Approximating Column-Restricted Covering IPs

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0,1 Covering Problems

Given: Elements $\{e_1,\ldots,e_m\}$ and sets $\{S_1,\ldots,S_n\}$ where set S_i has cost $c(S_i)$

Goal: Find a minimum-cost subset of all sets that covers all elements.

0,1 Covering IP's (CIP)

- **Input**: $n*m$ 0,1-matrix A of coefficients, vector b of demands, cost vector c
- **Goal**: Find 0,1-vector x that minimizes $c^T x$ s.t.

Elements

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} \geq \mathbb{1}$$

Sets

Approximating 0,1-CIPs

Problem is rather well understood in terms of approximability ...

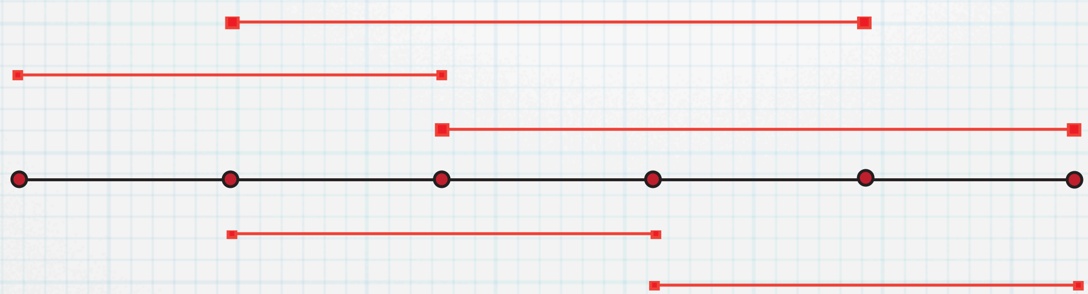
[Chvatal '79] O(log n) Greedy Apx

[Feige '98] (1-o(1)) ln(n) Hardness

But: Can often do better, exploiting problem-specific structure.

Example: Line Cover

- Elements are edges of a line; sets are line segments.



- Line segments have cost. Edge may have demand.
  Goal: find mincost set of segments that satisfies all demands.
- Constraint matrix is TU → Solvable in polytime.

Some more Examples...

[Column-sparse matrices] (<= α non-zeros per column) → $O(1+\log \alpha)$-apx
[Srinivasan '99], [Kollipoulos & Young '05]

[Row-sparse matrices] (<= β non-zeros per row) → β-apx
[Pritchard & Chakrabarty '09], ...

0,1-CIP - IP, LP & Gap

$$\mathrm{opt}_{A,b,c} := \min\{c^T x \;:\; Ax \geq b,\ 0 \leq x \leq \mathbb{1},\ x \text{ integer}\}$$

LP Relaxation ↓

$$\mathrm{lp}_{A,b,c} := \min\{c^T x \;:\; Ax \geq b,\ 0 \leq x \leq \mathbb{1}\}$$

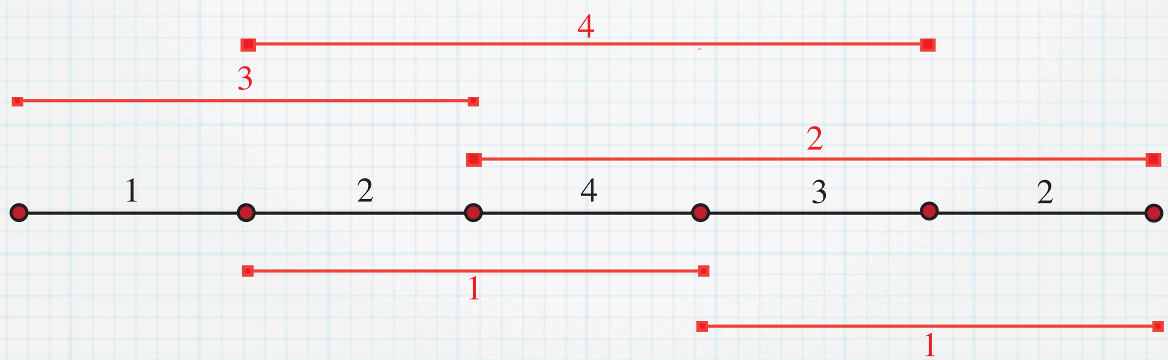
LP relaxation provides bound on optimum solution value. Quality depends on integrality gap:

$$\alpha(A) := \sup_{b,c} \frac{\mathrm{opt}_{A,b,c}}{\mathrm{lp}_{A,b,c}}$$

This Talk

- Significantly less is known for structured general CIPs than for structured 0,1-CIPs
- Present two generalizations of 0,1-CIPs
  - Column Restricted Covering Problems (CCIP)
    [natural model of capacitaties]
  - Priority Covering Problems (PCIP)
    [arise when modeling service requirements]
- Study how generality affects approximability, and integrality gap.

Column Restricted Line Cover



<u>More general version of line cover</u>

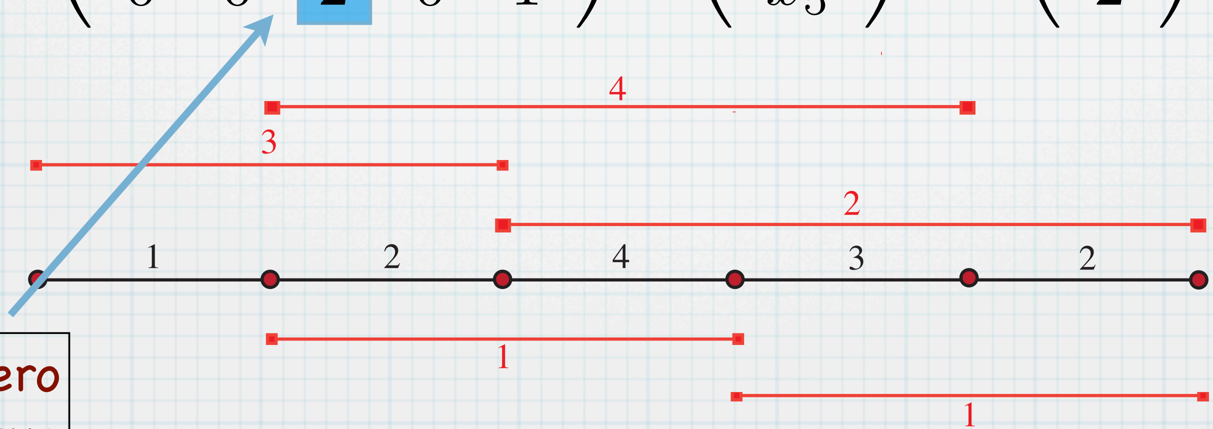
Every edge e has a demand b_e, and every segment i has a supply s_i

Goal: find mincost set of segments such that each edge receives total supply >= demand. [allow each segment at most once]

Column Restricted Line Cover

Want to find mincost integral $x \leq \mathbb{1}$ s.t.

$$\begin{pmatrix} 0 & 3 & 0 & 0 & 0 \\ 4 & 3 & 0 & 1 & 0 \\ 4 & 0 & 2 & 1 & 0 \\ 4 & 0 & 2 & 0 & 1 \\ 0 & 0 & 2 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} \geq \begin{pmatrix} 1 \\ 2 \\ 4 \\ 3 \\ 2 \end{pmatrix}$$



All non-zero entries have same value!

Column Restricted CIPs

Arises from 0,1-CIP by

- Giving each set i a supply s_i, and
- Giving each element e a demand b_e

New coefficient matrix:

$$\underbrace{\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \end{pmatrix}}_{A} \implies \underbrace{\begin{pmatrix} 0 & s_2 & 0 & 0 & 0 \\ s_1 & s_2 & 0 & s_4 & 0 \\ s_1 & 0 & s_3 & s_4 & 0 \\ s_1 & 0 & s_3 & 0 & s_5 \\ 0 & 0 & s_3 & 0 & s_5 \end{pmatrix}}_{A[s]}$$

Column Restricted Covering

In CCIP defined by (A,b,c), we want to find an integer solution to

$$
\begin{aligned}
\min \quad & c^T x \\
\text{s.t.} \quad & A[s] \cdot x \geq b \\
& 0 \leq x \leq \mathbb{1}
\end{aligned}
$$

In general: any d >= 0

Column Restricted Covering

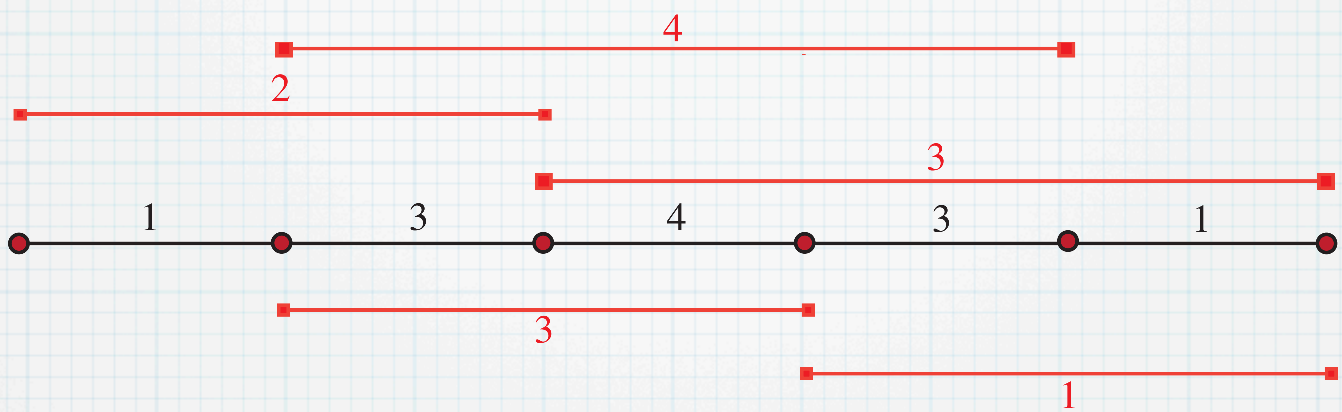
- Q: How do approximability of CCIP and corresponding 0,1-CIP relate?
- Q: How do integrality gap of CCIP and 0,1-CIP relate?

- Can it be that

$$\gamma(A) := \sup_{b,c,s} \frac{\mathrm{opt}_{A[s],b,c}}{\mathrm{lp}_{A[s],b,c}} \gg \alpha(A) := \sup_{b,c} \frac{\mathrm{opt}_{A,b,c}}{\mathrm{lp}_{A,b,c}}$$

integral

Priority Covering Problems

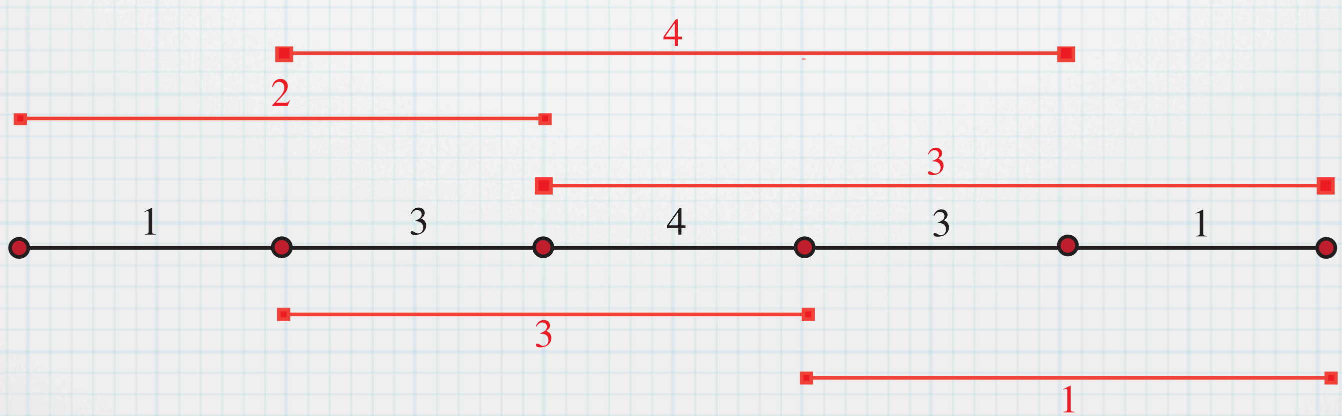


- Each edge has a priority demand b_e
  think: quality of service requirement
- Each segment has priority supply s_j
- A segment j covers an edge e if
  a) e is contained in j, and b) $s_j >= b_e$
- Goal: Find mincost feasible cover.

Priority Line Cover

Formally: find mincost integral $x \leq \mathbb{1}$ s.t.

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} \geq \mathbb{1}$$



Priority Covering Problems

- Given 0,1-CIP with constraint matrix A, demands b_i for every row and supplies s_j for every column, one defines the matrix A[b,s] as

$$A[b,s]_{ij} = \begin{cases} A_{ij} & : \quad \text{if } s_j \geq b_i \\ 0 & : \quad \text{otherwise} \end{cases}$$

- PCIP is then given by

$$\min \sum_{j=1}^{n} c_j x_j \quad : \quad A[b,s] \cdot x \geq \mathbb{1}, \; x_j \in \{0,1\}, \forall j$$

Priority Covering Problems

- How is the approximability of a PCIP related to that of the original 0,1-CIP?
- How is the integrality gap of the canonical relaxation of a PCIP related to that of the original (0,1)-CIP?

$$\beta(A) := \sup_{b,c,s} \frac{\mathrm{opt}_{A[b,s],c}}{\mathrm{lp}_{A[b,s],c}} \overset{?}{\gg} \alpha(A) := \sup_{b,c} \frac{\mathrm{opt}_{A,b,c}}{\mathrm{lp}_{A,b,c}}$$

Related Work: Packing LP

- Column Restricted Packing Integer Programs

$$\max \sum_{j=1}^{n} c_j x_j \quad : \quad A[s]x \leq b, \ 0 \leq x \leq \mathbb{1}$$

- No bottleneck assumption:

  s_j <= b_i for all i and j.

- With this, assumption, can show $\gamma = O(\alpha)$ [Kollipoulos & Stein '98], [Chekuri, Mydlarz, Shepherd '03]

- Without no bottleneck: much worse!

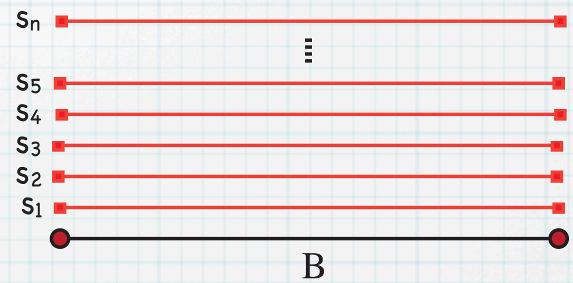
Back to Column-Restricted Cover

The CCIP maybe harder than underlying 0,1-CIP.

Example: Column-restricted Line Cover is NP-hard.

It encodes Knapsack.

- n elements with weights s_i and cost c_i
- Knapsack of capacity B
- Find minimum cost selection of elements of weight >= B.



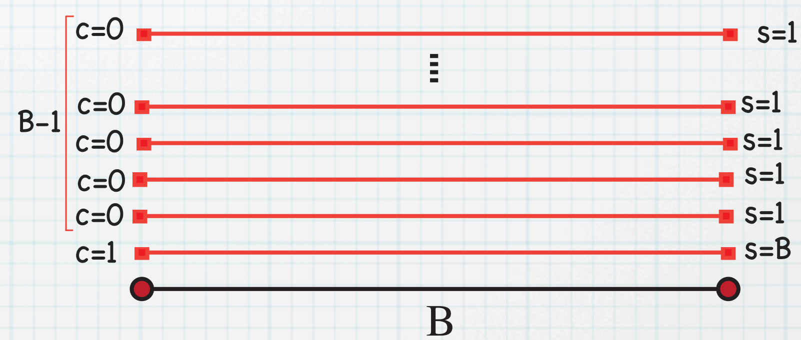
Column-Restricted Line Cover

- And indeed: using such examples, one can show that $\gamma >> \alpha$ is possible!

$$\min \quad \sum_{j=1}^{n} c_j x_j$$

$$\text{s.t.} \quad \sum_{j=1}^{n} s_j x_j \geq B$$

$$0 \leq x \leq \mathbb{1}$$



- opt=1 vs lp=1/B

Strengthening the LP

- X be any subset of segments (items)
- s(X) be the sum of supplies of segments in X
- The following must be satisfied by any integral solution.

$$\sum_{j \notin X} s_j x_j \geq B - s(X)$$

- Using integrality, we can strengthen this:

$$\sum_{j \notin X} s_j^X x_j \geq B - s(X)$$

$$s_j^X = \min(s_j, B - s(X))$$

Strengthening the LP

- Knapsack-Cover (KC) Inequality:

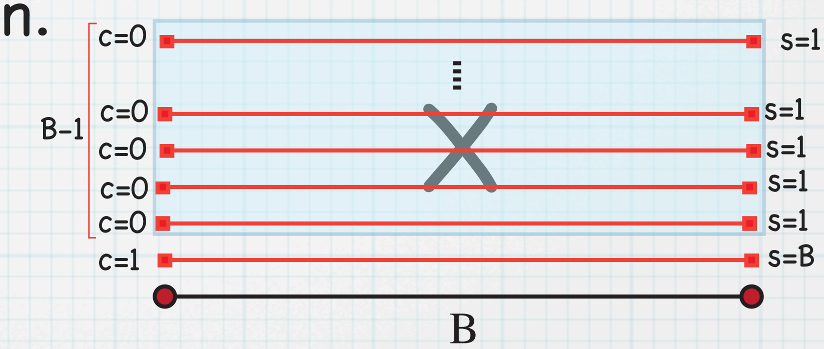
$$\sum_{j \notin X} s_j^X x_j \geq B - s(X)$$

$$s_j^X = \min(s_j, B - s(X))$$

- Let's consider example again.
  - X = all 0-cost segments
  - old LP solution doesn't satisfy KC-inequality for X:

    B - s(X) = 1, s^X_1 = 1

    → $x_1 = 1/B \not\geq 1$



Strengthening the LP

Stronger LP for Minimum Knapsack

$$\min \quad \sum_{j=1}^{n} c_j x_j$$

$$\forall X \subseteq [n] : \sum_{j \notin X} s_j^X x_j \geq B - s(X)$$

$$1 \geq x_j \geq 0$$

where $s_j^X = \min(s_j, B - s(X))$and

$$s(X) = \sum_{j \in X} s_j$$

Strengthening the LP

Applicable to any CIP giving the following stronger LP,

$$
\begin{aligned}
\min \quad & \sum_{j=1}^{n} c(S_j)x_j \\
\text{s.t.} \quad & \sum_{j \notin X} A_{ij}^X x_j \geq b_i - s_i(X) \quad \forall i \in [m], X \subseteq [n] \\
& 0 \leq x \leq \mathbb{1}
\end{aligned}
$$

LP'(A[s],b,c)

where $s_i(X) := \sum_{j \in X} A_{ij}$

and $A_{ij}^X = \min(A_{ij}, b_i - s_i(X))$

Strengthening the LP

- Integrality Gap of the stronger relaxation

$$\gamma(A) := \sup_{b,c,s} \frac{\text{opt}_{A[s],b,c}}{\text{lp'}_{A[s],b,c}}$$

- What is the relation between γ(A) and α(A)?

Connection

Theorem: Given a (0,1)-CIP with incidence matrix A

α, β, γ : integrality gaps of the canonical LP relax of the (0,1)-CIP, PCIP and the strengthened LP relaxation CCIP

→ $\gamma = O(\alpha + \beta)$

Remark: α, β are integrality gaps over all integer costs, demands and supplies.

Proof Idea

$$
\begin{aligned}
\min \quad & \sum_{j=1}^{n} c(S_j) x_j \\
\text{s.t.} \quad & \sum_{j \notin X} A_{ij}^{X} x_j \geq b_i - s_i(X) \quad \forall i \in [m], X \subseteq [n] \\
& 0 \leq x \leq \mathbb{1}
\end{aligned}
$$

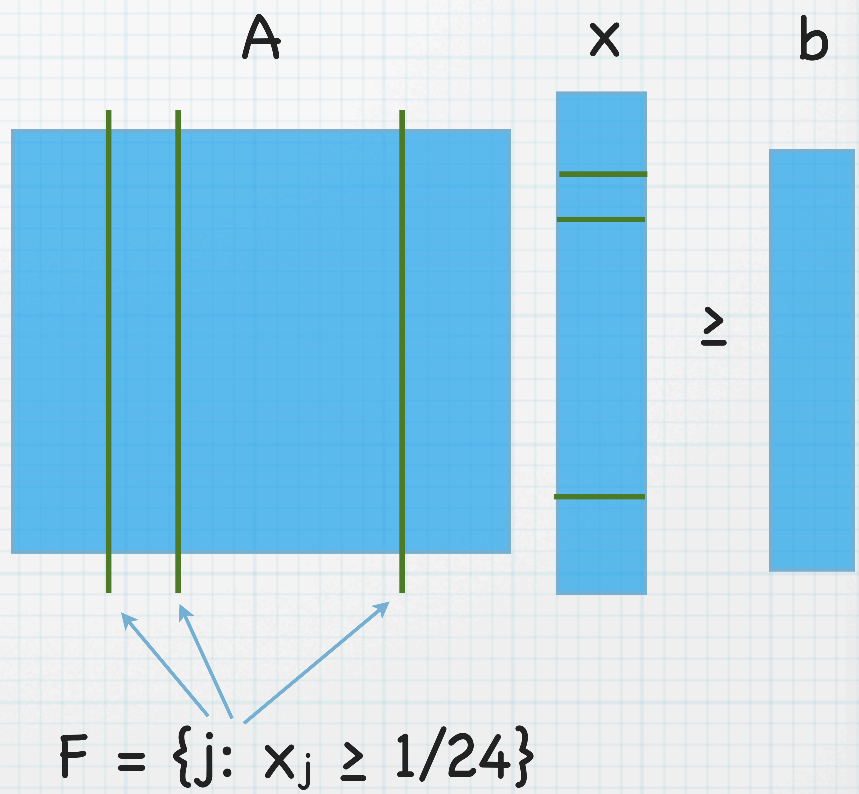
Step 0: Solve the LP to get solution x.

Small problem: don't know how to do it!

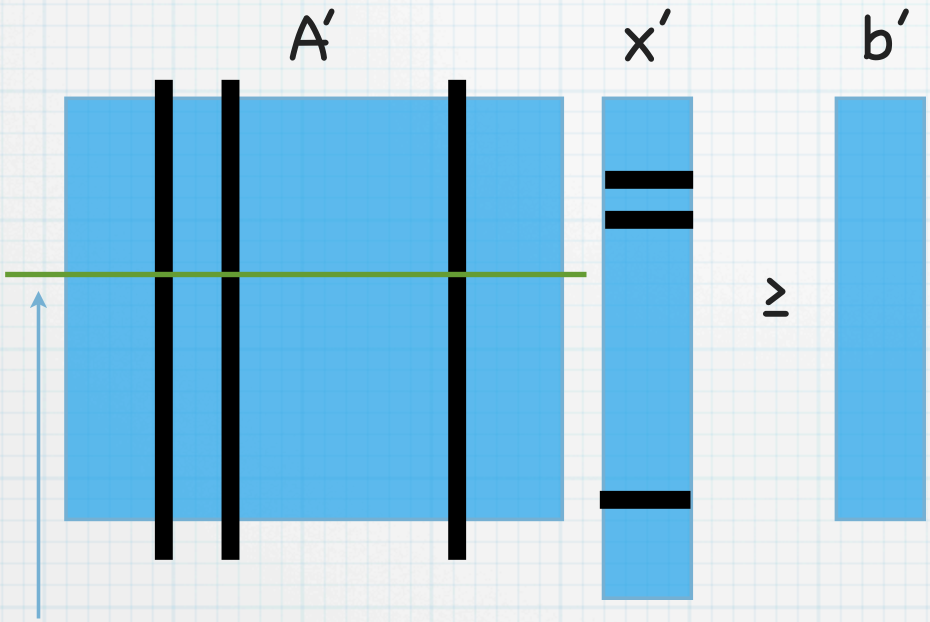
Proof Idea

Step 1: Filtering

- Pick set F of columns with large fractional value. [round up immediately]
- b'_i : residual demand of row i. Done if $b'_i = 0$, for all i.
- Construct matrix A′ by
  - taking all columns **not** in F
  - rows corresponding to Knapsack Cover inequalities for set F.



Proof Idea – Filtering



Observe that

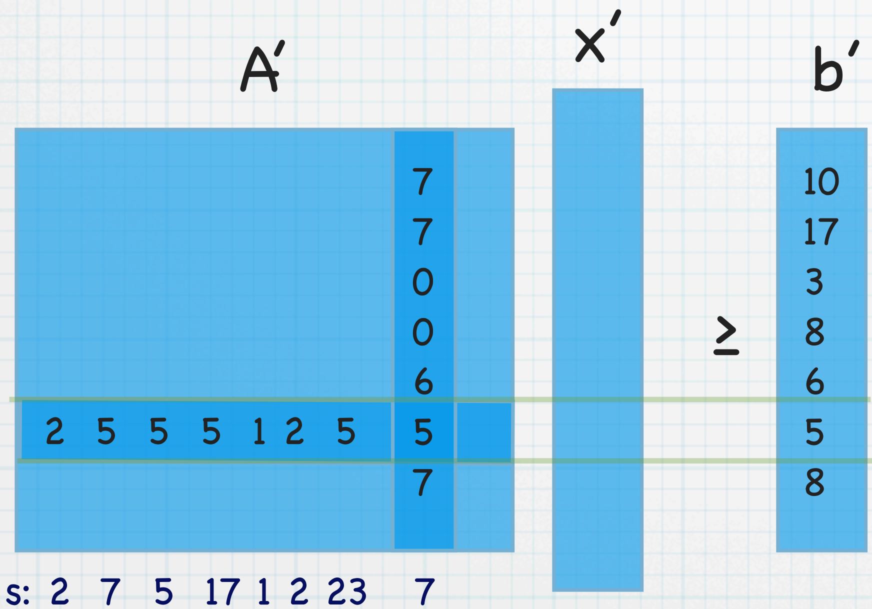
- b'_i is >= each entry in row i
- A′ is not column-restricted
- x'_j < 1/24 for all j

KC-inequality:

$$\sum_{j \notin F} A^F_{ij} x_j \geq b'_i$$

$$\left(A^F_{ij} = \min\{A_{ij}, b'_i\}\right)$$

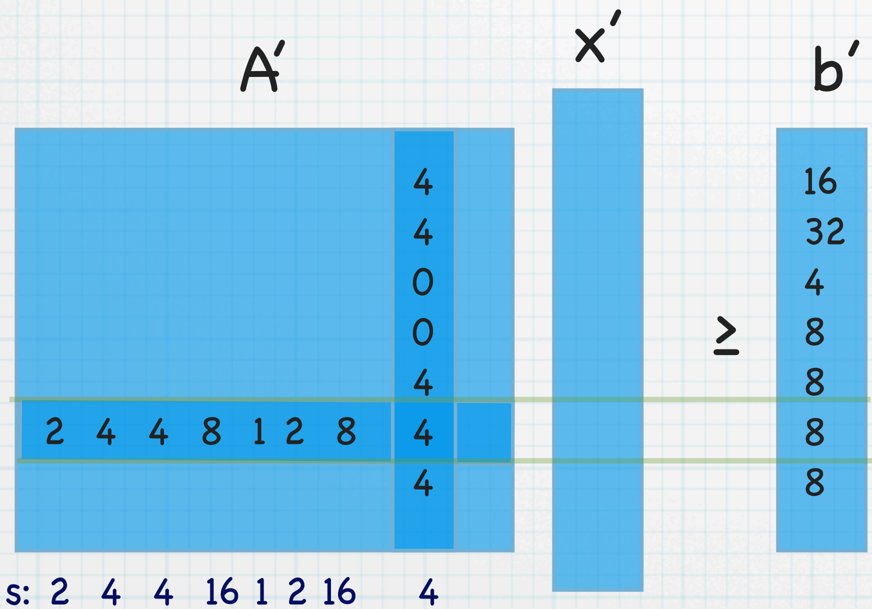
Proof Idea - Beautify Data



Step 2 : Powers of 2

- Increase all b′ entries to nearest power of 2
- Decrease all s_j's to nearest power of 2

Proof Idea – Beautify Data

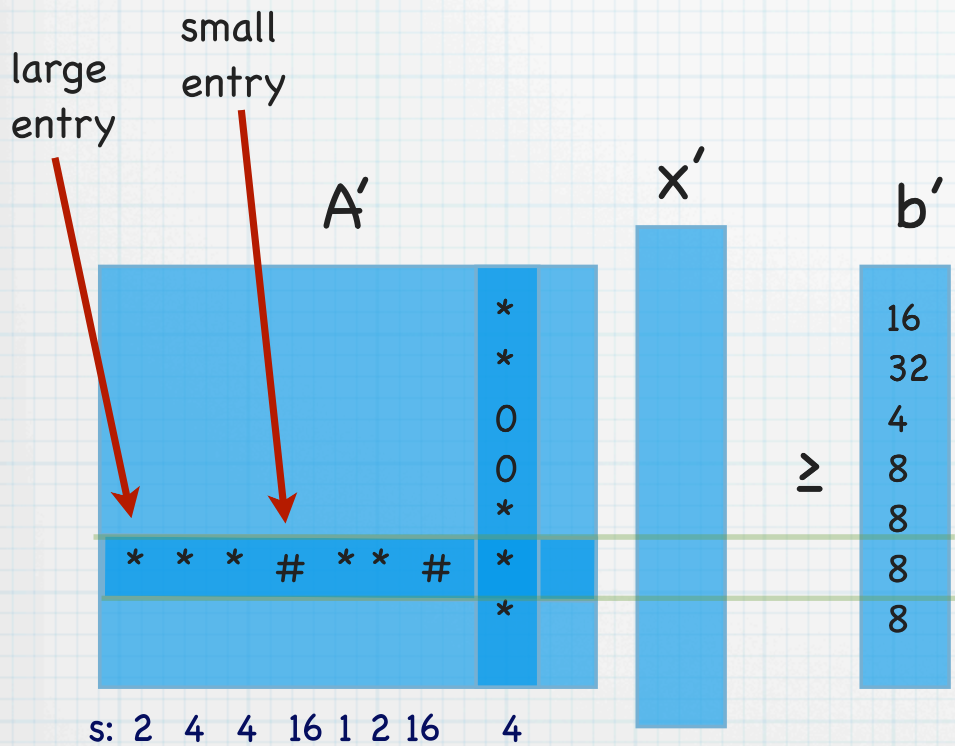


Step 2 : Powers of 2

- Increase all b′ entries to nearest power of 2
- Decrease all s_j's to nearest power of 2

Note: y=4x is valid solution.

Proof Idea – Row Partition



Step 3 : Partition Rows

* original value

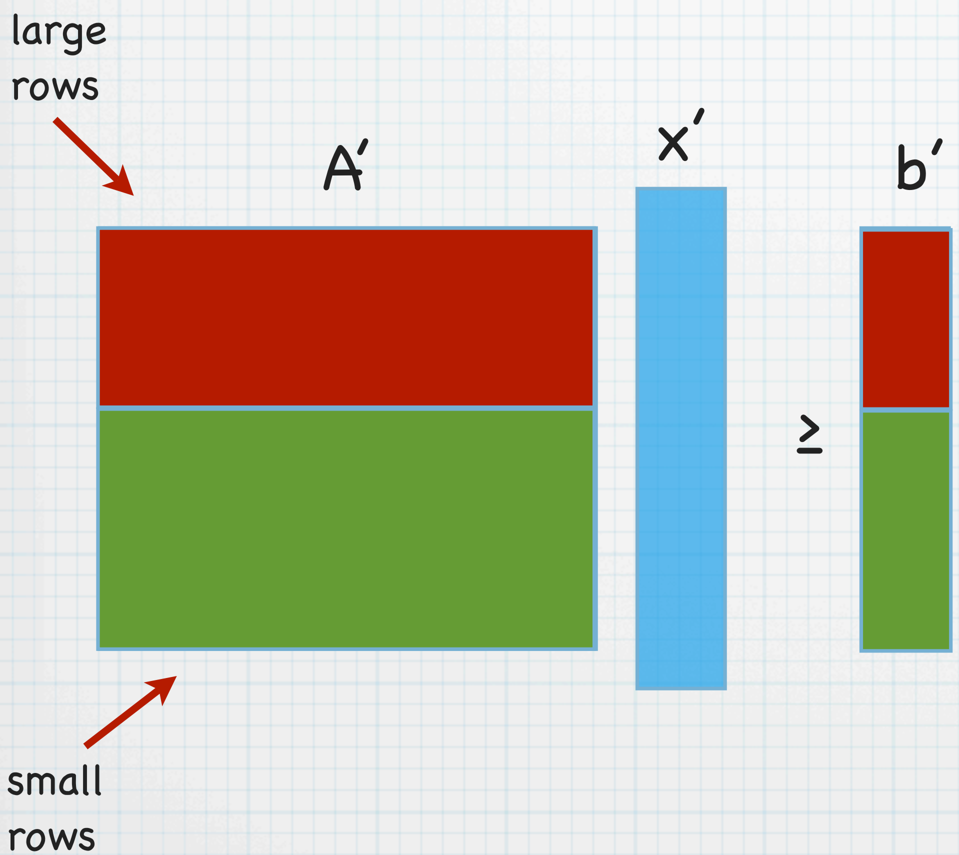
\# b′ value

A row i is

[large] if most of fractional coverage comes from *'s

[small] otherwise

Proof Idea – Row Partition



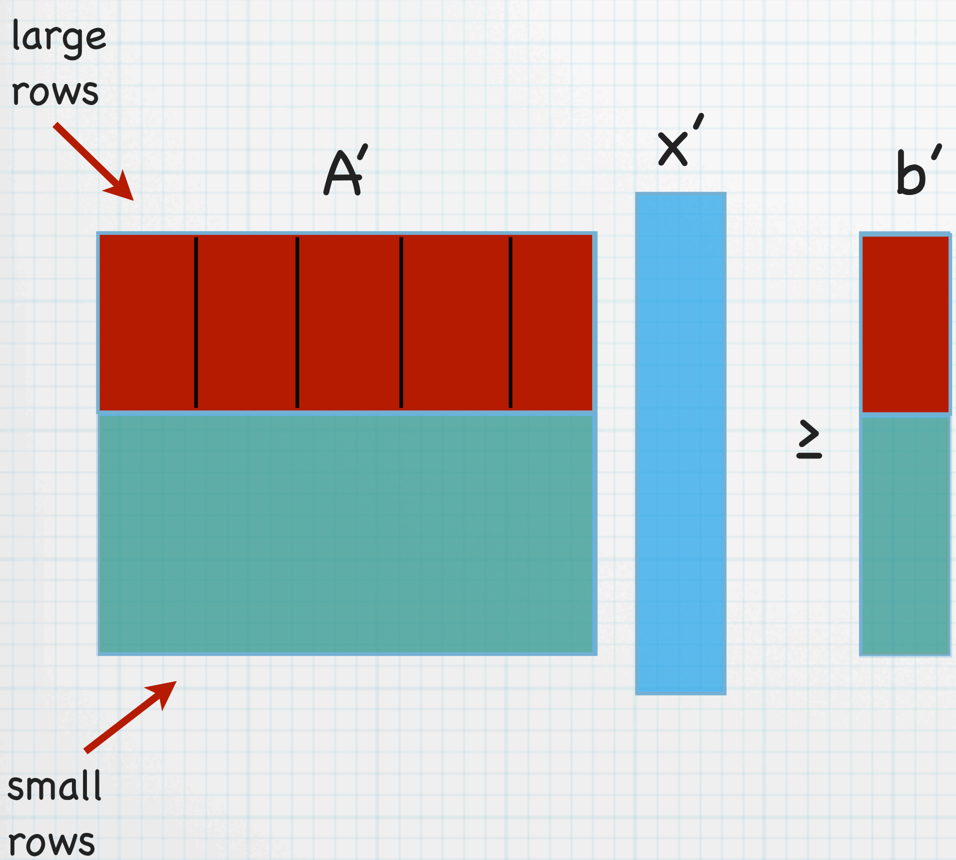
Step 3 : Partition Rows

Now look at large and small rows separately.

Compute two solutions, one that is feasible for large rows, and one for small!

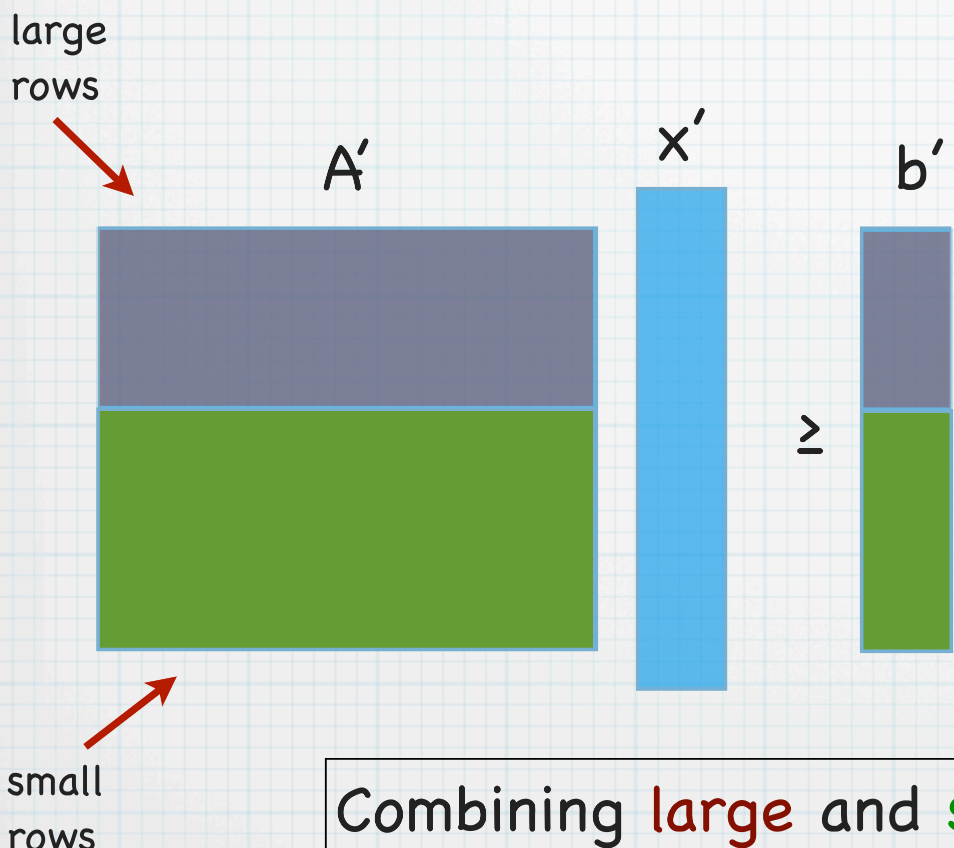
Combine in the end.

Proof Idea - Large Rows (4a)



1. 0-out all small entries
2. Grouping & Scaling [Kollipoulos & Stein]
   i) Group columns by supply into powers of 2
   ii) For each part, divide by the power to get 0,1-CIP → bounded integrality gap!
   iii) Find α-apx for each group

Proof Idea – Small Rows (4b)



1. 0-out all large entries
2. Resulting 0,1-CIP is a PCIP!
3. Has integrality gap β by assumption → find β-apx

Combining large and small solutions gives O(α+β)-approximation as wanted.

CCIPs and PCIPs

Have seen: $O(\alpha+\beta)$-apx

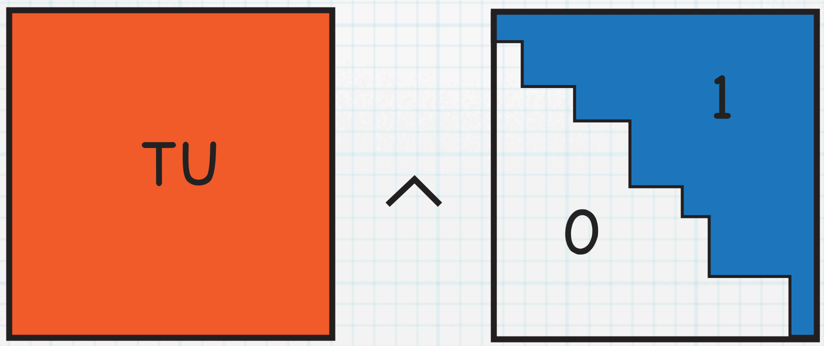
<u>Conjecture:</u> There is a $O(\alpha)$-apx for CCIP.
[e.g., column-restrictedness does not add much]

One way to show this:

Show that PCIPs have $O(\alpha)$-apx.;
i.e., adding priorities does not hurt.

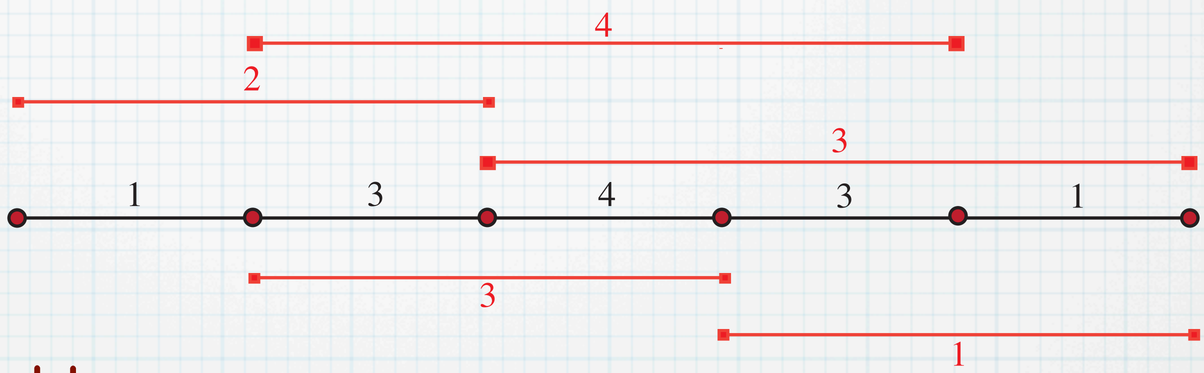
CCIPs and PCIPs

Concrete Q: What is the integrality gap of IP with coefficient matrix



[PCIP of a TU System in Disguise]

Specific Priority CIPs



Line Cover Problem

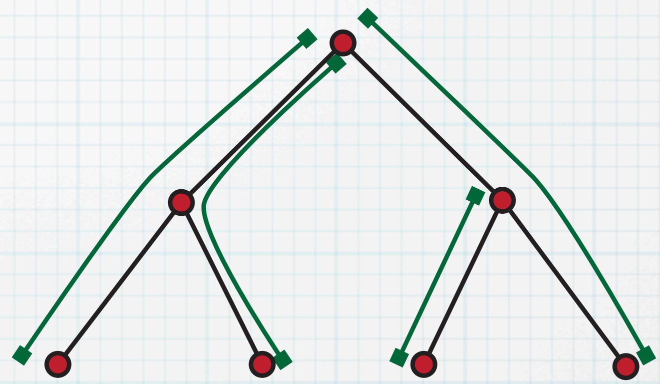
- 0,1-CIP matrix is TU
- PCIP has integrality gap >= 3/2
- PCIP has integrality gap <= 2
- CCIP for Line Cover has O(1) integrality gap.

Priority Line Cover is solvable in polynomial time. Is there an exact LP relaxation?

Specific Priority CIPs

Rooted Tree Cover problem

- Edges are edges of a rooted tree
- Segments are rooted paths in tree
- Coefficient matrix is TU



Our Results:

- PCIP has integrality gap at least e/(e-1)
- Unweighted PCIP has gap at most 6
- Unweighted CCIP has gap O(1)

Priority Tree Cover is APX-hard, and 2-approximable.

Conclusions

- Two generalizations of (0,1)-CIPs: CCIPs & PCIPs
- CCIP-approximability connected to integrality gap of underlying PCIP and 0,1-CIP.
  Unclear: PCIP ↔ 0,1-CIP Relationship
- PCIP: interesting to study in their own right
- Connections to geometric problems

Thank You!

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