# NONEXISTENCE OF MINIMIZERS FOR THE SECOND CONFORMAL EIGENVALUE NEAR THE ROUND SPHERE IN LOW DIMENSIONS

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Abstract. We consider the problem of minimizing the second conformal eigenvalue of the conformal Laplacian in a conformal class of metrics with renormalized volume. We prove, in dimensions $n \in \{3, \dots, 10\}$, that a minimizer for this problem does not exist for metrics sufficiently close to the round metric on the sphere. This is in striking contrast with the situation in dimensions $n \geq 11$, where Ammann and Humbert [1] obtained the existence of minimizers for the second conformal eigenvalue on any smooth closed non-locally conformally flat manifold. As a byproduct of our techniques, we also obtain a lower bound on the energy of sign-changing solutions of the Yamabe equation in dimensions 3, 4 and 5, which extends a result obtained by Weth [54] in the case of the round sphere.

## 1. Introduction and main result

We let $(M, g)$ be a smooth closed Riemannian manifold of dimension $n \geq 3$. We let $[g]$ be the conformal class of the metric $g$. For each metric $\hat{g} \in [g]$, we denote by $L_{\hat{g}}$ the conformal Laplacian of $(M, \hat{g})$, i.e.

$$L_{\hat{g}} := \Delta_{\hat{g}} + c_n \operatorname{Scal}_{\hat{g}},$$

where $\Delta_{\hat{g}} := -\operatorname{div}_{\hat{g}}(\nabla \cdot)$ is the Laplace–Beltrami operator of $(M, \hat{g})$, $\operatorname{Scal}_{\hat{g}}$ is the scalar curvature of $(M, \hat{g})$ and $c_n := \frac{n-2}{4(n-1)}$. Since $M$ is closed, for each $\hat{g} \in [g]$ the eigenvalues of $L_{\hat{g}}$ form a nondecreasing sequence $(\lambda_k(L_{\hat{g}}))_{k \in \mathbb{N}}$ such that

$$\lambda_1(L_{\hat{g}}) < \lambda_2(L_{\hat{g}}) \leq \cdots \leq \lambda_k(L_{\hat{g}}) \leq \cdots \to \infty.$$

For each $k \in \mathbb{N}$, the $k$-th conformal eigenvalue of $(M, [g])$ is defined as

$$\Lambda_k(M, [g]) := \inf_{\hat{g} \in [g]} \left( \lambda_k(L_{\hat{g}}) \operatorname{Vol}(M, \hat{g})^{\frac{2}{n}} \right),$$

where $\lambda_k(L_{\hat{g}})$ is the $k$-th eigenvalue of $L_{\hat{g}}$ and $\operatorname{Vol}(M, \hat{g})$ is the volume of $(M, \hat{g})$. This invariant was first introduced and studied by Ammann and Humbert [1] and further studied by El Sayed [17]. It is not difficult to see that if $\Lambda_1(M, [g]) \geq 0$, then $\Lambda_1(M, [g])$ coincides with the classical Yamabe invariant, i.e.

$$\Lambda_1(M, [g]) = \inf_{\hat{g} \in [g]} \left( \operatorname{Vol}(M, \hat{g})^{\frac{2-n}{n}} \int_M \operatorname{Scal}_{\hat{g}} \, dv_{\hat{g}} \right),$$

where $dv_{\hat{g}}$ is the volume element of $(M, \hat{g})$. In this paper, we consider the case of metrics with positive Yamabe invariant. In this case, it follows from the work

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of Trudinger [52], Aubin [3] and Schoen [48] that $\Lambda_1(M,[g])$ is always attained by some smooth positive function, and, moreover,

$$\Lambda_1(M,[g]) \le \Lambda_1(\mathbb{S}^n,[g_0])$$

with equality if and only if $(M,g)$ is conformally equivalent to the round $n$-sphere $(\mathbb{S}^n,g_0)$.

In this paper, we focus on the case where $k=2$ and $\Lambda_1(M,[g])>0$. In this case, Ammann and Humbert [1] obtained

$$2\Lambda_1(M,[g])^{\frac{n}{2}} \le \Lambda_2(M,[g])^{\frac{n}{2}} \le \Lambda_1(M,[g])^{\frac{n}{2}} + \Lambda_1(\mathbb{S}^n,[g_0])^{\frac{n}{2}}. \tag{1.1}$$

They also obtained that $\Lambda_2(M,[g])$ is attained provided the second inequality in (1.1) is strict, i.e.

$$\Lambda_2(M,[g])^{\frac{n}{2}} < \Lambda_1(M,[g])^{\frac{n}{2}} + \Lambda_1(\mathbb{S}^n,[g_0])^{\frac{n}{2}} \tag{1.2}$$

and that if (1.2) is satisfied, then $\Lambda_2(M,[g])$ is attained at a "generalized" metric $\hat{g} := |u|^{2^*-2} g$ for some $u \in C^{3,\vartheta}(M)$ for some $\vartheta < 2^*-2$ (see Section 2 for more details). Ammann and Humbert also obtained in [1] that, by test-functions computations, if $n \ge 11$ and $(M,g)$ has non-vanishing Weyl tensor somewhere, then (1.2) is satisfied and that, for each $n\ge 3$, $\Lambda_2(\mathbb{S}^n,[g_0]) = 2^{\frac{2}{n}}\Lambda_1(\mathbb{S}^n,[g_0])$ is never attained.

Except for the trivial case of the round sphere $(\mathbb{S}^n,g_0)$, the existence of minimizers for $\Lambda_2(M,[g])$ in dimensions 3 to 10 is an open problem. Our main result provides a partial negative answer to this question:

**Theorem 1.1.** *Assume that $3\le n\le 10$. Then there exist $\delta\in(0,\infty)$ and $m\in\mathbb{N}$ such that, for every smooth metric $g$ on $\mathbb{S}^n$, if $\|g-g_0\|_{C^m(\mathbb{S}^n)}<\delta$, then*

$$\Lambda_2(\mathbb{S}^n,[g])^{\frac{n}{2}} = \Lambda_1(\mathbb{S}^n,[g])^{\frac{n}{2}} + \Lambda_1(\mathbb{S}^n,[g_0])^{\frac{n}{2}}$$

*and $\Lambda_2(\mathbb{S}^n,[g])$ is not attained by any generalized metric.*

We recall that every smooth metric on $\mathbb{S}^n$ which is not conformally equivalent to $g_0$ is also not locally conformally flat. Theorem 1.1 establishes a striking dichotomy between the case where $3\le n\le 10$ and the case where $n\ge 11$. First, when $3\le n\le 10$, Theorem 1.1 has to be understood as a perturbative nonexistence result of extremals for $\Lambda_2(\mathbb{S}^n,[g])$ for $g$ close to the round metric $g_0$, the analogue of which fails when $n\ge 11$ by the results of [1]. Second, Theorem 1.1 establishes that, when $3\le n\le 10$, (1.2) cannot be guaranteed by solely enforcing local geometric assumptions on $g$. This again strongly contrasts with the results of [1] in dimensions $n\ge 11$, where a minimizer for $\Lambda_2(M,[g])$ is proven to exist for any $g$ which is not locally conformally flat. Determining whether (1.2) holds true and whether there exist extremals for $\Lambda_2(M,[g])$ when $3\le n\le 10$ therefore requires new ideas. Theorem 1.1 can be understood as a step forward in this direction : when $(M,g)=(\mathbb{S}^n,g)$ it reveals that a necessary condition for (1.2) to hold is that $g$ is sufficiently far from the round metric $g_0$ in a strong sense. Not being conformally diffeomorphic to $g_0$, in particular, is not enough, which is very surprising in view of the definition of $\Lambda_2$. How Theorem 1.1 may adapt on a general manifold is still unclear, but it seems to hint that global infomation on $(M,g)$ is needed to obtain (1.2).

Eigenvalue optimization problems in conformal classes have attracted a lot of attention in recent years. When $n\ge 3$ and in the case of the conformal Laplacian

$L_g$ which we consider here, the invariants $\Lambda_k(M,[g])$ that we define are the only meaningful ones when $\Lambda_1(M,[g])>0$. Indeed, it is for instance proven by Ammann and Jammes [2] that, if $\Lambda_k(M,[g])>0$, then

$$\sup_{\hat{g}\in[g]}\left(\lambda_k\left(L_{\hat{g}}\right)\operatorname{Vol}\left(M,\hat{g}\right)^{\frac{2}{n}}\right)=\infty.$$

On the other hand, in the case where $\Lambda_k(M,[g])<0$, it is natural to replace the infimum in the definition of $\Lambda_k$ by a supremum since otherwise $\Lambda_k(M,[g])=-\infty$ (see Proposition 8.1 in [1]). This therefore leads to a maximization problem, which is very different in nature. We refer to the work of Gursky and Pérez-Ayala [20] where this problem is studied in the case where $k=2$. Most of previous work on eigenvalue optimization problems in conformal classes of closed manifolds of dimension larger than or equal to 2 concern the Laplace–Beltrami operator $\Delta_g$, for which again only the maximization problem is interesting. In dimensions $n\geq 3$, the maximization of conformal eigenvalues of $\Delta_g$ was recently investigated by Pétrides [41]. In dimension 2, this problem was investigated by many authors. In this case, we refer for instance to the work of Nadirashvili and Sire [37], Petrides [39,40], Matthiesen and Siffert [32] and Karpukhin and Stern [24,25]. To the best of our knowledge, another remarkable feature of Theorem 1.1 is that it is the first nonexistence result of extremals for conformal eigenvalues of any kind (for any dimension $n\geq 2$ and any of the operators $\Delta_g$ and $L_g$) for metrics that are not conformal to the round metric on $\mathbb{S}^n$.

The structure of the paper is as follows. In Section 2, we discuss the connection between the second conformal eigenvalue and sign-changing solutions of the Yamabe equation of lowest energy. We also state a stronger result than Theorem 1.1, in dimensions 3, 4 and 5, namely Theorem 2.1. Section 3 is devoted to a sharp bubbling analysis of sign-changing solutions of the Yamabe equation whose energies converge to $\Lambda_2\left(\mathbb{S}^n,[g_0]\right)^{\frac{n}{2}}$. We prove bubble-tree convergence results as well as sharp pointwise asymptotics. We then prove Theorems 1.1 and 2.1 in Section 4.

## 2. The second conformal eigenvalue and sign-changing solutions of the Yamabe equation

The second conformal eigenvalue of the conformal Laplacian has a strong connection with sign-changing solutions of the Yamabe equation

$$L_g u=|u|^{2^*-2}u\quad\text{in } M,\tag{2.1}$$

where $2^*:=\frac{2n}{n-2}$ is the critical Sobolev exponent. Indeed, Ammann and Humbert [1] proved that if $\Lambda_1(M,[g])\geq 0$ and $\Lambda_2(M,[g])$ is attained, then there exists a sign-changing function $u\in L^{2^*}(M)$ such that the "generalized" metric $\hat{g}:=|u|^{2^*-2}g$ satisfies

$$\lambda_2\left(L_{\hat{g}}\right)=\Lambda_2(M,[g])\quad\text{and}\quad\operatorname{Vol}(M,\hat{g})=1,\ \text{i.e.}\int_M|u|^{2^*}dv_g=1$$

(see [1, Section 3.2] for the rigorous definition of $\lambda_2\left(L_{\hat{g}}\right)$ when $u\in L^{2^*}(M)\setminus\{0\}$). It is also shown in [1] that $u$ is a second "generalized" eigenvector associated to $\lambda_2\left(L_{\hat{g}}\right)$, which implies that $u\in C^{3,\vartheta}(M)$ for some $\vartheta<2^*-2$ and, up to a renormalization factor, $u$ can be made into a sign-changing solution of (2.1) with energy

$$\int_M|u|^{2^*}dv_g=\Lambda_2(M,[g])^{\frac{n}{2}}.$$

Furthermore, in this case $u$ is a sign-changing solution of (2.1) of least energy among all sign-changing solutions, and it has exactly two nodal domains. We again refer to [1, Section 3] for more details. As mentioned in the introduction, (1.2) is sufficient to ensure that $\Lambda_2(M,[g])$ is attained. Equation (2.1) has to be understood as the Euler-Lagrange equation for minimizers of $\Lambda_2(M,[g])$. Its hidden meaning is that, for a generalized metric $\hat{g} := |u|^{2^*-2} g$ attaining $\Lambda_2(M,[g])$, $\lambda_2(L_{\hat{g}})$ is simple. This unusual feature for an eigenvalue optimization problem is a direct consequence of the definition of $\Lambda_2(M,[g])$ as an infimum (see for instance [20, Remark 6.1] for a detailed explanation).

A consequence of Theorem 1.1 is that, for every smooth metric $g$ on $\mathbb{S}^n$ sufficiently close to $g_0$ in $C^m(\mathbb{S}^n)$ for some sufficiently large $m \in \mathbb{N}$, there does not exist any sign-changing solution $u$ of (2.1) such that

$$\int_{\mathbb{S}^n} |u|^{2^*} \, dv_g \le \Lambda_1(\mathbb{S}^n,[g])^{\frac{n}{2}} + \Lambda_1(\mathbb{S}^n,[g_0])^{\frac{n}{2}} .$$

Theorem 1.1 thus gives a lower bound on the energy of sign-changing solutions of the Yamabe equation (2.1) on the sphere of dimension lower than or equal to 10 when equipped with metrics sufficiently close to the round metric. In fact, in dimensions 3, 4 and 5, we obtain a stronger result:

**Theorem 2.1.** *Assume that $n \in \{3,4,5\}$. There exist $\delta, \varepsilon \in (0,\infty)$ and $m \in \mathbb{N}$ such that, for every smooth metric $g$ on $\mathbb{S}^n$, if $\|g-g_0\|_{C^m(\mathbb{S}^n)} < \delta$, then the energy of every sign-changing solution of the Yamabe equation*

$$L_g u = |u|^{2^*-2} u \quad \text{in } \mathbb{S}^n$$

*is greater than $2\Lambda_1(\mathbb{S}^n,[g_0])^{\frac{n}{2}} + \varepsilon$.*

Theorem 2.1 extends a result obtained by Weth [54] in the exact case of the round sphere. Notice, however, that the result of Weth [54] holds for all dimensions $n \ge 3$. This is specific to the exact case $g = g_0$. Indeed, as shown by the results of Ammann and Humbert [1], at least in the case where $n \ge 11$, Theorem 2.1 is false when $g$ is not conformal to $g_0$.

We prove Theorems 1.1 and 2.1 in the next two sections. By using (2.1) together with a contradiction argument, which is explained in details at the beginning of Section 3, the proof of Theorems 1.1 and 2.1 amounts to ruling out the existence of sequences $(u_k)_{k\in\mathbb{N}}$ of sign-changing solutions of (2.1) with $g = g_k$, where $g_k$ converges to $g_0$ in $C^m(\mathbb{S}^n)$ as $k \to \infty$ for all $m \in \mathbb{N}$ such that the energies of $(u_k)_k$ converge to $2\Lambda_1(\mathbb{S}^n,[g_0])^{\frac{n}{2}}$. In dimensions 6 to 10, we assume in addition that, for each $k \in \mathbb{N}$, $\Lambda_2(\mathbb{S}^n,[g_k])$ is attained by the generalized metric $|u_k|^{2^*-2} g_k$. The proof of Weth [54] in the case where $g_k = g_0$ for all $k \in \mathbb{N}$ relies on the symmetries of $(\mathbb{S}^n, g_0)$ and uses the action of the conformal group of the sphere on the sign-changing solutions of the Yamabe equation. In our setting, however, the metrics $(g_k)_k$ do not have any symmetries in general, and we need to perform a much finer asymptotic analysis of $(u_k)_k$. As a first result, in Lemma 3.1, we prove that $(u_k)_k$ behaves like the difference between two positive solutions of the Yamabe equation on the sphere (see (3.12)). This amounts to say that $\Lambda_2(\mathbb{S}^n,[g_k])$ is asymptotically attained by the disjoint union of two round spheres. The rest of Section 3 is devoted to obtaining sharp pointwise estimates for the blow-up of $(u_k)_k$, which, in particular, captures the local geometry of $(g_k)_k$. This refined blow-up analysis is based on iterated estimates and makes crucial use of arguments previously developed in the context of positive solutions (see the work of Chen and Lin [7], Schoen [49, 50], Li and Zhu [30], Druet [15], Marques [31], Li and Zhang [28, 29] and Khuri, Marques and Schoen [26]; see also the counterexamples in high dimensions of Brendle [4] and Brendle and Marques [5] and the survey article by Brendle and Marques [6]) and more recently in the context of sign-changing solutions (see Premoselli [42] and Premoselli and Vétois [44–46]). We finally prove Theorems 1.1 and 2.1 in Section 4 by using the analysis developed in Section 3.

Although similar at first glance, the settings of Theorems 1.1 and 2.1 are quite different from that of the celebrated compactness result of Khuri, Marques and Schoen [26]. We point out two crucial differences: first, since the functions $(u_k)_k$ change sign, the concentration points are neither isolated nor simple; second, since $g_k \to g_0$ as $k \to \infty$ in $C^m(\mathbb{S}^n)$ for all $m \in \mathbb{N}$, the Riemannian masses of the metrics $(g_k)_k$ converge to 0 at any point of $\mathbb{S}^n$, so that no local sign restriction argument is available to rule out blow-up. Therefore, and unlike in [26], our contradiction does not originate from a local sign restriction due to the Positive Mass Theorem. In dimensions 3, 4 and 5, instead, we obtain a contradiction by means of a Pohozaev-type identity in a region where we observe that a large virtual mass is created solely by the interaction between the two bubbles. In dimensions 6 to 10, the Pohozaev-type identity is not sufficient to conclude since additional lower-order terms appear which involve more of the geometry of the metrics $(g_k)_k$ at the concentration points. In this case, we still manage to obtain a sharp asymptotic estimate on $\Lambda_1(\mathbb{S}^n, [g_k])$ (see (4.2)). We then obtain a contradiction with this estimate by doing another estimation of $\Lambda_1(\mathbb{S}^n, [g_k])$ based on a better family of test-functions, which construction again relies on the analysis of Section 3. The contradiction when $6 \le n \le 10$ thus really comes from the minimality of $\Lambda_2(\mathbb{S}^n, [g_k])$.

There is an abundant literature on sign-changing solutions of the Yamabe equation. In addition to the above-mentioned articles of Ammann and Humbert [1], El Sayed [17] and Gursky and Perez-Ayala [20], we also refer on this topic to the historic work of Ding [14] and the more recent work of Clapp [8], Clapp, Pistoia and Weth [11], del Pino, Musso, Pacard and Pistoia [12, 13], Fernandez, Palmas and Petean [18], Fernandez and Petean [19], Medina, Musso and Wei [35], Musso and Medina [34], Musso and Wei [36], Premoselli and Vétois [44] and Weth [54] in the case of the sphere, Clapp and Fernandez [9] in the case of manifolds satisfying some symmetry assumptions and Clapp, Pistoia and Tavares [10], Premoselli and Robert [43] and Premoselli and Vétois [46] in the case of more general manifolds. Other results in this spirit have been obtained for some classes of sign-changing solutions of equations with different potential functions than the Yamabe equation (see our previous articles [44, 45, 53]). Theorems 1.1 and 2.1 can also be seen as a continuation of our study, initiated in [46], of minimal energy sign-changing blowing-up solutions of the Yamabe equation.

## 3. Asymptotic analysis and symmetry estimates

The section and the next are devoted to the proofs of Theorems 1.1 and 2.1. We begin with recalling some well-known facts about constant-sign solutions of the Yamabe equation in $\mathbb{R}^n$ and $\mathbb{S}^n$. By splitting the solutions into positive and negative parts, it is easy to see that there does not exist any sign-changing solution

of the Yamabe equation
$$L_{g_0} u = |u|^{2^*-2} u \quad \text{in } \mathbb{S}^n. \tag{3.1}$$
with energy smaller than or equal to $2\Lambda_1 (\mathbb{S}^n, [g_0])^{\frac{n}{2}}$. Moreover, according to the classification result of Obata [38], every nonzero nonnegative solution of (3.1) has energy equal to $\Lambda_1 (\mathbb{S}^n, [g_0])^{\frac{n}{2}}$ and is either constant or of the form
$$u = \left( \frac{2\sqrt{n(n-2)}\mu}{2\mu^2 + \left(4-\mu^2\right)\left(1 - \cos\left(\mathrm{d}_{g_0}(\cdot, x)\right)\right)} \right)^{\frac{n-2}{2}}$$
for some $x \in \mathbb{S}^n$ and $\mu \in (0,2)$. We also recall that, letting $\xi$ be the Euclidean metric on $\mathbb{R}^n$ and $\Delta_\xi := -\sum_{i=1}^n \partial_{y_i}^2$, the stereographic projection gives a bijection between the solutions of (3.1) and the solutions $u$ of the equation
$$\Delta_\xi u = |u|^{2^*-2} u \quad \text{in } \mathbb{R}^n, \tag{3.2}$$
which belong to the energy space $D^{1,2}(\mathbb{R}^n)$ defined as the closure of $C_c^\infty(\mathbb{R}^n)$ with respect to the norm $\|\nabla \cdot\|^2_{L^2(\mathbb{R}^n)}$. By using this bijection, we obtain that every nonzero nonnegative solution of (3.2) has energy equal to $\Lambda_1 (\mathbb{S}^n, [g_0])^{\frac{n}{2}}$ and is of the form
$$u = \left( \frac{\sqrt{n(n-2)}\widetilde{\mu}}{\widetilde{\mu}^2 + |\cdot - y|^2} \right)^{\frac{n-2}{2}}$$
for some $y \in \mathbb{R}^n$ and $\widetilde{\mu} \in (0, \infty)$, and that there does not exist any sign-changing solution $u \in D^{1,2}(\mathbb{R}^n)$ of (3.2) with energy smaller than or equal to $2\Lambda_1 (\mathbb{S}^n, [g_0])^{\frac{n}{2}}$. It is well-known that the positive solutions of (3.2) are the extremals for the Sobolev inequality in $\mathbb{R}^n$, i.e.
$$\Lambda_1 (\mathbb{S}^n, [g_0]) = \inf_{v \in C_c^\infty(\mathbb{R}^n)\setminus\{0\}} \frac{\displaystyle\int_{\mathbb{R}^n} |\nabla v|^2 \, \mathrm{d}y}{\left(\displaystyle\int_{\mathbb{R}^n} |v|^{2^*} \mathrm{d}y\right)^{\frac{n-2}{n}}},$$
where $\mathrm{d}y$ is the volume element of $(\mathbb{R}^n, \xi)$.

We prove Theorems 1.1 and 2.1 by contradiction. From now until the end of the paper, we assume that $3 \le n \le 10$. We assume that there exists a sequence $(g_k)_{k\in\mathbb{N}}$ of smooth metrics on $\mathbb{S}^n$ such that, for each $m \in \mathbb{N}$, $g_k \to g_0$ in $C^m(\mathbb{S}^n)$ as $k \to \infty$ and for each $k \in \mathbb{N}$, there exists a sign-changing solution $u_k \in C^{3,\vartheta}(\mathbb{S}^n)$, $\vartheta < 2^* - 2$, of the Yamabe equation
$$L_{g_k} u_k = |u_k|^{2^*-2} u_k \quad \text{in } \mathbb{S}^n. \tag{3.3}$$
In the case where $3 \le n \le 5$, in view of Theorem 2.1, we only assume that
$$\limsup_{k\to\infty} \int_{\mathbb{S}^n} |u_k|^{2^*} \mathrm{dv}_{g_k} \le 2\Lambda_1 (\mathbb{S}^n, [g_0])^{\frac{n}{2}}. \tag{3.4}$$
In the case where $6 \le n \le 10$, in view of Theorem 1.1, we assume moreover that, for each $k \in \mathbb{N}$, $\Lambda_2 (\mathbb{S}^n, [g_k])$ is attained by the generalized metric $|u_k|^{2^*-2} g_k$ and
$$\int_{\mathbb{S}^n} |u_k|^{2^*} \mathrm{dv}_{g_k} = \Lambda_2 (\mathbb{S}^n, [g_k])^{\frac{n}{2}}, \tag{3.5}$$
which implies that $u_k$ is a sign-changing solution of (3.3) of least-energy among all sign-changing solutions. We point out in passing that, since $u_k$ satisfies (3.3), the

celebrated results of Hardt and Simon [22] gives that $u_k$ vanishes on a set of measure zero and is therefore admissible in the definition of $\Lambda_2(\mathbb{S}^n,[g_k])$ (see [1, Section 3]). By putting together (1.1) and (3.4), we obtain

$$\lim_{k\to\infty}\int_{\mathbb{S}^n}|u_k|^{2^*}\,\mathrm{dv}_{g_k}=2\Lambda_1(\mathbb{S}^n,[g_0])^{\frac{n}{2}}\,. \tag{3.6}$$

A first simple remark which follows from the previous discussion is that the sequence $(u_k)_k$ blows up as $k\to\infty$, i.e.

$$\|u_k\|_{L^\infty(\mathbb{S}^n)}\to\infty\quad\text{as } k\to\infty. \tag{3.7}$$

The following result provides a first description of the blowing-up behavior of $(u_k)_k$:

**Lemma 3.1.** *Let $(g_k)_k$ and $(u_k)_k$ be as defined above. Then, up to a subsequence and a change of sign, there exist $x_1,x_2\in\mathbb{S}^n$ and sequences $(x_{1,k})_k$ and $(x_{2,k})_k$ in $\mathbb{S}^n$ and $(\mu_{1,k})_k$ and $(\mu_{2,k})_k$ in $(0,2)$ such that*

(i) $\mu_{2,k}\le\mu_{1,k}$ *for all $k\in\mathbb{N}$.*

(ii) *For each $i\in\{1,2\}$, $x_{i,k}\to x_i$ and $\mu_{i,k}\to 0$ as $k\to\infty$.*

(iii) $\dfrac{\mu_{1,k}}{\mu_{2,k}}+\dfrac{d_k^2}{\mu_{1,k}\mu_{2,k}}\to\infty$ *as $k\to\infty$, where $d_k:=\mathrm{d}_{g_0}(x_{1,k},x_{2,k})$.*

(iv) *For each $i\in\{1,2\}$ and $k\in\mathbb{N}$, define*

$$B_{i,k}:=\left(\frac{2\sqrt{n(n-2)}\mu_{i,k}}{2\mu_{i,k}^2+\left(4-\mu_{i,k}^2\right)\left(1-\cos\left(\mathrm{d}_{g_0}(\cdot,x_{i,k})\right)\right)}\right)^{\frac{n-2}{2}},$$

*where $\mathrm{d}_{g_0}$ is the distance function on $(\mathbb{S}^n,g_0)$. Then*

$$\left\|\frac{u_k-(B_{1,k}-B_{2,k})}{B_{1,k}+B_{2,k}}\right\|_{L^\infty(\mathbb{S}^n)}\to 0\qquad\text{as } k\to\infty. \tag{3.8}$$

(v) *For each $k\in\mathbb{N}$,*

$$u_k(x_{2,k})=\min_M u_k=-B_{2,k}(x_{2,k})=-\left(\frac{\sqrt{n(n-2)}}{\mu_{2,k}}\right)^{\frac{n-2}{2}}. \tag{3.9}$$

(vi) *If, moreover,*

$$\sqrt{\mu_{1,k}\mu_{2,k}}=\mathrm{o}(d_k)\qquad\text{as } k\to\infty, \tag{3.10}$$

*then for each $k\in\mathbb{N}$,*

$$u_k(x_{1,k})=\max_M u_k=B_{1,k}(x_{1,k})=\left(\frac{\sqrt{n(n-2)}}{\mu_{1,k}}\right)^{\frac{n-2}{2}}. \tag{3.11}$$

In what follows, for simplicity, we rewrite (3.8) as

$$u_k=B_{1,k}-B_{2,k}+\mathrm{o}(B_{1,k}+B_{2,k})\quad\text{in } C^0(\mathbb{S}^n)\text{ as } k\to\infty. \tag{3.12}$$

We point out that the assumption $n\le 10$ comes into play in this lemma. Indeed, as is explained in the proof below, it is crucial in order to obtain (3.8).

*Proof of Lemma 3.1.* Since $(u_k)_k$ blows-up with finite energy as $k\to\infty$ by (3.6) and (3.7), a celebrated result of Struwe [51] (see also the book of Druet, Hebey and Robert [16] and the article of Mazumdar [33] for versions in the Riemannian setting) shows that, up to a subsequence, there exist $m\in\{1,2\}$, $x_1,\dots,x_m\in\mathbb{S}^n$ and sequences $(\widetilde{x}_{1,k})_k,\dots,(\widetilde{x}_{m,k})_k$ in $\mathbb{S}^n$ and $(\widetilde{\mu}_{1,k})_k,\dots,(\widetilde{\mu}_{m,k})_k$ in $(0,\infty)$ such that

$$\widetilde{x}_{i,k}\to x_i\in\mathbb{S}^n\quad\text{and}\quad\widetilde{\mu}_{i,k}\to 0\quad\text{as }k\to\infty\quad\forall i\in\{1,\dots,n\}\tag{3.13}$$

and

$$u_k=B_0+\sum_{i=1}^m\pm\widetilde{B}_{i,k}+\mathrm{o}(1)\quad\text{in }H^1(\mathbb{S}^n)\text{ as }k\to\infty,\tag{3.14}$$

where $H^1(\mathbb{S}^n)=H^1(\mathbb{S}^n,g_0)$, $B_0$ is a constant-sign solution of (3.1), which may be equal to 0, and $\widetilde{B}_{i,k}$ is given by

$$\widetilde{B}_{i,k}:=\left(\frac{2\sqrt{n(n-2)}\widetilde{\mu}_{i,k}}{2\widetilde{\mu}_{i,k}^2+\left(4-\widetilde{\mu}_{i,k}^2\right)\left(1-\cos\left(\mathrm{d}_{g_0}(\cdot,\widetilde{x}_{i,k})\right)\right)}\right)^{\frac{n-2}{2}}.$$

Moreover, in the case where $m=2$, up to a subsequence, we may further assume that

$$\widetilde{\mu}_{2,k}\le\widetilde{\mu}_{1,k}\quad\text{and}\quad\frac{\widetilde{\mu}_{1,k}}{\widetilde{\mu}_{2,k}}+\frac{\mathrm{d}_{g_0}(\widetilde{x}_{1,k},\widetilde{x}_{2,k})^2}{\widetilde{\mu}_{1,k}\widetilde{\mu}_{2,k}}\to\infty\quad\text{as }k\to\infty\tag{3.15}$$

(see the remark at the end of Section 3.2 in [16]). A consequence of (3.14) is that

$$\|u_k\|_{H^1(\mathbb{S}^n)}=\|B_0\|_{H^1(\mathbb{S}^n)}+m\Lambda_1(\mathbb{S}^n,[g_0])^{\frac{n}{2}}+\mathrm{o}(1)\quad\text{as }k\to\infty.\tag{3.16}$$

It follows from (3.6) and (3.16) that either [$m=2$ and $B_0=0$] or [$m=1$ and $B_0$ is a non-zero constant-sign solution of (3.1)].

Assume first that $m=1$ and $B_0$ is a non-zero constant-sign solution of (3.1). Up to a change of sign, me may assume that $B_0$ is positive. Since the functions $(u_k)_k$ change sign, it then follows from (3.14) that

$$u_k=B_0-\widetilde{B}_{1,k}+\mathrm{o}(1)\quad\text{in }H^1(\mathbb{S}^n)\text{ as }k\to\infty.\tag{3.17}$$

By using the pointwise blow-up theory for sign-changing solutions developed by Premoselli [42] (see also [16, 21] in the case of positive solutions), it follows from (3.17) that

$$u_k=B_0-\widetilde{B}_{1,k}+\mathrm{o}\left(\widetilde{B}_{1,k}\right)+\mathrm{o}(1)\quad\text{in }C^0(\mathbb{S}^n)\text{ as }k\to\infty.\tag{3.18}$$

The proof of [42] is written for a fixed metric $g$ but adapts straightforwardly to the case of a strongly converging sequence of metrics $(g_k)_{k\in\mathbb{N}}$ as is the case here. By using (3.18), since $n\le 10$ and the Weyl tensor of $(\mathbb{S}^n,g_0)$ vanishes everywhere, Theorem 1.2 of Premoselli and Vétois [46] yields a contradiction with (3.18). The proof of [46] is again stated for a fixed metric but its arguments adapt straightforwardly since they only rely on (3.18) (see [46, Section 5] for more details).

We have thus proven that $m=2$ and $B_0=0$ hold in (3.14). Up to a change of sign, since the functions $(u_k)_k$ change sign, we then obtain

$$u_k=\widetilde{B}_{1,k}-\widetilde{B}_{2,k}+\mathrm{o}(1)\quad\text{in }H^1(\mathbb{S}^n)\text{ as }k\to\infty.\tag{3.19}$$

By using again the pointwise blow-up theory of [42], it follows from (3.19) that

$$u_k=\widetilde{B}_{1,k}-\widetilde{B}_{2,k}+\mathrm{o}\left(\widetilde{B}_{1,k}+\widetilde{B}_{2,k}\right)\quad\text{in }C^0(\mathbb{S}^n)\text{ as }k\to\infty.\tag{3.20}$$

By putting together (3.13), (3.15) and (3.20), we obtain (i) to (iv) in Lemma 3.1.

We now prove that the centers and weights of $(B_{1,k})_k$ and $(B_{2,k})_k$ can be chosen so that (v) and (vi) are also satisfied. For each $k \in \mathbb{N}$, we let $x_{1,k}$, $x_{2,k}$, $\mu_{1,k}$ and $\mu_{2,k}$ be such that (3.9) and (3.11) hold true. For each $i \in \{1,2\}$, by using (3.9), (3.11) and (3.20), we obtain

$$\begin{aligned}\mu_{i,k}^{\frac{2-n}{2}} &\le \left(\frac{2\widetilde{\mu}_{i,k}\left(1+\mathrm{o}\left(1\right)\right)}{2\widetilde{\mu}_{i,k}^2+\left(4-\widetilde{\mu}_{i,k}^2\right)\left(1-\cos\left(\mathrm{d}_{g_0}\left(x_{i,k},\widetilde{x}_{i,k}\right)\right)\right)}\right)^{\frac{n-2}{2}}\\ &\le \widetilde{\mu}_{i,k}^{\frac{2-n}{2}}\left(1+\mathrm{o}\left(1\right)\right) \quad \text{as } k\to\infty \end{aligned} \tag{3.21}$$

and

$$\begin{aligned}\mu_{i,k}^{\frac{2-n}{2}} \ge{}& \widetilde{\mu}_{i,k}^{\frac{2-n}{2}}\left(1+\mathrm{o}\left(1\right)\right)\\ &-\left(\frac{2\widetilde{\mu}_{3-i,k}\left(1+\mathrm{o}\left(1\right)\right)}{2\widetilde{\mu}_{3-i,k}^2+\left(4-\widetilde{\mu}_{3-i,k}^2\right)\left(1-\cos\left(\mathrm{d}_{g_0}\left(\widetilde{x}_{1,k},\widetilde{x}_{2,k}\right)\right)\right)}\right)^{\frac{n-2}{2}} \quad \text{as } k\to\infty. \end{aligned} \tag{3.22}$$

We now assume that

$$\sqrt{\widetilde{\mu}_{1,k}\widetilde{\mu}_{2,k}} = \mathrm{o}\left(\mathrm{d}_{g_0}\left(\widetilde{x}_{1,k},\widetilde{x}_{2,k}\right)\right) \quad \text{as } k\to\infty. \tag{3.23}$$

It follows from (3.15) and (3.23) that

$$\left(\frac{2\widetilde{\mu}_{3-i,k}}{2\widetilde{\mu}_{3-i,k}^2+\left(4-\widetilde{\mu}_{3-i,k}^2\right)\left(1-\cos\left(\mathrm{d}_{g_0}\left(\widetilde{x}_{1,k},\widetilde{x}_{2,k}\right)\right)\right)}\right)^{\frac{n-2}{2}} = \mathrm{o}\left(\widetilde{\mu}_{i,k}^{\frac{2-n}{2}}\right) \quad \text{as } k\to\infty,$$

which together with (3.21) and (3.22) give

$$\mu_{i,k}\sim\widetilde{\mu}_{i,k} \quad \text{and} \quad \mathrm{d}_{g_0}\left(x_{i,k},\widetilde{x}_{i,k}\right)=\mathrm{o}\left(\mu_{i,k}\right) \quad \text{as } k\to\infty. \tag{3.24}$$

By passing to a subsequence and exchanging $(B_{1,k})_k$ and $(B_{2,k})_k$ if necessary and using (3.13), (3.15), (3.20) and (3.24), we now obtain that the sequences $(x_{1,k})_k$, $(x_{2,k})_k$, $(\mu_{1,k})_k$ and $(\mu_{2,k})_k$ simultaneously satisfy (i) to (vi) in Lemma 3.1. □

Lemma 3.1 shows that the singular metric $|u_k|^{\frac{4}{n-2}} g_k$ decomposes asymptotically as the disjoint union of two round spheres centered at $x_{1,k}$ and $x_{2,k}$, respectively. The location of these two points is unknown. Lemma 3.1 does not claim, in particular, that $d_k = \mathrm{d}_{g_0}(x_{1,k}, x_{2,k})$ has a positive limit as $k \to \infty$. In order to prove Theorems 1.1 and 2.1, we need a more precise description of the blow-up behavior of $u_k$. As is often the case with the Yamabe equation, it is convenient to work with the conformal normal coordinate system introduced by Lee and Parker [27]. We define this coordinate system in the following:

**Lemma 3.2.** *Let $(g_k)_k$ be as in Lemma 3.1. Let $\varepsilon_0 \in (0,\infty)$ and $\varphi_0$ be a smooth positive function on $\mathbb{S}^n \times \mathbb{S}^n$ such that*

$$\varphi_0\left(x,y\right) := \left(\frac{2}{1+\cos\left(\mathrm{d}_{g_0}\left(x,y\right)\right)}\right)^{\frac{n-2}{2}} \qquad \forall x\in\mathbb{S}^n,\ y\in B_{g_0}\left(x,r_0\right), \tag{3.25}$$

*where*

$$r_0 := 2\tan^{-1}\left(\varepsilon_0/2\right).$$

*Then there exists a sequence $(\varphi_k)_k$ of smooth positive functions on $\mathbb{S}^n \times \mathbb{S}^n$ such that the following holds:*

(i) $\varphi_k \to \varphi_0$ *in* $C^m(\mathbb{S}^n \times \mathbb{S}^n)$ *as* $k \to \infty$ *for all* $m \in \mathbb{N}$.

(ii) *For each* $k \in \mathbb{N}$ *and* $x \in \mathbb{S}^n$,

$$\varphi_k(x,x) = 1 \quad \text{and} \quad \nabla \varphi_k(x,\cdot)(x) = 0. \tag{3.26}$$

(iii) *For each* $k \in \mathbb{N}$ *and* $x \in \mathbb{S}^n$, *let* $g_{k,x}$ *and* $\hat{g}_{k,x}$ *be the metrics on* $\mathbb{S}^n$ *and* $\mathbb{R}^n$, *respectively, defined as*

$$g_{k,x} := \varphi_k(x,\cdot)^{2^*-2} g_k \quad \text{and} \quad \hat{g}_{k,x} := {\exp_{k,x}}^* g_{k,x},$$

*where* $\exp_{k,x}$ *is the exponential map at* $x$ *with respect to* $g_{k,x}$ *and where we identify* $T_xM$ *with* $\mathbb{R}^n$. *Then*

$$\mathrm{dv}_{\hat{g}_{k,x}}(y) = \left(1 + \mathrm{o}\left(|y|^N\right)\right) \mathrm{dy} \quad \text{as } k \to \infty \tag{3.27}$$

*uniformly with respect to* $x \in \mathbb{S}^n$ *and* $y \in \mathrm{B}_\xi(0,\varepsilon_0)$, *where* $\xi$ *is the Euclidean metric on* $\mathbb{R}^n$, $\mathrm{dv}_{\hat{g}_{k,x}}$ *and* $\mathrm{dy}$ *are the volume elements of* $(\mathbb{R}^n, \hat{g}_{k,x})$ *and* $(\mathbb{R}^n, \xi)$, *respectively, and* $N \in \mathbb{N}$ *can be chosen arbitrarily large.*

*Proof of Lemma 3.2.* The results follow from Theorem 5.1 in [27] with again a simple adaptation here due to the facts that $g_k \to g_0$ in $C^m(\mathbb{S}^n)$ as $k \to \infty$ for all $m \in \mathbb{N}$ and

$${\exp_{0,x}}^* g_{0,x} = \xi \quad \text{in } \mathrm{B}_\xi(0,\varepsilon_0), \tag{3.28}$$

where

$$g_{0,x} := \varphi_0(x,\cdot)^{2^*-2} g_0$$

and $\exp_{0,x}$ is the exponential map at $x$ with respect to $g_{0,x}$. □

As observed by Khuri, Marques and Schoen [26], it is convenient to express the conformal normal coordinate system in exponential form, which gives the following:

**Lemma 3.3.** *Let* $(\hat{g}_{k,x})_{k,x}$ *and* $\varepsilon_0$ *be as in Lemma 3.2. Then*

(i) *For each* $k \in \mathbb{N}$ *and* $x \in \mathbb{S}^n$, *there exists a smooth symmetric 2-covariant tensor* $h_{k,x}$ *in* $\mathbb{R}^n$ *such that*

$$\hat{g}_{k,x} = \exp(h_{k,x}), \tag{3.29}$$

$$h_{k,x}(y)\, y = 0 \quad \forall y \in \mathbb{R}^n \tag{3.30}$$

*and*

$$\mathrm{tr}(h_{k,x}(y)) = \mathrm{o}\left(|y|^N\right) \quad \text{as } k \to \infty \tag{3.31}$$

*uniformly with respect to* $x \in \mathbb{S}^n$ *and* $y \in \mathrm{B}_\xi(0,\varepsilon_0)$, *where* exp *and* tr *are the matrix exponential and trace maps, respectively.*

(ii) *The tensor* $h_{k,x}$ *satisfies*

$$h_{k,x}(y) = H_{k,x}(y) + \mathrm{o}\left(|y|^{\max(n-3,2)}\right) \quad \text{as } k \to \infty \tag{3.32}$$

*uniformly with respect to* $x \in \mathbb{S}^n$ *and* $y \in \mathrm{B}_\xi(0,\varepsilon_0)$, *where* $H_{k,x}(y)$ *is of the form*

$$H_{k,x}(y) = \sum_{|\alpha|=2}^{n-4} h_{k,x,\alpha} y^\alpha$$

*for some trace-free symmetric real matrices* $h_{k,x,\alpha}$ *which do not depend on* $y$. *Moreover,*

$$H_{k,x}(y)\, y = 0 \quad \text{and} \quad \mathrm{tr}(H_{k,x}(y)) = 0 \quad \forall y \in \mathbb{R}^n \tag{3.33}$$

*and* (3.32) *can be differentiated.*

(iii) *The scalar curvature of* $\hat{g}_{k,x}$ *satisfies*

$$\begin{aligned}\mathrm{Scal}_{\hat{g}_{k,x}}(y) = \sum_{a,b=1}^{n} \partial_{y_a}\partial_{y_b}(h_{k,x})_{ab}(y) - \sum_{a,b,c=1}^{n} \Big(\partial_{y_b}\left((H_{k,x})_{ab}\,\partial_{y_c}(H_{k,x})_{ac}\right) \\ - \frac{1}{2}\partial_{y_b}(H_{k,x})_{ab}\,\partial_{y_c}(H_{k,x})_{ac} + \frac{1}{4}\left(\partial_{y_c}(H_{k,x})_{ab}\right)^2\Big)(y) \\ + \mathrm{O}\left(\sum_{|\alpha|=2}^{d_n} |h_{k,x,\alpha}|^2 |y|^{2|\alpha|}\right) + \mathrm{o}\left(|y|^{n-1}\right) \quad \text{as } k\to\infty \qquad (3.34)\end{aligned}$$

*and*

$$\begin{aligned}\mathrm{Scal}_{\hat{g}_{k,x}}(y) = \sum_{a,b=1}^{n} \partial_{y_a}\partial_{y_b}(h_{k,x})_{ab}(y) + \mathrm{O}\left(\sum_{|\alpha|=2}^{d_n} |h_{k,x,\alpha}|^2 |y|^{2|\alpha|-2}\right) \\ + \mathrm{o}\left(|y|^{\max(n-3,2)}\right) \quad \text{as } k\to\infty \qquad (3.35)\end{aligned}$$

*uniformly with respect to* $x\in\mathbb{S}^n$ *and* $y\in \mathrm{B}_\xi(0,\varepsilon_0)$*, where* $(h_{k,x})_{ab}$ *and* $(H_{k,x})_{ab}$ *are the coefficients of* $h_{k,x}$ *and* $H_{k,x}$*, respectively, and*

$$d_n := \left[\frac{n-2}{2}\right].$$

Remark that (3.30) and the first identity in (3.33) can also be written as

$$\sum_{b=1}^{n} h_{k,x}(y)_{ab}\, y_b = 0 \quad \text{and} \quad \sum_{b=1}^{n} H_{k,x}(y)_{ab}\, y_b = 0 \quad \forall a\in\{1,\dots,n\},\ y\in\mathbb{R}^n.$$

We also point out that in the case where $3\le n\le 5$, (3.32) and (3.35) simply give

$$h_{k,x}(y) = \mathrm{o}\left(|y|^2\right) \quad \text{and} \quad \mathrm{Scal}_{\hat{g}_{k,x}}(y) = \mathrm{o}\left(|y|^2\right) \quad \text{as } k\to\infty$$

uniformly with respect to $x\in\mathbb{S}^n$ and $y\in \mathrm{B}_\xi(0,\varepsilon_0)$.

*Proof of Lemma 3.3.* We refer to [26, Section 4] for the proofs of (3.29), (3.30) and (3.31). By using (3.30) and (3.31) together with simple linear algebra considerations, we then obtain (3.33). That the remainder terms in (3.34) and (3.35) are respectively $\mathrm{o}\left(|y|^{n-1}\right)$ and $\mathrm{o}\left(|y|^{\max(n-3,2)}\right)$ follows from (3.28), [4, Proposition 26] and the fact that $g_k\to g_0$ in $C^m(\mathbb{S}^n)$ as $k\to\infty$ for all $m\in\mathbb{N}$. ☐

For each $k\in\mathbb{N}$, $x\in\mathbb{S}^n$ and $d\in\{2,\dots,n-4\}$, we define

$$H_{k,x,d}(y) := \sum_{|\alpha|=d} h_{k,x,\alpha} y^\alpha \quad \forall y\in\mathbb{R}^n. \qquad (3.36)$$

It is easy to see that $H_{k,x,d}$ is a homogeneous polynomial of degree $d$ and

$$H_{k,x} = \sum_{d=2}^{n-4} H_{k,x,d}.$$

As a consequence of Lemma 3.3, we obtain

$$\sum_{b=1}^{n} (H_{k,x,d}(y))_{bb} = 0 \quad \text{and} \quad \sum_{b=1}^{n} (H_{k,x,d}(y))_{ab}\, y_b = 0 \quad \forall a\in\{1,\dots,n\},\ y\in\mathbb{R}^n. \qquad (3.37)$$

For each $d \in \{0, \dots, n-4\}$, we define

$$w_{k,x,d}(y) := \sum_{|\alpha|=d} \sum_{a,b=1}^{n} (h_{k,x,\alpha})_{ab} \, \partial_{y_a} \partial_{y_b} (y^\alpha) \quad \forall y \in \mathbb{R}^n. \tag{3.38}$$

It is easy to see that $w_{k,x,d}$ is a homogeneous polynomial of degree $d-2 \in \{0, \dots, n-6\}$, which by (3.35) captures the main term in the Taylor expansion of $\mathrm{Scal}_{\hat{g}_{k,x}}(y)$ at 0. We claim that, for each $k \in \mathbb{N}$ and $x \in \mathbb{S}^n$,

$$w_{k,x,d}(y) = 0 \quad \forall y \in \mathbb{R}^n, \, d \in \{0,1,2,3\}. \tag{3.39}$$

This is obvious in the case where $d \in \{0,1\}$. When $d \in \{2,3\}$, $w_{k,x,d}$ is a homogeneous polynomial of order 0 or 1, respectively. By using (3.35) and remarking that

$$\mathrm{Scal}_{\hat{g}_{k,x}}(y) = \mathrm{O}\left( |y|^2 \right)$$

uniformly with respect to $y \in \mathrm{B}_{g_0}(x, r_0)$ and $k \in \mathbb{N}$, which follows from properties of the conformal normal coordinates (see [27]), we obtain (3.39). As a consequence of (3.39), we obtain that if $3 \le n \le 7$, then

$$w_{k,x,d}(y) = 0 \quad \forall y \in \mathbb{R}^n, \, d \in \{0, \dots, n-4\}, \tag{3.40}$$

hence (3.38) is trivial in this case. In dimensions $n \ge 8$, another result we need from Khuri, Marques and Schoen [26] (see also Li and Zhang [28, 29]) is the following:

**Lemma 3.4.** *Assume that $n \ge 8$. Let $(h_{k,x,\alpha})_{k,x,\alpha}$ be as in Lemma 3.3. For each $k \in \mathbb{N}$, $x \in \mathbb{S}^n$ and $d \in \{4, \dots, n-4\}$, let $w_{k,x,d}$ be defined by* (3.38). *Then there exists a unique family of real numbers $(\gamma_{k,x,d,l,m})_{0 \le m \le [(d-2)/2], 0 \le l \le m+2}$ such that the function $v_{k,x,d} : \mathbb{R}^n \to \mathbb{R}$ defined by*

$$v_{k,x,d}(y) := \left(1 + |y|^2\right)^{-\frac{n}{2}} \sum_{m=0}^{[(d-2)/2]} \sum_{l=0}^{m+2} \gamma_{k,x,d,l,m} \, |y|^{2l} \, \Delta_\xi^m w_{k,x,d} \quad \forall y \in \mathbb{R}^n$$

*solves the equation*

$$\Delta_\xi v_{k,x,d} = (2^* - 1) \, U_0^{2^*-2} v_{k,x,d} + U_0 w_{k,x,d} \quad \text{in } \mathbb{R}^n, \tag{3.41}$$

*where*

$$U_0(y) := \left( \frac{\sqrt{n(n-2)}}{1 + |y|^2} \right)^{\frac{n-2}{2}} \qquad \forall y \in \mathbb{R}^n.$$

*Moreover,*

$$v_{k,x,d}(0) = |\nabla v_{k,x,d}(0)| = 0 \tag{3.42}$$

*and, for each $j \in \mathbb{N}$,*

$$\left| \nabla^j v_{k,x,d}(y) \right| = \mathrm{O}\left( \sum_{|\alpha|=d} \frac{|h_{k,x,\alpha}|}{(1 + |y|)^{n-d-2+j}} \right) \tag{3.43}$$

*uniformly with respect to $x \in \mathbb{S}^n$, $y \in \mathbb{R}^n$ and $k \in \mathbb{N}$.*

*Proof of Lemma 3.4.* It is easy to see that

$$w_{k,x,d}(y) = \sum_{a,b=1}^{n} \partial_{y_a} \partial_{y_b} (H_{k,x,d})_{ab}(y) = \mathrm{div}_\xi \, \mathrm{div}_\xi \, H_{k,x,d}(y) \quad \forall y \in \mathbb{R}^n, \tag{3.44}$$

so that, in particular, $w_{k,x,d}$ is a homogeneous polynomial of degree $d-2 \in \{2,\dots,n-6\}$. We recall that two homogeneous polynomials $p$ and $q$ in $\mathbb{R}^n$ are said to be orthogonal if

$$\int_{\mathbb{S}^{n-1}} pq d\sigma = 0,$$

where d$\sigma$ is the volume element of the round metric on the $(n-1)$-sphere $\mathbb{S}^{n-1}$. Since $w_{k,x,d}$ is homogeneous of degree $d-2$, we obtain

$$\int_{\mathbb{S}^{n-1}} w_{k,x,d}\, \mathrm{d}\sigma = (n+d-2)\int_{\mathbb{B}^n} w_{k,x,d}\, \mathrm{dy}, \tag{3.45}$$

where $\mathbb{B}^n := \mathrm{B}_\xi(0,1)$. On the other hand, by using (3.37) and (3.44) together with an integration by parts, we obtain

$$\begin{aligned}\int_{\mathbb{B}^n} w_{k,x,d}\, \mathrm{dy} &= \sum_{a,b=1}^n \int_{\mathbb{S}^{n-1}} \partial_{y_b}\left(H_{k,x,d}\right)_{ab}(y)\, y_a\, \mathrm{d}\sigma(y)\\ &= \sum_{b=1}^n \int_{\mathbb{S}^{n-1}} \left(\partial_{y_b}\left(\sum_{a=1}^n \left(H_{k,x,d}\right)_{ab}(y)\, y_a\right) - \left(H_{k,x,d}\right)_{bb}(y)\right)\mathrm{d}\sigma(y)\\ &= 0.\end{aligned} \tag{3.46}$$

It follows from (3.45) and (3.46) that $w_{k,x,d}$ is orthogonal to 1. In a similar way, for each $i \in \{1,\dots,n\}$, we obtain

$$\int_{\mathbb{S}^{n-1}} w_{k,x,d}(y)\, y_i\, \mathrm{d}\sigma(y) = (n+d-1)\int_{\mathbb{B}^n} w_{k,x,d}(y)\, y_i\, \mathrm{dy} \tag{3.47}$$

and

$$\begin{aligned}\int_{\mathbb{B}^n} w_{k,x,d}(y)\, y_i\, \mathrm{dy} &= \sum_{a=1}^n \int_{\mathbb{B}^n} \partial_{y_a}\left(\sum_{b=1}^n \partial_{y_b}\left(H_{k,x,d}\right)_{ab}(y)\, y_i - \left(H_{k,x,d}\right)_{ai}(y)\right)\mathrm{dy}\\ &= \sum_{a=1}^n \int_{\mathbb{S}^{n-1}} \left(\sum_{b=1}^n \partial_{y_b}\left(H_{k,x,d}\right)_{ab}(y)\, y_i - \left(H_{k,x,d}\right)_{ai}(y)\right) y_a\, \mathrm{d}\sigma(y)\\ &= 0.\end{aligned} \tag{3.48}$$

It follows from (3.47) and (3.48) that $w_{k,x,d}$ is orthogonal to $y_i$. We are now in position to apply [26, Proposition 4.1] (see also the remark below), from which Lemma 3.4 then follows. □

From now on, we let $(g_k)_k$, $(u_k)_k$, $(x_{1,k})_k$, $(x_{2,k})_k$, $(\mu_{1,k})_k$, $(\mu_{2,k})_k$ and $(d_k)_k$ be as in Lemma 3.1, $(\varphi_k)_k$, $(\hat{g}_{k,x})_{k,x}$, $\left(\exp_{k,x}\right)_{k,x}$, $\varepsilon_0$ be as in Lemma 3.2, $(h_{k,x,\alpha})_{k,x,\alpha}$ and $d_n$ be as in Lemma 3.3 and $(v_{k,x,d})_{k,x,d}$ be as in Lemma 3.4. We now introduce some additional notations of radii and rescaled functions. For each $k \in \mathbb{N}$, we define

$$\varrho_{1,k} := \frac{d_k}{\mu_{1,k}} \quad \text{and} \quad \varrho_{2,k} := \sqrt{\frac{\mu_{1,k}}{\mu_{2,k}} + \frac{d_k^2}{\mu_{1,k}\mu_{2,k}}}. \tag{3.49}$$

For each $k \in \mathbb{N}$ and $i \in \{1,2\}$, we define

$$\exp_{i,k} := \exp_{k,x_{i,k}}, \quad h_{i,k,\alpha} := h_{k,x_{i,k},\alpha} \quad \text{and} \quad v_{i,k,d} := v_{k,x_{i,k},d} \tag{3.50}$$

as well as

$$\hat{u}_{i,k}(y) := \mu_{i,k}^{\frac{n-2}{2}}\left(\varphi_k\left(x_{i,k},\cdot\right)^{-1} u_k\right)\left(\exp_{i,k}\left(\mu_{i,k} y\right)\right) \quad \forall y \in \mathbb{R}^n \tag{3.51}$$

and

$$\hat{v}_{i,k} := c_n \sum_{d=4}^{n-4} \mu_{i,k}^d v_{i,k,d}. \tag{3.52}$$

In the case where $3 \leq n \leq 7$, $\hat{v}_{i,k} = 0$ (see the discussion around (3.40)).

To prove Theorems 1.1 and 2.1, we need refined asymptotics on the functions $(u_k)_k$. In the following lemma, we improve the a priori estimates of Lemma 3.1 and obtain a sharp description of $u_k + (-1)^i B_{i,k}$ near $x_{i,k}$, which depends on the local geometry of $g_k$ near this point. After scaling, this amounts to obtaining refined pointwise estimates on $\hat{u}_{i,k} + (-1)^i U_0$ in Euclidean balls of radii of order $\varrho_{i,k}$. The analysis of [26] does not directly apply here since the functions $(u_k)_k$ change sign and, as a consequence, the blow-up points $(x_{1,k})_k$ and $(x_{2,k})_k$ are not isolated and simple (in particular, $d_k = \mathrm{d}_{g_0}(x_{1,k}, x_{2,k})$ may tend to 0 as $k \to \infty$). Our refined estimates are as follows:

**Lemma 3.5.** *Let $(g_k)_k$, $(u_k)_k$, $(x_{1,k})_k$, $(x_{2,k})_k$, $(\mu_{1,k})_k$, $(\mu_{2,k})_k$ and $(d_k)_k$ be as in Lemma 3.1, $(\varphi_k)_k$, $(\hat{g}_{k,x})_{k,x}$, $(\exp_{k,x})_{k,x}$, $\varepsilon_0$ be as in Lemma 3.2, $(h_{k,x,\alpha})_{k,x,\alpha}$ and $d_n$ be as in Lemma 3.3 and $(v_{k,x,d})_{k,x,d}$ be as in Lemma 3.4. Let $i \in \{1,2\}$ and $(\varrho_k)_{i,k}$, $(\hat{u}_{i,k})_k$ and $(\hat{v}_{i,k})_k$ be as in (3.49), (3.51) and (3.52), respectively. In the case where $i = 1$, assume that*

$$\mu_{1,k} = \mathrm{o}(d_k) \quad (\textit{i.e. } \varrho_{1,k} \to \infty) \quad \textit{as } k \to \infty \tag{3.53}$$

*and (3.11) holds true (observe that (3.53) implies (3.10) since $\mu_{2,k} \leq \mu_{1,k}$ for all $k \in \mathbb{N}$). In the case where $i = 2$, we do not make any additional assumptions. Then there exist $\delta_0 \in (0, \varepsilon_0/\pi)$ and $k_0 \in \mathbb{N}$ such that*

$$\sum_{j=0}^{2} (1+|y|)^j \left|\nabla^j\left(\hat{u}_{i,k} + (-1)^i (U_0 - \hat{v}_{i,k})\right)(y)\right|$$
$$= \mathrm{O}\left(\left\{\sum_{|\alpha|=2}^{d_n-1} \frac{|h_{i,k,\alpha}|^2 \mu_{i,k}^{2|\alpha|}}{(1+|y|)^{n-2|\alpha|-2}} + \frac{\mu_{i,k}^{n-3}}{1+|y|} \quad \textit{if } n \geq 6\right\} + \varrho_{i,k}^{2-n}\right) \tag{3.54}$$

*uniformly with respect to $y \in \mathrm{B}_\xi(0, \delta_0 \varrho_{i,k})$ and $k > k_0$.*

In the case where $n \in \{6,7\}$, the sum in the right-hand side of (3.54) is empty.

*Proof of Lemma 3.5.* We adapt the arguments in the proof of [26, Proposition 5.1], taking into account a general configuration for $(x_{1,k})_k$ and $(x_{2,k})_k$. In particular, we assume neither that $d_k \not\to 0$ as $k \to \infty$ nor that the blow-up points $(x_{1,k})_k$ and $(x_{2,k})_k$ are isolated and simple. Since $g_k \to g_0$ and $\varphi_k(x_{i,k}, x_{i,k}) = 1$, we obtain

$$\begin{aligned} \mathrm{d}_{g_0}\left(\exp_{i,k}(y), x_{i,k}\right) &= (1+\mathrm{o}(1))\, \mathrm{d}_{g_k}\left(\exp_{i,k}(y), x_{i,k}\right) \\ &= |y| + \mathrm{O}\left(|y|^2\right) + \mathrm{o}(|y|) \quad \text{as } k \to \infty \end{aligned} \tag{3.55}$$

uniformly with respect to $y$ in compact subsets of $\mathbb{R}^n$. Moreover, since $d_k \leq \pi$ and $\mu_{2,k} \leq \mu_{1,k} \to 0$, we obtain

$$\mu_{1,k}\varrho_{1,k} = d_k \leq \pi \tag{3.56}$$

and

$$\mu_{2,k}\varrho_{2,k} = \sqrt{\mu_{1,k}\mu_{2,k} + \frac{\mu_{2,k}}{\mu_{1,k}} d_k^2} \leq d_k + \mathrm{O}(\mu_{1,k}) \leq \pi + \mathrm{o}(1) \quad \text{as } k \to \infty. \tag{3.57}$$

Since $g_{k,x_{i,k}} \to g_{0,x_i}$ in $C^m(\mathbb{S}^n)$ as $k \to \infty$ for all $m \in \mathbb{N}$, it follows from (3.56) and (3.57) that there exist $\delta_0 \in (0, \varepsilon_0/\pi)$ and $k_0 \in \mathbb{N}$ such that, for each $k > k_0$, $\delta_0 \mu_{i,k} \varrho_{i,k}$ is smaller than the injectivity radius at $x_{i,k}$ of the metric $g_{k,x_{i,k}}$ or, equivalently, $\delta_0 \varrho_{i,k}$ is smaller than the injectivity radius at 0 of the rescaled metric

$$\hat{g}_{i,k} := \hat{g}_{k,x_{i,k}}(\mu_{i,k}\cdot). \tag{3.58}$$

By letting $k_0$ be smaller if necessary, (3.56) and (3.57) also give

$$\mu_{i,k}|y| \le \varepsilon_0 \quad \forall y \in \mathrm{B}_\xi(0, \delta_0\varrho_{i,k}). \tag{3.59}$$

We restrict ourselves to giving the proof of (3.54) for $j = 0$ as the estimates on the derivatives then follow by standard elliptic theory. Since $\varphi_k \to \varphi_0$ in $C^m(\mathbb{S}^n \times \mathbb{S}^n)$ for all $m \in \mathbb{N}$, $x_{i,k} \to x_i$ as $k \to \infty$ and $\delta_0 < \varepsilon_0/\pi$, it follows from (3.12), (3.25), (3.56) and (3.57) that

$$\begin{aligned}&\left|\left(\hat{u}_{i,k} + (-1)^i U_0\right)(y)\right|\\ &\quad = \mathrm{O}\left(\mu_{i,k}^{\frac{n-2}{2}} B_{3-i,k}\left(\exp_{i,k}(\mu_{i,k}y)\right)\right) + \mathrm{o}(U_0(y)) \quad \text{as } k \to \infty\end{aligned} \tag{3.60}$$

uniformly with respect to $y \in \mathrm{B}_\xi(0, \delta_0\varrho_{i,k})$. Moreover, by using (3.55), (3.56) and (3.57), we obtain

$$\begin{aligned}&\mathrm{d}_{g_0}\left(\exp_{i,k}(\mu_{i,k}y), x_{3-i,k}\right)\\ &\quad \ge d_k - \mathrm{d}_{g_0}\left(\exp_{i,k}(\mu_{i,k}y), x_{i,k}\right)\\ &\quad \ge (1 - \delta_0 + \mathrm{o}(1))\, d_k + \mathrm{O}\left((\delta_0 d_k)^2 + \delta_0\mu_{3-i,k}\right) \quad \text{as } k \to \infty\end{aligned} \tag{3.61}$$

uniformly with respect to $y \in \mathrm{B}_\xi(0, \delta_0\varrho_{i,k})$. By letting $\delta_0$ be smaller if necessary, it follows from (3.61) that

$$\mu_{i,k}^{\frac{n-2}{2}} B_{3-i,k}\left(\exp_{i,k}(\mu_{i,k}y)\right) = \mathrm{O}\left(\varrho_{i,k}^{2-n}\right) \tag{3.62}$$

uniformly with respect to $y \in \mathrm{B}_\xi(0, \delta_0\varrho_{i,k})$ and $k > k_0$. By combining (3.60) and (3.62), we obtain

$$\left|\left(\hat{u}_{i,k} + (-1)^i U_0\right)(y)\right| = \mathrm{O}\left(\varrho_{i,k}^{2-n}\right) + \mathrm{o}(U_0(y)) \quad \text{as } k \to \infty \tag{3.63}$$

uniformly with respect to $y \in \mathrm{B}_\xi(0, \delta_0\varrho_{i,k})$. Moreover, (3.43) gives

$$|\hat{v}_{i,k}(y)| = \mathrm{O}\left(\sum_{|\alpha|=4}^{n-4} \frac{|h_{i,k,\alpha}|\, \mu_{i,k}^{|\alpha|}\, |y|^{|\alpha|+2}}{(1+|y|)^n}\right) = \mathrm{o}(U_0(y)) \quad \text{as } k \to \infty \tag{3.64}$$

uniformly with respect to $y \in \mathrm{B}_\xi(0, \delta_0\varrho_{i,k})$. For each $k > k_0$, we define

$$m_{i,k} := \max_{\mathrm{B}_\xi(0,\delta_0\varrho_{i,k})} \left|\left(\hat{u}_{i,k} + (-1)^i(U_0 - \hat{v}_{i,k})\right)\right|$$

and

$$\psi_{i,k} := m_{i,k}^{-1}\left(\hat{u}_{i,k} + (-1)^i(U_0 - \hat{v}_{i,k})\right).$$

By using (3.53) for $i = 1$ and Lemma 3.1 (iii) for $i = 2$, we obtain $\varrho_{i,k} \to 0$ as $k \to \infty$ in both cases. By using (3.63) and (3.64), we then obtain $m_{i,k} \to 0$ as $k \to \infty$. By using the conformal invariance of the conformal Laplacian, we can rewrite (3.3) as

$$\Delta_{\hat{g}_{i,k}}\hat{u}_{i,k} + c_n\mu_{i,k}^2\,\mathrm{Scal}_{\hat{g}_{i,k}}\,\hat{u}_{i,k} = |\hat{u}_{i,k}|^{2^*-2}\,\hat{u}_{i,k} \quad \text{in } \mathbb{R}^n. \tag{3.65}$$

Moreover, by using (3.41), we obtain

$$\Delta_\xi \hat{v}_{i,k} = (2^*-1)\, U_0^{2^*-2} \hat{v}_{i,k} + U_0 \hat{w}_{i,k} \quad \text{in } \mathbb{R}^n, \tag{3.66}$$

where

$$\hat{w}_{i,k} := c_n \sum_{d=4}^{n-4} \mu_{i,k}^d w_{k,x_{i,k},d}. \tag{3.67}$$

By using (3.65) and (3.66) together with the equation $\Delta_\xi U_0 = U_0^{2^*-1}$, we obtain

$$\Delta_{\hat{g}_{i,k}} \psi_{i,k} + c_n \mu_{i,k}^2 \operatorname{Scal}_{\hat{g}_{i,k}} \psi_{i,k} = (2^*-1)\, U_0^{2^*-2} \psi_{i,k} + m_{i,k}^{-1} f_{i,k} \tag{3.68}$$

in $\mathrm{B}_\xi(0, \delta_0 \varrho_{i,k})$, where

$$\begin{aligned} f_{i,k} := {} & (-1)^i \left(\Delta_{\hat{g}_{i,k}} - \Delta_\xi\right)(U_0 - \hat{v}_{i,k}) - (-1)^i c_n \mu_{i,k}^2 \operatorname{Scal}_{\hat{g}_{i,k}} \hat{v}_{i,k} \\ & + (-1)^i U_0 \left(c_n \mu_{i,k}^2 \operatorname{Scal}_{\hat{g}_{i,k}} - \hat{w}_{i,k}\right) + |\hat{u}_{i,k}|^{2^*-2} \hat{u}_{i,k} + (-1)^i U_0^{2^*-1} \\ & - (2^*-1)\, U_0^{2^*-2} \big((-1)^i \hat{v}_{i,k} + m_{i,k} \psi_{i,k}\big). \end{aligned} \tag{3.69}$$

We now estimate the terms in the right-hand side of (3.69). Since $U_0$ is radially symmetric around 0, it follows from (3.27) that

$$\left(\Delta_{\hat{g}_{i,k}} - \Delta_\xi\right) U_0(y) = \mathrm{O}\left(\mu_{i,k}^N |y|^{N-1} |\nabla U_0(y)|\right) = \mathrm{O}\left(\frac{(\mu_{i,k}|y|)^N}{(1+|y|)^n}\right) \tag{3.70}$$

uniformly with respect to $y \in \mathrm{B}_\xi(0, \delta_0 \varrho_{i,k})$ and $k > k_0$. We recall that, in the case where $3 \le n \le 7$, $\hat{v}_{i,k} = 0$ for all $k \in \mathbb{N}$. When $n \ge 8$, by using Lemmas 3.3 and 3.4 together with straightforward estimates and the fact that $n - 4 \ge d_n$, we obtain

$$\begin{aligned} & \left(\Delta_{\hat{g}_{i,k}} - \Delta_\xi\right) \hat{v}_{i,k}(y) - (-1)^i c_n \mu_{i,k}^2 \operatorname{Scal}_{\hat{g}_{i,k}}(y)\, \hat{v}_{i,k}(y) \\ & = \mathrm{O}\left(|\nabla \hat{g}_{i,k}(y)|\, |\nabla \hat{v}_{i,k}(y)| + |(\hat{g}_{i,k} - \xi)(y)| \left|\nabla^2 \hat{v}_{i,k}(y)\right| + \mu_{i,k}^2 \left|\operatorname{Scal}_{\hat{g}_{i,k}}(y)\, \hat{v}_{i,k}(y)\right|\right) \\ & = \mathrm{O}\left(\sum_{|\alpha|=2}^{d_n-1} \frac{|h_{i,k,\alpha}|^2 \mu_{i,k}^{2|\alpha|}}{(1+|y|)^{n-2|\alpha|}}\right) + \mathrm{o}\left(\frac{\mu_{i,k}^{n-3}}{(1+|y|)^3}\right) \quad \text{as } k \to \infty \end{aligned} \tag{3.71}$$

uniformly with respect to $y \in \mathrm{B}_\xi(0, \delta_0 \varrho_{i,k})$. By using (3.32), (3.35) and (3.64), we obtain

$$\begin{aligned} & U_0(y) \left(c_n \mu_{i,k}^2 \operatorname{Scal}_{\hat{g}_{i,k}} - \hat{w}_{i,k}\right)(y) \\ & \quad = \left\{ \mathrm{O}\left(\sum_{|\alpha|=4}^{d_n-1} \frac{|h_{i,k,\alpha}|^2 \mu_{i,k}^{2|\alpha|}}{(1+|y|)^{n-2|\alpha|}}\right) + \mathrm{o}\left(\frac{\mu_{i,k}^{n-3}}{(1+|y|)^3}\right) \quad \text{if } n \ge 6 \right\} \\ & \qquad + \frac{\mu_{i,k}^2 (\mu_{i,k}|y|)^{\max(n-3.2)}}{(1+|y|)^{n-2}} \quad \text{as } k \to \infty \end{aligned} \tag{3.72}$$

and

$$\begin{aligned} & \left(|\hat{u}_{i,k}|^{2^*-2} \hat{u}_{i,k} + (-1)^i U_0^{2^*-1} - (2^*-1)\, U_0^{2^*-2} \big((-1)^i \hat{v}_{i,k} + m_{i,k} \psi_{i,k}\big)\right)(y) \\ & \quad = \mathrm{O}\left(U_0(y)^{2^*-3} \left(\hat{v}_{i,k}(y)^2 + (m_{i,k} \psi_{i,k}(y))^2\right)\right) \end{aligned}$$

$$= \left\{ \mathrm{O}\left( \sum_{|\alpha|=4}^{d_n-1} \frac{|h_{i,k,\alpha}|^2 \mu_{i,k}^{2|\alpha|}}{(1+|y|)^{n-2|\alpha|}} \right) + \mathrm{o}\left( \frac{\mu_{i,k}^{n-3}}{(1+|y|)^3} \right) \quad \text{if } n \geq 6 \right\}$$
$$+ \mathrm{o}\left( m_{i,k} U_0(y)^{2^*-2} |\psi_{i,k}(y)| \right) \quad \text{as } k \to \infty \tag{3.73}$$

uniformly with respect to $y \in \mathrm{B}_\xi(0, \delta_0 \varrho_{i,k})$. We now let $\widehat{G}_{i,k}$ be the Green's function of $\Delta_{\hat{g}_{i,k}} + c_n \mu_{i,k}^2 \operatorname{Scal}_{\hat{g}_{i,k}}$ in $\mathrm{B}_\xi(0, \delta_0 \varrho_{i,k})$ with zero Dirichlet boundary condition on $\partial \mathrm{B}_\xi(0, \delta_0 \varrho_{i,k})$. By definition of $\hat{g}_{i,k}$, it is easy to see that

$$\widehat{G}_{i,k}(x,y) = \mu_{i,k}^{n-2} \widetilde{G}_{i,k}(\mu_{i,k} x, \mu_{i,k} y) \quad \forall x, y \in \mathrm{B}_\xi(\delta_0 \varrho_{i,k}),\ x \neq y,$$

where $\widetilde{G}_{i,k}$ is the Green's function of $\Delta_{\hat{g}_{k,x_{i,k}}} + c_n \operatorname{Scal}_{\hat{g}_{k,x_{i,k}}}$ with Dirichlet boundary condition in $\mathrm{B}_\xi(0, \delta_0 \mu_{i,k} \varrho_{i,k})$. Since $g_k \to g_0$ in $C^m(\mathbb{S}^n)$ as $k \to \infty$ for all $m \in \mathbb{N}$, it follows from (3.28) and (3.59) that $\hat{g}_{k,x_{i,k}} \to \xi$ as $k \to \infty$ in $C^m(\mathrm{B}_\xi(0, \varepsilon_0))$ for all $m \in \mathbb{N}$. By using standard estimates for $\widetilde{G}_{i,k}$, which can be found for instance in [47], it is not difficult to see that $\widehat{G}_{i,k}$ satisfies

$$\widehat{G}_{i,k}(y,z) \leq C |y-z|^{2-n} \quad \forall y, z \in \mathrm{B}_\xi(0, \delta_0 \varrho_{i,k}) \tag{3.74}$$

and

$$\left| \partial_\nu \widehat{G}_{i,k}(y,z) \right| \leq C |y-z|^{1-n} \quad \forall y \in \mathrm{B}_\xi(0, \delta_0 \varrho_{i,k}),\ z \in \partial \mathrm{B}_\xi(0, \delta_0 \varrho_{i,k}) \tag{3.75}$$

for some constant $C$ independent of $k$. A representation formula for (3.68) now gives

$$\psi_{i,k}(y) = \int_{\mathrm{B}_\xi(0,\delta_0 \varrho_{i,k})} \widehat{G}_{i,k}(y,\cdot) \left( (2^*-1) U_0^{2^*-2} \psi_{i,k} + m_{i,k}^{-1} f_{i,k} \right) \mathrm{dv}_{\hat{g}_{i,k}}$$
$$- \int_{\partial \mathrm{B}_\xi(0,\delta_0 \varrho_{i,k})} \partial_\nu \widehat{G}_{i,k}(y,\cdot) \psi_{i,k} \, \mathrm{d}\sigma_{\hat{g}_{i,k}} \tag{3.76}$$

for all $y \in \mathrm{B}_\xi(0, \delta_0 \varrho_{i,k})$, where $\nu$ and $\mathrm{d}\sigma_{\hat{g}_{i,k}}$ are the outward unit normal vector and volume element, respectively, induced by $\hat{g}_{i,k}$ on $\partial \mathrm{B}_\xi(0, \delta_0 \varrho_{i,k})$. We observe that (3.63) and (3.64) give

$$\max_{\mathrm{B}_\xi(0,\delta_0 \varrho_{i,k}) \setminus \mathrm{B}_\xi(0,\delta_0 \varrho_{i,k}/2)} |\psi_{i,k}| = \mathrm{O}\left( m_{i,k}^{-1} \varrho_{i,k}^{2-n} \right) \tag{3.77}$$

uniformly with respect to $k > k_0$. First considering the case where $|y| \leq \delta_0 \varrho_{i,k}/2$, by putting together (3.70), (3.71), (3.72), (3.73), (3.74), (3.75), (3.76) and (3.77) and using straightforward integral estimates, we obtain

$$\psi_{i,k}(y) = \mathrm{O}\left( \int_{\mathrm{B}_\xi(0,\delta_0 \varrho_{i,k})} \frac{|\psi_{i,k}(z)| \, \mathrm{dz}}{|y-z|^{n-2} (1+|z|)^4} + m_{i,k}^{-1} \left( \left\{ \sum_{|\alpha|=2}^{d_n-1} \frac{|h_{i,k,\alpha}|^2 \mu_{i,k}^{2|\alpha|}}{(1+|y|)^{n-2|\alpha|-2}} \right. \right. \right.$$
$$\left. \left. \left. + \frac{\mu_{i,k}^{n-3}}{1+|y|} \quad \text{if } n \geq 6 \right\} + \varrho_{i,k}^{2-n} \left( 1 + (\mu_{i,k} \varrho_{i,k})^N \right) \right) \right)$$
$$= \mathrm{O}\left( \int_{\mathrm{B}_\xi(0,\delta_0 \varrho_{i,k})} \frac{|\psi_{i,k}(z)| \, \mathrm{dz}}{|y-z|^{n-2} (1+|z|)^4} + m_{i,k}^{-1} \left( \left\{ \sum_{|\alpha|=2}^{d_n-1} \frac{|h_{i,k,\alpha}|^2 \mu_{i,k}^{2|\alpha|}}{(1+|y|)^{n-2|\alpha|-2}} \right. \right. \right.$$
$$\left. \left. \left. + \frac{\mu_{i,k}^{n-3}}{1+|y|} \quad \text{if } n \geq 6 \right\} + \varrho_{i,k}^{2-n} \right) \right) \tag{3.78}$$

uniformly with respect to $y \in \mathrm{B}_\xi\left(0, \delta_0 \varrho_{i,k}/2\right)$ and $k > k_0$. It follows from (3.77) that (3.78) actually remains true for $y \in \mathrm{B}_\xi\left(0, \delta_0 \varrho_{i,k}\right)$. We now claim that

$$m_{i,k} = \mathrm{O}\left(\left\{\sum_{|\alpha|=2}^{d_n-1} \left|h_{i,k,\alpha}\right|^2 \mu_{i,k}^{2|\alpha|} + \mu_{i,k}^{n-3} \quad \text{if } n \geq 6\right\} + \varrho_{i,k}^{2-n}\right) \tag{3.79}$$

uniformly with respect to $k > k_0$. We assume by contradiction that (3.79) does not hold true, i.e. there exists a subsequence $(k_j)_{j\in\mathbb{N}}$ such that $k_j \to \infty$ and

$$\left\{\sum_{|\alpha|=2}^{d_n-1} \left|h_{i,k_j,\alpha}\right|^2 \mu_{i,k_j}^{2|\alpha|} + \mu_{i,k_j}^{n-3} \quad \text{if } n \geq 6\right\} + \varrho_{i,k_j}^{2-n} = \mathrm{o}\left(m_{i,k_j}\right) \quad \text{as } j \to \infty. \tag{3.80}$$

Since $\left|\psi_{i,k_j}\right| \leq 1$ in $\mathrm{B}_\xi\left(0, \delta_0 \varrho_{i,k_j}\right)$ and $\hat{g}_{i,k_j} \to \xi$ in $C^m_{\mathrm{loc}}\left(\mathbb{R}^n\right)$ as $j \to \infty$, it follows from (3.68), (3.70), (3.71), (3.72) and (3.73) and standard elliptic estimates that, up to a subsequence, $\left(\psi_{i,k_j}\right)_j$ converges in $C^1_{\mathrm{loc}}\left(\mathbb{R}^n\right)$ as $j \to \infty$ to a solution $\psi_0 \in C^\infty\left(\mathbb{R}^n\right)$ of the equation

$$\Delta_\xi \psi_0 = \left(2^* - 1\right) U_0^{2^*-2} \psi_0 \quad \text{in } \mathbb{R}^n. \tag{3.81}$$

On the other hand, by using (3.78) and (3.80), we obtain

$$\psi_{i,k_j}(y) = \mathrm{O}\left(\left(1 + |y|\right)^{-2}\right) + \mathrm{o}(1) \quad \text{as } j \to \infty \tag{3.82}$$

uniformly with respect to $y \in \mathrm{B}_\xi\left(0, \delta_0 \varrho_{i,k_j}\right)$. By passing to the limit as $j \to \infty$ into (3.82), we then obtain

$$\psi_0(y) = \mathrm{O}\left(\left(1 + |y|\right)^{-2}\right) \tag{3.83}$$

uniformly with respect to $y \in \mathbb{R}^n$. By applying Lemma 2.4 in [7], it follows from (3.81) and (3.83) that

$$\psi_0(y) = \lambda_0 \frac{1 - \frac{|y|^2}{n(n-2)}}{\left(1 + \frac{|y|^2}{n(n-2)}\right)^{\frac{n}{2}}} + \sum_{i=1}^n \lambda_i \frac{y_i}{\left(1 + \frac{|y|^2}{n(n-2)}\right)^{\frac{n}{2}}} \quad \forall y \in \mathbb{R}^n \tag{3.84}$$

for some $\lambda_0, \dots, \lambda_n \in \mathbb{R}$. On the other hand, by using (3.9), (3.11), (3.26), and (3.42), we obtain $\psi_{i,k}(0) = \left|\nabla \psi_{i,k}(0)\right| = 0$ for all $k \in \mathbb{N}$, which gives $\psi_0(0) = \left|\nabla \psi_0(0)\right| = 0$. It then follows from (3.83) and (3.84) that $\lambda_0 = \dots = \lambda_n = 0$, and so $\psi_0 = 0$. Independently, for each $k > k_0$, by definition of $m_{i,k}$, there exists a point $y_{i,k} \in \overline{\mathrm{B}_\xi\left(0, \delta_0 \varrho_{i,k}\right)}$ such that $\left|\psi_{i,k}\left(y_{i,k}\right)\right| = 1$. It follows from (3.82) that $\left(y_{i,k_j}\right)_j$ is bounded. This is in contradiction with the fact that $\psi_{i,k_j} \to \psi_0 = 0$ as $j \to \infty$ uniformly in compact subsets of $\mathbb{R}^n$. This proves that (3.79) holds true. Finally, (3.54) follows from (3.79) together with successive iterations of (3.78). This ends the proof of Lemma 3.5. □

By using (3.43), (3.54) and (3.59) and observing that $h_{k,x_{i,k}} \to 0$ in $C^m_{\mathrm{loc}}\left(\mathbb{R}^n\right)$ as $k \to \infty$ for all $m \in \mathbb{N}$ and $d_n \leq n - 4$ when $n \geq 6$, we obtain

$$\begin{aligned}&\sum_{j=0}^2 \left(1 + |y|\right)^j \left|\nabla^j \left(\hat{u}_{i,k} + (-1)^i U_0\right)(y)\right| \\ &\quad = \mathrm{O}\left(\left\{\sum_{|\alpha|=2}^{n-4} \frac{\left|h_{i,k,\alpha}\right| \mu_{i,k}^{|\alpha|}}{\left(1 + |y|\right)^{n-2-|\alpha|}} + \frac{\mu_{i,k}^{n-3}}{1 + |y|} \quad \text{if } n \geq 6\right\} + \varrho_{i,k}^{2-n}\right)\end{aligned} \tag{3.85}$$

uniformly with respect to $y \in \mathrm{B}_\xi\left(0, \delta_0 \varrho_{i,k}\right)$ and $k > k_0$. Here again, in the case where $n \in \{6,7\}$, the sum in the right-hand side of (3.85) is empty. We frequently use this estimate in what follows.

We now write a suitable Pohozaev-type identity:

**Lemma 3.6.** *Let $\left(\varrho_{2,k}\right)_k$, $\left(\hat{u}_{2,k}\right)_k$ and $\left(\hat{g}_{2,k}\right)_k$ be as in (3.49), (3.51) and (3.58), respectively, and $k_0$ and $\delta_0$ be as in Lemma 3.5. Then, for each $\delta \in (0, \delta_0)$ and $k > k_0$,*

$$\begin{aligned}\int_{\mathrm{B}_\xi(0,\delta\varrho_{2,k})}&\left(\langle\nabla \hat{u}_{2,k},\cdot\rangle_\xi+\frac{n-2}{2}\hat{u}_{2,k}\right)\left(\left(\Delta_{\hat{g}_{2,k}}-\Delta_\xi\right)\hat{u}_{2,k}+c_n\mu_{2,k}^2\operatorname{Scal}_{\hat{g}_{2,k}}\hat{u}_{2,k}\right)\mathrm{d}y\\&=\int_{\partial\,\mathrm{B}_\xi(0,\delta\varrho_{2,k})}\left(\frac{n-2}{2}\hat{u}_{2,k}\partial_\nu\hat{u}_{2,k}+\delta\varrho_{2,k}\left(\partial_\nu\hat{u}_{2,k}\right)^2-\frac{\delta\varrho_{2,k}}{2}\left|\nabla\hat{u}_{2,k}\right|_\xi^2\right.\\&\qquad\left.+\frac{\delta\varrho_{2,k}}{2^*}\left|\hat{u}_{2,k}\right|^{2^*}\right)\mathrm{d}\sigma,\end{aligned}\tag{3.86}$$

*where $\nu$ and $\mathrm{d}\sigma$ are the outward unit normal vector and volume element, respectively, of the metric induced by $\xi$ on $\partial\,\mathrm{B}_\xi\left(0,\delta\varrho_{2,k}\right)$.*

*Proof of Lemma 3.5.* See for example (2.7) in [31]. □

We first estimate the boundary term of (3.86). We obtain the following:

**Lemma 3.7.** *Let $\left(\varrho_{2,k}\right)_k$, $\left(\hat{u}_{2,k}\right)_k$ and $\left(\hat{g}_{2,k}\right)_k$ be as in (3.49), (3.51) and (3.58), respectively. Then*

$$\begin{aligned}\lim_{\delta\to0}\lim_{k\to\infty}&\left(\varrho_{2,k}^{n-2}\int_{\partial\,\mathrm{B}_\xi(0,\delta\varrho_{2,k})}\left(\frac{n-2}{2}\hat{u}_{2,k}\partial_\nu\hat{u}_{2,k}+\delta\varrho_{2,k}\left(\partial_\nu\hat{u}_{2,k}\right)^2\right.\right.\\&\qquad\left.\left.-\frac{\delta\varrho_{2,k}}{2}\left|\nabla\hat{u}_{2,k}\right|_\xi^2+\frac{\delta\varrho_{2,k}}{2^*}\left|\hat{u}_{2,k}\right|^{2^*}\right)\mathrm{d}\sigma\right)>0.\end{aligned}\tag{3.87}$$

*Proof of Lemma 3.7.* By letting

$$\check{u}_{2,k}(y):=\varrho_{2,k}^{n-2}\hat{u}_{2,k}\left(\varrho_{2,k}y\right)\quad\text{and}\quad\check{g}_{2,k}(y):=\hat{g}_{2,k}\left(\varrho_{2,k}y\right)\quad\forall y\in\mathbb{R}^n,$$

we obtain

$$\begin{aligned}\int_{\partial\,\mathrm{B}_\xi(0,\delta\varrho_{2,k})}&\left(\frac{n-2}{2}\hat{u}_{2,k}\partial_\nu\hat{u}_{2,k}+\delta\varrho_{2,k}\left(\partial_\nu\hat{u}_{2,k}\right)^2-\frac{\delta\varrho_{2,k}}{2}\left|\nabla\hat{u}_{2,k}\right|_\xi^2\right.\\&\left.+\frac{\delta\varrho_{2,k}}{2^*}\left|\hat{u}_{2,k}\right|^{2^*}\right)\mathrm{d}\sigma\\&=\varrho_{2,k}^{2-n}\int_{\partial\,\mathrm{B}_\xi(0,\delta)}\left(\frac{n-2}{2}\check{u}_{2,k}\partial_\nu\check{u}_{2,k}+\delta\left(\partial_\nu\check{u}_{2,k}\right)^2-\frac{\delta}{2}\left|\nabla\check{u}_{2,k}\right|_\xi^2\right.\\&\left.+\frac{\delta\varrho_{2,k}^{-2}}{2^*}\left|\check{u}_{2,k}\right|^{2^*}\right)\mathrm{d}\sigma\,.\end{aligned}\tag{3.88}$$

By recalling (3.12), (3.25), (3.55) and (3.57) and since $\delta_0 < \varepsilon_0/\pi$, $\varphi_k \to \varphi_0$ in $C^m\left(\mathbb{S}^n\times\mathbb{S}^n\right)$ for all $m\in\mathbb{N}$, $x_{2,k}\to x_2$, $\mu_{2,k}\le\mu_{1,k}\to0$ and $\varrho_{2,k}\to\infty$ as $k\to\infty$, we obtain

$$\mu_{2,k}^{\frac{n-2}{2}}\varrho_{2,k}^{n-2}\big(\varphi_k\left(x_{2,k},\cdot\right)^{-1}B_{2,k}\big)\big(\exp_{2,k}\left(\mu_{2,k}\varrho_{2,k}y\right)\big)$$

$$=\left(\frac{\left(1+\mathrm{o}\left(1\right)\right)\sqrt{n\left(n-2\right)}\left(\mu_{2,k}\varrho_{2,k}\right)^2}{\mu_{2,k}^2+\left(\delta\mu_{2,k}\varrho_{2,k}\right)^2}\right)^{\frac{n-2}{2}}$$

$$=\left(\frac{\sqrt{n\left(n-2\right)}}{\delta^2}\right)^{\frac{n-2}{2}}+\mathrm{o}\left(1\right)\quad\text{as }k\to\infty \tag{3.89}$$

and

$$\mu_{2,k}^{\frac{n-2}{2}}\varrho_{2,k}^{n-2}\big(\varphi_k\left(x_{2,k},\cdot\right)^{-1}B_{1,k}\big)\big(\exp_{2,k}\left(\mu_{2,k}\varrho_{2,k}y\right)\big)$$

$$=\left(\frac{\left(2+\mathrm{o}\left(1\right)\right)\sqrt{n\left(n-2\right)}\mu_{1,k}\mu_{2,k}\varrho_{2,k}^2}{2\mu_{1,k}^2+\left(4-\mu_{1,k}^2\right)\left(1-\cos\left(\mathrm{d}_{g_0}\left(x_{1,k},x_{2,k}\right)+\mathrm{O}\left(\delta\mu_{2,k}\varrho_{2,k}\right)\right)\right)}\right)^{\frac{n-2}{2}}$$

$$=\ell_0+\mathrm{O}\left(\delta\right)+\mathrm{o}\left(1\right)\quad\text{as }k\to\infty \tag{3.90}$$

uniformly with respect to $y\in\partial\,\mathrm{B}_\xi\left(0,\delta\right)$ and $\delta\in\left(0,\delta_0\right)$, where

$$\ell_0:=\begin{cases}\left(\dfrac{\sqrt{n\left(n-2\right)}\,\mathrm{d}_{g_0}\left(x_1,x_2\right)^2}{2\left(1-\cos\left(\mathrm{d}_{g_0}\left(x_1,x_2\right)\right)\right)}\right)^{\frac{n-2}{2}} & \text{if }\ \mathrm{d}_{g_0}\left(x_1,x_2\right)>0\\ \left(n\left(n-2\right)\right)^{\frac{n-2}{4}} & \text{if }\ \mathrm{d}_{g_0}\left(x_1,x_2\right)=0.\end{cases}$$

It follows from (3.12), (3.89) and (3.90) that

$$\check{u}_{2,k}\left(y\right)=\ell_0-\left(\frac{\sqrt{n\left(n-2\right)}}{\delta^2}\right)^{\frac{n-2}{2}}+\mathrm{O}\left(\delta\right)+\mathrm{o}\left(1\right)\quad\text{as }k\to\infty \tag{3.91}$$

uniformly with respect to $y\in\partial\,\mathrm{B}_\xi\left(0,\delta\right)$ and $\delta\in\left(0,\delta_0\right)$. On the other hand, by using (3.54) together with standard elliptic theory, we obtain

$$\check{u}_{2,k}\to\check{u}_{2,0}\quad\text{in }C^1_{\mathrm{loc}}\big(\overline{\mathrm{B}_\xi\left(0,\delta\right)}\backslash\left\{0\right\}\big)\quad\text{as }k\to\infty,$$

where, by (3.91),

$$\check{u}_{2,0}\left(y\right):=\ell_0-\left(\frac{\sqrt{n\left(n-2\right)}}{\left|y\right|^2}\right)^{\frac{n-2}{2}}\qquad\forall y\in\overline{\mathrm{B}_\xi\left(0,\delta\right)}\backslash\left\{0\right\}. \tag{3.92}$$

By using (3.91) and (3.92), we obtain

$$\limsup_{k\to\infty}\bigg|\int_{\partial\,\mathrm{B}_\xi\left(0,\delta\right)}\bigg(\frac{n-2}{2}\check{u}_{2,k}\partial_\nu\check{u}_{2,k}+\delta\left(\partial_\nu\check{u}_{2,k}\right)^2-\frac{\delta}{2}\left|\nabla\check{u}_{2,k}\right|_\xi^2$$

$$+\frac{\delta\varrho_{2,k}^{-2}}{2^*}\left|\check{u}_{2,k}\right|^{2^*}\bigg)\mathrm{d}\sigma-\frac{1}{2}n^{\frac{n-2}{4}}\left(n-2\right)^{\frac{n+6}{4}}\omega_{n-1}\ell_0\bigg|=\mathrm{O}\left(\delta\right) \tag{3.93}$$

where $\omega_{n-1}$ is the volume of the round $(n-1)$-sphere. Finally, (3.87) follows from (3.88) and (3.93). □

Now considering the interior term of (3.86), we obtain the following:

**Lemma 3.8.** *Let* $(\varrho_{2,k})_k$*,* $(\hat{u}_{2,k})_k$ *and* $(\hat{g}_{2,k})_k$ *be as in* (3.49)*,* (3.51) *and* (3.58)*, respectively, and* $k_0$ *and* $\delta_0$ *be as in Lemma 3.5. Then*

$$\int_{B_\xi(0,\delta\varrho_{2,k})}\left(\langle\nabla\hat{u}_{2,k},\cdot\rangle_\xi+\frac{n-2}{2}\hat{u}_{2,k}\right)\left(\left(\Delta_{\hat{g}_{2,k}}-\Delta_\xi\right)\hat{u}_{2,k}+c_n\mu_{2,k}^2\operatorname{Scal}_{\hat{g}_{2,k}}\hat{u}_{2,k}\right)dy$$
$$=O\left(\sum_{|\alpha|=2}^{d_n}\left|h_{2,k,\alpha}\right|^2\mu_{2,k}^{2|\alpha|}\left|\ln\mu_{2,k}\right|^{\vartheta(2|\alpha|,n-2)}+\delta\varrho_{2,k}^{2-n}\right) \tag{3.94}$$

*uniformly with respect to* $\delta\in(0,\delta_0)$ *and* $k>k_0$*, where*

$$\vartheta(s,t):=\begin{cases}1 & \text{if } s=t\\ 0 & \text{if } s\neq t\end{cases}\qquad\forall s,t\in\mathbb{R}.$$

Remark that by definition of $\vartheta$, the term involving $|\ln\mu_{2,k}|$ only appears when $n$ is even and $|\alpha|=\frac{n-2}{2}$.

*Proof of Lemma 3.8.* By using (3.27), we obtain

$$\left(\langle\nabla\hat{u}_{2,k}(y),y\rangle_\xi+\frac{n-2}{2}\hat{u}_{2,k}(y)\right)\left(\left(\Delta_{\hat{g}_{2,k}}-\Delta_\xi\right)\hat{u}_{2,k}+c_n\mu_{2,k}^2\operatorname{Scal}_{\hat{g}_{2,k}}\hat{u}_{2,k}\right)(y)$$
$$=\left(\langle\nabla U_0(y),y\rangle_\xi+\frac{n-2}{2}U_0(y)+O\left(\left|\left(\hat{u}_{2,k}+U_0\right)(y)\right|+|y|\left|\nabla\left(\hat{u}_{2,k}+U_0\right)(y)\right|\right)\right)$$
$$\times\left(\hat{w}_{2,k}U_0(y)+O\left(\left|\left(c_n\mu_{2,k}^2\operatorname{Scal}_{\hat{g}_{2,k}}-\hat{w}_{2,k}\right)(y)\right|\left|\hat{u}_{2,k}(y)\right|\right.\right.$$
$$+\left|\hat{w}_{2,k}(y)\right|\left|\left(\hat{u}_{2,k}+U_0\right)(y)\right|+\mu_{2,k}^N|y|^{N-1}\left|\nabla U_0(y)\right|$$
$$\left.\left.+\left|\nabla\hat{g}_{2,k}(y)\right|\left|\nabla\left(\hat{u}_{2,k}+U_0\right)(y)\right|+\left|\left(\hat{g}_{2,k}-\xi\right)(y)\right|\left|\nabla^2\left(\hat{u}_{2,k}+U_0\right)(y)\right|\right)\right). \tag{3.95}$$

uniformly with respect to $y\in B_\xi(0,\delta_0\varrho_{2,k})$ and $k>k_0$. By using (3.32) and (3.85), we obtain

$$\left|\nabla\hat{g}_{2,k}(y)\right|\left|\nabla\left(\hat{u}_{2,k}+U_0\right)(y)\right|+\left|\left(\hat{g}_{2,k}-\xi\right)(y)\right|\left|\nabla^2\left(\hat{u}_{2,k}+U_0\right)(y)\right|$$
$$=O\left(\sum_{|\alpha|=2}^{n-4}\frac{\left|h_{2,k,\alpha}\right|^2\mu_{2,k}^{2|\alpha|}}{(1+|y|)^{n-2|\alpha|}}+\frac{\mu_{2,k}^2\left(\mu_{2,k}|y|\right)^{\max(n-3,2)}}{(1+|y|)^{n-2}}+\mu_{2,k}^2\varrho_{2,k}^{2-n}\right) \tag{3.96}$$

uniformly with respect to $y\in B_\xi(0,\delta_0\varrho_{2,k})$ and $k>k_0$. Moreover, (3.85) gives

$$\left|\left(\hat{u}_{2,k}+U_0\right)(y)\right|+|y|\left|\nabla\left(\hat{u}_{2,k}+U_0\right)(y)\right|=O\left(U_0(y)\right) \tag{3.97}$$

uniformly with respect to $y\in B_\xi(0,\delta_0\varrho_{2,k})$ and $k>k_0$. By using (3.35) together with the fact that $d_n\le n-4$ when $n\ge 6$, straightforward estimates give

$$\left|\left(c_n\mu_{2,k}^2\operatorname{Scal}_{\hat{g}_{2,k}}-\hat{w}_{2,k}\right)(y)\right|\left|\hat{u}_{2,k}(y)\right|$$
$$=O\left(\sum_{|\alpha|=2}^{n-4}\frac{\left|h_{2,k,\alpha}\right|^2\mu_{2,k}^{2|\alpha|}}{(1+|y|)^{n-2|\alpha|}}+\frac{\mu_{2,k}^2\left(\mu_{2,k}|y|\right)^{\max(n-3,2)}}{(1+|y|)^{n-2}}\right) \tag{3.98}$$

uniformly with respect to $y\in B_\xi(0,\delta_0\varrho_{2,k})$ and $k>k_0$. Similarly straightforward estimates using (3.38) and (3.67) give

$$\left|\hat{w}_{2,k}(y)\right|\left|\left(\hat{u}_{2,k}+U_0\right)(y)\right|$$
$$=O\left(\left\{\sum_{|\alpha|=2}^{n-4}\frac{\left|h_{2,k,\alpha}\right|^2\mu_{2,k}^{2|\alpha|}}{(1+|y|)^{n-2|\alpha|}}+\frac{\mu_{2,k}^{n-1}}{1+|y|}\quad\text{if } n\ge 6\right\}+\mu_{2,k}^2\varrho_{2,k}^{2-n}\right), \tag{3.99}$$

uniformly with respect to $y \in \mathrm{B}_\xi\left(0, \delta_0 \varrho_{2,k}\right)$ and $k > k_0$. By plugging (3.96), (3.97), (3.98) and (3.99) into (3.95), we obtain

$$\begin{aligned}&\left(\left\langle \nabla \hat{u}_{2,k}\left(y\right), y\right\rangle_\xi + \frac{n-2}{2}\hat{u}_{2,k}\left(y\right)\right)\left(\left(\Delta_{\hat{g}_{2,k}} - \Delta_\xi\right)\hat{u}_{2,k} + c_n \mu_{2,k}^2 \operatorname{Scal}_{\hat{g}_{2,k}} \hat{u}_{2,k}\right)\left(y\right)\\ &\quad = \frac{n^{\frac{n-2}{2}}\left(n-2\right)^{\frac{n}{2}}\left(1-\left|y\right|^2\right)}{2\left(1+\left|y\right|^2\right)^{n-1}}\hat{w}_{2,k}\left(y\right) + \mathrm{O}\left(\sum_{\left|\alpha\right|=2}^{n-4}\frac{\left|h_{2,k,\alpha}\right|^2\mu_{2,k}^{2\left|\alpha\right|}}{\left(1+\left|y\right|\right)^{2n-2\left|\alpha\right|-2}}\right.\\ &\qquad \left. + \frac{\mu_{2,k}^2\left(\mu_{2,k}\left|y\right|\right)^{\max\left(n-3,2\right)}}{\left(1+\left|y\right|\right)^{2n-4}} + \frac{\mu_{2,k}^2\varrho_{2,k}^{2-n}}{\left(1+\left|y\right|\right)^{n-2}} + \frac{\left(\mu_{2,k}\left|y\right|\right)^N}{\left(1+\left|y\right|\right)^{2n-2}}\right) \qquad (3.100)\end{aligned}$$

uniformly with respect to $y \in \mathrm{B}_\xi\left(0, \delta \varrho_{2,k}\right)$, $\delta \in \left(0, \delta_0\right)$ and $k > k_0$. We now integrate (3.95) in $\mathrm{B}_\xi\left(0, \delta \varrho_{2,k}\right)$. By using (3.38), (3.46) and (3.67), we obtain

$$\int_{\mathrm{B}_\xi\left(0,\delta\varrho_{2,k}\right)} \frac{n^{\frac{n-2}{2}}\left(n-2\right)^{\frac{n}{2}}\left(1-\left|y\right|^2\right)}{2\left(1+\left|y\right|^2\right)^{n-1}}\hat{w}_{2,k}\left(y\right)\mathrm{d}y = 0. \qquad (3.101)$$

On the other hand, when $2 \le \left|\alpha\right| \le n-4$, straightforward estimates give

$$\int_{\mathrm{B}_\xi\left(0,\delta\varrho_{2,k}\right)} \frac{\mu_{2,k}^{2\left|\alpha\right|}\mathrm{d}y}{\left(1+\left|y\right|\right)^{2n-2\left|\alpha\right|-2}} = \mathrm{O}\left(\left\{\begin{array}{ll}\mu_{2,k}^{2\left|\alpha\right|} & \text{if } \left|\alpha\right| < d_n\\ \mu_{2,k}^{n-2}\left|\ln\mu_{2,k}\right| & \text{if } \left|\alpha\right| = d_n\\ \delta^{n-2}\varrho_{i,k}^{2-n} & \text{if } \left|\alpha\right| > d_n\end{array}\right\}\right) \qquad (3.102)$$

uniformly with respect to $\delta \in \left(0, \delta_0\right)$ and $k > k_0$. It follows from (3.100), (3.101) and (3.102) that

$$\begin{aligned}&\int_{\mathrm{B}_\xi\left(0,\delta\varrho_{2,k}\right)}\left(\left\langle \nabla \hat{u}_{2,k}, \cdot\right\rangle_\xi + \frac{n-2}{2}\hat{u}_{2,k}\right)\left(\left(\Delta_{\hat{g}_{2,k}} - \Delta_\xi\right)\hat{u}_{2,k} + c_n\mu_{2,k}^2\operatorname{Scal}_{\hat{g}_{2,k}}\hat{u}_{2,k}\right)\mathrm{d}y\\ &\quad = \mathrm{O}\left(\sum_{\left|\alpha\right|=2}^{d_n}\left|h_{2,k,\alpha}\right|^2\mu_{2,k}^{2\left|\alpha\right|}\left|\ln\mu_{2,k}\right|^{\vartheta\left(2\left|\alpha\right|,n-2\right)} + \delta\varrho_{2,k}^{2-n}\left(\delta^{1-n}\left(\delta\mu_{2,k}\varrho_{2,k}\right)^{\max\left(n-1,4\right)}\right.\right.\\ &\qquad \left.\left. + \delta\left(\mu_{2,k}\varrho_{2,k}\right)^2 + \delta^{N-n+1}\left(\mu_{2,k}\varrho_{2,k}\right)^N\right)\right) \qquad (3.103)\end{aligned}$$

uniformly with respect to $\delta \in \left(0, \delta_0\right)$ and $k > k_0$. Finally, (3.94) follows from (3.57) and (3.103). □

We point out that in the proofs of Lemmas 3.6, 3.7 and 3.8 we only used (3.54) with $i = 2$, namely for the most concentrated bubble. We recall that while this estimate holds true without any additional assumptions in the case where $i = 2$, we need to show that (3.53) holds true in order to use it in the case where $i = 1$. This is done in the next section.

## 4. Proofs of Theorems 1.1 and 2.1

In this section, we apply the analysis of Section 3 to prove Theorems 1.1 and 2.1. By using Lemmas 3.6, 3.7 and 3.8, we can first complete the proof of Theorem 2.1:

*End of proof of Theorem 2.1.* When $n \in \{3,4,5\}$, Lemma 3.8 gives

$$\int_{\mathrm{B}_\xi(0,\delta\varrho_{2,k})} \left(\langle \nabla \hat{u}_{2,k}, \cdot\rangle_\xi + \frac{n-2}{2}\hat{u}_{2,k}\right)\left(\left(\Delta_{\hat{g}_{2,k}} - \Delta_\xi\right)\hat{u}_{2,k} + c_n \mu_{2,k}^2 \operatorname{Scal}_{\hat{g}_{2,k}} \hat{u}_{2,k}\right) \mathrm{dy} \\ = \mathrm{O}\left(\delta \varrho_{2,k}^{2-n}\right)$$

uniformly with respect to $\delta \in (0,\delta_0)$ and $k > k_0$, which, together with Lemma 3.6, yields an obvious contradiction with Lemma 3.7 as $\delta \to 0$. □

In larger dimensions $n \in \{6,\dots,10\}$, the Pohozaev identity of Lemma 3.6 alone is not enough to conclude and we need to perform a more refined analysis. As a first result, we obtain a priori estimates on $(\varrho_{1,k})_k$ and $(\varrho_{2,k})_k$ as well as a sharp asymptotic expansion of $\Lambda_1(\mathbb{S}^n,[g_k])$ as $k \to \infty$. For each $k \in \mathbb{N}$, we define

$$\zeta_k(x,\mu) := \sum_{|\alpha|=2}^{d_n} |h_{k,x,\alpha}|^2 \mu^{2|\alpha|} |\ln \mu|^{\vartheta(2|\alpha|,n-2)} \quad \forall x \in \mathbb{S}^n, \mu > 0,$$

where $\vartheta$ is as in Lemma 3.8. By using the asymptotic analysis performed in Lemmas 3.1 to 3.8, we obtain the following:

**Lemma 4.1.** *Let $(\mu_{1,k})_k$ and $(\mu_{2,k})_k$ be as in Lemma 3.1, $d_n$ be as in Lemma 3.3, $(\varrho_{2,k})_k$ and $(h_{2,k,\alpha})_{k,\alpha}$ be as in (3.49) and (3.50), respectively, and $k_0$ be as in Lemma 3.5. If $6 \le n \le 10$, then*

$$\varrho_{1,k}^{2-n} + \varrho_{2,k}^{2-n} = \mathrm{O}\left(\max_{\mathbb{S}^n}\left(\zeta_k(\cdot,\mu_{1,k})\right)\right) \tag{4.1}$$

*and*

$$\Lambda_1(\mathbb{S}^n,[g_k]) = \Lambda_1(\mathbb{S}^n,[g_0]) + \mathrm{O}\left(\max_{\mathbb{S}^n}\left(\zeta_k(\cdot,\mu_{1,k})\right)\right) \tag{4.2}$$

*uniformly with respect to $k > k_0$.*

*Proof of Lemma 4.1.* We begin with proving that (4.1) holds true. It follows from Lemmas 3.6, 3.7 and 3.8 that

$$\varrho_{2,k}^{2-n} = \mathrm{O}\left(\zeta_k(x_{2,k},\mu_{2,k})\right) = \mathrm{o}\left(\begin{Bmatrix} \mu_{2,k}^4 |\ln \mu_{2,k}| & \text{if } n=6 \\ \mu_{2,k}^4 & \text{if } 7 \le n \le 10 \end{Bmatrix}\right) \text{ as } k \to \infty. \tag{4.3}$$

We assume by contradiction that, up to a subsequence,

$$d_k = \mathrm{O}(\mu_{1,k}) \tag{4.4}$$

uniformly with respect to $k > k_0$. It follows from (3.49) and (4.4) that

$$\frac{\mu_{2,k}}{\mu_{1,k}} = \mathrm{O}\left(\varrho_{2,k}^{-2}\right),$$

which, together with (4.3), gives

$$\frac{\mu_{2,k}}{\mu_{1,k}} = \mathrm{o}\left(\begin{Bmatrix} \mu_{2,k}^2 |\ln \mu_{2,k}|^{\frac{1}{2}} & \text{if } n=6 \\ \mu_{2,k}^{\frac{8}{n-2}} & \text{if } 7 \le n \le 10 \end{Bmatrix}\right) = \mathrm{o}(\mu_{2,k}) \quad \text{as } k \to \infty. \tag{4.5}$$

Clearly, (4.5) contradicts the fact that $\mu_{1,k} \to 0$ as $k \to \infty$. This proves that (4.1) holds true, and thus (3.10) and (3.11) hold true. Moreover, by using (4.1) and (4.3) together with (3.49) and the facts that $\mu_{2,k} \le \mu_{1,k}$, the function $\mu \mapsto \mu^{\frac{n-2}{2}} |\ln \mu|$ is decreasing and $\frac{n-2}{2} \le 4$ when $n \le 10$, we obtain

$$\begin{aligned} \varrho_{1,k}^{2-n} &= d_k^{2-n} \mu_{1,k}^{n-2} \\ &\sim \varrho_{2,k}^{2-n} \mu_{1,k}^{\frac{n-2}{2}} \mu_{2,k}^{-\frac{n-2}{2}} \\ &= \mathrm{O}\left( \mu_{1,k}^{\frac{n-2}{2}} \sum_{|\alpha|=2}^{d_n} |h_{2,k,\alpha}|^2 \mu_{2,k}^{2|\alpha| - \frac{n-2}{2}} |\ln \mu_{2,k}|^{\vartheta(2|\alpha|, n-2)} \right) \\ &= \mathrm{O}\left( \zeta_k \left( x_{2,k}, \mu_{1,k} \right) \right) \end{aligned} \tag{4.6}$$

uniformly with respect to $k > k_0$. Finally, (4.1) follows from (4.3) and (4.6).

We now prove (4.2) by using (1.1) and (3.5) and estimating the energy of $u_k$. We claim that

$$\int_{\mathbb{S}^n} |u_k|^{2^*} \mathrm{dv}_{g_k} = 2\Lambda_1 \left( \mathbb{S}^n, [g_0] \right)^{\frac{n}{2}} + \mathrm{O}\left( \max_{\mathbb{S}^n} \left( \zeta_k \left( \cdot, \mu_{1,k} \right) \right) \right) \tag{4.7}$$

uniformly with respect to $k > k_0$, which, together with (1.1) and (3.5), implies (4.2). We prove this claim. For each $i \in \{1, 2\}$ and $\delta \in (0, \delta_0)$, we let $\widetilde{g}_{i,k} := g_{k, x_{i,k}}$ be given by Lemma 3.2. By using (3.3) together with a rescaling argument and the conformal covariance of the conformal Laplacian, we obtain

$$\begin{aligned} &\int_{\mathrm{B}_{\widetilde{g}_{i,k}}(x_{i,k}, \delta\mu_{i,k}\varrho_{i,k})} |u_k|^{2^*} \mathrm{dv}_{\widetilde{g}_{i,k}} \\ &= \frac{n}{2} \int_{\mathrm{B}_{g_{k,x_{i,k}}}(x_{i,k}, \delta\mu_{i,k}\varrho_{i,k})} \left( L_{g_k} u_k - \frac{n-2}{n} |u_k|^{2^*-2} u_k \right) u_k \, \mathrm{dv}_{\widetilde{g}_{i,k}} \\ &= \frac{n}{2} \int_{\mathrm{B}_\xi(0, \delta\varrho_{i,k})} \left( \Delta_{\hat{g}_{i,k}} \hat{u}_{i,k} + c_n \mu_{i,k}^2 \operatorname{Scal}_{\hat{g}_{i,k}} \hat{u}_{i,k} - \frac{n-2}{n} |\hat{u}_{i,k}|^{2^*-2} \hat{u}_{i,k} \right) \hat{u}_{i,k} \, \mathrm{dv}_{\hat{g}_{i,k}} . \end{aligned} \tag{4.8}$$

By integrating by parts, we obtain

$$\begin{aligned} &\int_{\mathrm{B}_\xi(0, \delta\varrho_{i,k})} \hat{u}_{i,k} \Delta_{\hat{g}_{i,k}} \hat{u}_{i,k} \, \mathrm{dv}_{\hat{g}_{i,k}} \\ &\quad = \int_{\mathrm{B}_\xi(0, \delta\varrho_{i,k})} \Big( U_0 \Delta_{\hat{g}_{i,k}} U_0 + 2 \big( \hat{u}_{i,k} + (-1)^i U_0 \big) \Delta_{\hat{g}_{i,k}} \hat{u}_{i,k} \\ &\qquad - \left| \nabla \big( \hat{u}_{i,k} + (-1)^i U_0 \big) \right|_{\hat{g}_{i,k}}^2 \Big) \mathrm{dv}_{\hat{g}_{i,k}} + \int_{\partial \mathrm{B}_\xi(0, \delta\varrho_{i,k})} \Big( (-1)^i U_0 \partial_\nu \big( \hat{u}_{i,k} + (-1)^i U_0 \big) \\ &\qquad + \big( \hat{u}_{i,k} + (-1)^i U_0 \big) \partial_\nu \hat{u}_{i,k} \Big) \mathrm{d}\sigma_{\hat{g}_{i,k}} . \end{aligned} \tag{4.9}$$

We recall that, by (4.6), Lemma 3.5 now applies to both $i = 1$ and $i = 2$. By using (3.27), (3.54), (3.102), and (4.9), we obtain

$$\begin{aligned} &\int_{\mathrm{B}_\xi(0, \delta\varrho_{i,k})} \hat{u}_{i,k} \Delta_{\hat{g}_{i,k}} \hat{u}_{i,k} \, \mathrm{dv}_{\hat{g}_{i,k}} \\ &\quad = \int_{\mathrm{B}_\xi(0, \delta\varrho_{i,k})} U_0 \Delta_\xi U_0 \, \mathrm{dy} + 2 \int_{\mathrm{B}_\xi(0, \delta\varrho_{i,k})} \big( \hat{u}_{i,k} + (-1)^i U_0 \big) \Delta_{\hat{g}_{i,k}} \hat{u}_{i,k} \big) \mathrm{dv}_{\hat{g}_{i,k}} \end{aligned}$$

$$+\,\mathrm{O}\left(\int_{\mathrm{B}_\xi(0,\delta\varrho_{i,k})}\left(\left\{\sum_{|\alpha|=2}^{n-4}\frac{|h_{i,k,\alpha}|^2\mu_{i,k}^{2|\alpha|}}{(1+|y|)^{2n-2|\alpha|-2}}+\frac{\mu_{i,k}^{2n-6}}{(1+|y|)^4}\quad\text{if } n\geq 6\right\}\right.\right.$$
$$\left.\left.+\frac{\varrho_{i,k}^{4-2n}}{(1+|y|)^2}+\frac{(\mu_{i,k}|y|)^N}{(1+|y|)^{2n-2}}\right)\mathrm{d}y\right)$$
$$+\,\mathrm{O}\left(\int_{\partial\,\mathrm{B}_\xi(0,\delta\varrho_{i,k})}\left(\sum_{|\alpha|=2}^{n-4}\frac{|h_{i,k,\alpha}|\,\mu_{i,k}^{|\alpha|}}{(1+|y|)^{2n-|\alpha|-3}}+\frac{\varrho_{i,k}^{2-n}}{(1+|y|)^{n-1}}\right)\mathrm{d}\sigma\right)$$
$$=\int_{\mathbb{R}^n}U_0\Delta_\xi U_0\,\mathrm{d}y+2\int_{\mathrm{B}_\xi(0,\delta\varrho_{i,k})}\big(\hat{u}_{i,k}+(-1)^iU_0\big)\Delta_{\hat{g}_{i,k}}\hat{u}_{i,k}\,\mathrm{dv}_{\hat{g}_{i,k}}$$
$$+\,\mathrm{O}\Big(\zeta_k(x_{i,k},\mu_{i,k})+\varrho_{i,k}^{2-n}\Big)\tag{4.10}$$

uniformly with respect to $k>k_0$. We now estimate the last two terms in the right-hand side of (4.8). By using (3.27), (3.35), (3.85) and (3.102), we obtain

$$c_n\mu_{i,k}^2\int_{\mathrm{B}_\xi(0,\delta\varrho_{i,k})}\mathrm{Scal}_{\hat{g}_{i,k}}\,\hat{u}_{i,k}^2\,\mathrm{dv}_{\hat{g}_{i,k}}$$
$$=c_n\mu_{i,k}^2\int_{\mathrm{B}_\xi(0,\delta\varrho_{i,k})}\mathrm{Scal}_{\hat{g}_{i,k}}\Big(U_0^2+2\hat{u}_{i,k}\big(\hat{u}_{i,k}+(-1)^iU_0\big)$$
$$-\big(\hat{u}_{i,k}+(-1)^iU_0\big)^2\Big)\,\mathrm{dv}_{\hat{g}_{i,k}}$$
$$=\int_{\mathrm{B}_\xi(0,\delta\varrho_{i,k})}\Big(\hat{w}_{i,k}U_0^2+2c_n\mu_{i,k}^2\,\mathrm{Scal}_{\hat{g}_{i,k}}\,\hat{u}_{i,k}\big(\hat{u}_{i,k}+(-1)^iU_0\big)$$
$$+\,\mathrm{O}\Big(\big|c_n\mu_{i,k}^2\,\mathrm{Scal}_{\hat{g}_{i,k}}-\hat{w}_{i,k}\big|\,U_0^2+\mu_{i,k}^2\,\big|\mathrm{Scal}_{\hat{g}_{i,k}}\big|\,\big|\hat{u}_{i,k}+(-1)^iU_0\big|^2\Big)\Big)\,\mathrm{dv}_{\hat{g}_{i,k}}$$
$$=2c_n\mu_{i,k}^2\int_{\mathrm{B}_\xi(0,\delta\varrho_{i,k})}\mathrm{Scal}_{\hat{g}_{i,k}}\,\hat{u}_{i,k}\big(\hat{u}_{i,k}+(-1)^iU_0\big)\,\mathrm{dv}_{\hat{g}_{i,k}}$$
$$+\,\mathrm{O}\left(\int_{\mathrm{B}_\xi(0,\delta\varrho_{i,k})}\left(\sum_{|\alpha|=2}^{n-4}\frac{|h_{i,k,\alpha}|^2\mu_{i,k}^{2|\alpha|}}{(1+|y|)^{2n-2|\alpha|-2}}+\frac{\mu_{i,k}^{n-1}}{(1+|y|)^{n-1}}+\mu_{i,k}^4\varrho_{i,k}^{4-2n}|y|^2\right.\right.$$
$$\left.\left.+\frac{\mu_{i,k}^{N+4}|y|^{N+2}}{(1+|y|)^{2n-4}}\right)\mathrm{d}y\right)$$
$$=2c_n\mu_{i,k}^2\int_{\mathrm{B}_\xi(0,\delta\varrho_{i,k})}\mathrm{Scal}_{\hat{g}_{i,k}}\,\hat{u}_{i,k}\big(\hat{u}_{i,k}+(-1)^iU_0\big)\,\mathrm{dv}_{\hat{g}_{i,k}}$$
$$+\,\mathrm{O}\big(\zeta_k(x_{i,k},\mu_{i,k})+\varrho_{i,k}^{-n}\big).\tag{4.11}$$

uniformly with respect to $k>k_0$. By using again (3.85) and (3.102), we obtain

$$\int_{\mathrm{B}_\xi(0,\delta\varrho_{i,k})}|\hat{u}_{i,k}|^{2^*}\,\mathrm{dv}_{\hat{g}_{i,k}}$$
$$=\int_{\mathrm{B}_\xi(0,\delta\varrho_{i,k})}\Big(U_0^{2^*}+2^*|\hat{u}_{i,k}|^{2^*-2}\,\hat{u}_{i,k}\big(\hat{u}_{i,k}+(-1)^iU_0\big)$$
$$+\,\mathrm{O}\Big(U_0^{2^*-2}\big|\hat{u}_{i,k}+(-1)^iU_0\big|^2\Big)\Big)\,\mathrm{dv}_{\hat{g}_{i,k}}$$

$$
\begin{aligned}
&= \int_{\mathrm{B}_\xi(0,\delta\varrho_{i,k})} U_0^{2^*}\,\mathrm{dy} + 2^* \int_{\mathrm{B}_\xi(0,\delta\varrho_{i,k})} \left|\hat{u}_{i,k}\right|^{2^*-2} \hat{u}_{i,k}\big(\hat{u}_{i,k} + (-1)^i\, U_0\big)\,\mathrm{dv}_{\hat{g}_{i,k}} \\
&\quad + \mathrm{O}\left(\int_{\mathrm{B}_\xi(0,\delta\varrho_{i,k})} \left(\left\{\sum_{|\alpha|=2}^{n-4} \frac{\left|h_{i,k,\alpha}\right|^2 \mu_{i,k}^{2|\alpha|}}{(1+|y|)^{2n-2|\alpha|}} + \frac{\mu_{i,k}^{2n-6}}{(1+|y|)^6} \quad \text{if } n\geq 6\right\}\right.\right. \\
&\quad \left.\left. + \frac{\varrho_{i,k}^{4-2n}}{(1+|y|)^4} + \frac{(\mu_{i,k}\,|y|)^N}{(1+|y|)^{2n}}\right)\mathrm{dy}\right) \\
&= \int_{\mathbb{R}^n} U_0^{2^*}\,\mathrm{dy} + 2^* \int_{\mathrm{B}_\xi(0,\delta\varrho_{i,k})} \left|\hat{u}_{i,k}\right|^{2^*-2} \hat{u}_{i,k}\big(\hat{u}_{i,k} + (-1)^i\, U_0\big)\,\mathrm{dv}_{\hat{g}_{i,k}} \\
&\quad + \mathrm{O}\left(\zeta_k\left(x_{i,k},\mu_{i,k}\right) + \varrho_{i,k}^{2-n}\right) \qquad (4.12)
\end{aligned}
$$

uniformly with respect to $k > k_0$. By putting together (4.8), (4.9), (4.10), (4.11) and (4.12) and using the equations (3.65) and $\Delta_\xi U_0 = U_0^{2^*-1}$, we obtain

$$
\int_{\mathrm{B}_{\widetilde{g}_{i,k}}(x_{i,k},\delta\mu_{i,k}\varrho_{i,k})} \left|u_k\right|^{2^*} \mathrm{dv}_{\widetilde{g}_{i,k}} = \int_{\mathbb{R}^n} U_0^{2^*}\,\mathrm{dy} + \mathrm{O}\left(\zeta_k\left(x_{i,k},\mu_{i,k}\right) + \varrho_{i,k}^{2-n}\right) \qquad (4.13)
$$

uniformly with respect to $k > k_0$. We recall that

$$
\int_{\mathbb{R}^n} U_0^{2^*}\,\mathrm{dy} = \Lambda_1\left(\mathbb{S}^n,[g_0]\right)^{\frac{n}{2}}. \qquad (4.14)
$$

Moreover, by using (3.53) together with the definition of $\varrho_{1,k}$ and the fact that $\mu_{2,k} \leq \mu_{1,k}$, we obtain that there exists $\delta_1 \in (0,\delta_0)$ and $k_1 > k_0$ such that, for each $\delta \in (0,\delta_1)$ and $k > k_1$,

$$
\mathrm{B}_{\widetilde{g}_{1,k}}\left(x_{1,k},\delta\mu_{1,k}\varrho_{1,k}\right) \cap \mathrm{B}_{\widetilde{g}_{2,k}}\left(x_{2,k},\delta\mu_{2,k}\varrho_{2,k}\right) = \emptyset. \qquad (4.15)
$$

On the other hand, by using similar estimates as in the beginning of the proof of Lemma 3.5, we obtain

$$
\begin{aligned}
&\int_{\mathbb{S}^n\backslash \bigcup_{i=1}^{2} \mathrm{B}_{\widetilde{g}_{i,k}}\left(x_{i,k},\delta\mu_{i,k}\varrho_{i,k}\right)} \left|u_k\right|^{2^*} \mathrm{dv}_{\widetilde{g}_{i,k}} \\
&\quad = \mathrm{O}\left(\sum_{i=1}^{2} \int_{\mathbb{S}^n\backslash\, \mathrm{B}_{\widetilde{g}_{i,k}}\left(x_{i,k},\delta\mu_{i,k}\varrho_{i,k}\right)} \left(B_{3-i,k}\right)^{2^*} \mathrm{dv}_{\widetilde{g}_{i,k}}\right) \\
&\quad = \mathrm{O}\left(\varrho_{1,k}^{-n} + \varrho_{2,k}^{-n}\right) \qquad (4.16)
\end{aligned}
$$

uniformly with respect to $k > k_1$. By using (4.1), (4.13), (4.14), (4.15) and (4.16) together with the fact that $\mu_{2,k} \leq \mu_{1,k}$, we obtain (4.7), which completes the proof of Lemma 4.1. □

We are now in position to conclude the proof of Theorem 1.1. The idea to reach a contradiction consists in directly estimating $\Lambda_1\left(\mathbb{S}^n,[g_k]\right)$ with the help of suitable test-functions. These test-functions are modeled on the first-order expansion of the functions $\left(u_k\right)_k$ as in (3.54), centered at maximum points of the functions $\left(\zeta_k\right)_k$ in $M$ and more concentrated than the functions $\left(B_{1,k}\right)_k$. We prove that these test-functions provide better competitors for $\Lambda_1\left(\mathbb{S}^n,[g_k]\right)$ and yield a contradiction with (4.2). As we mentioned in Section 2, our contradiction argument comes from the very definition of $\Lambda_2\left(\mathbb{S}^n,[g_k]\right)$ and its minimality. We cannot use a local sign restriction argument as in [26] here since all local masses vanish due to the fact that $g_k \to g_0$ in $C^m(\mathbb{S}^n)$ for all $m \in \mathbb{N}$.

*End of proof of Theorem 1.1.* The form of the leading term of the test-functions we use is inspired from the historic work of Schoen [48]. We add a lower-order term inspired from the work of Li and Zhang [28, 29] and Khuri, Marques and Schoen [26] (see also the early work of Hebey and Vaugon [23] using this idea of adding a small correction term in the test-functions). For each $k > k_0$, we let $x_k \in \mathbb{S}^n$ and $\mu_k \in (0,\infty)$ be such that

$$\zeta_k(x_k, \mu_{1,k}) = \max_{\mathbb{S}^n}(\zeta_k(\cdot, \mu_{1,k})) \quad \text{and} \quad \mu_k = \lambda\mu_{1,k}, \tag{4.17}$$

where $\lambda \in (1,\infty)$ is some fixed number to be chosen large later on. We let $\widetilde{g}_k := g_{k,x_k}$ and $\exp_k := \exp_{k,x_k}$ be as in Lemma 3.2, $H_k := H_{k,x_k}$ and $h_{k,\alpha} := h_{k,x_k,\alpha}$ be as in Lemma 3.3, $w_{k,d} := w_{k,x_k,d}$ be as in (3.38) and $v_{k,d} := v_{k,x_k,d}$ be as in Lemma 3.4. Up to a subsequence, we may further assume that $x_k \to x_0 \in \mathbb{S}^n$ as $k \to \infty$. We then define $\widetilde{g}_0 := g_{0,x_0}$. We let $\widetilde{G}_0$ be the Green's function of $L_{\widetilde{g}_0}$ in $\mathbb{S}^n$. In particular, for each $x \in \mathbb{S}^n$, $\widetilde{G}_0(x,\cdot)$ is a positive function in $\mathbb{S}^n \backslash \{x\}$ satisfying the equation

$$L_{\widetilde{g}_0}\widetilde{G}_0(x,\cdot) = 0 \quad \text{in } \mathbb{S}^n \backslash \{x\}. \tag{4.18}$$

We let $\delta \in (0, \varepsilon_0/2)$, where $\varepsilon_0$ is as in Lemma 3.2. We let $\eta$ be a smooth function on $[0,\infty)$ such that $\eta = 1$ on $[0,1]$ and $\eta = 0$ on $[2,\infty)$. We define the functions

$$\begin{aligned}
B_k &:= \left(\frac{\sqrt{n(n-2)}\mu_k}{\mu_k^2 + \mathrm{d}_{\widetilde{g}_k}(x_k,\cdot)^2}\right)^{\frac{n-2}{2}},\\
\hat{w}_k &:= c_n \sum_{d=4}^{n-4} \mu_k^d w_{k,d},\\
\hat{v}_k &:= c_n \sum_{|\alpha|=4}^{n-4} \mu_k^{|\alpha|} v_{k,d},\\
v_k &:= \mu_k^{-\frac{n-2}{2}} \hat{v}_k \circ \left(\mu_k^{-1}\exp_k^{-1}\right),\\
\check{w}_k &:= c_n \mu_k^2 \sum_{a,b,c=1}^{n} \Bigg(\partial_{y_b}\left((H_k)_{ab}\,\partial_{y_c}(H_k)_{ac}\right)\\
&\qquad - \frac{1}{2}\partial_{y_b}(H_k)_{ab}\,\partial_{y_c}(H_k)_{ac} + \frac{1}{4}\left(\partial_{y_c}(H_k)_{ab}\right)^2\Bigg)(\mu_k\cdot),\\
\Gamma_k &:= n^{\frac{n-2}{4}}(n-2)^{\frac{n+2}{4}}\omega_{n-1}\mu_k^{\frac{n-2}{2}}\widetilde{G}_0(x_k,\cdot)
\end{aligned}$$

and

$$z_k := \eta\left(\delta^{-1}\mathrm{d}_{\widetilde{g}_k}(x_k,\cdot)\right)(B_k - v_k) + \left(1 - \eta\left(\delta^{-1}\mathrm{d}_{\widetilde{g}_k}(x_k,\cdot)\right)\right)\Gamma_k,$$

where $\omega_{n-1}$ is the volume of the round $(n-1)$-sphere. We also define the metric

$$\hat{g}_k := \exp_k^* \widetilde{g}_k(\mu_k\cdot).$$

We recall that, by Lemma 3.4, $\hat{v}_k$ satisfies

$$\Delta_\xi \hat{v}_k = (2^*-1)\,U_0^{2^*-2}\hat{v}_k + U_0\hat{w}_k \quad \text{in } \mathbb{R}^n. \tag{4.19}$$

We claim that there exists $C_n > 0$ depending only on $n$ such that

$$\Lambda_1\left(\mathbb{S}^n,[g_0]\right)-\Lambda_1\left(\mathbb{S}^n,[g_k]\right)\geq C_n\zeta_k\left(x_k,\mu_k\right)+\mathrm{o}\left(\mu_k^{n-2}\right)\quad\text{as } k\to\infty. \tag{4.20}$$

Before proving (4.20), we first show that this estimate yields a contradiction with (4.2) when $\lambda$ is chosen large enough, thus completing the proof of Theorem 1.1. Since $\mu_{1,k}=\mathrm{O}\left(\varrho_{1,k}^{-1}\right)$ by (3.56), it follows from (4.1) and (4.17) that

$$\mu_k^{n-2}=\mathrm{O}\left(\zeta_k\left(x_k,\mu_{1,k}\right)\right) \tag{4.21}$$

uniformly with respect to $k>k_0$, where the constant in $\mathrm{O}\left(\cdot\right)$ depends on $\lambda$, but this is not a problem since this term is multiplied by $\mathrm{o}\left(1\right)$ in (4.20). Moreover, since $\lambda>1$, by definition of $\zeta_k$, it is easy to see that

$$\zeta_k\left(x_k,\mu_k\right)\geq\lambda^2\zeta_k\left(x_k,\mu_{1,k}\right). \tag{4.22}$$

It follows from (4.20), (4.21) and (4.22) that

$$\Lambda_1\left(\mathbb{S}^n,[g_0]\right)-\Lambda_1\left(\mathbb{S}^n,[g_k]\right)\geq\left(\lambda^2C_n+\mathrm{o}\left(1\right)\right)\zeta_k\left(x_k,\mu_{1,k}\right)\quad\text{as } k\to\infty,$$

which contradicts (4.2) when $\lambda$ is chosen large enough.

We now prove (4.20). Since $\widetilde{g}_k\in[g_k]$ and $z_k\in C^\infty\left(\mathbb{S}^n\right)$, we obtain

$$\Lambda_1\left(\mathbb{S}^n,[g_k]\right)\leq\frac{\displaystyle\int_{\mathbb{S}^n}\left(\left|\nabla z_k\right|_{\widetilde{g}_k}^2+c_n\operatorname{Scal}_{\widetilde{g}_k}z_k^2\right)\mathrm{dv}_{\widetilde{g}_k}}{\left(\displaystyle\int_{\mathbb{S}^n}\left|z_k\right|^{2^*}\mathrm{dv}_{\widetilde{g}_k}\right)^{\frac{n-2}{n}}}. \tag{4.23}$$

By definition of $z_k$, it is easy to see that

$$\int_{\mathbb{S}^n}\left|z_k\right|^{2^*}\mathrm{dv}_{\widetilde{g}_k}=\int_{\mathrm{B}_{\widetilde{g}_k}(x_k,\delta)}\left|B_k-v_k\right|^{2^*}\mathrm{dv}_{\widetilde{g}_k}+\mathrm{o}\left(\mu_k^{n-2}\right)\quad\text{as } k\to\infty. \tag{4.24}$$

By using (3.27), (3.43) and (4.24) together with similar estimates as in (4.8)–(4.16), we obtain

$$\begin{aligned}\int_{\mathbb{S}^n}\left|z_k\right|^{2^*}\mathrm{dv}_{\widetilde{g}_k}&=\int_{\mathrm{B}_\xi(0,\delta/\mu_k)}\left|U_0-\hat{v}_k\right|^{2^*}\mathrm{dv}_{\hat{g}_k}+\mathrm{o}\left(\mu_k^{n-2}\right)\\&=\int_{\mathrm{B}_\xi(0,\delta/\mu_k)}\bigg(U_0^{2^*}-2^*U_0^{2^*-1}\hat{v}_k+\frac{2^*\left(2^*-1\right)}{2}U_0^{2^*-2}\hat{v}_k^2\\&\quad+\mathrm{O}\left(U_0^{2^*-3}\left|\hat{v}_k\right|^3\right)\bigg)\mathrm{dv}_{\hat{g}_k}+\mathrm{o}\left(\mu_k^{n-2}\right)\\&=\int_{\mathbb{R}^n}\left(U_0^{2^*}-2^*U_0^{2^*-1}\hat{v}_k+\frac{2^*\left(2^*-1\right)}{2}U_0^{2^*-2}\hat{v}_k^2\right)\mathrm{dy}\\&\quad+\mathrm{O}\left(\int_{\mathrm{B}_\xi(0,\delta/\mu_k)}\sum_{|\alpha|=4}^{n-4}\frac{\left|h_{k,\alpha}\right|^3\mu_k^{3|\alpha|}}{\left(1+|y|\right)^{2n-3|\alpha|}}\mathrm{dy}\right)+\mathrm{o}\left(\mu_k^{n-2}\right)\\&=\Lambda_1\left(\mathbb{S}^n,[g_0]\right)^{\frac{n}{2}}-\frac{2^*}{2}\int_{\mathbb{R}^n}\left(2U_0^{2^*-1}\hat{v}_k-\left(2^*-1\right)U_0^{2^*-2}\hat{v}_k^2\right)\mathrm{dy}\\&\quad+\mathrm{o}\left(\zeta_k\left(x_k,\mu_k\right)+\mu_k^{n-2}\right)\quad\text{as } k\to\infty.\end{aligned} \tag{4.25}$$

We now estimate the numerator in (4.23). By using (4.18) together with the definition of $z_k$ and the facts that $x_k\to x_0$ in $\mathbb{S}^n$ and $\widetilde{g}_k\to\widetilde{g}_0$ in $C^m\left(\mathbb{S}^n\right)$ as $k\to\infty$

for all $m \in \mathbb{N}$, we obtain

$$\begin{aligned}
&\int_{\mathbb{S}^n \setminus \mathrm{B}_{\widetilde{g}_k}(x_k,\delta)} z_k L_{\widetilde{g}_k} z_k \,\mathrm{dv}_{\widetilde{g}_k} \\
&\quad = \int_{\mathrm{B}_{\widetilde{g}_k}(x_k,2\delta)\setminus \mathrm{B}_{\widetilde{g}_k}(x_k,\delta)} z_k L_{\widetilde{g}_k}\left(z_k - \Gamma_k\right) \mathrm{dv}_{\widetilde{g}_k} \\
&\qquad + n^{\frac{n-2}{4}} \left(n-2\right)^{\frac{n+2}{4}} \omega_{n-1} \mu_k^{\frac{n-2}{2}} \left( \int_{\mathbb{S}^n \setminus \mathrm{B}_{\widetilde{g}_0}(x_0,2\delta)} z_k L_{\widetilde{g}_0} \widetilde{G}_0\left(x_0,\cdot\right) \mathrm{dv}_{\widetilde{g}_k} + \mathrm{o}\left(1\right) \right) \\
&\quad = \int_{\mathrm{B}_{\widetilde{g}_k}(x_k,2\delta)\setminus \mathrm{B}_{\widetilde{g}_k}(x_k,\delta)} z_k L_{\widetilde{g}_k}\left(z_k - \Gamma_k\right) \mathrm{dv}_{\widetilde{g}_k} + \mathrm{o}\left(\mu_k^{n-2}\right) \quad \text{as } k \to \infty. \qquad (4.26)
\end{aligned}$$

On the other hand, since $\widetilde{g}_0$ is flat in $\mathrm{B}_{\widetilde{g}_0}\left(x_0, \varepsilon_0\right)$ by (3.28), we obtain (see for instance [27]) that there exists a function $R \in C^2\left(\overline{\mathrm{B}_{\widetilde{g}_0}\left(x_0,\varepsilon_0\right)}\right)$ such that $R\left(x_0\right) = 0$ and

$$\widetilde{G}_0\left(x_0, y\right) = \frac{1}{\left(n-2\right)\omega_{n-1}\,\mathrm{d}_{\widetilde{g}_0}\left(x_0,y\right)^{n-2}} + R\left(y\right) \quad \forall y \in \mathrm{B}_{\widetilde{g}_0}\left(x_0,\varepsilon_0\right). \qquad (4.27)$$

Moreover, an easy consequence of (3.43) is that

$$\sum_{j=0}^{2} \delta^j \left|\nabla^j v_k\left(y\right)\right| = \mathrm{o}\left(\mu_k^{\frac{n-2}{2}}\right) \quad \text{as } k \to \infty \qquad (4.28)$$

uniformly with respect to $y \in \mathrm{B}_{\widetilde{g}_k}\left(x_k, 2\delta\right) \setminus \mathrm{B}_{\widetilde{g}_k}\left(x_k,\delta\right)$. By using (4.27) and (4.28) together with the definition of $z_k$ and the facts that $R\left(x_0\right) = 0$, $x_k \to x_0$ in $\mathbb{S}^n$ and $\widetilde{g}_k \to \widetilde{g}_0$ in $C^m\left(\mathbb{S}^n\right)$ as $k \to \infty$ for all $m \in \mathbb{N}$, we obtain

$$\sum_{j=0}^{2} \delta^j \left|\nabla^j \left(z_k - \Gamma_k\right)\left(y\right)\right| = \mathrm{O}\left(\delta \mu_k^{\frac{n-2}{2}}\right) + \mathrm{o}\left(\mu_k^{\frac{n-2}{2}}\right) \quad \text{as } k \to \infty \qquad (4.29)$$

uniformly with respect to $y \in \mathrm{B}_{\widetilde{g}_k}\left(x_k, 2\delta\right) \setminus \mathrm{B}_{\widetilde{g}_k}\left(x_k,\delta\right)$, where the term $\mathrm{O}\left(\delta\mu_k^{n-2}\right)$ is also uniform with respect to $\delta \in \left(0, \varepsilon/2\right)$ (the same holds true in the next estimates). By using (4.18) and (4.29), we then obtain

$$\int_{\mathrm{B}_{\widetilde{g}_k}(x_k,2\delta)\setminus \mathrm{B}_{\widetilde{g}_k}(x_k,\delta)} z_k L_{\widetilde{g}_k}\left(z_k - \Gamma_k\right) \mathrm{dv}_{\widetilde{g}_k} = \mathrm{O}\left(\delta \mu_k^{n-2}\right) + \mathrm{o}\left(\mu_k^{n-2}\right) \quad \text{as } k \to \infty. \qquad (4.30)$$

It follows from (4.30) together with an integration by parts and the definition of $z_k$ that

$$\begin{aligned}
&\int_{\mathbb{S}^n} \left(\left|\nabla z_k\right|_{\widetilde{g}_k}^2 + c_n \operatorname{Scal}_{\widetilde{g}_k} z_k^2\right) \mathrm{dv}_{\widetilde{g}_k} \\
&\quad = \int_{\mathrm{B}_{\widetilde{g}_k}(x_k,\delta)} \left(B_k L_{\widetilde{g}_k} B_k - 2 v_k L_{\widetilde{g}_k} B_k + v_k L_{\widetilde{g}_k} v_k\right) \mathrm{dv}_{\widetilde{g}_k} + \mathrm{O}\left(\delta \mu_k^{n-2}\right) \\
&\qquad + \mathrm{o}\left(\mu_k^{n-2}\right) \quad \text{as } k \to \infty. \qquad (4.31)
\end{aligned}$$

By using (3.27), (3.34), (3.35), (3.43) and (4.19) together with similar estimates as in (4.9)–(4.11), we obtain

$$\begin{aligned}
&\int_{\mathrm{B}_{\widetilde{g}_k}(x_k,\delta)} \left(B_k L_{\widetilde{g}_k} B_k - 2v_k L_{\widetilde{g}_k} B_k + v_k L_{\widetilde{g}_k} v_k\right) \mathrm{dv}_{\widetilde{g}_k}\\
&= \int_{\mathrm{B}_\xi(0,\delta/\mu_k)} \Big( U_0 \left(\Delta_{\hat{g}_k} + c_n \mu_k^2 \operatorname{Scal}_{\hat{g}_k}\right) U_0 - 2\hat{v}_k \left(\Delta_{\hat{g}_k} + c_n \mu_k^2 \operatorname{Scal}_{\hat{g}_k}\right) U_0\\
&\quad + \hat{v}_k \left(\Delta_{\hat{g}_k} + c_n \mu_k^2 \operatorname{Scal}_{\hat{g}_k}\right) \hat{v}_k \Big) \mathrm{dv}_{\hat{g}_k}\\
&= \int_{\mathrm{B}_\xi(0,\delta/\mu_k)} \Big( U_0 \left(\Delta_\xi + \hat{w}_k - \check{w}_k\right) U_0 - 2\hat{v}_k \left(\Delta_\xi + \hat{w}_k\right) U_0 + \hat{v}_k \Delta_\xi \hat{v}_k\\
&\quad + \mathrm{O}\Big( \left|c_n \mu_k^2 \operatorname{Scal}_{\hat{g}_k} - \hat{w}_k + \check{w}_k\right| U_0^2 + \left|c_n \mu_k^2 \operatorname{Scal}_{\hat{g}_k} - \hat{w}_k\right| U_0 \left|\hat{v}_k\right| + \mu_k^2 \left|\operatorname{Scal}_{\hat{g}_k}\right| \hat{v}_k^2\\
&\quad + \left(U_0 + \left|\hat{v}_k\right|\right) \left|\left(\Delta_{\hat{g}_k} - \Delta_\xi\right) U_0\right| + \left|\hat{v}_k\right| \left(\left|\nabla \hat{g}_k\right| \left|\nabla \hat{v}_k\right| + \left|\hat{g}_k - \xi\right| \left|\nabla^2 \hat{v}_k\right|\right) \Big) \Big) \mathrm{dv}_{\hat{g}_k}\\
&= \Lambda_1\left(\mathbb{S}^n, [g_0]\right)^{\frac{n}{2}} - \int_{\mathbb{R}^n} \left(2U_0^{2^*-1} \hat{v}_k - \left(2^*-1\right) U_0^{2^*-2} \hat{v}_k^2\right) \mathrm{dy}\\
&\quad - \int_{\mathrm{B}_\xi(0,\delta/\mu_k)} U_0 \left(U_0 \check{w}_k + \hat{w}_k \hat{v}_k\right) \mathrm{dy} + \mathrm{O}\Bigg( \sum_{|\alpha|=2}^{d_n-1} \left|h_{k,\alpha}\right|^2 \mu_k^{2|\alpha|+2} \left|\ln \mu_k\right|^{\vartheta(2|\alpha|,n-4)}\\
&\quad + \sum_{|\alpha|=2}^{[(n-2)/3]} \left|h_{k,\alpha}\right|^3 \mu_k^{3|\alpha|} \left|\ln \mu_k\right|^{\vartheta(3|\alpha|,n-2)} \Bigg) + \mathrm{o}\left(\mu_k^{n-2}\right)\\
&= \Lambda_1\left(\mathbb{S}^n, [g_0]\right)^{\frac{n}{2}} - \int_{\mathbb{R}^n} \left(2U_0^{2^*-1} \hat{v}_k - \left(2^*-1\right) U_0^{2^*-2} \hat{v}_k^2\right) \mathrm{dy}\\
&\quad - \int_{\mathrm{B}_\xi(0,\delta/\mu_k)} U_0 \left(U_0 \check{w}_k + \hat{w}_k \hat{v}_k\right) \mathrm{dy} + \mathrm{o}\left(\zeta_k\left(x_k, \mu_k\right) + \mu_k^{n-2}\right) \text{ as } k \to \infty. \qquad (4.32)
\end{aligned}$$

It follows from (4.23), (4.25), (4.31) and (4.32) that

$$\begin{aligned}
\Lambda_1\left(\mathbb{S}^n, [g_0]\right) - \Lambda_1\left(\mathbb{S}^n, [g_k]\right) &= \int_{\mathrm{B}_\xi(0,\delta/\mu_k)} U_0 \left(U_0 \check{w}_k + \hat{w}_k \hat{v}_k\right) \mathrm{dy} + \mathrm{O}\left(\delta \mu_k^{n-2}\right)\\
&\quad + \mathrm{o}\left(\zeta_k\left(x_k, \mu_k\right) + \mu_k^{n-2}\right) \quad \text{as } k \to \infty. \qquad (4.33)
\end{aligned}$$

We now estimate the integral in the right-hand side of (4.33), starting with the first term in this integral. We recall that

$$H_k = \sum_{p=2}^{n-4} H_{k,p},$$

where $H_{k,p} = H_{k,x_k,p}$ and $H_{k,x,p}$ is as in (3.36). By using polar coordinates and the definition of $\check{w}_k$, we obtain

$$\begin{aligned}
&\int_{\mathrm{B}_\xi(0,\delta/\mu_k)} U_0^2 \check{w}_k \, \mathrm{dy}\\
&\quad = c_n \sum_{p,q=2}^{n-4} \mu_k^{p+q} \sum_{a,b,c=1}^{n} \int_{\mathbb{S}^n} \left( \frac{1}{4} \partial_{y_c} \left(H_{k,p}(y)\right)_{ab} \partial_{y_c} \left(H_{k,q}(y)\right)_{ab}\right.
\end{aligned}$$

$$- \frac{1}{2} \partial_{y_b} \left(H_{k,p}(y)\right)_{ab} \partial_{y_c} \left(H_{k,q}(y)\right)_{ac} \Big) \mathrm{d}\sigma(y)$$
$$\times \int_0^{\frac{\delta}{\mu_k}} \left( \frac{\sqrt{n(n-2)}}{1+r^2} \right)^{n-2} r^{p+q+n-3} \, \mathrm{dr} + \mathrm{o}\left(\mu_k^{n-2}\right) \quad \text{as } k \to \infty. \tag{4.34}$$

To obtain (4.34), we also use that, for each $p, q \in \{2, \dots, n-4\}$,

$$\int_{\mathrm{B}_\xi(0,\delta/\mu_k)} \sum_{a,b,c=1}^{n} U_0^2 \partial_{y_b} \left( \left(H_{k,p}(y)\right)_{ab} \partial_{y_c} \left(H_{k,q}(y)\right)_{ac} \right) \mathrm{dy} = 0,$$

which follows from an integration by parts together with (3.37) and the fact that $U_0$ is radially symmetric around 0. We observe that, since $H_k \to 0$ in $C^m(\mathrm{B}_\xi(0, \varepsilon_0))$ as $k \to \infty$ for all $m \in \mathbb{N}$, for each $p, q \in \{2, \dots, n-4\}$ such that $q > d_n$,

$$\begin{aligned} |H_{k,p}| \, |H_{k,q}| \, \mu_k^{p+q} &= \mathrm{O}\left( |H_{k,p}|^2 \mu_k^{2p+2q+2-n} + |H_{k,q}|^2 \mu_k^{n-2} \right) \\ &= \mathrm{o}\left( |H_{k,p}|^2 \mu_k^{2p} + \mu_k^{n-2} \right) \quad \text{as } k \to \infty, \end{aligned} \tag{4.35}$$

where

$$|H_{k,p}| := \sum_{|\alpha|=p} |h_{k,\alpha}| \, .$$

Integrations by parts give that, for each $d \in \{4, \dots, n-3\}$,

$$\begin{aligned} &\int_0^{\frac{\varepsilon_0}{\mu_k}} \left( \frac{\sqrt{n(n-2)}}{1+r^2} \right)^{n-2} r^{n+d-3} \, \mathrm{dr} \\ &= \frac{n-2}{d} \int_0^{\infty} \left( \frac{\sqrt{n(n-2)}}{1+r^2} \right)^{n-2} \frac{r^2-1}{1+r^2} r^{n+d-3} \, \mathrm{dr} + \mathrm{O}\left(\mu_k^{n-d-2}\right) \end{aligned} \tag{4.36}$$

and straightforward estimates give that, for each $d \geq n-2$,

$$\begin{aligned} &\int_0^{\frac{\delta}{\mu_k}} \left( \frac{\sqrt{n(n-2)}}{1+r^2} \right)^{n-2} r^{n+d-3} \, \mathrm{dr} \\ &= \begin{cases} (n(n-2))^{\frac{n-2}{2}} |\ln \mu_k| + \mathrm{O}(1) & \text{if } d = n-2 \\ \mathrm{O}\left(\mu_k^{n-d-2}\right) & \text{if } d > n-2. \end{cases} \end{aligned} \tag{4.37}$$

Combining (4.34), (4.35), (4.36) and (4.37) we obtain

$$\begin{aligned} \int_{\mathrm{B}_\xi(0,\delta/\mu_k)} U_0^2 \check{w}_k \, \mathrm{dy} &= 2 c_n \left(n(n-2)\right)^{\frac{n-2}{2}} \int_0^{\mu_k} \mu^{-1} F_{k,1}(\mu) \, \mathrm{d}\mu \\ &\quad + \mathrm{o}\left( \zeta_k(x_k, \mu_k) + \mu_k^{n-2} \right) \quad \text{as } k \to \infty, \end{aligned} \tag{4.38}$$

where

$$F_{k,1}(\mu) := \frac{n-2}{2} \sum_{p,q=2}^{d_n} \mu^{p+q} \Bigg( \sum_{a,b,c=1}^{n} \int_{\mathbb{S}^n} \Bigg( \frac{1}{4} \partial_{y_c} (H_{k,p}(y))_{ab} \partial_{y_c} (H_{k,q}(y))_{ab} - \frac{1}{2} \partial_{y_b} (H_{k,p}(y))_{ab} \partial_{y_c} (H_{k,q}(y))_{ac} \Bigg) \mathrm{d}\sigma(y) \times \begin{Bmatrix} c_{p+q} & \text{if } p+q < n-2 \\ |\ln \mu| & \text{if } p = q = d_n \end{Bmatrix} \Bigg)$$

and

$$c_{p+q} := \int_0^\infty \left( \frac{1}{1+r^2} \right)^{n-2} \frac{r^2-1}{1+r^2} r^{n+p+q-3} \, \mathrm{d}r \, .$$

As regards the second term in the integral in the right-hand side of (4.33), by using again polar coordinates, we obtain

$$\int_{\mathrm{B}_\xi(0,\delta/\mu_k)} U_0 \hat{w}_k \hat{v}_k \, \mathrm{d}y = c_n^2 \sum_{p,q=2}^{n-4} \mu_k^{p+q} \int_{\mathrm{B}_\xi(0,\delta/\mu_k)} U_0 w_{k,p} v_{k,q} \, \mathrm{d}y + \mathrm{o}\left(\mu_k^{n-2}\right) \quad \text{as } k \to \infty. \tag{4.39}$$

By using (3.43) and since $H_k \to 0$ in $C^m(\mathrm{B}_\xi(0,\varepsilon_0))$ as $k \to \infty$ for all $m \in \mathbb{N}$, we obtain that, for each $p, q \in \{2, \dots, n-4\}$ such that $d := p+q \in \{4, \dots, n-3\}$,

$$\int_{\mathrm{B}_\xi(0,\delta/\mu_k)} U_0 w_{k,p} v_{k,q} \, \mathrm{d}y = \int_{\mathbb{R}^n} U_0 w_{k,p} v_{k,q} \, \mathrm{d}y + \mathrm{o}\left(\mu_k^{n-d-2}\right) \quad \text{as } k \to \infty. \tag{4.40}$$

We now consider the case where $p, q \in \{2, \dots, n-4\}$ are such that $d := p+q \geq n-2$. We recall that by Lemma 3.4,

$$v_{k,q}(y) = \frac{P_{k,q}(y)}{\left(1+|y|^2\right)^{\frac{n}{2}}} \quad \forall y \in \mathbb{R}^n,$$

where $P_{k,q}$ is the polynomial of degree $q+2$ given by

$$P_{k,q}(y) := \sum_{m=0}^{[(d-2)/2]} \sum_{l=0}^{m+2} \gamma_{k,x_k,d,l,m} |y|^{2l} \Delta_\xi^m w_{k,x_k,d}(y) \quad \forall y \in \mathbb{R}^n.$$

We let $P_{k,q}^{(q+2)}$ be the sum of terms of highest degree in $P_{k,q}$, i.e.

$$P_{k,q}^{(q+2)}(y) = \sum_{m=0}^{[(q-2)/2]} \gamma_{k,x_k,q,m+2,m} |y|^{2m+4} \Delta_\xi^m w_{k,q}(y) \quad \forall y \in \mathbb{R}^n.$$

Since $H_k \to 0$ in $C^m(\mathrm{B}_\xi(0,\varepsilon_0))$ as $k \to \infty$ for all $m \in \mathbb{N}$, straightforward estimates using polar coordinates then give

$$\int_{\mathrm{B}_\xi(0,\delta/\mu_k)} U_0 w_{k,p} v_{k,q} \, \mathrm{d}y = \begin{cases} (n(n-2))^{\frac{n-2}{2}} |\ln \mu_k| \int_{\mathbb{S}^{n-1}} w_{k,p}(y) P_{k,q}^{(q+2)}(y) \, \mathrm{d}\sigma(y) + \mathrm{o}(1) & \text{if } d = n-2 \\ \mathrm{o}\left(\mu_k^{n-d-2}\right) & \text{if } d > n-2. \end{cases} \tag{4.41}$$

By putting together (4.35), (4.39), (4.40) and (4.41), we obtain

$$\int_{\mathrm{B}_\xi(0,\delta/\mu_k)} U_0 \hat{w}_k \hat{v}_k \,\mathrm{dy} = 2c_n \left(n\left(n-2\right)\right)^{\frac{n-2}{2}} \int_0^{\mu_k} \mu^{-1} F_{k,2}\left(\mu\right) \mathrm{d}\mu + \mathrm{o}\left(\zeta_k\left(x_k,\mu_k\right) + \mu_k^{n-2}\right) \quad \text{as } k \to \infty, \tag{4.42}$$

where

$$F_{k,2}\left(\mu\right) := c_n \sum_{p,q=2}^{d_n} \mu^{p+q} \times \begin{cases} p \int_{\mathbb{R}^n} \left(1+|y|^2\right)^{2-n} w_{k,p}\left(y\right) v_{k,q}\left(y\right) \mathrm{dy} & \text{if } p+q<n-2 \\ d_n \left|\ln \mu_k\right| \int_{\mathbb{S}^{n-1}} w_{k,d_n}\left(y\right) P_{k,d_n}^{(d_n+2)}\left(y\right) \mathrm{d}\sigma\left(y\right) & \text{if } p=q=d_n. \end{cases}$$

The functions $F_{1,k}$ and $F_{2,k}$ have been investigated in [26]. In particular, since $n \leq 10$, Proposition A.4 of [26] applies[1] and gives that there exists $C'_n > 0$ depending only on $n$ such that

$$F_{1,k}\left(\mu\right) + F_{2,k}\left(\mu\right) \geq C'_n \zeta_k\left(x_k,\mu\right) \quad \forall \mu > 0. \tag{4.43}$$

It follows from (4.38), (4.42) and (4.43) that

$$\int_{\mathrm{B}_\xi(0,\varepsilon_0/\mu_k)} U_0\left(U_0 \check{w}_k + \hat{w}_k \hat{v}_k\right) \mathrm{dy} \geq C_n \zeta_k\left(x_k,\mu_k\right) + \mathrm{o}\left(\mu_k^{n-2}\right) \quad \text{as } k \to \infty \tag{4.44}$$

for some $C_n > 0$ depending only on $n$. By plugging (4.44) into (4.33), and passing to a subsequence in $\delta$, we obtain (4.20), which completes the proof of Theorem 1.1. ☐

## References


[1] B. Ammann and E. Humbert, *The second Yamabe invariant*, J. Funct. Anal. **235** (2006), no. 2, 377–412.

[2] B. Ammann and P. Jammes, *The supremum of first eigenvalues of conformally covariant operators in a conformal class*, Variational problems in differential geometry, London Math. Soc. Lecture Note Ser., vol. 394, Cambridge Univ. Press, Cambridge, 2012, pp. 1–23.

[3] Th. Aubin, *Équations différentielles non linéaires et problème de Yamabe concernant la courbure scalaire*, J. Math. Pures Appl. (9) **55** (1976), no. 3, 269–296 (French).

[4] S. Brendle, *Blow-up phenomena for the Yamabe equation*, J. Amer. Math. Soc. **21** (2008), no. 4, 951–979.

[5] S. Brendle and F. C. Marques, *Blow-up phenomena for the Yamabe equation. II*, J. Differential Geom. **81** (2009), no. 2, 225–250.

[6] ———, *Recent progress on the Yamabe problem*, Surveys in geometric analysis and relativity, Adv. Lect. Math., vol. 20, Int. Press, Somerville, MA, 2011, pp. 29–47.

[7] C.-C. Chen and C.-S. Lin, *Estimate of the conformal scalar curvature equation via the method of moving planes. II*, J. Differential Geom. **49** (1998), no. 1, 115–178.

[8] M. Clapp, *Entire nodal solutions to the pure critical exponent problem arising from concentration*, J. Differential Equations **261** (2016), no. 6, 3042–3060.

[9] M. Clapp and J. C. Fernández, *Multiplicity of nodal solutions to the Yamabe problem*, Calc. Var. Partial Differential Equations **56** (2017), no. 5, 145, 22 p.

[10] M. Clapp, A. Pistoia, and H. Tavares, *Yamabe systems, optimal partitions, and nodal solutions to the Yamabe equation*, J. Eur. Math. Soc. (to appear).


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[1] Our functions $F_{1,k}\left(\mu\right)$ and $F_{2,k}$ coincide with the functions $I_{1,\mu}^{(n)}\left(H_k,H_k\right)$ and $I_{2,\mu}^{(n)}\left(H_k,H_k\right)$ introduced in [26, Appendix A]. Our definition of $F_{2,k}$ seems to introduce a factor $-c_n$ with respect to $I_{2,\mu}^{(n)}\left(H_k,H_k\right)$ in [26], but this is a choice of convention: the function $Z\big(H_k^{(p)}\big)$ involved in the definition of $I_{2,\mu}^{(n)}\left(H_k,H_k\right)$ in [26] is equal to $-c_n w_{k,p}$ where $w_{k,p}$ is as in Lemma 3.4.

[11] M. Clapp, A. Pistoia, and T. Weth, *An upper bound for the least energy of a nodal solution to the Yamabe equation on the sphere*, Minimax Theory Appl. **7** (2022), no. 2.

[12] M. del Pino, M. Musso, F. Pacard, and A. Pistoia, *Large energy entire solutions for the Yamabe equation*, J. Differential Equations **251** (2011), no. 9, 2568–2597.

[13] ———, *Torus action on $\mathbb{S}^n$ and sign-changing solutions for conformally invariant equations*, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) **12** (2013), no. 1, 209–237.

[14] W. Y. Ding, *On a conformally invariant elliptic equation on $\mathbb{R}^n$*, Comm. Math. Phys. **107** (1986), no. 2, 331–335.

[15] O. Druet, *Compactness for Yamabe metrics in low dimensions*, Int. Math. Res. Not. **23** (2004), 1143–1191.

[16] O. Druet, E. Hebey, and F. Robert, *Blow-up theory for elliptic PDEs in Riemannian geometry*, Mathematical Notes, vol. 45, Princeton University Press, Princeton, NJ, 2004.

[17] S. El Sayed, *Second eigenvalue of the Yamabe operator and applications*, Calc. Var. Partial Differential Equations **50** (2014), no. 3-4, 665–692.

[18] J. C. Fernández, O. Palmas, and J. Petean, *Supercritical elliptic problems on the round sphere and nodal solutions to the Yamabe problem in projective spaces*, Discrete Contin. Dyn. Syst. **40** (2020), no. 4, 2495–2514.

[19] J. C. Fernández and J. Petean, *Low energy nodal solutions to the Yamabe equation*, J. Differential Equations **268** (2020), no. 11, 6576–6597.

[20] M. J. Gursky and S. Pérez-Ayala, *Variational properties of the second eigenvalue of the conformal Laplacian*, J. Funct. Anal. **282** (2022), no. 8, 109371.

[21] E. Hebey, *Compactness and stability for nonlinear elliptic equations*, Zurich Lectures in Advanced Mathematics, European Mathematical Society, Zürich, 2014.

[22] R. Hardt and L. Simon, *Nodal sets for solutions of elliptic equations*, J. Differential Geom. **30** (1989), no. 2, 505–522.

[23] E. Hebey and M. Vaugon, *Le problème de Yamabe équivariant*, Bull. Sci. Math. **117** (1993), no. 2, 241–286.

[24] M. Karpukhin and D. Stern, *Existence of harmonic maps and eigenvalue optimization in higher dimensions*, Invent. Math. **236** (2024), no. 2, 713–778.

[25] ———, *Min-max harmonic maps and a new characterization of conformal eigenvalues*, J. Eur. Math. Soc. (to appear).

[26] M. A. Khuri, F. C. Marques, and R. M. Schoen, *A compactness theorem for the Yamabe problem*, J. Differential Geom. **81** (2009), no. 1, 143–196.

[27] J. M. Lee and T. H. Parker, *The Yamabe problem*, Bull. Amer. Math. Soc. (N.S.) **17** (1987), no. 1, 37–91.

[28] Y. Li and L. Zhang, *Compactness of solutions to the Yamabe problem. II.*, Calc. Var. Partial Differential Equations **24** (2005), no. 2, 185–237.

[29] ———, *Compactness of solutions to the Yamabe problem. III.*, J. Funct. Anal. **245** (2007), no. 2, 438–474.

[30] Y. Li and M. Zhu, *Yamabe type equations on three-dimensional Riemannian manifolds*, Commun. Contemp. Math. **1** (1999), no. 1, 1–50.

[31] F. C. Marques, *A priori estimates for the Yamabe problem in the non-locally conformally flat case*, J. Differential Geom. **71** (2005), no. 2, 315–346.

[32] H. Matthiesen and A. Siffert, *Existence of metrics maximizing the first eigenvalue on non-orientable surfaces*, J. Spectr. Theory **11** (2021), no. 3, 1279–1296.

[33] S. Mazumdar, *Struwe's decomposition for a polyharmonic operator on a compact Riemannian manifold with or without boundary*, Commun. Pure Appl. Anal. **16** (2017), no. 1, 311–330.

[34] M. Medina and M. Musso, *Doubling nodal solutions to the Yamabe equation in $\mathbb{R}^n$ with maximal rank*, J Math Pures Appl. **152** (2021), no. 4, 145–188.

[35] M. Medina, M. Musso, and J. C. Wei, *Desingularization of Clifford torus and nonradial solutions to the Yamabe problem with maximal rank*, J. Funct. Anal. **276** (2019), no. 8, 2470–2523.

[36] M. Musso and J. C. Wei, *Nondegeneracy of nodal solutions to the critical Yamabe problem*, Commun. Math. Phys. **340** (2015), no. 3, 1049–1107.

[37] N. Nadirashvili and Y. Sire, *Conformal spectrum and harmonic maps*, Mosc. Math. J. **15** (2015), no. 1, 123–140.

[38] M. Obata, *The conjectures on conformal transformations of Riemannian manifolds*, J. Differential Geometry **6** (1971/72), 247–258.

[39] R. Petrides, *Existence and regularity of maximal metrics for the first Laplace eigenvalue on surfaces*, Geom. Funct. Anal. **24** (2014), no. 4, 1336–1376.

[40] ———, *On the existence of metrics which maximize Laplace eigenvalues on surfaces*, Int. Math. Res. Not. IMRN **14** (2018), 4261–4355.

[41] ———, *Maximizing one Laplace eigenvalue on n-dimensional manifolds* (2022). arXiv:2211.15636.

[42] B. Premoselli, *A priori estimates for finite-energy sign-changing blowing-up solutions of critical elliptic PDEs of second order* (2021). arXiv:2111.02470.

[43] B. Premoselli and F. Robert, *One-bubble nodal blow-up for asymptotically critical stationary Schrödinger-type equations* (2024). arXiv:2404.16384.

[44] B. Premoselli and J. Vétois, *Compactness of sign-changing solutions to scalar curvature-type equations with bounded negative part*, J. Differential Equations **266** (2019), no. 11, 7416–7458.

[45] ———, *Stability and instability results for sign-changing solutions to second-order critical elliptic equations*, J. Math. Pures Appl. (9) **167** (2022), 257–293.

[46] ———, *Sign-changing blow-up for the Yamabe equation at the lowest energy level*, Adv. Math. **410** (2022), 108769.

[47] F. Robert, *Existence et asymptotiques optimales des fonctions de Green des opérateurs elliptiques d'ordre deux*. http://www.iecn.u-nancy.fr/ frobert/ConstrucGreen.pdf.

[48] R. M. Schoen, *Conformal deformation of a Riemannian metric to constant scalar curvature*, J. Differential Geom. **20** (1984), no. 2, 479–495.

[49] ———, *Notes from graduate lectures in Stanford University*. http://www.math.washington.edu/pollack/research/Schoen-1988-notes.html.

[50] ———, *On the number of constant scalar curvature metrics in a conformal class*, Differential geometry, Pitman Monogr. Surveys Pure Appl. Math., vol. 52, Longman Sci. Tech., Harlow, 1991, pp. 311–320.

[51] M. Struwe, *A global compactness result for elliptic boundary value problems involving limiting nonlinearities*, Math. Z. **187** (1984), no. 4, 511–517.

[52] N. S. Trudinger, *Remarks concerning the conformal deformation of Riemannian structures on compact manifolds*, Ann. Scuola Norm. Sup. Pisa (3) **22** (1968), 265–274.

[53] J. Vétois, *Multiple solutions for nonlinear elliptic equations on compact Riemannian manifolds*, Internat. J. Math. **18** (2007), no. 9, 1071–1111.

[54] T. Weth, *Energy bounds for entire nodal solutions of autonomous superlinear equations*, Calc. Var. Partial Differential Equations **27** (2006), no. 4, 421–437.

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