UNIQUENESS OF CONFORMAL METRICS WITH CONSTANT Q-CURVATURE ON CLOSED EINSTEIN MANIFOLDS

JÉRÔME VÉTOIS

Abstract. On a smooth, closed Einstein manifold (M,g) of dimension $n\ge 3$ with positive scalar curvature and not conformally diffeomorphic to the standard sphere, we prove that the only conformal metrics to g with constant Q-curvature of order 4 are the metrics λg with $\lambda>0$ constant.

1. Introduction and main result

On a smooth, closed (i.e. compact and without boundary) Riemannian manifold of dimension $n\ge 3$, Branson's Q-curvature [3] is defined as

$$\mathrm{Q}_g := \frac{1}{2\left(n-1\right)}\Delta_g\,\mathrm{S}_g+\frac{n^3-4n^2+16n-16}{8\left(n-1\right)^2\left(n-2\right)^2}\,\mathrm{S}_g^2-\frac{2}{\left(n-2\right)^2}\left|\mathrm{Ric}_g\right|_g^2, \tag{1.1}$$

where $\Delta_g := -\operatorname{div}_g\nabla = -\operatorname{tr}_g\nabla^2$ is the Laplace–Beltrami operator, S_g is the scalar curvature and Ric_g is the Ricci curvature of the manifold.

In the case of the standard sphere, the conformal metrics with constant Q-curvature have been classified by Lin [27] by using the moving-plane method. In this case, there exists an explicit, multi-dimensional family of conformal metrics with constant Q-curvature. In this article, we examine the case of smooth, closed Einstein manifolds with positive scalar curvature and not conformally diffeomorphic to the standard sphere. In this case, we obtain the following:

Theorem 1.1. *Let (M,g) be a smooth, closed Einstein manifold of dimension $n\ge 3$ with positive scalar curvature and not conformally diffeomorphic to the standard sphere. Then the only conformal metrics to g with constant Q-curvature are the metrics λg with $\lambda>0$ constant.*

This result extends to the Q-curvature equation a result obtained by Obata [29] for the scalar curvature equation (see also the subsequent articles by Bidaut-Véron and Véron [2] and Gidas and Spruck [14] for extensions of Obata's result to more general second-order equations). The distinct σ_k-curvature case, on locally conformally flat metrics, has been addressed by Viaclovsky [33].

Date: October 13, 2022.

To appear in *Potential Analysis*.

The author was supported by the NSERC Discovery Grant RGPIN-2022-04213.

The author would like to thank very much Robin Graham, Matthew Gursky, Emmanuel Hebey and Frédéric Robert for helpful advice and suggestions as well as Matthew Gursky and Andrea Malchiodi for very interesting comments on the manuscript.

The transformation law for the Q-curvature under a conformal change of metric is given by the equations

$$\begin{cases} \mathrm{P}_g\, u + \mathrm{Q}_g = \mathrm{Q}_{e^{2u} g}\, e^{4u} \quad \text{in } M & \text{if } n = 4 \\ \mathrm{P}_g\, u = \dfrac{n-4}{2}\, \mathrm{Q}_{u^{\frac{4}{n-4}} g}\, u^{\frac{n+4}{n-4}},\ u > 0 \quad \text{in } M & \text{if } n \neq 4, \end{cases} \tag{1.2}$$

where P_g is the Paneitz–Branson operator [3, 30] defined as

$$\mathrm{P}_g := \Delta_g^2 - \operatorname{div}_g \left(\left(\frac{n^2 - 4n + 8}{2\,(n-1)\,(n-2)}\, \mathrm{S}_g\, g - \frac{4}{n-2}\, \mathrm{Ric}_g \right) \nabla u \right) + \frac{n-4}{2}\, \mathrm{Q}_g\,.$$

In particular, in the case where (M, g) is Einstein, we obtain

$$Q_g = \frac{(n+2)\,(n-2)}{8n\,(n-1)^2}\, \mathrm{S}_g^2 \quad \text{and} \quad \mathrm{P}_g = \Delta_g^2 + \frac{n^2 - 2n - 4}{2n\,(n-1)}\, \mathrm{S}_g\, \Delta_g + \frac{n-4}{2}\, \mathrm{Q}_g\,.$$

We mention in passing that the concept of Q-curvature and the corresponding operator P_g have been shown to have natural extensions to higher orders. Some references in this case are by Branson [4], Fefferman and Graham [12,13], Gover [15], Graham, Jenne, Mason and Sparling [16] and Juhl [23]. In particular, in the case of Einstein manifolds, explicit formulas can be found for all orders (see [13, Proposition 7.9] and also [15]).

The Q-curvature very rapidly became a subject in its own and has been intensively studied in the literature since the seminal and beautiful work we mentioned above. Surprisingly, despite all the studies which have been published on this topic in recent years and despite the fact it naturally extends from the scalar-curvature equation, Obata's result for the Q-curvature equation Theorem 1.1 has not been proven before. As regards the method, we use a similar approach as in Obata's proof [29] for the scalar curvature equation. Given a conformal metric $g_v := v^{-1} g$, where $v \in C^\infty(M)$, $v > 0$ in M, this approach consists in finding a suitable function $\Theta_g^{(k)}(v) \in C^\infty(M)$, with $k = 1$ for the scalar curvature equation and $k = 2$ for the Q-curvature equation, such that if S_{g_v} is constant for $k = 1$ and Q_{g_v} is constant for $k = 2$, then

$$\Theta_g^{(k)}(v) \geq 0 \text{ in } M,\ [\Theta_g^{(k)}(v) \equiv 0 \text{ in } M \iff \nabla v \equiv 0 \text{ in } M] \text{ and } \int_M \Theta_g^{(k)}(v)\, dv_g = 0.$$

The last equality is obtained by applying multiple integrations by parts. It easily follows from these properties that if such a function $\Theta_g^{(k)}(v)$ exists, then v must be constant. For $k = 1$, this is achieved by considering the function

$$\Theta_g^{(1)}(v) := v^{\frac{3-n}{2}}\, |\mathrm{E}_{g_v}|_g^2\,,$$

where E_{g_v} is the trace-free Ricci tensor of the metric g_v, i.e.

$$\mathrm{E}_{g_v} := \mathrm{Ric}_{g_v} - \frac{1}{n}\, \mathrm{S}_{g_v}\, g_v = \mathrm{Ric}_g - \frac{1}{n}\, \mathrm{S}_g\, g + (n-2)\, \sqrt{v}^{-1} \Big(\nabla^2 \sqrt{v} + \frac{1}{n} \Delta_g \sqrt{v}\, g \Big). \tag{1.3}$$

For $k = 2$, we use the function

$$\begin{aligned} \Theta_g^{(2)}(v) := v^{\frac{1-n}{2}} \Bigg(& \left| \nabla \mathrm{S}_{g_v} + \frac{3n-4}{2\,(n-2)}\, \mathrm{E}_{g_v}\, \nabla v \right|_g^2 - \frac{(3n-4)^2}{4\,(n-2)^2} \left| \mathrm{E}_{g_v}\, \nabla v \right|_g^2 \\ & + \frac{|\mathrm{E}_{g_v}|_g^2}{(n-2)^2} \big(2\,(n^2-2)\, \mathrm{S}_{g_v}\, v + 4\,(n-1)\, \mathrm{S}_g\, v^2 + n\,(n-1)^2\, |\nabla v|_g^2 \big) \Bigg). \end{aligned} \tag{1.4}$$

Under the conditions of Theorem 1.1, we can see that the function $\Theta_g^{(2)}(v)$ is nonnegative in M by observing that

$$(3n-4)^2 \left| \mathrm{E}_{g_v} \nabla v \right|_g^2 \leq (3n-4)^2 \left| \mathrm{E}_{g_v} \right|_g^2 \left| \nabla v \right|_g^2 \leq 4n(n-1)^2 \left| \mathrm{E}_{g_v} \right|_g^2 \left| \nabla v \right|_g^2 \qquad (1.5)$$

and using the positivity of the functions v, S_g and S_{g_v}, the latter following from a result obtained by Gursky and Malchiodi [18] (see Theorem 2.3 below). Finally, it is not difficult to see that $\Theta_g^{(2)}(v) \equiv 0$ in M if and only if $\nabla \mathrm{S}_{g_v} \equiv 0$ in M and either $\mathrm{E}_{g_v} \equiv 0$ or $\nabla v \equiv 0$ in M, which, by using Obata's result [29], implies that v is constant provided (M, g) is Einstein and not conformally equivalent to the standard sphere. In the case where (M, g) is conformally equivalent to the standard sphere, this approach also gives an alternative proof of the classification of conformal metrics with constant Q-curvature.

The strategy which lead us to discover formula (1.4) was to reduce the problem by first considering the case of the Euclidean space $\mathbb{R}^n$ with $u \in D^{1,2}(\mathbb{R}^n)$ (which corresponds to the case of the sphere by stereographic projection) and in this case, consider all possible linear combinations of divergence terms of fifth-order vector fields and look for these which can be written as a sum of nonnegative terms, i.e. squares or scalar curvature terms (the help of a computation software was used at this stage). Once the formula is found in this simpler case, it can easily be extended to the more general case of a smooth, closed Einsten manifold with positive scalar curvature as well as to the case of subcritical equations (see Theorems 2.1 and 2.2).

In the case of more general manifolds, the equation (1.2) has been studied by several authors. Existence results have been obtained by Brendle [5], Chang and Yang [7], Djadli and Malchiodi [9] and Li, Li and Liu [25] in dimension $n = 4$, Djadli, Hebey and Ledoux [8], Esposito and Robert [11], Gursky, Hang and Lin [17], Gursky and Malchiodi [18], Hang and Yang [19, 21] and Qing and Raske [32] in dimensions $n \geq 5$, and Hang and Yang [20] in dimension $n = 3$. Non-uniqueness results have been obtained by Bettiol, Piccione and Sire [1] in dimensions $n \geq 5$. The question of compactness of the set of solutions has also been studied by Druet and Robert [10], Malchiodi [28] and Weinstein and Zhang [35] in dimension $n = 4$, Hebey and Robert [22], Li [24], Li and Xiong [26] and Qing and Raske [31] in dimensions $n \geq 5$, and Hang and Yang [20] in dimension $n = 3$. Finally non-compactness results have been obtained in dimensions $n \geq 25$ by Wei and Zhao [34].

After this paper was posted on arXiv, Jeffrey Case [6] announced a generalization of the ideas we present here where the case of constant Q-curvature is replaced by the fact that $\mathrm{Q} + a\sigma_2$ has to be constant for $a \in \mathbb{R}$ suitably close to zero, where σ_2 is the second symmetric function of the Shouten tensor. In the case where $a = 0$, we are back to the present paper.

2. Extended versions and proof of Theorem 1.1

In this section, we prove the following results, which are slightly more general than Theorem 1.1:

Theorem 2.1. *Let $p \leq 4$ and (M, g) be a smooth, closed Einstein manifold of dimension $n = 4$ with positive scalar curvature. In the case where $p = 4$, assume that (M, g) is not conformally diffeomorphic to the standard sphere. Then there does not exist any non-constant solutions to the equation*

$$\mathrm{P}_g u + \mathrm{Q}_g = e^{pu} \quad \text{in } M. \qquad (2.1)$$

Theorem 2.2. *Let $n \geq 3$, $n \neq 4$, $\lambda = 1$ if $n \geq 4$, $\lambda = -1$ if $n = 3$, $\lambda p \leq \lambda \frac{2n}{n-4}$ and (M, g) be a smooth, closed Einstein manifold of dimension n with positive scalar curvature. In the case where $p = \frac{2n}{n-4}$, assume that (M, g) is not conformally diffeomorphic to the standard sphere. Then there does not exist any non-constant solutions to the equation*

$$\mathrm{P}_g\, u = \lambda u^{p-1}, \ u > 0 \quad \text{in } M. \tag{2.2}$$

We obtain Theorem 1.1 from Theorems 2.1 and 2.2 as follows:

Proof of Theorem 1.1. Let $\tilde{g}$ be a conformal metric to g with constant Q-curvature. Let $u \in C^\infty(M)$ be such that $\tilde{g} = e^{2u} g$ in M if $n = 4$ and $\tilde{g} = u^{\frac{4}{n-4}} g$ with $u > 0$ in M if $n \neq 4$. Since Q_g and $\mathrm{Q}_{\tilde{g}}$ are constant, integrating (1.2) in M gives

$$\mathrm{Q}_g \operatorname{Vol}_g(M) = \begin{cases} \mathrm{Q}_{\tilde{g}} \displaystyle\int_M e^{4u} dv_g & \text{if } n = 4 \\ \mathrm{Q}_{\tilde{g}} \displaystyle\int_M u^{\frac{n+4}{n-4}} dv_g & \text{if } n \neq 4, \end{cases} \tag{2.3}$$

where $\operatorname{Vol}_g(M)$ is the volume of (M, g) and dv_g is the volume element of (M, g). Since $Q_g > 0$ in M, it follows from (2.3) that $\mathrm{Q}_{\tilde{g}} > 0$ in M. We then define

$$\tilde{u} := \begin{cases} u + \dfrac{1}{4} \ln \mathrm{Q}_{\tilde{g}} & \text{if } n = 4 \\ \left(\dfrac{|n-4|}{2} \mathrm{Q}_{\tilde{g}} \right)^{\frac{n-4}{8}} u & \text{if } n \neq 4. \end{cases}$$

We can then rewrite (1.2) as

$$\begin{cases} \mathrm{P}_g\, \tilde{u} + \mathrm{Q}_g = e^{4\tilde{u}} \quad \text{in } M & \text{if } n = 4 \\ \mathrm{P}_g\, \tilde{u} = \lambda \tilde{u}^{\frac{n+4}{n-4}}, \ \tilde{u} > 0 \quad \text{in } M & \text{if } n \neq 4. \end{cases}$$

Applying Theorems 2.1 and 2.2, we then obtain that $\tilde{u}$ is constant, which implies that u is constant. This ends the proof of Theorem 1.1. □

The proofs of Theorems 2.1 and 2.2 use the following result, which is a straightforward variation of a result obtained by Gursky and Malchiodi [18]:

Theorem 2.3. *Let (M, g) be a smooth, closed Riemannian manifold of dimension $n \geq 3$ with positive scalar curvature and non-negative* Q*-curvature. Let $\tilde{g}$ be a conformal metric to g with non-negative* Q*-curvature. Then the scalar curvature of $\tilde{g}$ is positive.*

Proof of Theorem 2.3. We closely follow the proof of Theorem 2.2 in [18]. Let $u \in C^\infty(M)$ be such that $\tilde{g} = e^{2u} g$ in M if $n = 4$ and $\tilde{g} = u^{\frac{4}{n-4}} g$ with $u > 0$ in M if $n \neq 4$. Since $\mathrm{S}_g > 0$ in M, we can define

$$t_0 := \sup \left\{ t \in [0, 1] : \mathrm{S}_{g_s} > 0 \text{ for all } s \in [0, t] \right\},$$

where $g_s := e^{2su} g$ if $n = 4$ and $g_s = (1 - s + su)^{\frac{4}{n-4}} g$ in M if $n \neq 4$. Notice that by continuity, we obtain $\mathrm{S}_{g_{t_0}} \geq 0$ in M. On the other hand, since $\mathrm{Q}_g \geq 0$ and $\mathrm{Q}_{\tilde{g}} \geq 0$ in M, using (1.2), we obtain

$$\mathrm{Q}_{g_{t_0}} = \left. \begin{cases} e^{-4t_0 u} \left((1 - t_0) \mathrm{Q}_g + t_0 \mathrm{Q}_{\tilde{g}}\, e^{4u} \right) & \text{if } n = 4 \\ (1 - t_0 + t_0 u)^{-\frac{n+4}{n-4}} \left((1 - t_0) \mathrm{Q}_g + t_0 \mathrm{Q}_{\tilde{g}}\, u^{\frac{n+4}{n-4}} \right) & \text{if } n \neq 4 \end{cases} \right\} \geq 0 \quad \text{in } M. \tag{2.4}$$

It follows from (1.1) and (2.4) that

$$\Delta_{g_{t_0}} \mathrm{S}_{g_{t_0}} \ge -\frac{n^3-4n^2+16n-16}{4\left(n-1\right)\left(n-2\right)^2} \mathrm{S}_{g_{t_0}}^2 \quad \text{in } M. \tag{2.5}$$

Since $\mathrm{S}_{g_{t_0}} \ge 0$ in M, by the strong maximum principle, it follows from (2.5) that either $\mathrm{S}_{g_{t_0}} \equiv 0$ or $\mathrm{S}_{g_{t_0}} > 0$ in M. Since $\mathrm{S}_g > 0$ in M and a conformal class cannot contain both a metric with positive scalar curvature and a metric with zero scalar curvature, we then obtain that $\mathrm{S}_{g_{t_0}} > 0$ in M, which implies that $t_0 = 1$ and $\mathrm{S}_{\tilde{g}} > 0$ in M. This ends the proof of Theorem 2.3. □

Let us now set some notations and recall some preliminary formulas. We let $\left(\cdot,\cdot\right)_g$ be the multiple inner product induced by the metric g for the tensors of same rank, i.e. such that $\left(S,T\right)_g = S^{i_1\dots i_l}T_{j_1\dots j_l}$ for all tensors S and T of rank $l\in\mathbb{N}$ (with the standard convention on raising and lowering indices). We let $\left|\cdot\right|_g$ be the norm induced by $\left(\cdot,\cdot\right)_g$. We denote by $\overline{\Delta}_g$ the connection Laplacian and Δ_g the Hodge Laplacian on 1-forms. Given a smooth function u in M, the Weitzenbock identity gives

$$\Delta_g du = \overline{\Delta}_g du + \mathrm{Ric}_g \nabla u. \tag{2.6}$$

We also recall the Bochner–Lichnerowicz–Weitzenbock formula

$$\frac{1}{2}\Delta_g \left|\nabla u\right|_g^2 = \left(\nabla\Delta_g u, \nabla u\right)_g - \left|\nabla^2 u\right|_g^2 - \mathrm{Ric}_g\left(\nabla u,\nabla u\right). \tag{2.7}$$

We begin with proving Theorem 2.1.

Proof of Theorem 2.1. Let u be a solution of (2.1). Using (2.6) together with the fact that (M,g) is Einstein and $\Delta_g d = \nabla\Delta_g$, we then obtain

$$\begin{aligned}
A_0 := &\int_M e^{-u}\bigg(\left(\overline{\Delta}_g d\Delta_g u, \nabla u\right)_g + \frac{5}{12}\mathrm{S}_g\left(\nabla\Delta_g u,\nabla u\right)_g - 4\left|\nabla u\right|_g^2\Delta_g^2 u \\
&\quad - \frac{2}{3}\mathrm{S}_g\left|\nabla u\right|_g^2\Delta_g u - \frac{1}{6}\mathrm{S}_g^2\left|\nabla u\right|_g^2\bigg)dv_g \\
= &\int_M e^{-u}\bigg(\left(\nabla\Delta_g^2 u, \nabla u\right)_g + \frac{1}{6}\mathrm{S}_g\left(\nabla\Delta_g u,\nabla u\right)_g - 4\left|\nabla u\right|_g^2\Delta_g^2 u \\
&\quad - \frac{2}{3}\mathrm{S}_g\left|\nabla u\right|_g^2\Delta_g u - \frac{1}{6}\mathrm{S}_g^2\left|\nabla u\right|_g^2\bigg)dv_g \\
= &\int_M e^{-u}\big(\left(\nabla P_g u,\nabla u\right)_g - 4\left|\nabla u\right|_g^2\left(P_g u + Q_g\right)\big)dv_g \\
= &\left(p-4\right)\int_M e^{(p-1)u}\left|\nabla u\right|_g^2 dv_g.
\end{aligned} \tag{2.8}$$

Integrating by parts and using (2.6) and (2.7), we also obtain

$$\begin{aligned}
A_1 := &\int_M e^{-u}\big(\left(\overline{\Delta}_g d\Delta_g u,\nabla u\right)_g - \left(\nabla^2\Delta_g u, \nabla^2 u - \nabla u\otimes\nabla u\right)_g\big)dv_g \\
= &-\int_M \mathrm{div}_g\left(e^{-u}\nabla^2\left(\Delta_g u\right)\left(\nabla u\right)\right)dv_g \\
= &\ 0,
\end{aligned} \tag{2.9}$$

$$
\begin{aligned}
A_2 :=& \int_M e^{-u}\bigg(2\left(\nabla^2\Delta_g u, \nabla^2 u\right)_g - \left(\nabla\Delta_g u, \nabla\left|\nabla u\right|_g^2\right)_g - 2\left|\nabla\Delta_g u\right|_g^2 \\
&+ \frac{1}{2}\,\mathrm{S}_g\left(\nabla\Delta_g u, \nabla u\right)_g\bigg) dv_g \\
=& \int_M e^{-u}\big(2\left(\nabla^2\Delta_g u, \nabla^2 u\right)_g - \left(\nabla\Delta_g u, \nabla\left|\nabla u\right|_g^2\right)_g - 2\big(\overline{\Delta}_g du, \nabla\Delta_g u\big)_g\big) dv_g \\
=& \, 2\int_M \operatorname{div}_g\left(e^{-u}\nabla^2 u\left(\nabla\Delta_g u\right)\right) dv_g \\
=& \, 0, \qquad (2.10)
\end{aligned}
$$

$$
\begin{aligned}
A_3 :=& \int_M e^{-u}\big(2\left(\nabla^2\Delta_g u, \nabla u\otimes\nabla u\right)_g + \left(\nabla\Delta_g u, \nabla\left|\nabla u\right|_g^2\right)_g - 2\left(\nabla\Delta_g u, \nabla u\right)_g\Delta_g u \\
&- 2\left|\nabla u\right|_g^2\left(\nabla\Delta_g u, \nabla u\right)_g\big) dv_g \\
=& \, 2\int_M \operatorname{div}_g\big(e^{-u}\left(\nabla\Delta_g u, \nabla u\right)_g\nabla u\big) dv_g \\
=& \, 0, \qquad (2.11)
\end{aligned}
$$

$$
\begin{aligned}
A_4 :=& \int_M e^{-u}\big(\left|\nabla u\right|_g^2\Delta_g^2 u - \left(\nabla\Delta_g u, \nabla\left|\nabla u\right|_g^2\right)_g + \left|\nabla u\right|_g^2\left(\nabla\Delta_g u, \nabla u\right)_g\big) dv_g \\
=& -\int_M \operatorname{div}_g\big(e^{-u}\left|\nabla u\right|_g^2\nabla\Delta_g u\big) dv_g \\
=& \, 0, \qquad (2.12)
\end{aligned}
$$

$$
\begin{aligned}
A_5 :=& \int_M e^{-u}\bigg(\left(\nabla\Delta_g u, \nabla\left|\nabla u\right|_g^2\right)_g - 2\left(\nabla\Delta_g u, \nabla u\right)_g\Delta_g u + 2\left|\nabla^2 u\right|_g^2\Delta_g u \\
&- \left(\nabla\left|\nabla u\right|_g^2, \nabla u\right)_g\Delta_g u + \frac{1}{2}\,\mathrm{S}_g\left|\nabla u\right|_g^2\Delta_g u\bigg) dv_g \\
=& \int_M e^{-u}\big(\left(\nabla\Delta_g u, \nabla\left|\nabla u\right|_g^2\right)_g - \Delta_g\left|\nabla u\right|_g^2\Delta_g u - \left(\nabla\left|\nabla u\right|_g^2, \nabla u\right)_g\Delta_g u\big) dv_g \\
=& \int_M \operatorname{div}_g\big(e^{-u}\left(\Delta_g u\right)\nabla\left|\nabla u\right|_g^2\big) dv_g \\
=& \, 0, \qquad (2.13)
\end{aligned}
$$

$$
\begin{aligned}
A_6 :=& \int_M e^{-u}\big(2\big(\nabla\Delta_g u, \nabla u\big)_g\Delta_g u - \left(\Delta_g u\right)^3 - \left|\nabla u\right|_g^2\left(\Delta_g u\right)^2\big) dv_g \\
=& \int_M \operatorname{div}_g\big(e^{-u}\left(\Delta_g u\right)^2\nabla u\big) dv_g \\
=& \, 0, \qquad (2.14)
\end{aligned}
$$

$$
\begin{aligned}
A_7 :=& \int_M e^{-u}\bigg(2\left|\nabla u\right|_g^2\left(\nabla\Delta_g u, \nabla u\right)_g - 2\left|\nabla u\right|_g^2\left|\nabla^2 u\right|_g^2 \\
&- \left|\nabla\left|\nabla u\right|_g^2\right|_g^2 + \left|\nabla u\right|_g^2\left(\nabla\left|\nabla u\right|_g^2, \nabla u\right)_g - \frac{1}{2}\,\mathrm{S}_g\left|\nabla u\right|_g^4\bigg) dv_g \\
=& \int_M e^{-u}\big(\left|\nabla u\right|_g^2\Delta_g\left|\nabla u\right|_g^2 - \left|\nabla\left|\nabla u\right|_g^2\right|_g^2 + \left|\nabla u\right|_g^2\left(\nabla\left|\nabla u\right|_g^2, \nabla u\right)_g\big) dv_g \\
=& -\int_M \operatorname{div}_g\big(e^{-u}\left|\nabla u\right|_g^2\nabla\left|\nabla u\right|_g^2\big) dv_g \\
=& \, 0, \qquad (2.15)
\end{aligned}
$$

$$\begin{aligned}
A_8 :=& \int_M e^{-u}\big(|\nabla u|_g^2 \left(\nabla\Delta_g u, \nabla u\right)_g + \big(\nabla |\nabla u|_g^2, \nabla u\big)_g \Delta_g u - |\nabla u|_g^2 \left(\Delta_g u\right)^2 \\
&- |\nabla u|_g^4 \Delta_g u\big) dv_g \\
=& \int_M \operatorname{div}_g \big(e^{-u} |\nabla u|_g^2 \left(\Delta_g u\right) \nabla u\big) dv_g \\
=& \, 0, \qquad (2.16)\\
A_9 :=& \int_M e^{-u}\big(2 |\nabla u|_g^2 \big(\nabla |\nabla u|_g^2, \nabla u\big)_g - |\nabla u|_g^4 \Delta_g u - |\nabla u|_g^6\big) dv_g \\
=& \int_M \operatorname{div}_g \big(e^{-u} |\nabla u|_g^4 \nabla u\big) dv_g \\
=& \, 0, \qquad (2.17)\\
A_{10} :=& \int_M e^{-u}\left(2 \left(\nabla\Delta_g u, \nabla u\right)_g - 2 \left|\nabla^2 u\right|_g^2 + \big(\nabla |\nabla u|_g^2, \nabla u\big)_g - \frac{1}{2} \mathrm{S}_g |\nabla u|_g^2 \right) dv_g \\
=& \int_M e^{-u}\big(\Delta_g |\nabla u|_g^2 + \big(\nabla |\nabla u|_g^2, \nabla u\big)_g\big) dv_g \\
=& -\int_M \operatorname{div}_g \big(e^{-u} \nabla |\nabla u|_g^2\big) dv_g \\
=& \, 0, \qquad (2.18)\\
A_{11} :=& \int_M e^{-u}\big(\left(\nabla\Delta_g u, \nabla u\right)_g - \left(\Delta_g u\right)^2 - |\nabla u|_g^2 \Delta_g u\big) dv_g \\
=& \int_M \operatorname{div}_g \big(e^{-u} \left(\Delta_g u\right) \nabla u\big) dv_g \\
=& \, 0, \qquad (2.19)\\
A_{12} :=& \int_M e^{-u}\big(\big(\nabla |\nabla u|_g^2, \nabla u\big)_g - |\nabla u|_g^2 \Delta_g u - |\nabla u|_g^4\big) dv_g \\
=& \int_M \operatorname{div}_g \big(e^{-u} |\nabla u|_g^2 \nabla u\big) dv_g \\
=& \, 0. \qquad (2.20)
\end{aligned}$$

Combining (2.8)–(2.20), we then obtain

$$\begin{aligned}
& 36\,(p-4) \int_M e^{(p-1)u} |\nabla u|_g^2 \, dv_g \\
&= 36A_0 - 36A_1 - 18A_2 + 18A_3 + 144A_4 + 84A_5 + 42A_6 + 12A_7 - 60A_8 + 18A_9 \\
&\quad - 20\,\mathrm{S}_g A_{10} + 10\,\mathrm{S}_g A_{11} - 12\,\mathrm{S}_g A_{12} \\
&= \int_M e^{-u}\big(36 |\nabla\Delta_g u|_g^2 - 24\big(\nabla\Delta_g u, \nabla |\nabla u|_g^2\big)_g - 120\big(\nabla\Delta_g u, \nabla u\big)_g \Delta_g u \\
&\quad + 72 |\nabla u|_g^2 \left(\nabla\Delta_g u, \nabla u\right)_g - 12\big|\nabla |\nabla u|_g^2\big|_g^2 - 144\big(\nabla |\nabla u|_g^2, \nabla u\big)_g \Delta_g u \\
&\quad + 48 |\nabla u|_g^2 \big(\nabla |\nabla u|_g^2, \nabla u\big)_g - 24 |\nabla u|_g^2 \left|\nabla^2 u\right|_g^2 + 168 \left|\nabla^2 u\right|_g^2 \Delta_g u - 42 \left(\Delta_g u\right)^3 \\
&\quad + 18 |\nabla u|_g^2 \left(\Delta_g u\right)^2 + 42 |\nabla u|_g^4 \Delta_g u - 18 |\nabla u|_g^6 - 24\,\mathrm{S}_g \left(\nabla\Delta_g u, \nabla u\right)_g \\
&\quad - 32\,\mathrm{S}_g \big(\nabla |\nabla u|_g^2, \nabla u\big)_g + 40\,\mathrm{S}_g \left|\nabla^2 u\right|_g^2 - 10\,\mathrm{S}_g \left(\Delta_g u\right)^2 + 20\,\mathrm{S}_g |\nabla u|_g^2 \Delta_g u \\
&\quad + 6\,\mathrm{S}_g |\nabla u|_g^4 + 4\,\mathrm{S}_g^2 |\nabla u|_g^2\big) dv_g. \qquad (2.21)
\end{aligned}$$

On the other hand, the transformation law for the scalar curvature under a conformal change of metric gives

$$\mathrm{S}_{e^{2u}g} = e^{-3u}\left(6\Delta_g e^u + \mathrm{S}_g\, e^u\right) = e^{-2u}\big(6\Delta_g u - 6\left|\nabla u\right|_g^2 + \mathrm{S}_g\big). \tag{2.22}$$

Differentiating (2.22), we obtain

$$\nabla\, \mathrm{S}_{e^{2u}g} = e^{-2u}\big(6\nabla\Delta_g u - 12\Delta_g u \nabla u - 6\nabla\left|\nabla u\right|_g^2 + 12\left|\nabla u\right|_g^2 \nabla u - 2\,\mathrm{S}_g\, \nabla u\big). \tag{2.23}$$

Moreover, using (1.3), we obtain

$$\mathrm{E}_{e^{2u}g} = -2\nabla^2 u + 2\nabla u\otimes\nabla u - \frac{1}{2}\big(\Delta_g u + \left|\nabla u\right|_g^2\big) g, \tag{2.24}$$

$$\left|\mathrm{E}_{e^{2u}g}\right|_g^2 = 4\left|\nabla^2 u\right|_g^2 - 4\big(\nabla\left|\nabla u\right|_g^2, \nabla u\big)_g - \left(\Delta_g u\right)^2 - 2\left|\nabla u\right|_g^2 \Delta_g u + 3\left|\nabla u\right|_g^4 \tag{2.25}$$

and

$$\begin{aligned}\left|\, \mathrm{E}_{e^{2u}g}\,\nabla\left(e^{-2u}\right)\right|_g^2 &= e^{-4u}\big(4\big|\nabla\left|\nabla u\right|_g^2\big|_g^2 + 4\big(\nabla\left|\nabla u\right|_g^2, \nabla u\big)_g \Delta_g u\\ &\quad - 12\left|\nabla u\right|_g^2\left(\nabla\left|\nabla u\right|_g^2, \nabla u\right)_g + \left|\nabla u\right|_g^2\left(\Delta_g u\right)^2 - 6\left|\nabla u\right|_g^4\Delta_g u + 9\left|\nabla u\right|_g^6\big).\end{aligned} \tag{2.26}$$

It follows from (2.22)–(2.26) that

$$\begin{aligned}&\left|\nabla\, \mathrm{S}_{e^{2u}g} + 2\,\mathrm{E}_{e^{2u}g}\,\nabla\left(e^{-2u}\right)\right|_g^2 - 4\left|\,\mathrm{E}_{e^{2u}g}\,\nabla\left(e^{-2u}\right)\right|_g^2\\ &\quad = 4e^{-4u}\big(9\left|\nabla\Delta_g u\right|_g^2 - 6\big(\nabla\Delta_g u, \nabla\left|\nabla u\right|_g^2\big)_g - 30\big(\nabla\Delta_g u, \nabla u\big)_g\Delta_g u\\ &\qquad + 18\left|\nabla u\right|_g^2\left(\nabla\Delta_g u, \nabla u\right)_g - 3\big|\nabla\left|\nabla u\right|_g^2\big|_g^2 + 6\big(\nabla\left|\nabla u\right|_g^2, \nabla u\big)_g\Delta_g u\\ &\qquad + 6\left|\nabla u\right|_g^2\left(\nabla\left|\nabla u\right|_g^2, \nabla u\right)_g + 24\left|\nabla u\right|_g^2\left(\Delta_g u\right)^2 - 24\left|\nabla u\right|_g^4\Delta_g u\\ &\qquad - 6\,\mathrm{S}_g\left(\nabla\Delta_g u, \nabla u\right)_g + 2\,\mathrm{S}_g\left(\nabla\left|\nabla u\right|_g^2, \nabla u\right)_g + 10\,\mathrm{S}_g\left|\nabla u\right|_g^2\Delta_g u\\ &\qquad - 6\,\mathrm{S}_g\left|\nabla u\right|_g^4 + \mathrm{S}_g^2\left|\nabla u\right|_g^2\big)\end{aligned} \tag{2.27}$$

and

$$\begin{aligned}&\left|\mathrm{E}_{e^{2u}g}\right|_g^2\left(7\,\mathrm{S}_{e^{2u}g}\, e^{-2u} + 3\,\mathrm{S}_g\, e^{-4u} + 9\left|\nabla e^{-2u}\right|_g^2\right)\\ &\quad = e^{-4u}\big(-168\big(\nabla\left|\nabla u\right|_g^2, \nabla u\big)_g\Delta_g u + 24\left|\nabla u\right|_g^2\left(\nabla\left|\nabla u\right|_g^2, \nabla u\right)_g\\ &\qquad - 24\left|\nabla u\right|_g^2\left|\nabla^2 u\right|_g^2 + 168\left|\nabla^2 u\right|_g^2\Delta_g u - 42\left(\Delta_g u\right)^3 - 78\left|\nabla u\right|_g^2\left(\Delta_g u\right)^2\\ &\qquad + 138\left|\nabla u\right|_g^4\Delta_g u - 18\left|\nabla u\right|_g^6 - 40\,\mathrm{S}_g\left(\nabla\left|\nabla u\right|_g^2, \nabla u\right)_g + 40\,\mathrm{S}_g\left|\nabla^2 u\right|_g^2\\ &\qquad - 10\,\mathrm{S}_g\left(\Delta_g u\right)^2 - 20\,\mathrm{S}_g\left|\nabla u\right|_g^2\Delta_g u + 30\,\mathrm{S}_g\left|\nabla u\right|_g^4\big).\end{aligned} \tag{2.28}$$

Putting together (2.21), (2.27) and (2.28), we then obtain

$$36\left(p-4\right)\int_M e^{(p-1)u}\left|\nabla u\right|_g^2 dv_g = \int_M \Theta_g^{(2)}\left(e^{-2u}\right) dv_g. \tag{2.29}$$

Notice that (1.2) and (2.1) give $Q_{e^{2u}g} > 0$ in M. Moreover, since $\mathrm{S}_g > 0$ in M and (M,g) is Einstein, we obtain that $\mathrm{Q}_g > 0$ in M. Applying Theorem 2.3, it then follows that $\mathrm{S}_{e^{2u}g} > 0$ in M. Using (1.5) together with the positivity of the functions v, S_g and $\mathrm{S}_{e^{2u}g}$, we then obtain that $\Theta_g^{(2)}\left(e^{-2u}\right) \geq 0$ in M. It then follows from (2.29) that if $p<4$, then $\nabla u \equiv 0$ in M and if $p=4$, then $\Theta_g^{(2)}\left(e^{-2u}\right) \equiv 0$ in M. Since in the latter case we assumed that (M,g) is not conformally diffeomorphic to the standard sphere, as we explained in the introduction, we then obtain that u is constant. This ends the proof of Theorem 2.1. □

We now prove Theorem 2.2.

Proof of Theorem 2.2. Let u be a solution of (2.1). Integrating by parts, we then obtain

$$\begin{aligned} A_0 &:= \int_M u^{-\frac{2}{n-4}} \left(\Delta_g u - \frac{n+2}{n-4} u^{-1} |\nabla u|_g^2\right) \mathrm{P}_g\, u\, dv_g \\ &= \lambda \int_M \left(\left(p - \frac{2n}{n-4}\right) u^{p-\frac{2(n-3)}{n-4}} |\nabla u|_g^2 - \mathrm{div}_g \left(u^{p-\frac{n-2}{n-4}} \nabla u\right)\right) dv_g \\ &= \lambda \left(p - \frac{2n}{n-4}\right) \int_M u^{p-\frac{2(n-3)}{n-4}} |\nabla u|_g^2\, dv_g. \end{aligned} \tag{2.30}$$

Integrating by parts and using (2.7) together with the fact that (M, g) is Einstein, we also obtain

$$\begin{aligned} A_1 &:= \int_M u^{-\frac{2}{n-4}} \left(\Delta_g^2 u \Delta_g u - |\nabla \Delta_g u|_g^2 + \frac{2}{n-4} u^{-1} (\nabla \Delta_g u, \nabla u)_g\, \Delta_g u\right) dv_g \\ &= -\int_M \mathrm{div}_g \left(u^{-\frac{2}{n-4}} (\Delta_g u) \nabla \Delta_g u\right) dv_g \\ &= 0, \end{aligned} \tag{2.31}$$

$$\begin{aligned} A_2 &:= \int_M u^{-\frac{n-2}{n-4}} \Bigg(|\nabla u|_g^2\, \Delta_g^2 u - \left(\nabla \Delta_g u, \nabla |\nabla u|_g^2\right)_g \\ &\quad + \frac{n-2}{n-4} u^{-1} |\nabla u|_g^2 (\nabla \Delta_g u, \nabla u)_g \Bigg) dv_g \\ &= -\int_M \mathrm{div}_g \left(u^{-\frac{n-2}{n-4}} |\nabla u|_g^2\, \nabla \Delta_g u\right) dv_g \\ &= 0, \end{aligned} \tag{2.32}$$

$$\begin{aligned} A_3 &:= \int_M u^{-\frac{n-2}{n-4}} \Bigg(\left(\nabla \Delta_g u, \nabla |\nabla u|_g^2\right)_g - 2 (\nabla \Delta_g u, \nabla u)_g\, \Delta_g u + 2 \left|\nabla^2 u\right|_g^2 \Delta_g u \\ &\quad - \frac{n-2}{n-4} u^{-1} \left(\nabla |\nabla u|_g^2, \nabla u\right)_g \Delta_g u + \frac{2}{n} \mathrm{S}_g\, |\nabla u|_g^2\, \Delta_g u \Bigg) dv_g \\ &= \int_M u^{-\frac{n-2}{n-4}} \Bigg(\left(\nabla \Delta_g u, \nabla |\nabla u|_g^2\right)_g - \Delta_g |\nabla u|_g^2\, \Delta_g u \\ &\quad - \frac{n-2}{n-4} u^{-1} \left(\nabla |\nabla u|_g^2, \nabla u\right)_g \Delta_g u \Bigg) dv_g \\ &= \int_M \mathrm{div}_g \left(u^{-\frac{n-2}{n-4}} (\Delta_g u) \nabla |\nabla u|_g^2\right) dv_g \\ &= 0, \end{aligned} \tag{2.33}$$

$$\begin{aligned} A_4 &:= \int_M u^{-\frac{n-2}{n-4}} \left(2 (\nabla \Delta_g u, \nabla u)_g\, \Delta_g u - (\Delta_g u)^3 - \frac{n-2}{n-4} u^{-1} |\nabla u|_g^2 (\Delta_g u)^2 \right) dv_g \\ &= \int_M \mathrm{div}_g \left(u^{-\frac{n-2}{n-4}} (\Delta_g u)^2 \nabla u\right) dv_g \\ &= 0, \end{aligned} \tag{2.34}$$

$$\begin{aligned} A_5 &:= \int_M u^{-\frac{2(n-3)}{n-4}} \Bigg(2 |\nabla u|_g^2 (\nabla \Delta_g u, \nabla u)_g - 2 |\nabla u|_g^2 \left|\nabla^2 u\right|_g^2 - \left|\nabla |\nabla u|_g^2\right|_g^2 \\ &\quad + \frac{2(n-3)}{n-4} u^{-1} |\nabla u|_g^2 \left(\nabla |\nabla u|_g^2, \nabla u\right)_g - \frac{2}{n} \mathrm{S}_g\, |\nabla u|_g^4 \Bigg) dv_g \end{aligned}$$

$$\begin{aligned}
&= \int_M u^{-\frac{2(n-3)}{n-4}} \bigg(|\nabla u|_g^2 \, \Delta_g |\nabla u|_g^2 - \big|\nabla |\nabla u|_g^2\big|_g^2 \\
&\qquad + \frac{2(n-3)}{n-4} u^{-1} |\nabla u|_g^2 \left(\nabla |\nabla u|_g^2, \nabla u\right)_g \bigg) dv_g \\
&= -\int_M \operatorname{div}_g \left(u^{-\frac{2(n-3)}{n-4}} |\nabla u|_g^2 \, \nabla |\nabla u|_g^2 \right) dv_g \\
&= 0, \qquad (2.35) \\
A_6 &:= \int_M u^{-\frac{2(n-3)}{n-4}} \bigg(|\nabla u|_g^2 \left(\nabla \Delta_g u, \nabla u\right)_g + \left(\nabla |\nabla u|_g^2, \nabla u\right)_g \Delta_g u - |\nabla u|_g^2 \left(\Delta_g u\right)^2 \\
&\qquad - \frac{2(n-3)}{n-4} u^{-1} |\nabla u|_g^4 \, \Delta_g u \bigg) dv_g \\
&= \int_M \operatorname{div}_g \left(u^{-\frac{2(n-3)}{n-4}} |\nabla u|_g^2 \left(\Delta_g u\right) \nabla u \right) dv_g \\
&= 0, \qquad (2.36) \\
A_7 &:= \int_M u^{-\frac{3n-10}{n-4}} \bigg(2 |\nabla u|_g^2 \left(\nabla |\nabla u|_g^2, \nabla u\right)_g - |\nabla u|_g^4 \, \Delta_g u \\
&\qquad - \frac{3n-10}{n-4} u^{-1} |\nabla u|_g^6 \bigg) dv_g \\
&= \int_M \operatorname{div}_g \left(u^{-\frac{3n-10}{n-4}} |\nabla u|_g^4 \, \nabla u \right) dv_g \\
&= 0, \qquad (2.37) \\
A_8 &:= \int_M u^{-\frac{2}{n-4}} \bigg(\left(\nabla \Delta_g u, \nabla u\right)_g - \left(\Delta_g u\right)^2 - \frac{2}{n-4} u^{-1} |\nabla u|_g^2 \, \Delta_g u \bigg) dv_g \\
&= \int_M \operatorname{div}_g \left(u^{-\frac{2}{n-4}} \left(\Delta_g u\right) \nabla u \right) dv_g \\
&= 0, \qquad (2.38) \\
A_9 &:= \int_M u^{-\frac{2}{n-4}} \bigg(2 \left(\nabla \Delta_g u, \nabla u\right)_g - 2 \left|\nabla^2 u\right|_g^2 + \frac{2}{n-4} u^{-1} \left(\nabla |\nabla u|_g^2, \nabla u\right)_g \\
&\quad - \frac{2}{n} \mathrm{S}_g |\nabla u|_g^2 \bigg) dv_g \\
&= \int_M u^{-\frac{2}{n-4}} \bigg(\Delta_g |\nabla u|_g^2 + \frac{2}{n-4} u^{-1} \left(\nabla |\nabla u|_g^2, \nabla u\right)_g \bigg) dv_g \\
&= -\int_M \operatorname{div}_g \left(u^{-\frac{2}{n-4}} \nabla |\nabla u|_g^2 \right) dv_g \\
&= 0, \qquad (2.39) \\
A_{10} &:= \int_M u^{-\frac{n-2}{n-4}} \bigg(\left(\nabla |\nabla u|_g^2, \nabla u\right)_g - |\nabla u|_g^2 \, \Delta_g u - \frac{n-2}{n-4} u^{-1} |\nabla u|_g^4 \bigg) dv_g \\
&= \int_M \operatorname{div}_g \left(u^{-\frac{n-2}{n-4}} |\nabla u|_g^2 \, \nabla u \right) dv_g \\
&= 0, \qquad (2.40)
\end{aligned}$$

$$\begin{aligned}
A_{11} &:= \int_M u^{\frac{n-6}{n-4}} \left(\Delta_g u - \frac{n-6}{n-4} u^{-1} \left| \nabla u \right|_g^2 \right) dv_g \\
&= - \int_M \mathrm{div}_g \left(u^{\frac{n-6}{n-4}} \nabla u \right) dv_g \\
&= 0. \qquad (2.41)
\end{aligned}$$

Combining (2.30)–(2.41), we then obtain

$$\begin{aligned}
&\frac{16\left(n-1\right)^2}{\left(n-4\right)^2} \left(p - \frac{2n}{n-4} \right) \int_M u^{p-\frac{2(n-3)}{n-4}} \left| \nabla u \right|_g^2 dv_g \\
&= \frac{16\left(n-1\right)^2}{\left(n-4\right)^2} A_0 - \frac{16\left(n-1\right)^2}{\left(n-4\right)^2} A_1 + \frac{16\left(n-1\right)^2\left(n+2\right)}{\left(n-4\right)^3} A_2 \\
&\quad + \frac{16\left(n-1\right)\left(n^2-2\right)}{\left(n-4\right)^3} A_3 + \frac{32\left(n-1\right)\left(n^2-2\right)}{n\left(n-4\right)^3} A_4 + \frac{32\left(n-1\right)\left(n-2\right)}{\left(n-4\right)^4} A_5 \\
&\quad - \frac{16\left(n-1\right)\left(n-2\right)\left(n^3-n^2-4n+8\right)}{n\left(n-4\right)^4} A_6 + \frac{64\left(n-1\right)^2\left(n-2\right)^2}{n\left(n-4\right)^5} A_7 \\
&\quad + \frac{8n\left(n-2\right)}{\left(n-4\right)^2} \mathrm{S}_g A_8 - \frac{4\left(n^2+2n-4\right)}{\left(n-4\right)^2} \mathrm{S}_g A_9 - \frac{8\left(n-1\right)\left(n^2-12\right)}{\left(n-4\right)^3} \mathrm{S}_g A_{10} \\
&\quad - \frac{\left(n-2\right)\left(n+2\right)}{n\left(n-4\right)} \mathrm{S}_g^2 A_{11} \\
&= \frac{8}{\left(n-4\right)^2} \int_M u^{-\frac{2}{n-4}} \Bigg(2\left(n-1\right)^2 \left| \nabla \Delta_g u \right|_g^2 - \frac{2n\left(n-1\right)}{n-4} u^{-1} \left(\nabla \Delta_g u, \nabla \left| \nabla u \right|_g^2 \right)_g \\
&\quad - \frac{4\left(n-1\right)\left(n^3-n^2-3n+4\right)}{n\left(n-4\right)} u^{-1} \left(\nabla \Delta_g u, \nabla u \right)_g \Delta_g u \\
&\quad + \frac{4\left(n-1\right)^2\left(n-2\right)\left(n+4\right)}{n\left(n-4\right)^2} u^{-2} \left| \nabla u \right|_g^2 \left(\nabla \Delta_g u, \nabla u \right)_g \\
&\quad - \frac{4\left(n-1\right)\left(n-2\right)}{\left(n-4\right)^2} u^{-2} \left| \nabla \left| \nabla u \right|_g^2 \right|_g^2 \\
&\quad - \frac{2\left(n-1\right)\left(n-2\right)\left(n+2\right)\left(2n^2-5n+4\right)}{n\left(n-4\right)^2} u^{-2} \left(\nabla \left| \nabla u \right|_g^2, \nabla u \right)_g \Delta_g u \\
&\quad + \frac{8\left(n-1\right)\left(n-2\right)\left(3n^2-9n+4\right)}{n\left(n-4\right)^3} u^{-3} \left| \nabla u \right|_g^2 \left(\nabla \left| \nabla u \right|_g^2, \nabla u \right)_g \\
&\quad - \frac{8\left(n-1\right)\left(n-2\right)}{\left(n-4\right)^2} u^{-2} \left| \nabla u \right|_g^2 \left| \nabla^2 u \right|_g^2 + \frac{4\left(n-1\right)\left(n^2-2\right)}{n-4} u^{-1} \left| \nabla^2 u \right|_g^2 \Delta_g u \\
&\quad - \frac{4\left(n-1\right)\left(n^2-2\right)}{n\left(n-4\right)} u^{-1} \left(\Delta_g u \right)^3 \\
&\quad + \frac{2\left(n-1\right)\left(n-2\right)^2\left(n-3\right)\left(n+2\right)}{n\left(n-4\right)^2} u^{-2} \left| \nabla u \right|_g^2 \left(\Delta_g u \right)^2 \\
&\quad + \frac{4\left(n-1\right)\left(n-2\right)\left(n^4-4n^3-3n^2+26n-28\right)}{n\left(n-4\right)^3} u^{-3} \left| \nabla u \right|_g^4 \Delta_g u \\
&\quad - \frac{8\left(n-1\right)^2\left(n-2\right)^2\left(3n-10\right)}{n\left(n-4\right)^4} u^{-4} \left| \nabla u \right|_g^6 - 4\left(n-1\right) \mathrm{S}_g \left(\nabla \Delta_g u, \nabla u \right)_g
\end{aligned}$$

$$-\frac{n^3-10n+8}{n-4}\,\mathrm{S}_g\,u^{-1}\big(\nabla\left|\nabla u\right|_g^2,\nabla u\big)_g+\left(n^2+2n-4\right)\mathrm{S}_g\left|\nabla^2 u\right|_g^2$$
$$-\frac{n^2+2n-4}{n}\,\mathrm{S}_g\left(\Delta_g u\right)^2+\frac{2\left(n^2-2n+2\right)}{n-4}\,\mathrm{S}_g\,u^{-1}\left|\nabla u\right|_g^2\Delta_g u$$
$$+\frac{(n-1)(n-2)\left(n^3-12n-8\right)}{n(n-4)^2}\,\mathrm{S}_g\,u^{-2}\left|\nabla u\right|_g^4+2\,\mathrm{S}_g^2\left|\nabla u\right|_g^2\Bigg)dv_g. \tag{2.42}$$

On the other hand, the transformation law for the scalar curvature under a conformal change of metric gives

$$\begin{aligned}\mathrm{S}_{u^{\frac{4}{n-4}}g}&=u^{-\frac{n+2}{n-4}}\left(\frac{4(n-1)}{n-2}\Delta_g\left(u^{\frac{n-2}{n-4}}\right)+\mathrm{S}_g\,u^{\frac{n-2}{n-4}}\right)\\&=u^{-\frac{n}{n-4}}\left(\frac{4(n-1)}{n-4}\Delta_g u-\frac{8(n-1)}{(n-4)^2}u^{-1}\left|\nabla u\right|_g^2+\mathrm{S}_g\,u\right).\end{aligned} \tag{2.43}$$

Differentiating (2.43), we obtain

$$\begin{aligned}\nabla\,\mathrm{S}_{u^{\frac{4}{n-4}}g}=\frac{4}{n-4}u^{-\frac{n}{n-4}}\Bigg(&(n-1)\,\nabla\Delta_g u-\frac{n(n-1)}{n-4}u^{-1}\Delta_g u\nabla u\\&-\frac{2(n-1)}{n-4}u^{-1}\nabla\left|\nabla u\right|_g^2+\frac{4(n-1)(n-2)}{(n-4)^2}u^{-2}\left|\nabla u\right|_g^2\nabla u-\mathrm{S}_g\,\nabla u\Bigg).\end{aligned} \tag{2.44}$$

Moreover, using (1.3), we obtain

$$\begin{aligned}\mathrm{E}_{u^{\frac{4}{n-4}}g}=-\frac{2(n-2)}{n-4}u^{-1}\Bigg(&\nabla^2 u-\frac{n-2}{n-4}u^{-1}\nabla u\otimes\nabla u\\&+\frac{1}{n}\left(\Delta_g u+\frac{n-2}{n-4}u^{-1}\left|\nabla u\right|_g^2\right)g\Bigg),\end{aligned} \tag{2.45}$$

$$\begin{aligned}\left|\mathrm{E}_{u^{\frac{4}{n-4}}g}\right|_g^2=\frac{4(n-2)^2}{(n-4)^2}u^{-2}\Bigg(&\left|\nabla^2 u\right|_g^2-\frac{n-2}{n-4}u^{-1}\big(\nabla\left|\nabla u\right|_g^2,\nabla u\big)_g-\frac{1}{n}\left(\Delta_g u\right)^2\\&-\frac{2(n-2)}{n(n-4)}u^{-1}\left|\nabla u\right|_g^2\Delta_g u+\frac{(n-1)(n-2)^2}{n(n-4)^2}u^{-2}\left|\nabla u\right|_g^4\Bigg)\end{aligned} \tag{2.46}$$

and

$$\begin{aligned}\left|\mathrm{E}_{u^{\frac{4}{n-4}}g}\nabla\left(u^{-\frac{4}{n-4}}\right)\right|_g^2=&\frac{16(n-2)^2}{(n-4)^4}u^{-\frac{4(n-2)}{n-4}}\Bigg(\left|\nabla\left|\nabla u\right|_g^2\right|_g^2\\&+\frac{4}{n}\big(\nabla\left|\nabla u\right|_g^2,\nabla u\big)_g\Delta_g u-\frac{4(n-1)(n-2)}{n(n-4)}u^{-1}\left|\nabla u\right|_g^2\big(\nabla\left|\nabla u\right|_g^2,\nabla u\big)_g\\&+\frac{4}{n^2}\left|\nabla u\right|_g^2\left(\Delta_g u\right)^2-\frac{8(n-1)(n-2)}{n^2(n-4)}u^{-1}\left|\nabla u\right|_g^4\Delta_g u\\&+\frac{4(n-1)^2(n-2)^2}{n^2(n-4)^2}u^{-2}\left|\nabla u\right|_g^6\Bigg).\end{aligned} \tag{2.47}$$

It follows from (2.43)–(2.47) that

$$\left|\nabla\,\mathrm{S}_{u^{\frac{4}{n-4}}g}+\frac{3n-4}{2(n-2)}\,\mathrm{E}_{u^{\frac{4}{n-4}}g}\nabla\left(u^{-\frac{4}{n-4}}\right)\right|_g^2-\frac{(3n-4)^2}{4(n-2)^2}\left|\mathrm{E}_{u^{\frac{4}{n-4}}g}\nabla\left(u^{-\frac{4}{n-4}}\right)\right|_g^2$$
$$=\frac{16}{(n-4)^2}u^{-\frac{2n}{n-4}}\Bigg((n-1)^2\left|\nabla\Delta_g u\right|_g^2-\frac{n(n-1)}{n-4}u^{-1}\big(\nabla\Delta_g u,\nabla\left|\nabla u\right|_g^2\big)_g$$

$$
\begin{aligned}
&- \frac{2(n-1)\left(n^3-n^2-3n+4\right)}{n(n-4)} u^{-1}\left(\nabla \Delta_g u, \nabla u\right)_g \Delta_g u \\
&+ \frac{2(n-1)^2(n-2)(n+4)}{n(n-4)^2} u^{-2}\left|\nabla u\right|_g^2\left(\nabla \Delta_g u, \nabla u\right)_g \\
&- \frac{2(n-1)(n-2)}{(n-4)^2} u^{-2}\left|\nabla\left|\nabla u\right|_g^2\right|_g^2 \\
&+ \frac{(n-1)(n-2)^2(n+4)}{n(n-4)^2} u^{-2}\left(\nabla\left|\nabla u\right|_g^2, \nabla u\right)_g \Delta_g u \\
&+ \frac{4(n-1)(n-2)\left(2n^2-7n+4\right)}{n(n-4)^3} u^{-3}\left|\nabla u\right|_g^2\left(\nabla\left|\nabla u\right|_g^2, \nabla u\right)_g \\
&+ \frac{(n-1)(n-2)\left(n^2+n-4\right)}{(n-4)^2} u^{-2}\left|\nabla u\right|_g^2\left(\Delta_g u\right)^2 \\
&- \frac{2(n-1)(n-2)\left(n^3+3n^2-16n+16\right)}{n(n-4)^3} u^{-3}\left|\nabla u\right|_g^4 \Delta_g u \\
&- \frac{8(n-1)^2(n-2)^2}{n(n-4)^3} u^{-4}\left|\nabla u\right|_g^6 - 2(n-1)\,\mathrm{S}_g\left(\nabla \Delta_g u, \nabla u\right)_g \\
&+ \frac{n}{n-4}\,\mathrm{S}_g\, u^{-1}\left(\nabla\left|\nabla u\right|_g^2, \nabla u\right)_g + \frac{2\left(n^3-n^2-3n+4\right)}{n(n-4)}\,\mathrm{S}_g\, u^{-1}\left|\nabla u\right|_g^2 \Delta_g u \\
&- \frac{2(n-1)(n-2)(n+4)}{n(n-4)^2}\,\mathrm{S}_g\, u^{-2}\left|\nabla u\right|_g^4 + \mathrm{S}_g^2\left|\nabla u\right|_g^2\Bigg)
\end{aligned}
\tag{2.48}
$$

and

$$
\begin{aligned}
&\left|\mathrm{E}_{u^{\frac{4}{n-4}} g}\right|_g^2\left(2\left(n^2-2\right)\mathrm{S}_{e^{2u}g}\, u^{-\frac{4}{n-4}} + 4(n-1)\,\mathrm{S}_g\, u^{-\frac{8}{n-4}} + n(n-1)^2\left|\nabla\left(u^{-\frac{4}{n-4}}\right)\right|_g^2\right) \\
&= \frac{8(n-2)^2}{(n-4)^2} u^{-\frac{2n}{n-4}}\Bigg(-\frac{4(n-1)(n-2)\left(n^2-2\right)}{(n-4)^2} u^{-2}\left(\nabla\left|\nabla u\right|_g^2, \nabla u\right)_g \Delta_g u \\
&+ \frac{8(n-1)(n-2)^2}{(n-4)^3} u^{-3}\left|\nabla u\right|_g^2\left(\nabla\left|\nabla u\right|_g^2, \nabla u\right)_g \\
&- \frac{8(n-1)(n-2)}{(n-4)^2} u^{-2}\left|\nabla u\right|_g^2\left|\nabla^2 u\right|_g^2 \\
&+ \frac{4(n-1)\left(n^2-2\right)}{n-4} u^{-1}\left|\nabla^2 u\right|_g^2 \Delta_g u - \frac{4(n-1)\left(n^2-2\right)}{n(n-4)} u^{-1}\left(\Delta_g u\right)^3 \\
&- \frac{8(n-1)(n-2)\left(n^2-3\right)}{n(n-4)^2} u^{-2}\left|\nabla u\right|_g^2\left(\Delta_g u\right)^2 \\
&+ \frac{4(n-1)(n-2)^2\left(n^3-n^2-2n+6\right)}{n(n-4)^3} u^{-3}\left|\nabla u\right|_g^4 \Delta_g u \\
&- \frac{8(n-1)^2(n-2)^3}{n(n-4)^4} u^{-4}\left|\nabla u\right|_g^6 \\
&- \frac{(n-2)\left(n^2+2n-4\right)}{n-4}\,\mathrm{S}_g\, u^{-1}\left(\nabla\left|\nabla u\right|_g^2, \nabla u\right)_g
\end{aligned}
$$

$$+\left(n^2+2n-4\right)\mathrm{S}_g\left|\nabla^2 u\right|_g^2-\frac{n^2+2n-4}{n}\mathrm{S}_g\left(\Delta_g u\right)^2$$
$$-\frac{2\left(n-2\right)\left(n^2+2n-4\right)}{n\left(n-4\right)}\mathrm{S}_g\,u^{-1}\left|\nabla u\right|_g^2\Delta_g u$$
$$+\frac{\left(n-1\right)\left(n-2\right)^2\left(n^2+2n-4\right)}{n\left(n-4\right)^2}\mathrm{S}_g\,u^{-2}\left|\nabla u\right|_g^4\Bigg). \tag{2.49}$$

Putting together (2.42), (2.48) and (2.49), we then obtain

$$\frac{16\left(n-1\right)^2}{\left(n-4\right)^2}\lambda\left(p-\frac{2n}{n-4}\right)\int_M u^{p-\frac{2(n-3)}{n-4}}\left|\nabla u\right|_g^2 dv_g=\int_M \Theta_g^{(2)}\left(u^{-\frac{4}{n-4}}\right)dv_g.$$

We then conclude as in the proof of Theorem 2.1. □

References

[1] R. G. Bettiol, P. Piccione, and Y. Sire, *Nonuniqueness of conformal metrics with constant Q-curvature*, Int. Math. Res. Not. IMRN **9** (2021), 6967–6992.

[2] M.-F. Bidaut-Véron and L. Véron, *Nonlinear elliptic equations on compact Riemannian manifolds and asymptotics of Emden equations*, Invent. Math. **106** (1991), no. 3, 489–539.

[3] T. P. Branson, *Differential operators canonically associated to a conformal structure*, Math. Scand. **57** (1985), no. 2, 293–345.

[4] ———, *Sharp inequalities, the functional determinant, and the complementary series*, Trans. Amer. Math. Soc. **347** (1995), no. 10, 3671–3742.

[5] S. Brendle, *Global existence and convergence for a higher order flow in conformal geometry*, Ann. of Math. (2) **158** (2003), no. 1, 323–343.

[6] J. Case, *The Obata-Vétois argument and its applications*, arXiv:2309.12431 (2023).

[7] S.-Y. A. Chang and P. C. Yang, *Extremal metrics of zeta function determinants on 4-manifolds*, Ann. of Math. (2) **142** (1995), no. 1, 171–212.

[8] Z. Djadli, E. Hebey, and M. Ledoux, *Paneitz-type operators and applications*, Duke Math. J. **104** (2000), no. 1, 129–169.

[9] Z. Djadli and A. Malchiodi, *Existence of conformal metrics with constant Q-curvature*, Ann. of Math. (2) **168** (2008), no. 3, 813–858.

[10] O. Druet and F. Robert, *Bubbling phenomena for fourth-order four-dimensional PDEs with exponential growth*, Proc. Amer. Math. Soc. **134** (2006), no. 3, 897–908.

[11] P. Esposito and F. Robert, *Mountain pass critical points for Paneitz-Branson operators*, Calc. Var. Partial Differential Equations **15** (2002), no. 4, 493–517.

[12] C. Fefferman and C. R. Graham, *The ambient metric*, Annals of Mathematics Studies, vol. 178, Princeton University Press, Princeton, NJ, 2012.

[13] ———, *Juhl's formulae for GJMS operators and Q-curvatures*, J. Amer. Math. Soc. **26** (2013), no. 4, 1191–1207.

[14] B. Gidas and J. Spruck, *Global and local behavior of positive solutions of nonlinear elliptic equations*, Comm. Pure Appl. Math. **34** (1981), no. 4, 525–598.

[15] A. R. Gover, *Laplacian operators and Q-curvature on conformally Einstein manifolds*, Math. Ann. **336** (2006), no. 2, 311–334.

[16] C. R. Graham, R. Jenne, L. J. Mason, and G. A. J. Sparling, *Conformally invariant powers of the Laplacian. I. Existence*, J. London Math. Soc. (2) **46** (1992), no. 3, 557–565.

[17] M. J. Gursky, F. Hang, and Y.-J. Lin, *Riemannian manifolds with positive Yamabe invariant and Paneitz operator*, Int. Math. Res. Not. IMRN **5** (2016), 1348–1367.

[18] M. J. Gursky and A. Malchiodi, *A strong maximum principle for the Paneitz operator and a non-local flow for the Q-curvature*, J. Eur. Math. Soc. (JEMS) **17** (2015), no. 9, 2137–2173.

[19] F. Hang and P. C. Yang, *Sign of Green's function of Paneitz operators and the Q curvature*, Int. Math. Res. Not. IMRN **19** (2015), 9775–9791.

[20] ———, *Q curvature on a class of 3-manifolds*, Comm. Pure Appl. Math. **69** (2016), no. 4, 734–744.

[21] ———, *Q-curvature on a class of manifolds with dimension at least 5*, Comm. Pure Appl. Math. **69** (2016), no. 8, 1452–1491.

[22] E. Hebey and F. Robert, *Compactness and global estimates for the geometric Paneitz equation in high dimensions*, Electron. Res. Announc. Amer. Math. Soc. **10** (2004), 135–141.
[23] A. Juhl, *Explicit formulas for GJMS-operators and Q-curvatures*, Geom. Funct. Anal. **23** (2013), no. 4, 1278–1370.
[24] G. Li, *A compactness theorem on Branson's Q-curvature equation*, Pacific J. Math. **302** (2019), no. 1, 119–179.
[25] J. Li, Y. Li, and P. Liu, *The Q-curvature on a 4-dimensional Riemannian manifold* (M,g) *with* $\int_M Q dV_g = 8\pi^2$, Adv. Math. **231** (2012), no. 3-4, 2194–2223.
[26] Y.Y. Li and J. Xiong, *Compactness of conformal metrics with constant Q-curvature. I*, Adv. Math. **345** (2019), 116–160.
[27] C.-S. Lin, *A classification of solutions of a conformally invariant fourth order equation in* $\mathbf{R}^n$, Comment. Math. Helv. **73** (1998), no. 2, 206–231.
[28] A. Malchiodi, *Compactness of solutions to some geometric fourth-order equations*, J. Reine Angew. Math. **594** (2006), 137–174.
[29] M. Obata, *The conjectures on conformal transformations of Riemannian manifolds*, J. Differential Geometry **6** (1971/72), 247–258.
[30] S. M. Paneitz, *A quartic conformally covariant differential operator for arbitrary pseudo-Riemannian manifolds (summary)*, SIGMA Symmetry Integrability Geom. Methods Appl. **4** (2008), no. 36, 3 pp.
[31] J. Qing and D. Raske, *Compactness for conformal metrics with constant Q curvature on locally conformally flat manifolds*, Calc. Var. Partial Differential Equations **26** (2006), no. 3, 343–356.
[32] ———, *On positive solutions to semilinear conformally invariant equations on locally conformally flat manifolds*, Int. Math. Res. Not. **2006** (2006), no. 94172, 20 pp.
[33] J. A. Viaclovsky, *Conformal geometry, contact geometry, and the calculus of variations*, Duke Math. J. **101** (2000), no. 2, 283–316.
[34] J. Wei and C. Zhao, *Non-compactness of the prescribed Q-curvature problem in large dimensions*, Calc. Var. Partial Differential Equations **46** (2013), no. 1-2, 123–164.
[35] G. Weinstein and L. Zhang, *The profile of bubbling solutions of a class of fourth order geometric equations on 4-manifolds*, J. Funct. Anal. **257** (2009), no. 12, 3895–3929.

Jérôme Vétois, Department of Mathematics and Statistics, McGill University, 805 Sherbrooke Street West, Montreal, Quebec H3A 0B9, Canada
Email address: `jerome.vetois@mcgill.ca`