An upper bound for the chromatic number of line graphs

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Abstract

It was conjectured by Reed [12] that for any graph G, the graph's chromatic number $\chi(G)$ is bounded above by $\left\lceil \frac{\Delta(G)+1+\omega(G)}{2} \right\rceil$, where $\Delta(G)$ and $\omega(G)$ are the maximum degree and clique number of G, respectively. In this paper we prove that this bound holds if G is the line graph of a multigraph. The proof yields a polynomial time algorithm that takes a line graph G and produces a colouring that achieves our bound.

1 Introduction

The *chromatic number* of a graph G, denoted by $\chi(G)$, is the minimum number of colours required to colour the vertex set of G so that no two adjacent vertices are assigned the same colour. That is, the vertices of a given colour form a *stable set*. Determining the exact chromatic number of a graph efficiently is very difficult, and for this reason it has proven fruitful to explore the relationships between $\chi(G)$ and other invariants of G. The *clique number* of G, denoted by $\omega(G)$, is the largest set of mutually adjacent vertices in G and the *degree* of a vertex v, written $\deg(v)$, is the number of vertices to which v is adjacent; the maximum degree over all vertices in G is denoted by $\Delta(G)$. It is easy to see that $\omega(G) \leq \chi(G) \leq \Delta(G) + 1$, so for $\chi(G)$ we have a trivial upper bound in terms of $\Delta(G)$ and a trivial lower bound in terms of $\omega(G)$. We are interested in exploring upper bounds on $\chi(G)$ in terms of a convex combination of $\Delta(G)+1$ and $\omega(G)$.

In [12], Reed conjectured two such bounds on the chromatic number of any graph G, one with a round-up and one without:

Conjecture 1. *For any graph G, $\chi(G) \leq \left\lceil \frac{\Delta(G)+1+\omega(G)}{2} \right\rceil$.*

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Conjecture 2. *For any connected graph G that is not an odd cycle,* $\chi(G) \leq \frac{2(\Delta(G)+1)+\omega(G)}{3}$.

Note that when $\omega(G)$ is equal to $\Delta(G)+1$ or $\Delta(G)-1$, the round-up in the first bound is not used, so we can see that the second bound is only better than the first bound if $\Delta(G) = \omega(G)$; in this case the second bound is better by one. Brooks' Theorem (see [1]) implies that if $\Delta(G) = \omega(G)$ then either G is an odd cycle or $\chi(G) \leq \Delta(G)$:

Brooks' Theorem. $\chi(G) \leq \Delta(G)$ *unless G contains a clique of size* $\Delta(G)+1$ *or* $\Delta(G) = 2$ *and G contains an odd cycle.*

Either way, both conjectures are true, so Conjecture 2 is, in fact, strictly weaker than Conjecture 1.

Several related results exist. In the same paper, Reed proved that Conjecture 1 holds if $\Delta(G)$ is sufficiently large and $\omega(G)$ is sufficiently close to $\Delta(G)$. Using this, he proved that there exists a positive constant α such that $\chi(G) \leq \alpha(\omega(G)) + (1-\alpha)(\Delta(G)+1)$ for all graphs. Some results are also known for generalizations of the chromatic number.

A *fractional vertex c-colouring* of a graph G can be described as a set $\{S_1, S_2, \ldots S_l\}$ of stable sets with associated non-negative real weights $\{w_1, w_2, \ldots w_l\}$ such that for every vertex v, $\sum_{S_i : v \in S_i} w_i = 1$ and $\sum_{i=1}^{l} w_i = c$. The *fractional chromatic number* of G, written $\chi^f(G)$, is the smallest c for which G has a *fractional vertex c-colouring*. Note that it is always bounded above by the chromatic number. The *list chromatic number* of a graph G, written $\chi^l(G)$, is the smallest r such that for any assignment of a list of r colours to each vertex v, the graph has a colouring in which every vertex is coloured with a colour on its list. For any graph we clearly have $\chi^f(G) \leq \chi(G) \leq \chi^l(G)$.

In [10], Molloy and Reed proved the fractional analogue of Conjecture 1 for all graphs:

Theorem 3. *For any graph G,* $\chi^f(G) \leq \left\lceil \frac{\Delta(G)+1+\omega(G)}{2} \right\rceil$.

In fact, the round-up is not needed in the fractional case. In this paper we prove that Conjecture 1 (and hence Conjecture 2) holds for line graphs, which are defined in the next section.

2 Fractional and Integer Colourings in Line Graphs of Multigraphs

A *multigraph* is a graph in which multiple edges are permitted between any pair of vertices – all multigraphs in this paper are loopless. Given a multigraph $H = (V, E)$, the *line graph* of H, denoted by $L(H)$, is a graph with vertex set E; two vertices of $L(H)$ are adjacent if and only if the corresponding edges in H share at least one endpoint. We say that G is a line graph if there is a multigraph H for which $G = L(H)$.

The *chromatic index* of H, written $\chi_e(H)$, is the chromatic number of $L(H)$. Similarly, the *fractional chromatic index* $\chi_e^f(H)$ is equal to the fractional chromatic number of $L(H)$. In [6], Holyer proved that determining the chromatic index of an arbitrary graph is NP-complete, so practically speaking we are bound to the task of approximating the chromatic index of multigraphs and hence the chromatic number of line graphs.

Vizing's Theorem (see [14]) bounds the chromatic index of a multigraph in terms of its maximum degree, stating that $\Delta(H) \leq \chi_e(H) \leq \Delta(H)+d$, where d is the maximum number of edges between any two vertices in H. Both bounds are achievable, but a more meaningful bound should consider other invariants of H.

Of course, $\chi_e(H)$ is always bounded below by $\chi_e^f(H)$. Let w be a nonnegative weighting on the edges of H. Duality tells us that given a nonnegative weighting w on the edges of H such that for every matching M in H, $\sum_{e\in M} w(e) \leq 1$, $\chi_e^f(H) \geq \sum_{e\in E(H)} w(e)$. Two such weightings give us lower bounds. In the first, we assign a weight of 1 to each edge incident to a given vertex v of maximum degree; every other edge is assigned weight 0. In the second, we take an induced subgraph W of H and assign to each edge of W a weight of $1/\lfloor |V(W)|/2 \rfloor$; other edges of H are assigned weight 0. Edmond's theorem for matching polytopes (presented in [3], also mentioned in [8]) tells us that the greater of these lower bounds is tight, so setting

$$\Gamma(H) = \max\left\{ \frac{2|E(W)|}{|V(W)|-1} : W \subseteq H, |V(W)| \text{ is odd} \right\},$$

we have

$$\chi_e^f(H) = \max\{\Delta(H), \Gamma(H)\}. \tag{1}$$

Does this necessarily translate into a good upper bound on the chromatic index of a multigraph? The following long-standing conjecture, proposed by Goldberg [4] and Seymour [13], implies that $\chi_e^f(H) \leq \chi_e(H) \leq \chi_e^f(H) + 1$:

Goldberg-Seymour Conjecture. *For a multigraph H for which $\chi_e(H) > \Delta(H)+1$, $\chi_e(H) = \lceil \Gamma(H) \rceil$.*

Asymptotic results are known: Kahn [7] proved that the fractional chromatic index asymptotically agrees with the integral chromatic index, i.e. that $\chi_e(H) \leq (1+o(1))\chi_e^f(H)$. This implies the Goldberg-Seymour Conjecture asymptotically. He later proved that in fact, the fractional chromatic index asymptotically agrees with the list chromatic index [8].

Another result that supports the Goldberg-Seymour Conjecture is the following theorem:

Theorem 4 (Caprara and Rizzi [2])**.** *For any multigraph H, $\chi_e(H) \leq \max\{\lfloor 1.1\Delta(H)+0.7 \rfloor, \lceil \Gamma(H) \rceil\}$.*

This theorem is a slight improvement of an earlier result of Nishizeki and Kashiwagi [11], lowering the additive factor from 0.8 to 0.7. Note that this implies the Goldberg-Seymour Conjecture for any multigraph H with $\Delta(H) \leq 12$, since in this case we have $\lfloor 1.1\Delta(H) + 0.7 \rfloor \leq \Delta(H) + 1$.

3 The Main Result

We will now prove our main result:

Theorem 5. *For any line graph G, $\chi(G) \leq \left\lceil \frac{\Delta(G)+1+\omega(G)}{2} \right\rceil$.*

Consider a multigraph H for which $G = L(H)$. The proof consists of two cases: the case where $\Delta(G)$ is large compared to $\Delta(H)$, and the case where $\Delta(G)$ is close to $\Delta(H)$. In both cases we use the fact that $\omega(G) \geq \Delta(H)$. The first case is given by the following lemma, which follows easily from Theorem 4.

Lemma 6. *If G is the line graph of a multigraph H, and $\Delta(G) \geq \frac{3}{2}\Delta(H) - 1$, then $\chi(G) \leq \left\lceil \frac{\Delta(G)+1+\omega(G)}{2} \right\rceil$.*

Proof. Combining Equation 1 with Theorem 4 gives us $\chi(G) \leq \max\{\lfloor 1.1\Delta(H)+0.7 \rfloor, \lceil \chi^f(G) \rceil\}$, so by Theorem 3 we have $\chi(G) \leq \max\{\lfloor 1.1\Delta(H)+0.7 \rfloor, \left\lceil \frac{\Delta(G)+1+\omega(G)}{2} \right\rceil\}$. Since $\Delta(G) \geq \frac{3}{2}\Delta(H) - 1$, $\left\lceil \frac{\Delta(G)+1+\omega(G)}{2} \right\rceil \geq \left\lceil \frac{5}{4}\Delta(H) \right\rceil \geq \lfloor 1.1\Delta(H) + 0.7 \rfloor$, so we are done. □

We can now proceed to the case where $\Delta(G) < \frac{3}{2}\Delta(H) - 1$.

Proof of Theorem 5. Consider a counterexample $G = L(H)$ such that the theorem holds for every line graph on fewer vertices. By Lemma 6, we know that $\Delta(G) < \frac{3}{2}\Delta(H) - 1$. Our approach is as follows: We find a maximal stable set $S \subset V(G)$ that has a vertex in every maximum clique in G, and let G' be the subgraph of G induced on $V(G) \setminus S$. We can see that $\Delta(G') \leq \Delta(G) - 1$ (since S is maximal) and $\omega(G') = \omega(G) - 1$, and that the theorem holds for G', as any induced subgraph of a line graph is clearly a line graph. So we know that $\chi(G') \leq \left\lceil \frac{\Delta(G)+1+\omega(G)}{2} \right\rceil - 1$. We can now construct a proper $\chi(G') + 1$-colouring of $V(G)$ by taking a proper $\chi(G')$-colouring of G' and letting S be the final colour class, hence $\chi(G) \leq \left\lceil \frac{\Delta(G)+1+\omega(G)}{2} \right\rceil$, a contradiction.

It suffices, then, to show the existence of such a stable set S in G. We actually need only find a stable set that hits all the maximum cliques of G, as we can extend any such stable set until it is maximal. We will do this in terms of a *matching* in H, i.e. a set of edges in $E(H)$, no two of which share an endpoint – a matching in H exactly represents a stable set in G. We need some notation first. For a pair of vertices $u, v \in V(H)$, the *multiplicity* of uv is the number of edges in $E(H)$ between u and v; we denote it by $\mu(u, v)$. A *triangle* in H is a set of three mutually adjacent vertices, and the edges of a triangle are those edges in $E(H)$ joining the triangle's vertices. We denote the maximum number of edges of any triangle in H by $\text{tri}(H)$. Note the following facts that relate invariants of H and G:

Fact 1. $\Delta(G) = \max_{uv \in E(H)}\{\deg(u) + \deg(v) - \mu(u, v) - 1\}$.

Fact 2. $\omega(G) = \max\{\Delta(H), \text{tri}(H)\}$.

We say that a matching *hits* a vertex v if v is an endpoint of an edge in the matching. We will find a maximal matching M in H which corresponds to a desired stable set because it hits every vertex of maximum degree in H and contains an edge of every triangle with $\max\{\Delta(H), \mathrm{tri}(H)\}$ edges in H.

To this end, let S_Δ be the set of vertices of degree $\Delta(H)$ in H and let T be the set of triangles in H that contain $\max\{\Delta(H), \mathrm{tri}(H)\}$ edges. It is instructive to consider how the elements of T interact.

Observation 1. *If two triangles of T intersect in exactly the vertices a and b then ab has multiplicity greater than $\Delta(H)/2$.*

Proof. For any edge e of H between a and b, the degree of the corresponding vertex of G is at least $2\Delta(H) - \mu(a, b)$. □

Observation 2. *If abc is a triangle of T intersecting another triangle ade of T in exactly the vertex a then $\mu(b, c)$ is greater than $\Delta(H)/2$.*

Proof. The degree of a vertex of G corresponding to an edge between a and d is at least $2\Delta(H) - \mu(b, c) - 1$. □

Observation 3. *If there is an edge of H joining two vertices a and b of S_Δ then $\mu(a, b) > \Delta(H)/2$.*

Guided by these observations, we let T' be those triangles in T that contain no pair of vertices of multiplicity $> \Delta(H)/2$ and S'_Δ be those elements of S_Δ which are in no pair of vertices of multiplicity greater than $\Delta(H)/2$. We treat $T' \cup S'_\Delta$ and $(T \setminus T') \cup (S_\Delta \setminus S'_\Delta)$ separately. A few more observations regarding S'_Δ and T' will serve us well. For a set S of vertices we denote the union of the vertices' neighbourhoods by $N(S)$.

Observation 4. *For any $S \subseteq S'_\Delta$, $|N(S)| \geq |S|$.*

Proof. It follows from Observation 3 that S'_Δ is a stable set. This means that S and $N(S)$ are disjoint, and given $S \subseteq S'_\Delta$ there are $|S|\Delta(H)$ edges between S and $N(S)$; the result follows from the fact that no vertex in $N(S)$ has degree $\geq \Delta(H)$. □

Observation 5. *If an edge ab in H has exactly one endpoint in a triangle bcd of T', then the degree of a is less than $\Delta(H)$.*

Proof. Any vertex in G corresponding to an edge between a and b has degree at least $\deg(a) - 1 + \Delta(H) - \mu(c, d)$, and $\mu(c, d) \leq \Delta(H)/2$. □

Observation 6. *If an edge ab in H has exactly one endpoint in a triangle bcd of T', then $\mu(a, b) \leq \Delta(H)/2$.*

Proof. The degree of any vertex in G corresponding to an edge between b and c has degree at least $\mu(a, b) + \Delta(H) - 1$. □

Observation 7. *For any vertex v with two neighbours u and w, $\deg(u) + \mu(vw) \leq \frac{3}{2}\Delta(H)$.*

Proof. An edge between u and v is incident to at least $\deg(u) + \mu(vw) - 1$ other edges. □

Finally, we state Hall's Theorem (see [5]), a fundamental result on matchings in bipartite graphs.

Hall's Theorem. *Let G be a bipartite graph with vertex set $V = (A, B)$. There is a matching that hits every vertex in A precisely if for every $S \subseteq A$, $|N(S)| \geq |S|$.*

It is now straightforward to show that the desired matching exists. We begin with a matching M consisting of one edge between each vertex pair with multiplicity greater than $\Delta(H)/2$ – this hits $S_\Delta \setminus S'_\Delta$ and contains an edge of each triangle in $T \setminus T'$. Observation 4 tells us that we can apply Hall's Theorem to get a matching in H that hits S'_Δ; Observation 7 dictates that this matching cannot hit M, so the union M' of these two matchings is a matching in H that hits S_Δ and contains an edge of each triangle in $T \setminus T'$. Every edge in this matching either hits a maximum-degree vertex in H or has endpoints with multiplicity greater than $\Delta(H)/2$.

What, then, can prevent us from extending this M' to contain an edge of every triangle in T'? Observations 1 and 2 tell us that any two triangles in T' are vertex-disjoint, so our only worry is that M' already hits two vertices of some triangle in T'. Observations 3, 5 and 6 guarantee that at most one such vertex in a given triangle is hit, and if there is such a vertex, it has degree $\Delta(H)$. We can therefore extend M' to contain an edge of every triangle in T'. The result is a matching that satisfies all of our requirements, so the proof of the theorem is complete. □

4 Algorithmic Considerations

We have presented a new upper bound for the chromatic number of line graphs, i.e. $\chi(G) \leq \left\lceil \frac{\Delta(G)+1+\omega(G)}{2} \right\rceil$. Our proof of the bound yields an algorithm for constructing a colour class in G but we have an initial condition in the proof (i.e. $\Delta(G) < \frac{3}{2}\Delta(H) - 1$) that does not necessarily remain if we remove these vertices. However, the bound given by Caprara and Rizzi in Theorem 4 can be achieved in $O(|E(H)|(|V(H)| + \Delta(H)))$ time [2]. It is easy to see that in the proof of Theorem 5 we can find our matching in polynomial time, so we can formulate a polytime algorithm for $\left\lceil \frac{\Delta(G)+1+\omega(G)}{2} \right\rceil$-colouring a line graph G with root graph H as follows.

1. While $\Delta(L(H)) < \frac{3}{2}\Delta(H) - 1$, remove a matching M from H as in the proof of Theorem 5 (and let it be a colour class).

2. Employ Caprara and Rizzi's algorithm to complete the edge colouring of H.

This, of course, assumes that we have the root graph H such that $G = L(H)$. Lehot provides an $O(|E(G)|)$ algorithm that detects whether or not G is the line graph of a simple graph H and outputs H if possible [9]. Two vertices u and v in G are *twins* if they are adjacent and their neighbourhoods are otherwise identical. We can extend Lehot's algorithm to line graphs of multigraphs by contracting each set of k mutually twin vertices in G into a single vertex, which we say has multiplicity k. This can be done trivially in $O(|E(G)|\Delta(G))$ time. The resulting graph G' is the line graph of a simple graph H' if and only if G is the line graph of a multigraph H; we can generate H from H' by considering the multiplicities of the vertices in G' and duplicating edges in H' accordingly.

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