189-235A: Basic Algebra I
Assignment 4
Due: Friday, October 14

1. Let R be a commutative ring. Is the set S of matrices of the form $\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$, with $a, b, c \in R$, a subring of $M_2(R)$?

2. Same question as 1, but with S replaced by the set of matrices of the form $\begin{pmatrix} a & b \\ 0 & a \end{pmatrix}$, with $a, b \in R$.

3. Show that the ring $\mathbf{Q}$ of rational numbers has no subrings which are finite sets.

4. Let A be a commutative ring, let $R = A \times A$, and let S be the subset of elements of R of the form $(a, 0)$, with $a \in A$. Show that S is a ring which is isomorphic to A.

5. Let R be a ring, and let a be an element of R which is not a zero divisor. Show that the cancellation law can be applied to a, i.e., for all $x, y \in R$, if $ax = ay$ then $x = y$.

6. Let $R = \mathbf{Q}(\sqrt{2})$ be the ring of elements of the form $a + b\sqrt{2}$, with $a, b \in \mathbf{Q}$. Show that the function which sends $a + b\sqrt{2}$ to $a - b\sqrt{2}$ is an isomorphism from R to itself.

7. Let R be a ring. Show that there is a *unique* ring homomorphism f from $\mathbf{Z}$ to R.

8. Given a ring R, let $R[i]$ denote the set of pairs (a, b) with $(a, b) \in R$. Define an addition and multiplication on $R[i]$ by the rules:

$$(a_1, b_1) + (a_2, b_2) = (a_1 + a_2, b_1 + b_2).$$

$$(a_1, b_1)(a_2, b_2) = (a_1a_2 - b_1b_2, a_1b_2 + b_1a_2).$$

Show that these rules equip $R[i]$ with the structure of a ring.

9. Show that the subset S of the ring $R[i]$ of exercise 8 consisting of elements of the form $(r, 0)$ with $r \in R$ is a subring of $R[i]$ which is isomorphic to R. Produce an element i in $R[i]$ satisfying $i^2 = -1$.

10. Show that the rings $\mathbf{Z}_{48}$ and $\mathbf{Z}_6 \times \mathbf{Z}_8$ are *not* isomorphic.

Optional questions.

11. Keeping the notations of exercise 8, show that the ring $\mathbf{C}[i]$ is isomorphic to $\mathbf{C} \times \mathbf{C}$.

12. Show that a ring R which is a finite set and an integral domain (has no zero-divisors) is necessarily a field.