189-235A: Basic Algebra I
Assignment 8
Due: Wednesday, November 23

1. Let F be a field, and define a binary composition law on $G = F - \{1\}$ by the rule

$$a * b = a + b - ab.$$

Show that G, with this operation, is a group. (In particular, write down the neutral element for $*$, and give a formula for the inverse of $a \in G$.

2. List all the elements of order 3 in S_3. How many are there?

3. List all the elements of order 6 in S_5. How many such elements are there?

4. Given an example of non-abelian groups of order 12 and 30.

5. Suppose that G is a group in which $x^2 = 1$, for all $x \in G$. Show that G is abelian. Give an example of a **non-abelian** group G of order 27 in which $x^3 = 1$ for all $x \in G$.

6. Show that the groups S_3 and $\mathbf{GL}_2(\mathbf{Z}_2)$ are isomorphic, by writing down an isomorphism between them.

7. A *transposition* in the symmetric group S_n is a permutation of the form (ab) (i.e., a permutation that interchanges two elements $a, b \in \{1, \ldots, n\}$, leaving all other $n-2$ elements fixed. Show that every permutation in S_n can be expressed as a product of transpositions.