POSITIVITY AND NON-POSITIVITY RESULTS FOR THE SIXTH-ORDER Q-CURVATURE OF CONFORMAL METRICS IN $\mathbb{R}^n$

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Abstract. Given $n, m \in \mathbb{N}$ such that $n \geq 2m \geq 4$, letting g be a conformally Euclidean metric on $\mathbb{R}^n$, we consider the question of positivity of the lower-order Q-curvatures $Q_g^{(2k)}$ for $k \in \{1, \dots, m-1\}$ when $Q_g^{(2m)}$ is assumed to be nonnegative and not identically zero. We assume moreover that the scalar curvature of the metric g is nonnegative near infinity if $n = 2m$ or that $Q_g^{(2m)}$ satisfies a slow decay barrier condition near infinity if $n > 2m$. Positive results for this question have been obtained by Gursky and Malchiodi [20] for $m = 2$ and $k = 1$ in the context of closed manifolds with nonnegative scalar curvature and by Li and Xu [32] and Li, Wei, and Xu [30] for $m \geq 2$ and $k \in \{1, \min(m-1, 2)\}$ in the context of conformally Euclidean metrics on $\mathbb{R}^n$. These results hold for all $n \geq 2m$. Considering the case where $m \geq 4$ and $k = 3$, we obtain a positive result for this question when $n \in \{2m, 2m+1, \dots, 4m-6\}$, namely for these dimensions, we obtain that if $Q_g^{(2m)} \geq 0$ and $Q_g^{(2m)} \not\equiv 0$ in $\mathbb{R}^n$, then $Q_g^{(6)} > 0$. On the other hand, in surprising contrast with the results of [20, 30, 32], we find that the answer to this question is negative when $k = 3$ and $n \geq N_m$ for some $N_m \in \mathbb{R}$. In this case, we are able to construct examples of conformally Euclidean metrics such that $Q_g^{(2m)}$ is positive everywhere, but $Q_g^{(6)}$ is negative at some point. By stereographic projection, our examples extend to metrics conformal to the standard metric on $\mathbb{S}^n$.

1. Introduction and main result

Let φ be a smooth and positive function on $\mathbb{R}^n$ and $g = \varphi^2 |dx|^2$ be a conformally Euclidean metric on $\mathbb{R}^n$, $n \geq 2$. For every $m \in \mathbb{N}$ such that $n \geq 2m$, the Q-curvature of order $2m$ of the metric g is given by

$$Q_g^{(2m)} = \begin{cases} \dfrac{2}{n-2m} \varphi^{-\frac{n+2m}{2}} (-\Delta)^m \left(\varphi^{\frac{n-2m}{2}}\right) & \text{if } n > 2m \\ \varphi^{-n} (-\Delta)^m (\ln \varphi) & \text{if } n = 2m. \end{cases} \tag{1.1}$$

These curvatures were introduced by Branson and Ørsted [3] for $n = 2m = 4$ and extended by Branson [2] in general dimensions. They constitute a widely studied family of conformal invariants which naturally extend the scalar curvature to higher orders. Their construction is based on Fefferman and

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Graham's ambient metric [13,14], as well as on Graham, Jenne, Mason, and Sparling's GJMS operators [18].

The case where $Q_g^{(2m)} \equiv 1$ has received special attention. In the context we are considering, this case is well understood: up to a finite total Q-curvature condition in the case where $n = 2m$, the conformally Euclidean metrics solving this equation are known to be radially symmetric and have been fully classified. On this subject, we refer to the historical articles by Obata [40] and Caffarelli, Gidas, and Spruck [4] for $m = 1$ and to the later articles by Lin [34], Wei and Xu [44], and Martinazzi [35] for higher orders.

In this article, we are interested in the question of positivity of the lower-order Q-curvatures $Q_g^{(2k)}$ for $k \in \{1, \dots, m-1\}$ when $Q_g^{(2m)}$ is assumed to be nonnegative. In the case where $m = 2$ and $k = 1$, this question was answered positively by Gursky and Malchiodi [20] in the context of a closed manifold with non-negative scalar curvature (see also in this case the article by Fazly, Wei, and Xu [12] where this result was obtained for a class of Q-curvature-type equations in $\mathbb{R}^n$). This result has proven to be useful in establishing several fundamental results. A brief discussion of these and related results is presented at the end of the introduction.

Gursky and Malchiodi's curvature positivity result was recently shown to remain valid in the context of conformally Euclidean metrics on $\mathbb{R}^n$ by Li and Xu [32] for $n > 2m$, $m \geq 2$ and $k \in \{1, \min(m-1, 2)\}$ and by Li, Wei, and Xu [30] for $n = 2m$, $m \geq 2$ and $k \in \{1, \min(m-1, 2)\}$, under a suitable assumption which differs for $n > 2m$ and $n = 2m$ (see (A) below). This remarkable improvement naturally led the authors of these articles to conjecture that this result should remain valid in the case where $1 \leq k \leq m-1$ and $n \geq 2m$. In this article, however, we show that the situation is surprisingly different for $k = 3$. In this case, our results show that the positivity results of [20,30,32] remain valid when n is not too large relative to m, but is not true for all sufficiently large n.

Throughout our work, as in the articles by Li and Xu [32] and Li, Wei, and Xu [30], we make the following assumption:

(A) If $n > 2m$, then we assume that there exists $s \in (-2m, 0)$ and $c, R \in (0, \infty)$ such that $Q_g^{(2m)} \geq c\,|x|^s$ for all $x \in \mathbb{R}^n \setminus B(0, R)$. If $n = 2m$, then we assume that there exists $R \in (0, \infty)$ such that $Q_g^{(2)} \geq 0$ and $Q_g^{(n)} \geq 0$ in $\mathbb{R}^n \setminus B(0, R)$. If $n = 2m$, then we assume moreover that $Q_g^{(n)} \not\equiv 0$ in $\mathbb{R}^n$.

The last point in (A) is not assumed in [30], but it allows us to rule out a trivial case. Assuming (A), we can use an integral representation formula for the conformal factor of the metric g. More precisely, by letting

$$u := \begin{cases} \varphi^{\frac{n-2m}{2}} & \text{if } n > 2m \\ \ln \varphi & \text{if } n = 2m \end{cases}$$

and applying Li and Xu's Theorem 2.6 [32] for $n > 2m$ and Li, Wei, and Xu's Theorems 3.8 and 3.9 [30] for $n = 2m$, we obtain

$$u(x) = \begin{cases} \displaystyle\int_{\mathbb{R}^n} |x-y|^{2m-n} \, d\mu_g^{(2m)}(y) & \text{if } n > 2m \\ \displaystyle\int_{\mathbb{R}^n} \ln\left(\frac{|y|}{|x-y|}\right) d\mu_g^{(n)}(y) + C & \text{if } n = 2m \end{cases} \quad \forall x \in \mathbb{R}^n, \tag{1.2}$$

where $C \in \mathbb{R}$ is a constant, $d\mu_g^{(2m)}$ is the measure defined by

$$d\mu_g^{(2m)}(y) := \frac{\Gamma\left(\frac{n}{2} - m + 1\right)}{2^{2m-1}(m-1)!\,\Gamma\left(\frac{n}{2}\right)\omega_{n-1}} Q_g^{(2m)}(y)\, u(y)^{\frac{n+2m}{n-2m}} dy \quad \forall y \in \mathbb{R}^n$$

if $n > 2m$ and

$$d\mu_g^{(n)}(y) := \frac{2}{(n-1)!\,\omega_n} Q_g^{(n)}(y)\, e^{nu(y)} dy \quad \forall y \in \mathbb{R}^n$$

if $n = 2m$. Moreover, we can differentiate up to $2m-1$ times the integral in (1.2) under the integral sign. In the case where $n = 2m$, this formula relates to the concept of normal metrics introduced by Finn [15] for $m = 1$, extended to higher orders by Chang, Qing, and Yang [7], Fang [11], and Ndiaye and Xiao [39], and subsequently studied in several articles including the recent articles by Li [26–28], Li, Wei, and Xu [30], and Li and Xu [33].

We obtain the following positivity result:

Theorem 1.1. *For every $m \geq 4$, if $n \in \{2m, 2m+1, \dots, 4m-6\}$ and g is a conformally Euclidean metric on $\mathbb{R}^n$ satisfying (A) and $Q_g^{(2m)} \geq 0$ in $\mathbb{R}^n$, then $Q_g^{(6)} > 0$ in $\mathbb{R}^n$.*

On the other hand, for large values of n, we obtain the following non-positivity result:

Theorem 1.2. *For every $m \in \mathbb{N}$ such that $m \geq 4$, let N_m be the largest real root of the polynomial*

$$\begin{aligned} \gamma_m(n) :=& n^5 - 19(m-1)n^4 + \left(117m^2 - 212m + 92\right)n^3 - \left(337m^3 - 851m^2\right. \\ & \left. + 652m - 140\right)n^2 + 2(m-1)\left(241m^3 - 518m^2 + 264m - 24\right)n \\ & - 4m(m-1)\left(70m^3 - 187m^2 + 133m - 24\right). \end{aligned} \tag{1.3}$$

Then for every $n > N_m$, there exists a conformally Euclidean metric g on $\mathbb{R}^n$ such that $Q_g^{(6)}(0) < 0$ and $C^{-1} \leq Q_g^{(2m)} \leq C$ in $\mathbb{R}^n$ for some constant $C > 0$. Additionally, by stereographic projection, g extends to a smooth metric on $\mathbb{S}^n$. Furthermore, $N_m \sim \Lambda m$ as $m \to \infty$ and $N_m < \Lambda m$ for all $m \in \mathbb{N}$ with $m \geq 4$, where $\Lambda \simeq 10.55$ is the largest real root of the polynomial $X^4 - 17X^3 + 83X^2 - 171X + 140$.

For small values of m, numerical computations give $N_4 \simeq 29.1$, $N_5 \simeq 39.7$, $N_6 \simeq 50.3$, $N_7 \simeq 60.9$, $N_8 \simeq 71.4$, $N_9 \simeq 82$, and $N_{10} \simeq 92.6$.

Both the proofs of Theorems 1.1 and 1.2 rely on an integral formula for $Q_g^{(6)}$ which we derive in Section 2. We then prove Theorem 1.1 in Section 3 and Theorem 1.2 in Section 4. Some comments are in order. The proof of Theorem 1.1 reduces to studying the positivity of a cubic polynomial of multiple nonnegative variables. Note that this is clearly more difficult than in the cases of $Q_g^{(2)}$ and $Q_g^{(4)}$, where the polynomial to study is of degree 1 and 2, respectively. The proof is then divided into several cases. When $2m+1 \leq n \leq \frac{5m-3}{2}$ or $n \in \left\{\frac{8m-6}{3}, 3m-3, 4m-6\right\}$, all the coefficients of the polynomial are nonnegative. When $\frac{5m-3}{2} < n < \frac{8m-6}{3}$ or $\frac{8m-6}{3} < n < 3m-3$, only one coefficient is negative. The most difficult case that we deal with in Theorem 1.1 is the case where $3m-3 < n < 4m-6$, in which four coefficients are negative. In the remaining case where $n > 4m-6$, three coefficients are negative. While we do not think that the dimension $4m-6$ is optimal in Theorem 1.1, it is clear from Theorem 1.2 that the result does not remain valid for all $n > 4m-6$. The proof of Theorem 1.2 relies again on the study of the cubic polynomial obtained in Section 2. For large values of n, we manage to obtain our non-positivity result for $Q_g^{(6)}$ by constructing a suitable family of conformal metrics with peaks of different heights at two different points.

We conclude this introduction by briefly mentioning some results in the context of closed manifolds which have been obtained thanks to Gursky and Malchiodi's curvature positivity result, as well as some related results. First, Gursky and Malchiodi [20] used this result to obtain positivity results on the associated Paneitz–Branson operator and on its Green's function, which in turn allowed them to obtain an existence result for the constant fourth-order Q-curvature problem. In this regard, we also mention the article by Hang and Yang [21] where a different approach is used for these results and the positive scalar curvature assumption is relaxed. More generally, positivity properties of GJMS operators and of their Green's functions are known to have applications to several fundamental results including existence, quantitative stability, and compactness results of metrics with prescribed Q-curvature or, more generally, positive solutions to Q-curvature-type equations. In addition to the already mentioned articles [20, 21], earlier and later references on existence results include articles by Djadli, Hebey and Ledoux [9], Esposito and Robert [10], Robert [41, 42], Hang and Yang [22], Gursky, Hang, and Lin [19], Chen and Hou [8], Mazumdar and Vétois [38], and Mazumdar and Ndiaye [37]; see also the article by Hyder and Sire [25] on the existence of singular solutions. A reference on quantitative result is by Andrade, König, Ratzkin, and Wei [1]. As regards compactness results, we refer to the articles by Hebey and Robert [23], Hebey, Robert, and Wen [24], Li and Xiong [31], Gong, Kim, and Wei [17], and Mazumdar and Premoselli [36]. We now mention another type of applications of Gursky and Malchiodi's curvature positivity result. As can be seen in the articles by Vétois [43], Case [5], and Li and Wei [29], this result has also proven to be useful in establishing uniqueness results of conformal metrics with prescribed Q-curvature or, again, more generally, positive solutions to Q-curvature-type equations. We refer to the article by Case and Gover [6] for a recent survey on GJMS operators and Q-curvatures. We also mention a recent article by Ge, Wang, and Wei [16] where analogues of the above-mentioned existence and uniqueness results have been obtained for a problem involving the quotient between the Q-curvature of order 4 and the scalar curvature.

2. An integral formula for $Q_g^{(6)}$

Let $n, m \in \mathbb{N}$ be such that $n \geq 2m \geq 8$ and g be a conformally Euclidean metric on $\mathbb{R}^n$ satisfying (A). Throughout this section and the next, we use the notation

$$d\mu_g^{(2m)}(y) := d\mu_g^{(2m)}(y_1) \times \cdots \times d\mu_g^{(2m)}(y_l)$$

for all $l \in \mathbb{N}$ and $y = (y_1, \ldots, y_l) \in \mathbb{R}^{ln}$. We denote $y = y_1$ if $l = 1$ and $y = (y_1, \ldots, y_l)$ if $l \geq 2$. For conciseness, we use the notation

$$|x-y|^{[\theta_1, \ldots, \theta_l]} := \prod_{i=1}^{l} |x-y_i|^{\theta_i}$$

for all $l \in \mathbb{N}$, $\theta_1, \ldots, \theta_l \in \mathbb{R}$, $x \in \mathbb{R}^n$, and $y = (y_1, \ldots, y_l) \in \mathbb{R}^{ln}$. We also define, for $i, j \in \mathbb{N}$ with $i \neq j$,

$$Z_{i,j}(x,y) := \frac{|y_i - y_j|^2}{|x-y_i|^2 |x-y_j|^2} \quad \text{and} \quad Z_i(x,y) := \frac{1}{|x-y_i|^2}$$

for all $x, y_i, y_j \in \mathbb{R}^n$. We note

$$\frac{\langle x-y_i, x-y_j \rangle}{|x-y_i|^2 |x-y_j|^2} = \frac{1}{2}\left(Z_i(x,y) + Z_j(x,y) - Z_{i,j}(x,y)\right),$$

which gives (assuming we can differentiate under the integral sign)

$$\begin{aligned}
&\Delta\left(\int_{\mathbb{R}^{ln}} |x-y|^{[\theta_1,\ldots,\theta_l]} f(y)\, d\mu_g^{(2m)}(y)\right) \\
&\quad = \int_{\mathbb{R}^{ln}} |x-y|^{[\theta_1,\ldots,\theta_l]} f(y) \left(\left(\sum_{i=1}^{l} \theta_i + n - 2\right)\left(\sum_{i=1}^{l} \theta_i Z_i(x,y)\right)\right. \\
&\qquad \left. - \sum_{1 \leq i < j \leq l} \theta_i \theta_j Z_{i,j}(x,y)\right) d\mu_g^{(2m)}(y), \qquad (2.1)
\end{aligned}$$

$$\Delta u(x) = -(2m-2)\max(n-2m, 1) \int_{\mathbb{R}^n} |x-y|^{2m-n-2}\, d\mu_g^{(2m)}(y), \quad (2.2)$$

and

$$\left\langle \nabla u\left(x\right), \nabla \left(\int_{\mathbb{R}^{ln}} \left|x-y\right|^{\left[\theta_1,\dots,\theta_l\right]} f\left(y\right) d\mu_g^{(2m)}\left(y\right)\right)\right\rangle$$
$$= -\frac{1}{2}\max\left(n-2m,1\right)\int_{\mathbb{R}^{(l+1)n}} \left|x-y\right|^{\left[\theta_1,\dots,\theta_l,2m-n\right]} f\left(y\right)\sum_{i=1}^{l}\theta_i\left(Z_i\left(x,y\right)\right.$$
$$\left.+Z_{l+1}\left(x,y\right)-Z_{i,l+1}\left(x,y\right)\right)d\mu_g^{(2m)}\left(y\right), \tag{2.3}$$

where u is as in (1.2).

This section is devoted to proving the following result:

Proposition 2.1. *Let $n,m \in \mathbb{N}$ be such that $n \geq 2m \geq 8$ and g be a conformally Euclidean metric on $\mathbb{R}^n$ satisfying (A). If $n > 2m$, then*

$$Q_g^{(6)}\left(x\right) = 2\left(m-3\right)\int_{\mathbb{R}^{6n}}\prod_{i=1}^{6}\left|x-y_i\right|^{2m-n} P_{n,m}\left(Z\left(x,y\right)\right)d\mu_g^{(2m)}\left(y\right), \tag{2.4}$$

where $Z := \left(Z_{i,j},Z_i\right)_{1\leq i<j\leq 6}$ and $P_{n,m}$ is the cubic polynomial defined by

$$\begin{aligned}
&P_{n,m}\left(Z\right)\\
&:= Z_{1,2}\Big(\left(2m-n-2\right)^2\left(2m-n-4\right)^2 Z_{1,2}^2\\
&\quad -6\left(4m-n-6\right)\left(2m-n-2\right)^2\left(2m-n-4\right)Z_{1,2}Z_{1,3}\\
&\quad +3\left(4m-n-6\right)\left(3m-n-3\right)\left(2m-n-2\right)^2 Z_{1,2}Z_{3,4}\\
&\quad +4\left(4m-n-6\right)\left(3m-n-3\right)\left(2m-n-2\right)\left(2m-n-4\right)Z_{1,3}Z_{1,4}\\
&\quad -2\left(4m-n-6\right)\left(2m-n-2\right)^3 Z_{1,3}Z_{2,3}\\
&\quad +12\left(4m-n-6\right)\left(3m-n-3\right)\left(2m-n-2\right)^2 Z_{1,3}Z_{3,4}\\
&\quad -6\left(4m-n-6\right)\left(3m-n-3\right)\left(8m-3n-6\right)\left(2m-n-2\right)Z_{1,3}Z_{5,6}\\
&\quad +\left(4m-n-6\right)\left(3m-n-3\right)\left(5m-2n-3\right)\left(8m-3n-6\right)Z_{3,4}Z_{5,6}\Big).
\end{aligned}$$

If $n = 2m$, then

$$Q_g^{(6)}\left(x\right) = e^{-6u}\int_{\mathbb{R}^{6n}} P_{n,m}\left(Z\left(x,y\right)\right)d\mu_g^{(n)}\left(y\right), \tag{2.5}$$

where $Z := \left(Z_{i,j},Z_i\right)_{1\leq i<j\leq 6}$ and $P_{n,m}$ is defined by

$$\begin{aligned}
&P_{n,m}\left(Z\right)\\
&:= \frac{4\left(n-6\right)\left(2-\lambda\right)}{\lambda^5}\left(n-2-\frac{n-6}{2}\lambda\right)\left(n-4-\frac{n-6}{2}\lambda\right)Z_1^3\\
&\quad+\frac{6\left(n-6\right)^2\left(2-\lambda\right)}{\lambda^4}\left(n-4-\frac{n-6}{2}\lambda\right)Z_1^2 Z_{1,2}\\
&\quad+\frac{3\left(n-6\right)^2\left(2-\lambda\right)}{2\lambda^4}\left(n-2-\frac{n-6}{2}\lambda\right)\left(n-4-\frac{n-6}{2}\lambda\right)Z_1^2 Z_2
\end{aligned}$$

$$
\begin{aligned}
&+\frac{3(n-6)^3(2-\lambda)}{8\lambda^3}\left(n-4-\frac{n-6}{2}\lambda\right)Z_1^2Z_{2,3}+\frac{12(n-6)^2(2-\lambda)}{\lambda^4}Z_1Z_{1,2}^2\\
&+\frac{3(n-6)^3(2-\lambda)}{\lambda^3}Z_1Z_{1,2}Z_{1,3}\\
&+\frac{3(n-6)^2(2-\lambda)}{\lambda^4}\left(n-4-\frac{n-6}{2}\lambda\right)Z_1Z_2Z_{1,2}\\
&+\frac{3(n-6)^3(2-\lambda)}{\lambda^3}Z_1Z_{1,2}Z_{2,3}+\frac{3(n-6)^4(2-\lambda)}{8\lambda^2}Z_1Z_{1,2}Z_{3,4}\\
&+\frac{3(n-6)^3(2-\lambda)}{2\lambda^3}\left(n-4-\frac{n-6}{2}\lambda\right)Z_1Z_2Z_{1,3}\\
&+\frac{(n-6)^3(2-\lambda)}{8\lambda^3}\left(n-2-\frac{n-6}{2}\lambda\right)\left(n-4-\frac{n-6}{2}\lambda\right)Z_1Z_2Z_3\\
&+\frac{3(n-6)^4(2-\lambda)}{32\lambda^2}\left(n-4-\frac{n-6}{2}\lambda\right)Z_1Z_2Z_{3,4}\\
&+\frac{3(n-6)^3(2-\lambda)}{4\lambda^3}Z_1Z_{2,3}^2+\frac{3(n-6)^4(2-\lambda)}{8\lambda^2}Z_1Z_{2,3}Z_{2,4}\\
&+\frac{3(n-6)^5(2-\lambda)}{128\lambda}Z_1Z_{2,3}Z_{4,5}+\frac{16(n-6)}{\lambda^4}Z_{1,2}^3+\frac{12(n-6)^2}{\lambda^3}Z_{1,2}^2Z_{1,3}\\
&+\frac{3(n-6)^3}{8\lambda^2}Z_{1,2}^2Z_{3,4}+\frac{(n-6)^3}{\lambda^2}Z_{1,2}Z_{1,3}Z_{1,4}+\frac{2(n-6)^2}{\lambda^3}Z_{1,2}Z_{1,3}Z_{2,3}\\
&+\frac{3(n-6)^3}{2\lambda^2}Z_{1,2}Z_{1,3}Z_{3,4}+\frac{3(n-6)^4}{16\lambda}Z_{1,2}Z_{1,3}Z_{5,6}+\frac{(n-6)^5}{256}Z_{1,2}Z_{3,4}Z_{5,6},
\end{aligned}
$$

where $\lambda := \mu_g^{(2m)}(\mathbb{R}^n)$.

We note that for $n = 2m$, we have $0 < \lambda \leq 2$ as a consequence of (A) together with Li, Wei, and Xu's Theorem 3.8 [30].

Proof of Proposition 2.1. We begin with assuming $n > 2m$. Let φ be a smooth and positive function on $\mathbb{R}^n$, $g = \varphi^2 |dx|^2$ be a conformally Euclidean metric on $\mathbb{R}^n$, and u be as in (1.2). We begin by computing $\Delta^2(u^t)$, where

$$t := \frac{n-6}{n-2m}.$$

We note

$$t-1 = \frac{2m-6}{n-2m}.$$

Simple computations show

$$\Delta\left(u^t\right) = \operatorname{div}\left(tu^{t-1}\nabla u\right) = t(t-1)u^{t-2}|\nabla u|^2 + tu^{t-1}\Delta u.$$

Plugging (2.2) and (2.3) into this formula gives

$$u(x)^{2-t}\,\Delta\left(u^t\right)(x) = -(n-6)\int_{\mathbb{R}^{2n}} |x-y_1|^{2m-n}\,|x-y_2|^{2m-n} \times \left(4Z_1(x,y) + (m-3)\,Z_{1,2}(x,y)\right) d\mu_g^{(2m)}(y). \tag{2.6}$$

We now write

$$t-2 = \frac{4m-n-6}{n-2m} \quad \text{and} \quad t-3 = \frac{6m-2n-6}{n-2m}.$$

Similarly as in (2.6), we calculate

$$\begin{aligned} &u(x)^{4-t}\,\Delta\left(u^{t-2}\right)(x) \\ &\quad = (4m-n-6)\int_{\mathbb{R}^{2n}} |x-y_3|^{2m-n}\,|x-y_4|^{2m-n}\left((4m-2n-4)\,Z_3(x,y)\right. \\ &\qquad \left. -(3m-n-3)\,Z_{3,4}(x,y)\right) d\mu_g^{(2m)}(y). \end{aligned} \tag{2.7}$$

We then have

$$\begin{aligned} \Delta^2\left(u^t\right) &= \Delta\left(u^{t-2}\cdot u^{2-t}\Delta\left(u^t\right)\right) \\ &= u^{2-t}\Delta\left(u^{t-2}\right)\Delta\left(u^t\right) + 2\left\langle \nabla\left(u^{t-2}\right), \nabla\left(u^{2-t}\Delta\left(u^t\right)\right)\right\rangle \\ &\quad + u^{t-2}\Delta\left(u^{2-t}\Delta\left(u^t\right)\right). \end{aligned} \tag{2.8}$$

Applying (2.1), (2.3), (2.6), (2.7), and (2.8) gives

$$\begin{aligned} &u(x)^{4-t}\,\Delta^2\left(u^t\right)(x) \\ &\quad = -(n-6)\int_{\mathbb{R}^{4n}} \prod_{i=1}^{4} |x-y_i|^{2m-n}\Big((4m-n-6)\,(4Z_1 + (m-3)\,Z_{1,2}) \\ &\qquad \times ((4m-2n-4)\,Z_3 - (3m-n-3)\,Z_{3,4}) - 4\,(4m-n-6)\,Z_1 \\ &\qquad \times \big((2m-n-2)\,Z_1 + (2m-n)\,Z_2 + (4m-2n-2)\,Z_3 \\ &\qquad - (2m-n-2)\,Z_{1,3} - (2m-n)\,Z_{2,3}\big) - (m-3)\,(4m-n-6) \\ &\qquad \times (2m-n-2)\,Z_{1,2}\,(Z_1 + Z_2 + 2Z_3 - Z_{1,3} - Z_{2,3}) \\ &\qquad + 4Z_1\big((4m-n-4)\,((2m-n-2)\,Z_1 + (2m-n)\,Z_2) \\ &\qquad - (2m-n)\,(2m-n-2)\,Z_{1,2}\big) + (m-3)\,(2m-n-2)\,Z_{1,2} \\ &\qquad \times ((4m-n-6)\,(Z_1+Z_2) - (2m-n-2)\,Z_{1,2})\Big)(x,y)\,d\mu_g^{(2m)}(y). \end{aligned}$$

By interchanging the order of integration, we then obtain

$$\begin{aligned} &u(x)^{4-t}\,\Delta^2\left(u^t\right)(x) \\ &\quad = (n-6)\int_{\mathbb{R}^{4n}} \prod_{i=1}^{4} |x-y_i|^{2m-n}\left(c_1 Z_1^2 + c_2 Z_1 Z_2 + c_3 Z_1 Z_{1,2} + c_4 Z_1 Z_{2,3}\right. \\ &\qquad \left. + c_5 Z_{1,2}^2 + c_6 Z_{1,2} Z_{1,3} + c_7 Z_{1,2} Z_{3,4}\right)(x,y)\,d\mu_g^{(2m)}(y), \end{aligned} \tag{2.9}$$

where

$$\begin{aligned} c_1 &:= 4(4m-n-6)(2m-n-2) - 4(4m-n-4)(2m-n-2) \\ &= -8(2m-n-2), \\ c_2 &:= -4(4m-n-6)(4m-2n-4) + 4(4m-n-6)(6m-3n-2) \\ &\quad - 4(4m-n-4)(2m-n) \\ &= 16(m-3), \\ c_3 &:= -4(4m-n-6)(2m-n-2) + 2(m-3)(4m-n-6)(2m-n-2) \\ &\quad + 4(2m-n)(2m-n-2) - 2(m-3)(2m-n-2)(4m-n-6) \\ &= -8(m-3)(2m-n-2), \\ c_4 &:= -(4m-n-6)(m-3)(4m-2n-4) + 4(4m-n-6)(3m-n-3) \\ &\quad - 4(4m-n-6)(2m-n) + 2(m-3)(4m-n-6)(2m-n-2) \\ &= 4(m-3)(4m-n-6), \\ c_5 &:= (m-3)(2m-n-2)^2, \\ c_6 &:= -2(m-3)(4m-n-6)(2m-n-2), \\ c_7 &:= (m-3)(4m-n-6)(3m-n-3). \end{aligned}$$

To compute $(-\Delta)^3\left(u^t\right)$, we write

$$\begin{aligned} \Delta^3\left(u^t\right) &= \Delta\left(u^{t-4}\cdot u^{4-t}\Delta^2\left(u^t\right)\right) \\ &= u^{4-t}\Delta\left(u^{t-4}\right)\Delta^2\left(u^t\right) + 2\left\langle \nabla\left(u^{t-4}\right), \nabla\left(u^{4-t}\Delta^2\left(u^t\right)\right)\right\rangle \\ &\quad + u^{t-4}\Delta\left(u^{4-t}\Delta^2\left(u^t\right)\right). \end{aligned} \tag{2.10}$$

Similarly to (2.11), we now write

$$t-4 = \frac{8m-3n-6}{n-2m} \quad \text{and} \quad t-5 = \frac{10m-4n-6}{n-2m},$$

and calculate

$$\begin{aligned} &u(x)^{6-t}\,\Delta\left(u^{t-4}\right)(x) \\ &\quad = (8m-3n-6)\int_{\mathbb{R}^{2n}} |x-y_5|^{2m-n}\,|x-y_6|^{2m-n}\,((8m-4n-4)\,Z_5(x,y) \\ &\qquad - (5m-2n-3)\,Z_{5,6}(x,y))\,d\mu_g^{(2m)}(y). \end{aligned} \tag{2.11}$$

Applying (2.1), (2.3), (2.9), and (2.11) gives

$$\begin{aligned} &\frac{1}{n-6}\,u(x)^{10-2t}\,\Delta\left(u^{t-4}\right)(x)\,\Delta^2\left(u^t\right)(x) \\ &\quad = (8m-3n-6)\int_{\mathbb{R}^{6n}}\prod_{i=1}^{6}|x-y_i|^{2m-n}\left(c_1Z_1^2 + c_2Z_1Z_2 + c_3Z_1Z_{1,2}\right. \\ &\qquad \left. + c_4Z_1Z_{2,3} + c_5Z_{1,2}^2 + c_6Z_{1,2}Z_{1,3} + c_7Z_{1,2}Z_{3,4}\right)((8m-4n-4)\,Z_5 \\ &\qquad - (5m-2n-3)\,Z_{5,6})\,(x,y)\,d\mu_g^{(2m)}(y), \end{aligned}$$

$$\begin{aligned}
&\frac{2}{n-6}\, u\left(x\right)^{6-t} \left\langle \nabla\left(u^{t-4}\right)\left(x\right), \nabla\left(u^{4-t}\Delta^2\left(u^t\right)\right)\left(x\right)\right\rangle \\
&\quad = \left(3n-8m+6\right)\int_{\mathbb{R}^{6n}}\prod_{i=1}^{6}\left|x-y_i\right|^{2m-n}\Big(c_1 Z_1^2\Big(\left(2m-n-4\right)\left(Z_1+Z_5\right. \\
&\qquad \left.-Z_{1,5}\right)+\left(2m-n\right)\left(Z_2+Z_3+Z_4+3Z_5-Z_{2,5}-Z_{3,5}-Z_{4,5}\right)\Big) \\
&\qquad + c_2 Z_1 Z_2\Big(\left(2m-n-2\right)\left(Z_1+Z_2+2Z_5-Z_{1,5}-Z_{2,5}\right)+\left(2m-n\right) \\
&\qquad \times\left(Z_3+Z_4+2Z_5-Z_{3,5}-Z_{4,5}\right)\Big)+c_3 Z_1 Z_{1,2}\Big(\left(2m-n-4\right)\left(Z_1+Z_5\right. \\
&\qquad \left.-Z_{1,5}\right)+\left(2m-n-2\right)\left(Z_2+Z_5-Z_{2,5}\right)+\left(2m-n\right)\left(Z_3+Z_4+2Z_5\right. \\
&\qquad \left.-Z_{3,5}-Z_{4,5}\right)\Big)+c_4 Z_1 Z_{2,3}\Big(\left(2m-n-2\right)\left(Z_1+Z_2+Z_3+3Z_5-Z_{1,5}\right. \\
&\qquad \left.-Z_{2,5}-Z_{3,5}\right)+\left(2m-n\right)\left(Z_4+Z_5-Z_{4,5}\right)\Big)+c_5 Z_{1,2}^2\Big(\left(2m-n-4\right) \\
&\qquad \times\left(Z_1+Z_2+2Z_5-Z_{1,5}-Z_{2,5}\right)+\left(2m-n\right)\left(Z_3+Z_4+2Z_5-Z_{3,5}\right. \\
&\qquad \left.-Z_{4,5}\right)\Big)+c_6 Z_{1,2} Z_{1,3}\Big(\left(2m-n-4\right)\left(Z_1+Z_5-Z_{1,5}\right)+\left(2m-n-2\right) \\
&\qquad \times\left(Z_2+Z_3+2Z_5-Z_{2,5}-Z_{3,5}\right)+\left(2m-n\right)\left(Z_4+Z_5-Z_{4,5}\right)\Big) \\
&\qquad +\left(2m-n-2\right)c_7 Z_{1,2}Z_{3,4}\left(Z_1+Z_2+Z_3+Z_4+4Z_5-Z_{1,5}-Z_{2,5}\right. \\
&\qquad \left.-Z_{3,5}-Z_{4,5}\right)\Big)\left(x,y\right)d\mu_g^{(2m)}\left(y\right),
\end{aligned}$$

and

$$\begin{aligned}
&\frac{1}{n-6}\, u\left(x\right)^2 \Delta\left(u^{4-t}\Delta^2\left(u^t\right)\right)\left(x\right) \\
&\quad = \int_{\mathbb{R}^{6n}}\prod_{i=1}^{6}\left|x-y_i\right|^{2m-n}\Big(c_1 Z_1^2\Big(\left(8m-3n-6\right)\left(\left(2m-n-4\right)Z_1\right. \\
&\qquad \left.+\left(2m-n\right)\left(Z_2+Z_3+Z_4\right)\right)-\left(2m-n\right)\left(2m-n-4\right)\left(Z_{1,2}+Z_{1,3}\right. \\
&\qquad \left.+Z_{1,4}\right)-\left(2m-n\right)^2\left(Z_{2,3}+Z_{2,4}+Z_{3,4}\right)\Big)+c_2 Z_1 Z_2\Big(\left(8m-3n-6\right) \\
&\qquad \times\left(\left(2m-n-2\right)\left(Z_1+Z_2\right)+\left(2m-n\right)\left(Z_3+Z_4\right)\right)-\left(2m-n-2\right)^2 Z_{1,2} \\
&\qquad -\left(2m-n\right)\left(2m-n-2\right)\left(Z_{1,3}+Z_{1,4}+Z_{2,3}+Z_{2,4}\right)-\left(2m-n\right)^2 Z_{3,4}\Big) \\
&\qquad + c_3 Z_1 Z_{1,2}\Big(\left(8m-3n-8\right)\left(\left(2m-n-4\right)Z_1+\left(2m-n-2\right)Z_2\right. \\
&\qquad \left.+\left(2m-n\right)\left(Z_3+Z_4\right)\right)-\left(2m-n-2\right)\left(2m-n-4\right)Z_{1,2}-\left(2m-n\right) \\
&\qquad \times\left(2m-n-4\right)\left(Z_{1,3}+Z_{1,4}\right)-\left(2m-n\right)\left(2m-n-2\right)\left(Z_{2,3}+Z_{2,4}\right) \\
&\qquad -\left(2m-n\right)^2 Z_{3,4}\Big)+c_4 Z_1 Z_{2,3}\Big(\left(8m-3n-8\right)\left(\left(2m-n-2\right)\left(Z_1+Z_2\right.\right. \\
&\qquad \left.\left.+Z_3\right)+\left(2m-n\right)Z_4\right)-\left(2m-n-2\right)^2\left(Z_{1,2}+Z_{1,3}+Z_{2,3}\right)-\left(2m-n\right) \\
&\qquad \times\left(2m-n-2\right)\left(Z_{1,4}+Z_{2,4}+Z_{3,4}\right)\Big)+c_5 Z_{1,2}^2\Big(\left(8m-3n-10\right) \\
&\qquad \times\left(\left(2m-n-4\right)\left(Z_1+Z_2\right)+\left(2m-n\right)\left(Z_3+Z_4\right)\right)-\left(2m-n-4\right)^2 Z_{1,2} \\
&\qquad -\left(2m-n\right)\left(2m-n-4\right)\left(Z_{1,3}+Z_{1,4}+Z_{2,3}+Z_{2,4}\right)-\left(2m-n\right)^2 Z_{3,4}\Big)
\end{aligned}$$

$$
\begin{aligned}
&+ c_6 Z_{1,2} Z_{1,3} \big((8m-3n-10)\left((2m-n-4) Z_1 + (2m-n-2)\right. \\
&\times (Z_2+Z_3) + (2m-n) Z_4\big) - (2m-n-2)(2m-n-4)(Z_{1,2}+Z_{1,3}) \\
&- (2m-n)(2m-n-4) Z_{1,4} - (2m-n-2)^2 Z_{2,3} - (2m-n) \\
&\times (2m-n-2)(Z_{2,4}+Z_{3,4}) \big) + (2m-n-2) c_7 Z_{1,2} Z_{3,4} \\
&\times \big((8m-3n-10)(Z_1+Z_2+Z_3+Z_4) - (2m-n-2)(Z_{1,2}+Z_{1,3} \\
&+Z_{1,4}+Z_{2,3}+Z_{2,4}+Z_{3,4}) \big) \big) (x,y)\, d\mu_g^{(2m)}(y).
\end{aligned}
$$

Interchanging the order of integration and plugging these formulas into (2.10) then gives

$$
\begin{aligned}
Q_g^{(6)}(x) &= \frac{2}{n-6} u(x)^{6-t} (-\Delta)^3 \left(u^t\right)(x) \\
&= 2 \int_{\mathbb{R}^{6n}} \prod_{i=1}^{6} |x-y_i|^{2m-n} Z_{1,2}(x,y) \big(c_1' Z_{1,2}^2 + c_2' Z_{1,2} Z_{1,3} \\
&\quad + c_3' Z_{1,2} Z_{3,4} + c_4' Z_{1,3} Z_{1,4} + c_5' Z_{1,3} Z_{2,3} + c_6' Z_{1,3} Z_{3,4} \\
&\quad + c_7' Z_{1,3} Z_{5,6} + c_8' Z_{3,4} Z_{5,6} \big)(x,y)\, d\mu_g^{(2m)}(y),
\end{aligned}
$$

where

$$
\begin{aligned}
c_1' &:= (2m-n-4)^2 c_5 \\
&= (m-3)(2m-n-2)^2 (2m-n-4)^2, \\
c_2' &:= 2(3n-8m+6)(2m-n-4) c_5 + 4(2m-n)(2m-n-4) c_5 \\
&\quad + 2(2m-n-2)(2m-n-4) c_6 \\
&= -6(m-3)(4m-n-6)(2m-n-2)^2 (2m-n-4), \\
c_3' &:= (8m-3n-6)(5m-2n-3) c_5 + 2(3n-8m+6)(2m-n) c_5 \\
&\quad + (2m-n)^2 c_5 + 2(2m-n-2)^2 c_7 \\
&= 3(m-3)(4m-n-6)(3m-n-3)(2m-n-2)^2, \\
c_4' &:= (3n-8m+6)(2m-n-4) c_6 + (2m-n)(2m-n-4) c_6 \\
&= 4(m-3)(4m-n-6)(3m-n-3)(2m-n-2)(2m-n-4), \\
c_5' &:= (2m-n-2)^2 c_6 \\
&= -2(m-3)(4m-n-6)(2m-n-2)^3, \\
c_6' &:= 2(3n-8m+6)(2m-n-2) c_6 + 2(2m-n)(2m-n-2) c_6 \\
&\quad + 4(2m-n-2)^2 c_7 \\
&= 12(m-3)(4m-n-6)(3m-n-3)(2m-n-2)^2, \\
c_7' &:= (8m-3n-6)(5m-2n-3) c_6 + (3n-8m+6)(2m-n) c_6 \\
&\quad + 4(3n-8m+6)(2m-n-2) c_7 \\
&= -6(m-3)(4m-n-6)(3m-n-3)(8m-3n-6)(2m-n-2),
\end{aligned}
$$

$$\begin{aligned}c'_8 &:= (8m-3n-6)(5m-2n-3)\,c_7\\&= (m-3)(4m-n-6)(3m-n-3)(8m-3n-6)(5m-2n-3)\,.\end{aligned}$$

Note that no term containing $Z_1,\dots,Z_6$ appears in this formula. Indeed, computations similar to those above show that the coefficients of all these terms are zero.

Now we assume $n=2m$. In this case, in addition to (2.1), (2.2), and (2.3), we note the formula

$$\int_{\mathbb{R}^{(k+l)n}} f(y)\,d\mu_g^{(n)}(y) = \lambda^k \int_{\mathbb{R}^{ln}} f(y)\,d\mu_g^{(n)}(y) \tag{2.12}$$

for all $k,l\in\mathbb{N}$, $y=(y_1,\dots,y_l)\in\mathbb{R}^{ln}$ and $f\in L^1\big(\mathbb{R}^{ln},\mu_g^{(n)}\big)$. We compute

$$\begin{aligned}&-e^{-\frac{n-6}{2}u(x)}\Delta\left(e^{\frac{n-6}{2}u}\right)(x)\\&= -\frac{n-6}{2}\left(\Delta u(x)+\frac{n-6}{2}\left|\nabla u(x)\right|^2\right)\\&= \frac{n-6}{2}\left((n-2)\int_{\mathbb{R}^n} Z_1\,d\mu_g^{(n)} - \frac{n-6}{4}\int_{\mathbb{R}^{2n}}(Z_1+Z_2-Z_{1,2})\,d\mu_g^{(n)}\right)\\&= a_1\int_{\mathbb{R}^n} Z_1\,d\mu_g^{(2m)} + a_2\int_{\mathbb{R}^{2n}} Z_{1,2}\,d\mu_g^{(n)},\end{aligned} \tag{2.13}$$

where

$$a_1 := \frac{n-6}{2}\left(n-2-\frac{n-6}{2}\lambda\right) \quad\text{and}\quad a_2 := \frac{(n-6)^2}{8}.$$

We then compute

$$\begin{aligned}&e^{-\frac{n-6}{2}u(x)}\Delta^2\left(e^{\frac{n-6}{2}u}\right)(x)\\&= e^{-(n-6)u(x)}\left(\Delta\left(e^{\frac{n-6}{2}u}\right)(x)\right)^2 + \Delta\left(e^{-\frac{n-6}{2}u}\Delta\left(e^{\frac{n-6}{2}u}\right)(x)\right)\\&\quad+(n-6)\left\langle\nabla u(x),\nabla\left(e^{-\frac{n-6}{2}u}\Delta\left(e^{\frac{n-6}{2}u}\right)\right)(x)\right\rangle\\&= a_1^2\int_{\mathbb{R}^{2n}} Z_1Z_2\,d\mu_g^{(n)} + 2a_1\int_{\mathbb{R}^{3n}} Z_1Z_{2,3}\,d\mu_g^{(n)} + a_2^2\int_{\mathbb{R}^{4n}} Z_{1,2}Z_{3,4}\,d\mu_g^{(n)}\\&\quad+2(n-4)\,a_1\int_{\mathbb{R}^n} Z_1^2\,d\mu_g^{(n)} + a_2\int_{\mathbb{R}^{2n}} Z_{1,2}\,(2(n-6)(Z_1+Z_2)\\&\quad+4Z_{1,2})\,d\mu_g^{(n)} - (n-6)\left(a_1\int_{\mathbb{R}^{2n}} Z_1(Z_1+Z_2-Z_{1,2})\,d\mu_g^{(n)}\right.\\&\quad\left.+a_2\int_{\mathbb{R}^{3n}} Z_{1,2}(Z_1+Z_2+2Z_3-Z_{1,3}-Z_{2,3})\,d\mu_g^{(n)}\right)\\&= b_1\int_{\mathbb{R}^n} Z_1^2\,d\mu_g^{(n)} + b_2\int_{\mathbb{R}^{2n}} Z_1Z_2\,d\mu_g^{(n)} + b_3\int_{\mathbb{R}^{2n}} Z_1Z_{1,2}\,d\mu_g^{(n)}\end{aligned}$$

$$+ b_4 \int_{\mathbb{R}^{2n}} Z_{1,2}^2 \, d\mu_g^{(n)} + b_5 \int_{\mathbb{R}^{3n}} Z_1 Z_{2,3} \, d\mu_g^{(n)} + b_6 \int_{\mathbb{R}^{3n}} Z_{1,2} Z_{1,3} \, d\mu_g^{(n)}$$
$$+ b_7 \int_{\mathbb{R}^{4n}} Z_{1,2} Z_{3,4} \, d\mu_g^{(n)}, \tag{2.14}$$

where

$$\begin{aligned}
b_1 &:= (2(n-4) - (n-6)\lambda) a_1 \\
&= (n-6)\left(n-2-\frac{n-6}{2}\lambda\right)\left(n-4-\frac{n-6}{2}\lambda\right), \\
b_2 &:= a_1^2 - (n-6) a_1 \\
&= \frac{(n-6)^2}{4}\left(n-2-\frac{n-6}{2}\lambda\right)\left(n-4-\frac{n-6}{2}\lambda\right), \\
b_3 &:= (n-6) a_1 + (n-6)(4-2\lambda) a_2 \\
&= (n-6)^2\left(n-4-\frac{n-6}{2}\lambda\right), \\
b_4 &:= 4a_2 \\
&= \frac{(n-6)^2}{2}, \\
b_5 &:= 2a_1 a_2 - 2(n-6) a_2 \\
&= \frac{(n-6)^3}{8}\left(n-4-\frac{n-6}{2}\lambda\right), \\
b_6 &:= 2(n-6) a_2 \\
&= \frac{(n-6)^3}{4}, \\
b_7 &:= a_2^2 \\
&= \frac{(n-6)^4}{64}.
\end{aligned}$$

Applying (2.1), (2.3), and (2.14) then gives

$$2\left\langle \nabla u(x), \nabla\left(e^{-\frac{n-6}{2}u} \Delta^2\left(e^{\frac{n-6}{2}u}\right)\right)(x)\right\rangle$$
$$= 4b_1 \int_{\mathbb{R}^{2n}} Z_1^2 (Z_1 + Z_2 - Z_{1,2}) \, d\mu_g^{(n)} + \int_{\mathbb{R}^{3n}} \big(2b_2 Z_1 Z_2 (Z_1 + Z_2 + 2Z_3$$
$$- Z_{1,3} - Z_{2,3}) + b_3 Z_1 Z_{1,2} (4(Z_1 + Z_3 - Z_{1,3}) + 2(Z_2 + Z_3 - Z_{2,3}))$$
$$+ 4b_4 Z_{1,2}^2 (Z_1 + Z_2 + 2Z_3 - Z_{1,3} - Z_{2,3})\big) \, d\mu_g^{(n)} + \int_{\mathbb{R}^{4n}} \big(2b_5 Z_1 Z_{2,3}$$
$$\times (Z_1 + Z_2 + Z_3 + 3Z_4 - Z_{1,4} - Z_{2,4} - Z_{3,4}) + b_6 Z_{1,2} Z_{1,3} (4(Z_1 + Z_4$$

$$-Z_{1,4}) + 2\left(Z_2 + Z_3 + 2Z_4 - Z_{2,4} - Z_{3,4}\right)\big)\big)\, d\mu_g^{(n)} + 2b_7 \int_{\mathbb{R}^{5n}} Z_{1,2} Z_{3,4}$$
$$\times \left(Z_1 + Z_2 + Z_3 + Z_4 + 4Z_5 - Z_{1,5} - Z_{2,5} - Z_{3,5} - Z_{4,5}\right) d\mu_g^{(n)}$$

and

$$\begin{aligned}
&\Delta\left(e^{-\frac{n-6}{2}u}\Delta^2\left(e^{\frac{n-6}{2}u}\right)\right)(x)\\
&\quad = -4\left(n-6\right) b_1 \int_{\mathbb{R}^n} Z_1^3\, d\mu_g^{(n)} - \int_{\mathbb{R}^{2n}} \big(b_2 Z_1 Z_2 \left(2\left(n-6\right)\left(Z_1+Z_2\right) + 4Z_{1,2}\right)\\
&\qquad + b_3 Z_1 Z_{1,2}\left(\left(n-8\right)\left(4Z_1 + 2Z_2\right) + 8Z_{1,2}\right) + b_4 Z_{1,2}^2\left(4\left(n-10\right)\left(Z_1+Z_2\right)\right.\\
&\qquad \left.+16Z_{1,2}\right)\big)\, d\mu_g^{(n)} - \int_{\mathbb{R}^{3n}} \big(b_5 Z_1 Z_{2,3}\left(2\left(n-8\right)\left(Z_1+Z_2+Z_3\right)\right.\\
&\qquad \left.+4\left(Z_{1,2}+Z_{1,3}+Z_{2,3}\right)\right) + b_6 Z_{1,2} Z_{1,3}\left(\left(n-10\right)\left(4Z_1+2Z_2+2Z_3\right)\right.\\
&\qquad \left.+8Z_{1,2}+8Z_{1,3}+4Z_{2,3}\right)\big)\, d\mu_g^{(n)} - b_7 \int_{\mathbb{R}^{4n}} \big(Z_{1,2} Z_{3,4}\left(2\left(n-10\right)\left(Z_1+Z_2\right.\right.\\
&\qquad \left.\left.+Z_3+Z_4\right) + 4\left(Z_{1,2}+Z_{1,3}+Z_{1,4}+Z_{2,3}+Z_{2,4}+Z_{3,4}\right)\right)\big)\, d\mu_g^{(n)}.
\end{aligned}$$

Finally, applying these formulas together with (2.12) and interchanging the order of integration give

$$\begin{aligned}
&(-\Delta)^3\left(e^{\frac{n-6}{2}u}\right)(x)\\
&\quad = -e^{-\frac{n-6}{2}u(x)}\Delta\left(e^{\frac{n-6}{2}u}\right)(x)\,\Delta^2\left(e^{\frac{n-6}{2}u}\right)(x)\\
&\qquad - \left(n-6\right) e^{\frac{n-6}{2}u(x)}\left\langle \nabla u\left(x\right), \nabla\left(e^{-\frac{n-6}{2}u}\Delta^2\left(e^{\frac{n-6}{2}u}\right)\right)(x)\right\rangle\\
&\qquad - e^{\frac{n-6}{2}u(x)}\Delta\left(e^{-\frac{n-6}{2}u}\Delta^2\left(e^{\frac{n-6}{2}u}\right)\right)(x)\\
&\quad = e^{\frac{n-6}{2}u(x)} \int_{\mathbb{R}^{6n}} \big(b_1' Z_1^3 + b_2' Z_1^2 Z_{1,2} + b_3' Z_1^2 Z_2 + b_4' Z_1^2 Z_{2,3} + b_5' Z_1 Z_{1,2}^2\\
&\qquad + b_6' Z_1 Z_{1,2} Z_{1,3} + b_7' Z_1 Z_2 Z_{1,2} + b_8' Z_1 Z_{1,2} Z_{2,3} + b_9' Z_1 Z_{1,2} Z_{3,4}\\
&\qquad + b_{10}' Z_1 Z_2 Z_{1,3} + b_{11}' Z_1 Z_2 Z_3 + b_{12}' Z_1 Z_2 Z_{3,4} + b_{13}' Z_1 Z_{2,3}^2\\
&\qquad + b_{14}' Z_1 Z_{2,3} Z_{2,4} + b_{15}' Z_1 Z_{2,3} Z_{4,5} + b_{16}' Z_{1,2}^3 + b_{17}' Z_{1,2}^2 Z_{1,3}\\
&\qquad + b_{18}' Z_{1,2}^2 Z_{3,4} + b_{19}' Z_{1,2} Z_{1,3} Z_{1,4} + b_{20}' Z_{1,2} Z_{1,3} Z_{2,3} + b_{21}' Z_{1,2} Z_{1,3} Z_{3,4}\\
&\qquad + b_{22}' Z_{1,2} Z_{1,3} Z_{5,6} + b_{23}' Z_{1,2} Z_{3,4} Z_{5,6}\big)\, d\mu_g^{(n)},
\end{aligned}$$

where

$$\begin{aligned}
b_1' &:= 2\left(n-6\right)\left(2-\lambda\right)\lambda^{-5} b_1,\\
b_2' &:= \lambda^{-4}\left(2\left(n-6\right) b_1 + \left(4\left(n-8\right) - 2\left(n-6\right)\lambda\right) b_3\right),\\
b_3' &:= \lambda^{-4}\left(\left(a_1 - 2\left(n-6\right)\right) b_1 + \left(n-6\right)\left(4-2\lambda\right) b_2\right),\\
b_4' &:= \lambda^{-3}\left(a_2 b_1 + \left(2\left(n-8\right) - \left(n-6\right)\lambda\right) b_5\right),\\
b_5' &:= \lambda^{-4}\left(8 b_3 + \left(8\left(n-10\right) - 4\left(n-6\right)\lambda\right) b_4\right),
\end{aligned}$$

$$
\begin{aligned}
b_6' &:= \lambda^{-3}\left(2\left(n-6\right)b_3+\left(4\left(n-10\right)-2\left(n-6\right)\lambda\right)b_6\right),\\
b_7' &:= \lambda^{-4}\left(4b_2+\left(2\left(n-8\right)-\left(n-6\right)\lambda\right)b_3\right),\\
b_8' &:= \lambda^{-3}\left(\left(n-6\right)b_3+8b_5+\left(4\left(n-10\right)-2\left(n-6\right)\lambda\right)b_6\right),\\
b_9' &:= \lambda^{-2}\left(a_2b_3+\left(n-6\right)b_5+\left(8\left(n-10\right)-4\left(n-6\right)\lambda\right)b_7\right),\\
b_{10}' &:= \lambda^{-3}\left(2\left(n-6\right)b_2+\left(a_1-3\left(n-6\right)\right)b_3+\left(4\left(n-8\right)-2\left(n-6\right)\lambda\right)b_5\right),\\
b_{11}' &:= \left(a_1-2\left(n-6\right)\right)\lambda^{-3}b_2,\\
b_{12}' &:= \lambda^{-2}\left(a_2b_2+\left(a_1-3\left(n-6\right)\right)b_5\right),\\
b_{13}' &:= \lambda^{-3}\left(\left(a_1-4\left(n-6\right)\right)b_4+4b_5\right),\\
b_{14}' &:= \lambda^{-2}\left(2\left(n-6\right)b_5+\left(a_1-4\left(n-6\right)\right)b_6\right),\\
b_{15}' &:= \lambda^{-1}\left(a_2b_5+\left(a_1-4\left(n-6\right)\right)b_7\right),\\
b_{16}' &:= 16\lambda^{-4}b_4,\\
b_{17}' &:= \lambda^{-3}\left(4\left(n-6\right)b_4+16b_6\right),\\
b_{18}' &:= \lambda^{-2}\left(a_2b_4+8b_7\right),\\
b_{19}' &:= 2\left(n-6\right)\lambda^{-2}b_6,\\
b_{20}' &:= 4\lambda^{-3}b_6,\\
b_{21}' &:= \lambda^{-2}\left(2\left(n-6\right)b_6+16b_7\right),\\
b_{22}' &:= \lambda^{-1}\left(a_2b_6+4\left(n-6\right)b_7\right),\\
b_{23}' &:= a_2b_7.
\end{aligned}
$$

Plugging the expressions of $b_1,\dots,b_7$ into those of $b_1',\dots,b_{23}'$ and factorizing then give (2.5). This ends the proof of Proposition 2.1. $\square$

3. Proof of Theorem 1.1

In this section, we prove Theorem 1.1 by using the integral formula for $Q_g^{(6)}$ obtained in the previous section:

Proof of Theorem 1.1. Let $m\geq 4$, $n\in\{2m,2m+1,\dots,4m-6\}$, g be a conformally Euclidean metric on $\mathbb{R}^n$ satisfying (A), and $P_{n,m}$ be as in Proposition 2.1. In the case where $n=2m$, we observe that the last eight coefficients of $P_{2m,m}$ are positive and the first fifteen coefficients are nonnegative since $0<\lambda\leq 2$. Since $Q_g^{(2m)}\not\equiv 0$ in $\mathbb{R}^n$ as assumed in (A), we then obtain $Q_g^{(6)}>0$ as a direct consequence of Proposition 2.1. We now consider the case $n>2m$. In this case, we have $Q_g^{(2m)}>0$ in $\mathbb{R}^n\setminus B\left(0,R\right)$ as a consequence of (A). Analyzing the sign changes of the terms of $P_{n,m}$ reveals three nontrivial cases to consider: $\frac{5m-3}{2}<n<\frac{8m-6}{3}$, $\frac{8m-6}{3}<n<3m-3$, and

$3m-3<n<4m-6$. We label these cases a), b), and c) respectively. Let

$$Y_{i,j}\left(x,y\right):=\frac{x-y_i}{\left|x-y_i\right|^2}-\frac{x-y_j}{\left|x-y_j\right|^2}$$

for all $i,j\in\{1,\dots,6\}$ and $x,y_1,\dots,y_6\in\mathbb{R}^n$. It is easy to see $\left|Y_{i,j}\right|^2=Z_{i,j}$.

For case a), by expanding and interchanging the order of integration, we easily see

$$\begin{aligned}0&\le\int_{\mathbb{R}^{6n}}\prod_{i=1}^{6}\left|x-y_i\right|^{2m-n}\left|Y_{1,2}\right|^2\left(\left|Y_{3,4}\right|^2-\left|Y_{5,6}\right|^2\right)^2d\mu_g^{(2m)}\left(y\right)\\&=2\int_{\mathbb{R}^{6n}}\prod_{i=1}^{6}\left|x-y_i\right|^{2m-n}\left|Y_{1,2}\right|^2\left|Y_{3,4}\right|^2\left(\left|Y_{1,2}\right|^2-\left|Y_{5,6}\right|^2\right)d\mu_g^{(2m)}\left(y\right).\end{aligned}\tag{3.1}$$

Therefore, analyzing the coefficients of $P_{n,m}$, to obtain $Q_g^{(6)}>0$ in $\mathbb{R}^n$, it suffices to show

$$3\left(n-2m+2\right)^2\ge\left(2n-5m+3\right)\left(8m-3n-6\right).$$

This is easy to see by noting the discriminant of

$$\begin{aligned}&3\left(2m-n-2\right)^2-\left(2n-5m+3\right)\left(8m-3n-6\right)\\&\quad=9n^2+\left(33-43m\right)n+52m^2-78m+30\end{aligned}$$

as a quadratic in n is negative for $m\ge1$.

For case b), we use the inequality

$$\begin{aligned}0&\le\int_{\mathbb{R}^{6n}}\prod_{i=1}^{6}\left|x-y_i\right|^{2m-n}\left|Y_{1,2}\right|^2\left(\left|Y_{1,3}\right|^2-\left|Y_{5,6}\right|^2\right)^2d\mu_g^{(2m)}\left(y\right)\\&=\int_{\mathbb{R}^{6n}}\prod_{i=1}^{6}\left|x-y_i\right|^{2m-n}\left|Y_{1,2}\right|^2\Big(\left|Y_{1,2}\right|^2\left|Y_{1,3}\right|^2+\left|Y_{1,2}\right|^2\left|Y_{3,4}\right|^2\\&\qquad-2\left|Y_{1,3}\right|^2\left|Y_{5,6}\right|^2\Big)d\mu_g^{(2m)}\left(y\right)\end{aligned}$$

obtained by Fubini's theorem similarly to (3.1). To obtain $Q_g^{(6)}>0$ in $\mathbb{R}^n$, it therefore suffices to show the inequalities

$$2\left(n-2m+4\right)\left(n-2m+2\right)\ge\left(3n-8m+6\right)\left(3m-n-3\right)$$

and

$$n-2m+2\ge3n-8m+6.$$

The first is obtained by arguing as in case a) and the second is immediate from $3m-3>n$.

For case c), we have four negative terms to consider. We therefore rearrange the terms of $P_{n,m}$ as

$$\begin{aligned}
&P_{n,m}(Z)\\
&\quad= Z_{1,2}\Big((n-2m+2)^2 (n-2m+4)^2 Z_{1,2}^2 - (4m-n-6)(n-3m+3)\\
&\qquad\times \big(9(n-2m+2)^2 + (3n-8m+6)(2n-5m+3) - 3(n-2m+2)\\
&\qquad\times (3n-8m+6)\big) Z_{1,2}Z_{3,4} + (4m-n-6)(n-2m+2)(6(m+1)\\
&\qquad\quad\times (n-2m+2) Z_{1,2}Z_{1,3} - 4(n-3m+3)(n-2m+4) Z_{1,3}Z_{1,4})\\
&\qquad+ 2(4m-n-6)(n-2m+2)^3 Z_{1,3}Z_{2,3} + 6(4m-n-6)(n-3m+3)\\
&\qquad\times (n-2m+2)^2 (Z_{1,2}Z_{3,4} + Z_{1,2}Z_{1,3} - 2Z_{1,3}Z_{3,4}) + 3(4m-n-6)\\
&\qquad\times (3m-n-3)(8m-3n-6)(n-2m+2)(2Z_{1,3}Z_{5,6} - Z_{1,2}Z_{3,4})\\
&\qquad+ (4m-n-6)(n-3m+3)(3n-8m+6)(2n-5m+3)\\
&\qquad\times (Z_{1,2}Z_{3,4} - Z_{3,4}Z_{5,6})\Big).
\end{aligned}$$

We apply the inequalities

$$\begin{aligned}
0 &\le \int_{\mathbb{R}^{6n}} \prod_{i=1}^{6} |x-y_i|^{2m-n} |Y_{1,2}|^2 \left(|Y_{1,3}|^2 - |Y_{3,4}|^2\right)^2 d\mu_g^{(2m)}(y)\\
&= \int_{\mathbb{R}^{6n}} \prod_{i=1}^{6} |x-y_i|^{2m-n} |Y_{1,2}|^2 \Big(|Y_{1,2}|^2 |Y_{1,3}|^2 + |Y_{1,2}|^2 |Y_{3,4}|^2\\
&\quad - 2|Y_{1,3}|^2 |Y_{3,4}|^2 \Big) d\mu_g^{(2m)}(y),\\
0 &\le \int_{\mathbb{R}^{6n}} \prod_{i=1}^{6} |x-y_i|^{2m-n} |Y_{1,2}|^2 \left(|Y_{1,3}|^2 - |Y_{1,4}|^2\right)^2 d\mu_g^{(2m)}(y)\\
&= 2\int_{\mathbb{R}^{6n}} \prod_{i=1}^{6} |x-y_i|^{2m-n} |Y_{1,2}|^2 |Y_{1,3}|^2 \left(|Y_{1,2}|^2 - |Y_{1,4}|^2\right) d\mu_g^{(2m)}(y),\\
0 &\le \int_{\mathbb{R}^{6n}} \prod_{i=1}^{6} |x-y_i|^{2m-n} |Y_{1,2}|^2 \left(|Y_{1,2}|^2 - |Y_{3,4}|^2\right)^2 d\mu_g^{(2m)}(y)\\
&= \int_{\mathbb{R}^{6n}} \prod_{i=1}^{6} |x-y_i|^{2m-n} |Y_{1,2}|^4 \left(|Y_{1,2}|^2 - |Y_{3,4}|^2\right) d\mu_g^{(2m)}(y),
\end{aligned}$$

and

$$0 \le \int_{\mathbb{R}^{6n}} \prod_{i=1}^{6} |x-y_i|^{2m-n} |Y_{1,2}|^2 \left(|Y_{3,4}|^2 |Y_{5,6}|^2 - \langle Y_{3,4}, Y_{5,6}\rangle^2\right) d\mu_g^{(2m)}(y)$$

$$
\begin{aligned}
&= \int_{\mathbb{R}^{6n}} \prod_{i=1}^{6} |x - y_i|^{2m-n} |Y_{1,2}|^2 \Bigg(|Y_{3,4}|^2 |Y_{5,6}|^2 \\
&\quad - \frac{1}{4} \left(|Y_{3,5}|^2 + |Y_{4,6}|^2 - |Y_{3,6}|^2 - |Y_{4,5}|^2 \right)^2 \Bigg) d\mu_g^{(2m)}(y) \\
&= \int_{\mathbb{R}^{6n}} \prod_{i=1}^{6} |x - y_i|^{2m-n} |Y_{1,2}|^2 \left(2 |Y_{1,3}|^2 |Y_{5,6}|^2 - |Y_{1,2}|^2 |Y_{3,4}|^2 \right) d\mu_g^{(2m)}(y)
\end{aligned}
$$

as well as (3.1). To obtain $Q_g^{(6)} > 0$ in $\mathbb{R}^n$, it therefore suffices to prove the inequalities

$$
\begin{aligned}
&(n - 2m + 2)^2 (n - 2m + 4)^2 \\
&\quad \geq (4m - n - 6)(n - 3m + 3) \big(9 (n - 2m + 2)^2 \\
&\qquad + (3n - 8m + 6)(2n - 5m + 3) - 3 (n - 2m + 2)(3n - 8m + 6) \big) \\
&\quad = (4m - n - 6)(n - 3m + 3) \left(6n^2 - (25m - 21) n + 28m^2 - 42m + 18 \right)
\end{aligned}
$$

and

$$
6 (m + 1)(n - 2m + 2) \geq 4 (n - 2m + 4)(n - 3m + 3).
$$

To obtain the first inequality, it is easy to see

$$
6n^2 - 25mn + 21n + 28m^2 - 42m + 18 \leq 7 (n - 2m + 4)^2
$$

in the range $3m - 3 < n < 4m - 6$ by writing

$$
\begin{aligned}
&7 (n - 2m + 4)^2 - \left(6n^2 - 25mn + 21n + 28m^2 - 42m + 18 \right) \\
&\quad = n^2 - (3m - 35) n - 70m + 94 \\
&\quad = \int_{3m-3}^{n} (2t - 3m + 35) \, dt + 26m - 2.
\end{aligned}
$$

We then conclude by noting

$$
\begin{aligned}
&(n - 2m + 2)^2 - 7 (4m - n - 6)(n - 3m + 3) \\
&\quad = 8n^2 - (53m - 67) n + 88m^2 - 218m + 130 \\
&\quad > 0,
\end{aligned}
$$

as its discriminant as a quadratic form in n is negative for $m \geq 4$. For the second inequality, since $3m - 3 < n < 4m - 6$ and $m \geq 4$, we have

$$
\begin{aligned}
&6 (m + 1)(n - 2m + 2) - 4 (n - 2m + 4)(n - 3m + 3) \\
&\quad > 6 (m + 1)(n - 2m + 2) - 4 (m - 3)(n - 2m + 4) \\
&\quad = 2 (m + 9) n - 4m^2 - 40m + 60 \\
&\quad > 2 (m + 9)(3m - 3) - 4m^2 - 40m + 60 \\
&\quad = 2 (m + 1)(m + 3) \\
&\quad > 0.
\end{aligned}
$$

Therefore, in all cases a), b), and c), we obtain $Q_g^{(6)} > 0$ in $\mathbb{R}^n$. This ends the proof of Theorem 1.1. $\square$

4. Proof of Theorem 1.2

In this section, we prove Theorem 1.2 by again using the integral formula for $Q_g^{(6)}$ from Section 2:

Proof of Theorem 1.2. Let $n, m \in \mathbb{N}$ be such that $n > 2m \geq 8$ and $P_{n,m}$ be as in (2.4). For $\varepsilon > 0$, let $\rho_\varepsilon \in C_c^\infty(\mathbb{R}^n)$ be such that $\rho_\varepsilon \geq 0$, $\int_{\mathbb{R}^n} \rho_\varepsilon(x)\, dx = 1$ and $\rho_\varepsilon \mathcal{L} \overset{*}{\rightharpoonup} \delta_0$ weakly in the sense of measures as $\varepsilon \to 0$, where $\mathcal{L}$ is the Lebesgue measure in $\mathbb{R}^n$. For $r, \varepsilon > 0$ and $x \in \mathbb{R}^n$, let

$$h_{r,\varepsilon}(x) := \varepsilon\big(1 + |x|^2\big)^{-\frac{n+2m}{2}} + r\rho_\varepsilon(x - e_1) + \rho_\varepsilon(x + e_1),$$

where $e_1 := (1, 0, \dots, 0)$. In particular, it is easy to see that

$$h_{r,\varepsilon}\mathcal{L} \overset{*}{\rightharpoonup} r\delta_{e_1} + \delta_{-e_1}$$

as $\varepsilon \to 0$ weakly in the sense of measures, where $\delta_{\pm e_1}$ is the Dirac measure at $\pm e_1$. Now let

$$u_{r,\varepsilon}(x) := \frac{\Gamma\left(\frac{n}{2} - m + 1\right)}{2^{2m-1}(m-1)!\,\Gamma\left(\frac{n}{2}\right)\omega_{n-1}} \int_{\mathbb{R}^n} |x - y|^{2m-n} h_{r,\varepsilon}(y)\, dy.$$

Then $u_{r,\varepsilon}$ is smooth and satisfies

$$(-\Delta)^m u_{r,\varepsilon} = \frac{n - 2m}{2} h_{r,\varepsilon} \quad \text{in } \mathbb{R}^n$$

and

$$C_{r,\varepsilon}^{-1}\big(1 + |x|^2\big)^{\frac{2m-n}{2}} \leq u_{r,\varepsilon}(x) \leq C_{r,\varepsilon}\big(1 + |x|^2\big)^{\frac{2m-n}{2}}$$

for some constant $C_{r,\varepsilon} > 0$ depending on n, m, r and ε. Let $g_{r,\varepsilon}$ be the metric on $\mathbb{R}^n$ defined by

$$g_{r,\varepsilon} := u_{r,\varepsilon}^{\frac{4}{n-2m}} |dx|^2.$$

Then $Q_{g_{r,\varepsilon}}^{(2m)} = u_{r,\varepsilon}^{-\frac{n+2m}{n-2m}} h_{r,\varepsilon}$ satisfies

$$C_{r,\varepsilon}^{-1} \leq Q_{g_{r,\varepsilon}}^{(2m)} \leq C_{r,\varepsilon},$$

for some constant $C_{r,\varepsilon} > 0$ depending on n, m, r and ε. In particular, $g_{r,\varepsilon}$ satisfies (A) for all $\varepsilon > 0$. Thus Proposition 2.1 gives,

$$\begin{aligned} Q^{(6)}_{g_{r,\varepsilon}}(0) &= 2(m-3)\int_{\mathbb{R}^{6n}}\prod_{i=1}^{6}|y_i|^{2m-n}P_{n,m}(Z(0,y))\,d\mu^{(2m)}_{g_{r,\varepsilon}}(y) \\ &\to \left(\frac{\Gamma\left(\frac{n}{2}-m+1\right)}{2^{2m-1}(m-1)!\,\Gamma\left(\frac{n}{2}\right)\omega_{n-1}}\right)^6 \sum_{\alpha\in\{e_1,-e_1\}^6} r^{\theta_\alpha}P_{n,m}(Z(0,\alpha)) \\ &=: \left(\frac{\Gamma\left(\frac{n}{2}-m+1\right)}{2^{2m-1}(m-1)!\,\Gamma\left(\frac{n}{2}\right)\omega_{n-1}}\right)^6 K_{n,m}(r) \end{aligned} \tag{4.1}$$

as $\varepsilon \to 0$, where

$$\theta_\alpha := |\{i\in\{1,\dots,6\}:\ \alpha(i)=e_1\}|.$$

We must show $K_{n,m}(r) < 0$ for some $r > 0$. Let $a_1,\dots,a_8$ be the coefficients of the polynomial $P_{n,m}$ in the order stated in (2.4). Since $Z_{i,j}(0,\alpha) = 4$ if $\alpha(i) \neq \alpha(j)$ and $Z_{i,j}(0,\alpha) = 0$ otherwise, due to symmetry, we note $K_{n,m}(r)$ is a quintic polynomial of the form

$$K_{n,m}(r) = 64r\left(b_0 + b_1 r + b_2 r^2 + b_1 r^3 + b_0 r^4\right),$$

where

$$\begin{aligned} b_0 &:= 2a_1 + a_2 + a_4, \\ b_1 &:= 8a_1 + 4a_2 + 4a_3 + 2a_4 + 2a_6 + 2a_7, \\ b_2 &:= 12a_1 + 6a_2 + 8a_3 + 2a_4 + 4a_6 + 4a_7 + 8a_8. \end{aligned}$$

For $r > 0$, we use the standard method of reducing the degree of a palindromic polynomial by considering

$$\begin{aligned} r^{-3}K_{n,m}(r) &= 64\left(b_0\left(r^2+r^{-2}\right) + b_1\left(r+r^{-1}\right) + b_2\right) \\ &= 64\left(b_0\left(r+r^{-1}\right)^2 + b_1\left(r+r^{-1}\right) + b_2 - 2b_0\right). \end{aligned}$$

It therefore suffices to analyze whether this quadratic in $r + r^{-1}$ attains a negative value in $[2,\infty)$, which is the case if

$$b_1^2 - 4b_0(b_2 - 2b_0) > 0 \quad \text{and} \quad -\frac{b_1}{2b_0} \geq 2, \quad \text{i.e.} \quad b_1 + 4b_0 \leq 0 \tag{4.2}$$

since

$$b_0 = 8(m-1)(m-2)(n-2m+2)(n-2m+4) > 0.$$

We compute

$$\begin{aligned} b_1 + 4b_0 = -8(n-2m+2)\big(&n^3 - (12m-13)n^2 + \left(39m^2 - 79m + 38\right)n \\ &- 2(m-1)\left(22m^2 - 37m + 4\right)\big) \end{aligned}$$

and

$$b_1^2 - 4b_0(b_2 - 2b_0) = 64(n-2m+2)(n-3m+3)(n-4m+6)\gamma_m(n),$$

where γ_m is as in (1.3). We let N_m be the largest real root of γ_m. We compute

$$\gamma_m(4m-6) = -96(m-1)^2(m-2)(m-3)^2 < 0.$$

Since the leading coefficient of γ_m is positive and $2m-2 < 3m-3 < 4m-6$ for $m \geq 4$, we then obtain $N_m > 4m-6$ and $b_1^2 - 4b_0(b_2 - 2b_0) > 0$ for all $n > N_m$. To show $b_1 + 4b_0 \leq 0$, we consider the largest real root N_m' of

$$n^3 - (12m-13)n^2 + \left(39m^2 - 79m + 38\right)n - 2(m-1)\left(22m^2 - 37m + 4\right)$$

as a polynomial in n. A simple Euclidean division gives

$$\begin{aligned}\gamma_m(n) = {} & \left(n^2 - (7m-6)n - 6(m-1)(m-4)\right)\big(n^3 - (12m-13)n^2 \\ & + \left(39m^2 - 79m + 38\right)n - 2(m-1)\left(22m^2 - 37m + 4\right)\big) \\ & - 4(m-1)(m-2)\big((23m-27)n^2 - 2\left(51m^2 - 106m + 57\right)n \\ & + 4\left(34m^3 - 89m^2 + 63m - 6\right)\big).\end{aligned}$$

Moreover, the remainder polynomial in this division is negative for all $n \in \mathbb{N}$, which can be seen by noting $23m - 27 > 0$ and

$$\begin{aligned} & 4\left(51m^2 - 106m + 57\right)^2 - 16(23m-27)\left(34m^3 - 89m^2 + 63m - 6\right) \\ & \quad = -4\left(527m^4 - 1048m^3 - 1642m^2 + 4728m - 2601\right) \\ & \quad = -4\big(527(m-2)^4 + 3168(m-2)^3 + 4718(m-2)^2 + 2448(m-2) \\ & \qquad + 335\big) \\ & \quad < 0.\end{aligned}$$

Therefore, we obtain $\gamma_m(N_m') < 0$, which implies $N_m' < N_m$, and therefore if $n > N_m$, then both inequalities in (4.2) are satisfied, which in turn implies $K_{n,m}(r) < 0$ for some $r > 0$. Applying (4.1) then gives $Q^{(6)}_{g_{r,\varepsilon}}(0) < 0$ provided ε is sufficiently small.

We now show $N_m \sim \Lambda m$ as $m \to \infty$ and $N_m \leq \Lambda m$ for all $m \in \mathbb{N}$ with $m \geq 4$, where $\Lambda \simeq 10.55$ is the largest real root of the polynomial $n^4 - 17n^3 + 83n^2 - 171n + 140$. We let $\tilde{n} := n/m$. We then write

$$\begin{aligned}\gamma_m(n) = {} & (\tilde{n}-2)\left(\tilde{n}^4 - 17\tilde{n}^3 + 83\tilde{n}^2 - 171\tilde{n} + 140\right)m^5 + \big(19\tilde{n}^4 - 212\tilde{n}^3 \\ & + 851\tilde{n}^2 - 1518\tilde{n} + 1028\big)m^4 + 4\left(23\tilde{n}^3 - 163\tilde{n}^2 + 391\tilde{n} - 320\right)m^3 \\ & + 4\left(35\tilde{n}^2 - 144\tilde{n} + 157\right)m^2 + 48(\tilde{n}-2)m.\end{aligned}$$

In particular, considering the leading term, we obtain $N_m \sim \Lambda m$ as $m \to \infty$. Moreover, after analyzing the polynomials in $\tilde{n}$ appearing in the other terms, it turns out that all of them are positive for all $\tilde{n} \geq \Lambda$, which implies $N_m < \Lambda m$ for all $m \in \mathbb{N}$ with $m \geq 4$.

Finally, we remark that for every $r, \varepsilon > 0$, by stereographic projection, $g_{r,\varepsilon}$ extends to a smooth metric on $\mathbb{S}^n$. Indeed, if we let $\Phi_{x_0} : \mathbb{S}^n \setminus \{x_0\} \to \mathbb{R}^n$

be the stereographic projection from some pole $x_0 \in \mathbb{S}^n$, then a standard computation using conformal invariance gives

$$u_{r,\varepsilon} \circ \Phi_{x_0} = \left(\frac{2}{1+|\Phi_{x_0}|^2}\right)^{\frac{n-2m}{2}} \int_{\mathbb{S}^n} G_{g_0}^{(2m)}(\cdot, y)\, \widehat{h}_{r,\varepsilon}(y)\, dv_{g_0}(y)$$

and

$$(\Phi_{x_0})^* g_{r,\varepsilon} = \left(\int_{\mathbb{S}^n} G_{g_0}^{(2m)}(\cdot, y)\, \widehat{h}_{r,\varepsilon}(y)\, dv_{g_0}(y)\right)^{\frac{4}{n-2m}} g_0,$$

where g_0 is the standard metric on $\mathbb{S}^n$, $G_{g_0}^{(2m)}$ is the Green's function of the GJMS operator of order $2m$ on $(\mathbb{S}^n, g_0)$, dv_{g_0} is the volume element of $(\mathbb{S}^n, g_0)$, and $\widehat{h}_{r,\varepsilon}$ is the smooth function on $\mathbb{S}^n$ defined by

$$\begin{aligned}\widehat{h}_{r,\varepsilon} &:= \left(\frac{1+|\Phi_{x_0}|^2}{2}\right)^{\frac{n+2m}{2}} h_{r,\varepsilon} \circ \Phi_{x_0} \\ &= 2^{-\frac{n+2m}{2}} \Big(\varepsilon + \big(1+|\Phi_{x_0}|^2\big)^{\frac{n+2m}{2}} \left(r\rho_\varepsilon\left(\Phi_{x_0}(\cdot) - e_1\right) + \rho_\varepsilon\left(\Phi_{x_0}(\cdot) + e_1\right)\right)\Big).\end{aligned}$$

This ends the proof of Theorem 1.2. □

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