A GENERAL THEOREM FOR THE CONSTRUCTION OF BLOWING-UP SOLUTIONS TO SOME ELLIPTIC NONLINEAR EQUATIONS VIA LYAPUNOV-SCHMIDT'S FINITE-DIMENSIONAL REDUCTION

FRÉDÉRIC ROBERT AND JÉRÔME VÉTOIS

ABSTRACT. We prove a general finite-dimensional reduction theorem for critical equations of scalar curvature type. Solutions of these equations are constructed as a sum of peaks. The use of this theorem reduces the proof of existence of multi-peak solutions to some test-functions estimates and to the analysis of the interactions of peaks.

1. Introduction and statement of the result

Let (M,g) be a compact Riemannian manifold of dimension $n \geq 3$ without boundary. We let $H_1^2(M)$ be the completion of $C^\infty(M)$ for the norm $\|\cdot\|_{H_1^2} := \|\cdot\|_2 + \|\nabla \cdot\|_2$. We let $h \in L^\infty(M)$ be such that the operator $\Delta_g + h$ is coercive, that is $\lambda_1(\Delta_g + h) > 0$, where $\Delta_g := -\text{div}_g(\nabla)$ is the Laplace-Beltrami operator. Non-positive examples of such h's are after the theorem. We define $2^\star := \frac{2n}{n-2}$ and $H : \mathbb{R} \to \mathbb{R}$ such that $\{H(x) = |x| \text{ for all } x \in \mathbb{R}\}$ or $\{H(x) = x_+ := \max\{x, 0\} \text{ for all } x \in \mathbb{R}\}$. Given $f \in C^0(M)$, $q \in (2, 2^\star]$, and $G \in C^2(H_1^2(M))$, we give a general theorem to construct solutions $v \in H_1^2(M)$ to the equation

$$\Delta_g v + hv = fH(v)^{q-2}v + G'(v) \text{ in the distributional sense on } M \tag{1}$$

of the form

$$v = u_0 + \sum_{i=1}^k W_{\kappa_i,\delta_i,\xi_i} + \text{ remainder },$$

where $k \in \mathbb{N}$, $(\kappa_i)_{i=1,\cdots,k} \in \{-1,+1\}$, $(\delta_i)_{i=1,\cdots,k} \in (0,+\infty)$, $(\xi_i)_{i=1,\cdots,k} \in M$ are the parameters, and the $W_{\kappa,\delta,\xi}$'s are peaks defined in (12) below and are C^1 with respect to the parameters. The function $u_0 \in H_1^2(M)$ is a distributional solution to

$$\Delta_g u_0 + h_0 u_0 = f_0 H(u_0)^{2^\star-2} u_0 + G_0'(u_0), \tag{2}$$

where $h_0 \in L^\infty(M)$ is such that $\lambda_1(\Delta_g + h_0) > 0$, $f_0 \in C^0(M)$, and $G_0 \in C^2(H_1^2(M))$ is of *subcritical type*, see Definition 2.1 below. Examples of nonlinearities of *subcritical type* are maps like $u \mapsto \int_M a(x)|u|^r\,dx$, where $a \in L^\infty(M)$ and $2 \leq r < 2^\star$. Solutions to (1) and (2) are critical points respectively for the functionals

$$J(v) := \frac{1}{2}\int_M \left(|\nabla v|_g^2 + hv^2\right)\,dv_g - F(v)\,;\; J_0(v) := \frac{1}{2}\int_M \left(|\nabla v|_g^2 + h_0 v^2\right)\,dv_g - F_0(v),$$

Date: January 30, 2013.

Published in *Concentration Analysis and Applications to PDE* (ICTS Workshop, Bangalore, 2012), Trends in Mathematics, Birkhäuser/Springer Basel, 2013, 85–116.

where dv_g is the Riemannian element of volume, and

$$F(v) := \frac{1}{q}\int_M f H(v)^q\, dv_g + G(v) \text{ and } F_0(v) := \frac{1}{2^\star}\int_M f_0 H(v)^{2^\star}\, dv_g + G_0(v)$$

for all $v \in H_1^2(M)$. We introduce the kernel of the linearization of (2) by

$$K_0 := \left\{\varphi \in H_1^2(M)/\, \Delta_g\varphi + h_0\varphi = F_0''(u_0)\varphi\right\}. \tag{3}$$

We get that $d := \dim_{\mathbb{R}} K_0 < +\infty$ since the operator $\varphi \mapsto (\Delta_g + h_0)^{-1}(F_0''(u_0)\varphi)$ is compact on $H_1^2(M)$. We let $u \in C^1(B_1(0) \subset \mathbb{R}^d, H_1^2(M))$ be such that $u(0) = u_0$, and we assume that

$$K_0 = \text{Span}\{\Pi_{K_0}^{h_0}(\partial_{z_i} u(0))/\, i = 1, \cdots, d\}, \tag{4}$$

where $\Pi_{K_0}^{h_0}$ is the orthogonal projection on K_0 with respect to the scalar product $(u,v) \mapsto (u,v)_{h_0} := \int_M ((\nabla u, \nabla v)_g + h_0 uv)\, dv_g$. We consider a finite covering $(U_\gamma)_{\gamma\in\mathcal{C}}$ of M of *parallel type* (see Definition 2.2), and we choose a correspondance $i \mapsto \gamma_i \in \mathcal{C}$ for all $i \in \{1, \cdots, k\}$. For any $\varepsilon > 0$, $N > 0$, and $k \in \mathbb{N}$, we define

$$\mathcal{D}_k(\varepsilon, N) := \left\{ ((\delta_i)_i, (\xi_i)_i) \in (0,\varepsilon)^k \times M^k \text{ s.t. } \left\{ \begin{array}{c} \xi_i \in U_{\gamma_i} \\ |\delta_i^{2^\star - q} - 1| < \varepsilon \text{ and} \\ \frac{\delta_i}{\delta_j} + \frac{\delta_j}{\delta_i} + \frac{d_g(\xi_i,\xi_j)^2}{\delta_i\delta_j} > N \\ \text{for all } i \neq j \in \{1,\cdots,k\} \end{array} \right\} \right\}.$$

We define the error term

$$R(z,(\delta_i)_i,(\xi_i)_i) := \left\| u(z) + \sum_{i=1}^k W_{\kappa_i,\delta_i,\xi_i} - (\Delta_g + h)^{-1}\Big(F'\Big(u(z) + \sum_{i=1}^k W_{\kappa_i,\delta_i,\xi_i}\Big)\Big)\right\|_{H_1^2}. \tag{5}$$

Theorem 1.1. *We fix $k \in \mathbb{N}$, $\nu_0, C_0 > 0$, $\theta \in (0,1)$, $h_0 \in L^\infty(M)$ such that $\lambda_1(\Delta_g + h_0) > 0$, $f_0 \in C^0(M)$, $u_0 \in H_1^2(M)$, and $G_0 \in C^{2,\theta}_{loc}(H_1^2(M))$ of subcritical type. We define K_0 as in (3), we let d be its dimension and β_0 be a basis of K_0. We fix $(\kappa_i)_i \in \{-1,+1\}^k$. Then there exist $N > 0$ and $\varepsilon > 0$ such that for any $q \in (2, 2^\star]$, $h \in L^\infty(M)$, $f \in C^0(M)$, $G \in C^{2,\theta}_{loc}(H_1^2(M))$, and $u \in C^1(B_1(0), H_1^2(M))$ such that*

$$u(0) = u_0\,,\ \|u\|_{C^1(B_1(0),H_1^2)} \le C_0\,,\ f_0(\xi_i) \ge \nu_0 \text{ for all } i = 1,\cdots,k\,, \tag{6}$$

$$\|h - h_0\|_\infty + \|f - f_0\|_{C^0(M)} + d_{C_B^{2,\theta}}(G, G_0) + (2^\star - q) < \varepsilon\,, \tag{7}$$

(see Definition 2.3 for the distance $d_{C_B^{2,\theta}}$) and for any $z \in B_1(0)$, if

$$\Big|\det_{\beta_0}(\Pi_{K_0}^{h_0}(\partial_{z_1}u(z)), \cdots, \Pi_{K_0}^{h_0}(\partial_{z_d}u(z)))\Big| \ge \nu_0 \prod_{i=1}^d \|\partial_{z_i}u(z)\|_{H_1^2} > 0\,, \tag{8}$$

then there exists $\phi \in C^1(B_\varepsilon(0) \times \mathcal{D}_k(\varepsilon,N), H_1^2(M))$ such that $u(z,(\delta_i)_i,(\xi_i)_i) := u(z) + \sum_{i=1}^k W_{\kappa_i,\delta_i,\xi_i} + \phi(z,(\delta_i)_i,(\xi_i)_i)$ is a critical point of J iff $(z,(\delta_i)_i,(\xi_i)_i)$ is a critical point of $(z,(\delta_i)_i,(\xi_i)_i) \mapsto J(u(z,(\delta_i)_i,(\xi_i)_i))$ in $B_\varepsilon(0) \times \mathcal{D}_k(\varepsilon,N)$. Moreover, we have that

$$\|\phi(z,(\delta_i)_i,(\xi_i)_i)\|_{H_1^2} \le C \cdot R(z,(\delta_i)_i,(\xi_i)_i),$$

where C is a constant depending on (M,g), k, ν_0, θ, C_0, u_0, h_0, f_0, and G_0.

Miscellaneous remarks

1. The implicit definition of $\phi(z,(\delta_i)_i,(\xi_i)_i)$ is in (61) of Proposition 5.1.

2. In addition to Theorem 1.1, we have that

$$\left|J(u(z,(\delta_i)_i,(\xi_i))) - J\Big(u(z) + \sum_{i=1}^{k} W_{\kappa_i,\delta_i,\xi_i}\Big)\right| \leq C\cdot R(z,(\delta_i)_i,(\xi_i)_i)^2$$

for all $(z,(\delta_i)_i,(\xi_i)_i)) \in B_\varepsilon(0)\times\mathcal{D}_k(\varepsilon,N)$, where C is a constant depending on (M,g), k, ν_0, C_0, u_0, G_0.

3. Theorem 1.1 is valid under a little more general hypothesis on G. There exists $\tilde{R}>0$ depending only on (M,g), u_0, and k such that the same conclusion of the theorem holds if G_0 and G satisfy the following

$$G_0\in C^2(B_{\tilde{R}}(0)),\, G\in C^{2,\theta}(B_{\tilde{R}}(0)),\, \|G\|_{C^{2,\theta}(B_{\tilde{R}}(0))}\leq C_0,\, \|G-G_0\|_{C^2(B_{\tilde{R}}(0))}<\varepsilon\,.$$

4. We have assumed for convenience a L^∞–control of the potentials h_0 and h. If they are only controled in $L^{n/2}$, it suffices to include them in the perturbations G_0 and G.

5. As one checks, if $h\geq 0$ and $h\not\equiv 0$, one gets $\lambda_1(\Delta_g+h)>0$. As a consequence, $\lambda_1(\Delta_g+h-\alpha)>0$ for such h and all $\alpha<\lambda_1(\Delta_g+h)$.

As a consequence of Theorem 1.1, finding solutions to (1) reduces to computing the expansion of $J(u(z,(\delta_i)_i,(\xi_i)))$ and controling the rest $R(z,(\delta_i)_i,(\xi_i)_i)$. In particular, Theorem 1.1 covers the general reduction theory in the recent articles Esposito-Pistoia–Vétois [6], Micheletti–Pistoia–Vétois [9], Pistoia–Vétois [10], and Robert–Vétois [12]. This finite-reduction method is very classical and has proved to be very powerful in the last decades to find blowing-up solutions to critical equations. The litterature on this issue is abundant: here, we refer to the early reference Rey [11], and to Brendle [3], Brendle–Marques [4], del Pino–Musso–Pacard–Pistoia [5], and Guo–Li–Wei [8] for more recent references. The list of constributions above does not pretend to exhaustivity: we refer to the references of the above papers and also to the monograph [1] by Ambrosetti–Malchiodi for further bibliographic complements. A general reference on Lyapunov-Schmidt's reduction, including the group action point of view, is the monograph [7] by Falaleev–Loginov–Sidorov–Sinitsyn.

2. Definitions and notations

2.1. Nonlinearities of subcritical type.

Definition 2.1. *Let $G_0\in C^2(H_1^2(M))$. We say that G_0 is of subcritical type if for all sequences $(u_p)_p,(v_p)_p,(w_p)_p\in H_1^2(M)$ converging weakly respectively to $u,v,w\in H_1^2(M)$, we have that*

$$G_0(u_p)\to G_0(u)\,,\; G_0'(u_p)(v_p)\to G_0'(u)(v),\ \text{and } G_0''(u_p)(v_p,w_p)\to G_0''(u)(v,w)$$

when $p\to+\infty$.

2.2. Covering of parallel type.

Definition 2.2. *We say that $(U_\gamma)_{\gamma\in\mathcal{C}}$ is a covering of parallel type if $\cup_\gamma U_\gamma = M$ and for any $\gamma\in\mathcal{C}$, U_γ is open and there exists n smooth vector fields $e_i^{(\gamma)}:U_\gamma\to TM$ such that for any $\xi\in U_\gamma$, $\{e_1^{(\gamma)}(\xi),\cdots,e_n^{(\gamma)}(\xi)\}$ is an orthonormal basis of $T_\xi M$, the tangent space at the point ξ.*

Since (M,g) is compact, it follows from the Gram-Schmidt orthogonalisation procedure that a finite covering of *parallel type* always exists. A manifold is parallelizable if there exists a smooth global orthonormal basis.

In the sequel, we let $(U_\gamma)_{\gamma\in\mathcal{C}}$ be a fixed finite covering of parallel type of M. With a slight abuse of notation, for any $\gamma\in\mathcal{C}$, and any $\xi\in U_\gamma$, we define $e_j(\xi)=e_j^{(\gamma)}(\xi)$ for $j=1,\cdots,n$, where $e_j^{(\gamma)}$ is as in Definition 2.2. In other words, for any $\gamma\in\mathcal{C}$, there exists n smooth maps $e_1,\cdots,e_j : U_\gamma\to TM$ such that for any $\xi\in U_\gamma$, $(e_1(\xi),\cdots,e_n(\xi))$ is an orthonormal basis of $T_\xi M$. We can then assimilate smoothly the tangent space $T_\xi M$ at $\xi\in U_\gamma$ to $\mathbb{R}^n$ via the map

$$\begin{array}{cccc}\Phi_\xi : & \mathbb{R}^n & \to & T_\xi M\\ & X & \mapsto & \sum_{j=1}^n X^j e_j(\xi).\end{array} \tag{9}$$

2.3. The distance on $C_B^{2,\theta}$.

Definition 2.3. *Let E be a Banach space. We define $C_B^{2,\theta}(E)$ as the set of functions that are in $C^{2,\theta}(B)$ for any bounded open set $B\subset E$: we endow $C_B^{2,\theta}(E)$ with the topology inherited from the natural associated family of semi-norms. This topology is metrizable with the distance*

$$d_{C_B^{2,\theta}}(G_1,G_2):=\sup_{p\in\mathbb{N}}\frac{\|G_1-G_2\|_{C^{2,\theta}(B_p(0))}}{2^p(1+\|G_1-G_2\|_{C^{2,\theta}(B_p(0))})}\text{ for all } G_1,G_2\in C_B^{2,\theta}(E).$$

2.4. **The peaks $W_{\kappa,\delta,\xi}$.** We consider a function $\Lambda\in C^\infty(M\times M)$ such that, defining $\Lambda_\xi:=\Lambda(\xi,\cdot)$ for all $\xi\in M$, we have that

$$\Lambda_\xi>0 \text{ and } \Lambda_\xi(\xi)=1 \text{ for all } \xi\in M. \tag{10}$$

We then define a metric $g_\xi:=\Lambda_\xi^{\frac{4}{n-2}}g$ for all $\xi\in M$ conformal to g. Since Λ is continuous, there exists $C>0$ such that

$$\frac{1}{C}g\le g_\xi\le Cg \tag{11}$$

for all $\xi\in M$. The compactness of M yields the existence of $r_0>0$ such that the injectivity radius of the metric g_ξ satisfies $i_{g_\xi}(M)\ge r_0$ for all $\xi\in M$. We let $\chi\in C^\infty(\mathbb{R})$ be such that $\chi(t)=1$ for $t\le r_0/3$, $\chi(t)=0$ for all $t\ge r_0/2$ and $0\le\chi\le 1$.

For $\kappa\in\{-1,+1\}$, $\delta>0$, and $\xi\in M$ such that $f_0(\xi)>0$, a bubble is defined as

$$W_{\kappa,\delta,\xi}(x):=\kappa\chi(d_{g_\xi}(x,\xi))\Lambda_\xi(x)\left(\frac{\delta\sqrt{\frac{n(n-2)}{f_0(\xi)}}}{\delta^2+d_{g_\xi}(x,\xi)^2}\right)^{\frac{n-2}{2}}+B_{\delta,\xi}(x) \tag{12}$$

for all $x\in M$, where $(\delta,\xi)\mapsto B_{\delta,\xi}$ is C^1 from $(0,+\infty)\times M$ to $H_1^2(M)$ and

$$\|B_{\delta,\xi}\|_{H_1^2}+\delta\|\partial_\delta B_{\delta,\xi}\|_{H_1^2}+\delta\|\nabla_\xi B_{\delta,\xi}\|_{H_1^2}\le\epsilon(\delta) \tag{13}$$

for all $\delta>0$ and $\xi\in M$, where $\lim_{\delta\to0}\epsilon(\delta)=0$. If $H=(\cdot)_+$, we require that $\kappa=1$.

2.5. **Derivatives of the peaks.** We let $D_1^2(\mathbb{R}^n)$ be the completion of $C_c^\infty(\mathbb{R}^n)$ for the norm $u \mapsto \|\nabla u\|_2$. Given $a > 0$, we are interested in solutions $U \in D_1^2(\mathbb{R}^n)$ to the equation

$$\Delta_{\text{Eucl}} U_a = a U_a^{2^\star - 1} \text{ in } \mathbb{R}^n, \tag{14}$$

where Eucl is the Euclidean metric. As one checks, the Lie group $(0, +\infty) \times \mathbb{R}^n$ (with the relevant structure) leaves the solution to (14) invariant via the action

$$(\delta, x_0) \in (0, +\infty) \times \mathbb{R}^n \mapsto \delta^{-\frac{n-2}{2}} U_a(\delta^{-1}(\cdot - x_0)). \tag{15}$$

For $a > 0$, we define

$$U_a(x) := \left(\frac{\sqrt{\frac{n(n-2)}{a}}}{1 + |x|^2} \right)^{\frac{n-2}{2}} \text{ for all } x \in \mathbb{R}^n.$$

As easily checked, we have that $U_a \in D_1^2(\mathbb{R}^n)$ is a solution to (14). Therefore, the action of the Lie algebra of $(0, +\infty) \times \mathbb{R}^n$ yields elements of the set K_{BE} of solutions $V \in D_1^2(\mathbb{R}^n)$ of the linearized equation

$$\Delta_{\text{Eucl}} V = (2^\star - 1) a U_a^{2^\star - 2} V \text{ in } \mathbb{R}^n. \tag{16}$$

Conversely, it follows from Bianchi–Egnell [2] that this actions is onto, that is

$$K_{BE} = \text{Span}\{V_j /\, j = 0, \cdots, n\},$$

where

$$V_0 := \frac{2}{n-2} \left(\frac{a}{n(n-2)} \right)^{(n-2)/4} \frac{\partial}{\partial \delta} (\delta^{-\frac{n-2}{2}} U_a(\delta^{-1} \cdot))_{\delta=1} = \frac{|x|^2 - 1}{(1 + |x|^2)^{\frac{n}{2}}},$$

$$V_j := \frac{-1}{n-2} \left(\frac{a}{n(n-2)} \right)^{(n-2)/4} \partial_{x_j} U_a = \frac{x_j}{(1 + |x|^2)^{\frac{n}{2}}} \text{ for all } j = 1, \cdots, n.$$

The functions V_j form an orthonormal basis of K_{BE} for the scalar product $(u, v) \mapsto \int_{\mathbb{R}^n} (\nabla u, \nabla v)\, dx$. Rescaling and pulling-back on M, for any $\delta > 0$, $\xi \in M$, and $X \in T_\xi M$, we define

$$Z_{\delta,\xi}(x) := \chi(d_{g_\xi}(x, \xi)) \Lambda_\xi(x) \delta^{\frac{n-2}{2}} \frac{d_{g_\xi}(x, \xi)^2 - \delta^2}{(\delta^2 + d_{g_\xi}(x, \xi)^2)^{\frac{n}{2}}}, \tag{17}$$

$$Z_{\delta,\xi,X}(x) := \chi(d_{g_\xi}(x, \xi)) \Lambda_\xi(x) \delta^{\frac{n}{2}} \frac{\langle (\exp_\xi^{g_\xi})^{-1}(x), X \rangle_{g_\xi(\xi)}}{(\delta^2 + d_{g_\xi}(x, \xi)^2)^{\frac{n}{2}}} \tag{18}$$

for all $x \in M$. We let $(U_\gamma)_\gamma$ be as in Definition 2.2. Here and in the sequel, $\exp_\xi^{g_\xi}$ denotes the exponential map at the point $\xi \in M$ with respect to the metric g_ξ. For $\gamma \in \mathcal{C}$, $\xi \in U_\gamma$, and $\delta > 0$, we define

$$Z_{\delta,\xi,j} := Z_{\delta,\xi,e_j(\xi_i)} \text{ for } j = 1, \cdots, n \text{ and } Z_{\delta,\xi,0} := Z_{\delta,\xi}, \tag{19}$$

where the $e_j(\xi)$'s are defined in Definition 2.2: we have omitted the index γ for clearness. Since the isometric assimilation (9) of the tangent space to $\mathbb{R}^n$ is smooth with respect to $\xi \in U_\gamma$, we define

$$\begin{array}{cccc} \tilde{\exp}_\xi^{g_\xi} : & \mathbb{R}^n & \to & M \\ & X & \mapsto & \exp_\xi^{g_\xi}(\Phi_\xi(X)). \end{array} \tag{20}$$

2.6. **Sobolev inequalities.** It follows from Sobolev's Theorem that $D_1^2(\mathbb{R}^n)$ is embedded continuously in $L^{2^\star}(\mathbb{R}^n)$ and that for any $\varphi\in D_1^2(\mathbb{R}^n)$, we have that

$$\|\varphi\|_{2^\star}\leq K(n,2)\|\nabla\varphi\|_2 \text{ with } K(n,2):=2(n\,(n-2)\,\omega_n^{2/n})^{-1/2}. \tag{21}$$

On the compact manifold (M,g), $H_1^2(M)$ is embedded in $L^{2^\star}(M)$ and there exists $A>0$ such that for any $\phi\in H_1^2(M)$, we have that

$$\|\phi\|_{2^\star}\leq A\|\phi\|_{H_1^2}\,. \tag{22}$$

2.7. **Riesz correspondence.** We let $\epsilon_0>0$ be such that for $h\in L^\infty(M)$ such that $\|h-h_0\|_\infty<\varepsilon_0$, we have that $\|h\|_\infty\leq\|h_0\|_\infty+1$ and $\lambda_1(\Delta_g+h)\geq\lambda_1(\Delta_g+h_0)/2>0$. With a slight abuse of notation, we define

$$\left\{\begin{array}{lccc}\Delta_g+h: & H_1^2(M) & \to & (H_1^2(M))'\\ & u & \mapsto & \left(v\mapsto(u,v)_h:=\int_M((\nabla u,\nabla v)_g+huv)\,dv_g\right)\end{array}\right\},$$

and its inverse is denoted as $(\Delta_g+h)^{-1}$. For $\tau\in L^{\frac{2n}{n+2}}(M)$, it follows from Sobolev's Theorem (see (22)) that the map $T_\tau:v\mapsto\int_M\tau v\,dv_g$ is defined and continuous for $v\in H_1^2(M)$: we will then write $(\Delta_g+h)^{-1}(\tau):=(\Delta_g+h)^{-1}(T_\tau)$. It then follows from regularity theory that for $\|h-h_0\|<\varepsilon_0$, we have that

$$\|(\Delta_g+h)^{-1}(\tau)\|_{H_1^2}\leq C(h_0,\varepsilon_0)\|\tau\|_{\frac{2n}{n+2}}\,, \tag{23}$$

where $C(h_0,\varepsilon_0)>0$ depends only on (M,g), $h_0\in L^\infty(M)$, and $\varepsilon_0>0$.

2.8. **Notation.** In the sequel, $C,C_1,C_2,\cdots$ will denote positive constants depending only on (M,g), k, ν_0, θ, C_0, u_0, h_0, f_0, and G_0. We will often use the same notation C or C_i $(i\geq1)$ for different constants from line to line, and even in the same line.

The notation $\omega_{a,b,\cdots}(x)$ will denote a constant depending on $a,b,\cdots$, x, (M,g), k, ν_0, θ, C_0, u_0, h_0, f_0, and G_0 and such that $\lim_{x\to l}\omega_{a,b,\cdots}(x)=0$, where $l\in\{0,+\infty\}$ will be explicit for each statement.

3. Preliminary computations 1: rescaling and pull-back

The objective of this section and the following is to express qualitatively the transfer of the action of $(0,+\infty)\times\mathbb{R}^n$ on $D_1^2(\mathbb{R}^n)$ to an infinitesimal action on $H_1^2(M)$. We fix $\gamma\in\mathcal{C}$, where $\mathcal{C}$ is as in Definition 2.2. We choose a function $F\in C^\infty(M\times M)$ such that $F(\xi,x)=0$ if $d_{g_\xi}(\xi,x)\geq r_0$ for $\xi,x\in M$. For $\varphi\in D_1^2(\mathbb{R}^n)$, we define for $\xi\in U_\gamma$ and $\delta>0$

$$\mathrm{Resc}^F_{\delta,\xi}(\varphi)(x):=F(\xi,x)\delta^{-\frac{n-2}{2}}\varphi\big(\delta^{-1}(\tilde{\exp}_\xi^{g_\xi})^{-1}(x)\big) \tag{24}$$

for all $x\in M$. This transformation is the infinitesimal transfer via the exponential map of the action $(0,+\infty)\times\mathbb{R}^n$ on $D_1^2(\mathbb{R}^n)$ defined in (15). As a preliminary remark, it follows from (19) that

$$Z_{\delta_i,\xi_i,j}=\mathrm{Resc}^{F^{(1)}}_{\delta_i,\xi_i}(V_j) \text{ and } W_{\kappa_i,\xi_i,\delta_i}=\mathrm{Resc}^{F^{(2)}}_{\delta_i,\xi_i}(U_1)+B_{\delta_i,\xi_i}\,, \tag{25}$$

with $F^{(1)}(\xi,x):=\chi(d_{g_\xi}(x,\xi))\Lambda_\xi(x)$, $F^{(2)}(\xi,x):=\kappa_if_0(\xi)^{-(n-2)/4}\chi(d_{g_\xi}(x,\xi))\Lambda_\xi(x)$ for all $\xi,x\in M$.

Proposition 3.1. *For all $\varphi \in D_1^2(\mathbb{R}^n)$, $\delta > 0$, and $\xi \in U_\gamma$, there hold $Resc_{\delta,\xi}(\varphi) \in H_1^2(M)$ and*

$$\|Resc^F_{\delta,\xi}(\varphi)\|_{H_1^2} \leq C_1(F)\|\varphi\|_{D_1^2}\,, \tag{26}$$

where $C_1(F) > 0$ is independent of $\xi \in U_\gamma$, $\delta > 0$, and $\varphi \in D_1^2(\mathbb{R}^n)$. Moreover, for all $\varphi, \psi \in D_1^2(\mathbb{R}^n)$, and for all $\delta, R > 0$ and $\xi \in U_\gamma$, we have that

$$\left|\int_{B^{g_\xi}_{R\delta}(\xi)} \left(\nabla Resc^F_{\delta,\xi}(\varphi), \nabla Resc^F_{\delta,\xi}(\psi)\right)_{g_\xi} dv_{g_\xi} - F(\xi,\xi)^2 \int_{\mathbb{R}^n} (\nabla\varphi, \nabla\psi)_{\mathrm{Eucl}}\, dx\right| \leq \omega_{1,F,\varphi,\psi}(R) + \omega_{2,F,\varphi,\psi}(\delta), \tag{27}$$

$$\int_{M\setminus B^{g_\xi}_{R\delta}(\xi)} |\nabla Resc^F_{\delta,\xi}(\varphi)|^2_{g_\xi}\, dv_{g_\xi} \leq \omega_{1,F,\varphi,\psi}(R) + \omega_{2,\varphi,\psi}(\delta), \tag{28}$$

$$\int_M |Resc^F_{\delta,\xi}(\varphi)|^2\, dv_{g_\xi} \leq \omega_{3,F,\varphi}(\delta), \tag{29}$$

where $\lim_{R\to+\infty} \omega_{1,F,\varphi,\psi}(R) = \lim_{\delta\to 0} \omega_{2,F,\varphi,\psi}(\delta) = \lim_{\delta\to 0} \omega_{3,F,\varphi}(\delta) = 0$.

Proof of Proposition 3.1: We fix $\varphi, \psi \in D_1^2(\mathbb{R}^n)$. We consider a domain $D \subset M$. A change of variable yields

$$\int_D \left(\nabla \mathrm{Resc}^F_{\delta,\xi}(\varphi), \nabla \mathrm{Resc}^F_{\delta,\xi}(\psi)\right)_{g_\xi} dv_{g_\xi} = \int_{D_{\delta,\xi}} (\nabla(\phi_{\delta,\xi}\varphi), \nabla(\phi_{\delta,\xi}\psi))_{g_{\delta,\xi}}\, dv_{g_{\delta,\xi}}\,, \tag{30}$$

where $D_{\delta,\xi} := \delta^{-1}(\tilde{\exp}^{g_\xi}_\xi)^{-1}(D \cap B^{g_\xi}_{r_0}(\xi))$, $g_{\delta,\xi}(x) := ((\tilde{\exp}^{g_\xi}_\xi)^\star g)(\delta x)$ and

$$\phi_{\delta,\xi}(x) := F(\xi, \tilde{\exp}^{g_\xi}_\xi(\delta x))$$

for all $x \in \mathbb{R}^n$. Integrating (30) by parts yields

$$\int_D \left(\nabla \mathrm{Resc}^F_{\delta,\xi}(\varphi), \nabla \mathrm{Resc}^F_{\delta,\xi}(\psi)\right)_{g_\xi} dv_{g_\xi} = \int_{D_{\delta,\xi}} \left(\phi^2_{\delta,\xi}(\nabla\varphi, \nabla\psi)_{g_{\delta,\xi}} + \phi_{\delta,\xi}(\Delta_{g_{\delta,\xi}}\phi_{\delta,\xi})\varphi\psi\right) dv_{g_{\delta,\xi}}\,. \tag{31}$$

Since F is smooth, there exists $C(F) > 0$ such that

$$\begin{cases} |\phi_{\delta,\xi}(x) - \phi_{\delta,\xi}(0)| \leq C(F)\delta|x|\,,\ |\phi_{\delta,\xi}\Delta_{g_{\delta,\xi}}\phi_{\delta,\xi}(x)| \leq C(F)\delta^2, \\ |g_{\delta,\xi}(x) - g_{\delta,\xi}(0)| \leq C(F)\delta\,\mathrm{Eucl}\,,\ \text{and } |dv_{g_{\delta,\xi}}(x) - dx| \leq C(F)\delta\, dx \end{cases} \tag{32}$$

for all $x \in B_{r_0/\delta}(0) \subset \mathbb{R}^n$. Since $\phi_{\delta,\xi}(0) = F(\xi,\xi)$ and $g_{\delta,\xi}(0) = \mathrm{Eucl}$ the Euclidean metric in $\mathbb{R}^n$, plugging (32) into (31) yields

$$\begin{aligned} &\left|\int_D \left(\nabla \mathrm{Resc}^F_{\delta,\xi}(\varphi), \nabla \mathrm{Resc}^F_{\delta,\xi}(\psi)\right)_{g_\xi} dv_{g_\xi} - F(\xi,\xi)^2 \int_{D_{\delta,\xi}} (\nabla\varphi, \nabla\psi)_{\mathrm{Eucl}}\, dx\right| \\ &\leq C(F)\int_{D_{\delta,\xi}} (\delta|\nabla\varphi|\cdot|\nabla\psi| + \delta^2|\varphi|\cdot|\psi|)\, dx \\ &\leq C(F)\delta\|\nabla\varphi\|_2\|\nabla\psi\|_2 + C(F)\sqrt{\delta^2\int_{B_{r_0/\delta}(0)} \varphi^2\, dx}\cdot\sqrt{\delta^2\int_{B_{r_0/\delta}(0)} \psi^2\, dx}\,. \end{aligned} \tag{33}$$

Independently, for any $R>0$, we have that

$$\begin{aligned}\delta^2\int_{B_{r_0/\delta}(0)}\varphi^2\,dx &\leq \delta^2\int_{B_{r_0/\delta}(0)\setminus B_R(0)}\varphi^2\,dx+\delta^2\int_{B_R(0)}\varphi^2\,dx\\ &\leq \delta^2\cdot\left(\int_{B_{r_0/\delta}(0)\setminus B_R(0)}dx\right)^{\frac{2}{n}}\left(\int_{B_{r_0/\delta}(0)\setminus B_R(0)}|\varphi|^{2^\star}\,dx\right)^{\frac{2}{2^\star}}\\ &\quad+\delta^2\cdot\left(\int_{B_R(0)}dx\right)^{\frac{2}{n}}\left(\int_{B_R(0)}|\varphi|^{2^\star}\,dx\right)^{\frac{2}{2^\star}}\\ &\leq Cr_0^2\left(\int_{\mathbb{R}^n\setminus B_R(0)}|\varphi|^{2^\star}\,dx\right)^{\frac{2}{2^\star}}+C\delta^2R^2\left(\int_{\mathbb{R}^n}|\varphi|^{2^\star}\,dx\right)^{\frac{2}{2^\star}}.\end{aligned}\tag{34}$$

Since $\varphi\in D_1^2(\mathbb{R}^n)$, it follows from Sobolev's inequality (21) that $\varphi\in L^{2^\star}(\mathbb{R}^n)$ and

$$\lim_{\delta\to0}\delta^2\int_{B_{r_0/\delta}(0)}\varphi^2\,dx=0\,.\tag{35}$$

As a consequence, for all $\xi\in U_\gamma$, all $\delta>0$, and all domain $D\subset M$, we have that

$$\left|\int_D\left(\nabla\mathrm{Resc}^F_{\delta,\xi}(\varphi),\nabla\mathrm{Resc}^F_{\delta,\xi}(\psi)\right)_{g_\xi}dv_{g_\xi}-F(\xi,\xi)^2\int_{D_{\delta,\xi}}(\nabla\varphi,\nabla\psi)_{\mathrm{Eucl}}\,dx\right|\\ \leq\omega_{4,F,\varphi,\psi}(\delta),\tag{36}$$

where $\lim_{\delta\to0}\omega_{4,F,\varphi,\psi}(\delta)=0$. Taking alernatively $D:=B^{g_\xi}_{R\delta}(\xi)$ or $D:=M\setminus B^{g_\xi}_{R\delta}(\xi)$, and letting $R\to+\infty$ yields (27) and (28). Taking $R=0$ in (34), taking $D:=M$ and $\psi=\varphi$ in (33), and using Sobolev's inequality (21), we get that

$$\int_M\left|\nabla\mathrm{Resc}^F_{\delta,\xi}(\varphi)\right|^2_{g_\xi}dv_{g_\xi}\leq C(F)\int_{\mathbb{R}^n}|\nabla\varphi|^2\,dx+C(F)\left(\int_{\mathbb{R}^n}|\varphi|^{2^\star}\,dx\right)^{\frac{2}{2^\star}}\tag{37}$$

for $\delta<1$. A change of variable and Hölder's inequality yields

$$\int_M\mathrm{Resc}_{\delta,\xi}(\varphi)^2\,dv_{g_\xi}=\delta^2\int_{B_{r_0/\delta}(0)}|\phi_{\delta,\xi}\varphi|^2\,dv_{g_{\delta,\xi}}\leq C(F)\delta^2\int_{B_{r_0/\delta}(0)}\varphi^2\,dx\,.$$

Assertion (29) follows from inequality (11), (35), and the latest inequality. Assertion (26) follows from (37), inequality (11), Sobolev's inequality (21), and (29). ☐

As a consequence, we get the following orthogonality property:

Proposition 3.2. *Let $\varphi,\psi\in D_1^2(\mathbb{R}^n)$ be two functions and $h\in L^\infty(M)$ such that $\|h\|_\infty<C_1$. Then for any $((\delta_i)_i,(\xi_i)_i)\in\mathcal{D}_k(\varepsilon,N)$, we have that*

$$\left|(Resc^F_{\delta_i,\xi_i}(\varphi),Resc^F_{\delta_j,\xi_j}(\psi))_h-\delta_{i,j}F(\xi,\xi)\int_{\mathbb{R}^n}(\nabla\varphi,\nabla\psi)\,dx\right|\leq\omega_{5,F,C_1,\varphi,\psi}(\varepsilon,N)$$

for all $i,j\in\{1,..,k\}$, where $\lim_{\varepsilon\to0\,;\,N\to+\infty}\omega_{5,F,C_1,\varphi,\psi}(\varepsilon,N)=0$. Here, $\delta_{i,j}=1$ if $i=j$ and 0 otherwise.

Proof of Proposition 3.2: We let $R>0$ be a positive number. We have that

$$\left|\int_M (\nabla \mathrm{Resc}^F_{\delta_i,\xi_i}(\varphi), \nabla \mathrm{Resc}^F_{\delta_j,\xi_j}(\psi))_g\, dv_g\right| \leq \int_{M\setminus B^g_{R\delta_i}(\xi_i)} \cdots + \int_{M\setminus B^g_{R\delta_j}(\xi_j)} \cdots + \int_{B^g_{R\delta_i}(\xi_i)\cap B^g_{R\delta_j}(\xi_j)} \cdots \tag{38}$$

It follows from (11), and assertions (28) and (26) of Proposition 3.1 that

$$\begin{aligned}\int_{M\setminus B^g_{R\delta_i}(\xi)} &\left|(\nabla \mathrm{Resc}^F_{\delta_i,\xi_i}(\varphi), \nabla \mathrm{Resc}^F_{\delta_j,\xi_j}(\psi))_g\right| dv_g \\ &\leq \sqrt{\int_{M\setminus B^{g_\xi}_{R\delta_i}(\xi)} |\nabla \mathrm{Resc}^F_{\delta_i,\xi_i}(\varphi)|^2_{g_\xi}\, dv_{g_\xi}} \cdot \|\mathrm{Resc}^F_{\delta_j,\xi_j}(\psi)\|_{H_1^2} \leq \omega_{6,F,\varphi,\psi}(R),\end{aligned} \tag{39}$$

where $\lim_{R\to+\infty}\omega_{6,F,\varphi,\psi}(R)=0$.
We first assume that $i\neq j$. If $B^g_{R\delta_i}(\xi_i)\cap B^g_{R\delta_j}(\xi_j)=\emptyset$, we get (42) from (38) and (39). We assume that $B^g_{R\delta_i}(\xi_i)\cap B^g_{R\delta_j}(\xi_j)\neq\emptyset$ and $i\neq j$. Then we have that $d_g(\xi_i,\xi_j)<R(\delta_i+\delta_j)$. Since $((\delta_i)_i,(\xi_i)_i)\in\mathcal{D}_k(\varepsilon,N)$, exchanging i and j if necessary, we then get that for N large enough and ε small enough that

$$\frac{\delta_i}{\delta_j} < \frac{2(1+R^2)}{N}\,. \tag{40}$$

Therefore, using (26), we get that

$$\begin{aligned}\int_{B^g_{R\delta_i}(\xi_i)\cap B^g_{R\delta_j}(\xi_j)} &\left|(\nabla \mathrm{Resc}^F_{\delta_i,\xi_i}(\varphi), \nabla \mathrm{Resc}^F_{\delta_j,\xi_j}(\psi))_g\right| dv_g \\ &\leq \|\mathrm{Resc}^F_{\delta_i,\xi_i}(\varphi)\|_{H_1^2(M)}\cdot\sqrt{\int_{B^{g_{\xi_j}}_{R\delta_i}(\xi_i)\cap B^{g_{\xi_j}}_{R\delta_j}(\xi_j)} |\nabla \mathrm{Resc}^F_{\delta_j,\xi_j}(\psi)|_{g_{\xi_j}}\, dv_{\xi_j}} \\ &\leq C(F)\|\varphi\|_{D_1^2}\cdot\sqrt{\int_{\delta_j^{-1}\tilde{\exp}^{-1}_{\xi_j}(B^{g_{\xi_j}}_{R\delta_i}(\xi_i))\cap B_R(0)} |\nabla\psi|^2_{\mathrm{Eucl}}\, dx}\,.\end{aligned} \tag{41}$$

Via Lebesque's theorem, it follows from (40) that for $R>0$ fixed, the right-hand-side above is as small as desired for $N>0$ large. Plugging together (38), (39), and (41), we get that for $i\neq j$,

$$\left|\int_M (\nabla \mathrm{Resc}^F_{\delta_i,\xi_i}(\varphi), \nabla \mathrm{Resc}^F_{\delta_j,\xi_j}(\psi))_g\, dv_g\right| \leq \omega_{7,F,\varphi,\psi}(N), \tag{42}$$

where $\lim_{N\to+\infty}\omega_{7,F,\varphi,\psi}(N)=0$.
We now assume that $i=j$. For $R>0$ fixed, we have that $|g_{\xi_i}-g|\leq C(R)\delta_i g$ on $B_{R\delta_i}(\xi_i)$ since $\Lambda_{\xi_i}(\xi_i)=1$. We then get with (26) that

$$\begin{aligned}&\left|\int_{B^{g_{\xi_i}}_{R\delta_i}(\xi_i)} (\nabla \mathrm{Resc}^F_{\delta_i,\xi_i}(\varphi), \nabla \mathrm{Resc}^F_{\xi_i,\delta_i}(\psi))_g\, dv_g\right. \\ &\qquad\left. - \int_{B^{g_{\xi_i}}_{R\delta_i}(\xi_i)} (\nabla \mathrm{Resc}^F_{\delta_i,\xi_i}(\varphi), \nabla \mathrm{Resc}^F_{\xi_i,\delta_i}(\psi))_{g_{\xi_i}}\, dv_{g_{\xi_i}}\right| \\ &\qquad\leq C(F,R)\delta_i\|\mathrm{Resc}^F_{\delta_i,\xi_i}(\varphi)\|_{H_1^2}\|\mathrm{Resc}^F_{\delta_j,\xi_j}(\psi)\|_{H_1^2}\leq C(F,R,\varphi,\psi)\delta_i\,.\end{aligned} \tag{43}$$

Proposition 3.2 then follows from (42), (39), (27), (29), and (43). □

As a corollary, we get an orthogonality property for the $Z_{\delta_i,\xi_i,j}$'s defined in (19):

Corollary 3.3. *Let $h\in L^\infty(M)$ be such that $\|h\|_\infty\leq\tilde{C}_1$. For any $i,i'\in\{1,..,k\}$ and any $j,j'\in\{0,..,n\}$, we have that*

$$\left|(Z_{\delta_i,\xi_i,j},Z_{\delta_{i'},\xi_{i'},j'})_h-\delta_{i,i'}\delta_{j,j'}\|\nabla V_j\|_2^2\right|\leq\omega_{8,\tilde{C}_1}(\varepsilon,N),$$

where $\lim_{\varepsilon\to 0\,,\,N\to+\infty}\omega_{8,\tilde{C}_1}(\varepsilon,N)=0$. Here, $\delta_{i,i'}=1$ if $i=i'$ and 0 otherwise.

Proof of Corollary 3.3: Taking $F(\xi,x):=\chi(d_{g_\xi}(x,\xi))\Lambda_\xi(x)$, the corollary is a direct consequence of (25), Proposition 3.4 above, and the fact that the V_j's form an orthogonal family of $D_1^2(\mathbb{R}^n)$. □

We now deal with the nonlinear interactions of different rescalings:

Proposition 3.4. *Let $\varphi,\psi\in D_1^2(\mathbb{R}^n)$ be two functions. Then for any $i\neq j\in\{1,..,k\}$ and all $r,s\geq 0$ such that $1\leq r+s\leq 2^\star$, we have that*

$$\int_M|Resc^F_{\delta_i,\xi_i}(\varphi)|^r|Resc^F_{\delta_j,\xi_j}(\varphi)|^s\,dv_g\leq\omega_{9,F,\varphi,\psi}(\varepsilon,N),\tag{44}$$

where $\lim_{\varepsilon\to 0\,,\,N\to+\infty}\omega_{9,F,\varphi,\psi}(\varepsilon,N)=0$.

Proof of Proposition 3.4: We let $R>0$ be a positive number. We have that

$$\int_M|\mathrm{Resc}^F_{\delta_i,\xi_i}(\varphi)|^r|\mathrm{Resc}^F_{\delta_j,\xi_j}(\psi)|^s\,dv_g\leq\int_{M\setminus B^{g_{\xi_i}}_{R\delta_i}(\xi_i)}\cdots+\int_{M\setminus B^{g_{\xi_j}}_{R\delta_j}(\xi_j)}\cdots+\int_{B^{g_{\xi_i}}_{R\delta_i}(\xi_i)\cap B^{g_{\xi_j}}_{R\delta_j}(\xi_j)}\cdots\tag{45}$$

It follows from (11), Hölder's inequality, (26), and Sobolev's embedding (22) that

$$\begin{aligned}&\int_{M\setminus B^{g_{\xi_i}}_{R\delta_i}(\xi_i)}|\mathrm{Resc}^F_{\delta_i,\xi_i}(\varphi)|^r|\mathrm{Resc}^F_{\delta_j,\xi_j}(\psi)|^s\,dv_g\\&\leq(Vol_g(M))^{\frac{2^\star-(r+s)}{2^\star}}\left(\int_{M\setminus B^{g_{\xi_i}}_{R\delta_i}(\xi_i)}|\mathrm{Resc}^F_{\delta_i,\xi_i}(\varphi)|^{2^\star}\,dv_{g_\xi}\right)^{\frac{r}{2^\star}}\cdot\|\mathrm{Resc}^F_{\delta_j,\xi_j}(\psi)\|^s_{2^\star}\\&\leq(Vol_g(M))^{\frac{2^\star-(r+s)}{2^\star}}\left(\int_{\mathbb{R}^n\setminus B_R(0)}|\varphi|^{2^\star}\,dx\right)^{\frac{r}{2^\star}}\cdot C(F)\|\mathrm{Resc}^F_{\delta_j,\xi_j}(\psi)\|^s_{H_1^2}\end{aligned}$$

$$\leq C(F)\cdot(1+Vol_g(M))\cdot\left(\int_{\mathbb{R}^n\setminus B_R(0)}|\varphi|^{2^\star}\,dx\right)^{\frac{r}{2^\star}}\cdot\|\psi\|^s_{D_1^2}\leq\omega_{10,F,\varphi,\psi}(R),\tag{46}$$

where $\lim_{R\to+\infty}\omega_{10,F,\varphi,\psi}(R)=0$.

We now assume that $B^{g_{\xi_i}}_{R\delta_i}(\xi_i)\cap B^{g_{\xi_j}}_{R\delta_j}(\xi_j)\neq\emptyset$ and $i\neq j$. Then we have that $d_g(\xi_i,\xi_j)<C_1R(\delta_i+\delta_j)$, where $C_1>0$. Since $((\delta_i)_i,(\xi_i)_i)\in\mathcal{D}_k(\varepsilon,N)$, up to exchanging i and j if necessary, we then get that

$$\frac{\delta_i}{\delta_j}<\frac{2(1+C_1^2R^2)}{N}\,.\tag{47}$$

Therefore, using the comparison between g_{ξ_i} and g_{ξ_j} given by (11), we get that

$$\begin{aligned}
&\int_{B^{g_{\xi_i}}_{R\delta_i}(\xi_i)\cap B^{g_{\xi_j}}_{R\delta_j}(\xi_j)} |\mathrm{Resc}^F_{\delta_i,\xi_i}(\varphi)|^r |\mathrm{Resc}^F_{\delta_j,\xi_j}(\psi)|^s\, dv_g \\
&\leq (Vol_g(M))^{\frac{2^\star-(r+s)}{2^\star}} \|\mathrm{Resc}^F_{\delta_i,\xi_i}(\varphi)\|^r_{L^{2^\star}(M)} \\
&\quad\times\left(\int_{B^{g_{\xi_j}}_{C_8R\delta_i}(\xi_i)\cap B^{g_{\xi_j}}_{R\delta_j}(\xi_j)} |\mathrm{Resc}^F_{\delta_j,\xi_j}(\psi)|^{2^\star}\, dv_{g_\xi}\right)^{\frac{s}{2^\star}} \\
&\leq C(F)\|\mathrm{Resc}^F_{\delta_i,\xi_i}(\varphi)\|_{H^2_1}\cdot\left(\int_{\delta_j^{-1}\tilde{\exp}^{-1}_{\xi_j}(B^{g_{\xi_j}}_{C_2R\delta_i}(\xi_i))\cap B_R(0)} |\psi|^{2^\star}\, dv_{g_\xi}\right)^{\frac{s}{2^\star}} \\
&\leq C(F)\|\varphi\|_{D^2_1}\cdot\left(\int_{\delta_j^{-1}\tilde{\exp}^{-1}_{\xi_j}(B^{g_{\xi_j}}_{C_2R\delta_i}(\xi_i))\cap B_R(0)} |\psi|^{2^\star}\, dv_{g_\xi}\right)^{\frac{s}{2^\star}}. \qquad (48)
\end{aligned}$$

Via Lebesque's theorem, it follows from (47) that for $R>0$ fixed, the right-hand-side above is as small as desired for $N>0$ large. Plugging (46) and (48) into (45) yields (44). This ends the proof of Proposition 3.4. □

The last tool introduced here is the inverse rescaling. Let $\tilde{F}\in C^\infty(M\times\mathbb{R}^n)$ be such that $\tilde{F}(\xi,z)=0$ if $|z|\geq r_0$. Let $\phi\in H^2_1(M)$ be a function. For $\xi\in U_\gamma$ and $\delta>0$, we define

$$\tilde{\mathrm{Resc}}^{\tilde{F}}_{\delta,\xi}(\phi)(x) := \tilde{F}(\xi,\delta|x|)\delta^{\frac{n-2}{2}}\phi\circ\tilde{\exp}_\xi(\delta x) \qquad (49)$$

for all $x\in\mathbb{R}^n$.

Proposition 3.5. *For any $\phi\in H^2_1(M)$, $\xi\in U_\gamma$, and $\delta>0$, then $\tilde{Resc}^{\tilde{F}}_{\delta,\xi}(\phi)\in D^2_1(\mathbb{R}^n)$. In addition, if $\|h\|_\infty\leq\tilde{C}_1$, then*

$$\|\tilde{Resc}^{\tilde{F}}_{\delta,\xi}(\phi)\|_{D^2_1}\leq C(\tilde{C}_1)\|\phi\|_{H^2_1(M)}.$$

Proof of Proposition 3.5: By density, it is enough to prove the result for $\phi\in C^\infty(M)$. Then, $\tilde{\mathrm{Resc}}^{\tilde{F}}_{\delta,\xi}(\phi)\in C^\infty_c(\mathbb{R}^n)$. A change of variable yields

$$\begin{aligned}
&\int_{\mathbb{R}^n}|\nabla\tilde{\mathrm{Resc}}^{\tilde{F}}_{\delta,\xi}(\phi)|^2_{\mathrm{Eucl}}\,dx \\
&\qquad = \int_{B^{g_\xi}_{r_0}(\xi)}|\nabla(\tilde{F}(\xi,(\tilde{\exp}^{g_\xi}_\xi)^{-1}))\phi|^2_{(\tilde{\exp}^{g_\xi}_\xi)^{-1})^\star\,\mathrm{Eucl}}\,dv_{(\tilde{\exp}^{g_\xi}_\xi)^{-1})^\star\,\mathrm{Eucl}}.
\end{aligned}$$

Since $F\in C^\infty(M\times\mathbb{R}^n)$ and (11) holds, we have that $(\tilde{\exp}^{g_\xi}_\xi)^{-1})^\star\,\mathrm{Eucl}\leq Cg$ and

$$\int_{\mathbb{R}^n}|\nabla\tilde{\mathrm{Resc}}^{\tilde{F}}_{\delta,\xi}(\phi)|^2_{\mathrm{Eucl}}\,dx\leq C(\tilde{F})\|\phi\|^2_{H^2_1(M)}$$

for all $\phi\in C^\infty(M)$. Proposition 3.5 then follows by density. □

4. Preliminary computations 2: estimates of derivatives

This section is devoted to the proof of the following estimates:

Proposition 4.1. *For $\gamma \in \mathcal{C}$, for any $\xi \in U_\gamma$ and $\delta > 0$, we have that*

$$\partial_\delta W_{\kappa,\delta,\xi} = \kappa \frac{n-2}{2}\left(\frac{n(n-2)}{f_0(\xi)}\right)^{\frac{n-2}{4}} \cdot \frac{1}{\delta} \cdot \left(Z_{\xi,\delta,0} + o(1)\right), \tag{50}$$

$$\partial_{(\xi)_j} W_{\kappa,\delta,\xi} = \kappa \frac{n-2}{2}\left(\frac{n(n-2)}{f_0(\xi)}\right)^{\frac{n-2}{4}} \cdot \frac{1}{\delta} \cdot \left(Z_{\xi,\delta,j} + o(1)\right) \tag{51}$$

for all $j = 1, \cdots, n$, where $\|o(1)\|_{H_1^2} \leq \omega_{11}(\delta)$ and $\lim_{\delta\to 0}\omega_{11}(\delta) = 0$. Moreover, we have that

$$\delta\|\partial_\delta Z_{\xi,\delta,j}\|_{H_1^2} \leq C \text{ and } \delta\|\nabla_\xi Z_{\xi,\delta,j}\|_{H_1^2} \leq C\,, \tag{52}$$

where $C > 0$ is independent of $\xi \in U_\gamma$ and $\delta \in (0,1)$. The partial derivatives along the center $\xi \in U_\gamma$ in (51) *are defined in* (56) *below.*

In other words, the differentiation of the rescaling along $(0,+\infty)\times M$ is essentially the rescaling of the differentiation of U_1 along the Lie algebra of $(0,+\infty)\times\mathbb{R}^n$ for the action (15).

Proof of Proposition 4.1: Straightforward computations yield

$$\partial_\delta W_{\kappa,\delta,\xi} = \kappa \frac{n-2}{2}\left(\frac{n(n-2)}{f_0(\xi)}\right)^{\frac{n-2}{4}} \cdot \frac{1}{\delta} \cdot Z_{\xi,\delta,0} + \partial_\delta B_{\delta,\xi}\,, \tag{53}$$

$$\partial_\delta Z_{\xi,\delta,j} = \frac{1}{\delta} \cdot \mathrm{Resc}_{\delta,\xi}^{F^{(1)}}(\Phi_j) \text{ for all } j = 0, \cdots, n, \tag{54}$$

where $F^{(1)}(\xi,x) := \chi(d_{g_\xi}(x,\xi))\Lambda_\xi(x)$ for $\xi, x \in M$ and $\Phi_j \in D_1^2(\mathbb{R}^n)$ are such that

$$\Phi_0(x) := \frac{\frac{n-2}{2}|x|^4 - (n+2)|x|^2 + \frac{n-2}{2}}{(1+|x|^2)^{\frac{n+2}{2}}} \text{ and } \Phi_j(x) := \frac{n(|x|^2-1)x_j}{2(1+|x|^2)^{\frac{n+2}{2}}} \text{ for } j = 1, .., n$$

for all $x \in \mathbb{R}^n$. It then follows from (13), (53), (26), (54) that (50) and the first inequality of (52) hold.

We now focus on the derivatives along the center ξ. Since the $W_{\delta,\xi}$'s and the $Z_{\delta,\xi,j}$'s enjoy the same representation (25), we work with the function

$$\mathcal{W}_{\delta,\xi}(x) := \Psi(\xi,x)\delta^{-\frac{n-2}{2}} V\big(\delta^{-1}(\tilde{\exp}_\xi^{g_\xi})^{-1}(x)\big)$$

for all $x \in M$, where $V \in D_1^2(\mathbb{R}^n)$ is such that $\partial_j V \in D_1^2(\mathbb{R}^n)$ for all $j = 1, \cdots, n$ and $\Psi \in C^\infty(U_\gamma \times M)$ is such that $\Psi(\xi,x) = 0$ if $d_{g_\xi}(x,\xi) \geq r_0$. For $\vec{\tau} \in \mathbb{R}^n$, we define $\vec{\tau}(t) := \tilde{\exp}_\xi^{g_\xi}(t\vec{\tau})$, and we consider

$$\mathcal{W}_{\delta,\vec{\tau}(t)} := \Psi(\vec{\tau}(t),\cdot)\delta^{-\frac{n-2}{2}} V\left(\delta^{-1}\Theta_\xi\right), \tag{55}$$

where $\Theta_\xi := (\tilde{\exp}_\xi^{g_\xi})^{-1}$ with definition (20). We then define

$$\partial_{(\xi)_j}\mathcal{W}_{\delta,\xi} := \frac{d}{dt}\left(\mathcal{W}_{\delta,\vec{\tau}(t)}\right)_{|t=0} \tag{56}$$

for $\vec{\tau} = (0, \cdots, 0, 1, 0, \cdots, 0)$ being the j^{th} vector of the canonical basis of $\mathbb{R}^n$. Straightforward computations yield

$$\frac{d}{dt}\left(\mathcal{W}_{\delta,\vec{\tau}(t)}\right)_{|t=0} = \mathrm{Resc}_{\delta,\xi}^{\Psi_0}(V) + \sum_{k=1}^n \delta^{-1}\mathrm{Resc}_{\delta,\xi}^{\Psi_k}(\partial_k V),$$

where $\Psi_0(\xi,x) := d\Psi_{(\xi,x)}(d(\tilde{\exp}_\xi^{g_\xi})_0(\vec{\tau}),0)$ and $\Psi_k(\xi,x) := \Psi(\xi,x)\frac{d}{dt}\left(\Theta_{\vec{\tau}(t)}(x)\right)^k_{|t=0}$ for $k=1,..,n$ and $\xi\in U_\gamma$, $x\in M$. We define

$$\tilde{\Psi}_k(\xi,x) := \Psi_k(\xi,x) - \Psi_k(\xi,\xi)F^{(1)}(\xi,x), \text{ where } F^{(1)}(\xi,x) := \chi(d_{g_\xi}(x,\xi))\Lambda_\xi(x)$$

for all $k=1,\cdots,n$ and $\xi,x\in M$. We then have that

$$\frac{d}{dt}\left(\mathcal{W}_{\delta,\vec{\tau}(t)}\right)_{|t=0} = \sum_{k=1}^n \delta^{-1}\Psi_k(\xi,\xi)\mathrm{Resc}_{\delta,\xi}^{F^{(1)}}(\partial_k V) + \mathrm{Resc}_{\delta,\xi}^{\Psi_0}(V) + \sum_{k=1}^n \delta^{-1}\mathrm{Resc}_{\delta,\xi}^{\tilde{\Psi}_k}(\partial_k V).$$

Since $\tilde{\Psi}_k(\xi,\xi)=0$, it follows from Proposition 3.1 that

$$\left\|\frac{d}{dt}\left(\mathcal{W}_{\delta,\vec{\tau}(t)}\right)_{|t=0} - \sum_{k=1}^n \delta^{-1}\Psi_k(\xi,\xi)\mathrm{Resc}_{\delta,\xi}^{F^{(1)}}(\partial_k V)\right\|_{H_1^2} \leq \delta^{-1}\omega_{12,V}(\delta), \tag{57}$$

where $\lim_{\delta\to 0}\omega_{12}(\delta)=0$. We are left with computing $\Psi_k(\xi,\xi)$. We define $X(t) := \Theta_{\vec{\tau}(t)}(\xi)$ and for t small. In particular, X and ξ are smooth with respect to t small. The definition of Θ, Taylor expansions, and the fact that $X(0)=0$ yield

$$\begin{aligned}
0 &= \left(\tilde{\exp}_\xi^{g_\xi}\right)^{-1}\circ\tilde{\exp}_{\vec{\tau}(t)}^{g_{\vec{\tau}(t)}}(X(t)) = \left(\tilde{\exp}_\xi^{g_\xi}\right)^{-1}\circ\tilde{\exp}_{\vec{\tau}(t)}^{g_{\vec{\tau}(t)}}(tX'(0)+o(t))\\
&= \left(\tilde{\exp}_\xi^{g_\xi}\right)^{-1}\circ\tilde{\exp}_{\vec{\tau}(t)}^{g_{\vec{\tau}(t)}}(0) + td\Big(\left(\tilde{\exp}_\xi^{g_\xi}\right)^{-1}\circ\tilde{\exp}_{\vec{\tau}(t)}^{g_{\vec{\tau}(t)}}\Big)_0(X'(0)) + o(t)\\
&= \left(\tilde{\exp}_\xi^{g_\xi}\right)^{-1}(\vec{\tau}(t)) + td\Big(\left(\tilde{\exp}_\xi^{g_\xi}\right)^{-1}\circ\tilde{\exp}_\xi^{g_\xi}\Big)_0(X'(0)) + o(t)\\
&= t\vec{\tau} + tX'(0) + o(t)
\end{aligned}$$

when $t\to 0$. Therefore $X'(0) = -\vec{\tau}$. Since $\vec{\tau} = (0,\cdots,0,1,0,\cdots,)$ (the j^{th} vector), we then get that $\Psi_k(\xi,\xi) = -\Psi(\xi,\xi)$ if $k=j$ and 0 otherwise. Then (57) rewrites

$$\left\|\frac{d}{dt}\left(\mathcal{W}_{\delta,\tilde{\exp}_\xi^{g_\xi}(t\vec{\tau})}\right)_{|t=0} - \delta^{-1}\Psi(\xi,\xi)\mathrm{Resc}_{\delta,\xi}^{F^{(1)}}(\partial_j V)\right\|_{H_1^2} \leq \delta^{-1}\omega_{12,V}(\delta), \tag{58}$$

where $\lim_{\delta\to 0}\omega_{12}(\delta)=0$. The assertion (51) and the second assertion of (52) follow from the expressions (25) and (55), and from (13) and (58). □

5. Inversion and fixed-point argument

For $((\delta_i)_i,(\xi_i)_i)\in(0,+\infty)^k\times M^k$, we define

$$K_{(\delta_i)_i,(\xi_i)_i} := \mathrm{Span}\left\{Z_{\delta_i,\xi_i}\,;\,Z_{\delta_i,\xi_i,\omega_i}\,;\,\varphi\,/\,i=1,\cdots,k\,,\,\omega_i\in T_{\xi_i}M\,,\,\varphi\in K_0\right\}.$$

We let $\{\varphi_1,\cdots,\varphi_d\}$ be an orthonormal basis of K_0 for $(\cdot,\cdot)_{h_0}$. We then have that

$$K_{(\delta_i)_i,(\xi_i)_i} = \mathrm{Span}\left\{Z_{\delta_i,\xi_i,j}\,;\,\varphi_l\,/\,i=1,\cdots,k\,,\,j=0,\cdots,n\,,\,l=1,\cdots,d\right\}. \tag{59}$$

It follows from (25), (26), and (29) that the $Z_{\delta_i,\xi_i,j}$'s go weakly to 0 in $H_1^2(M)$ when $\delta_i\to 0$ uniformly with respect to $\xi_i\in U_{\gamma(i)}$. It follows from Corollary 3.3 that for $\varepsilon>0$ small enough and $N>0$ large enough, the $Z_{\delta_i,\xi_i,j}$'s ($i=1,\cdots,k$ and $j=0,\cdots,n$) form an "almost" orthogonal family. Therefore, the generating family in (59) is "almost" orthogonal for $(\delta_i)_i,(\xi_i)_i\in\mathcal{D}_k(\varepsilon,N)$ for $\varepsilon>0$ small and $N>0$ large, and therefore, $\dim_{\mathbb{R}}K_{(\delta_i)_i,(\xi_i)_i} = k(n+1)+d$. We define $K_{(\delta_i)_i,(\xi_i)_i}^\perp$ as the orthogonal of $K_{(\delta_i)_i,(\xi_i)_i}$ in $H_1^2(M)$ for the scalar product $(\cdot,\cdot)_h$.

We define $\Pi_{K(\delta_i)_i,(\xi_i)_i} : H_1^2(M) \to H_1^2(M)$ and $\Pi_{K^\perp_{(\delta_i)_i,(\xi_i)_i}} : H_1^2(M) \to H_1^2(M)$ respectively as the orthogonal projection on $K_{(\delta_i)_i,(\xi_i)_i}$ and $K^\perp_{(\delta_i)_i,(\xi_i)_i}$ with respect to the scalar product $(\cdot,\cdot)_h$. As easily checked, $v \in H_1^2(M)$ is a solution to (1) iff

$$\begin{aligned} &\Pi_{K^\perp_{(\delta_i)_i,(\xi_i)_i}} \left(v - (\Delta_g + h)^{-1}(F'(v))\right) = 0 \\ &\text{and } \Pi_{K_{(\delta_i)_i,(\xi_i)_i}} \left(v - (\Delta_g + h)^{-1}(F'(v))\right) = 0\,. \end{aligned} \tag{60}$$

In this section, we solve the first equation of (60):

Proposition 5.1. *Under the hypotheses of Theorem 1.1, there exists $N > 0$ and $\varepsilon > 0$ such that for any $h \in L^\infty(M)$, $f \in C^0(M)$, $G \in C^{2,\theta}_{loc}(H_1^2(M))$, and $u \in C^1(B_1(0), H_1^2(M))$ such that (6), (7), and (8) hold, there exists $\phi \in C^1(B_\varepsilon(0) \times \mathcal{D}_k(\varepsilon, N), H_1^2(M))$ such that*

$$u(z,(\delta_i)_i,(\xi_i)_i) := u(z) + \sum_{i=1}^k W_{\kappa_i,\delta_i,\xi_i} + \phi(z,(\delta_i)_i,(\xi_i)_i)$$

is a solution to

$$\Pi_{K^\perp_{(\delta_i)_i,(\xi_i)_i}} \left(u(z,(\delta_i)_i,(\xi_i)_i) - (\Delta_g + h)^{-1}(F'(u(z,(\delta_i)_i,(\xi_i)_i)))\right) = 0 \tag{61}$$

for all $(z,(\delta_i)_i,(\xi_i)_i) \in B_\varepsilon(0) \times \mathcal{D}_k(\varepsilon, N)$. In addition, we have that

$$\phi(z,(\delta_i)_i,(\xi_i)_i) \in K^\perp_{(\delta_i)_i,(\xi_i)_i} \text{ and } \|\phi(z,(\delta_i)_i,(\xi_i)_i)\|_{H_1^2} \le C \cdot R(z,(\delta_i)_i,(\xi_i)_i),$$

where C is a constant depending on (M,g), k, ν_0, θ, C_0, u_0, h_0, f_0, and G_0. The remainder $R(z,(\delta_i)_i,(\xi_i)_i)$ is defined in (5). Moreover, we have that

$$R(z,(\delta_i)_i,(\xi_i)_i) \le \omega_{13}(\varepsilon, N)$$

for all $z \in B_\varepsilon(0)$ and $((\delta_i)_i,(\xi_i)_i) \in \mathcal{D}_k(\varepsilon, N)$, where $\lim_{\varepsilon\to 0, N\to+\infty} \omega_{13}(\varepsilon, N) = 0$.

5.1. **Inversion of the linearized operator.**

Proposition 5.2. *Under the hypotheses of Theorem 1.1, there exists $c > 0$, there exists $N > 0$ and $\varepsilon > 0$ such that for $h \in L^\infty(M)$, $f \in C^0(M)$, $G \in C^{2,\theta}_{loc}(H_1^2(M))$, and $u \in C^1(B_1(0), H_1^2(M))$ such that (6), (7), and (8) hold, then there exists $c > 0$ such that for any $z \in B_\varepsilon(0)$ and $((\delta_i)_i,(\xi_i)_i) \in \mathcal{D}_k(\varepsilon, N)$, we have that*

$$\|L_{z,(\delta_i)_i,(\xi_i)_i}(\varphi)\|_{H_1^2} \ge c\|\varphi\|_{H_1^2} \tag{62}$$

for all $\varphi \in H_1^2(M)$, where

$$L_{z,(\delta_i)_i,(\xi_i)_i)} : \left\{ \begin{array}{ccc} K^\perp_{(\delta_i)_i,(\xi_i)_i} & \to & K^\perp_{(\delta_i)_i,(\xi_i)_i} \\ \varphi & \mapsto & \Pi_{K^\perp_{(\delta_i)_i,(\xi_i)_i}} \left(\varphi - (\Delta_g + h)^{-1}(F''(u(z,(\delta_i)_i,(\xi_i)_i)\varphi)\right) \end{array} \right\}.$$

In particular, $L_{z,(\delta_i)_i,(\xi_i)_i)}$ is a bi-continuous isomorphism.

Proof of Proposition 5.2: We prove (62) by contradiction. We assume that there exist $(q_\alpha)_\alpha \in (2, 2^\star]$, $(h_\alpha)_\alpha \in L^\infty(M)$, $(f_\alpha)_\alpha \in C^0(M)$, $(z_\alpha)_\alpha \in B_1(0)$, $(u_\alpha)_\alpha \in C^1(B_1(0); H_1^2(M))$, $(G_\alpha)_\alpha \in C^{2,\theta}_{loc}(H_1^2(M))$, $(\delta_{i,\alpha})_\alpha$, and $(\xi_{i,\alpha})_\alpha$ for $i = 1, \cdots, k$ and $(\phi_\alpha)_\alpha \in H_1^2(M)$ such that

$$\lim_{\alpha\to+\infty} \|h_\alpha - h_0\|_\infty + \|f_\alpha - f_0\|_{C^0} + d_{C^{2,\theta}_B}(G_\alpha, G_0) = 0\,, \tag{63}$$

$$\lim_{\alpha\to+\infty}\delta_{i,\alpha}=0\,,\ \lim_{\alpha\to+\infty}\delta_{i,\alpha}^{2^\star-q_\alpha}=1\,,\ \lim_{\alpha\to+\infty}z_\alpha=0\,,\ \lim_{\alpha\to+\infty}q_\alpha=2^\star\,, \tag{64}$$

$$u_\alpha(0)=u_0\,,\ \|u_\alpha\|_{C^1(B_1(0),H_1^2)}\le C_0\,,\ f_0(\xi_{i,\alpha})\ge\nu_0 \text{ for all } i=1,\cdots,k, \tag{65}$$

$$\lim_{\alpha\to+\infty}\left(\frac{\delta_{i,\alpha}}{\delta_{j,\alpha}}+\frac{\delta_{j,\alpha}}{\delta_{i,\alpha}}+\frac{d_g(\xi_{i,\alpha},\xi_{j,\alpha})^2}{\delta_{i,\alpha}\delta_{j,\alpha}}\right)=+\infty\,, \tag{66}$$

$$\|\phi_\alpha\|_{H_1^2}=1\,,\ \phi_\alpha\in K_\alpha^\perp\,, \tag{67}$$

and

$$L_\alpha(\phi_\alpha)=o(1), \tag{68}$$

where $\lim_{\alpha\to+\infty}o(1)=0$ in $H_1^2(M)$ and

$$L_\alpha:=L_{z_\alpha,(\delta_{i,\alpha})_i,(\xi_{i,\alpha})_i} \text{ and } K_\alpha:=K_{(\delta_{i,\alpha})_i,(\xi_{i,\alpha})_i}\,.$$

In the sequel, all convergences are with respect to a subsequence of α. It follows from the boundedness of $(\phi_\alpha)_\alpha$ in (67) that there exists $\phi\in H_1^2(M)$ such that

$$\phi_\alpha\rightharpoonup\phi \text{ weakly in } H_1^2(M) \text{ when } \alpha\to+\infty\,. \tag{69}$$

It follows from (68) that there exist $(\lambda_\alpha^{ij})_\alpha\in\mathbb{R}$ and $(\mu_\alpha^l)_\alpha\in\mathbb{R}$ for $i\in\{1,\cdots,k\}$, $j\in\{0,\cdots,n\}$, and $l\in\{1,\cdots,d\}$ such that

(70)
$$\phi_\alpha-(\Delta_g+h_\alpha)^{-1}\Big(F_\alpha''\Big(u_\alpha(z_\alpha)+\sum_{i=1}^k W_{i,\alpha}\Big)\phi_\alpha\Big)=o(1)+\sum_{i,j}\lambda_\alpha^{ij}Z_{i,j,\alpha}+\sum_{l=1}^d\mu_\alpha^l\varphi_l\,,$$

where $\lim_{\alpha\to0}o(1)=0$ in $H_1^2(M)$ and

$$F_\alpha(v):=\frac{1}{q_\alpha}\int_M f_\alpha H(v)^{q_\alpha}\,dv_g+G_\alpha(v) \tag{71}$$

for all $v\in H_1^2(M)$ and

$$W_{i,\alpha}:=W_{\kappa_i,\delta_{i,\alpha},\xi_{i,\alpha}} \text{ and } Z_{i,j,\alpha}:=Z_{\delta_{i,\alpha},\xi_{i,\alpha},j} \text{ for all } i\in\{1,\cdots,k\},\ j\in\{0,\cdots,n\}.$$

It follows from Proposition 3.1 that for any $i\in\{1,\cdots,k\}$ and $j\in\{0,\cdots,n\}$, we have that

$$W_{i,\alpha}\rightharpoonup0 \text{ and } Z_{i,j,\alpha}\rightharpoonup0 \text{ weakly in } H_1^2(M) \text{ when } \alpha\to+\infty\,. \tag{72}$$

Since $\phi_\alpha\in K_\alpha^\perp$, for any $i=1,\cdots,k$ and any $j=0,\cdots,n$, we have that

$$(\phi_\alpha,Z_{i,j,\alpha})_{h_\alpha}=0 \text{ and } (\phi_\alpha,\varphi)_{h_\alpha}=0 \text{ for all } \varphi\in K_0\,. \tag{73}$$

It follows from the local C^2-convergence (63) of G_α to G_0, from the continuity properties of G_0 (see Definition 2.1) and from (69) that

$$\begin{aligned}\phi_\alpha-(\Delta_g+h_\alpha)^{-1}\Big((q_\alpha-1)f_\alpha H\Big(u_\alpha(z_\alpha)+\sum_{i=1}^k W_{i,\alpha}\Big)^{q_\alpha-2}\phi_\alpha\Big)\\ -(\Delta_g+h_0)^{-1}(G_0''(u_0)\phi)=o(1)+\sum_{i,j}\lambda_\alpha^{ij}Z_{i,j,\alpha}+\sum_{l=1}^d\mu_\alpha^l\varphi_l\end{aligned} \tag{74}$$

where $\lim_{\alpha\to 0} o(1) = 0$ in $H_1^2(M)$. We define

$$\Lambda_\alpha := \sum_{i=1}^{k}\sum_{j=0}^{n} |\lambda_\alpha^{ij}| + \sum_{l=1}^{d} |\mu_\alpha^l| \text{ for all } \alpha.$$

We fix $\varphi \in H_1^2(M)$. It then follows from (74) that

$$(75)\quad (\phi_\alpha, \varphi)_{h_\alpha} - (q_\alpha - 1)\int_M f_\alpha H\Big(u_\alpha(z_\alpha) + \sum_{i=1}^{k} W_{i,\alpha}\Big)^{q_\alpha-2}\phi_\alpha\varphi\, dv_g$$

$$-G_0''(u_0)(\phi,\varphi) = o_\alpha(1)\,(1+\Lambda_\alpha))\,(\|\varphi\|_{H_1^2}) + \sum_{i,j}\lambda_\alpha^{ij}(Z_{i,j,\alpha},\varphi)_{h_\alpha} + \sum_{l=1}^{d}\mu_\alpha^l(\varphi_l,\varphi)_{h_0}\,.$$

Here and in the sequel, $\lim_{\alpha\to+\infty} o_\alpha(1) \to 0$ uniformly with respect to $\varphi \in H_1^2(M)$.

Step 1: We first bound the μ_α^l's. We fix $\varphi \in H_1^2(M)$. It follows from (63), (64) (65), (25), and (26) that the family $(f_\alpha H(u_\alpha(z_\alpha)+\sum_{i=1}^k W_{i,\alpha})^{q_\alpha-2}\phi_\alpha)_\alpha$ is uniformly bounded in $L^{2n/(n+2)}(M)$ and converges a.e. to $f_0 H(u_0)^{2^\star-2}\phi$ when $\alpha \to +\infty$. It then follows from integration theory that the convergence holds weakly in $L^{2^\star}(M)'$. Therefore, passing to the limit $\alpha \to +\infty$ in (75) for $\varphi \in H_1^2(M)$ fixed, we get that

$$(76)\quad (\phi,\varphi)_{h_0} - (2^\star - 1)\int_M f_0 H(u_0)^{2^\star-2}\phi\varphi\, dv_g - G_0''(u_0)(\phi,\varphi)$$

$$= (\phi,\varphi)_{h_0} - F_0''(u_0)(\phi,\varphi) = o_\alpha(1)\,(1+\Lambda_\alpha) + \sum_{l=1}^{d}\mu_\alpha^l(\varphi_l,\varphi)_{h_0}\,.$$

Passing to the limit $\alpha \to +\infty$ in the second equality of (73) yields $(\phi,\varphi)_{h_0} = 0$ for all $\varphi \in K_0$. It then follows from (3) that $F_0''(u_0)(\varphi,\phi) = (\phi,\varphi)_{h_0} = 0$, and then

$$\sum_{l=1}^{d}\mu_\alpha^l(\varphi_l,\varphi)_{h_0} = o_\alpha(1)\,(1+\Lambda_\alpha)$$

for all $\varphi \in K_0$. Since $\{\varphi_l /\, l = 1,\cdots,d\}$ is an orthonormal basis of K_0, we get that

$$(77)\qquad \sum_{l=1}^{d}|\mu_\alpha^l| = o_\alpha(1)\,(1+\Lambda_\alpha)\,,$$

where $\lim_{\alpha\to+\infty} o_\alpha(1) = 0$.

Step 2: We bound the λ_α^{ij}'s. We fix $i_0 \in \{1,\cdots,k\}$. We fix $\varphi \in D_1^2(\mathbb{R}^n)$ and define

$$\varphi_{i_0,\alpha} := \mathrm{Resc}^F_{\xi_{i_0,\alpha},\delta_{i_0,\alpha}}(\varphi), \text{ where } F(\xi,x) := \chi(d_{g_\xi}(\xi,x))\Lambda_\xi(x) \text{ for } \xi, x \in M.$$

In particular, it follows from Proposition 3.1 that

$$(78)\qquad \varphi_{i_0,\alpha} \rightharpoonup 0 \text{ weakly in } H_1^2(M) \text{ when } \alpha \to +\infty.$$

We define

$$\tilde\phi_{i_0,\alpha} := \tilde{\mathrm{Resc}}^{\tilde F}_{\xi_{i_0,\alpha},\delta_{i_0,\alpha}}(\phi_\alpha), \text{ where } \tilde F(\xi,x) := \frac{\chi(|x|)}{\Lambda_\xi(\tilde{\exp}_\xi^{g_\xi}(x))} \text{ for } \xi \in M,\, x \in \mathbb{R}^n.$$

Note that it follows from Proposition 3.5 that $(\tilde\phi_{i_0,\alpha})_\alpha$ is bounded in $D_1^2(\mathbb{R}^n)$, and then, there exists $\tilde\phi_{i_0} \in D_1^2(\mathbb{R}^n)$ such that

$$(79)\qquad \tilde\phi_{i_0,\alpha} \rightharpoonup \tilde\phi_{i_0} \text{ in } D_1^2(\mathbb{R}^n) \text{ when } \alpha \to +\infty.$$

As easily checked, $\mathrm{Resc}^F_{\xi_{i_0,\alpha},\delta_{i_0,\alpha}}(\tilde{\phi}_{i_0,\alpha}) = \phi_\alpha + \tau_\alpha$, where $(\tau_\alpha)_\alpha$ is bounded in $H_1^2(M)$ with support in $M \setminus B^{g_{\xi_{i_0,\alpha}}}_{\xi_{i_0,\alpha}}(r_0/3)$. It then follows from (28) and (29) that

(80)
$$\left(\mathrm{Resc}^F_{\xi_{i_0,\alpha},\delta_{i_0,\alpha}}(\tilde{\phi}_{i_0,\alpha}), \mathrm{Resc}^F_{\xi_{i_0,\alpha},\delta_{i_0,\alpha}}(\varphi)\right)_{h_\alpha} = \left(\phi_\alpha, \mathrm{Resc}^F_{\xi_{i_0,\alpha},\delta_{i_0,\alpha}}(\varphi)\right)_{h_\alpha} + o_\alpha(1),$$

where $\lim_{\alpha\to+\infty} o_\alpha(1) = 0$. Applying (75) to $\varphi_{i_0,\alpha}$ yields

$$\begin{aligned}(\phi_\alpha, \varphi_{i_0,\alpha})_{h_\alpha} - (q_\alpha - 1)\int_M f_\alpha H\left(u_\alpha(z_\alpha) + \sum_{i=1}^k W_{i,\alpha}\right)^{q_\alpha-2} \phi_\alpha \varphi_{i_0,\alpha}\, dv_g \\ -G_0''(u_0)(\phi, \varphi_{i_0,\alpha}) = o_\alpha(1)\,(1+\Lambda_\alpha)\,(\|\varphi_{i_0,\alpha}\|_{H_1^2}) \\ + \sum_{i,j} \lambda_\alpha^{ij} (Z_{i,j,\alpha}, \varphi_{i_0,\alpha})_{h_\alpha} + \sum_{l=1}^d \mu_\alpha^l (\varphi_l, \varphi_{i_0,\alpha}))_{h_0}\,.\end{aligned}$$

It then follows from (80), (33), (43), the properties of G_0 (see Definition 2.1), (78), and Proposition 3.2 that

$$(\tilde{\phi}_{i_0,\alpha}, \varphi)_{\mathrm{Eucl}} - (q_\alpha - 1)\int_M f_\alpha H\left(u_\alpha(z_\alpha) + \sum_{i=1}^k W_{i,\alpha}\right)^{q_\alpha-2} \phi_\alpha \varphi_{i_0,\alpha}\, dv_g = o_\alpha(1)\,(1+\Lambda_\alpha) + \sum_j \lambda_\alpha^{i_0 j} (V_j, \varphi)_{\mathrm{Eucl}}\,. \tag{81}$$

Without loss of generality, we can assume that $\theta < \min\{1, 2^\star - 2\}$. Then, there exists $C(\theta) > 0$ such that

$$\left|H\left(\sum_{i=0}^k X_i\right)^{q_\alpha-2} - H(X_0)^{q_\alpha-2}\right| \le C(\theta)|X_0|^\theta \sum_{i\neq 0}^k |X_i|^{q_\alpha-2-\theta} + C(\theta)\sum_{i\neq 0}|X_i|^{q_\alpha-2}$$

for all $X_i \in \mathbb{R}$, $i = 0, \cdots, k$. As a consequence, we get that

$$\begin{aligned}&\left|\int_M f_\alpha\left(H\left(u_\alpha(z_\alpha) + \sum_{i=1}^k W_{i,\alpha}\right)^{q_\alpha-2} - H(W_{i_0,\alpha})^{q_\alpha-2}\right)\phi_\alpha\varphi_{i_0,\alpha}\, dv_g\right| \\ &\le C\int_M \left(|W_{i_0,\alpha}|^\theta |u_\alpha(z_\alpha)|^{q_\alpha-2-\theta} + \sum_{i\neq i_0} |W_{i_0,\alpha}|^\theta |W_{i,\alpha}|^{q_\alpha-2-\theta}\right)|\phi_\alpha|\cdot|\varphi_{i_0,\alpha}|\, dv_g \\ &\quad + C\int_M \left(|u_\alpha(z_\alpha)|^{q_\alpha-2} + \sum_{i\neq i_0}|W_{i,\alpha}|^{q_\alpha-2}\right)|\phi_\alpha|\cdot|\varphi_{i_0,\alpha}|\, dv_g \\ &\le C\int_M |W_{i_0,\alpha}|^\theta |\varphi_{i_0,\alpha}|\cdot|\phi_\alpha|\cdot|u_\alpha(z_\alpha)|^{q_\alpha-2-\theta}\, dv_g \\ &\quad + C\int_M |\varphi_{i_0,\alpha}|\cdot|\phi_\alpha|\cdot|u_\alpha(z_\alpha)|^{q_\alpha-2}\, dv_g \\ &\quad + C\sum_{i\neq i_0} \||W_{i_0,\alpha}|^\theta |W_{i,\alpha}|^{(q_\alpha-2-\theta)}\|_{2^\star/(2^\star-2)} \|\varphi_{i_0,\alpha}\|_{2^\star}\|\phi_\alpha\|_{2^\star} \\ &\quad + C\sum_{i\neq i_0}\||W_{i,\alpha}|^{q_\alpha-2}\varphi_{i_0,\alpha}\|_{2^\star/(2^\star-1)}\|\phi_\alpha\|_{2^\star}.\end{aligned}$$

Since $(|\varphi_{i_0,\alpha}|\cdot|\phi_\alpha|)_\alpha$ goes to 0 almost everywhere and is bounded in $L^{2^\star/2}(M)$, since $(|W_{i_0,\alpha}|^\theta|\varphi_{i_0,\alpha}|\cdot|\phi_\alpha|)_\alpha$ goes to 0 almost everywhere and is bounded in $L^{2^\star/(2+\theta)}(M)$, it follows from standard integration theory and Proposition 3.4 that

$$\int_M f_\alpha\bigg(H\bigg(u(z_\alpha)+\sum_{i=1}^k W_{i,\alpha}\bigg)^{q_\alpha-2}-H(W_{i_0,\alpha})^{q_\alpha-2}\bigg)\phi_\alpha\varphi_{i_0,\alpha}\,dv_g\to 0$$

when $\alpha\to+\infty$. Plugging this limit in (81) yields

$$(\tilde\phi_{i_0,\alpha},\varphi)_{\mathrm{Eucl}}-(q_\alpha-1)\int_M f_\alpha H\left(W_{i_0,\alpha}\right)^{q_\alpha-2}\phi_\alpha\varphi_{i_0,\alpha}\,dv_g = o_\alpha(1)\left(1+\Lambda_\alpha\right)+\sum_j\lambda_\alpha^{i_0j}(V_j,\varphi)_{\mathrm{Eucl}}. \tag{82}$$

For any $R>0$, we have that

$$\begin{aligned}&\bigg|\int_{M\backslash B^{g_{\xi_{i_0,\alpha}}}_{R\delta_{i_0,\alpha}}(\xi_{i_0,\alpha})} f_\alpha H\left(W_{i_0,\alpha}\right)^{q_\alpha-2}\phi_\alpha\varphi_{i_0,\alpha}\,dv_g\bigg|\\
&\le C\|W_{i_0,\alpha}\|_{2^\star}^{q_\alpha-2}\|\phi_\alpha\|_{2^\star}\bigg(\int_{M\backslash B^{g_{\xi_{i_0,\alpha}}}_{R\delta_{i_0,\alpha}}(\xi_{i_0,\alpha})}|\varphi_{i_0,\alpha}|^{2^\star}\,dv_g\bigg)^{1/2^\star}\\
&\le C\|\varphi\|_{L^{2^\star}(\mathbb{R}^n\backslash B_R(0))}\,.\end{aligned}$$

Since $\varphi\in L^{2^\star}(\mathbb{R}^n)$, we get that

$$\lim_{R\to+\infty}\lim_{\alpha\to+\infty}\int_{M\backslash B^{g_{\xi_{i_0,\alpha}}}_{R\delta_{i_0,\alpha}}(\xi_{i_0,\alpha})} f_\alpha H\left(W_{i_0,\alpha}\right)^{q_\alpha-2}\phi_\alpha\varphi_{i_0,\alpha}\,dv_g=0\,. \tag{83}$$

A change of variable, (63) and (64) yield

$$\begin{aligned}&\int_{B^{g_{\xi_{i_0,\alpha}}}_{R\delta_{i_0,\alpha}}(\xi_{i_0,\alpha})} f_\alpha H\left(W_{i_0,\alpha}\right)^{q_\alpha-2}\phi_\alpha\varphi_{i_0,\alpha}\,dv_g\\
&=\int_{B_R(0)} f_\alpha(\tilde{\exp}^{g_{\xi_{i_0,\alpha}}}_{\xi_{i_0,\alpha}}(\delta_{i_0,\alpha}\cdot))\left(\delta_{i_0,\alpha}^{2^\star-q_\alpha}\right)^{\frac{n-2}{2}}H\left(\kappa_i U_{f_0(\xi_{i_0,\alpha})}\right)^{q_\alpha-2}\tilde\phi_{i_0,\alpha}\varphi\,dv_{g_\alpha}\\
&=\int_{B_R(0)}U_1^{2^\star-2}\tilde\phi_{i_0,\alpha}\varphi\,dx+o(1),\end{aligned} \tag{84}$$

where $g_\alpha:=(\tilde{\exp}^{g_{\xi_{i_0,\alpha}}}_{\xi_{i_0,\alpha}})^\star g(\delta_{i_0,\alpha}\cdot)$, and we have used that $\kappa_i=1$ if $H=(\cdot)_+$. Moreover, it follows from Hölder's inequality that

$$\left|\int_{\mathbb{R}^n\backslash B_R(0)}U_1^{2^\star-2}\tilde\phi_{i_0,\alpha}\varphi\,dx\right|\le C\|U_1\|^{2^\star-2}_{L^{2^\star}(\mathbb{R}^n\backslash B_R(0))}\|\tilde\phi_{i_0,\alpha}\|_{2^\star}\|\varphi\|_{2^\star}\,. \tag{85}$$

Plugging (83), (84), and (85) into (82), and using (79) yields

$$(\tilde\phi_{i_0},\varphi)_{\mathrm{Eucl}}-(2^\star-1)\int_{\mathbb{R}^n}U_1^{2^\star-2}\tilde\phi_{i_0}\varphi\,dx = o_\alpha(1)\left(1+\Lambda_\alpha\right)+\sum_j\lambda_\alpha^{i_0j}(V_j,\varphi)_{\mathrm{Eucl}}\,. \tag{86}$$

It follows from (73), from (80), (33), and (43) that

$$(\tilde\phi_{i_0},V_j)_{\mathrm{Eucl}}=0\text{ for all }j=0,\cdots,n. \tag{87}$$

Since the V_j's are solutions to (16), we then get that $\int_{\mathbb{R}^n} U_1^{2^\star-2}\tilde{\phi}_{i_0} V_j\,dx = 0$ for all $j = 0, \cdots, n$. Since the V_j's are orthogonal in $D_1^2(\mathbb{R}^n)$, taking $\varphi := V_j$ in (86) yields

$$\lambda^{i_0,j} = o_\alpha(1)\,(1+\Lambda_\alpha) \text{ for all } i_0 = 1,\cdots,k \text{ and } j = 0,\cdots,n. \tag{88}$$

Step 3: It follows from (77) and (88) that $\Lambda_\alpha = o_\alpha(1)\,(1+\Lambda_\alpha)$, and then $\Lambda_\alpha = o_\alpha(1)$ when $\alpha \to 0$. As a consequence, (76) rewrites $\Delta_g \phi + h_0\phi = F_0''(u_0)\phi$, and then $\phi \in K_0$. Moreover, passing to the limit $\alpha \to +\infty$ in the second equation of (73) yields $\phi \in K_0^\perp$. Therefore $\phi = 0$, and then (69) rewrites

$$\phi_\alpha \rightharpoonup 0 \text{ weakly in } H_1^2(M) \text{ when } \alpha \to +\infty.$$

Similarly, (86) rewrites $\Delta_{\text{Eucl}}\tilde{\phi}_{i_0} = (2^\star-1)U_1^{2^\star-2}\tilde{\phi}_{i_0}$ with $\tilde{\phi}_{i_0} \in D_1^2(\mathbb{R}^n)$. Then $\tilde{\phi}_{i_0} \in K_{BE}$ (see Subsection 2.5). On the other hand, (87) yields $\tilde{\phi}_{i_0} \in K_{BE}^\perp$. Therefore $\tilde{\phi}_{i_0} \equiv 0$, and then (79) rewrites

$$\tilde{\phi}_{i_0,\alpha} \rightharpoonup 0 \text{ weakly in } D_1^2(\mathbb{R}^n) \text{ when } \alpha \to +\infty$$

for any $i_0 = 1, \cdots, k$. Since $\phi \equiv 0$, taking $\varphi := \phi_\alpha$ in (75) yields

$$\begin{aligned}\|\phi_\alpha\|_{h_\alpha}^2 &= (q_\alpha - 1)\int_M f_\alpha H\left(u_\alpha(z_\alpha) + \sum_{i=1}^k W_{i,\alpha}\right)^{q_\alpha-2}\phi_\alpha^2\,dv_g + o(1)\\ &\le C\int_M |u_\alpha(z_\alpha)|^{q_\alpha-2}\phi_\alpha^2\,dv_g + C\sum_{i=1}^k\int_M |W_{i,\alpha}|^{q_\alpha-2}\phi_\alpha^2\,dv_g + o_\alpha(1),\end{aligned} \tag{89}$$

where $\lim_{\alpha\to+\infty} o_\alpha(1) = 0$. Since $\phi_\alpha \rightharpoonup 0$ when $\alpha \to +\infty$, it follows from integration theory that $\int_M |u_\alpha(z_\alpha)|^{q_\alpha-2}\phi_\alpha^2\,dv_g \to 0$ when $\alpha \to +\infty$. For any $i \in \{1,\cdots,k\}$, on the one hand, for any $R > 0$, we have that

$$\begin{aligned}\int_{M\setminus B^{g_{\xi_{i,\alpha}}}_{R\delta_{i,\alpha}}(\xi_{i,\alpha})} |W_{i,\alpha}|^{q_\alpha-2}\phi_\alpha^2\,dv_g &\le C\|\phi_\alpha\|_{2^\star}^2\left(\int_{M\setminus B^{g_{\xi_{i,\alpha}}}_{R\delta_{i,\alpha}}(\xi_{i,\alpha})} |W_{i,\alpha}|^{2^\star}\,dv_g\right)^{\frac{q_\alpha-2}{2^\star}}\\ &\le C\|\phi_\alpha\|_{H_1^2}^2\left(\int_{\mathbb{R}^n\setminus B_R(0)} U_1^{2^\star}\right)^{(q_\alpha-2)/2^\star},\end{aligned} \tag{90}$$

and then

$$\lim_{R\to+\infty}\lim_{\alpha\to+\infty}\int_{M\setminus B^{g_{\xi_{i_0,\alpha}}}_{R\delta_{i_0,\alpha}}(\xi_{i_0,\alpha})} |W_{i,\alpha}|^{q_\alpha-2}\phi_\alpha^2\,dv_g = 0\,.$$

On the other hand, we have that

$$\int_{B^{g_{\xi_{i_0,\alpha}}}_{R\delta_{i_0,\alpha}}(\xi_{i_0,\alpha})} |W_{i,\alpha}|^{q_\alpha-2}\phi_\alpha^2\,dv_g \le C\int_{B_R(0)} U_1^{q_\alpha-2}\tilde{\phi}_{i,\alpha}\,dx\,. \tag{91}$$

Since $\tilde{\phi}_{i,\alpha} \rightharpoonup 0$ when $\alpha \to +\infty$, it follows from integration theory that the right-hand side in (91) above goes to 0 as $\alpha \to +\infty$. Plugging this latest result and (90) into (89) yields $\|\phi_\alpha\|_{h_\alpha} = o(1)$ when $\alpha \to +\infty$. A contradiction with (67). This proves (62).

We write $L_{z,(\delta_i)_i,(\xi_i)_i} := Id - \tilde{L}$, where $\tilde{L}$ is a compact operator. It then follows from (62) and Fredholm theory that $L_{z,(\delta_i)_i,(\xi_i)_i}$ is a bi-continuous isomorphism. This ends the proof of Proposition 5.2 ∎

5.2. **Rough control of the rest.** We prove the following proposition:

Proposition 5.3. *We have that*

$$R(z,(\delta_i)_i,(\xi_i)_i)\leq \omega_{14}(\varepsilon,N) \tag{92}$$

for all $(z,(\delta_i)_i,(\xi_i)_i)\in B_\varepsilon(0)\times \mathcal{D}_k(\varepsilon,N)$, *where* $\lim_{\varepsilon\to 0,N\to+\infty}\omega_{14}(\varepsilon,N)=0$.

Proof of Proposition 5.3: We argue by contradiction. We assume that there exist $(q_\alpha)_\alpha\in(2,2^\star]$, $(h_\alpha)_\alpha\in L^\infty(M)$, $(f_\alpha)_\alpha\in C^0(M)$, $(z_\alpha)_\alpha\in B_1(0)$, $(u_\alpha)_\alpha\in C^1(B_1(0);H_1^2(M))$, $(G_\alpha)_\alpha\in C^{2,\theta}_{loc}(H_1^2(M))$, $(\delta_{i,\alpha})_\alpha$ and $(\xi_{i,\alpha})_\alpha$ for $i=1,\cdots,k$ and $c_0>0$ such that

$$\lim_{\alpha\to+\infty}\|h_\alpha-h_0\|_\infty+\|f_\alpha-f_0\|_{C^0}+d_{C^{2,\theta}_B}(G_\alpha,G_0)=0\,, \tag{93}$$

$$\lim_{\alpha\to+\infty}\delta_{i,\alpha}=0\,,\ \lim_{\alpha\to+\infty}\delta_{i,\alpha}^{2^\star-q_\alpha}=1\,,\ \lim_{\alpha\to 0}z_\alpha=0\,,\ \lim_{\alpha\to+\infty}q_\alpha=2^\star\,, \tag{94}$$

$$u_\alpha(0)=u_0\,,\ \|u_\alpha\|_{C^1(B_1(0),H_1^2)}\leq C_0\,,\ f_0(\xi_{i,\alpha})\geq\nu_0 \text{ for all } i=1,\cdots,k, \tag{95}$$

$$\lim_{\alpha\to+\infty}\left(\frac{\delta_{i,\alpha}}{\delta_{j,\alpha}}+\frac{\delta_{j,\alpha}}{\delta_{i,\alpha}}+\frac{d_g(\xi_{i,\alpha},\xi_{j,\alpha})^2}{\delta_{i,\alpha}\delta_{j,\alpha}}\right)=+\infty\,, \tag{96}$$

and

$$R_\alpha:=R(z_\alpha,(\delta_{i,\alpha})_i,(\xi_{i,\alpha})_i)\geq c_0 \text{ for all } \alpha\in\mathbb{N}. \tag{97}$$

We define $W_{i,\alpha}:=W_{\kappa_i,\delta_{i,\alpha},\xi_{i,\alpha}}$. In particular, Proposition 3.1 yields

$$W_{i,\alpha}\rightharpoonup 0 \text{ weakly in } H_1^2(M) \text{ when } \alpha\to+\infty. \tag{98}$$

Defining $\tilde{H}_{q_\alpha}(x)=H(x)^{q_\alpha-2}x$ and F_α as in (71), we have that

$$\begin{aligned}
R_\alpha&=\left\|u_\alpha(z_\alpha)+\sum_{i=1}^k W_{i,\alpha}-(\Delta_g+h_\alpha)^{-1}\left(F_\alpha'\left(u_\alpha(z_\alpha)+\sum_{i=1}^k W_{i,\alpha}\right)\right)\right\|_{H_1^2}\\
&\leq\left\|u_\alpha(z_\alpha)-(\Delta_g+h_\alpha)^{-1}\left(F_\alpha'(u_\alpha(z_\alpha))\right)\right\|_{H_1^2}\\
&\quad+\sum_{i=1}^k\left\|W_{i,\alpha}-(\Delta_g+h_\alpha)^{-1}(f_\alpha\tilde{H}_{q_\alpha}(W_{i,\alpha}))\right\|_{H_1^2}\\
&\quad+\left\|(\Delta_g+h_\alpha)^{-1}\left(F_\alpha'\left(u_\alpha(z_\alpha)+\sum_{i=1}^k W_{i,\alpha}\right)-F_\alpha'(u_\alpha(z_\alpha))\right)\right.\\
&\quad\left.-\sum_{i=1}^k(\Delta_g+h_\alpha)^{-1}(f_\alpha\tilde{H}_{q_\alpha}(W_{i,\alpha}))\right\|_{H_1^2}.
\end{aligned}$$

The control (95) yields $\lim_{\alpha\to+\infty}u_\alpha(z_\alpha)=u_0$ in $H_1^2(M)$. The convergence (93) then yields

$$\begin{aligned}
&\lim_{\alpha\to+\infty}\left\|u_\alpha(z_\alpha)-(\Delta_g+h_\alpha)^{-1}(F_\alpha'(u_\alpha(z_\alpha)))\right\|_{H_1^2}\\
&\quad=\left\|u_0-(\Delta_g+h_0)^{-1}(F_0'(u_0))\right\|_{H_1^2}=0\,.
\end{aligned}$$

Since $(u_\alpha(z_\alpha)+\sum_{i=1}^k W_{i,\alpha})_\alpha \rightharpoonup u_0$ weakly in $H_1^2(M)$, it follows from the convergence (93) of $(G_\alpha)_\alpha$ and the Definition 2.1 of subcriticality of G_0 that

$$G'_\alpha\left(u_\alpha(z_\alpha)+\sum_{i=1}^k W_{i,\alpha}\right)\to G'_0(u_0) \text{ strongly in } H_1^2(M) \text{ when } \alpha\to+\infty. \tag{99}$$

As a consequence, we get with the Riesz correspondence (23) that

$$R_\alpha \le o(1)+\sum_{i=1}^k \left\|W_{i,\alpha}-(\Delta_g+h_\alpha)^{-1}(f_\alpha \tilde{H}_{q_\alpha}(W_{i,\alpha})\right\|_{H_1^2}+C\|A_\alpha\|_{\frac{2n}{n+2}}\,,$$

where

$$A_\alpha := \tilde{H}_{q_\alpha}\left(u_\alpha(z_\alpha)+\sum_{i=1}^k W_{i,\alpha}\right)-\tilde{H}_{q_\alpha}(u_\alpha(z_\alpha))-\sum_{i=1}^k \tilde{H}_{q_\alpha}(W_{i,\alpha}).$$

As easily checked, for all family $(X_i)_{i=0,\cdots,k}\in\mathbb{R}$, we have that

$$\left|\tilde{H}_{q_\alpha}\left(\sum_{i=0}^k X_i\right)-\sum_{i=0}^k \tilde{H}_{q_\alpha}(X_i)\right|\le C\sum_{i\neq j}|X_i|\cdot|X_j|^{q_\alpha-2}$$

for all $\alpha\in\mathbb{N}$ large. Therefore, with Hölder's inequality, we get that

$$\begin{aligned}\|A_\alpha\|_{\frac{2n}{n+2}}^{\frac{2n}{n+2}} &\le C\sum_i\int_M |u_\alpha(z_\alpha)|^{\frac{2n}{n+2}}|W_{i,\alpha}|^{(q_\alpha-2)\frac{2n}{n+2}}\,dv_g\\ &+C\sum_i\int_M |u_\alpha(z_\alpha)|^{(q_\alpha-2)\frac{2n}{n+2}}|W_{i,\alpha}|^{\frac{2n}{n+2}}\,dv_g\\ &+C\sum_{i\neq j}\int_M |W_{i,\alpha}|^{\frac{2n}{n+2}}|W_{j,\alpha}|^{(q_\alpha-2)\frac{2n}{n+2}}\,dv_g\,.\end{aligned}$$

Since $u_\alpha(z_\alpha)\to u_0$ in $L^{2^\star}(M)$, $(|W_{i,\alpha}|^{(q_\alpha-2)\frac{2n}{n+2}})_\alpha$ is bounded in $L^{(n+2)/4}(M)$ and goes to zero a.e. on M when $\alpha\to+\infty$, integration theory yields the convergence to 0 of the first term of the right-hand side when $\alpha\to+\infty$. Similarly, the second term goes to 0 as $\alpha\to+\infty$. The expression (25), the property (13), and Proposition 3.4 yield the convergence to 0 of the third term when $\alpha\to+\infty$. Therefore, we get that $(\|A_\alpha\|_{\frac{2n}{n+2}})_\alpha\to 0$ and then

$$R_\alpha\le o(1)+\sum_{i=1}^k\left\|W_{i,\alpha}-(\Delta_g+h_\alpha)^{-1}\left(f_\alpha\tilde{H}_{q_\alpha}(W_{i,\alpha})\right)\right\|_{H_1^2}\,.$$

We define

$$W_{i,\alpha}^0 := \chi(d_{g_{\xi_{i,\alpha}}}(\cdot,\xi_{i,\alpha}))\Lambda_{\xi_{i,\alpha}}\left(\frac{\delta_{i,\alpha}\sqrt{\frac{n(n-2)}{f_0(\xi_{i,\alpha})}}}{\delta_{i,\alpha}^2+d_{g_{\xi_{i,\alpha}}}(\cdot,\xi_{i,\alpha})^2}\right)^{\frac{n-2}{2}}$$

so that $W_{i,\alpha} = \kappa_i W_{i,\alpha}^0 + o(1)$ when $\alpha \to +\infty$ (see (12)). Therefore, since $\kappa_i = 1$ if $H = (\cdot)_+$, (23) yields

$$
\begin{aligned}
R_\alpha &\le o(1) + \sum_{i=1}^k \left\| \kappa_i W_{i,\alpha}^0 - (\Delta_g + h_\alpha)^{-1}(f_\alpha \tilde{H}_{q_\alpha}(\kappa_i W_{i,\alpha}^0) \right\|_{H_1^2} \\
&\le o(1) + \sum_{i=1}^k \|(\Delta_g + h_\alpha) W_{i,\alpha}^0 - f_\alpha (W_{i,\alpha}^0)^{q_\alpha - 1}\|_{\frac{2n}{n+2}} . \qquad (100)
\end{aligned}
$$

In the sequel, $o(1)_{\frac{2n}{n+2}}$ denotes a function going to 0 in $L^{\frac{2n}{n+2}}(M)$ when $\alpha \to +\infty$. We define $c_n := \frac{n-2}{4(n-1)}$, and we let R_g be the scalar curvature of g. We denote $L_g := \Delta_g + c_n R_g$ the conformal Laplacian. If $g' := \varpi^{2^\star - 2} g$ and $\varpi \in C^2(M)$ positive, the conformal invariance properties of L_g yields

$$
L'_g(\varphi) := \varpi^{1-2^\star} L_g(\varpi \varphi)
$$

for all $\varphi \in C^2(M)$. Using the expression of the Laplacian in radial coordinates, omitting the index i and writing $r := d_{g_\xi}(x, \xi)$, we get that

$$
\begin{aligned}
(\Delta_g + h_\alpha) W_{i,\alpha}^0 &= L_g W_{i,\alpha}^0 + (h_\alpha - c_n R_g) W_{i,\alpha}^0 \\
&= L_{\Lambda_\xi^{2-2^\star} g_\xi} W_{i,\alpha}^0 + o(1)_{\frac{2n}{n+2}} = \Lambda_\xi^{2^\star - 1} L_{g_\xi} (\Lambda_\xi^{-1} W_{i,\alpha}^0) + o(1)_{\frac{2n}{n+2}} \\
&= \Lambda_\xi^{2^\star - 1} \Delta_{g_\xi} \left(\chi(r) \delta^{-\frac{n-2}{2}} U_{f_0(\xi)}(\delta^{-1} r) \right) + o(1)_{\frac{2n}{n+2}} \\
&= \Lambda_\xi^{2^\star - 1} \Delta_{\mathrm{Eucl}} \left(\chi(r) \delta^{-\frac{n-2}{2}} U_{f_0(\xi)}(\delta^{-1} r) \right) \\
&\quad - \partial_r \ln \sqrt{|g_\xi|} \partial_r \left(\chi(r) \delta^{-\frac{n-2}{2}} U_{f_0(\xi)}(\delta^{-1} r) \right) + o(1)_{\frac{2n}{n+2}} \\
&= \Lambda_\xi^{2^\star - 1} \Delta_{\mathrm{Eucl}} \left(\chi(r) \delta^{-\frac{n-2}{2}} U_{f_0(\xi)}(\delta^{-1} r) \right) \\
&\quad + O\left(\frac{\delta^{\frac{n-2}{2}}}{(\delta^2 + r^2)^{\frac{n-2}{2}}} \right) + o(1)_{\frac{2n}{n+2}} \\
&= \Lambda_\xi^{2^\star - 1} \chi(r) \Delta_{\mathrm{Eucl}} \left(\delta^{-\frac{n-2}{2}} U_{f_0(\xi)}(\delta^{-1} r) \right) + o(1)_{\frac{2n}{n+2}} \\
&= \Lambda_\xi^{2^\star - 1} \chi(r) f_0(\xi) \left(\delta^{-\frac{n-2}{2}} U_{f_0(\xi)}(\delta^{-1} r) \right)^{2^\star - 1} + o(1)_{\frac{2n}{n+2}} \\
&= f_0(\xi) \left(\chi(r) \Lambda_\xi \delta^{-\frac{n-2}{2}} U_{f_0(\xi)}(\delta^{-1} r) \right)^{2^\star - 1} + o(1)_{\frac{2n}{n+2}} \\
&= f_0(\xi) (W_{i,\alpha}^0)^{2^\star - 1} + o(1)_{\frac{2n}{n+2}} \\
&= f_0 (W_{i,\alpha}^0)^{2^\star - 1} + o(1)_{\frac{2n}{n+2}} = f_\alpha (W_{i,\alpha}^0)^{2^\star - 1} + o(1)_{\frac{2n}{n+2}} .
\end{aligned}
$$

Therefore, it follows from (100) that

$$
R_\alpha \le o(1) + C \sum_{i=1}^k \|(W_{i,\alpha}^0)^{2^\star - 1} - (W_{i,\alpha}^0)^{q_\alpha - 1}\|_{\frac{2n}{n+2}} . \qquad (101)
$$

We fix $i \in \{1, \cdots, k\}$. For any $R > 0$, a change of variable and (94) yields

$$\begin{aligned}
&\int_{B^{g_{\xi_{i,\alpha}}}_{R\delta_{i,\alpha}}(\xi_{i,\alpha})} \left|(W^0_{i,\alpha})^{2^\star-1} - (W^0_{i,\alpha})^{q_\alpha-1}\right|^{\frac{2n}{n+2}} dv_g \\
&\quad \le C \int_{B^{g_{\xi_{i,\alpha}}}_{R\delta_{i,\alpha}}(\xi_{i,\alpha})} \left|(W^0_{i,\alpha})^{2^\star-1} - (W^0_{i,\alpha})^{q_\alpha-1}\right|^{\frac{2n}{n+2}} dv_{g_{\xi_{i,\alpha}}} \\
&\quad \le C \int_{B_R(0)} \left|U^{2^\star-1}_{f_0(\xi_{i,\alpha})} - \delta_{i,\alpha}^{\frac{n-2}{2}(2^\star-q_\alpha)} U^{q_\alpha-1}_{f_0(\xi_{i,\alpha})}\right|^{\frac{2n}{n+2}} dx = o(1)
\end{aligned} \tag{102}$$

when $\alpha \to +\infty$. Independently, we have that

$$\begin{aligned}
&\int_{M\setminus B^{g_{\xi_{i,\alpha}}}_{R\delta_{i,\alpha}}(\xi_{i,\alpha})} \left|(W^0_{i,\alpha})^{2^\star-1} - (W^0_{i,\alpha})^{q_\alpha-1}\right|^{\frac{2n}{n+2}} dv_g \\
&\quad \le C \int_{M\setminus B^{g_{\xi_{i,\alpha}}}_{R\delta_{i,\alpha}}(\xi_{i,\alpha})} \left|W^0_{i,\alpha}\right|^{2^\star} dv_g + C \left(\int_{M\setminus B^{g_{\xi_{i,\alpha}}}_{R\delta_{i,\alpha}}(\xi_{i,\alpha})} \left|W^0_{i,\alpha}\right|^{2^\star} dv_g\right)^{\frac{q_\alpha-1}{2^\star-1}} \\
&\quad \le C \int_{\mathbb{R}^n\setminus B_R(0)} U_1^{2^\star} dx + C \left(\int_{\mathbb{R}^n\setminus B_R(0)} U_1^{2^\star} dx\right)^{\frac{q_\alpha-1}{2^\star-1}}.
\end{aligned}$$

Then

$$\lim_{R\to+\infty} \lim_{\alpha\to+\infty} \int_{M\setminus B^{g_{\xi_{i,\alpha}}}_{R\delta_{i,\alpha}}(\xi_{i,\alpha})} \left|(W^0_{i,\alpha})^{2^\star-1} - (W^0_{i,\alpha})^{q_\alpha-1}\right|^{\frac{2n}{n+2}} dv_g = 0\,. \tag{103}$$

Plugging (102) and (103) into (101) yields $R_\alpha = o(1)$ when $\alpha \to +\infty$, a contradiction with (97). This proves Proposition 5.3. □

5.3. **Proof of Proposition 5.1 via a fixed-point argument.** We let $\varepsilon, N > 0$ satisfy the hypotheses of Proposition 5.2 to be fixed later, and we let h, f, G, u satisfy the hypotheses of Theorem 1.1. We consider $(z, (\delta_i)_i, (\xi_i)_i) \in B_\varepsilon(0) \times \mathcal{D}_k(\varepsilon, N)$, and we define $K := K_{(\delta_i)_i,(\xi_i)_i}$. For any $\phi \in K^\perp \subset H^2_1(M)$, we have that

$$\Pi_{K^\perp}\left(u(z) + \sum_{i=1}^k W_{\kappa_i,\delta_i,\xi_i} + \phi - (\Delta_g + h)^{-1}\left(F'\left(u(z) + \sum_{i=1}^k W_{\kappa_i,\delta_i,\xi_i} + \phi\right)\right)\right) = 0 \tag{104}$$

if and only if

$$\phi = T(\phi),$$

where $T : K^\perp \to K^\perp$ is such that

$$T(\phi) := L^{-1} \circ \Pi_{K^\perp} \circ (\Delta_g + h)^{-1} (N(\phi)) - L^{-1} \circ \Pi_{K^\perp}(R),$$

where $L := L_{(\delta_i)_i,(\xi_i)_i}$,

$$\begin{aligned}
N(\phi) := {} & F'\left(u(z) + \sum_{i=1}^k W_{\kappa_i,\delta_i,\xi_i} + \phi\right) - F'\left(u(z) + \sum_{i=1}^k W_{\kappa_i,\delta_i,\xi_i}\right) \\
& - F''\left(u(z) + \sum_{i=1}^k W_{\kappa_i,\delta_i,\xi_i}\right)\phi
\end{aligned}$$

and

$$R := u(z) + \sum_{i=1}^{k} W_{\kappa_i,\delta_i,\xi_i} - (\Delta_g + h)^{-1}\left(F'\left(u(z) + \sum_{i=1}^{k} W_{\kappa_i,\delta_i,\xi_i}\right)\right).$$

We prove the existence of a solution to (104) via Picard's Fixed Point Theorem. We let $\phi_1, \phi_2 \in K^\perp$ be two test-functions. Since $\Pi_{K^\perp} : H_1^2(M) \to H_1^2(M)$ is 1-Lipschitz continuous, it follows from (62) that

$$\begin{aligned}\|T(\phi_1) - T(\phi_2)\|_{H_1^2} &\leq C\|N(\phi_1) - N(\phi_2)\|_{H_1^2}\\ &\leq C\left\|F'\left(u(z) + \sum_{i=1}^{k} W_{\kappa_i,\delta_i,\xi_i} + \phi_1\right) - F'\left(u(z) + \sum_{i=1}^{k} W_{\kappa_i,\delta_i,\xi_i} + \phi_2\right)\right.\\ &\qquad \left. -F''\left(u(z) + \sum_{i=1}^{k} W_{\kappa_i,\delta_i,\xi_i}\right)(\phi_1 - \phi_2)\right\|_{(H_1^2)'}.\end{aligned}$$

It then follows from the mean value inequality that

$$\|T(\phi_1) - T(\phi_2)\|_{H_1^2(M)} \leq C \cdot S \cdot \|\phi_1 - \phi_2\|_{H_1^2}, \tag{105}$$

where

$$\begin{aligned} S &:= \sup_{t\in[0,1]} \left\|F''\left(u(z) + \sum_{i=1}^{k} W_{\kappa_i,\delta_i,\xi_i} + \phi_1 + t(\phi_2 - \phi_1)\right)\right.\\ &\qquad \left. - F''\left(u(z) + \sum_{i=1}^{k} W_{\kappa_i,\delta_i,\xi_i}\right)\right\|_{H_1^2 \to (H_1^2)'}\\ &\leq \sup_{t\in[0,1]} \left(\|F\|_{C^{2,\theta}(B_{\tilde{R}}(0))} \cdot \|\phi_1 + t(\phi_2 - \phi_1)\|_{H_1^2}^\theta\right)\\ &\leq C\|F\|_{C^{2,\theta}(B_{\tilde{R}}(0))} \cdot \left(\|\phi_1\|_{H_1^2}^\theta + \|\phi_2\|_{H_1^2}^\theta\right), \end{aligned} \tag{106}$$

with $\tilde{R} = \tilde{R}(z, (\delta_i)_i, (\xi_i)_i) := \|u(z)\|_{H_1^2} + \sum_{i=1} \|W_{\kappa_i,\delta_i,\xi_i}\|_{H_1^2} + 1$, $\|\phi_1\|_{H_1^2}, \|\phi_1\|_{H_1^2} \leq 1$. It then follows from (6) and Proposition 3.1 that $\tilde{R}(z, (\delta_i)_i, (\xi_i)_i) \leq C$. As easily checked, for $2 < q \leq 2^\star$, we have that

$$F''(v)(\psi_1, \psi_2) = (q-1)\int_M fH(v)^{q-2}\psi_1\psi_2\, dv_g + G''(v)(\psi_1, \psi_2)$$

for all $v \in H_1^2(M)$ and all $\psi_1, \psi_2 \in H_1^2(M)$. Without loss of generality, we may assume that $0 < \theta < 2^\star - 2$. Requiring that $\varepsilon < 1$ and using (6), we then get that

$$\|F\|_{C^{2,\theta}(B_{\tilde{R}}(0))} \leq C(\tilde{R}, \theta) \tag{107}$$

for $\epsilon > 0$ small enough. Plugging together (105), (106), and (107) yields

$$\|T(\phi_1) - T(\phi_2)\|_{H_1^2(M)} \leq C_1 \cdot \left(\|\phi_1\|_{H_1^2}^\theta + \|\phi_2\|_{H_1^2}^\theta\right) \cdot \|\phi_1 - \phi_2\|_{H_1^2} \tag{108}$$

for $\phi_1, \phi_2 \in K^\perp$ such that $\|\phi_1\|_{H_1^2}, \|\phi_1\|_{H_1^2} \leq 1$. Moreover, it follows from (23) and (5) that

$$\|T(0)\|_{H_1^2} \leq C\|R\|_{H_1^2} \leq C_2 R(z, (\delta_i)_i, (\xi_i)_i). \tag{109}$$

We define

$$c := 2C_2 R(z, (\delta_i)_i, (\xi_i)_i).$$

We let $\phi_1, \phi_2 \in K^\perp \cap \overline{B}_c(0)$: it then follows from (108) and (109) that

$$\|T(\phi_1) - T(\phi_2)\|_{H_1^2(M)} \le 2C_1(2C_2)^\theta R(z, (\delta_i)_i, (\xi_i)_i)^\theta \cdot \|\phi_1 - \phi_2\|_{H_1^2}$$

and

$$\begin{aligned}\|T(\phi_1)\|_{H_1^2} &\le C_2 R(z, (\delta_i)_i, (\xi_i)_i) + C_1(2C_2)^{1+\theta} R(z, (\delta_i)_i, (\xi_i)_i)^{1+\theta} \\ &\le \left(C_2 + C_2(2C_2)^{1+\theta} R(z, (\delta_i)_i, (\xi_i)_i)^\theta\right) \cdot R(z, (\delta_i)_i, (\xi_i)_i).\end{aligned}$$

It follows from Proposition 5.3, that there exists $\varepsilon > 0$ and $N > 0$ such that

$$R(z, (\delta_i)_i, (\xi_i)_i)^\theta \le \min\left\{\frac{1}{C_1 C_2^\theta 2^{1+\theta}}\ ;\ \frac{1}{4C_2(2C_2)^\theta}\right\}$$

for all $z \in B_\varepsilon(0)$ and $((\delta_i)_i, (\xi_i)_i) \in \mathcal{D}_k(\varepsilon, N)$. It then follows that for such a choice, the map T is $1/2-$Lipschitz from $\overline{B}_c(0)$ onto itself. It then follows from Picard's fixed point theorem that there exists a unique solution $\phi(z, (\delta_i)_i, (\xi_i)_i) \in \overline{B}_c(0) \cap K^\perp$ to $T(\phi(z, (\delta_i)_i, (\xi_i)_i)) = \phi(z, (\delta_i)_i, (\xi_i)_i)$, in particular

$$\|\phi(z, (\delta_i)_i, (\xi_i)_i)\|_{H_1^2} \le 2C_2 R(z, (\delta_i)_i, (\xi_i)_i).$$

We are left with proving the C^1-regularity of ϕ. We define the map

$$\begin{array}{cccc}\mathcal{F}: & B_\varepsilon(0) \times \mathcal{D}_k(\varepsilon, N) \times H_1^2(M) & \to & H_1^2(M) \\ & (z, (\delta_i)_i, (\xi_i)_i, \phi) & \mapsto & \mathcal{F}(z, (\delta_i)_i, (\xi_i)_i, \phi),\end{array}$$

where

$$\begin{aligned}\mathcal{F}(z, (\delta_i)_i, (\xi_i)_i, \phi) := \Pi_K(\phi) + \Pi_{K^\perp}\Bigg(&u(z) + \sum_{i=1}^k W_{\kappa_i, \delta_i, \xi_i} + \Pi_{K^\perp}(\phi) \\ &- (\Delta_g + h)^{-1}(F'(u(z) + \sum_{i=1}^k W_{\kappa_i, \delta_i, \xi_i} + \Pi_{K^\perp}(\phi)))\Bigg).\end{aligned}$$

It follows from Proposition 5.2 that the differential with respect to ϕ is an isomorphism of $H_1^2(M)$ for all $(z, (\delta_i)_i, (\xi_i)_i, \phi) \in B_\varepsilon(0) \times \mathcal{D}_k(\varepsilon, N) \times H_1^2(M)$, with $\|\phi\|_{H_1^2} < c_0$ for some $c_0 > 0$ small. Since $\mathcal{F}(z, (\delta_i)_i, (\xi_i)_i, \phi(z, (\delta_i)_i, (\xi_i)_i)) = 0$ for all $(z, (\delta_i)_i, (\xi_i)_i) \in B_\varepsilon(0) \times \mathcal{D}_k(\varepsilon, N)$, it follows from the implicit functions theorem that $(z, (\delta_i)_i, (\xi_i)_i) \mapsto \phi(z, (\delta_i)_i, (\xi_i)_i)$ is C^1 on $B_\varepsilon(0) \times \mathcal{D}_k(\varepsilon, N)$. This ends the proof of Proposition 5.1. ☐

6. Equivalence of the critical points

We prove Theorem 1.1 in this section. With Proposition 5.1 above, this amounts to prove the equivalence of the critical points for $\varepsilon > 0$ small and $N > 0$ large. For $\varepsilon, N > 0$ satisfying the hypothesis of Proposition 5.1, there exists $\phi \in C^1(B_\varepsilon(0) \times \mathcal{D}_k(\varepsilon, N), H_1^2(M)$ such that

$$\Pi_{K^\perp_{(\delta_i)_i,(\xi_i)_i}}\left(u(z, (\delta_i)_i, (\xi_i)_i) - (\Delta_g + h)^{-1}(F'(u(z, (\delta_i)_i, (\xi_i)_i)))\right) = 0\,, \tag{110}$$

where

$$u(z, (\delta_i)_i, (\xi_i)_i) := u(z) + \sum_{i=1}^k W_{\kappa_i, \delta_i, \xi_i} + \phi(z, (\delta_i)_i, (\xi_i)_i),$$

and

$$\phi(z, (\delta_i)_i, (\xi_i)_i) \in K^\perp_{(\delta_i)_i,(\xi_i)_i} \text{ and } \|\phi(z, (\delta_i)_i, (\xi_i)_i)\|_{H_1^2} \le C \cdot R(z, (\delta_i)_i, (\xi_i)_i) \tag{111}$$

for $(z,(\delta_i)_i,(\xi_i)_i)\in\mathcal{D}_k(\varepsilon,N)$. By (59), it follows that there exist $\lambda^{ij}(z,(\delta_i)_i,(\xi_i)_i)\in\mathbb{R}$ $(i=1,\cdots,k$ and $j=0,\cdots,n)$ and $\mu^l(z,(\delta_i)_i,(\xi_i)_i)\in\mathbb{R}$ $(l=1,\cdots,d)$ such that

$$\begin{aligned}(112)\quad \Pi_{K_{(\delta_i)_i,(\xi_i)_i}}&\left(u(z,(\delta_i)_i,(\xi_i)_i)-(\Delta_g+h)^{-1}(F'(u(z,(\delta_i)_i,(\xi_i)_i)))\right)\\&=\sum_{i=1}^k\sum_{j=0}^n\lambda^{ij}(z,(\delta_i)_i,(\xi_i)_i)Z_{\delta_i,\xi_i,j}+\sum_{l=1}^d\mu^l(z,(\delta_i)_i,(\xi_i)_i)\varphi_l\,.\end{aligned}$$

It then follows from (110) and (112) that for any $\varphi\in H_1^2(M)$, we have that

$$\begin{aligned}&DJ(u(z,(\delta_i)_i,(\xi_i)_i))\varphi\\&\quad=\left(u(z,(\delta_i)_i,(\xi_i)_i)-(\Delta_g+h)^{-1}(F'(u(z,(\delta_i)_i,(\xi_i)_i))),\varphi\right)_h\\(113)\quad&\quad=\sum_{i=1}^k\sum_{j=0}^n\lambda^{ij}(z,(\delta_i)_i,(\xi_i)_i)\left(Z_{\delta_i,\xi_i,j},\varphi\right)_h+\sum_{l=1}^d\mu^l(z,(\delta_i)_i,(\xi_i)_i)(\varphi_l,\varphi)_h\,.\end{aligned}$$

If $u(z,(\delta_i)_i,(\xi_i)_i)$ is a critical point of J, then $(z,(\delta_i)_i,(\xi_i)_i)\in B_\varepsilon(0)\times\mathcal{D}_k(\varepsilon,N)$ is a critical point for $(z,(\delta_i)_i,(\xi_i)_i)\mapsto J(u(z,(\delta_i)_i,(\xi_i)_i))$. Conversely, we assume that $(z,(\delta_i)_i,(\xi_i)_i)\in B_\varepsilon(0)\times\mathcal{D}_k(\varepsilon,N)$ is a critical point for the map $(z,(\delta_i)_i,(\xi_i)_i)\mapsto J(u(z,(\delta_i)_i,(\xi_i)_i))$. We then get that

$$\begin{aligned}(114)\quad 0&=\frac{\partial}{\partial z_{l_0}}J(u(z,(\delta_i)_i,(\xi_i)_i))\\&=DJ(u(z,(\delta_i)_i,(\xi_i)_i)).\left(\partial_{l_0}u(z)+\partial_{z_{l_0}}\phi(z,(\delta_i)_i,(\xi_i)_i)\right),\\(115)\quad 0&=\frac{\partial}{\partial\delta_{i_0}}J(u(z,(\delta_i)_i,(\xi_i)_i))\\&=DJ(u(z,(\delta_i)_i,(\xi_i)_i)).\left(\partial_{\delta_{i_0}}W_{\alpha_{i_0},\delta_{i_0},\xi_{i_0}}+\partial_{\delta_{i_0}}\phi(z,(\delta_i)_i,(\xi_i)_i)\right),\\(116)\quad 0&=\frac{\partial}{\partial(\xi_{i_0})_{j_0}}J(u(z,(\delta_i)_i,(\xi_i)_i))\\&=DJ(u(z,(\delta_i)_i,(\xi_i)_i)).(\partial_{(\xi_{i_0})_{j_0}}W_{\alpha_{i_0},\delta_{i_0},\xi_{i_0}}+\partial_{(\xi_{i_0})_{j_0}}\phi(z,(\delta_i)_i,(\xi_i)_i))\end{aligned}$$

for all $l_0=1,\cdots,d$, $i_0=1,\cdots,k$, and $j_0=0,\cdots,n$. From now on, for the sake of clearness, we omit the variables $(z,(\delta_i)_i,(\xi_i)_i)$. We define

$$\Lambda:=\sum_{i=1}^k\sum_{j=0}^n|\lambda^{ij}|+\sum_{l=1}^d|\mu^l|\,.$$

We are going to prove that $\Lambda=0$, which will imply that $u(z,(\delta_i)_i,(\xi_i)_i)$ is a critical point of J due to (113).

6.1. **Consequences of** (114)**.** It follows from (113) and (114) that

$$\begin{aligned}(117)\quad&\sum_{i=1}^k\sum_{j=0}^n\lambda^{ij}\left((Z_{\delta_i,\xi_i,j},\partial_{l_0}u)_h+(Z_{\delta_i,\xi_i,j},\partial_{z_{l_0}}\phi)_h\right)\\&\qquad+\sum_{l=1}^d\mu^l\left((\varphi_l,\partial_{l_0}u)_h+(\varphi_l,\partial_{z_{l_0}}\phi)_h\right)=0\,.\end{aligned}$$

It follows from (111) that

$$(118)\qquad (Z_{\delta_i,\xi_i,j},\phi)_h=(\varphi_l,\phi)_h=0\,.$$

Differentiating (118) with respect to z_{l_0} yields $(Z_{\delta_i,\xi_i,j}, \partial_{z_{l_0}}\phi)_h = (\varphi_l, \partial_{z_{l_0}}\phi)_h = 0$, and therefore (117) rewrites

$$\sum_{i=1}^{k}\sum_{j=0}^{n}\lambda^{ij}(Z_{\delta_i,\xi_i,j}, \partial_{l_0}u)_h + \sum_{l=1}^{d}\mu^l(\varphi_l, \partial_{l_0}u)_h = 0\,, \tag{119}$$

and therefore, since $\|h-h_0\|_\infty < \varepsilon$, for all $l_0 = 1, .., d$, (25), and (26) yield

$$\left|\sum_{l=1}^{d}\mu^l\left(\varphi_l, \frac{\Pi_{K_0}^{h_0}(\partial_{l_0}u)}{\|\partial_{l_0}u\|_{H_1^2}}\right)_{h_0}\right| \le C\cdot\varepsilon\cdot\Lambda + C\sup_{i,j}|\lambda^{ij}|\,, \tag{120}$$

where $\Pi_{K_0}^{h_0}$ is the orthogonal projection on K_0 (see (3) and (4)) with respect to the Hilbert structure $(\cdot,\cdot)_{h_0}$. We define the matrix $(A(z))_{ll'} := (\varphi_l, \Pi_{K_0}^{h_0}(\partial_{l'}u))_{h_0}$ for all $l, l' \in \{1, \cdots, d\}$. With no loss of generality, we can assume that the basis β_0 is $\{\varphi_1, \cdots, \varphi_d\}$: it then follows from (8) and Cramer's explicit formula that the coefficients of the inverse of the matrix $A(z)$ are bounded from above by a constant C. Therefore, it follows from (120) that

$$\sum_{l=1}^{d}|\mu^l| \le C\cdot\varepsilon\cdot\Lambda + C\sup_{i,j}|\lambda^{ij}| \tag{121}$$

for $(z, (\delta_i)_i, (\xi_i)_i) \in B_\varepsilon(0)\times\mathcal{D}_k(\varepsilon, N)$.

6.2. **Consequences of** (115)**.** It follows from (113) and (115) that

$$\begin{aligned}\sum_{i=1}^{k}\sum_{j=0}^{n}\lambda^{ij}&\left((Z_{\delta_i,\xi_i,j}, \partial_{\delta_{i_0}}W_{\delta_{i_0},\xi_{i_0}})_h + (Z_{\delta_i,\xi_i,j}, \partial_{\delta_{i_0}}\phi)_h\right)\\ &+\sum_{l=1}^{d}\mu^l\left((\varphi_l, \partial_{\delta_{i_0}}W_{\delta_{i_0},\xi_{i_0}})_h + (\varphi_l, \partial_{\delta_{i_0}}\phi)_h\right) = 0\,.\end{aligned} \tag{122}$$

Differentiating (118) with respect to δ_{i_0}, we get that $(\varphi_l, \partial_{\delta_{i_0}}\phi)_h = 0$ and also $(\partial_{\delta_{i_0}}Z_{\delta_i,\xi_i,j}, \phi)_h + (Z_{\delta_i,\xi_i,j}, \partial_{\delta_{i_0}}\phi)_h = 0$, and therefore (122) rewrites

$$\begin{aligned}&\left|\sum_{i=1}^{k}\sum_{j=0}^{n}\lambda^{ij}\left(Z_{\delta_i,\xi_i,j}, \partial_{\delta_{i_0}}W_{\delta_{i_0},\xi_{i_0}}\right)_h\right|\\ &\quad\le\sum_{l=1}^{d}|\mu^l|\cdot\left|(\varphi_l, \partial_{\delta_{i_0}}W_{\delta_{i_0},\xi_{i_0}})_h\right| + \sum_{i=1}^{k}\sum_{j=0}^{n}\left|\lambda^{ij}\right|\cdot\|\partial_{\delta_{i_0}}Z_{\delta_i,\xi_i,j}\|_{H_1^2}\|\phi\|_{H_1^2}\end{aligned} \tag{123}$$

For any $i = 1, \cdots, k$ and $j = 0, \cdots, n$, it follows from (50) and Corollary 3.3 that

$$\begin{aligned}&\left(Z_{\delta_i,\xi_i,j}, \partial_{\delta_{i_0}}W_{\kappa_{i_0}\delta_{i_0},\xi_{i_0}}\right)_h\\ &\quad=\kappa_{i_0}\frac{n-2}{2}\left(\frac{n(n-2)}{f_0(\xi_{i_0})}\right)^{\frac{n-2}{4}}\cdot\frac{1}{\delta_{i_0}}\cdot\left((Z_{\delta_i,\xi_i,j}, Z_{\delta_{i_0},\xi_{i_0},0})_h + o(1)\right)\\ &\quad=\kappa_{i_0}\frac{n-2}{2}\left(\frac{n(n-2)}{f_0(\xi_{i_0})}\right)^{\frac{n-2}{4}}\cdot\frac{1}{\delta_{i_0}}\cdot\left(\delta_{i,i_0}\delta_{j,0}\|\nabla V_0\|_2 + o(1)\right),\end{aligned} \tag{124}$$

where $|o(1)| \le \omega_{15}(\varepsilon, N)$ and $\lim_{\varepsilon\to 0, N\to+\infty} \omega_{15}(\varepsilon, N) = 0$. Plugging (124) into (123) and using (50) yield

$$\begin{aligned}
&|\lambda^{i_0,0}| \cdot \frac{n-2}{2}\left(\frac{n(n-2)}{f_0(\xi_{i_0})}\right)^{\frac{n-2}{4}} \cdot \frac{1}{\delta_{i_0}}\|\nabla V_0\|_2 \\
&\le \Bigg(\sum_{l=1}^{d} \frac{n-2}{2}\left(\frac{n(n-2)}{f_0(\xi_{i_0})}\right)^{\frac{n-2}{4}} \frac{1}{\delta_{i_0}} \left|(\varphi_l, Z_{\delta_{i_0},\xi_{i_0},0})_h\right| \\
&\quad + \sum_{j=0}^{n} \|\partial_{\delta_{i_0}} Z_{\delta_{i_0},\xi_{i_0},j}\|_{H_1^2} \|\phi\|_{H_1^2} + \delta_{i_0}^{-1}\omega_{12}(\varepsilon, N)\Bigg) \cdot \Lambda\,.
\end{aligned}$$

It then follows from (52) and (111), Proposition 3.1, and the expression (25) that

$$|\lambda^{i_0,0}| \le \omega_{16}(\varepsilon, N)\cdot\Lambda \tag{125}$$

for all $i_0 = 1, \cdots, k$, where $\lim_{\varepsilon\to 0, N\to+\infty} \omega_{16}(\varepsilon, N) = 0$.

6.3. **Conclusion for the equivalence.** Arguing as above for (116), we get that

$$|\lambda^{i_0,j}| \le \omega_{17}(\varepsilon, N)\cdot\Lambda \tag{126}$$

for all $i_0 = 1, \cdots, k$ and $j = 1, \cdots, n$, where $\lim_{\varepsilon\to 0, N\to+\infty} \omega_{17}(\varepsilon, N) = 0$. Summing (121), (125), and (126) yields

$$\Lambda = \sum_{i=1}^{k}\sum_{j=0}^{n} |\lambda^{ij}| + \sum_{l=1}^{d} |\mu^l| \le \omega_{18}(\varepsilon, N)\cdot\Lambda\,,$$

where $\lim_{\varepsilon\to 0, N\to+\infty} \omega_{18}(\varepsilon, N) = 0$. Therefore, there exists $\varepsilon, N > 0$ such that $\omega_{15}(\varepsilon, N) < 1/2$, and therefore, we get that $\Lambda = 0$. As mentioned earlier, this implies that $DJ(u(z,(\delta_i)_i,(\xi_i)_i)) = 0$, and then $u(z,(\delta_i)_i,(\xi_i)_i)$ is a critical point of J for $(z,(\delta_i)_i,(\xi_i)_i) \in B_\varepsilon(0)\times\mathcal{D}_k(\varepsilon, N)$. This ends the proof of Theorem 1.1. □

References

[1] A. Ambrosetti and A. Malchiodi, *Perturbation methods and semilinear elliptic problems on* $\mathbb{R}^n$, Progress in Mathematics, vol. 240, Birkhäuser Verlag, Basel, 2006.

[2] G. Bianchi and H. Egnell, *A note on the Sobolev inequality*, J. Funct. Anal. **100** (1991), no. 1, 18–24.

[3] S. Brendle, *Blow-up phenomena for the Yamabe equation*, J. Amer. Math. Soc. **21** (2008), no. 4, 951–979.

[4] S. Brendle and F. C. Marques, *Blow-up phenomena for the Yamabe equation. II*, J. Differential Geom. **81** (2009), no. 2, 225–250.

[5] M. del Pino, M. Musso, F. Pacard, and A. Pistoia, *Large energy entire solutions for the Yamabe equation*, J. Differential Equations **251** (2011), no. 9, 2568–2597.

[6] P. Esposito, A. Pistoia, and J. Vétois, *The effect of linear perturbations on the Yamabe problem*, Math. Ann. doi:10.1007/S00208-013-0971-9 (to appear in print).

[7] M. Falaleev, B. Loginov, N. Sidorov, and A. Sinitsyn, *Lyapunov-Schmidt methods in nonlinear analysis and applications*, Mathematics and its Applications, vol. 550, Kluwer Academic Publishers, Dordrecht, 2002.

[8] Y. Guo, B. Li, and J. Wei, *Large energy entire solutions for the Yamabe type problem of polyharmonic operator*, J. Differential Equations **254** (2013), no. 1, 199–228.

[9] A. M. Micheletti, A. Pistoia, and J. Vétois, *Blow-up solutions for asymptotically critical elliptic equations*, Indiana Univ. Math. J. **58** (2009), no. 4, 1719–1746.

[10] A. Pistoia and J. Vétois, *Sign-changing bubble towers for asymptotically critical elliptic equations on Riemannian manifolds*, J. Differential Equations **254** (2013), no. 11, 4245–4278.

[11] O. Rey, *The role of the Green's function in a nonlinear elliptic equation involving the critical Sobolev exponent*, J. Funct. Anal. **89** (1990), no. 1, 1–52.

[12] F. Robert and J. Vétois, *Sign-Changing Blow-Up for Scalar Curvature Type Equations*, Comm. in Partial Differential Equations **38** (2013), no. 8, 1437–1465.

Frédéric Robert, Institut Élie Cartan, Université de Lorraine, BP 70239, F-54506 Vandœuvre-lès-Nancy, France
E-mail address: frederic.robert@univ-lorraine.fr

Jérôme Vétois, Université de Nice Sophia Antipolis, Laboratoire J. A. Dieudonné, CNRS UMR 6621, Parc Valrose, F-06108 Nice Cedex 2, France
E-mail address: vetois@unice.fr