give the definitions of: *maximal ideal* and *prime ideal* in a commutative ring R. Show that if I is a prime ideal then R/I is an integral domain.

8. State the isomorphism theorem for rings. Show that the ring $R_1 = \mathbf{Z}[x]/(3x-2)$ is isomorphic to the subring R_2 of $\mathbf{Q}$ consisting of all rational numbers whose denominator is a power of 3. (Hint: construct a surjective homomorphism from $\mathbf{Z}[x]$ to R_2 whose kernel is the ideal $(3x-2)\mathbf{Z}[x]$.)

9. For each of the following rings R, state whether or not there exists a homomorphism from R to $\mathbf{Z}_3$, and, if so, how many there are.

9a) $R = \mathbf{Z}_7$;
9b) $R = \mathbf{Z}_3[x]$;
9c) $R = \mathbf{Z}_3[x]/(x^2+2)$;
9d) $R = \mathbf{Z}_3[x]/(x^2-2)$.

10. Show that the symmetric group S_5 contains elements of order 6, and that they are all conjugate. How many such elements are there?

Extra Credit Problem

11. Let $G = \mathbf{GL}_3(\mathbf{Z}_2)$ be the group of invertible 3×3 matrices with entries in $\mathbf{Z}_2$. What is the cardinality of this group? Write down representatives for each of the distinct conjugacy classes, and give their orders, ie., compute the class equation for G. Use this to show that G has no non-trivial normal subgroups.