A Polynomial Time Algorithm for finding Nash Equilibria in Planar Win-Lose Games

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Abstract. Two-player win-lose games have a simple directed graph representation. Exploiting this, we develop graph theoretic techniques for finding Nash equilibria in such games. In particular, we give a polynomial time algorithm for finding a Nash equilibrium in a two-player win-lose game whose graph representation is planar.

1. Introduction

A win-lose game is a game in which the payoff to every player is either zero or one. In this paper we consider two-player win-lose games. Here payoffs are given by two $m \times n$ zero-one matrices A and B for players I and II, respectively. If player I plays the pure strategy row r_i, $1 \leq i \leq m$, and player II plays the pure strategy column c_j, $1 \leq j \leq n$, then player I receives the payoff a_{ij} and player II receives b_{ij}.

Our interest in two-player win-lose games is motivated by recent ground-breaking work regarding the complexity of finding Nash equilibria[1]. Specifically, Daskalakis, Goldberg and Papadimitriou [11], [8] instigated a series of papers investigating the hardness of finding equilibria in k-player normal-form games. The cumulation of this work was the result of Chen and Deng [6] showing that the Nash equilibrium problem in two-player games is PPAD-complete. Furthermore, Abbott, Kane, and Valiant [1] showed that finding a Nash equilibrium in a two-player win-lose game is as hard as in general games. In fact, a recent result of Chen, Teng and Valiant [7] shows that even approximating Nash equilibria to a logarithmic number of bits is hard. Codenotti and Stefankovic [4] show that determining whether there are at least two Nash equilibria is NP-complete for win-lose games.

So solving win-lose games is hard even in the two-player case. This immediately leads to the following question: are there win-lose games for which polynomial time algorithms exist for finding a Nash equilibrium? In particular, what structural properties of a win-lose game are sufficient to guarantee the existence of a polynomial time algorithm?

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[1] Briefly: a mixed strategy for a player is a probability distribution over the set of pure strategies for that player. A pure strategy is a best response to an opponent's mixed strategy if it maximises the player's expected payoff. A pair of mixed strategies is a Nash equilibrium if each pure strategy that is played with non-zero probability is a best response to the opponent's mixed strategy.

Interestingly, the structural properties that we are interested in may be viewed as graph theoretic properties. To see this, observe that there is a very simple bipartite digraph representation[2] of a two-player win-lose game $\mathcal{G}$. We have one vertex for each pure strategy; that is, our digraph G has one vertex for each row r_i and one vertex for each column c_j. We have an arc (r_i, c_j) if the entry $a_{ij} = 1$; observe that in this case the pure strategy r_i is a best response for player I to the pure strategy c_j. Similarly, we have an arc (c_j, r_i) if the entry $b_{ij} = 1$.

Given this digraph representation, in order to design good algorithms for finding equilibria, we need to answer the following questions:

(i) What combinatorial structures correspond to Nash equilibria?

(ii) Can we search for these combinatorial structures in an efficient manner?

This paper, therefore, considers these two questions. Firstly, we present a range of combinatorial structures that produce Nash equilibria. For example, one of the main structures we look for relates to induced cycles (see Section 2.2 for details). With regards to the second question, we show how to efficiently find a Nash equilibria if the graph representation is planar. The key to this is a proof that one of the desired combinatorial structures must arise in a planar graph. A polynomial time algorithm then follows by applying basic network connectivity algorithms and standard planar graph drawing techniques.

We remark that the restriction to planar graphs is a strong one. In particular, it is certainly not clear that planarity is a common property amongst games. However, we believe our approach is useful for three reasons. Firstly, it is crucial to try to understand what games can be solved in polynomial time. There are very few classes of games with polytime algorithms and so obtaining non-trivial examples is an important task. Secondly, interpreting equilibria combinatorially is of interest in its own right; moreover, this combinatorial viewpoint could also have wider application. Thirdly, our basic approach will actually work on most graphs. Specifically, for the approach to fail, a graph must not exhibit a relevant combinatorial structure *anywhere* within it. Typically a graph will have such structures; for example, a random graph will have one of the desired structures with high probability.

1.1. **Related Work.** Independently of this work, Codenotti, Leoncini and Resta [3] also consider a restricted subset of win-lose games. They show that in the case where the number of winning entries in each row and column of both payoff matrices is at most two, a Nash can be found in polynomial (in fact linear) time. Their approach is similar in flavour to ours; they also use structural results on a digraph computed from the game, and rely on finding certain structures that correspond to Nash

[2]This representation differs from the more complex strategy profile graph.

equilibria. In particular, their definition of a "bump-free cycle" corresponds almost exactly to undominated cycles in our paper (the only difference being that the arrows in their representation have the opposite orientation to ours). However, the techniques they use rely on the digraph having maximal indegree and outdegree at most two, and do not carry over to the class of planar graphs. In addition, the structures our algorithm searches for (namely, *induced* undominated cycles) need not exist in their case, and so their algorithm has to search for a broader class of structures.

2. Basic Operations and Combinatorial Structures

Take the digraph representation G, with vertex bipartition $R \cup C$, of our game $\mathcal{G}$.

2.1. A Reduction to Strongly Connected Digraphs.

In this section we show that we can reduce our problem to searching for Nash equilibria in strongly connected digraphs (this observation was also made in [3]). We begin by presenting some results which give very simple structures corresponding to Nash equilibria.

Claim 1. *A pair of vertices $r_i \in R, c_j \in C$ satisfying $\delta^-(\{r_i\}) = \emptyset$ and $\delta^-(\{c_j\}) = \emptyset$ forms a pure strategy Nash equilibrium.*

Proof. If $\delta^-(\{r_i\}) = \emptyset$ then $b_{ij} = 0$ for all strategies c_j. Thus, every column is a best response for player II to row r_i. Similarly, if $\delta^-(\{c_j\}) = \emptyset$ then every row is a best response for player I to column c_j. Therefore, the pair r_i and c_j is a pure strategy Nash equilibrium. □

Claim 2. *A pair of vertices u, v with $(u, v) \in A$ and $\delta^-(\{u\}) = \emptyset$ forms a pure strategy Nash equilibrium.*

Proof. We may assume that $u = r_i$ and $v = c_j$. Since (r_i, c_j) is an arc we see that r_i is a best response for player I against row c_j. Every column is a best response to row r_i, so the pair forms a pure strategy Nash equilibrium. □

Clearly, in the first stage of any algorithm, we can efficiently search in linear time for the equilibria described in Claims 1 and 2. Henceforth, we may assume that our digraph G contains no pair of vertices with the corresponding properties. Consequently, G must consist of weakly connected components (components whose underlying undirected graph is connected) plus singleton vertices all on the same side of the bipartition.

We denote by $\Gamma^+(v)$ the outneighbours of a vertex v, and by $\Gamma^+_S(v)$ the outneighbours that are elements of $S \subseteq V$. In a win-lose game, a vertex (pure strategy) v is *weakly dominated* if there is another vertex u for which $\Gamma^+(v) \subset \Gamma^+(u)$. Thus, we also have the following simple claim.

Claim 3. *A vertex v with $\delta^+(\{v\}) = \emptyset$ is weakly dominated.* $\square$

Observe that G contains at least one non-singleton weakly connected component, otherwise we have a Nash equilibria by Claim 1. In addition, there is at least one vertex in G that weakly dominates every isolated vertex otherwise we have a Nash equilibria by Claim 2. Since at least one Nash equilibrium survives if we iteratively delete weakly dominated strategies, we may discard isolated vertices and then look for a Nash equilibrium in the resultant graph.

Now let $G[S]$ be the subgraph induced by $S \subseteq V$. Clearly this corresponds to a two-player win-lose game whose pure strategies are the elements of S. Making the assumptions on G noted above (namely that G has no Nash equilibria of the form considered in Claims 1 and 2, and has no isolated vertices), we obtain

Lemma 2.1. *If $S \subseteq V$ is a weakly connected induced subgraph of G with $\delta^-(S) = \emptyset$ then a Nash equilibrium in $G[S]$ is a Nash equilibrium in G.*

Proof. We first note that S must contain both row and column vertices. For suppose not; then since S is weakly connected, it must be a singleton. But this means we must have either an isolated vertex, or a Nash of the form considered in Claim 2, a contradiction.

So take a Nash equilibrium in $G[S]$ consisting of a probability distribution p on the rows of S and a probability distribution q on the columns of S. Extend these to distributions p' and q' on the row and columns in V in the obvious way; that is, $p'_{r_i} = p_{r_i}$ if $r_i \in S$ and $p'_{r_i} = 0$ if $r_i \notin S$ (define q' in a similar fashion). Then p' and q' form a Nash equilibrium in G. This follows simply from the observation that any pure strategy $c_j \notin S$ (respectively $r_i \notin S$) has zero expected payoff if player I (respectively player II) used the mixed strategy p' (respectively q') since $\delta^-(S) = \emptyset$. $\square$

Now if G is not strongly connected, we must be able to find a weakly connected $S \subseteq V$ with $\delta^-(S) = \emptyset$, and so we need consider only $G[S]$ by the above lemma. So we may assume that G is strongly connected. Implementing this phase of the algorithm to reduce the problem size is also easy. We just need to find the strongly connected components of G; this can be done in linear time ([13], [5]).

2.2. **Nash equilibria and Induced Cycles.** So far we have seen some very simple graph structures that correspond to Nash equilibria. Here we will see one more. We use the following notation. We say that a bipartite digraph is (α, β)-outregular if each vertex $r_i \in R$ has outdegree exactly α, and each vertex $c_j \in C$ has outdegree exactly β. Then

Lemma 2.2. *Let $S \subseteq V$ induce an (α, β)-outregular graph. Suppose $|\Gamma_S^+(r_i)| \leq \alpha$ for all $r_i \notin S$ and $|\Gamma_S^+(c_j)| \leq \beta$ for all $c_j \notin S$. Then the uniform distributions on $S \cap R$ and $S \cap C$ give a Nash equilibrium.*

Proof. Clearly if player I uses the uniform distribution p on $S \cap R$ then each pure strategy in $S \cap C$ gives player II an expected payoff of $\frac{\beta}{|S \cap R|}$. However, any pure strategy c_j not in $S \cap C$ gives player II an expected payoff of at most $\frac{\beta}{|S \cap R|}$. Thus the uniform distribution q on $S \cap C$ is a best response to p. Similarly, p is a best response to q. □

We will call a cycle C *dominated* if we can find a vertex v *not on the cycle* such that there are at least two arcs originating from v and terminating on C. So, for the simplest case $\alpha = \beta = 1$, Lemma 2.2 gives the following corollary.

Corollary 2.3. *Let $G[S]$ be an induced cycle. If the cycle is not dominated by any vertex $v \notin S$ then S corresponds to a Nash equilibrium.* □

We remark that if an induced cycle $G[S]$ *is* dominated then the pair of uniform distributions on the respective bipartitions of the cycle will *not* produce a Nash equilibrium. So our goal is to find an *undominated* induced cycle. In addition, observe that a digon, by bipartiteness, cannot be dominated. Thus, if $|S| = 2$ then the induced cycle corresponds to a pure strategy Nash equilibria. Searching for undominated induced cycles will be an important tool in finding Nash equilibria in planar digraphs.

3. Planar Graphs

We now present the polynomial time algorithm for win-lose games in which the auxiliary graph is planar. Our proof relies on a structural result regarding induced cycles in planar graphs. Specifically we will show that any strongly connected, planar, bipartite graph contains an undominated induced cycle.

We will call a graph *non-trivial* if it has at least one edge.

Theorem 3.1. *Any non-trivial, strongly connected, bipartite, planar graph has an undominated induced cycle.*

Before proving Theorem 3.1, let us see that its conditions cannot be relaxed. The planarity assumption is necessary as non-planar counterexamples exists. Figure 1, an orientation of $K_{3,3}$, is such an example.

Bipartiteness is also necessary. See Figure 2 for a planar, nonbipartite counter-example. This is irrelevant from the perspective of games as non-bipartite graphs have no clean game theoretic interpretation. It is perhaps interesting, though, that the existence of this graph theoretic structure does rely upon this basic game theoretic assumption.

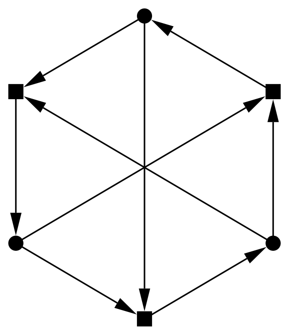

FIGURE 1. A non-planar counterexample.

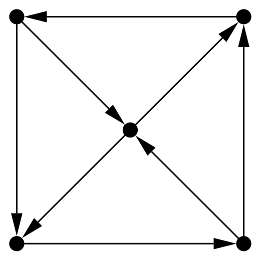

FIGURE 2. A non-bipartite counterexample.

Let's move towards the proof of Theorem 3.1. We begin with the following lemma.

Lemma 3.2. *Any planar embedding of a non-trivial strongly connected graph G has at least two facial cycles.*

Proof. Consider the (directed) planar dual G^*. Observe that a directed cut in G corresponds to a cycle in G^*. Therefore, as G is strongly connected, G^* must be acyclic. So G^* has at least one source and at least one sink. Moreover, a source or sink in G^* corresponds to a facial cycle in G. Thus, there are at least two facial cycles. □

For the next lemma we use the following notation. Given a cycle C in G, let the removal of C partition the plane into two regions R_1 and R_2. Let G_i, $i = 1, 2$, be the graph consisting of vertices and arcs in the closure of R_i, denoted by $\mathrm{cl}(R_i)$.

Lemma 3.3. *The graphs G_1 and G_2 are both strongly connected.*

Proof. Take any pair of vertices u, v in G_1. There is a path from P_{uv} from u to v in G. This may use vertices in $G - G_1$. Let $a_1 = (c_1, x_1)$ and $a_2 = (x_2, c_2)$ be, respectively, the first and last arcs of P_{uv} that are in R_2. Then c_1 and c_2 are on C, so we may replace the sub-path from a_1 to a_2 using the path from c_1 to c_2 in C. Thus G_1 is strongly connected. Similarly, G_2 is strongly connected. □

Theorem 3.4. *Any non-trivial, strongly connected, bipartite, planar graph contains an undominated facial cycle.*

Proof. Let G be a minimal counterexample. By Lemma 3.2, G contains a facial cycle C. It must be dominated by some vertex v; observe that, by bipartiteness, this implies that C is not a digon. Take $x, y \in V(C)$ such that (v, x) and (v, y) are arcs in G. The set $\{(v, x), (v, y)\} \cup C$ divides the plane into 3 regions: the face F surrounded by C, and two other regions A_1 and A_2. Without loss of generality, we take F to be the outer face; see Figure 3.

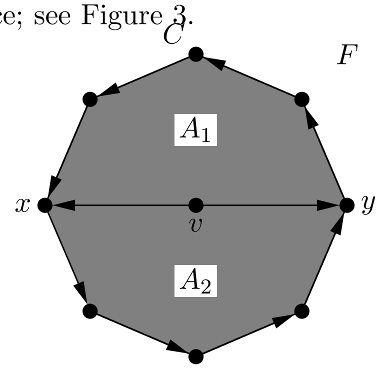

FIGURE 3. A dominated cycle. Note that there may be more (or less) vertices than shown on C.

Since G is strongly connected, we can find a path to v from one of x, y that does not use the other. Such a path must be contained in $\mathrm{cl}(A_1)$ or $\mathrm{cl}(A_2)$, because $\{v, x, y\}$ is a separator in G. As C also provides a path between x and y in either direction, it follows that there are paths from x and y to v both contained in either $\mathrm{cl}(F \cup A_1)$ or $\mathrm{cl}(F \cup A_2)$. Without loss of generality, suppose it is the former. Applying similar arguments to those of Lemma 3.3, we have that the subgraph G' induced by $\mathrm{cl}(F \cup A_1)$ is strongly connected. Clearly G' is also bipartite and planar; we claim it has no undominated facial cycles. Suppose not, then there must be a facial cycle D that is dominated in G but undominated in G'. This is only possible if D contains both x and y, since $\{v, x, y\}$ is a separator in G, and v is on the other side of the bipartition to x and y. D must contain v also, else v itself would dominate D. Let P_x be the subgraph contained in A_1 bounded by D and (v, x), and similarly let P_y be the subgraph bounded by D and (v, y); see Figure 4. Note that at least one of P_x and P_y is non-empty and has an outer facial cycle, since one of (v, x) and (v, y) is counter to the direction of D; by Lemma 3.3, it follows that it is strongly connected. It also cannot be dominated from outside itself, because the removal of $\{v, x\}$ (respectively $\{v, y\}$) separates P_x (respectively P_y) from the rest of the graph, and v lies on the other side of the bipartition from x and y. This contradicts the minimality of G. So G' is also a counterexample, and therefore $G = G'$ by minimality.

Hence, the region A_2 is empty. Now let C_i be the part of C bordering A_i, for $i = 1, 2$. Let H be the graph obtained by removing C_2 from G (that is, removing all of its edges, and all of its vertices other than x and y). Now take the strongly

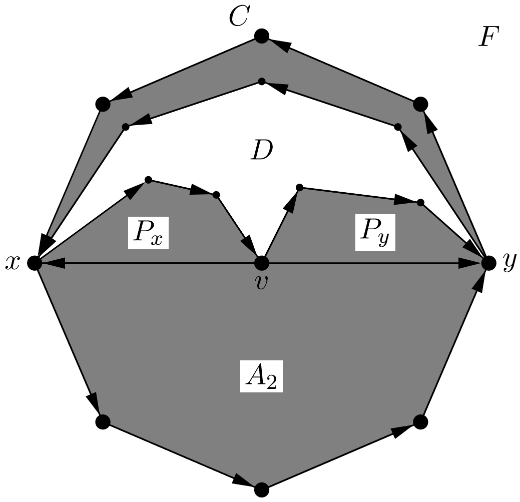

FIGURE 4. Finding a smaller counterexample

connected component S of H containing v. It contains at least two vertices, since there is a path from either y or x to v in G not using C_2, and so either x or y is in the same component. Note that there is a path from v to any other vertex in H, so any vertex in H with an outneighbour in S is also in S. Thus, any cycle that is not dominated in S is not dominated in G because the vertices in $C_2 - \{x, y\}$ have outdegree one. This implies that S is a smaller counterexample, contradicting the minimality of G. □

Lemma 3.5. *An undominated facial cycle in a strongly connected, bipartite, planar graph has an undominated induced subcycle.*

Proof. Denote the undominated facial cycle by C; without loss of generality, take the related face F to be the outer face. We need only show that the graph induced by C, say $H = G[C]$, has an undominated induced cycle, since C is undominated from outside.

By Lemma 3.2, H has another facial cycle D aside from the outer facial cycle. Observe that D must consist of edges in C and chords between vertices of C. Since there can be no chords inside D, nor outside D (unless H has a digon, in which case we're done), D is an induced cycle. Now take any $v \in H$, $v \notin D$. There must be a chord $(p, q) \in D$ such that v and $D \backslash \{p, q\}$ are in different components of $H \backslash \{p, q\}$; see Figure 5. Thus the only vertices of D where v could dominate are p and q; but since these are on different sides of the bipartition, this is impossible. Thus D is undominated. □

Thus we obtain our main structural result.

Proof of Theorem 3.1. The theorem follows immediately from Theorem 3.4 and Lemma 3.5. □

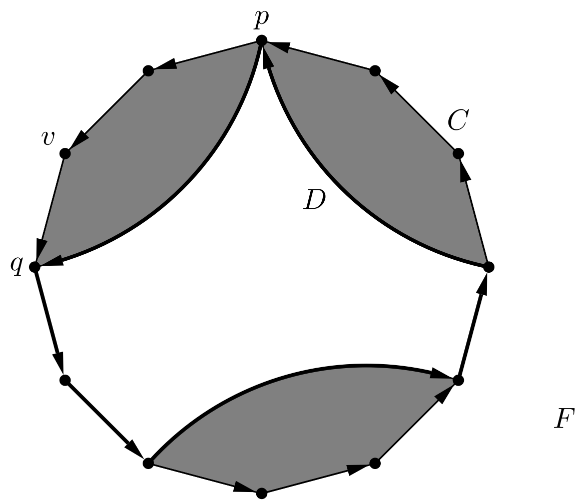

FIGURE 5. Finding an undominated induced cycle.

This also gives a polynomial time algorithm for finding a Nash equilibrium in two-player planar win-lose games. As we have seen we can implement the techniques of Section 2 efficiently; thus, we may assume that G is strongly connected. The proof of Theorem 3.1 is constructive. The faces of a planar embedding correspond to vertices of the dual; the dual can be also be found in polynomial time. Then consider each face (including the outer face) of any planar embedding in turn, until we find an undominated facial cycle (which we know exists). Finally, find an induced cycle of this facial cycle; this is an undominated induced cycle, and hence a Nash equilibrium. So we have our main result.

Theorem 3.6. *There is a polynomial time algorithm for finding a Nash equilibrium in a two-player planar win-lose game.* □

4. Conclusion

Two natural questions arise. On the positive side, on what other classes of graphs can Nash equilibria be efficiently obtained? On the negative side, for which classes of graphs is the problem hard?

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References

[1] T. Abbott, D. Kane and P. Valiant, "On the complexity of two-player win-lose games", *FOCS*, pp113-122, 2005.

[2] I. Barany, S. Vempala and A. Vetta, "Nash equilibria in random games", *FOCS*, pp123-131, 2005.

[3] B. Codenotti, M. Leoncini and G. Resta, "Efficient computation of Nash equilibria for very sparse win-lose games", *ESA*, 2006.

[4] B. Codenotti, D. Stefankovic, "On the computational complexity of Nash equilibria for (0,1) bimatrix games", *Information Processing Letters*, **94(3)**, pp145-150, 2005.

[5] T. Cormen, C. Leiserson, R. Rivest and C. Stein, *Introduction to algorithms*, McGraw-Hill, second edition, 2001.

[6] X. Chen and X. Deng, "Settling the complexity of 2-player Nash-equilibrium", *FOCS*, pp261-272, 2006.

[7] X. Chen, S. Teng and P. Valiant, "The approximation complexity of win-lose games", *SODA*, 2007.

[8] C. Daskalakis, P. Goldberg and C. Papadimitriou, "The complexity of computing a Nash equilibria", *STOC*, pp71-78, 2006.

[9] A. Fabrikant and C. Papadimitriou and K. Talwar, "On the complexity of pure equilibria", *STOC*, pp604-612, 2004.

[10] M. Goemans, V. Mirrokni, and A. Vetta, "Sink equilibria and convergence", *FOCS*, pp142-151, 2005.

[11] P. Goldberg and C. Papadimitriou, "Reducability among equilibrium problems", *STOC*, pp61-70, 2006.

[12] E. Koutsoupias and C. Papadimitriou, "Worst-case equilibria", *STACS*, pp404-413, 1999.

[13] R. Tarjan, "Depth first search and linear graph algorithms", *SIAM Journal on Computing*, **1(2)**, pp146-160, 1972.