THE RESIDUAL FINITENESS OF NEGATIVELY CURVED POLYGONS OF FINITE GROUPS

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Abstract. A subgroup $M \subset G$ is *almost malnormal* provided that for each $g \in G - M$, the intersection $M^g \cap M$ is finite. It is proven that the free product of two virtually free groups amalgamating a finitely generated almost malnormal subgroup, is residually finite. A consequence of a generalization of this result is that an acute-angled n-gon of finite groups is residually finite if $n \geq 4$. Another consequence is that if G acts properly discontinuously and cocompactly on a 2-dimensional hyperbolic building whose chambers have acute angles and at least 4 sides, then G is residually finite.

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1. Introduction

In [12], Gromov revolutionized infinite group theory by treating groups as geometric objects. From this viewpoint, the most prominent class of groups are the word-hyperbolic groups which are characterized by being coarsely negatively curved. In the last ten years, word-hyperbolic groups have received a tremendous amount of attention, and many of their properties have been understood. However, one striking problem which remains open is:

Problem 1.1 (Gromov [12]). Is every word-hyperbolic group residually finite?

Recall that a group G is residually finite if for each nontrivial $g \in G$, there is a finite quotient $G \to \bar{G}$ such that $\bar{g}$ is nontrivial. Problem 1.1 is considered a very difficult problem, and most workers in the field are expecting a negative answer. However, in this paper we prove a result

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that indicates that at least in certain natural cases, there is indeed a connection between negative curvature and residual finiteness.

The main result of this paper is Theorem 11.3 which asserts that groups which split as certain graphs of virtually free groups are residually finite. We will state a special case of Theorem 11.3 after the following definition:

Definition 1.2 (Almost Malnormal)**.** A subgroup $M \subset G$ is *malnormal* provided that for each $g \in G - M$, the intersection $M \cap M^g$ is trivial. Similarly, M is *almost malnormal* provided that for each $g \in G - M$, the intersection $M \cap M^g$ is finite.

Theorem 1.3. *Let $G = A *_M B$ be an amalgamated free product, where A and B are virtually free, and M is a finitely generated almost malnormal subgroup of A and B. Then G is residually finite.*

Theorem 1.3 is a special case of the main result, Theorem 11.3, which establishes residual finiteness if G splits as a graph of virtually free groups satisfying a similar condition. Theorem 1.3 is distinguished from previous residual finiteness results in two ways. Firstly, the known residual finiteness results involving graphs of residually finite groups typically rely upon (often quite simple) algebraic rather than geometric techniques. For instance, the free product of two residually finite groups amalgamating a finite group is residually finite. We refer the reader to [22] for a survey of some typical results. Another more recent theorem is that the graph product of finite groups is residually finite which was proven by E. Green in her thesis [11]. This was reproven in [17] where these group were actually shown to be linear.

Secondly, word-hyperbolicity plays a circumstantial role in the proof of residual finiteness. Bestvina and Feighn gave a combination theorem in [4] which allows one to show that certain graphs of word-hyperbolic groups are themselves word-hyperbolic (see also [19]). One of the simplest cases of their theorem is that the groups of Theorem 1.3 are word-hyperbolic.

In this paper we also provide several simple applications of Theorem 11.3 towards negatively curved-polygons of finite groups, groups acting on buildings, and negatively curved squared complexes. Other applications of Theorem 11.3 to the residual finiteness of small-cancellation groups and one-relator groups are given in [40], [34] and [35].

Recall that a group G is *Hopfian* provided that any surjective endomorphism $G \to G$ is an automorphism. It is an old result of Malćev [21, IV.4.10] that finitely generated residually finite groups are Hopfian. Therefore, some evidence for an affirmative answer to Problem 1.1 is provided by Sela [29] who proved the remarkable result that every torsion-free word-hyperbolic group is Hopfian. Examples were provided in [37] of compact nonpositively curved 2-complexes whose fundamental groups are not Hopfian. This highlights the following theme which we propound below: word-hyperbolic groups have stronger residual properties than nonpositively curved groups.

Problem 1.1 largely subsumes the following related problem which predates Gromov's introduction of word-hyperbolic groups.

Problem 1.4 (Schupp [27])**.** Is every finitely presented small-cancellation group residually finite?

We note that Problem 1.4 is in fact a special case of Problem 1.1 because finitely presented small-cancellation groups are word-hyperbolic except for the limiting cases which are: $C(3)$–$T(6)$, $C(4)$–$T(4)$, and $C(6)$ (see [21] for more about small-cancellation groups). It turns out that residual finiteness need not hold in these non-word-hyperbolic limiting cases. This is discussed below after the following definition which demarcates the most important subclasses of $C(4)$–$T(4)$ and $C(4)$–$T(5)$ small-cancellation complexes:

Definition 1.5 (curvature of squared complexes)**.** A *squared* 2*-complex* is a 2-complex whose 2-cells are attached by combinatorial paths of length 4. Thus, any squared 2-complex can be obtained from a graph by attaching Euclidean squares through the identification of their edges with edges in the graph. A squared 2-complex X is *nonpositively curved* provided that the link of each vertex of X does not contain any cycles of length < 4. Similarly, X is *negatively curved* provided that there are no cycles of length < 5.

It turns out that Problem 1.4 has a negative solution for small-cancellation groups which are not word-hyperbolic. The first non-residually finite finitely presented small-cancellation groups were constructed in [38] where the following theorem was proven.

Theorem 1.6. *There is a compact nonpositively curved squared* 2*-complex* X*, such that* $\pi_1 X$ *has no finite quotients. Furthermore, the universal cover* $\tilde{X}$ *of* X *is the direct product of two trees.*

We note that X can be subdivided so that $\pi_1 X$ is a $C(3)$–$T(6)$ group as well. However, it remains an open question whether $C(6)$ groups are residually finite. More recently, Burger and Mozes [8] have announced that there are groups as in Theorem 1.6 which are even simple.

Definition 1.7. A squared 2-complex X is *locally-*$\mathcal{VH}$ provided that the link of each vertex of X is a bipartite graph.

The examples in Theorem 1.6 are locally-$\mathcal{VH}$ complexes. In fact they are examples of $\mathcal{VH}$-complexes which will be discussed later in Definition 4.1. As an application of Theorem 11.3 we obtain the following:

Theorem 1.8. *Let* X *be a compact negatively curved locally-*$\mathcal{VH}$ *complex, then* $\pi_1 X$ *is residually finite.*

Again the residual finiteness result in the negatively curved case stands in stark contrast with the non-residually finite counterexamples in the nonpositively curved case, which we described above in Theorem 1.6. We emphasize that in both cases the 2-complexes are $\mathcal{VH}$.

We refer the reader to [31] for the notion of a quasiconvex subgroup of a word-hyperbolic group. Roughly speaking, a subgroup is quasiconvex if it corresponds to a convex subspace of the group - in a coarse sense. We note that the trivial subgroup and every cyclic subgroup of a word-hyperbolic group is quasiconvex. Problem 1.1 can be generalized to the following problem to which it is probably equivalent (see Remark 11.8 for the rationale for this guess).

Problem 1.9. Is every quasiconvex subgroup of a word-hyperbolic group separable?

Recall that a subgroup $H \subset G$ is *separable* provided that H is the intersection of finite index subgroups of G. Observe that G is residually finite if and only if the trivial subgroup of G is separable. Therefore the property that every quasiconvex subgroup be separable is a much stronger property than residual finiteness. By combining results in this paper with a separability result from [36] which is stated in Theorem 4.6, we demonstrate that Problem 1.9 has an affirmative answer in the following case.

Theorem 1.10. *Let* X *be a compact negatively curved locally-*$\mathcal{VH}$ *complex. Then every quasiconvex subgroup of* $\pi_1 X$ *is separable.*

Polygons of Groups

We now turn our attention to the residual properties of polygons of finite groups. Let $\tilde{P}$ be a simply connected 2-complex, and let G act on $\tilde{P}$ cellularly and with the property that if

$g \in G$ stabilizes a cell, then g fixes it pointwise. Suppose that $G \backslash \tilde{P}$ consists of a single 2-cell P. By identifying P with some 2-cell of $\tilde{P}$, we obtain a polygonal amalgam of groups, which is a commutative diagram consisting of the stabilizers of cells in P and the various inclusions between them. For instance, when P is a square we obtain the following diagram:

$$\begin{array}{ccccc} V_1 & \leftarrow & E_1 & \rightarrow & V_2 \\ \uparrow & & \uparrow & & \uparrow \\ E_4 & \leftarrow & F & \rightarrow & E_2 \\ \downarrow & & \downarrow & & \downarrow \\ V_4 & \leftarrow & E_3 & \rightarrow & V_3 \end{array}$$

The group G is the direct limit of this diagram and G is referred to as $\pi_1 P$.

Of course, one can begin with any (polygonal) diagram of inclusions but it is not always the case that the corresponding amalgam is *developable* meaning that it arises from an action on $\tilde{P}$ as above. See [33] for a surprising examples of polygons of nontrivial groups whose direct limit is the trivial group.

We now briefly describe the criterion given by Gersten and Stallings [33] for a polygon of groups to be developable. We note that only the case of a triangle is treated explicitly in [33] but the same proof works for any n-gon. We refer the reader to [9, 13] for more general treatments.

Given subgroups $A \subset D$, $B \subset D$, and $C \subset (A \cap B)$, the *Gersten-Stallings angle* between A and B with respect to C is $\frac{2\pi}{n}$ where n is the length of the shortest normal form in the kernel of $A *_C B \to D$. Accordingly, at each corner of P, there is an angle assigned where D is the vertex group, A and B are the edge groups, and C is the face group. The polygon is *nonpositively* [*negatively*] *curved* provided that the sum of the angles of P is $\leq$ [$<$] the sum of a Euclidean polygon with the same number of sides. Gersten and Stallings then prove that:

Theorem 1.11. *Let P be a nonpositively [negatively] curved polygon of groups, then P is developable, and the corresponding $\tilde{P}$ can be metrized as a $CAT(0)$ [$CAT(-1)$] space.*

Examples of nonpositively curved polygons of finite groups which are not residually finite, were obtained by Hsu and Wise [16] (see also [15] for an improved construction). Our method was to embed the non-residually finite groups of Theorem 1.6 into nonpositively curved polygons of groups. The result follows because a subgroup of a residually finite group must itself be residually finite. Specifically we showed that:

Theorem 1.12. *There is a nonpositively curved square of finite groups which is not residually finite. There is a nonpositively curved triangle of finite groups which is not residually finite.*

In contrast to Theorem 1.12 we prove (see Theorem 9.1) that:

Theorem 1.13. *Let P be an n-gon of finite groups where $n \geq 4$. Suppose that the Gersten-Stalling's angle at each vertex of P is acute. Then every quasiconvex subgroup of $\pi_1 P$ is separable, and in particular, $\pi_1 P$ is residually finite.*

For $n \geq 4$, the fundamental group of a nonpositively curved n-gon of groups has a well-known splitting. This splitting occurs along an internal line segment in the polygon which divides the vertices into two sets, each of which contains at least 2 vertices. By using this splitting, Theorem 1.13 can be deduced from Theorem 1.3. The contrast between Theorem 1.12 and Theorem 1.13 is striking and suggests that there may be a relationship between negative curvature and residual finiteness.

Another application along the same lines as Theorem 1.13 is the following (see Theorem 10.1 for a more general statement):

Theorem 1.14. *Let B be a 2-dimensional hyperbolic building and let C be a chamber of B. Suppose that C has at least 4 sides and suppose that each corner of C is acute. Then G is residually finite, and each quasiconvex subgroup of G is separable.*

Technical overview

A graph of groups is *clean* provided that each vertex group is free, and each edge group is a free factor in its vertex groups. The proof that a group which splits as a clean graph of finitely generated free groups is residually finite is easy and is given in Section 3. Roughly speaking, a graph of groups is *thin* provided that distinct conjugates of edge groups have finite intersection. (see Section 2 for a more rigorous discussion). Technically, the main result of the paper is Theorem 11.3 which states that if a group splits as a finite thin graph of finitely generated virtually free groups then there is a finite index subgroup whose induced splitting is clean.

The results and methods in this paper are closely related to [39] where we obtained Theorem 1.15 below. Note that if the second possibility in Theorem 1.15 holds then the graph of groups is not thin.

Theorem 1.15. *Let G split as a finite graph of free groups with cyclic edge groups. Then exactly one of the following holds:*

1. *G has a subgroup of finite index whose induced splitting is clean.*
2. *There is a nontrivial element $g \in G$ such that g^m is conjugate to g^n for nonzero m, n with $m \neq \pm n$.*

A special case of Theorem 11.3 is proven in Theorem 7.3 in a more geometric context. Roughly speaking, a *non-positively curved $\mathcal{VH}$-complex* is a space which splits as a graph of spaces where the vertex spaces are graphs, the edge spaces are products of graphs and intervals, and the attaching maps are immersions of graphs. There is a corresponding notion of *thin* $\mathcal{VH}$-complex, and a notion of *clean* $\mathcal{VH}$-complex which means that every attaching map is an embedding. A clean $\mathcal{VH}$-complex therefore has a splitting as a clean graph of groups. Theorem 7.3 states that any compact thin $\mathcal{VH}$-complex has a clean finite cover. The subgroup separability results are derived by combining Theorem 7.3 with Theorem 4.6 which was proven in [36] and applies to clean $\mathcal{VH}$-complexes.

The proof of Theorem 7.3 is more concrete than the proof of Theorem 11.3 which employs an extra layer of notation. However, all the crucial ideas are already present in Theorem 7.3, and the reader will more readily understand the technique by first concentrating exclusively on this geometric case. To further aid the reader, the proof of Theorem 11.3 is written so that the steps and notation are parallel (wherever possible) to the steps and notation employed in the proof of Theorem 7.3.

The applications in Sections 8, 9 and 10 which involve groups with torsion, employ Theorem 8.1.1, whose statement is a generalization of Theorem 7.3 to the case of groups with torsion. Unfortunately, the proof of Theorem 8.1.1 employs Theorem 11.3. However, I believe the reader will be comfortable reading the sections in the given order particularly because the theorems and applications involving $\mathcal{VH}$-complexes are gathered together.

In Section 5 we describe the *canonical completion* which is the main technical tool employed in the proofs of Theorems 7.3 and 11.3. The canonical completion of an immersion of finite graphs $A \to B$ is a specific way to extend A to a covering space of B by adding finitely many vertices and edges. The canonical completion (and canonical retraction) plays a critical role in constructing finite covering spaces of graphs which have desired properties.

The basic plan we use to prove Theorem 7.3 is as follows: Let L, R and E be finite graphs, let $E \to L$ and $E \to R$ be immersions, and let $E \times I$ be the product of E with an interval. Let X be a $\mathcal{VH}$-complex formed by attaching the two ends of $E \times I$ to L and R using the maps $E \to L$ and $E \to R$. We use the canonical completion and retraction to produce finite regular covers $\hat{L}$ and $\hat{R}$ which induce the same cover $\hat{E}$, such that furthermore, $\hat{E} \to \hat{L}$ and $\hat{E} \to \hat{R}$ are embeddings. We then form a clean finite cover of X by gluing together copies of $\hat{L}$, $\hat{R}$ and $\hat{E} \times I$.

2. Thin Graphs of Groups and Graphs of Spaces

For $n \geq 1$, let I_n denote the graph consisting of n edges whose underlying space is $[0, n]$ and whose vertices are $\{0, 1, \ldots, n\}$. We let $I = I_1$.

Following Scott and Wall [28], most of our arguments about graphs of groups will be expressed in terms of graphs of spaces. We collect the basic notions and terminology regarding graphs of spaces in the following definition:

Definition 2.1. A graph Γ is a *graph of groups* provided that the following holds: Associated to each vertex $v \in \Gamma$ there is a vertex group G_v, and associated to each edge e of Γ, there is an edge group G_e. Associated to the attaching maps of e to its initial vertex $\iota(v)$ and its terminal vertex $\tau(v)$, there are monomorphisms $G_e \to G_{\iota(v)}$ and $G_v \to G_{\tau(v)}$.

A *graph of spaces* X_Γ corresponding to Γ is formed as follows: For each vertex $v \in \Gamma$, there is a connected based space X_v called a *vertex space* with $\pi_1 X_v = \pi_1 G_v$. For each edge e of Γ, there is a connected based space X_e with $\pi_1 X_v = \pi_1 G_e$. The space $X_e \times I$ is then called an *edge space.* Furthermore, corresponding to the monomorphisms $G_e \to G_{\iota(e)}$ and $G_e \to G_{\tau(e)}$ we have *attaching maps* $X_e \times \{0\} \to X_{\iota(e)}$ and $X_e \times \{1\} \to X_{\tau(e)}$ which induce monomorphisms $\pi_1 X_e \to \pi_1 X_{\iota(e)}$ and $\pi_1 X_e \to \pi_1 X_{\tau(e)}$ which are identical with $G_e \to G_{\iota(e)}$ and $G_e \to G_{\tau(e)}$. Finally, we form $X = X_\Gamma$ by attaching each edge space $X_e \times I$ to its vertex spaces $X_{\iota(e)}$ and $X_{\tau(e)}$ using the attaching maps. We will refer to edges of the form $\{v\} \times I$ where v is a vertex of X_e, as *horizontal* edges of X. Since we are primarily interested in studying graphs of groups, when we consider a graph of spaces we will always assume that all edge spaces are connected and that all attaching maps are π_1-injective.

Note that a covering space $\hat{X}$ of X inherits a structure as a graph of spaces. The vertex spaces of $\hat{X}$ are the connected components of the preimages of the vertex spaces of X, and the edge spaces of $\hat{X}$ correspond to the connected components in the complement of the vertex spaces in $\hat{X}$.

Definition 2.2 (Thin)**.** An action of a group G on a tree T is *thin* provided that the stabilizer of any embedded length 2 path in T is finite.

A graph of groups Γ is *thin* if the action of $\pi_1 \Gamma$ on $\tilde{\Gamma}$ is thin.

The proof of the following is left to the reader:

Lemma 2.3 (Characterizing thin)**.** *A graph of groups Γ is thin if and only if the following two conditions hold:*

(1) *For each end of an edge incident at v, the corresponding subgroup of G_v is almost malnormal.*
(2) *For each pair of ends of edges of Γ incident at v, any conjugates of the subgroups of G_v corresponding to these ends of edges have finite intersection.*

3. Clean Graphs of Groups and Graphs of Spaces

In this section we define a clean graph of groups. We then show that in the case of greatest interest, a group which splits as a clean graph of groups is residually finite. Next we say what it means for a graph of groups to be virtually clean, and finally we characterize virtually clean graphs of groups in terms of the separability of their edge groups.

Definition 3.1 (Clean graph of groups)**.** A graph of groups is *clean* provided that each vertex group is free and each edge group is a free factor of its vertex groups. A graph of spaces is *clean* provided that the corresponding graph of groups is clean.

The motivating example is a graph of spaces X where each vertex space is a graph, and each edge space is the direct product of a graph and I, and both attaching maps of each edge space are embeddings. We refer to this special type of clean graph of spaces as *geometrically clean.*

Geometrically clean graphs of spaces were studied extensively in [39] and [36] with applications to subgroup separability, and in [38] with applications towards understanding the relationship between immersed flat planes and $\mathbb{Z} \times \mathbb{Z}$ subgroups.

Lemma-Definition 3.2 (Characteristic)**.** A subgroup $K \subset G$ is said to be *characteristic* provided that for each automorphism ϕ of G, $\phi(K) = K$. If H is a finite index subgroup of a finitely generated group G, then H contains a finite index characteristic subgroup of G. Indeed, for each n, the subgroup $K_n = \cap_{[G:J]\leq n} J$ is obviously characteristic. Furthermore, K_n is of finite index because there are finitely many subgroups of a given index in a finitely generated group, and the intersection of finitely many finite index subgroups is of finite index. (See [21, IV.4.7]).

The following lemma is a simple case of a more general condition implying residual finiteness of certain graphs of groups satisfying a more elaborate residual condition on the embeddings of the edge groups. This sort of graph of group statement is implicit in [1] where amalgamated free products are treated. However, it is important to note that Baumslag's conditions are generally extremely difficult to verify in practice.

Lemma 3.3 (Residual finiteness criterion)**.** *Let $G = F*_{H_i^{t_i}=H_i'}$ be a multiple HNN extension of a group F where for each i, the stable letter t_i conjugates the subgroup H_i to the subgroup H_i'. Then G is residually finite provided that the following conditions hold:*

1. *F is finitely generated and residually finite.*
2. *For each i, H_i is separable.*
3. *For each i there is an automorphism ϕ_i of F, such that the isomorphism $H_i \to H_i'$ of edge subgroups that is determined by conjugation by t_i, is induced by ϕ_i.*

Proof. Observe that if C is a characteristic subgroup of F, then any quotient $F \to F/C = \bar{F}$ extends to a quotient $G = F*_{H_i^{t_i}=H_i'} \to \bar{F}*_{\bar{H}_i^{\bar{t}_i}=\bar{H}_i'} = \bar{G}$ where we let conjugation by $\bar{t}_i$ be the edge group isomorphism $\bar{H}_i \to \bar{H}_i'$ induced by $\bar{\phi}_i$.

We will now use this to show that any nontrivial element $g \in G$ survives in some finite quotient of G. In case g has a normal form of length 0, then we choose C to be a finite index characteristic subgroup such that $g \notin C$. Otherwise, g may be written in normal form as: $g = f_0 t_{i_1}^{\epsilon_1} f_1 \cdots t_{i_r}^{\epsilon_r} f_r$. We choose C to be a characteristic finite index subgroup of F such that for any subword $t_{i_k}^{-1} f_k t_{i_{k+1}}$ (respectively $t_{i_k} f_k t_{i_{k+1}}^{-1}$) of the word above, if $i_k = i_{k+1}$ then $\bar{f}_k \notin \bar{H}_{i_k}$ (respectively $\bar{f}_k \notin \bar{H}_{i_k}'$).

Note that since the word representing g is reduced, $f_k \notin H_{i_k}$ (respectively $f_k \notin H_{i_k}'$). And so, using the fact that H_i and H_i' are separable for each i, we can find finite index subgroups N_i and

N_i' separating H_i and H_i' from the finitely many elements, and we can let C be a characteristic finite index subgroup contained in the intersection of the N_i and N_i'.

The image of g in $\bar{G}$ is nontrivial because the normal form for g is mapped to a normal form for $\bar{g}$. Moreover, the quotient $\bar{G}$ is a multiple HNN extension of a finite group, and thus residually finite (it is even virtually free) and therefore $\bar{g}$ survives in some finite quotient. □

Theorem 3.4 (Clean ⇒ residually finite)**.** *Let G split as a graph of groups Γ, and suppose that Γ is finite, Γ is clean, and each edge group of Γ is finitely generated. Then G is residually finite.*

Proof. Without loss of generality, we may assume that the vertex groups of G are finitely generated, for otherwise G is a free product $G' * F_\infty$ where G' satisfies this additional assumption, and F_∞ is a free group of infinite rank. Thus if G' is residually finite then G is residually finite.

Observe that G is a free factor of a multiple HNN extension of a finitely generated free group where the edge groups are free factors in the vertex group. Specifically, consider a graph of spaces for Γ, and identify the basepoints of the vertex spaces of Γ to obtain a new graph of spaces with a single vertex space, whose vertex group is a finitely generated free group, and whose edge groups are free factors.

Finally, the hypotheses of Lemma 3.3 are satisfied for this multiple HNN extension, and so the group is residually finite. □

Example 3.5. We now show by example that the conclusion of Theorem 3.4 does not hold under the weaker assumption that G is finitely generated. In [2], G. Baumslag gave an example of an infinitely generated group $F_\infty \rtimes \langle a \rangle$ which is not residually finite. We form the group $F_\infty \rtimes \langle a, b \rangle$ where conjugation by a acts on F_∞ as above, and conjugation by b acts on F_∞ by the subscript shift. This group has a clean splitting and it is finitely generated. However, it cannot be residually finite because it contains $F_\infty \rtimes \langle a \rangle$ as a subgroup.

Definition 3.6 (Virtually clean)**.** A splitting of a group G as a graph of groups Γ_G is *virtually clean* provided that G has a finite index subgroup H whose induced splitting Γ_H is clean. Similarly, we say that a graph of spaces is virtually clean provided that the corresponding graph of groups is virtually clean.

Remark 3.7. As defined in Definition 3.6, being virtually clean is a property of the splitting and not a property of the underlying group. In fact, an example was given in [20] of a group G which splits in two ways as a graph of free groups. The first splitting of G is as a semidirect product of a free group and an infinite cyclic group, and so this splitting is clean. The second splitting of G is as an ascending HNN extension of a free group, which is an HNN extension were conjugation by the stable letter maps the base group properly into itself. It is easy to check that an ascending HNN extension is not virtually clean.

We close this section with the statement of the following theorem which relates virtually clean splittings with a residual property.

Theorem 3.8 (Characterizing Virtually Clean)**.** *Let G split as a finite graph Γ_G of virtually free groups with finitely generated edge groups. Then Γ_G is virtually clean if and only if*

(1) *Each edge group is separable.*
(2) *G is virtually torsion-free.*

We omit the proof, which is analogous to the proof of [36, Lemma 16.2] where the $\mathcal{VH}$-case is treated. I do not know if the second condition can be dropped.

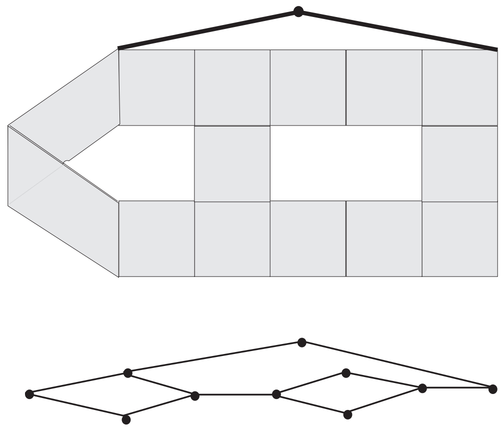

FIGURE 1. The $\mathcal{VH}$-complex X above consists of 14 squares and 2 isolated edges. The vertical edges are drawn vertically and the remaining edges are horizontal. The graph Γ_X is illustrated directly beneath X.

4. $\mathcal{VH}$-COMPLEXES

Definition 4.1 ($\mathcal{VH}$-complexes)**.** A squared 2-complex X is a $\mathcal{VH}$*-complex* if the edges of X are partitioned into two classes V and H called *vertical* and *horizontal* edges respectively, and the attaching map of each 2-cell of X alternates between edges in V and H. See Figure 1 for a very simple example of a $\mathcal{VH}$-complex. We always regard a cartesian product $A \times B$ of graphs to be a $\mathcal{VH}$-complex where edges parallel to the first factor are vertical and the other edges are horizontal. A map $Y \to X$ between $\mathcal{VH}$-complexes is a $\mathcal{VH}$*-map* if it sends vertical [horizontal] edges of Y to vertical [horizontal] edges of X.

As in Definition 1.7, the squared 2-complex X is *locally* $\mathcal{VH}$ provided that the link of each 0-cell of X has a bipartite structure. It is easy to prove that any locally $\mathcal{VH}$-complex X has a double cover $\hat{X}$ which is a $\mathcal{VH}$-complex (see [38]).

Consider the quotient $X \to \Gamma_X$ which is induced by quotienting each square in X to an edge using the map $I \times I \to I$ which projects onto the second (horizontal) factor. We refer the reader to Figure 1 for a very simple example of X and Γ_X. The preimage of each point of Γ_X is a graph called a *leaf of* X whose edges are parallel to vertical edges of X. A leaf $L \subset X$ is *nonsingular* provided that for some $\epsilon > 0$, L has a neighborhood in X which is isomorphic to $L \times (-\epsilon, \epsilon)$, otherwise L is *singular*. It is easy to see that if a singular leaf does not lie inside the vertical 1-skeleton of X then it must pass through the centers of squares of X. We say that X is *singular* if X contains a singular leaf which is not contained in the vertical 1-skeleton of X, otherwise we say that X is *nonsingular*. The space Γ_X is a graph whose vertices are the images of singular leaves and leaves contained in the 1-skeleton of X.

We collect several additional basic properties of $\mathcal{VH}$-complexes in the following Proposition (see [36]).

Proposition 4.2. *A singular $\mathcal{VH}$-complex has a nonsingular double cover.*

If X is a nonsingular $\mathcal{VH}$-complex then X is isomorphic to a graph of spaces with the following properties, whose underlying graph is Γ_X.

1. *Each vertex space of X is a graph.*
2. *Each edge space of X is the product of a graph and an interval.*

(3) *Each attaching map of each edge space is combinatorial, that is, it sends vertices to vertices and edges to edges.*

Furthermore, if X is nonpositively curved then each attaching map of X is an immersion.

The nonsingular double cover in Proposition 4.2 is actually the quotient of $\tilde{X}$ by the subgroup G' given by Lemma 4.5.

Note that each vertex space is a leaf of X, and each edge space is the product of a leaf and an interval. Let V denote the set of vertical edges of X. If X is nonsingular then the vertex spaces of X are the components of $V \cup X^{(0)}$ and the edge spaces correspond to the components of $X - (V \cup X^{(0)})$.

Definition 4.3 (Clean and Thin $\mathcal{VH}$-complexes)**.** A $\mathcal{VH}$-complex X is *clean* provided that it is nonsingular, and the induced graph of spaces is geometrically clean. X is *virtually clean* if it has a finite cover which is clean.

A $\mathcal{VH}$-complex X is *thin* provided that for some n there does not exist a $\mathcal{VH}$-immersion $I_n \times I_2 \to X$. If X is nonpositively curved then X is thin if and only if for some n, there does not exist a $\mathcal{VH}$-embedding $I_n \times I_2 \hookrightarrow \tilde{X}$ where $\tilde{X}$ is the universal cover. Our use of the term 'thin' is justified by the torsion-free special case of Lemma 4.5 below.

It was shown in [36] that a non-positively curved $\mathcal{VH}$-complex X is virtually clean (that is, its induced splitting as a graph of spaces is virtually *geometrically* clean) if and only if the induced splitting of $\pi_1 X$ is virtually clean. It is partially because of this equivalence that we are being sloppy about using the word 'clean' to mean something slightly more restrictive for (graphs of spaces arising from) $\mathcal{VH}$-complexes than for arbitrary graphs of free groups.

Not every geometrically clean graph of spaces arises as a clean $\mathcal{VH}$-complex. For example, one can consider the graph of spaces whose unique vertex space is the complete bipartite graph $K(3,3)$ and let the edge spaces correspond to the homeomorphisms between pairs of simple cycles in $K(3,3)$. Note that some of these cycles have length 4 and some have length 6. There does not exist a way of subdividing the edges of $K(3,3)$ such that all these homeomorphisms can be chosen to be graph isomorphisms.

Lemma 4.4. *Let G act on a tree T, then there is a subgroup $G' \subset G$ of index ≤ 2 such that G' acts on T without* inversions *in the sense that any automorphism which stabilizes an edge of T fixes that edge.*

Proof. Let $G' \subset G$ be a subgroup of index ≤ 2 with the property that G' preserves the bipartite structure on T. Then G' acts on $\Gamma_{\tilde{X}}$ without inversions. □

Lemma 4.5. *Let G act properly discontinuously and cocompactly by $\mathcal{VH}$-maps on a nonpositively curved simply connected $\mathcal{VH}$-complex $\tilde{X}$. Suppose that for some m there is no $\mathcal{VH}$-embedding of $I_m \times I_2$ in $\tilde{X}$. Then G has a subgroup G' of index ≤ 2 such that the action of G' on $\Gamma_{\tilde{X}}$ is thin, and the resulting graph of groups is a finite graph of groups with finitely generated edge and vertex groups.*

Proof. Since $\tilde{X}$ is a simply connected nonpositively curved $\mathcal{VH}$-complex, $\Gamma_{\tilde{X}}$ is a tree, each vertex space is a tree, and each edge space is the product of a tree and an interval.

Since G acts on $\tilde{X}$ by maps which preserve the $\mathcal{VH}$-structure, there is an induced action of G on $\Gamma_{\tilde{X}}$. By Lemma 4.4, there is a subgroup $G' \subset G$ of index ≤ 2 such that G' acts on $\Gamma_{\tilde{X}}$ without inversions. Now the tree structure of $\tilde{X}$ gives a splitting of G'. The vertex groups are the stabilizers of vertex spaces of $\tilde{X}$, and the edge groups are stabilizers of edge spaces of $\tilde{X}$. Consequently, G' splits as a graph of virtually free groups.

Since G and hence G' acts cocompactly, the resulting graph of groups is a finite graph of groups, and furthermore, the edge groups are finitely generated. We will show that the splitting is thin by using the hypothesis that G' acts properly discontinuously and that there is no $\mathcal{VH}$-embedding of $I_m \times I_2$. Let $\mathrm{Stab}(S)$ denote the stabilizer of a set S. Let D and E be distinct edge spaces incident at a vertex space V. If D and E are disjoint in V, then since V is a tree, $\big(\mathrm{Stab}(D) \cap \mathrm{Stab}(E)\big) \subset \mathrm{Stab}(\sigma)$ where σ is the unique shortest path between D and E. If D and E intersect then $\big(\mathrm{Stab}(D) \cap \mathrm{Stab}(V)\big) \subset \mathrm{Stab}(D \cap E)$. Now $D \cap E$ must be a tree of diameter $\leq m$, because otherwise there would be a $\mathcal{VH}$-embedding $I_m \times I_2 \hookrightarrow \tilde{X}$, which is impossible by hypothesis. In either case, we see that the intersection of the stabilizers of D and E is contained in the stabilizer of a compact set, and therefore since G' acts properly discontinuously, the intersection is finite. □

Finally, we will make use of the following result proven in [36].

Theorem 4.6. *Let X be a compact clean $\mathcal{VH}$-complex and suppose that $\pi_1 X$ is word-hyperbolic. Then every quasiconvex subgroup of $\pi_1 X$ is separable.*

5. Canonical Completions and Retractions

The main goal of this section is to define the canonical completion and retraction of an immersion of graphs. We then derive several properties of the canonical completion that will play a fundamental role in the proofs of Theorem 7.3 and Theorem 11.3. Canonical completions were employed in [39] but the theory developed there was catered to a more specific situation. We use the term "canonical" to mean a uniform recipe for extending the immersion to a covering space. There are other such uniform recipes together with associated retraction maps that would also fulfill the roll of the canonical completion. In fact, the only properties of this uniform procedure that we will need are Corollary 5.3 and Corollary 5.4.

Definition 5.1 (Canonical Completion)**.** Let $\phi : A \to B$ be an immersion of a compact graph A. We will define a finite covering space $\mathsf{C}(A, B) \to B$ called the *canonical completion* of $A \to B$, such that there is a commutative diagram:

$$\begin{tikzcd} A \arrow[r, hook] \arrow[dr] & \mathsf{C}(A,B) \arrow[d] \\ & B \end{tikzcd}$$

We first consider the case where B is a bouquet of circles. For each circle c in B, consider its preimage $\phi^{-1}(c)$ in A. Each component of $\phi^{-1}(c)$ is either a finite connected cover of c, or can be completed to a finite connected cover of c by the addition of a single edge. Thus we can extend $\phi^{-1}(c)$ to a finite cover $\hat{c}$ of c. Note that the 0-skeleton of $\hat{c}$ is $A^{(0)}$.

Now we define $\mathsf{C}(A, B)$ to be the union of the covers $\hat{c}$ amalgamated along $A^{(0)}$. Observe that $\mathsf{C}(A, B) \to B$ is a covering space because it is a local isomorphism of graphs. Furthermore, A embeds in $\mathsf{C}(A, B)$ because A is the union of the graphs $\phi^{-1}(c)$ amalgamated along $A^{(0)}$. We refer the reader to the example illustrated in Figure 2.

We now define the *canonical retraction* map $\mathsf{C}(A, B) \to A$. For each circle c in B, there is an obvious retraction map $\hat{c} \to \phi^{-1}(c)$, which sends each added edge to the path that it completes to a circuit. We define the *canonical retraction* $\mathsf{C}(A, B) \to A$ to be induced by the retractions $\hat{c} \to \phi^{-1}(c)$. Note that this is well defined because these maps fix the 0-skeleton of $\mathsf{C}(A, B)$.

We will now define the canonical completion $\mathsf{C}(A, B)$ without assuming that B is a bouquet of circles. It will be sufficient to define $\mathsf{C}(A, B)$ in case A is connected, for if the connected components of A are $A_1, \ldots, A_r$ then we can define $\mathsf{C}(A, B)$ to be the disjoint union $\mathsf{C}(A_1, B) \cup \ldots \cup \mathsf{C}(A_r, B)$.

Letting $B^\star = B/B^{(0)}$ which is the bouquet of circles obtained by identifying all the vertices of B, the composition $A \to B \to B^\star$ gives an immersion of A to the bouquet of circles $B^\star$. We now define $\mathsf{C}(A, B) \to B$ to be the fiber-product of the maps $B \to B^\star$ and $\mathsf{C}(A, B^\star) \to B^\star$. That is, $\mathsf{C}(A, B)$ is the preimage of the diagonal of $B^\star \times B^\star$ in the map $B \times \mathsf{C}(A, B^\star) \to B^\star \times B^\star$. Stallings [32] refers to this fiber-product as the *pullback* of $B \to B^\star$ and $\mathsf{C}(A, B^\star) \to B^\star$.

A more concrete way of obtaining $\mathsf{C}(A, B)$ from $\mathsf{C}(A, B^\star)$ is as follows: First note that the set of ends of edges at the 0-cell of $B^\star$ is partitioned into classes corresponding to $B^{(0)}$ where two ends are in the same class if and only if the corresponding ends of edges in B are attached to the same 0-cell of B. We now *refine* $\mathsf{C}(A, B^\star)$ by first replacing each 0-cell p of $\mathsf{C}(A, B^\star)$, by a copy $B^{(0)}_p$ of $B^{(0)}$, and then regluing the ends of edges which were at p to the appropriate 0-cell of the copy of $B^{(0)}_p$. Observe that ends of edges belonging in the same class, remain glued to the same vertex. It is easy to see that A still embeds in the refinement of $\mathsf{C}(A, B^\star)$ because ends of edges in A are obviously incident at the same vertex of B. Furthermore, the refinement is clearly a cover of B. We note that even when A is connected, if B has more than one 0-cell, then $\mathsf{C}(A, B)$ may not be connected. For example if $A \to B$ is an isomorphism, then $\mathsf{C}(A, B)$ is only connected if A and B are bouquets of circles.

Finally, we define the *canonical retraction* $\mathsf{C}(A, B) \to A$ to be the composition $\mathsf{C}(A, B) \to \mathsf{C}(A, B^\star) \to A$ where the first map is the projection of the fiber-product and the second map is the canonical retraction that was defined above in the special case of a bouquet of circles.

Somehow, having a canonical procedure for extending any immersion to a cover gives unexpectedly complicated extensions of straightforward immersions. For instance, the canonical completion $\mathsf{C}(A, A)$ of an identity map $A \to A$ will not be connected unless A has a single 0-cell. Indeed, the degree of $\mathsf{C}(A, A) \to A$ equals the degree of $\mathsf{C}(A, A^\star) \to A^\star$ which equals the number of 0-cells in A. We encourage the reader to construct $\mathsf{C}(A, A)$ when A is a polygon. The consistent behavior of the extra components of $\mathsf{C}(A, A)$ will play an important role in proving Theorem 7.3 and Theorem 11.3. We now collect the essential properties of the canonical completion and retraction which will be used to prove these theorems.

Given a subgraph $C \subset B$ and a canonical completion $\mathsf{C}(A, B) \to B$ of some immersion $A \to B$, it will be very important to understand the preimage $\hat{C}$ of C in $\mathsf{C}(A, B)$ and to understand the effect of the canonical retraction on $\hat{C}$. This is expressed in the following lemma whose consequences, Corollary 5.3 and Corollary 5.4, will play a fundamental role in the proofs of Theorem 7.3 and Theorem 11.3.

Lemma 5.2 (Completion of preimage). *Let $A \to B$ be an immersion of graphs. Let C be a subgraph of B, let $\hat{C}$ denote the preimage of C in $\mathsf{C}(A, B)$ and let D be the preimage of C in A. Then $\hat{C}$ is isomorphic to $\mathsf{C}(D, C) \cup \big((A^{(0)} - D) \times C\big)$. And we have the following commutative diagram, where each $\downarrow$ is a canonical cover; the right $\uparrow$ is a canonical retraction; and the left $\uparrow$ is determined by the canonical retraction on $\mathsf{C}(D, C)$, together with the projection map $\big((A^{(0)} -$*

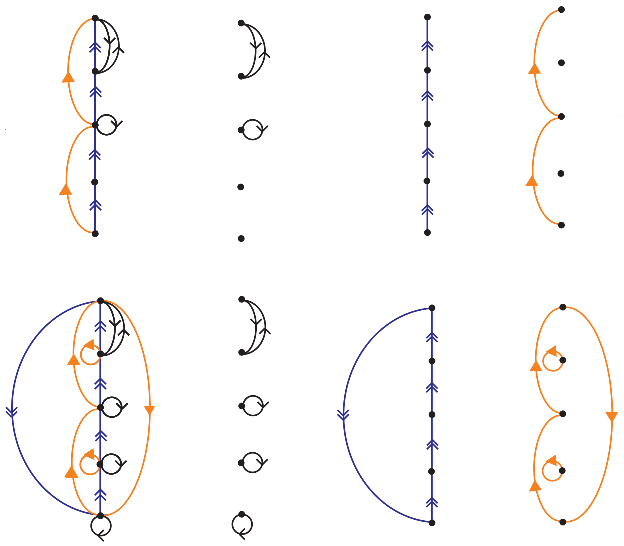

FIGURE 2. Let A denote the top left graph, which is immersed $\phi : A \to B$ in a bouquet B of 3 circles labelled by a, b, and c corresponding to single, double, and solid arrows respectively. The three graphs to the right of A are $\phi^{-1}(a)$, $\phi^{-1}(b)$, and $\phi^{-1}(c)$. Beneath them are $\hat{a}$, $\hat{b}$ and $\hat{c}$. The graph beneath A is $\mathsf{C}(A, B)$ which is the union of $\hat{a}$, $\hat{b}$ and $\hat{c}$ amalgamated along $A^{(0)}$.

$D) \times C\big) \to (A^{(0)} - D)$.

$$\begin{array}{rcccc} & D \cup (A^{(0)} - D) & \hookrightarrow & A \\ & \uparrow & & \uparrow \\ \hat{C} = & \mathsf{C}(D, C) \cup \big((A^{(0)} - D) \times C\big) & \hookrightarrow & \mathsf{C}(A, B) \\ & \downarrow & & \downarrow \\ & C & \hookrightarrow & B \end{array}$$

Proof. The lemma is clear in case B is a bouquet of circles. Indeed, following Definition 5.1, $\mathsf{C}(D, C)$ is just the union of the covers $\hat{c}$ where c is a circle of C, amalgamated along $D^{(0)} = A^{(0)}$, and $\mathsf{C}(A, B)$ is the union of the covers $\hat{c}$ where c is a circle of B, amalgamated along $A^{(0)}$. So there is an obvious inclusion $\mathsf{C}(D, C) \subset \mathsf{C}(A, B)$. Note that in this case, since $D^{(0)} = A^{(0)}$ the set $A^{(0)} - D$ is empty, and so we have $\hat{C} = \mathsf{C}(D, C) = (\mathsf{C}(D, C) \cup (A^{(0)} - D) \times C)$. Furthermore, the retraction map $\mathsf{C}(A, B) \to A$ obviously restricts to the retraction map $\mathsf{C}(D, C) \to D$.

In case B is not a bouquet of circles, first note that the corresponding statement is true for $C^\star \subset B^\star$. Now the preimage of $C^\star$ in A is the disjoint union $D \cup (A^{(0)} - D)$. Its canonical completion with respect to $C^\star$ is $\mathsf{C}(D, C^\star) \cup \mathsf{C}((A^{(0)} - D), C^\star)$.

When we refine $\mathsf{C}(A, B^\star)$ to obtain $\mathsf{C}(A, B)$ we see that each copy of $C^\star$ in $\mathsf{C}((A^{(0)} - D), C^\star) \subset \mathsf{C}(A, B^\star)$ is refined to a copy of C, and $\mathsf{C}(D, C^\star)$ is refined to $\mathsf{C}(D, C)$. This is because, since $C \subset B$, the partition of the ends of edges of $C^\star$ is consistent with the partition of the ends of edges of $B^\star$.

Finally, the commutativity with the canonical retraction map follows from the case where B is a bouquet of circles. $\square$

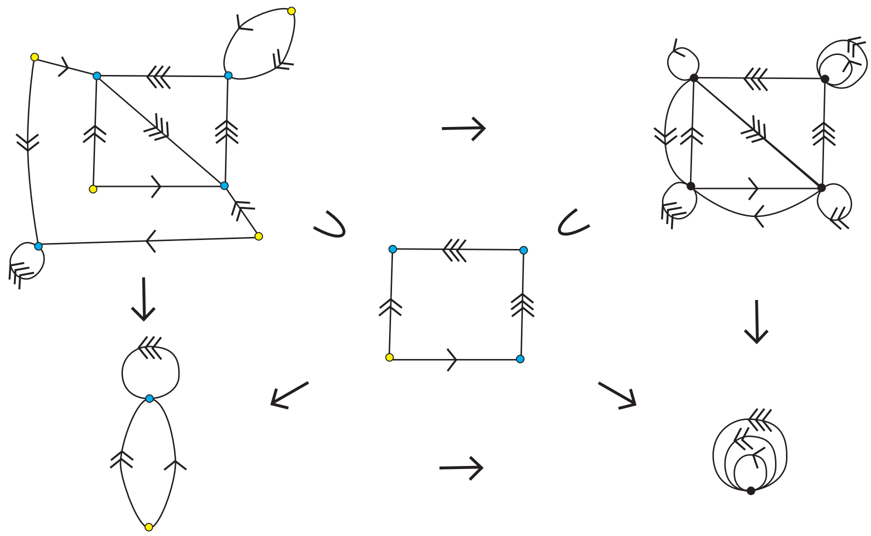

FIGURE 3. Forming a canonical completion where B has more than one vertex: Illustrated above is a diagram of maps between graphs corresponding to the following commutative diagram:

$$\begin{array}{ccccc}
\mathsf{C}(A,B) & & \rightarrow & & \mathsf{C}(A,B^\star) \\
& \nwarrow & & \nearrow & \\
\downarrow & & A & & \downarrow \\
& \swarrow & & \searrow & \\
B & & \rightarrow & & B^\star
\end{array}$$

Corollary 5.3 (Completing a subgraph)**.** *Let $A \subset B$ be a subgraph. Let $\hat{A}$ denote the full preimage of A in the covering space $\mathsf{C}(A,B)$. Then $\hat{A}$ is isomorphic to $\mathsf{C}(A,A)$ and we have the following commutative diagram where each $\downarrow$ is a canonical covering and each $\uparrow$ is a canonical retraction.*

$$\begin{array}{ccccc}
& A & = & A \\
& \uparrow & & \uparrow \\
\hat{A} = & \mathsf{C}(A,A) & \subset & \mathsf{C}(A,B) \\
& \downarrow & & \downarrow \\
& A & \subset & B
\end{array}$$

Proof. This is the special case of Lemma 5.2 where A is a subgraph of B and $C = A$. $\square$

Corollary 5.4 (Forest intersection $\Rightarrow$ null-homotopic in retraction)**.** *Let A and C be subgraphs of B and suppose that $A \cap C$ is a forest. Then each component $\breve{C}$ of the preimage of C in $\mathsf{C}(A,B)$ is null-homotopic in the canonical retraction map.*

Proof. Observe that the preimage D of C in A is equal to the intersection $A \cap C$. By Lemma 5.2, the image of $\breve{C}$ must lie in $(A \cap C) \cup (A^{(0)} - C)$, which is a forest. $\square$

6. Elevations

In this section we fix some language concerning lifts to covering spaces that will be employed extensively in Sections 7 and 11. We will work in the category of CW complexes.

Definition 6.1 (Induced covers and elevations)**.** Let $(A, a) \to (B, b)$ be a basepoint preserving map where A is connected, and let $(\hat{B}, \hat{b}) \to (B, b)$ be a based covering space. Consider a commutative diagram

$$\begin{array}{ccc} (\hat{A}, \hat{a}) & \to & (\hat{B}, \hat{b}) \\ \downarrow & & \downarrow \\ (A, a) & \to & (B, b) \end{array}$$

where $(\hat{A}, \hat{a}) \to (A, a)$ is a based connected covering space such that $\pi_1(\hat{A}, \hat{a})$ maps to the preimage of $\pi_1(\hat{B}, \hat{b})$ in $\pi_1(A, a)$. The map $(\hat{A}, \hat{a}) \to (A, a)$ is a *covering space of* (A, a) *induced by* $(\hat{B}, \hat{b}) \to (B, b)$. The map $(\hat{A}, \hat{a}) \to (\hat{B}, \hat{b})$ is an *elevation* of $(A, a) \to (B, b)$ at $\hat{b}$. We note that commutative diagrams with the above property always exist and are unique up to an appropriate isomorphism. Indeed, an induced cover $(\hat{A}, \hat{a})$ exists because of the correspondence between isomorphism classes of based covers and subgroups of π_1, and an elevation $(\hat{A}, \hat{a}) \to (\hat{B}, \hat{b})$ exists by an application of the unique lifting property to the composition $(\hat{A}, \hat{a}) \to (A, a) \to (B, b)$. Because of the uniqueness up to isomorphism, we will often refer to *the* elevation and *the* induced cover at $\hat{b}$.

Let $\hat{B} \to B$ be a connected based cover, and let $\hat{b}_1$ and $\hat{b}_2$ be vertices in $\hat{B}$ that map to b. We say that an elevation of $(A, a) \to (B, b)$ at $\hat{b}_1$ is *equivalent* to an elevation at $\hat{b}_2$ provided there is a path τ in $\hat{B}$ from $\hat{b}_1$ to $\hat{b}_2$ which projects to an element in the image of $\pi_1(A, a) \to \pi_1(B, b)$. Observe that an elevation $(\hat{A}_1, \hat{a}_1) \to (\hat{B}, \hat{b}_1)$ at $\hat{b}_1$ is equivalent to an elevation $(\hat{A}_2, \hat{a}_2) \to (\hat{B}, \hat{b}_2)$ at $\hat{b}_2$ if and only if there is a covering space isomorphism $\begin{array}{c} \hat{A}_1 \\ \downarrow \\ \hat{A}_2 \end{array} \begin{array}{c} \searrow \\ \nearrow \end{array} A$ such that we have a commutative diagram $\begin{array}{c} \hat{A}_1 \\ \downarrow \\ \hat{A}_2 \end{array} \begin{array}{c} \searrow \\ \nearrow \end{array} \hat{B}$.

We will generally suppress the basepoints and refer to an elevation $\hat{A} \to \hat{B}$ of $A \to B$ to $\hat{B}$ without mentioning the specific choices of basepoints in A and $\hat{B}$ that map to the basepoint of B. We say that an elevation of $A \to B$ at the vertex $\hat{b}$ of $\hat{B}$ is *injective* provided that the map $\hat{A} \to \hat{B}$ is injective. For an injective elevation $\hat{A} \hookrightarrow \hat{B}$, we will often identify $\hat{A}$ with its image in $\hat{B}$. This will be especially convenient in the proof of Theorem 7.3.

The equivalence of elevations has a natural algebraic interpretation. To simplify matters, assume that $A \to B$ is π_1-injective, identify $\pi_1(A, a)$ with its image in $\pi_1(B, b)$, and let $\hat{B} \to B$ be a connected cover with basepoint $\hat{b}$. Then an elevation of $A \to B$ at $\hat{b}_1$ is equivalent to an elevation at $\hat{b}_2$ if and only if the double cosets $\pi_1(\hat{B}, \hat{b})\sigma_1\pi_1(A, a)$ and $\pi_1(\hat{B}, \hat{b})\sigma_2\pi_1(A, a)$ are equal, where for each i the path $\sigma_i \to \hat{B}$ begins at $\hat{b}$ and ends at $\hat{b}_i$.

7. Thin ⇒ Virtually Clean (Geometric case)

In this section we prove the main theorem in the important geometric special case where the graph of spaces is a non-positively curved $\mathcal{VH}$-complex. We begin with the following simple construction which will be used here and in the proof of Theorem 11.3 to reduce consideration to amalgamated free products.

Construction 7.1. Let X be a finite graph of spaces with underlying graph Γ_X. We construct a certain (single or) double cover $\bar{X} \to X$ and an embedding $\bar{X} \hookrightarrow Y$ where Y is a graph of spaces with $\Gamma_Y \cong I$.

Let $\bar{\Gamma}_X \to \Gamma_X$ be a bipartite finite cover. Note that if Γ_X is not already bipartite then we can choose a bipartite double cover corresponding to the kernel of the groupoid homomorphism which sends each edge of Γ_X to the generator of $\mathbb{Z}_2$. Let $\bar{X}$ denote the induced cover of X, so that $\Gamma_{\bar{X}} = \bar{\Gamma}_X$. Since $\bar{\Gamma}$ is bipartite, we can orient its edges so that for each vertex of $\bar{\Gamma}$, either all of its edges are incoming or all of its edges are outgoing. We will refer to the corresponding vertex spaces as outgoing and incoming vertex spaces.

Suppose that the edges of $\bar{\Gamma}$ are $\{e_1, \ldots, e_n\}$. For $1 \leq i \leq n$, let h_i denote a horizontal edge in the edge space corresponding to e_i. Let T_n denote a compact tree with exactly n vertices which we denote by $\{p_1, \ldots, p_n\}$. Consider the 2-complex $T_n \times I$ and for $1 \leq i \leq n$ let t_i denote the 1-cell $\{p_i\} \times I$ which we orient from $(p_i, 0)$ to $(p_i, 1)$. We now form the complex $Y = (T_n \times I) \cup_{\{h_i = t_i : 1 \leq i \leq n\}} \bar{X}$ which is obtained by gluing $T_n \times I$ to $\bar{X}$ by identifying t_i with h_i in an orientation preserving manner for each i.

Observe that Y has the structure of a graph of spaces where $\Gamma_Y \cong I$. The first vertex space of Y is the union of the outgoing vertex spaces of $\bar{X}$ and $\{0\} \times T_n$, the second vertex space of Y is the union of the incoming vertex spaces of $\bar{X}$ and $\{1\} \times T_n$, and the edge space of Y is the union of the edge spaces of X and $T_n \times I$.

Lemma 7.2. *Let X be a $\mathcal{VH}$-complex and let Y be as in Construction 7.1.*

1. *Y is a $\mathcal{VH}$-complex.*
2. *If X is nonpositively curved then Y is nonpositively curved.*
3. *If X is compact then Y is compact.*
4. *If X is thin then Y is thin.*
5. *If Y is virtually geometrically clean then so is X.*

Proof. If X is a $\mathcal{VH}$-complex, then the construction yields a $\mathcal{VH}$-complex Y because we are identifying horizontal edges in the $\mathcal{VH}$-complex $T_n \times I$ with horizontal edges in the $\mathcal{VH}$-complex $\bar{X}$. Furthermore, if X is nonpositively curved then so is Y, because the only simple cycles in the link of a vertex of Y are simple cycles in the link of a vertex in $\bar{X}$. This is because each vertical edge in $T_n \times I$ is a free face in Y, and therefore no corner of a square in $T_n \times I$ belongs to a simple cycle in the link of a vertex of Y.

Since the vertical edges in $T_n \times I$ are free faces, any immersion $I_k \times I_2 \to Y$ must restrict to an immersion $I_k \times I_2 \to \bar{X}$. Consequently if Y is not thin then X is not thin.

If $\hat{Y}$ is a geometrically clean cover of Y then $\hat{X}$ is geometrically clean for the simple reason that each attaching map of an edge space of $\hat{X}$ is an injection because it is the restriction of an attaching map of an edge space of $\hat{Y}$ and each such attaching map is injective because $\hat{Y}$ is geometrically clean. □

Theorem 7.3. *Let X be a compact nonpositively curved $\mathcal{VH}$-complex. If X is thin then X has a clean finite cover.*

Proof. **Step 1:** By Construction 7.1 and Lemma 7.2, if X is a compact thin nonpositively curved $\mathcal{VH}$-complex then there is a compact thin nonpositively curved $\mathcal{VH}$-complex Y with the property that $\Gamma_Y = I$ and with the property that if Y is virtually clean, then X is virtually clean. It is therefore sufficient to show that X is virtually clean in the special case where $\Gamma_X = I$. In the remaining steps we will denote the based vertex spaces of X by L and R. We will denote the based edge space of X by $E \times I$. And without risk of confusion, we will refer to its two attaching maps by the pair of maps $L \leftarrow E \rightarrow R$.

Step 2: There exist finite based covers $\bar{L} \rightarrow L$ and $\bar{R} \rightarrow R$, with the following properties:

(1) Each elevation of $E \rightarrow L$ to $\bar{L}$ and of $E \rightarrow R$ to $\bar{R}$ is injective.
(2) Any two elevations of $E \rightarrow L$ to $\bar{L}$ (or of $E \rightarrow R$ to $\bar{R}$) are either equivalent, or intersect in a (possibly empty) forest.
(3) The based covers of E induced by the based covers $\bar{L}$ and $\bar{R}$ are isomorphic in the sense that we have the following commutative diagram, where $\bar{L} \hookleftarrow \bar{E}$ and $\bar{E} \hookrightarrow \bar{R}$ are the elevations of $L \leftarrow E$ and $E \rightarrow R$ at the basepoints of $\bar{L}$ and $\bar{R}$.

$$\begin{array}{ccccc} \bar{L} & \hookleftarrow & \bar{E} & \hookrightarrow & \bar{R} \\ \downarrow & & \downarrow & & \downarrow \\ L & \leftarrow & E & \rightarrow & R \end{array}$$

Let $L_1 \rightarrow L$ be the smallest connected regular cover factoring through each component of $\mathsf{C}(E,L) \rightarrow L$. Observe that the canonical inclusion $E \hookrightarrow \mathsf{C}(E,L)$ is an injective elevation of $E \rightarrow L$. Furthermore, each component of the preimage of $E \hookrightarrow \mathsf{C}(E,L)$ in L_1 is the image of an injective elevation of E to L_1. Since L_1 is regular, we see that every elevation of $E \rightarrow L$ to L_1 is injective. We define $R_1 \rightarrow R$ similarly.

Let $E_L \hookrightarrow L_1$ and $E_R \hookrightarrow R_1$ denote elevations of $E \rightarrow L$ and $E \rightarrow R$ at the basepoints of L_1 and R_1. Since E_L is a connected subgraph of the graph L_1, there exists a retraction $L_1 \rightarrow E_L$. Similarly, there exists a retraction $R_1 \rightarrow E_R$. Let $\bar{E} \rightarrow E$ denote the smallest based cover factoring through both $E_L \rightarrow E$ and $E_R \rightarrow E$. Let $\bar{L} \rightarrow L_1$ [respectively $\bar{R} \rightarrow R_1$] be the based cover induced by the cover $\bar{E} \rightarrow E_L$ [respectively $\bar{E} \rightarrow E_R$] and the retraction map to E_L [respectively E_R] as indicated by the diagrams below:

$$\begin{array}{ccc} \bar{L} & \rightarrow & \bar{E} \\ \downarrow & & \downarrow \\ L_1 & \rightarrow & E_L \end{array} \qquad\qquad \begin{array}{ccc} \bar{R} & \rightarrow & \bar{E} \\ \downarrow & & \downarrow \\ R_1 & \rightarrow & E_R \end{array}$$

Finally observe that the based cover of E induced by the based cover $\bar{L}$ is isomorphic to $\bar{E}$ and the based cover of E induced by the based cover $\bar{R}$ is isomorphic to $\bar{E}$ so we obtain (3).

To see that each elevation of $E \rightarrow L$ to $\bar{L}$ is injective, observe that the image of each elevation of $E \rightarrow L$ to $\bar{L}$ is just a component of the preimage in $\bar{L}$, of the image of an injective elevation of $E \rightarrow L$ to L_1. Similarly, each elevation of $E \rightarrow R$ to $\bar{R}$ is injective, and we see that (1) holds.

We will now prove that (2) holds for $\bar{L}$. An identical proof holds for $\bar{R}$. Let $\bar{\ell}_1$ and $\bar{\ell}_2$ be vertices in $\bar{L}$ that map to the basepoint of L. Let $E_1 \hookrightarrow \bar{L}$ and $E_2 \hookrightarrow \bar{L}$ be the elevations of $E \rightarrow L$ at $\bar{\ell}_1$ and $\bar{\ell}_2$. If $E_1 \cap E_2$ is not a forest then there is a closed simple cycle $\theta \rightarrow E_1 \cap E_2$ beginning and ending at a vertex $v \in E_1 \cap E_2$. For each $i \in \{1,2\}$ let σ_i be a path from $\bar{\ell}_i$ to v in E_i, and observe that $\sigma_i \theta \sigma_i^{-1}$ is an essential closed based path in E_i and hence projects to a closed based path in E. Observe that

$$(\sigma_1 \sigma_2^{-1})(\sigma_2 \theta \sigma_2^{-1})(\sigma_1 \sigma_2^{-1})^{-1} \simeq \sigma_1 \theta \sigma_1^{-1}$$

and therefore since $\pi_1 E \hookrightarrow \pi_1 L$ is malnormal, we see that $\sigma_1 \sigma_2^{-1}$ must represent an element of $\pi_1 E$. But then $\bar{\ell}_1$ and $\bar{\ell}_2$ are connected by a path which projects to an element of $\pi_1 E$, so E_1 and E_2 are equivalent and we are done.

Step 3: We now form the canonical completions of $\bar{E}$ with respect to $\bar{E}$, $\bar{L}$ and $\bar{R}$, and obtain the following commutative diagram by applying Corollary 5.3 twice.

$$\begin{array}{ccccc} \mathsf{C}(\bar{E},\bar{L}) & \hookleftarrow & \mathsf{C}(\bar{E},\bar{E}) & \hookrightarrow & \mathsf{C}(\bar{E},\bar{R}) \\ \downarrow & & \downarrow & & \downarrow \\ \bar{L} & \leftarrow & \bar{E} & \rightarrow & \bar{R} \end{array}$$

Step 4: Let $\hat{E} \to E$ be a finite connected cover of E which factors through all covers of E induced by $\mathsf{C}(\bar{E},\bar{L})$ and $\mathsf{C}(\bar{E},\bar{R})$ (the covers vary as the choice of basepoint varies). Let $\widehat{\mathsf{C}(\bar{E},\bar{E})}$, $\widehat{\mathsf{C}(\bar{E},\bar{L})}$ and $\widehat{\mathsf{C}(\bar{E},\bar{R})}$ be the fiber-products indicated by the three diagrams below where the map at the bottom of each square is a canonical retraction.

$$\begin{array}{ccc} \widehat{\mathsf{C}(\bar{E},\bar{E})} & \rightarrow & \hat{E} \\ \downarrow & & \downarrow \\ \mathsf{C}(\bar{E},\bar{E}) & \rightarrow & \bar{E} \end{array} \qquad \begin{array}{ccc} \widehat{\mathsf{C}(\bar{E},\bar{L})} & \rightarrow & \hat{E} \\ \downarrow & & \downarrow \\ \mathsf{C}(\bar{E},\bar{L}) & \rightarrow & \bar{E} \end{array} \qquad \begin{array}{ccc} \widehat{\mathsf{C}(\bar{E},\bar{R})} & \rightarrow & \hat{E} \\ \downarrow & & \downarrow \\ \mathsf{C}(\bar{E},\bar{R}) & \rightarrow & \bar{E} \end{array}$$

Observe that we have the following commutative diagram:

$$\begin{array}{ccccc} \widehat{\mathsf{C}(\bar{E},\bar{L})} & \hookleftarrow & \widehat{\mathsf{C}(\bar{E},\bar{E})} & \hookrightarrow & \widehat{\mathsf{C}(\bar{E},\bar{R})} \\ \downarrow & & \downarrow & & \downarrow \\ \mathsf{C}(\bar{E},\bar{L}) & \hookleftarrow & \mathsf{C}(\bar{E},\bar{E}) & \hookrightarrow & \mathsf{C}(\bar{E},\bar{R}) \end{array}$$

Step 5: Let $\overset{\bowtie}{L} \to L$ be the smallest regular cover of L factoring through each component of $\widehat{\mathsf{C}(\bar{E},\bar{L})}$. Similarly, let $\overset{\bowtie}{E}$ and $\overset{\bowtie}{R}$ be the smallest regular covers of E and R which factor through each component of $\widehat{\mathsf{C}(\bar{E},\bar{E})}$ and $\widehat{\mathsf{C}(\bar{E},\bar{R})}$. We claim that each cover of E induced by $\overset{\bowtie}{L}$ and $\overset{\bowtie}{R}$ is isomorphic to $\overset{\bowtie}{E}$.

It is sufficient to prove that each cover of E induced by $\overset{\bowtie}{L}$ is isomorphic to $\overset{\bowtie}{E}$, because a symmetric argument proves the same for $\overset{\bowtie}{R}$. Let $\check{E} \to E$ be a covering space induced by $\overset{\bowtie}{L}$, and note that $\check{E}$ does not depend on the choice of basepoint in $\overset{\bowtie}{L}$ because $\overset{\bowtie}{L}$ is regular.

To prove that $\check{E} \cong \overset{\bowtie}{E}$ we will show that $\check{E}$ and $\overset{\bowtie}{E}$ factor through each other. We first show that $\check{E}$ factors through $\overset{\bowtie}{E}$. To see this, first observe that $\widehat{\mathsf{C}(\bar{E},\bar{E})} \hookrightarrow \widehat{\mathsf{C}(\bar{E},\bar{L})}$, and therefore each component of $\widehat{\mathsf{C}(\bar{E},\bar{E})}$ is a cover of E induced by $\widehat{\mathsf{C}(\bar{E},\bar{L})}$. Consequently, $\check{E}$ factors through each component of $\widehat{\mathsf{C}(\bar{E},\bar{E})}$ and therefore, since $\check{E}$ is regular, $\check{E}$ must factor through $\overset{\bowtie}{E}$ which is the smallest regular cover factoring through these components.

Now we show that $\overset{\bowtie}{E}$ factors through $\check{E}$. To do this, we will show that $\overset{\bowtie}{E}$ factors through each cover of E induced by $\widehat{\mathsf{C}(\bar{E},\bar{L})}$. This is sufficient because $\check{E}$ is the smallest regular cover of E which factors through each cover of E induced by $\widehat{\mathsf{C}(\bar{E},\bar{L})}$. Indeed, since $\overset{\bowtie}{L}$ is the cover corresponding to the intersection of fundamental groups as the basepoint varies appropriately in $\widehat{\mathsf{C}(\bar{E},\bar{L})}$, we see that $\check{E}$ is the cover corresponding to the intersection of fundamental groups of induced covers of E at these basepoints.

We will divide our analysis of induced covers $\hat{K}$ of E into two cases according to the behavior of their associated elevations $\hat{K} \to \widehat{\mathsf{C}(\bar{E},\bar{L})}$:

(1) Elevations $\hat{K} \to \widehat{\mathsf{C}(\bar{E},\bar{L})}$ of $E \to L$ that factor through $\bar{E} \hookrightarrow \bar{L}$.

(2) Elevations $\hat{K} \to \widehat{\mathsf{C}(\bar{E}, \bar{L})}$ of $E \to L$ that do not factor through $\bar{E} \hookrightarrow \bar{L}$.

Case 1: If $K \to \bar{L}$ is an elevation of $E \to L$ which is equivalent to $\bar{E} \hookrightarrow \bar{L}$, then any elevation $\hat{K} \to \widehat{\mathsf{C}(\bar{E}, \bar{L})}$ of $E \to L$ which factors through $K \to \bar{L}$ is an elevation of $E \to L$ to $\widehat{\mathsf{C}(\bar{E}, \bar{E})}$. Indeed, the preimage of $\bar{E} \hookrightarrow \bar{L}$ in the cover $\widehat{\mathsf{C}(\bar{E}, \bar{L})}$ is $\widehat{\mathsf{C}(\bar{E}, \bar{E})} \subset \widehat{\mathsf{C}(\bar{E}, \bar{L})}$. Consequently $\overset{\bowtie}{E}$ factors through $\hat{K}$.

Case 2: If $K \hookrightarrow \bar{L}$ is an elevation of $E \to L$ which is not equivalent to $\bar{E} \hookrightarrow \bar{L}$, then according to the second property verified in Step 2, $K \hookrightarrow \bar{L}$ intersects $\bar{E}$ in a (possibly empty) forest. Let $\check{K} \hookrightarrow \mathsf{C}(\bar{E}, \bar{L})$ be an elevation of $E \to L$ which factors through $K \hookrightarrow \bar{L}$. By Corollary 5.4, $\check{K}$ is null-homotopic in the canonical retraction $\mathsf{C}(\bar{E}, \bar{L}) \to \bar{E}$. Consequently, for any elevation $\hat{K} \hookrightarrow \widehat{\mathsf{C}(\bar{E}, \bar{L})}$ of $E \to L$ which factors through $\check{K} \hookrightarrow \mathsf{C}(\bar{E}, \bar{L})$, the covers $\hat{K}$ and $\check{K}$ are isomorphic. Since $\hat{E} \to E$ was chosen so that $\hat{E}$ factors through each cover of E induced by $\mathsf{C}(\bar{E}, \bar{L})$, we see that $\hat{E}$ factors through $\check{K}$. But $\hat{E}$ is isomorphic to the based component of $\widehat{\mathsf{C}(\bar{E}, \bar{E})}$, and so since $\overset{\bowtie}{E}$ factors through $\hat{E}$ we see that $\overset{\bowtie}{E}$ factors through $\hat{K} \cong \check{K}$.

Step 6: In this final step we will form the clean cover $\hat{X}$ by gluing together vertex spaces which are copies of $\overset{\bowtie}{R}$ and $\overset{\bowtie}{L}$, and edge spaces which are copies of $\overset{\bowtie}{E} \times I$. Let ℓ, r and e denote the degrees of $\overset{\bowtie}{L} \to L$, $\overset{\bowtie}{R} \to R$ and $\overset{\bowtie}{E} \to E$. Let $\ell(\overset{\bowtie}{R})$ denote the cover of R consisting of ℓ copies of $\overset{\bowtie}{R}$, and let $r(\overset{\bowtie}{L})$ denote the cover of L consisting of r copies of $\overset{\bowtie}{L}$.

There are exactly $\frac{\ell}{e}$ equivalence classes of elevations of $E \to L$ to $\overset{\bowtie}{L}$ and consequently $\frac{r\ell}{e}$ equivalence classes of elevations of E to $r(\overset{\bowtie}{L})$. Similarly, there are $\frac{r\ell}{e}$ equivalence classes of elevations of $E \to R$ to $\ell(\overset{\bowtie}{R})$. By Step 5, the induced cover of E which is the domain of each such elevation is isomorphic to $\overset{\bowtie}{E}$.

Choose a one-to-one correspondence between the equivalence classes of these elevations to $r(\overset{\bowtie}{L})$ and the equivalence classes of these elevations to $\ell(\overset{\bowtie}{R})$. Finally we form a cover $\widehat{X} \to X$ as follows: The vertex spaces of $\widehat{X}$ are the connected components in the union $r(\overset{\bowtie}{L}) \cup \ell(\overset{\bowtie}{R})$. There are $\frac{r\ell}{e}$ edge spaces each of which is a copy of $\overset{\bowtie}{E} \times I$, and these edge spaces are attached along the elevations using the one-to-one correspondence. The resulting map $\hat{X} \to X$ is a covering map because it is a local isomorphism of complexes. $\square$

8. Groups acting on $\mathcal{VH}$-complexes

In this section we prove Theorem 8.1 which deals with groups acting properly discontinuously on certain $\mathcal{VH}$-complexes. The statement of Theorem 8.1.1 is of intermediate strength between Theorem 7.3 and Theorem 11.3. In subsequent sections, Theorem 8.1 will be applied to prove residual properties of certain negatively curved polygons of finite groups, and certain groups acting on buildings.

Theorem 8.1. *Let G act properly discontinuously and cocompactly by $\mathcal{VH}$-maps on a non-positively curved, simply-connected, $\mathcal{VH}$-complex $\tilde{X}$. Suppose that for some n there is no $\mathcal{VH}$-embedding of $I_n \times I_2$ in $\tilde{X}$. Then*

1. *There is a subgroup of finite index $H \subset G$ such that $H \backslash \tilde{X}$ is a compact clean $\mathcal{VH}$-complex.*
2. *G is residually finite, and in fact, every quasiconvex subgroup of G is separable.*

Proof. By Lemma 4.5, there is a finite index subgroup G' of G which splits as a thin finite graph of finitely generated virtually free groups. Consequently, by Corollary 11.5, G' has a finite index torsion-free subgroup J. The quotient $J\backslash\tilde{X}$ is a compact thin $\mathcal{VH}$-complex. It then follows from Theorem 7.3 that there is a finite index subgroup $H \subset J$ such that $H\backslash\tilde{X}$ is geometrically clean.

Theorem 3.4 implies that H and therefore G is residually finite. To see that every quasiconvex subgroup K of G is separable, it would be sufficient to show that $K \cap H$ is separable in H, because then $K \cap H$ would be separable in G, and so K would be separable because it contains a finite index separable subgroup. Now H is the fundamental group of the compact clean complex $H\backslash\tilde{X}$ and H is word-hyperbolic because $\tilde{X}$ contains no $I_m \times I_2$ and therefore no flat plane, and so it is δ-hyperbolic by [5]. The result then follows from Theorem 4.6. ☐

Remark 8.2. The essential difference between Theorem 8.1.1 and Theorem 7.3 is that the former allows actions which are not free. The interested reader will find that Theorem 8.1.1 can be proven in an almost identical manner to Theorem 7.3, except that one must replace the notion of an immersion between graphs with the notion of an immersion between graphs of finite groups. Unfortunately, the current proof depends upon Theorem 11.3 by using Corollary 11.5 to produce a torsion free subgroup of finite index, after which Theorem 7.3 produces the geometrically clean cover.

We now deduce Theorems 1.8 and 1.10 which were mentioned in the introduction without proof. Let X be a compact locally-$\mathcal{VH}$ complex. As in Definition 4.1, X has a double cover $\hat{X}$ which is a $\mathcal{VH}$-complex. Furthermore, the negative curvature hypothesis implies that $I_2 \times I_2$ does not embed in $\hat{X}$ and hence in $\tilde{X}$. We can therefore apply Theorem 8.1 to the action of $\pi_1\hat{X}$ on $\tilde{X}$ to complete the proof.

9. Polygons of finite groups

Theorem 9.1 (Residually finite polygons)**.** *Let P be an n-gon of finite groups such that all of the angles of P are $\leq \frac{\pi}{2}$. Then every quasiconvex subgroup of P is separable, and in particular P is residually finite provided that either:*

1. *P is a 4-gon with at least 2 acute angles;*
2. *P is a 5-gon with at least 1 acute angle;*
3. *P has at least 6 sides.*

Proof. The statement holds because $\pi_1 P$ splits as an amalgamated free product of two virtually free groups amalgamating a finitely generated almost malnormal subgroup. The splitting occurs along an edge of groups corresponding to a line segment $J \subset P$ whose endpoints are in the interiors of disjoint edges of ∂P.

Furthermore, as illustrated in Figure 4 we choose J so that:

1. if P is a 4-gon, then an acute angle of P lies on each side of J;
2. if P is a 5-gon, then there are three corners of P on one side of J, and an acute angle on the other side of J;
3. if P has at least 6 sides, then there are at least three corners of P on each side of J.

Now we claim that $\pi_1 P$ splits as an amalgamated free product where the two factors are generated by the vertex groups in the two components of $P - J$, and the amalgamated subgroup is generated by the edge groups corresponding to the endpoints of J. We give a sketch before providing a detailed proof: Let $\tilde{J}$ denote the preimage of J in $\tilde{P}$, and we observe that $\tilde{P}$ has a tree structure, whose vertices correspond to the connected components of $\tilde{P} - \tilde{J}$ and whose edges

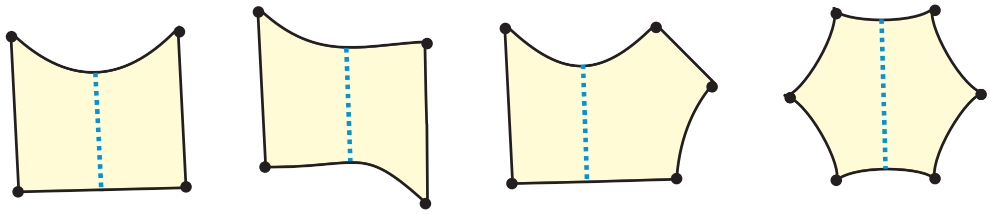

FIGURE 4. ¿From left to right are two 4-gons with two acute angles illustrating the two possible cases that can arise, a 5-gon with a single acute angle, and a 6-gon with no acute angles. In each polygon, the line segment J is indicated by a dotted line.

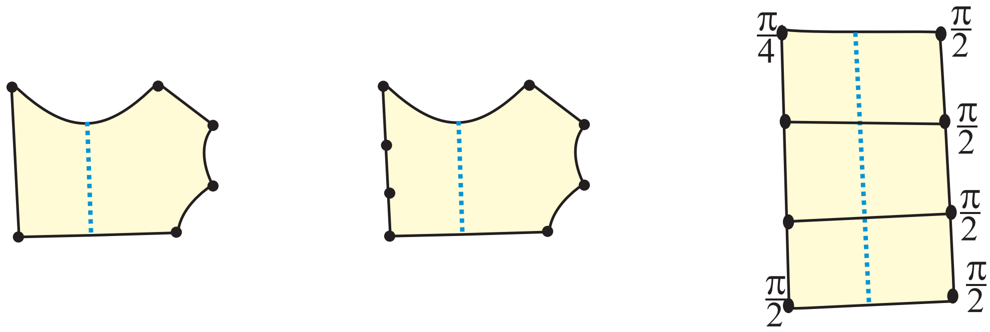

FIGURE 5.

correspond to components of $\tilde{P}$. To see that the vertex and edge groups are virtually free, one need only observe that $\tilde{J}$ is a tree, and each component of $\tilde{P} - \tilde{J}$ is a thickened tree. That the stabilizer of a component of $\tilde{J}$ is almost malnormal can be deduced as follows: If two components of $\tilde{J}$ are incident at the same vertex space, then when we thicken up these components, the angle condition implies that they have compact intersection, and therefore their stabilizers, which correspond to the conjugates of $\pi_1 J$ have finite intersection.

We now provide a detailed argument in a slightly different context. By possibly subdividing the edges of P, we can assume that there are the same number of 0-cells in each component of $P - J$. Consequently, we can subdivide P so that $P \cong I_n \times I$ and $J \cong I_n \times \{\frac{1}{2}\}$. From now on we will denote a point $p \in P$ by an ordered pair (a, b). However, if p is a vertex of P, then we will regard the angle at (a, b) to be the Gersten-Stallings angle at $p \in P$. For example, the polygon P on the left in Figure 5 is a 6-gon, and J separates two of the corners of P from the other four. As indicated in the center, we subdivide P by adding vertices (and hence corners with angle π) so that there are four vertices on each side of J. Finally, we regard this polygon as being $I_3 \times I$ as indicated on the right.

By hypothesis, one of the following possibilities holds: Either the left side of $P - J$ has a right-angled corner both of whose edges are contained in the left side of $P - J$, and consequently for some ℓ with $0 < \ell < n$ the vertex $(\ell, 0)$ has a right angle. Or some vertex of the left side of $P - J$ has an acute angle, and so for some ℓ with $0 \leq \ell \leq n$, the vertex $(\ell, 0)$ has an acute angle. A similar statement holds for the right side of P to obtain a vertex $(r, 1)$. In either case, the link of a preimage of the point $(\ell, 0)$ [respectively $(r, 1)$] has no cycles of length 4.

Since $P \cong I_n \times I$ is a $\mathcal{VH}$-complex, so is $\tilde{P}$, and $\pi_1 P$ acts on $\tilde{P}$ by $\mathcal{VH}$-isometries. Finally, suppose that $I_{2n} \times I_2$ embeds in $\tilde{P}$, we will show by contradiction that this is impossible. Indeed, consider the way $I_{2n} \times I_2$ is tessellated by the fundamental domain P of $\tilde{P}$, The pigeonhole principle guarantees that some lift of $(\ell, 0)$ [respectively $(r, 1)$] lies in the interior of $I_{2n} \times I_2$. But this contradicts the fact that such vertices have no cycles of length 4 in their links. The result now follows from Theorem 8.1 □

The following example shows we cannot drop the requirement in Theorem 9.1 that the vertex groups be finite:

Example 9.2. Consider the following presentation:

$$\langle a_1, a_2, a_3, a_4, t_1, t_2, t_3, t_4 \mid (a_4^2)^{t_4} = a_1^3,\ a_1^{t_1} = a_2,\ a_2^{t_2} = a_3,\ a_3^{t_3} = a_4 \rangle$$

This presentation corresponds naturally to a square of groups where each vertex group is a free group of rank 2 generated by $\langle a_i, t_i \rangle$. The edge groups are cyclic and correspond to the four relators. The face group is trivial. The Gersten-Stallings angle between the edge groups is 0 at each vertex, because in each vertex group, the two cyclic edge groups generate a free subgroup of rank 2. By successively substituting and remove the generators a_4, a_3, and then a_2, we obtain the following presentation

$$\langle a_1, t_1, t_2, t_3, t_4 \mid (a_1^2)^{t_1 t_2 t_3 t_4} = a_1^3 \rangle.$$

Letting $t = t_1 t_2 t_3 t_4$, and removing the generator t_1, we obtain the presentation

$$\langle a_1, t, t_2, t_3, t_4 \mid (a_1^2)^t = a_1^3 \rangle.$$

Finally we see that the group is isomorphic to the free product of $F_3 \cong \langle t_2, t_3, t_4 \rangle$ and the non-Hopfian, Baumslag-Solitar group $\langle a_1, t \mid (a_1^2)^t = a_1^3 \rangle$ (see [3]). Consequently, the group is not Hopfian, and therefore not residually finite. We note that there is a similar example if we substitute any natural number for 4.

J. Meier showed in [23, 24] that certain polygons of finite groups are Hopfian. Specifically he proved this for negatively curved polygons of finite groups and for polygons of finite groups arising from an action of the group on a building.

Prior to the introduction of nonpositive curvature techniques, an old problem of B. H. Neumann and H. Neumann [25], asked whether every developable triangle of finite groups is actually *embeddable* in a finite group, meaning that the diagram corresponds to a diagram of inclusions among subgroups (and their intersections) in some finite group. K. Brown solved this problem in [7] by providing an example of a (positively curved) triangle of finite groups whose direct limit is infinite simple, and therefore cannot be homomorphed to any nontrivial finite group. However, in view of Theorem 1.11, it is natural to restate this embeddability problem for nonpositively [negatively] curved polygons of finite groups. It is shown in [33] that any torsion element in a nonpositively curved polygon P of finite groups is in the stabilizer of a vertex of $\tilde{P}$ and thus conjugate into a vertex group of P. As a consequence, the embeddability problem is equivalent to the problem of whether $\pi_1 P$ is virtually torsion-free. Since there are only finitely many torsion elements up to conjugacy, we have the following corollary of Theorem 9.1 (see Corollary 11.5):

Theorem 9.3. *Let P be an n-gon of finite groups which satisfies the conditions of Theorem 9.1. Then $\pi_1 P$ is virtually torsion-free.*

Considering the examples mentioned in Theorem 1.12, it seems likely that there exist nonpositively curved polygons of finite groups that are not virtually torsion-free, but no examples are currently known. Furthermore, examples were given in [38] of groups G acting properly discontinuously and cocompactly on nonpositively curved 2-complexes, such that G is not virtually torsion-free.

10. Groups acting on 2-dimensional hyperbolic buildings

In this section we apply Theorem 8.1 to groups acting on 2-dimensional hyperbolic buildings. For more about buildings see [6] or [26].

Theorem 10.1. *Let B be a locally finite building and let C be a chamber of B. Suppose that C is an n-gon satisfying one of the following conditions:*

(1) $n = 4$ *and C has at least two acute angles.*
(2) $n = 5$ *and C has at least one acute angle.*
(3) $n \geq 6$.

Let G act properly discontinuously and cocompactly on B. Then the quasiconvex subgroups of G are separable, and in particular G is residually finite.

Proof. First observe that there is a finite index subgroup $H \subset G$ which is type preserving. Let C' be a copy of C, let H act on C' by the trivial action, and observe that there is an equivariant combinatorial map $B \to C'$. The proof is now similar to the proof of Theorem 9.1. We can subdivide C' so that it has a cell structure isomorphic to the $\mathcal{VH}$-complex $I_k \times I$ where $k = \lceil \frac{n-2}{2} \rceil$. We then have the following three cases where by *vertical side* we mean either $I_k \times \{0\}$ or $I_k \times \{1\}$:

(1) $k = 1$ and each vertical side contains at least one acute-angled corner of C'.
(2) $k = 2$ and one vertical side contains an acute-angled corner of C', and the other vertical side contains three corners of C'.
(3) $k \geq 2$ and each vertical side contains at least three corners of C'

Note that the $\mathcal{VH}$-structure on C' induces a $\mathcal{VH}$-structure on B, the group H acts by $\mathcal{VH}$-maps on B, and B does not contain an embedded $I_{2k} \times I_2$. Therefore the result follows from Theorem 8.1. □

We close this section by noting that examples were give in [14] of 2-dimensional hyperbolic buildings whose type-preserving automorphism groups are uncountable, simple and nonlinear.

11. Thin ⇒ Virtually Clean (General Case)

The main goal of this section is to prove Theorem 11.3 which is the main result of this paper.

Lemma 11.1. *Let X be a graph of spaces and let Y be as in Construction 7.1.*

(1) *If Γ_X is compact and each edge and vertex group of $\pi_1 X$ is finitely generated, then each edge and vertex group of $\pi_1 Y$ is finitely generated.*
(2) *If each edge and vertex group of $\pi_1 X$ is virtually free then each edge and vertex group of $\pi_1 Y$ is virtually free.*
(3) *If X is thin then Y is thin.*
(4) *If Y is virtually clean, then X is virtually clean.*

Proof. The edge group of $\pi_1 Y$ is the free product of the edge groups of $\pi_1\bar{X}$. Similarly, each vertex group of $\pi_1 Y$ is a free product of vertex groups of $\pi_1\bar{X}$ and a free group of rank $< n$. It follows that if the edge and vertex groups of $\pi_1 X$ are finitely generated, then the edge and vertex groups of $\pi_1 Y$ are finitely generated. It is easy to see that the free product of finitely many virtually free groups is virtually free. It follows that if the edge and vertex groups of $\pi_1 X$ are virtually free, then the edge and vertex groups of $\pi_1 Y$ are virtually free.

One shows that Y is thin provided that X is thin by using the normal form theorem for conjugacy classes in graphs of groups. We leave this to the reader (see [21, IV.2.5 and IV.2.8]).

Suppose that $\hat{Y} \to Y$ is a clean finite cover. Let $\hat{X}$ be a component of the preimage of $\bar{X}$ in $\hat{Y}$. We will show that $\hat{X}$ is clean and therefore X has a clean finite cover. We will use the following simple fact which the reader can verify: If A is a free factor of B and C is a subgroup of B then $A \cap C$ is a free factor of C. Any attaching map $E \to U$ of an edge space of $\hat{X}$ to a vertex space of $\hat{X}$ is the restriction of the attaching map $F \to V$ of an edge space of $\hat{Y}$ to a vertex space of $\hat{Y}$. Choose a basepoint for E and let the basepoints of F, U, and V be the images of this basepoint. Since $\hat{Y}$ is clean, $\pi_1 F$ embeds as a free factor of $\pi_1 V$. It is easy to see that $\pi_1 U \cap \pi_1 F = \pi_1 E$ and so according to the fact above, $\pi_1 E$ is a free factor of $\pi_1 U$. Applying this to each attaching map of each edge space of $\hat{X}$ shows that $\hat{X}$ is clean. □

Lemma 11.2. *Let B be a compact graph, and for $1 \leq i \leq n$ let $A_i \to B$ be an immersion of graphs where A_i is compact. There exists a finite cover $\hat{B} \to B$ such that for each i, every elevation of $A_i \to B$ to $\hat{B}$ is injective.*

Proof. Let $\hat{B}$ be the smallest connected regular cover that factors through all components of the finite covers $\mathsf{C}(A_i, B)$. Then for each i, every elevation of $A_i \to B$ to $\hat{B}$ is injective because (after taking a covering space translation of $\hat{B} \to B$) it corresponds to a component of the preimage of $A_i \subset \mathsf{C}(A_i, B)$. □

As explained in the introduction, the proof of Theorem 11.3 is written so that its steps are parallel to the proof of Theorem 7.3. The main exception to this parallelism is the insertion of an additional step called 'graphical representation' which contains the crucial additional idea enabling the parallelism, and in fact, enabling the proof.

Theorem 11.3. *Every thin finite graph of finitely generated virtually free groups is virtually clean.*

Proof. **Step 1:** Let X be a thin finite graph of spaces such that the vertex and edge groups of X are finitely generated and virtually free. Let Y be the graph of spaces provided by Construction 7.1. By Lemma 11.1, $\pi_1 Y$ splits as a free product of two finitely generated virtually free groups amalgamating an almost malnormal finitely generated virtually free subgroup. Furthermore, if Y is virtually clean then X is virtually clean. Consequently, it is sufficient to consider the special case where $\Gamma_X = I$ and we assume that the induced splitting is $\pi_1 X = A *_M B$.

Let $\mathcal{L}$, $\mathcal{E}$, and $\mathcal{R}$ be based Eilenberg-MacLane spaces whose fundamental groups are A, M, and B. Furthermore, let $\mathcal{E} \to \mathcal{L}$ and $\mathcal{E} \to \mathcal{R}$ be based maps which induce the inclusions $M \hookrightarrow A$ and $M \hookrightarrow B$. We can assume without loss of generality that X is formed from $\mathcal{L}$, $\mathcal{R}$ and $\mathcal{E} \times I$ using the attaching maps $\mathcal{E} \to \mathcal{L}$ and $\mathcal{E} \to \mathcal{R}$.

Graphical Representation: We first form based finite covers $\mathcal{L}_0 \to \mathcal{L}$ and $\mathcal{R}_0 \to \mathcal{R}$ with the following properties:

(1) $\pi_1 \mathcal{L}_0$ and $\pi_1 \mathcal{R}_0$ are free.

(2) $\mathcal{L}_0 \to \mathcal{L}$ and $\mathcal{R}_0 \to \mathcal{R}$ induce the same based cover $\mathcal{E}_0 \to \mathcal{E}$ so we have the following diagram:

$$\begin{array}{ccccc} \mathcal{L}_0 & \leftarrow & \mathcal{E}_0 & \rightarrow & \mathcal{R}_0 \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{L} & \leftarrow & \mathcal{E} & \rightarrow & \mathcal{R} \end{array}$$

(3) The subgroups $\pi_1\mathcal{E}_0 \hookrightarrow \pi_1\mathcal{L}_0$ and $\pi_1\mathcal{E}_0 \hookrightarrow \pi_1\mathcal{R}_0$ are free factors.

This can be done as follows: Let P be a finite index free subgroup of $\pi_1\mathcal{L}$ such that $P' = P \cap \pi_1\mathcal{E}$ is a free factor of P. Note that the subgroup P can be obtained by applying M. Hall's Theorem to the intersection between $\pi_1\mathcal{E}$ and a finite index free subgroup of $\pi_1\mathcal{L}$. Similarly, let Q be a finite index free subgroup of $\pi_1\mathcal{R}$ such that $Q' = Q \cap \pi_1\mathcal{E}$ is a free factor of $\pi_1\mathcal{R}$. Choose retractions $P \to P'$ and $Q \to Q'$. Let $C \subset \pi_1\mathcal{E}$ be the finite index subgroup $P' \cap Q'$. Let $\mathcal{E}_0$ be the based cover of $\mathcal{E}$ corresponding to C. Finally, the preimages of C in P and Q are finite index subgroups whose corresponding finite based covers $\mathcal{L}_0$ and $\mathcal{R}_0$ have the desired properties.

We will now choose maps from $\mathcal{L}_0$ and $\mathcal{R}_0$ to graphs, such that these maps induce π_1-isomorphisms and such that the induced maps on $\pi_1\mathcal{E}_0$ are consistent in a certain sense. Let E_0 be a bouquet of circles such that the number of circles in E_0 is the same as the rank of $\pi_1\mathcal{E}_0$. Define L_0 and R_0 similarly and choose inclusions $E_0 \hookrightarrow L_0$ and $E_0 \hookrightarrow R_0$.

Let $\mathcal{E}_0 \to E_0$ be a basepoint preserving map which induces a π_1-isomorphism. Using the fact that $\pi_1\mathcal{E}_0$ maps to a free factor in $\pi_1\mathcal{L}_0$, choose a based map $\mathcal{L}_0 \to L_0$ which induces a π_1-isomorphism, and such that we have the commutative diagram on the left in Equation (1). We are essentially, choosing a basis for $\pi_1\mathcal{L}_0$ which extends the chosen basis for the free factor subgroup which is the image of $\pi_1\mathcal{E}_0$. Similarly, define a bouquet of circles R_0, an inclusion $E_0 \subset R_0$, and a based map $\mathcal{R}_0 \to R_0$ which induces a π_1-isomorphism, so that we obtain the commutative diagram on the right in Equation (1).

$$\begin{array}{ccc} \pi_1\mathcal{E}_0 & \rightarrow & \pi_1\mathcal{L}_0 \\ \downarrow & & \downarrow \\ \pi_1 E_0 & \rightarrow & \pi_1 L_0 \end{array} \qquad \begin{array}{ccc} \pi_1\mathcal{E}_0 & \rightarrow & \pi_1\mathcal{R}_0 \\ \downarrow & & \downarrow \\ \pi_1 E_0 & \rightarrow & \pi_1 R_0 \end{array} \tag{1}$$

We will use these compatible graphical representations of $\mathcal{L}_0$ and $\mathcal{R}_0$ in order to employ the canonical completion and retraction technique as in the proof of Theorem 7.3. For the duration of the proof, we will use decorated calligraphic symbols to denote spaces, and decorated regular symbols to denote the corresponding graphs. For instance $\mathcal{L}_1 \leftrightarrow L_1$, $\bar{\mathcal{R}} \leftrightarrow \bar{R}$ etc.

Step 2: There exist finite based covers $\bar{L} \to L_0$ and $\bar{R} \to R_0$ and corresponding covers $\bar{\mathcal{L}} \to \mathcal{L}_0$ and $\bar{\mathcal{R}} \to \mathcal{R}_0$, so that the following hold:

(1) For each elevation $\mathcal{F} \to \bar{\mathcal{L}}$ of $\mathcal{E} \to \mathcal{L}$, the composition $\mathcal{F} \to \bar{\mathcal{L}} \to \bar{L}$ is a homotopy equivalence with a subgraph of $\bar{L}$. Similarly, each elevation of $\mathcal{E} \to \mathcal{R}$ to $\bar{\mathcal{R}}$ projects to a homotopy equivalence with a subgraph of $\bar{R}$.
(2) If two elevations of $\mathcal{E} \to \mathcal{L}$ to $\bar{\mathcal{L}}$ (respectively $\mathcal{E} \to \mathcal{R}$ to $\bar{\mathcal{R}}$) are equivalent then their corresponding subgraphs in $\bar{L}$ (respectively of $\bar{R}$) are identical. If the two elevations are not equivalent then their corresponding subgraphs intersect in a (possibly empty) forest.
(3) The based covers $\bar{L}$ and $\bar{R}$ induce the same based cover $\bar{E} \to E_0$. Similarly, the corresponding covers $\bar{\mathcal{L}}$ and $\bar{\mathcal{R}}$ induce the same cover $\bar{\mathcal{E}} \to \mathcal{E}_0$. We therefore have the following

two corresponding commutative diagrams:

$$\begin{array}{ccccc} \bar{\mathcal{L}} & \leftarrow & \bar{\mathcal{E}} & \rightarrow & \bar{\mathcal{R}} \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{L}_0 & \leftarrow & \mathcal{E}_0 & \rightarrow & \mathcal{R}_0 \end{array} \qquad \begin{array}{ccccc} \bar{L} & \hookleftarrow & \bar{E} & \hookrightarrow & \bar{R} \\ \downarrow & & \downarrow & & \downarrow \\ L_0 & \leftarrow & E_0 & \rightarrow & R_0 \end{array} \tag{2}$$

Each elevation $\breve{\mathcal{E}} \to \mathcal{L}_0$ of $\mathcal{E} \to \mathcal{L}$, gives rise to a map $\breve{\mathcal{E}} \to \mathcal{L}_0 \to L_0$. This can be represented by an immersion to L_0 of a compact finite graph $\breve{E}$, in the sense that there is a homotopy equivalence $\breve{\mathcal{E}} \to \breve{E}$, and an immersion $\breve{E} \to L_0$, such that the map which is the composition $\breve{\mathcal{E}} \to \mathcal{L}_0 \to L_0$ is homotopic to the composition $\breve{\mathcal{E}} \to \breve{E} \to L_0$. Note that we used the hypothesis that $\pi_1\mathcal{E}$ is finitely generated to guarantee that the free group $\pi_1\breve{\mathcal{E}}$ is finitely generated, and therefore as in [32], each of the elevations can be represented by an immersion of a compact graph.

Since there are finitely many distinct elevations, there are finitely many such immersions, and therefore by Lemma 11.2, there is a based finite cover $L_1 \to L_0$ such that for each of these immersions, all of its elevations are embeddings. Consequently, the corresponding cover $\mathcal{L}_1 \to \mathcal{L}_0$ has the property that for any elevation $\mathcal{E}' \to \mathcal{L}_1$ of $\mathcal{E} \to \mathcal{L}$, the induced map $\mathcal{E}' \to \mathcal{L}_1 \to L_1$ is a homotopy equivalence onto a subgraph $E' \subset L_1$. We define the based finite cover $R_1 \to R_0$ similarly.

Let $E_L \hookrightarrow L_1$ denote the elevation of $E_0 \to L_0$ to L_1 at the basepoint of L_1. Similarly, let $E_R \hookrightarrow R_1$ denote the elevation of $E_0 \to R_0$ at the basepoint of R_1. Choose retraction maps $L_1 \to E_L$ and $R_1 \to E_R$. Choose a based finite cover $\bar{E} \to E_0$ which factors through the based covers $E_L \to E_0$ and $E_R \to E_0$. Let $\bar{L} \to L_0$ and $\bar{R} \to R_0$ denote the based covers induced by the retraction maps and the covers $\bar{E} \to E_L$ and $\bar{E} \to E_R$. So we have the commutative diagram on the right in Equation (2). Let $\bar{\mathcal{L}}$, $\bar{\mathcal{R}}$ and $\bar{\mathcal{E}}$ be the covers induced by $\bar{L}$ and $\bar{R}$ and $\bar{E}$, and note that the commutative diagram on the left in Equation (2), is induced by the one on the right.

Observe that because of the way we chose L_1, each elevation $\breve{\mathcal{E}} \to \bar{\mathcal{L}}$ of $\mathcal{E} \to \mathcal{L}$ has the property that the composition $\breve{\mathcal{E}} \to \bar{\mathcal{L}} \to \bar{L}$ is a homotopy equivalence with a subgraph of $\bar{L}$. The analogous statement holds for elevations to $\bar{\mathcal{R}}$.

Because $\pi_1\mathcal{E}$ is almost malnormal in $\pi_1\mathcal{L}$, subgraphs of $\bar{L}$ corresponding to distinct elevations of $\mathcal{E}$ to $\bar{\mathcal{L}}$ must intersect in forests. This is proven in a similar manner to the analogous statement in Step 2 of the proof of Theorem 7.3.

Step 3: We now form the canonical completions of $\bar{E}$ with respect to $\bar{E}$, $\bar{L}$ and $\bar{R}$, and obtain the following commutative diagram by applying Corollary 5.3 twice.

$$\begin{array}{ccccc} \mathsf{C}(\bar{E},\bar{L}) & \hookleftarrow & \mathsf{C}(\bar{E},\bar{E}) & \hookrightarrow & \mathsf{C}(\bar{E},\bar{R}) \\ \downarrow & & \downarrow & & \downarrow \\ \bar{L} & \leftarrow & \bar{E} & \rightarrow & \bar{R} \end{array}$$

Corresponding to this commutative diagram is the following commutative diagram. Note that $\mathsf{C}(\bar{\mathcal{E}},\bar{\mathcal{L}})$ is undefined because we have not assumed that $\bar{\mathcal{E}} \to \bar{\mathcal{L}}$ is an immersion of graphs. However, in order to maintain the parallelism between the geometric and general cases, we are using the notation $\mathsf{C}(\bar{\mathcal{E}},\bar{\mathcal{L}})$ to mean the covering space of $\bar{\mathcal{L}}$ induced by $\mathsf{C}(\bar{E},\bar{L}) \to \bar{L}$.

$$\begin{array}{ccccc} \mathsf{C}(\bar{\mathcal{E}},\bar{\mathcal{L}}) & \hookleftarrow & \mathsf{C}(\bar{\mathcal{E}},\bar{\mathcal{E}}) & \hookrightarrow & \mathsf{C}(\bar{\mathcal{E}},\bar{\mathcal{R}}) \\ \downarrow & & \downarrow & & \downarrow \\ \bar{\mathcal{L}} & \leftarrow & \bar{\mathcal{E}} & \rightarrow & \bar{\mathcal{R}} \end{array}$$

Step 4: Let $\widehat{\mathcal{E}} \to \mathcal{E}$ denote a finite connected regular cover of $\mathcal{E}$ which factors through all the covers of $\mathcal{E}$ induced by $\mathsf{C}(\bar{\mathcal{E}},\bar{\mathcal{L}})$ and $\mathsf{C}(\bar{\mathcal{E}},\bar{\mathcal{R}})$ (as the basepoint varies in these covers). Let $\widehat{E} \to \bar{E}$

denote the cover of $\bar{E}$ corresponding to $\widehat{\mathcal{E}} \to \bar{\mathcal{E}}$. Let $\widehat{\mathsf{C}(\bar{E},\bar{L})}$ and $\widehat{\mathsf{C}(\bar{E},\bar{R})}$ and $\widehat{\mathsf{C}(\bar{E},\bar{E})}$ be the covers induced by the canonical retraction maps and the cover $\widehat{E} \to \bar{E}$. Let $\widehat{\mathsf{C}(\bar{\mathcal{E}},\bar{\mathcal{L}})}$ and $\widehat{\mathsf{C}(\bar{\mathcal{E}},\bar{\mathcal{R}})}$ and $\widehat{\mathsf{C}(\bar{\mathcal{E}},\bar{\mathcal{E}})}$ be the corresponding covers.

We obtain the following commutative diagram:

$$\begin{array}{ccccc} \widehat{\mathsf{C}(\bar{E},\bar{L})} & \supset & \widehat{\mathsf{C}(\bar{E},\bar{E})} & \subset & \widehat{\mathsf{C}(\bar{E},\bar{R})} \\ \downarrow & & \downarrow & & \downarrow \\ \mathsf{C}(\bar{E},\bar{L}) & \supset & \mathsf{C}(\bar{E},\bar{E}) & \subset & \mathsf{C}(\bar{E},\bar{R}) \end{array}$$

Similarly, we obtain the following corresponding diagram:

$$\begin{array}{ccccc} \widehat{\mathsf{C}(\bar{\mathcal{E}},\bar{\mathcal{L}})} & \leftarrow & \widehat{\mathsf{C}(\bar{\mathcal{E}},\bar{\mathcal{E}})} & \rightarrow & \widehat{\mathsf{C}(\bar{\mathcal{E}},\bar{\mathcal{R}})} \\ \downarrow & & \downarrow & & \downarrow \\ \mathsf{C}(\bar{\mathcal{E}},\bar{\mathcal{L}}) & \leftarrow & \mathsf{C}(\bar{\mathcal{E}},\bar{\mathcal{E}}) & \rightarrow & \mathsf{C}(\bar{\mathcal{E}},\bar{\mathcal{R}}) \end{array}$$

Step 5: Let $\overset{\bowtie}{\mathcal{L}} \to \mathcal{L}$ be the smallest regular cover of $\mathcal{L}$ factoring through each component of $\widehat{\mathsf{C}(\bar{\mathcal{E}},\bar{\mathcal{L}})}$. Similarly, let $\overset{\bowtie}{\mathcal{E}}$ and $\overset{\bowtie}{\mathcal{R}}$ be the smallest regular covers of $\mathcal{E}$ and $\mathcal{L}$ which factor through each component of $\widehat{\mathsf{C}(\bar{\mathcal{E}},\bar{\mathcal{E}})}$ and $\widehat{\mathsf{C}(\bar{\mathcal{E}},\bar{\mathcal{R}})}$. We claim that each cover of $\mathcal{E}$ induced by $\overset{\bowtie}{\mathcal{L}}$ or $\overset{\bowtie}{\mathcal{R}}$ is isomorphic to $\overset{\bowtie}{\mathcal{E}}$.

It is sufficient to prove that each cover of $\mathcal{E}$ induced by $\overset{\bowtie}{\mathcal{L}}$ is isomorphic to $\overset{\bowtie}{\mathcal{E}}$, because an analogous argument proves the same for $\overset{\bowtie}{\mathcal{R}}$. Let $\check{\mathcal{E}} \to \mathcal{E}$ be a covering space induced by $\overset{\bowtie}{\mathcal{L}} \to \mathcal{L}$ and note that $\check{\mathcal{E}}$ does not depend on the choice of basepoint since $\overset{\bowtie}{\mathcal{L}}$ is regular.

To prove that $\check{\mathcal{E}} \cong \overset{\bowtie}{\mathcal{E}}$ we will show that $\check{\mathcal{E}}$ and $\overset{\bowtie}{\mathcal{E}}$ factor through each other. We first show that $\check{\mathcal{E}}$ factors through $\overset{\bowtie}{\mathcal{E}}$. First observe that the restriction of the map $\widehat{\mathsf{C}(\bar{\mathcal{E}},\bar{\mathcal{E}})} \to \widehat{\mathsf{C}(\bar{\mathcal{E}},\bar{\mathcal{L}})}$ to each component of $\widehat{\mathsf{C}(\bar{\mathcal{E}},\bar{\mathcal{E}})}$, is an elevation of $\mathcal{E} \to \mathcal{L}$ to $\widehat{\mathsf{C}(\bar{\mathcal{E}},\bar{\mathcal{L}})}$. Consequently, $\check{\mathcal{E}}$ factors through each component of $\widehat{\mathsf{C}(\bar{\mathcal{E}},\bar{\mathcal{E}})}$ and therefore, since $\check{\mathcal{E}}$ is regular, $\check{\mathcal{E}}$ must factor through $\overset{\bowtie}{\mathcal{E}}$ which is the smallest regular cover factoring through these components.

Now we show that $\overset{\bowtie}{\mathcal{E}}$ factors through $\check{\mathcal{E}}$. As in the proof of Theorem 7.3, this will follow by showing that $\overset{\bowtie}{\mathcal{E}}$ factors through each cover of $\mathcal{E}$ induced by $\widehat{\mathsf{C}(\bar{\mathcal{E}},\bar{\mathcal{L}})}$. We will divide our analysis of elevations of $\mathcal{E}$ into two cases:

(1) Elevations $\mathcal{E} \to \widehat{\mathsf{C}(\bar{\mathcal{E}},\bar{\mathcal{L}})}$ of $\mathcal{E} \to \mathcal{L}$ that factor through $\bar{\mathcal{E}} \to \bar{\mathcal{L}}$.

(2) Elevations $\mathcal{E} \to \widehat{\mathsf{C}(\bar{\mathcal{E}},\bar{\mathcal{L}})}$ of $\mathcal{E} \to \mathcal{L}$ that do not factor through $\bar{\mathcal{E}} \to \bar{\mathcal{L}}$.

Case 1: If $\mathcal{K} \to \bar{\mathcal{L}}$ is an elevation of $\mathcal{E} \to \mathcal{L}$ which is equivalent to $\bar{\mathcal{E}} \to \bar{\mathcal{L}}$, then any elevation $\hat{\mathcal{K}} \to \widehat{\mathsf{C}(\bar{\mathcal{E}},\bar{\mathcal{L}})}$ of $\mathcal{E} \to \mathcal{L}$ which factors through $\mathcal{K} \to \bar{\mathcal{L}}$ is an elevation of $\mathcal{E} \to \mathcal{L}$ to $\widehat{\mathsf{C}(\bar{\mathcal{E}},\bar{\mathcal{E}})}$. Consequently $\overset{\bowtie}{\mathcal{E}}$ factors through $\hat{\mathcal{K}}$.

Case 2: Suppose that $\mathcal{K} \to \bar{\mathcal{L}}$ is an elevation of $\mathcal{E} \to \mathcal{L}$ which is not equivalent to $\bar{\mathcal{E}} \to \bar{\mathcal{L}}$, and let $K \hookrightarrow \bar{L}$ and $\bar{E} \hookrightarrow \bar{L}$ be the corresponding embedded graphs. By the second property listed in Step 2, $K \hookrightarrow \bar{L}$ intersects $\bar{E} \hookrightarrow \bar{L}$ in a (possibly empty) forest. Let $\check{\mathcal{K}} \to \mathsf{C}(\bar{\mathcal{E}},\bar{\mathcal{L}})$ be an elevation of $\mathcal{E} \to \mathcal{L}$ which factors through $\mathcal{K} \to \bar{\mathcal{L}}$, and let $\check{K} \hookrightarrow \mathsf{C}(\bar{E},\bar{L})$ be the corresponding graph. By Corollary 5.4, $\check{K}$ is null-homotopic in the canonical retraction $\mathsf{C}(\bar{E},\bar{L}) \to \bar{E}$. Consequently, for any elevation $\hat{K} \hookrightarrow \widehat{\mathsf{C}(\bar{E},\bar{L})}$ which factors through $\check{K} \hookrightarrow \mathsf{C}(\bar{E},\bar{L})$, the covers $\hat{K} \to K$ and

$\check{K} \to K$ are isomorphic. It follows that for any elevation $\hat{\mathcal{K}} \to \widehat{\mathsf{C}(\bar{\mathcal{E}}, \bar{\mathcal{L}})}$ of $\mathcal{E} \to \mathcal{L}$ which factors through $\check{\mathcal{K}} \to \mathsf{C}(\bar{\mathcal{E}}, \bar{\mathcal{L}})$, the covers $\check{\mathcal{K}} \to \mathcal{E}$ and $\hat{\mathcal{K}} \to \mathcal{E}$ are isomorphic. Since $\hat{\mathcal{E}} \to \mathcal{E}$ was chosen so that $\hat{\mathcal{E}}$ factors through each elevation of $\mathcal{E}$ to $\mathsf{C}(\bar{\mathcal{E}}, \bar{\mathcal{L}})$, we see that $\hat{\mathcal{E}}$ factors through $\check{\mathcal{K}}$. But $\hat{\mathcal{E}}$ is isomorphic to the based component of $\widehat{\mathsf{C}(\bar{\mathcal{E}}, \bar{\mathcal{E}})}$, and so since $\overset{\bowtie}{\mathcal{E}}$ factors through $\hat{\mathcal{E}}$ we see that $\overset{\bowtie}{\mathcal{E}}$ factors through $\hat{\mathcal{K}} \cong \check{\mathcal{K}}$.

Step 6: We form the clean cover $\widehat{X}$ out of vertex spaces isomorphic to $\overset{\bowtie}{\mathcal{L}}$ and $\overset{\bowtie}{\mathcal{R}}$ and edge spaces isomorphic to $\overset{\bowtie}{\mathcal{E}}$. This is done in precisely the same way as in Step 6 of the proof of Theorem 7.3. □

Theorem 11.4 (Thin ⇒ residually finite)**.** *Let G split as a thin finite graph of finitely generated virtually free groups, then G is residually finite.*

Proof. This follows by combining Theorem 11.3 with Theorem 3.4. □

Corollary 11.5 (Thin ⇒ virtually torsion-free)**.** *Let G split as a thin finite graph of finitely generated virtually free groups, then G is virtually torsion-free.*

Proof. By Theorem 11.3, G has a finite index subgroup H whose induced splitting is clean. But H is torsion-free because a graph of torsion-free groups is torsion-free [21].

It is worth noting that the result follows from Theorem 11.4. Indeed, G has finitely many torsion elements up to conjugacy, and therefore using residual finiteness, one can choose a finite index normal subgroup which does not contain any of these conjugacy classes. □

Example 11.6. We now give an example demonstrating that the conclusion of Theorem 1.3 fails to hold without assuming that the amalgamated subgroup is finitely generated. Indeed, the example G below is an amalgamated free product of two finitely generated free groups amalgamating a malnormal subgroup, but G has no nontrivial finite quotients. We note that it is also not difficult to construct examples which are not Hopfian.

Let G be the group obtained from the following presentation whose relations belong to four indexed families with $n \geq 1$.

$$\left\langle a, b, x, y \;\middle|\; \begin{array}{l} \{a = a^n b^{101n} a^n b^{102n} \cdots a^n b^{200n} x^n y^{101n} x^n y^{102n} \cdots x^n y^{200n}\}, \\ \{b = a^n b^{201n} a^n b^{202n} \cdots a^n b^{300n} x^n y^{201n} x^n y^{202n} \cdots x^n y^{300n}\}, \\ \{x = x^n y^{301n} x^n y^{302n} \cdots x^n y^{400n} a^n b^{301n} a^n b^{302n} \cdots a^n b^{400n}\}, \\ \{y = x^n y^{401n} x^n y^{402n} \cdots x^n y^{500n} a^n b^{401n} a^n b^{402n} \cdots a^n b^{500n}\} \end{array} \right\rangle$$

To see that G has no nontrivial finite quotient observe that the n-th relation in the first family of relations states that a is a product of n-th powers and therefore a maps to the identity element in any group of order n. Similarly, the other three families of relations imply that b, x, and y are mapped to the identity in any finite quotient.

Finally we claim that, as suggested by the presentation, G splits in an obvious way as a free product of free groups on $\{a, b\}$ and $\{x, y\}$ amalgamating infinite rank subgroups. One way to check that these subgroups are malnormal is to use the small-cancellation method developed in [40].

Remark 11.7. It appears likely that as in Theorem 8.1, all quasiconvex subgroups are separable. This would be the case if a generalized version of Theorem 4.6 held for clean graphs of groups instead of just geometrically clean $\mathcal{VH}$-complexes.

Remark 11.8. I now explain my reason for guessing that Problem 1.1 is equivalent to Problem 1.9. First of all the following theorem is implicit in [38]: *Every word-hyperbolic group is residually finite if and only if every malnormal quasiconvex subgroup of a word-hyperbolic group is separable.* Indeed, suppose that $M \subset G$ is malnormal and quasiconvex but not separable. Then the amalgamated free product $G *_M G$ obtained by doubling G along M is word-hyperbolic by [4], but $G *_M G$ is not residually finite (see [38, Corollary5.3]).

Furthermore, it seems highly plausible that if every malnormal quasiconvex subgroup is separable then every quasiconvex subgroup is separable. Indeed, if the quasiconvex subgroup Q of G contained a finite index subgroup M which was malnormal in a finite index subgroup of H of G, then the separability of M in H implies the separability of M in G and hence the separability of Q in G. On the other hand, *if the quasiconvex subgroup Q is separable, then Q is malnormal in a finite index subgroup H of G.* To see this, first note that by [10], there is a finite set of distinct cosets $Q = Qg_0, Qg_1, \ldots, Qg_r$ such that $Q^x \cap Q = \{1_G\}$ unless $x \in Qg_i$ for some i. We then use the separability of Q to choose a finite index subgroup H of G such that $Q \subset H$, but $g_i \notin H$ for $1 \leq i \leq r$. Finally, if $Q^x \cap Q \neq \{1_G\}$ then since $x \notin Qg_i$ for $1 \leq i \leq r$ by our choice of H, we see that $x \in Q$, and so Q is malnormal in H.

The statement that every word-hyperbolic group is residually finite is equivalent to two seemingly weaker statements. It is equivalent to the statement that every word-hyperbolic group has a proper finite index subgroup (see [18]), and to the statement that every word-hyperbolic group is virtually torsion-free (see [38, Proposition 6.2]).

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