Applications of nonlinear network flow models to market equilibria

László Végh
School of Computer Science, Georgia Tech

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Linear Fisher markets

* $B$: buyers, $G$: goods.
* Buyer $i$ has budget $m_i$, 1 divisible unit of each good $j$.
* Utility $U_{ij}$ for buyer $i$ on 1 unit of good $j$.
* Market clearing: prices $p_j$ and allocations $x_{ij}$ if:
  * everything is sold
  * all money is spent
  * only best bang-per-buck purchases: max. $U_{ij}/p_j$.

\[
U_{ij} = 5, \quad p_j = \$3
\]
\[
U_{ij} = 4, \quad p_j = \$2
\]
\[
5/3 < 4/2
\]
Linear Fisher markets

- Formulated by Fisher in 1891.
- Special case of the Arrow-Debreu model.
  - An equilibrium exists under very general conditions (Arrow, Debreu, 1954).
  - Nonconstructive proof based on Kakutani’s fixed point theorem.
- The linear Fisher model can be captured by the convex program by Eisenberg and Gale ’59.
Eisenberg-Gale convex program, 1959

\[
\max \sum_{i \in B} m_i \log U_i \\
U_i \leq \sum_{j \in G} U_{ij} x_{ij} \quad \forall i \in B \\
\sum_{i \in B} x_{ij} \leq 1 \quad \forall j \in G \\
x_{ij} \geq 0 \quad \forall i \in B, j \in G
\]

\text{prices: optimal Lagrange multipliers}

\[\ast\] Optimal solution corresponds to equilibrium prices.

\[\ast\] There exists a rational optimal solution.
Combinatorial algorithms for linear Fisher markets

- Devanur, Papadimitriou, Saberi, Vazirani ’02: polynomial time combinatorial algorithm.
- Several extensions and generalizations studied during the last decade.
- Fisher’s market with separable piecewise linear concave utilities: PPAD-complete (Vazirani & Yannakakis ’11).
## Market equilibria with rational convex programs and combinatorial algorithms

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Convex extensions of classical flow models:

- Concave generalized flows (CGF):
  - Truemper ’78, Shigeno ’06
  - We give the first combinatorial polytime algorithm.

- Minimum cost flows with separable convex objectives (MCCF):
  - Combinatorial polytime algorithms: Minoux ’86, Hochbaum&Shantikumar ’92, Karzanov&McCormick ’97
  - We give a strongly poly algorithm for certain classes of objectives.
Market equilibria with (rational) convex programs and combinatorial algorithms

| Linear Fisher | DPSV ’02/’08  
| Orlin ’10: strongly poly. | CGF  
| MCCF: strongly poly |
| Perfect price discrimination | Goel&Vazirani ’10 | CGF |
| Spending constraint utilities | Devanur&Vazirani ’04/’10 | MCCF: strongly poly |
| Arrow-Debreu Nash bargaining | Vazirani ’11 | CGF |
| Nonsymmetric ADNB | ? | CGF |

CGF: concave generalized flows V. ‘12b
MCCF: min. cost separable convex flows V. ‘12a
Generalized Flows

- Network flow model, with **gain factors** on the arcs.

- Maximize the flow amount reaching the sink $t$.

- Introduced by **Kantorovich ’39, Dantzig ’62**.

- Several applications: financial analysis, transportation, management, etc.
Currency conversion with bounds: obtain the most £ from 1000$.

\[ C = 500 \]
\[ C = 1000 \]
\[ \gamma = 0.641 \]
\[ \gamma = 0.641 \]
Generalized Flows

- Linear program.
- Early combinatorial algorithms: Onaga ’66, Truemper ’77.
- First polynomial time combinatorial algorithm: Goldberg, Plotkin, Tardos ’91.
- Followed by Cohen & Megiddo ’94, Goldfarb & Jin ’96, Goldfarb, Jin & Orlin ’97, Tardos & Wayne ’98, Wayne ’02, Radzik ’04, Restrepo & Williamson ’09, etc.
Concave Generalized Flows

Instead of gain factors, concave increasing gain functions.

\[
\alpha \rightarrow \Gamma(\cdot) \rightarrow \Gamma(\alpha)
\]
Convex Program

\[
\begin{align*}
\max & \quad \sum_{j:jt \in E} \Gamma_{jt}(f_{jt}) - \sum_{j: tj \in E} f_{tj} \\
& \quad \sum_{j: ji \in E} \Gamma_{ji}(f_{ji}) - \sum_{j: ij \in E} f_{ij} \geq b_i \quad \forall i \in V - t \\
& \quad \ell_{ij} \leq f_{ij} \leq u_{ij} \quad \forall ij \in E
\end{align*}
\]
Concave Generalized Flows

- First defined by Truemper ’78.
- Solvable via general purpose convex solver.
- Shigeno ’06 gave a combinatorial algorithm that is polynomial for some special classes of gain functions, including piecewise linear.
- We give a polynomial combinatorial algorithm for finding an \( \varepsilon \)-approximate solution in running time

\[
O(m(m + \log n) \log(MUm/\varepsilon))
\]

- For problems with a rational optimal solution, we can find it in polynomial time with a final rounding.
Eisenberg-Gale convex program, 1959

\[
\begin{align*}
\max & \quad \sum_{i \in B} m_i \log U_i \\
\text{subject to} & \quad U_i \leq \sum_{j \in G} U_{ij} x_{ij} \quad \forall i \in B \\
& \quad \sum_{i \in B} x_{ij} \leq 1 \quad \forall j \in G \\
& \quad x_{ij} \geq 0 \quad \forall i \in B, j \in G
\end{align*}
\]
Reduction for linear Fisher market

\[ \Gamma_{ji}(\alpha) = U_{ij} \alpha \]

\[ \Gamma_{ii}(\alpha) = m_i \log \alpha \]
Extensions of linear Fisher markets

- Goel, Vazirani ’10: perfect price discrimination
- (Piecewise linear) concave increasing utilities.
- Middleman between buyers and firms. He charges different costumers at different rates they are capable of paying.
- Replace $\Gamma_{ji}(\alpha) = U_{ij} \alpha$ by a piecewise linear concave function.
- Using our model, it can be replaced by arbitrary concave!
Nash bargaining, 1950

* $n$ players, set of possible outcomes $S \subseteq \mathbb{R}_+^n$

* In outcome $s = (s_1, \ldots, s_n) \in S$, player $i$ gets utility $s_i$.

* Disagreement point (status quo): $\sigma \in S$

* The players have to agree together in an outcome. If they cannot agree, the status quo remains.
Nash bargaining, 1950

Which is the best outcome?

Four criteria:

- Pareto optimality
- Invariance under affine transformations
- Symmetry
- Indifference of independent alternatives
Nash bargaining, 1950

**Theorem** (Nash, 1950)

For a convex feasible region, there exists a unique optimal solution, the one maximizing

\[ \sum_{i \in [n]} \log(s_i - \sigma_i) \]
Arrow-Debreu Nash bargaining: Vazirani ’12

* Nash bargaining between agents, each of them having an initial endowment of goods, giving utility $c_i$ to player $i$.

* Possible outcomes: distributions of goods.

$$\max_{i\in B} \sum \log(U_i - c_i)$$

$$U_i \leq \sum_{j\in G} U_{ij}x_{ij} \quad \forall i \in B$$

$$\sum_{i\in B} x_{ij} \leq 1 \quad \forall j \in G$$

$$x_{ij} \geq 0 \quad \forall i \in B, j \in G$$
Arrow-Debreu Nash bargaining: Vazirani ’12

* Vazirani ’12: sophisticated two phase algorithm, first deciding feasibility, then optimality.

\[
\begin{align*}
\max_{i \in B} & \sum \log(U_i - c_i) \\
U_i & \leq \sum_{j \in G} U_{ij} x_{ij} \quad \forall i \in B \\
\sum_{i \in B} x_{ij} & \leq 1 \quad \forall j \in G \\
x_{ij} & \geq 0 \quad \forall i \in B, j \in G
\end{align*}
\]
Reduction to Concave Generalized Flows

\[ \Gamma_{ji}(\alpha) = U_{ij} \alpha \]

\[ \Gamma_{ii}(\alpha) = m_i \log \alpha \]
Arrow-Debreu Nash bargaining: Vazirani, ’12

- Nonsymmetric Nash bargaining: Kalai ’77
- Different weights $m_i$ for player $i$.
- Finding a combinatorial algorithm was left open. Our model also captures this, solving in

$$\max \sum_{i \in B} m_i \log(U_i - c_i)$$

$$U_i \leq \sum_{j \in G} U_{ij}x_{ij} \quad \forall i \in B$$

$$\sum_{i \in B} x_{ij} \leq 1 \quad \forall j \in G$$

$$x_{ij} \geq 0 \quad \forall i \in B, j \in G$$

$O(m^2(\log C_{\max} + n \log(nU_{\max}M_{\max})))$

- Vazirani ’12 for symmetric:

$O(n^8 \log U_{\max} + n^4 \log C_{\max})$
Further applications of concave generalized flows

- Jain, Vazirani ’10: single source multiple sink flow markets.
- Jain ’11: online matching with concave utilities (offline optimum)
Linear and convex flow problems I.

**Linear**
- Minimum cost circulations
  - Edmonds & Karp ’72

**Convex**
- Minimum cost circulations w. separable convex cost
  - Minoux ’86

**Generalized flows**
- Generalized flows
  - Goldberg, Plotkin, Tardos’91
- Concave generalized flows
  - V. ‘12b
Linear minimum cost flow problem

- $G = (V, E)$ directed graph
- On each arc $ij$, lower and upper capacities $l_{ij}, u_{ij}$.
- On each node $i$, node demand $b_i$: incoming flow minus outgoing flow should be $b_i$.

Minimum cost flow problem: for a cost function $c$ on the arcs, find a minimum cost feasible flow

- First weakly polynomial algorithm: Edmonds, Karp ’72
- Strongly polynomial algorithms: Tardos ’85, Goldberg, Tarjan ’88, Orlin ’93, ...
Strongly polynomial algorithms

* Problem given by $N$ integers in the input, each at most $C$.

* (Weakly) polynomial algorithm: the running time is $\text{poly}(N, \log C)$.

* Strongly polynomial algorithm: the algorithm consists of $\text{poly}(N)$ elementary arithmetic operations, independent from $C$.

* The numbers in the operations are at most $\text{poly}(C)$.

* Alternatively, we may allow computation with real numbers, assuming we can perform basic arithmetic operations in $O(1)$ time.
Minimum cost flows with separable convex objectives

- $G=(V,E)$ directed graph
- On each arc $ij$, lower and upper capacities $l_{ij}, u_{ij}$.
- On each node $i$, node demand $b_i$: incoming flow minus outgoing flow should be $b_i$.
- We want to minimize $\sum C_{ij}(f_{ij})$ over feasible flows, where on each arc $ij$, $C_{ij}$ is a convex function.
- Convex program with several applications: traffic management, matrix balancing, stick percolation...
Minimum cost flows with separable convex objectives

* Selfish routing in urban traffic networks: transition time on a road is an increasing function of the traffic amount.
Minimum cost flows with separable convex objectives

- **Selfish routing in urban traffic networks**: transition time on a road is an increasing function of the traffic amount.

- **Nash equilibrium**: no car may find a shorter route if the others don’t change.
- Computing a Nash-equilibrium is a separable convex cost flow problem.
Linear Fisher market: convex formulations

* Eisenberg&Gale, ’59

\[ \begin{align*}
    x_{ij} & : \text{amount of good } j \text{ purchased by } i \\
    \max & \sum_{i \in B} m_i \log U_i \\
    U_i & \leq \sum_{j \in G} U_{ij} \cdot x_{ij} \quad \forall i \in B \\
    \sum_{i \in B} x_{ij} & \leq 1 \quad \forall j \in G \\
    x & \geq 0
\end{align*} \]

* Shmyrev; Devanur ’09

\[ \begin{align*}
    y_{ij} & : \text{amount of money payed by } i \text{ for } j \\
    \min & \sum_{i \in G} p_j \cdot (\log p_j - 1) - \sum_{ij \in E} y_{ij} \log U_{ij} \\
    \sum_{j \in G} y_{ij} & = m_i \quad \forall i \in B \\
    \sum_{i \in B} y_{ij} & = p_j \quad \forall j \in G \\
    y & \geq 0
\end{align*} \]

* concave generalized flow

* flow with separable convex objective
Reduction for linear Fisher market

\[ \Gamma_{ij}(\alpha) = u_{ij} \alpha \]

\[ \Gamma_{ii}(\alpha) = m_i \log \alpha \]
Reduction for linear Fisher market

\[-y_{ij} \log U_{ij}\]

\[p_j (\log p_j - 1)\]

\[-\sum m_i\]
Extensions of linear Fisher markets

- Devanur & Vazirani '04: spending constraint utilities
  - The utility of the buyers is a piecewise linear concave function of the *amount of money spent on the good*.

- Vazirani '10: combinatorial algorithm (extension of DPSV'02)

- Devanur et al. '11: discovered the convex programming relaxation.

- V. ‘12a: strongly polynomial algorithm
When is there a strongly polynomial algorithm for MCCF?

\[ G = (V, E) \text{ directed graph} \]
\[ C_{ij} : [\ell_{ij}, u_{ij}] \rightarrow \mathbb{R} \text{ convex, (differentiable)} \]
\[ \min \sum_{ij \in E} C_{ij}(f_{ij}) \]
\[ \sum_{j: ji \in E} f_{ji} - \sum_{j: ij \in E} f_{ij} = b_i \quad \forall i \in V \]
\[ \ell_{ij} \leq f_{ij} \leq u_{ij} \quad \forall ij \in E \]
Previously known cases

* Linear costs \((C_{ij}(x)=c_{ij}x)\)

* Quadratic costs \((C_{ij}(x)=c_{ij}x^2+d_{ij}x, c_{ij}\geq0)\)

  * Series parallel graphs: Tamir ’93.

  * Transportation problem with fixed number of sources: Cosares&Hochbaum ’94.

* Other nonlinear

  * Fisher’s market with linear utilities: Orlin ’10.
Negative results for strongly polynomiality

- Optimal solution can be irrational (even non-algebraic!)
- Q: is it possible to find an $\epsilon$-approximate solution in time polynomial in the input size and $\log 1/\epsilon$?
- Even this is impossible if the $C_{ij}$’s are polynomials of degree $\geq 3$ (Hochbaum ’94)
- Reason: impossible to $\epsilon$-approximate roots of polynomials in strongly polynomial time (Renegar ’87)
- This does not apply for quadratic objectives!
Our result 
(STOC 2012)

- Strongly polynomial algorithm under certain oracle assumptions on the objective.

- **Key assumption:** *we can compute an optimal solution, provided its support.*

- Special cases include:
  - Convex quadratic objectives.
  - Fisher’s market with linear and with spending constraint utilities.
Linear and convex flow problems I.

**Linear**
- Minimum cost circulations
  - Edmonds & Karp '72

**Convex**
- Minimum cost circulations w. separable convex cost
  - Minoux '86

**Generalized flows**
- Generalized flows
  - Goldberg, Plotkin, Tardos '91
- Concave generalized flows
  - V. '12b
Linear and convex flow problems I.

**Weakly polynomial**
- Minimum cost circulations
  - Edmonds & Karp ’72

**Strongly polynomial**
- Orlin ’93
- V. ’12a

**Linear**
- Minimum cost circulations
  - Edmonds & Karp ’72

**Convex**
- Minimum cost circulations w. separable convex cost
  - Minoux ’86
Main algorithmic ideas

- **Edmonds, Karp ’72:** capacity scaling algorithm:
  - Successive shortest paths method, first transporting the huge parts of the excesses.

- **Minoux ’86:** naturally extends to convex costs, with linearizing the cost in $\Delta$ chunks in the $\Delta$-phase.

\[
\frac{C_{ij}(f_{ij} + \Delta) - C_{ij}(f_{ij})}{\Delta}
\]
Main algorithmic ideas

- \textit{V'12a:} apply Minoux’s algorithm, and maintain a subset \( F \) of arcs guaranteed to be in the support of (the) optimal solution. \( F \) shall be extended in every \( O(\log n) \) iterations.

- In certain phases, we make a guess: \textit{maybe} \( F \) is already optimal? We compute an optimal solution based on the assumption that it’s support is \( F \):
  - if yes: \textit{great!}
  - if not: either it still gives a better solution than the current one: \( \Delta \) decreases radically;
  - or it gives a guarantee that \( F \) must soon be extended.
### Market equilibria with rational convex programs and combinatorial algorithms

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**CGF:** concave generalized flows V. ‘12b  
**MCCF:** min. cost separable convex flows V. ‘12a
Further questions

- Concave generalized flows: the algorithm is not strongly polynomial
- No strongly polynomial algorithm exists for **linear generalized flows**!
- Solving that could help develop strongly poly. alg. for certain concave gain functions.
- **Linear Arrow-Debreu markets**: no combinatorial algorithm known. Convex programming formulation: Nenakov&Primak ’83, Jain ’06.
Thank you for your attention!