SIGN-CHANGING BLOW-UP FOR THE YAMABE EQUATION AT THE LOWEST ENERGY LEVEL

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ABSTRACT. We investigate the blow-up behavior of sequences of sign-changing solutions for the Yamabe equation on a Riemannian manifold (M, g) of positive Yamabe type. For each dimension $n \geq 11$, we describe the value of the minimal energy threshold at which blow-up occurs. In dimensions $11 \leq n \leq 24$, where the set of *positive* solutions is known to be compact, we show that the set of *sign-changing* solutions is not compact and that blow-up already occurs at the lowest possible energy level. We prove this result by constructing a smooth, non-locally conformally flat metric on space forms $\mathbb{S}^n/\Gamma$, $\Gamma \neq \{1\}$, whose Yamabe equation admits a family of sign-changing blowing-up solutions. As a counterpart of this result, we also prove a sharp compactness result for sign-changing solutions at the lowest energy level, in small dimensions or under strong geometric assumptions.

1. Introduction

1.1. **Introduction and statements of the main results.** Let (M, g) be a smooth, closed (i.e. compact and without boundary) Riemannian manifold of dimension $n \geq 3$. In this paper, we are interested in the existence of sequences of sign-changing blowing-up solutions $(u_k)_{k\in\mathbb{N}}$ in $C^2(M)$ to the nodal (or sign-changing) Yamabe equation

$$\Delta_g u_k + c_n \operatorname{Scal}_g u_k = |u_k|^{2^*-2} u_k \quad \text{in } M, \tag{1.1}$$

where $\Delta_g := -\operatorname{div}_g \nabla$ is the Laplace–Beltrami operator, $c_n := \frac{n-2}{4(n-1)}$, Scal_g is the scalar curvature of the manifold and $2^* = \frac{2n}{n-2}$ is the critical exponent for the embeddings of the Sobolev space $H^1(M)$ into Lebesgue's spaces. Solutions of (1.1) are in $C^{3,\alpha}(M)$ for $0 < \alpha < \min(2^* - 2, 1)$ by Trudinger's result [56] and standard elliptic theory. We recall that $(u_k)_{k\in\mathbb{N}}$ is said to *blow up* as $k \to \infty$ if $\|u_k\|_{L^\infty(M)} \to \infty$ as $k \to \infty$.

The *Yamabe invariant* of the conformal class $[g]$ is defined as

$$\begin{aligned} Y(M,[g]) &:= \inf_{\hat{g}\in[g]} \left(\operatorname{Vol}_{\hat{g}}(M)^{\frac{2-n}{n}} \int_M \operatorname{Scal}_{\hat{g}}\, dv_{\hat{g}} \right) \\ &= \frac{4(n-1)}{n-2} \cdot \inf_{u\in C^\infty(M)} \frac{\int_M \left(|\nabla u|_g^2 + c_n \operatorname{Scal}_g u^2 \right) dv_g}{\left(\int_M |u|^{2^*} dv_g \right)^{\frac{2}{2^*}}}, \end{aligned}$$

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where $\mathrm{Vol}_{\hat{g}}(M)$ is the volume of $(M,\hat{g})$. Letting $(\mathbb{S}^n, g_{std})$ be the standard unit sphere of dimension $n \geq 3$, by conformal invariance, we also have

$$Y(\mathbb{S}^n,[g_{std}]) = \inf_{u\in C_c^\infty(\mathbb{R}^n)} \frac{\int_{\mathbb{R}^n} |\nabla u|^2 dx}{\left(\int_{\mathbb{R}^n} |u|^{2^*} dx\right)^{\frac{2}{2^*}}},$$

so that $Y(\mathbb{S}^n,[g_{std}])^{-\frac{1}{2}}$ is the optimal constant for the Sobolev inequality in $\mathbb{R}^n$. We say that (M,g) is of *positive Yamabe type* if $Y(M,[g])>0$, i.e. $\triangle_g + c_n \mathrm{Scal}_g$ is coercive. In this case, when (M,g) is not conformally diffeomorphic to the standard sphere $(\mathbb{S}^n, g_{std})$ (which we denote in what follows by $(M,g)\not\approx(\mathbb{S}^n,g_{std})$), we have $Y(M,[g]) < Y(\mathbb{S}^n,[g_{std}])$, and the existence of a positive solution u_0 to the Yamabe equation

$$\Delta_g u_0 + c_n \mathrm{Scal}_g\, u_0 = u_0^{2^*-1} \quad \text{in } M \tag{1.2}$$

attaining the Yamabe invariant $Y(M,[g])$ is known since the work of Trudinger [56], Aubin [3] and Schoen [50].

In this paper, we are interested in determining the value of the minimal energy level at which *sign-changing* blow-up occurs for (1.1). For every $u\in H^1(M)$, we define the energy of u as

$$E(u) := \int_M |u|^{2^*}\, dv_g.$$

We define $I(M,[g])\subset(0,\infty]$ as the set of numbers $E\in(0,+\infty]$ such that (1.1) admits a *blowing-up* sequence of solutions $(u_k)_{k\in\mathbb{N}}$ with $\limsup_{k\to\infty} E(u_k) = E$. We then define

$$E(M,[g]) := \inf I(M,[g]).$$

Similarly, we let $I_+(M,[g])\subset(0,+\infty]$ be the set of numbers $E\in(0,+\infty]$ such that (1.1) admits a *blowing-up* sequence of *positive* solutions $(u_k)_{k\in\mathbb{N}}$ satisfying $\limsup_{k\to\infty} E(u_k) = E$, and we define

$$E_+(M,[g]) := \inf I_+(M,[g]).$$

The value $+\infty$ is allowed in the definitions of $E(M,[g])$ and $E_+(M,[g])$ and corresponds to sequences of solutions of (1.1) with diverging energies. When $I(M,[g])$ (resp. $I_+(M,[g])$) is empty, it means that (1.1) does not admit any blowing-up solutions (resp. positive blowing-up solutions), and thus the set of solutions (resp. positive solutions) of (1.1) is compact in $C^2(M)$ by standard elliptic theory. In this case, we let $E(M,[g]) := -\infty$ (resp $E_+(M,[g]) := -\infty$). If $I(M,[g])\neq\emptyset$ and $(u_k)_{k\in\mathbb{N}}$ is a sequence of solutions of (1.1) satisfying $\limsup_{k\to\infty}E(u_k) < E(M,[g])$, then by definition $(u_k)_{k\in\mathbb{N}}$ does not blow-up, and is thus precompact in $C^2(M)$.

As long as *positive* solutions are considered, $E_+(M,[g])$ is well understood: in the proof of the compactness of positive solutions of the Yamabe equation, assuming the validity of the Positive Mass Theorem when $n\geq 8$, Khuri–Marques–Schoen [25] (with previous contributions by Schoen [51, 52], Li–Zhu [29], Druet [19], Marques [30] and Li–Zhang [27, 28]) proved that

$$E_+(M,[g]) = -\infty \text{ when } \begin{cases} 3\leq n\leq 24 \text{ and } (M,g)\not\approx(\mathbb{S}^n,g_{std}), \\ (M,g) \text{ is locally conformally flat or} \\ n\geq 6 \text{ and } \displaystyle\sum_{k=0}^{[\frac{n-6}{2}]} |\nabla^k \mathrm{Weyl}_g(x)|^2 > 0 \ \ \forall x\in M, \end{cases} \tag{1.3}$$

where Weyl_g is the Weyl curvature tensor of the manifold. On the other side, Brendle [7] and Brendle–Marques [8] proved that there exists a non-locally conformally flat metric g on $\mathbb{S}^n$ such that

$$E_+(\mathbb{S}^n, [g]) = Y(\mathbb{S}^n, [g_{std}])^{\frac{n}{2}} \quad \text{when } n \geq 25. \tag{1.4}$$

The blow-up behavior of *sign-changing* solutions of (1.1) is far less understood. A simple application of Struwe's celebrated H^1-compactness result [53] (see Proposition 5.1 below for a proof) shows that if (M, g) is of positive Yamabe type, then any *sign-changing* blowing-up sequence $(u_k)_{k\in\mathbb{N}}$ of solutions of (1.1) satisfies

$$\liminf_{k\to\infty} E(u_k) \geq Y(\mathbb{S}^n, [g_{std}])^{\frac{n}{2}} + Y(M, [g])^{\frac{n}{2}}. \tag{1.5}$$

As a consequence, if g is the Brendle [7] and Brendle–Marques [8] metric, then, by (1.4), we obtain

$$E(\mathbb{S}^n, [g]) = E_+(\mathbb{S}^n, [g]) = Y(\mathbb{S}^n, [g_{std}])^{\frac{n}{2}} \quad \text{when } n \geq 25.$$

In dimensions $n \geq 25$, blow-up for (1.1) thus occurs at the lowest energy level for positive solutions. Our aim in this paper is to investigate the situation for *sign-changing* solutions and in particular, the value of $E(M, [g])$ in dimensions $n \leq 24$, where the set of *positive* solutions of (1.1) is compact, i.e. $E_+(M, [g]) = -\infty$. As follows from (1.5), a lower bound on $E(M, [g])$ is given by $Y(\mathbb{S}^n, [g_{std}])^{\frac{n}{2}} + Y(M, [g])^{\frac{n}{2}}$. Our main result shows that this lower bound is attained in dimensions $11 \leq n \leq 24$:

Theorem 1.1. *Assume that $11 \leq n \leq 24$ and let Γ be a finite subgroup of isometries of $(\mathbb{S}^n, g_0)$, $\Gamma \neq \{Id\}$, acting freely and smoothly on $\mathbb{S}^n$. There exists a smooth, non-locally conformally flat Riemannian metric g on $\mathbb{S}^n/\Gamma$ of positive Yamabe type and such that*

$$E(M, [g]) = Y(\mathbb{S}^n, [g_{std}])^{\frac{n}{2}} + Y(\mathbb{S}^n/\Gamma, [g])^{\frac{n}{2}}.$$

The metric g in Theorem 1.1 is not the quotient metric g_Γ but can be chosen arbitrarily close to it. Theorem 1.1 is a special case of a more general result, Theorem 2.2 below, which holds true in any dimension $n \geq 11$ and for a larger class of manifolds than the spherical space forms $\mathbb{S}^n/\Gamma$. We refer to Section 2 for more details on this regard.

By (1.3), the set of *positive* solutions of (1.1) on $(\mathbb{S}^n/\Gamma, g)$, where g is given by Theorem 1.1, is compact in $C^2(M)$. As Theorem 1.1 shows, however, in this case, the set of *sign-changing* solutions is not compact and blow-up already occurs at the minimal energy level. This phenomenon of loss of compactness for sign-changing solutions in situations where the set of positive solutions is compact was recently highlighted in Premoselli–Vétois [42] for critical Schrödinger-type equations in M. We prove Theorem 1.1 by constructing a sign-changing blowing-up sequence $(u_k)_{k\in\mathbb{N}}$ of solutions of (1.1) such that

$$\lim_{k\to\infty} E(u_k) = Y(\mathbb{S}^n, [g_{std}])^{\frac{n}{2}} + Y(M, [g])^{\frac{n}{2}}. \tag{1.6}$$

As a counterpart of Theorem 1.1, we also prove the following result, which provides a lower bound for $E(M, [g])$ in smaller dimensions or under strong geometric assumptions:

Theorem 1.2. *Let (M, g) be a smooth, closed Riemannian manifold of dimension $n \geq 3$ and positive Yamabe type which is not conformally diffeomorphic to the standard sphere $(\mathbb{S}^n, [g_{std}])$. Assume that one of the following conditions is satisfied:*

- (M,g) *is locally conformally flat,*
- $n \le 9$,
- $n = 10$ *and* $u_0 \neq \frac{5}{567}|\operatorname{Weyl}_g|_g^2$ *for all points in* M *and all solutions* u_0 *of* (1.2) *attaining* $Y(M,[g])$, *or*
- $n \ge 11$ *and* $\operatorname{Weyl}_g \neq 0$ *for all points in* M.

Then

$$E(M,[g]) > Y(\mathbb{S}^n,[g_{std}])^{\frac{n}{2}} + Y(M,[g])^{\frac{n}{2}}.$$

Under the assumptions of Theorem 1.2 and by (1.3), the set of *positive* solutions of (1.1) is compact in $C^2(M)$, and thus blow-up for (1.1) can only occur for *sign-changing* solutions. If sign-changing blow-up occurs it happens at a possibly infinite energy level that is, at least, strictly higher than the minimal one given by (1.5). In particular, if the assumptions of Theorem 1.2 are satisfied, then there exists a constant $\varepsilon_0 > 0$ such that the set of all solutions $u \in C^2(M)$ of (1.1) satisfying

$$E(u) \le Y(M,[g])^{\frac{n}{2}} + Y(\mathbb{S}^n,[g_{std}])^{\frac{n}{2}} + \varepsilon_0$$

is compact in $C^2(M)$. The contrapositive of Theorem 1.2 provides necessary conditions for sign-changing blowing-up solutions to exist at the minimal energy level. In dimensions $n \ge 11$, this condition is that Weyl_g vanishes at some (but not all) points in M. This is consistent with Theorem 1.1 since the sequence $(u_k)_{k\in\mathbb{N}}$ that we construct to prove Theorem 1.1 blows up at a point where Weyl_g vanishes. Hence, Theorems 1.1 and 1.2 are sharp in every dimension $11 \le n \le 24$. The condition arising when $n = 10$ is purely analytical (see (5.44) below). The exact value of $E(M,[g])$ when $3 \le n \le 10$ is not yet known: as Theorem 1.2 shows, at least when $n \le 9$, it is strictly larger than $Y(M,[g])^{\frac{n}{2}} + Y(\mathbb{S}^n,[g_{std}])^{\frac{n}{2}}$. Computing $E(M,[g])$ when $3 \le n \le 10$ will be the focus of forthcoming work.

We conclude this subsection by mentioning an important additional motivation for investigating the value of $E(M,[g])$ and, more generally, (non-)compactness issues for sign-changing solutions of (1.1). Let g be a Riemannian metric in M of positive Yamabe type. For $k \in \mathbb{N}$ and $\tilde{g} \in [g]$, we denote by $\lambda_k(\tilde{g})$ the k-th eigenvalue (counted with multiplicity) of the conformal Laplacian $\triangle_{\tilde{g}} + c_n \operatorname{Scal}_{\tilde{g}}$ in M. Ammann–Humbert introduced in [2] the so-called k-th Yamabe invariant:

$$\mu_k(M,[g]) = \inf_{\tilde{g}\in[g]} \lambda_k(\tilde{g}) \operatorname{Vol}_{\hat{g}}(M)^{\frac{2}{n}}.$$

In the case where $k = 2$, test functions computations show that

$$\mu_2(M,[g])^{\frac{n}{2}} \le Y(M,[g])^{\frac{n}{2}} + Y(\mathbb{S}^n,[g_{std}])^{\frac{n}{2}} \tag{1.7}$$

holds. For $k \ge 2$, extremal metrics attaining $\mu_k(M,[g])$, when they exist, are not smooth in general. When (M,g) is of positive Yamabe type, Amman–Humbert [2] established the existence of extremal metrics attaining $\mu_2(M,[g])$ provided (M,g) is not locally conformally flat and $n \ge 11$. Moreover, Amman–Humbert [2] obtained that if $\mu_2(M,[g])$ is attained by a generalized metric $\tilde{g} = u^{\frac{4}{n-2}} g$ with $u \ge 0$ and $\|u\|_{L^{2^*}(M)} = 1$, then there exists a generalized eigenvector $\tilde{w}$ associated to $\mu_2(M,[g])$ such that $u = |\tilde{w}|$ and $w := \mu_2(M,[g])^{\frac{n-2}{4}} \tilde{w}$ is a sign-changing solution of (1.1) satisfying $E(w) = \mu_2(M,[g])^{\frac{n}{2}}$. With (1.7), this shows that extremal metrics for $\mu_2(M,[g])$ give rise to sign-changing solutions of (1.1) whose energies lie below the minimal blow-up level given by (1.5). Theorem 1.2 thus provides a compactness result for a range of energy levels including $\mu_2(M,[g])$. Such a result is generally perceived as a strong indication that $\mu_2(M,[g])$ is attained (at least for analogous problems in the two-dimensional case, see for example Matthiesen–Siffert [31] or Pétrides [37]): Theorem 1.2 can therefore also be seen as a first step in a more systematic investigation of $\mu_k(M,[g])$.

1.2. **Review of the literature and outline of the paper.** Existence results for the nodal Yamabe equation (1.1) have been the subject of several work in the last decades. On the standard sphere $(\mathbb{S}^n, g_{std})$, existence results of large-energy sign-changing solutions of (1.1) are in Ding [18], del Pino–Musso–Pacard–Pistoia [15, 16], Musso–Wei [35], Medina–Musso–Wei [33] and Medina–Musso [32]; other existence results at lower energy levels are in Clapp [11] and Fernandez–Petean [22]. For more general manifolds, existence and multiplicity results of sign-changing solutions of (1.1) have been obtained by Ammann–Humbert [2], Vétois [57], Clapp–Fernández [12], Clapp–Pistoia–Tavares [13] and Gursky–Pérez-Ayala [23]. Multiplicity results of sign-changing solutions of Yamabe–Schrödinger-type equations with more general potential functions can also be found in Vétois [57] and Clapp–Fernández [12].

Theorem 1.1 is both a non-compactness and an existence result: it shows in particular the existence of infinitely many solutions of (1.1) on $(\mathbb{S}^n/\Gamma, g)$. Theorem 1.2, on the contrary, is a compactness result for (1.1) below the energy level $Y(\mathbb{S}^n,[g_{std}])^{\frac{n}{2}} + Y(M,[g])^{\frac{n}{2}}$. Compactness and non-compactness results for sign-changing solutions of Yamabe–Schrödinger-type equations have been obtained by Vétois [57] and recently by Premoselli–Vétois [41, 42] (see also Robert–Vétois [46, 48], Pistoia–Vétois [38] and Deng–Musso–Wei [17] for existence results of sign-changing blowing-up solutions to equations of type (1.1) with asymptotically critical nonlinearities). A general pointwise description of finite-energy blowing-up sequences of solutions of such equations, including the geometric case of (1.1), has recently been obtained by Premoselli [39]. To the best of the authors' knowledge, Theorem 1.1 is the first constructive result of sign-changing *blowing-up* solutions for the geometric equation (1.1) on a different manifold than the standard sphere.

The paper is organised as follows. In Section 2, we state Theorem 2.2 which is a generalization of Theorem 1.1 in dimensions $n \geq 11$. We then prove Theorem 2.2 in Sections 3 and 4. The proof relies on a constructive Lyapunov–Schmidt reduction method. Our approach is inspired from the constructions on the sphere by Brendle [7] and Brendle–Marques [8] (see also Ambrosetti–Malchiodi [1] and Berti–Malchioldi [5]). In Section 3, we perform the Lyapunov–Schmidt reduction and construct a blowing-up sequence of approximate solutions of (1.1) of the form $u_k = u_0 - B_k+$ lower order terms, where $u_0 > 0$ solves (1.1) and B_k is a bubbling profile modeled on the positive standard bubble (see (3.18) below). In Section 4, we reduce the proof of Theorem 2.2 to finding a critical point of an energy function in $\mathbb{R}^{n+1}$ (see (4.3) below). The main difference with the constructions of Brendle [7] and Brendle–Marques [8] for positive solutions is that the critical point of F that we find is of saddle-type: this allows us to conclude up to dimension 11 but in turn forces us to work with greater precision and expand the reduced energy to the fourth order (see (4.23) below). Finally, we prove Theorem 1.2 in Section 5 by using a Pohozaev-type identity together with the pointwise blow-up description for sign-changing solutions of (1.1) recently obtained by Premoselli [39], which we refine here by using an approach based on iterated estimates, in the spirit of the method introduced by Chen–Lin [10] in the case of positive solutions (see also Marques [30] for applications to the Yamabe equation).

2. Y-non-degenerate metrics and a refined version of Theorem 1.1

Let (M, g) be a smooth, closed Riemannian manifold of dimension $n \geq 3$ and positive Yamabe type. By the resolution of the Yamabe problem (see Trudinger [56], Aubin [3] and Schoen [50]) there exists a smooth positive function u_0 in M that attains $Y(M,[g])$ and solves (1.2). In particular $\int_M u_0^{2^*} dv_g = Y(M,[g])^{\frac{n}{2}}$. Let $\varphi \in C^\infty(M), \varphi > 0$. By the conformal invariance property of the conformal Laplacian, $\hat{u}_0 = u_0/\varphi$ solves

$$\triangle_{g_0} \hat{u}_0 + c_n \operatorname{Scal}_{g_0} \hat{u}_0 = \hat{u}_0^{2^*-1} \quad \text{in } M,$$

where we have let $g_0 := \varphi^{2^*-2} g$. Assume that one of the positive minimizers u_0 achieving $Y(M,[g])$ is non-degenerate as a solution of the Yamabe equation. This means that

$$\operatorname{Ker}\big(\triangle_g + c_n \operatorname{Scal}_g - (2^*-1) u_0^{2^*-2}\big) = \{0\}, \tag{2.1}$$

where this kernel is regarded as a subset of $H^1(M)$. A simple application of the Implicit Function Theorem shows that for any metric g_* close enough to g in some C^p topology, $p \geq 3$, there exists a unique positive $u_* \in C^2(M)$ close to u_0 in $C^2(M)$ that solves

$$\triangle_{g_*} u_* + c_n \operatorname{Scal}_{g_*} u_* = u_*^{2^*-1} \quad \text{in } M \tag{2.2}$$

and that is also non-degenerate. Generically with respect to perturbations of the metric, at least in dimensions $n \leq 24$ (see Theorem 10.3 of Khuri–Marques–Schoen [25]), all positive solutions of (1.2) are non-degenerate. In the locally conformally flat case, concrete examples of situations where u_0 is non-degenerate are given by $u_0 := ((n-2)/2)^{\frac{n-2}{2}}$ on $\mathbb{S}^1(r) \times \mathbb{S}^{n-1}$, where $\mathbb{S}^1(r)$ is the circle of radius $r \in (0,\infty) \setminus \{i/\sqrt{n-2} : i \in \mathbb{N}\}$ and $\mathbb{S}^{n-1}$ is the unit $(n-1)$-sphere, both equipped with their standard metrics (see Proposition 3.4 of Robert–Vétois [46]).

We introduce the following definition:

Definition 2.1. *Let (M, g) be a smooth, closed Riemannian manifold of dimension $n \geq 3$ and positive Yamabe type. We say that g is Y-non-degenerate if*

- *one of the positive minimizers u_0 achieving $Y(M,[g])$ is non-degenerate*
- *and there exists a constant $\varepsilon_g > 0$ such that for any metric g_* satisfying $\|g_* - g\|_{C^3(M)} \leq \varepsilon_g$, the unique function u_* close to u_0 in $C^2(M)$ satisfying* (2.2) *still attains $Y(M,[g_*])$, i.e. satisfies*

$$\int_M u_*^{2^*} dv_{g_*} = Y(M,[g_*])^{\frac{n}{2}}.$$

By the conformal invariance of $\triangle_g + c_n \operatorname{Scal}_g$, if g is Y-non-degenerate in M then any metric in the conformal class of g is still Y-non-degenerate. This notion allows us to state a generalization of Theorem 1.1:

Theorem 2.2. *Let $(M, \hat{g})$ be a smooth, closed, locally conformally flat Riemannian manifold of dimension $n \geq 11$ which is* Y-non-degenerate *in the sense of Definition 2.1. Then there exist a smooth, non-locally conformally flat metric g in M and a* *sequence of blowing-up sign-changing solutions* $(u_k)_{k\in\mathbb{N}}$, $u_k \in C^{3,\alpha}(M)$ *for some* $0<\alpha<1$, *to the nodal Yamabe equation for* g*:*

$$\triangle_g u_k + \frac{n-2}{4(n-1)}\,\mathrm{Scal}_g\, u_k = |u_k|^{2^*-2}u_k \quad \text{in } M,$$

which satisfies

$$\int_M |u_k|^{2^*}dv_g \nearrow Y(M,[g])^{\frac{n}{2}} + Y(\mathbb{S}^n,[g_{std}])^{\frac{n}{2}}$$

as $k\to\infty$.

The metric g in Theorem 2.2 can be chosen arbitrarily close to the original metric $\hat{g}$ in $C^p(M)$ for any $p\geq 3$. Theorem 2.2 is proven in Sections 3 and 4. In the rest of this section, we prove that spherical space forms and their locally conformally flat perturbations are Y-non-degenerate and that Theorem 1.1 follows from Theorem 2.2. The first result is as follows:

Proposition 2.3. *Let* $n\geq 3$ *and let* Γ *be a finite subgroup of isometries of* $(\mathbb{S}^n, g_{std})$, $\Gamma\neq\{Id\}$ *acting freely and smoothly on* $\mathbb{S}^n$*. The quotient manifold* $M_\Gamma := \mathbb{S}^n/\Gamma$ *endowed with the quotient metric* g_Γ *is locally conformally flat and* Y*-non-degenerate in the sense of Definition 2.1.*

Proof. By definition, (M_Γ, g_Γ) is locally isometric to $(\mathbb{S}^n, g_{std})$, hence it is locally conformally flat, has constant scalar curvature equal to $n(n-1)$ and is thus of positive Yamabe type.

The constant function $u_0 \equiv (n(n-2)/4)^{\frac{n-2}{4}}$ is a solution of the Yamabe equation on $(\mathbb{S}^n, g_{std})$, so u_0 descends to M_Γ as a constant positive solution, that we still denote by u_0, of the Yamabe equation

$$\triangle_{g_\Gamma} u_0 + \frac{n(n-2)}{4}u_0 = u_0^{2^*-1} \quad \text{in } M_\Gamma. \tag{2.3}$$

We claim that the linearized operator at u_0 in M_Γ, which is given by $L = \triangle_{g_\Gamma} - n$, has zero kernel. Indeed, if $L\varphi = 0$ for some $\varphi\in H^1(M_\Gamma)$ then φ lifts to $\mathbb{S}^n$ as a function $\tilde\varphi \in H^1(\mathbb{S}^n)$ which is Γ-invariant and satisfies $\triangle_{g_{std}}\tilde\varphi - n\tilde\varphi = 0$. But the kernel of $\triangle_{g_{std}} - n$ in $\mathbb{S}^n$ consists of the restrictions of the coordinate functions $(x_i)_{0\leq i\leq n}$ of $\mathbb{R}^{n+1}$ to $\mathbb{S}^n$, that are not Γ-invariant. Hence $\varphi\equiv 0$ and u_0 is a non-degenerate solution of the Yamabe equation in M_Γ. This proves the first point in Definition 2.1.

We now prove the second point. Since (M_Γ, g_Γ) is Einstein, a celebrated theorem of Obata [36] shows that u_0 is the only positive solution of (2.3), so in particular u_0 satisfies

$$\int_{M_\Gamma} u_0^{2^*}dv_{g_\Gamma} = Y(M_\Gamma,[g_\Gamma])^{\frac{n}{2}} = \frac{1}{|\Gamma|}Y(\mathbb{S}^n,[g_{std}])^{\frac{n}{2}}.$$

By the Implicit Function Theorem there is $\eta_0>0$ such that, for any ε small enough and any metric g on M_Γ such that $\|g-g_\Gamma\|_{C^3(M_\Gamma)}\leq\varepsilon$, there is a unique positive function $u_g\in C^2(M_\Gamma)\cap B_{H^1(M_\Gamma)}(u_0,\eta_0)$ satisfying

$$\triangle_g u_g + c_n\,\mathrm{Scal}_g\, u_g = u_g^{2^*-1} \quad \text{in } M. \tag{2.4}$$

We also have $\|u_g-u_0\|_{C^2(M_\Gamma)}\leq C\varepsilon$ for some $C>0$ independent of ε, so u_g is still non-degenerate for ε small enough. We claim that for ε small enough, we again have

$$\int_{M_\Gamma} u_g^{2^*}dv_g = Y(M_\Gamma,[g])^{\frac{n}{2}}$$

for any g with $\|g - g_\Gamma\|_{C^3(M_\Gamma)} \leq \varepsilon$. Assume by contradiction that, for a sequence of real numbers $(\varepsilon_j)_{j\in\mathbb{N}}$ such that $\varepsilon_j \to 0$ as $j \to \infty$ and a sequence of metrics $(g_j)_{j\in\mathbb{N}}$ with $\|g_j - g_\Gamma\|_{C^3(M_\Gamma)} \leq \varepsilon_j$, the latter equality does not hold. Then

$$\int_{M_\Gamma} u_j^{2^*} dv_{g_j} > Y(M_\Gamma, [g_j])^{\frac{n}{2}} \quad \forall j \in \mathbb{N},$$

where we have let $u_j := u_{g_j}$. Since $g_j \to g_\Gamma$ in $C^3(M_\Gamma)$, we have $Y(M_\Gamma, [g_j]) \to Y(M_\Gamma, [g_\Gamma]) < Y(\mathbb{S}^n, [g_{std}])$ and thus, by the resolution of the Yamabe problem, there exists a sequence $(v_j)_{j\in\mathbb{N}}$ of positive solutions of (2.4) with $g = g_j$ satisfying

$$\int_{M_\Gamma} v_j^{2^*} dv_{g_j} = Y(M_\Gamma, [g_j])^{\frac{n}{2}} \leq \frac{1}{|\Gamma|} Y(\mathbb{S}^n, [g_{std}])^{\frac{n}{2}} + \mathrm{o}(1) \tag{2.5}$$

as $j \to \infty$. In particular, $u_j \neq v_j$ for all $j \in \mathbb{N}$. Struwe's H^1-compactness result [53] together with (2.5) show that the sequence $(v_j)_{j\in\mathbb{N}}$ strongly converges in $H^1(M_\Gamma)$ as $j \to \infty$ towards a positive solution of (2.3). By the uniqueness result of Obata [36], $v_j \to u_0$, and hence the local uniqueness shows that $v_j = u_j$ for large j, a contradiction. □

Proof of Theorem 1.1. Thanks to Proposition 2.3, we can apply Theorem 2.2 to $(M, \hat{g}) = (M_\Gamma, g_\Gamma)$ for some finite subgroup $\Gamma \neq \{Id\}$ of isometries of $(\mathbb{S}^n, g_{std})$ acting freely and smoothly on $\mathbb{S}^n$. We can then let g be as in the statement of Theorem 2.2. By definition of $E(M, [g])$, Theorem 2.2 then gives

$$E(M, [g]) \leq Y(M, [g])^{\frac{n}{2}} + Y(\mathbb{S}^n, [g_{std}])^{\frac{n}{2}}.$$

The other inequality follows from Proposition 5.1 below, which concludes the proof of Theorem 1.1. □

The arguments developed in the proof of Proposition 2.3 similarly show that if a metric g in M attains $Y(M, [g])$ at a unique positive minimizer u_0, that is also non-degenerate in the sense of (2.1), then (M, g) is Y-non-degenerate. With this observation, we can prove the following result that provides additional Y-non-degenerate examples to which Theorem 2.2 applies:

Proposition 2.4. *Let $n \geq 3$, let Γ be a finite subgroup of isometries of $(\mathbb{S}^n, g_{std})$, $\Gamma \neq \{Id\}$ acting freely and smoothly on $\mathbb{S}^n$ and let g be a locally conformally flat metric in M_Γ, $\Gamma \neq \{Id\}$, that is close to g_Γ in $C^p(M_\Gamma)$ for some p large enough. Then g is Y-non-degenerate.*

Proof. This is a consequence of the uniqueness result of de Lima–Piccione–Zedda [14] (Theorem 5). We provide some additional details here since the result of [14] is not stated in this way. The analysis in [14] leading to Theorem 5 applies as long as sequences of Yamabe metrics close to g_Γ can be made to converge strongly up to a subsequence. This is the case for sequences of locally conformally flat metrics $(g_k)_{k\in\mathbb{N}}$ on M_Γ close to g_Γ in $C^p(M_\Gamma)$ for some p large enough. Local arguments indeed show that any metric g_k has positive Riemannian mass, with a positive uniform bound from below, at every point of M_Γ. This is a purely local argument that works for any dimension $n \geq 3$ and does not rely on the positive mass theorem. The convergence of $(g_k)_{k\in\mathbb{N}}$ up to a subsequence then follows from the arguments of Schoen [52] and Li–Zhu [29]. Hence for any metric g on M_Γ with $|\Gamma| \geq 2$ that is locally conformally flat and C^p-close to g_Γ for p large enough, the equation (2.4) has a unique positive solution u_g. This solution u_g remains non-degenerate for g close enough to g_Γ. $\square$

Remark 2.5. *A natural question connected with Definition 2.1, and with the observation before Proposition 2.4, is whether it is possible that the Yamabe equation admits multiple minimizers. This is indeed the case. Examples of such situations can be obtained by considering manifolds with a nontrivial isometry group. In particular, Schoen* [52] *gave a detailed study of the multiplicity of positive solutions to the Yamabe equation in the case of the product manifold* $\mathbb{S}^1(r) \times \mathbb{S}^{n-1}$ *with* $r > 0$. *In this case, if* r *is chosen large enough, then there exists a family of distinct (degenerate) minimizers parametrized by* $\mathbb{S}^1$. *This example can be extended to more general manifolds with different isometry groups (see Hebey–Vaugon* [24]*).*

3. Proof of Theorem 2.2 – Part 1: A Lyapunov–Schmidt reduction

3.1. **The geometric setting.** In this section and the next, we prove Theorem 2.2. Throughout the paper, we denote by δ_0 the Euclidean metric in $\mathbb{R}^n$.

Let $(M, \hat{g})$ be a smooth, closed, locally conformally flat Riemannian manifold of positive Yamabe type that is Y-non-degenerate in the sense of Definition 2.1. In this section and the next, we always assume that $n := \dim(M) \geq 11$. Fix $x_0 \in M$ once and for all, and let $\delta > 0$ and $\varphi \in C^\infty(B_{\hat{g}}(x_0, 8\delta))$, $\varphi > 0$, be such that $g_0 = \varphi^{2^*-2}\hat{g}$ is flat in $B_{\hat{g}}(x_0, 8\delta)$. By decreasing δ if necessary and picking a local chart Φ that sends x_0 to 0, we can assume that $B_{\hat{g}}(x_0, 6\delta)$ contains $\Phi^{-1}(B(0, 4\delta))$ and that $\Phi_* g_0$ is the Euclidean metric in $B(0, 4\delta) \subset \mathbb{R}^n$, where $B(0, 4\delta)$ is an Euclidean ball.

For any $k \in \mathbb{N}$, we let $y_k = (\delta/k, 0, \ldots, 0) \in \mathbb{R}^n$ and we let $(r_k)_{k\in\mathbb{N}}$ be a decreasing sequence of positive numbers converging to 0 such that $r_0 \leq \delta$ and $4r_k \leq |y_k - y_{k+1}|$ for all $k \in \mathbb{N}$. Any two Euclidean balls $B(y_k, 2r_k)$ and $B(y_\ell, 2r_\ell)$ are thus disjoint for $k \neq \ell$. Let $(\varepsilon_k)_{k\in\mathbb{N}}$ be a sequence of positive numbers converging to 0 such that $\varepsilon_k = o(r_k^p)$ for any $p \geq 1$. Define, for any $k \in \mathbb{N}$,

$$\mu_k = \varepsilon_k^{\frac{4}{n-10}}. \tag{3.1}$$

Since $n \geq 11$ and $\varepsilon_k = o(r_k^p)$, we have $\mu_k = o(r_k^p)$ as $k \to \infty$ for any $p \geq 1$. Let h be a smooth, symmetric bilinear form in $\mathbb{R}^n$ that satisfies

$$\operatorname{tr} h(x) = 0, \quad \operatorname{div} h(x)_i = 0 \quad \text{and} \quad \sum_{j=1}^n x_j h_{ij}(x) = 0 \tag{3.2}$$

for all $x \in \mathbb{R}^n$ and $1 \leq i \leq n$, where we have let $\operatorname{tr} h := \sum_{j=1}^n h_{jj}$ and $\operatorname{div} h_i := \sum_{j=1}^n \partial_j h_{ji}$. These are respectively the trace and the divergence of h with respect to the Euclidean metric δ_0, but the subscript δ_0 will be omitted for clarity. We assume that for any $1 \leq i, j \leq n$, $x \mapsto h_{ij}(x)$ is a homogeneous polynomial of second-order in $\mathbb{R}^n$. Examples of such h satisfying (3.2) are given in (4.2) below. Let $\chi \in C_c^\infty(\mathbb{R})$ be such that $\chi \equiv 1$ on $[0,1]$ and $\chi \equiv 0$ on $\mathbb{R}\backslash[0,2]$. We define a new metric in $B(0, 4\delta)$ by

$$\tilde{g}(x) := \exp\left(\sum_{k=1}^\infty \varepsilon_k \chi\left(\frac{|x - y_k|}{r_k}\right) h\left(x - y_k\right)\right). \tag{3.3}$$

We assume in addition that $\sum_{k\in\mathbb{N}} \varepsilon_k r_k^{-p} < +\infty$ for all $p \geq 0$. Since the components of h are homogeneous polynomials of second order, $\tilde{g}$ is thus a smooth metric in

$B(0,4\delta)$, is flat in $B(0,4\delta)\backslash B(0,3\delta)$ and satisfies

$$\tilde{g}(x) = \exp\left(\varepsilon_k \chi\left(\frac{|x-y_k|}{r_k}\right) h\left(x-y_k\right)\right) \tag{3.4}$$

in $B(y_k, 2r_k)$ for any $k \in \mathbb{N}$. When extended and pulled back to M, $\Phi^* \tilde{g}$ defines a metric in $B_{\hat{g}}(x_0, 8\delta)$, equal to g_0 in $B_{\hat{g}}(x_0,8\delta)\backslash B_{\hat{g}}(x_0,6\delta)$. The metric $g = \varphi^{2-2^*}\Phi^*\tilde{g}$ thus defines a smooth metric in $B_{\hat{g}}(x_0,8\delta)$, equal to the original metric $\hat{g}$ in $B_{\hat{g}}(x_0,8\delta)\backslash B_{\hat{g}}(x_0,6\delta)$, that we extend to be equal to $\hat{g}$ on $M\backslash B_{\hat{g}}(x_0,8\delta)$. We still call this new metric g. We now define $\breve{g} = \varphi^{2^*-2} g$, which is a smooth metric in M such that $\Phi_* \breve{g} = \tilde{g}$ as in (3.3) in $B(0,4\delta)$. By (3.2), we have $\det \tilde{g} \equiv 1$ in $B(0,4\delta)$. Note that $\tilde{g}$ can be chosen to be arbitrarily close to the Euclidean metric δ_0 in $C^p(B(0,4\delta))$ for any $p \geq 3$ by assuming that $\sum_{k\in\mathbb{N}} \varepsilon_k r_k^{-p}$ is small enough. Hence g can be chosen arbitrarily close to the initial metric $\hat{g}$ in $C^p(M)$ for any $p \geq 3$.

Since $\hat{g}$ is Y-non-degenerate, so is $g_0 = \varphi^{2^*-2}\hat{g}$, and we can let $\hat{u}_0$ be a non-degenerate positive solution of

$$\triangle_{g_0} \hat{u}_0 + c_n \operatorname{Scal}_{g_0} \hat{u}_0 = \hat{u}_0^{2^*-1} \quad \text{in } M,$$

that also satisfies

$$\int_M \hat{u}_0^{2^*} dv_{g_0} = Y(M,[g_0])^{\frac{n}{2}} = Y(M,[\hat{g}])^{\frac{n}{2}}.$$

Definition 2.1 then yields the existence of a unique positive function $\breve{u}_0 \in C^2(M)$ solving

$$\triangle_{\breve{g}} \breve{u}_0 + c_n \operatorname{Scal}_{\breve{g}} \breve{u}_0 = \breve{u}_0^{2^*-1} \quad \text{in } M, \tag{3.5}$$

that is still non-degenerate in the sense of (2.1) and satisfies

$$\int_M \breve{u}_0^{2^*} dv_{\breve{g}} = Y(M,[\breve{g}])^{\frac{n}{2}} = Y(M,[g])^{\frac{n}{2}}.$$

In the rest of this section and in the following one, we construct, when $n \geq 11$, a sequence of sign-changing solutions $(u_k)_{k\in\mathbb{N}}$ of class $C^{3,\alpha}(M)$, $0 < \alpha \leq 2^* - 2$, to

$$\triangle_{\breve{g}} u_k + c_n \operatorname{Scal}_{\breve{g}} u_k = |u_k|^{2^*-2} u_k \quad \text{in } M, \tag{3.6}$$

where $c_n := \frac{n-2}{4(n-1)}$, that satisfies

$$\int_M |u_k|^{2^*} dv_{\breve{g}} \nearrow Y(M,[\breve{g}])^{\frac{n}{2}} + Y(\mathbb{S}^n,[g_{std}])^{\frac{n}{2}}.$$

Since $\breve{g} = \varphi^{2^*-2} g$, and by the conformal invariance of the conformal Laplacian, by replacing u_k with φu_k, this will prove Theorem 2.2.

3.2. **The *ansatz* of the construction.** Define $D^{1,2}(\mathbb{R}^n)$ to be the completion of $C_c^\infty(\mathbb{R}^n)$ for the norm $u \mapsto \|\nabla u\|_{L^2(\mathbb{R}^n)}$, that we endow with the associated scalar product. We fix $A > 0$ to be chosen later, and for $(t,z) \in [1/A, A] \times B(0,1)$, we let

$$\mu_k(t) := \mu_k t \quad \text{and} \quad \xi_k(z) := y_k + \mu_k z, \tag{3.7}$$

where y_k is as in the previous subsection, and for $x \in \mathbb{R}^n$,

$$B_{k,t,z}(x) = \frac{\mu_k(t)^{\frac{n-2}{2}}}{\left(\mu_k(t)^2 + \frac{|x-\xi_k(z)|^2}{n(n-2)}\right)^{\frac{n-2}{2}}}.$$

For any $(t,z) \in [1/A, A] \times B(0,1)$, $B_{k,t,z}$ solves

$$\triangle_{\delta_0} B_{k,t,z} = B_{k,t,z}^{2^*-1} \quad \text{in } \mathbb{R}^n,$$

where δ_0 is the Euclidean metric in $\mathbb{R}^n$ and $\triangle_{\delta_0} := -\text{div}_{\delta_0}\nabla$. For $x \in \mathbb{R}^n$, we let

$$V_0(x) := \frac{\frac{|x|^2}{n(n-2)} - 1}{\left(1 + \frac{|x|^2}{n(n-2)}\right)^{\frac{n}{2}}} \quad \text{and, for } 1 \le j \le n, \quad V_j(x) := \frac{x_j}{\left(1 + \frac{|x|^2}{n(n-2)}\right)^{\frac{n}{2}}}. \tag{3.8}$$

We then let

$$Z_{0,k,t,z}(x); = \mu_k(t)^{1-\frac{n}{2}} V_0\left(\frac{x - \xi_k(z)}{\mu_k(t)}\right) = \frac{2}{n-2} t\partial_t B_{k,t,z} \tag{3.9}$$

and, for $1 \le j \le n$,

$$Z_{j,k,t,z}(x); = \mu_k(t)^{1-\frac{n}{2}} V_j\left(\frac{x - \xi_k(z)}{\mu_k(t)}\right) = -nt\partial_{z_j} B_{k,t,z}, \tag{3.10}$$

and we let

$$K_{k,t,z}; = \text{span}\left\{Z_{j,k,t,z} : 0 \le j \le n\right\},$$

which is a finite-dimensional subspace of $D^{1,2}(\mathbb{R}^n)$. We denote by $K_{k,t,z}^{\perp}$ its orthogonal complement in $D^{1,2}(\mathbb{R}^n)$. The functions $Z_{i,k,t,z}$ satisfy

$$\triangle_{\delta_0} Z_{i,k,t,z} = (2^* - 1) B_{k,t,z}^{2^*-2} Z_{i,k,t,z} \quad \text{in } \mathbb{R}^n$$

for all $0 \le j \le n$, and by a result of Rey [44] and Bianchi-Egnell [6], they form an orthogonal basis of the set of solutions of this equation in $D^{1,2}(\mathbb{R}^n)$. Letting h be as in the previous subsection, we define, for $k \in \mathbb{N}$ and $x \in \mathbb{R}^n$,

$$h_k(x) := h\left(x - y_k\right). \tag{3.11}$$

By (3.2), h_k is trace-free and divergence-free in $\mathbb{R}^n$. As a first result, we obtain the following lemma:

Lemma 3.1. *For any $(t,z) \in [1/A, A] \times B(0,1)$ and $k \in \mathbb{N}$, there exists a unique $R_{k,t,z} \in K_{k,t,z}^{\perp}$ that satisfies*

$$\triangle_{\delta_0} R_{k,t,z} - (2^*-1) B_{k,t,z}^{2^*-2} R_{k,t,z} = -\varepsilon_k \sum_{p,q=1}^{n} (h_k)_{pq} \partial^2_{pq} B_{k,t,z} \quad \text{in } \mathbb{R}^n. \tag{3.12}$$

This function $R_{k,t,z}$ also satisfies, for $i \in \{0,1,2\}$,

$$|\nabla^i R_{k,t,z}(x)| \le C\varepsilon_k \mu_k(t)^{\frac{n+2}{2}} \frac{\ln\left(\frac{2\mu_k(t) + |x - \xi_k(z)|}{\mu_k(t)}\right)}{\left(\mu_k(t) + |x - \xi_k(z)|\right)^{n-2+i}} \tag{3.13}$$

for all $x \in \mathbb{R}^n$, for some $C > 0$ independent of k, t, z.

Proof. Let $u \in C^2(\mathbb{R}^n)$ be such that $u \in L^2(\mathbb{R}^n)$, $(1+|x|)|\nabla u(x)| \in L^2(\mathbb{R}^n)$ and $(1+|x|)^2|\nabla^2 u(x)| \in L^2(\mathbb{R}^n)$. Define then

$$G_k(u) := \frac{1}{2}\int_{\mathbb{R}^n} \sum_{p,q=1}^{n} (h_k)_{pq} \partial_p u\, \partial_q u\, dx.$$

Integrating by parts and since h_k is divergence free, we get that

$$G_k(u) = -\frac{1}{2}\int_{\mathbb{R}^n} \sum_{p,q=1}^{n} (h_k)_{pq} u\, \partial^2_{pq} u\, dx.$$

Choose now $u = B_{k,t,z}$ for $t > 0$ and $z \in B(0,1)$, which is admissible for G_k since $n \geq 11$. Explicit computations show that

$$B_{k,t,z}\, \partial^2_{pq} B_{k,t,z} = -\frac{1}{n} \frac{\mu_k(t)^{n-2}\delta_{pq}}{\left(\mu_k(t)^2 + \frac{|x-\xi_k(z)|^2}{n(n-2)}\right)^{n-1}} + \frac{n}{n-2}\, \partial_p B_{k,t,z}\, \partial_q B_{k,t,z},$$

which gives, since h_k is trace-free, $G_k(B_{k,t,z}) = -\frac{n}{n-2} G_k(B_{k,t,z})$, and so $G_k(B_{k,t,z}) = 0$. Differentiating with respect to t and z, using (3.9) and (3.10) and integrating by parts shows that for any $j \in \{0, \dots, n\}$,

$$\int_{\mathbb{R}^n} Z_{j,k,t,z} \sum_{p,q=1}^n (h_k)_{pq}\, \partial^2_{pq} B_{k,t,z}\, dx = 0$$

holds. Since, by Bianchi–Egnell [6], $\triangle_{\delta_0} - (2^* - 1) B_{k,t,z}^{2^*-2}$ is Fredholm and injective on $K_{k,t,z}^{\perp}$ the existence of a unique $R_{k,t,z} \in K_{k,t,z}^{\perp}$ satisfying (3.12) follows. By standard elliptic theory $R_{k,t,z}$ is smooth.

By (3.2), h_k is trace-free and satisfies $\sum_{j=1}^n (x - y_k)_j (h_k)_{ij}(x) = 0$ for all $1 \leq j \leq n$ and $x \in \mathbb{R}^n$. Direct computations then show that

$$\sum_{p,q=1}^n (h_k)_{pq}\, \partial^2_{pq} B_{k,t,z} = -\frac{\mu_k^2}{n(n-2)} \sum_{p,q=1}^n (h_k)_{pq} z_p z_q \frac{\mu_k(t)^{\frac{n-2}{2}}}{\left(\mu_k(t)^2 + \frac{|\cdot - \xi_k(z)|^2}{n(n-2)}\right)^{\frac{n+2}{2}}}.$$

It is then easily seen that, for all $x \in \mathbb{R}^n$,

$$R_{k,t,z}(x) = \varepsilon_k \mu_k^{3-\frac{n}{2}} R_{t,z}\left(\frac{x - y_k}{\mu_k}\right) \tag{3.14}$$

holds, where $R_{t,z}$ is the unique solution in $K_{t,z}^{\perp}$ of

$$\triangle_{\delta_0} R_{t,z} - (2^* - 1) B_{t,z}^{2^*-2} R_{t,z} = -\frac{t^{\frac{n-2}{2}}}{n(n-2)} \frac{\sum_{p,q=1}^n h_{pq} z_p z_q}{\left(t^2 + \frac{|\cdot - z|^2}{n(n-2)}\right)^{\frac{n+2}{2}}} \quad \text{in } \mathbb{R}^n, \tag{3.15}$$

where we have let

$$B_{t,z}(x) := t^{\frac{n-2}{2}} \left(t^2 + \frac{|x - z|^2}{n(n-2)}\right)^{-\frac{n-2}{2}} \tag{3.16}$$

and

$$K_{t,z} := \operatorname{span}\left\{\partial_t B_{t,z}, \partial_{z_1} B_{t,z}, \dots, \partial_{z_n} B_{t,z}\right\}.$$

Since $t \in [1/A, A]$ and $z \in B(0,1)$ and since the h_{ij} are homogeneous of degree 2, we can again write

$$R_{t,z}(x) = t^{1-\frac{n}{2}} S_{0,z}\left(\frac{x - z}{t}\right)$$

for some smooth function $S_{0,z} \in K_{1,0}^{\perp}$ that satisfies

$$\left|\triangle_{\delta_0} S_{0,z}(y) - (2^* - 1) B_{1,0}(y)^{2^*-2} S_{0,z}(y)\right| \leq C|z|^2 \left(1 + |y|\right)^{-n}$$

for all $y \in \mathbb{R}^n$, for some $C > 0$ independent of t and z. We can now write a representation formula for $\triangle_{\delta_0} - (2^* - 1) B_{1,0}^{2^*-2}$ (see for example Lemma 3.3 of Premoselli [40]) that shows that, for any $x \in \mathbb{R}^n$,

$$|S_{0,z}(x)| \leq C|z|^2 \int_{\mathbb{R}^n} |x - y|^{2-n} (1 + |y|)^{-n} dy \leq C|z|^2 \frac{\ln(2 + |x|)}{(1 + |x|)^{n-2}}$$

for some constant $C > 0$ independent of z. Differentiating the representation formula similarly yields

$$|\nabla S_{0,z}(x)| \le C|z|^2 \frac{\ln(2+|x|)}{(1+|x|)^{n-1}} \quad \text{and} \quad |\nabla S_{0,z}(x)|^2 \le C|z|^2 \frac{\ln(2+|x|)}{(1+|x|)^{n}}$$

for any $x \in \mathbb{R}^n$. Going back to $R_{k,t,z}$, this proves (3.13). $\square$

Let again $\chi \in C_c^\infty(\mathbb{R})$ be such that $\chi \equiv 1$ on $[0,1]$, $0 \le \chi \le 1$ and $\chi \equiv 0$ on $\mathbb{R}\backslash[0,2]$. For $t \in [1/A, A]$ and $z \in B(0,1)$, we let

$$U_{k,t,z}(x) := \chi\left(\frac{x-y_k}{r_k}\right)\left(B_{k,t,z}(x) + R_{k,t,z}(x)\right) \tag{3.17}$$

for any $x \in \mathbb{R}^n$, and

$$W_{k,t,z}(x) := U_{k,t,z}\left(\Phi(x)\right) - \check{u}_0(x) \tag{3.18}$$

for any $x \in M$, where Φ is the chart around x_0 introduced in the previous subsection, and where $\check{u}_0$ is as in (3.5). We let

$$f(s) := |s|^{2^*-2}s \quad \forall s \in \mathbb{R}. \tag{3.19}$$

We also let for $k \in \mathbb{N}$ and $(t,z) \in [1/A, A] \times B(0,1)$,

$$E_{k,t,z} := \left(\triangle_{\check{g}} + c_n \operatorname{Scal}_{\check{g}}\right) W_{k,t,z} - f(W_{k,t,z}). \tag{3.20}$$

Until the end of this section, C will denote a positive constant independent of k, t and z, that might change from one line to the other.

Lemma 3.2. *For any $k \in \mathbb{N}$ and $(t,z) \in [1/A, A] \times B(0,1)$, we have*

$$\|E_{k,t,z}\|_{L^{\frac{2n}{n+2}}(M)} \le C\mu_k^{\frac{n+2}{4}}. \tag{3.21}$$

Proof. By (3.1) and (3.13), we have $\|R_{k,t,z}/B_{k,t,z}\|_{L^\infty(B(y_k,2r_k))} \to 0$ as $k \to \infty$ uniformly with respect to t, z. As a consequence, and since $\mu_k^2 = \mathrm{o}\left(r_k\right)$,

$$\left|f(W_{k,t,z}) - f(U_{k,t,z}\circ\Phi) + f(\check{u}_0)\right| \le C\begin{cases}\left(B_{k,t,z}\circ\Phi\right)^{2^*-2} & \text{in } \Phi\big(B(\xi_k(z), \sqrt{\mu_k(t)})\big)\\ \left(B_{k,t,z}\circ\Phi\right) & \text{otherwise.}\end{cases}$$

Since $\check{u}_0$ satisfies (3.5) and $\tilde{g} = \Phi_*\check{g}$ in $B(0,4\delta)$, we obtain

$$\|E_{k,t,z}\|_{L^{\frac{2n}{n+2}}(M)} \le \|(\triangle_{\tilde{g}} + c_n \operatorname{Scal}_{\tilde{g}})U_{k,t,z} - f(U_{k,t,z})\|_{L^{\frac{2n}{n+2}}(B(y_k,2r_k))} + C\mu_k^{\frac{n+2}{4}}, \tag{3.22}$$

where $\tilde{g}$ is given by (3.3). First, by using (3.13) together with straightforward computations, we obtain

$$(\triangle_{\tilde{g}} + c_n \operatorname{Scal}_{\tilde{g}})U_{k,t,z} - f(U_{k,t,z}) = \mathrm{O}\left(\mu_k^{\frac{n-2}{2}} r_k^{-n}\right) \tag{3.23}$$

in $B(y_k, 2r_k)\backslash B(y_k, r_k)$. In $B(y_k, r_k)$, by (3.4), we have

$$\tilde{g} = \exp\left(\varepsilon_k h_k(x)\right),$$

where h_k is defined in (3.11). Since the components of h are homogeneous of degree two, $|\varepsilon_k h_k(x)| \le C\varepsilon_k r_k^2 = \mathrm{o}\left(1\right)$ holds for all $x \in B(y_k, r_k)$. The definition of $\tilde{g}$ therefore allows to expand its inverse as

$$\tilde{g}^{ij}(x) = \delta_{ij} - \varepsilon_k (h_k)_{ij}(x) + \frac{\varepsilon_k^2}{2}\sum_{p=1}^n (h_k)_{ip}(x)(h_k)_{pj}(x) + \mathrm{O}\left(\varepsilon_k^3 |x-y_k|^6\right) \tag{3.24}$$

for $i, j \in \{1, \dots, n\}$. Similarly, the Christoffel symbols of $\tilde{g}$ expand as

$$\Gamma_{ij}^{\ell}(\tilde{g})(x) = \frac{\varepsilon_k}{2} \left(\partial_i (h_k)_{j\ell}(x) + \partial_j (h_k)_{i\ell}(x) - \partial_\ell (h_k)_{ij}(x) \right) + \mathrm{O}\left(\varepsilon_k^2 |x - y_k|^3 \right) \quad (3.25)$$

for $i, j, \ell \in \{1, \dots, n\}$. Using Proposition 26 of Brendle [7], and since h_k is trace-free and divergence-free, the scalar curvature of $\tilde{g}$ expands as

$$\mathrm{Scal}_{\tilde{g}}(x) = -\frac{1}{4} \varepsilon_k^2 \sum_{i,j,\ell=1}^{n} \left(\partial_i (h_k)_{j\ell}(x) \right)^2 + \mathrm{O}\left(\varepsilon_k^3 |x - y_k|^4 \right). \quad (3.26)$$

Remark finally that, by definition of $\xi_k(z)$ in (3.7), there exists $C > 1$ such that for any $x \in B(y_k, r_k)$,

$$\frac{1}{C} \le \frac{\mu_k + |x - y_k|}{\mu_k + |x - \xi_k(z)|} \le C$$

holds true. Using (3.24), (3.25) and (3.26), we thus have, for $x \in B(y_k, r_k)$,

$$\begin{aligned}
&(\triangle_{\tilde{g}} + c_n \,\mathrm{Scal}_{\tilde{g}}) B_{k,t,z} - f(B_{k,t,z}) \\
&\quad = (\triangle_{\tilde{g}} - \triangle_{\delta_0}) B_{k,t,z} + \mathrm{O}\left(\varepsilon_k^2 \mu_k^{\frac{n-2}{2}} (\mu_k + |x - y_k|)^{4-n} \right) \\
&\quad = \varepsilon_k \sum_{p,q=1}^{n} (h_k)_{pq} \, \partial_{pq}^2 B_{k,t,z} + \mathrm{O}\left(\varepsilon_k^2 \mu_k^{\frac{n-2}{2}} (\mu_k + |x - y_k|)^{4-n} \right). \qquad (3.27)
\end{aligned}$$

Since $U_{k,t,z}(x) = B_{k,t,z}(x) + R_{k,t,z}(x)$, by (3.13),

$$|f(U_{k,t,z}(x)) - f(B_{k,t,z}(x)) - f'(B_{k,t,z}(x)) R_{k,t,z}(x)| \le C B_{k,t,z}(x)^{2^*-3} R_{k,t,z}(x)^2$$

holds for any $x \in B(y_k, r_k)$. With (3.1), (3.13), (3.24), (3.25), (3.26) and (3.27), we obtain that, in $B(y_k, r_k)$,

$$\begin{aligned}
&(\triangle_{\tilde{g}} + c_n \,\mathrm{Scal}_{\tilde{g}}) U_{k,t,z} - f(U_{k,t,z}) \\
&\quad = (\triangle_{\tilde{g}} + c_n \,\mathrm{Scal}_{\tilde{g}}) B_{k,t,z} - f(B_{k,t,z}) + \triangle_{\delta_0} R_{k,t,z} - f'(B_{k,t,z}(x)) R_{k,t,z} \\
&\qquad + (\triangle_{\tilde{g}} - \triangle_{\delta_0}) R_{k,t,z} + \mathrm{O}\left(\varepsilon_k^2 (\mu_k + |x - y_k|)^2 |R_{k,t,z}| \right) + \mathrm{O}\left(B_{k,t,z}^{2^*-3} R_{k,t,z}^2 \right) \\
&\quad = \mathrm{O}\left(\varepsilon_k^2 \mu_k^{\frac{n-2}{2}} (\mu_k + |x - y_k|)^{4-n} \right)
\end{aligned}$$

holds. With (3.1), (3.22) and (3.23), this finally shows that

$$\|E_{k,t,z}\|_{L^{\frac{2n}{n+2}}(M)} \le C \left(\mu_k^{\frac{n+2}{4}} + \mu_k^{\frac{n-2}{2}} r_k^{1-\frac{n}{2}} + \varepsilon_k^2 \mu_k^4 \right) \le C \mu_k^{\frac{n+2}{4}},$$

where the last inequality follows since $n \ge 11$ and $\mu_k = \mathrm{o}\left(r_k^p \right)$ for any $p \ge 1$. $\square$

3.3. **The Lyapunov–Schmidt reduction.** We endow $H^1(M)$ with the norm

$$\|u\|_{H^1(M)} := \sqrt{\int_M \left(|\nabla u|_{\breve{g}}^2 + c_n \,\mathrm{Scal}_{\breve{g}} \, u^2 \right) dv_{\breve{g}}},$$

and for $u \in H^1(M)$, we let

$$I(u) := \frac{1}{2} \int_M \left(|\nabla u|_{\breve{g}}^2 + c_n \,\mathrm{Scal}_{\breve{g}} \, u^2 \right) dv_{\breve{g}} - \frac{1}{2^*} \int_M |u|^{2^*} dv_{\breve{g}}.$$

For $(t, z) \in [1/A, A] \times B(0, 1)$, $1 \le j \le n$ and $x \in M$, we let

$$\hat{Z}_{0,k,t,z}(x) := \frac{2}{n-2} \, t \, \partial_t W_{k,t,z}(x) \quad \text{and} \quad \hat{Z}_{j,k,t,z}(x) := -n \, t \, \partial_{z_j} W_{k,t,z}(x), \quad (3.28)$$

and we let

$$\hat{K}_{k,t,z} := \operatorname{span}\{\hat{Z}_{j,k,t,z} : 0 \le j \le n\},$$

which is regarded as a subset of $H^1(M)$. We denote by $\hat{K}^{\perp}_{k,t,z}$ its orthogonal complement in $H^1(M)$. The following proposition shows the existence of a canonical solution of (3.6) in $\hat{K}^{\perp}_{k,t,z}$:

Proposition 3.3. *There exists $C > 0$ and, for large $k \in \mathbb{N}$, a function $\varphi_k : [1/A, A] \times B(0,1) \to H^1(M)$ of class C^1 which is the only solution of*

$$\Pi_{\hat{K}^{\perp}_{k,t,z}} \left(u_{k,t,z} - \left(\triangle_{\breve{g}} + c_n \operatorname{Scal}_{\breve{g}} \right)^{-1} \left(f(u_{k,t,z}) \right) \right) = 0$$

in the set

$$\left\{ \varphi \in \hat{K}^{\perp}_{k,t,z} : \|\varphi\|_{H^1(M)} \le C \|E_{k,t,z}\|_{L^{\frac{2n}{n+2}}(M)} \right\},$$

where we have let $u_{k,t,z} := W_{k,t,z} + \varphi_k(t,z)$. In particular,

$$\|\varphi_k(t,z)\|_{H^1(M)} \le C \|E_{k,t,z}\|_{L^{\frac{2n}{n+2}}(M)} \tag{3.29}$$

for some $C > 0$ independent of k, t, z. In addition, for large $k \in \mathbb{N}$, the function $u_{k,t,z}$ is a critical point of I (hence a solution of (3.6)) if and only if (t,z) is a critical point of the mapping $(t,z) \mapsto I(u_{k,t,z})$.

Proof. By (3.18), we have, for $0 \le j \le n$ and $x \in B(y_k, 2r_k)$,

$$\hat{Z}_{j,k,t,z}\left(\Phi^{-1}(x)\right) = \chi\left(\frac{x - y_k}{r_k}\right)\left(Z_{j,k,t,z}(x) + Q_{j,k,t,z}(x)\right), \tag{3.30}$$

where we have let $Q_{0,k,t,z} := \frac{2}{n-2} t\, \partial_t R_{k,t,z}$ and $Q_{j,k,t,z} := -nt\, \partial_{z_j} R_{k,t,z}$. By differentiating (3.12) and using (3.13), one gets that, for $0 \le j \le n$, $k \in \mathbb{N}$, $x \in B(y_k, 2r_k)$ and any $(t,z) \in [1/A, A] \times B(0,1)$, $Q_{j,k,t,z}$ satifies

$$\left| \triangle_{\delta_0} Q_{j,k,t,z} - (2^* - 1) B^{2^*-2}_{k,t,z} Q_{j,k,t,z} \right| \le C \varepsilon_k \mu_k^{\frac{n+2}{2}} \left(\mu_k + |\xi_k(z) - \cdot|\right)^{-n}. \tag{3.31}$$

As in the proof of (3.13), a representation formula yields once again that, for $i \in \{0,1,2\}$,

$$|\nabla^i Q_{j,k,t,z}(x)| \le C \varepsilon_k \mu_k(t)^{\frac{n+2}{2}} \frac{\ln\left(\frac{2\mu_k(t) + |x - \xi_k(z)|}{\mu_k(t)}\right)}{\left(\mu_k(t) + |x - \xi_k(z)|\right)^{n-2+i}} \tag{3.32}$$

for $x \in \mathbb{R}^n$. With (3.9), (3.10) and (3.28), we then obtain, for $0 \le j \le n$ and $x \in B(y_k, 2r_k)$, that

$$\hat{Z}_{j,k,t,z}\left(\Phi^{-1}(x)\right) = \chi\left(\frac{x - y_k}{r_k}\right) Z_{j,k,t,z}(x) + \mathrm{O}\left(\varepsilon_k \mu_k(t)^{\frac{n+2}{2}} \frac{\ln\left(\frac{2\mu_k(t) + |x - \xi_k(z)|}{\mu_k(t)}\right)}{\left(\mu_k(t) + |x - \xi_k(z)|\right)^{n-2}} \right), \tag{3.33}$$

$$\nabla\left(\hat{Z}_{j,k,t,z} \circ \Phi^{-1}\right)(x) = \nabla\left(\chi\left(\frac{\cdot - y_k}{r_k}\right) Z_{j,k,t,z}\right)(x) + \mathrm{O}\left(\varepsilon_k \mu_k(t)^{\frac{n+2}{2}} \frac{\ln\left(\frac{2\mu_k(t) + |x - \xi_k(z)|}{\mu_k(t)}\right)}{\left(\mu_k(t) + |x - \xi_k(z)|\right)^{n-1}} \right). \tag{3.34}$$

With these estimates, we can easily adapt the proof of Proposition 5.1 of Robert–Vétois [47] (see also Esposito–Pistoia–Vétois [21]), and Proposition 3.3 follows. □

We let K_n be the optimal constant of the Sobolev embedding $D^{1,2}(\mathbb{R}^n) \hookrightarrow L^{2^*}(\mathbb{R}^n)$. As explained in the introduction, $K_n^{-2} = Y(\mathbb{S}^n, [g_{std}])$. Since the latter is attained by the stereographic projection of the bubbles $B_{t,z}$ defined in (3.16), for all $(t,z) \in [1/A, A] \times B(0,1)$, we have

$$\int_{\mathbb{R}^n} B_{t,z}^{2^*} \, dx = \int_{\mathbb{R}^n} |\nabla B_{t,z}|^2 dx = K_n^{-n}.$$

The explicit value of K_n is known (see Aubin [4] and Talenti [54]). The next result is an expansion of $(t,z) \mapsto I(W_{k,t,z})$ as $k \to \infty$.

Proposition 3.4. *We have, as $k \to \infty$:*

$$\begin{aligned} I(W_{k,t,z}) &= \frac{1}{n} K_n^{-n} + I(\check{u}_0) + \mu_k^{\frac{n-2}{2}} \Bigg(\Lambda(n) \check{u}_0(x_0) t^{\frac{n-2}{2}} - \frac{1}{2} \int_{\mathbb{R}^n} |\nabla R_{t,z}|^2 dx \\ &+ \frac{1}{4} \int_{\mathbb{R}^n} \sum_{i,j,p=1}^n h_{ip} \, h_{pj} \, \partial_i B_{t,z} \, \partial_j B_{t,z} \, dx - \frac{n-2}{32(n-1)} \int_{\mathbb{R}^n} \sum_{i,j,\ell=1}^n (\partial_i h_{j\ell})^2 B_{t,z}^2 \, dx + \mathrm{o}(1) \Bigg), \end{aligned} \tag{3.35}$$

where $\check{u}_0$ is as in (3.5)*, $R_{t,z}$ is as in* (3.14) *and where $\Lambda(n)$ is a positive dimensional constant given by* (3.38) *below. This expansion holds true in $C^1([1/A, A] \times B(0,1))$.*

Proof. First, by (3.13) and (3.17), it is easily seen that $0 \le U_{k,t,z} \le C B_{k,t,z}$ in $B(y_k, 2r_k)$. By (3.18), we can then write, for some $0 < \theta < 2^* - 2$, that

$$\begin{aligned} \Big| |W_{k,t,z}|^{2^*} - (U_{k,t,z} \circ \Phi)^{2^*} - \check{u}_0^{2^*} + 2^* (U_{k,t,z} \circ \Phi)^{2^*-1} \check{u}_0 + 2^* (U_{k,t,z} \circ \Phi) \check{u}_0^{2^*-1} \Big| \\ \le C \left((B_{k,t,z} \circ \Phi)^{2^*-1-\theta} + (B_{k,t,z} \circ \Phi)^{1+\theta} \right) \end{aligned}$$

holds in M. As a consequence, and since $\check{u}_0$ solves (3.5), straightforward computations show that, for $0 < \theta < \frac{2}{n-2}$,

$$\begin{aligned} I(W_{k,t,z}) &= I(U_{k,t,z} \circ \Phi) + I(\check{u}_0) + \int_M (U_{k,t,z} \circ \Phi)^{2^*-1} \check{u}_0 dv_{\check{g}} \\ &\quad + \mathrm{O} \left(\int_M \left((B_{k,t,z} \circ \Phi)^{2^*-1-\theta} + (B_{k,t,z} \circ \Phi)^{1+\theta} \right) dv_{\check{g}} \right) \\ &= I(U_{k,t,z} \circ \Phi) + I(\check{u}_0) + \int_M (U_{k,t,z} \circ \Phi)^{2^*-1} \check{u}_0 \, dv_{\check{g}} + \mathrm{O} \left(\mu_k^{\frac{n-2}{2}(1+\theta)} \right). \end{aligned} \tag{3.36}$$

Independently, by (3.13) and (3.17), and since $\check{u}_0$ is of class C^2, we have

$$\int_M (U_{k,t,z} \circ \Phi)^{2^*-1} \check{u}_0 \, dv_{\check{g}} = \Lambda(n) \check{u}_0(x_0) \mu_k(t)^{\frac{n-2}{2}} + \mathrm{o} \left(\mu_k^{\frac{n-2}{2}} \right), \tag{3.37}$$

where $\mu_k(t) := \mu_k t$ is as in (3.7) and where we have let

$$\Lambda(n) := \int_{\mathbb{R}^n} \left(1 + \frac{|x|^2}{n(n-2)} \right)^{-\frac{n+2}{2}} dx. \tag{3.38}$$

We have thus proven that

$$I(W_{k,t,z}) = I(U_{k,t,z} \circ \Phi) + I(\check{u}_0) + \Lambda(n) \check{u}_0(x_0) \mu_k(t)^{\frac{n-2}{2}} + \mathrm{o} \left(\mu_k^{\frac{n-2}{2}} \right). \tag{3.39}$$

It remains to expand $I\left(U_{k,t,z}\circ\Phi\right)$. By definition, we have $\check{g}=\Phi^*\tilde{g}$ in $B_g(x_0,6\delta)$, where $\tilde{g}$ is given by (3.3), so by (3.17) and since $\det\tilde{g}=1$, we have

$$I\left(U_{k,t,z}\circ\Phi\right)=\frac{1}{2}\int_{\mathbb{R}^n}\Big(\left|\nabla\left(B_{k,t,z}+R_{k,t,z}\right)\right|^2_{\tilde{g}}+c_n\operatorname{Scal}_{\tilde{g}}\left(B_{k,t,z}+R_{k,t,z}\right)^2\Big)\,dx\\ -\frac{1}{2^*}\int_{\mathbb{R}^n}\left|B_{k,t,z}+R_{k,t,z}\right|^{2^*}dx+\mathrm{o}\left(\mu_k^{\frac{n-2}{2}}\right),\quad(3.40)$$

where $R_{k,t,z}$ is given by Lemma 3.1. In the latter equality, we implicitly assumed, in accordance with (3.3), that $\tilde{g}$ has been extended as a metric in $\mathbb{R}^n$ that coincides with the Euclidean metric in $\mathbb{R}^n\backslash B(0,4\delta)$. First, by (3.13) and (3.24), we have

$$\frac{1}{2}\int_{\mathbb{R}^n}\left|\nabla\left(B_{k,t,z}+R_{k,t,z}\right)\right|^2_{\tilde{g}}\,dx=\frac{1}{2}\int_{\mathbb{R}^n}|\nabla B_{k,t,z}|^2dx+\int_{\mathbb{R}^n}\langle\nabla B_{k,t,z},\nabla R_{k,t,z}\rangle\,dx\\ +\frac{1}{2}\int_{\mathbb{R}^n}|\nabla R_{k,t,z}|^2dx-\frac{\varepsilon_k}{2}\int_{\mathbb{R}^n}\sum_{i,j=1}^n(h_k)_{ij}\,\partial_iB_{k,t,z}\,\partial_jB_{k,t,z}\,dx\\ +\frac{\varepsilon_k^2}{4}\int_{\mathbb{R}^n}\sum_{i,j,p=1}^n(h_k)_{ip}(h_k)_{pj}\,\partial_iB_{k,t,z}\,\partial_jB_{k,t,z}\,dx\\ -\varepsilon_k\int_{\mathbb{R}^n}\sum_{i,j=1}^n(h_k)_{ij}\,\partial_iB_{k,t,z}\,\partial_jR_{k,t,z}\,dx+\mathrm{o}\left(\mu_k^{\frac{n-2}{2}}\right),\quad(3.41)$$

where h_k is as in (3.11). Similarly, using (3.1), (3.13) and (3.26), we have

$$\frac{c_n}{2}\int_{\mathbb{R}^n}\operatorname{Scal}_{\tilde{g}}\left(B_{k,t,z}+R_{k,t,z}\right)^2dx=-\frac{c_n}{8}\varepsilon_k^2\int_{\mathbb{R}^n}\sum_{i,j,\ell=1}^n\left(\partial_i(h_k)_{j\ell}\right)^2B_{k,t,z}^2\,dx\\ +\mathrm{o}\left(\mu_k^{\frac{n-2}{2}}\right).\quad(3.42)$$

Finally, using (3.13), we have

$$\frac{1}{2^*}\int_{\mathbb{R}^n}\left|B_{k,t,z}+R_{k,t,z}\right|^{2^*}dx=\frac{1}{2^*}\int_{\mathbb{R}^n}B_{k,t,z}^{2^*}dx+\int_{\mathbb{R}^n}B_{k,t,z}^{2^*-1}R_{k,t,z}\,dx\\ +(2^*-1)\int_{\mathbb{R}^n}B_{k,t,z}^{2^*-2}R_{k,t,z}^2\,dx+\mathrm{o}\left(\mu_k^{\frac{n-2}{2}}\right).\quad(3.43)$$

As shown in the proof of Lemma 3.1, we have

$$\int_{\mathbb{R}^n}\sum_{i,j=1}^n(h_k)_{ij}\,\partial_iB_{k,t,z}\,\partial_jB_{k,t,z}\,dx=0\quad(3.44)$$

for any $(t,z)\in[1/A,A]\times B(0,1)$. We now integrate (3.12) against $R_{k,t,z}$: after integrating by parts and using that h is divergence-free, we find that

$$\int_{\mathbb{R}^n}|\nabla R_{k,t,z}|^2dx-(2^*-1)\int_{\mathbb{R}^n}B_{k,t,z}^{2^*-2}R_{k,t,z}^2\,dx\\ =\varepsilon_k\int_{\mathbb{R}^n}\sum_{i,j=1}^n(h_k)_{ij}\,\partial_iB_{k,t,z}\,\partial_jR_{k,t,z}\,dx.\quad(3.45)$$

It remains to plug (3.41), (3.42), (3.43), (3.44) and (3.45) into (3.40). This gives

$$\begin{aligned} I\left(U_{k,t,z}\circ\Phi\right) = \frac{1}{n}K_n^{-n} - \frac{1}{2}\int_{\mathbb{R}^n}|\nabla R_{k,t,z}|^2 dx \\ + \frac{\varepsilon_k^2}{4}\int_{\mathbb{R}^n}\sum_{i,j,p=1}^n (h_k)_{ip}(h_k)_{pj}\,\partial_i B_{k,t,z}\,\partial_j B_{k,t,z}\,dx \\ - \frac{n-2}{32(n-1)}\varepsilon_k^2\int_{\mathbb{R}^n}\sum_{i,j,\ell=1}^n(\partial_i(h_k)_{j\ell})^2 B_{k,t,z}^2\,dx + \mathrm{o}\left(\mu_k^{\frac{n-2}{2}}\right). \end{aligned}$$

Simple changes of variables using (3.11) and (3.15) now yield

$$\begin{aligned} &\frac{\varepsilon_k^2}{4}\int_{\mathbb{R}^n}\sum_{i,j,p=1}^n (h_k)_{ip}(h_k)_{pj}\,\partial_i B_{k,t,z}\,\partial_j B_{k,t,z}\,dx \\ &\qquad - \frac{n-2}{32(n-1)}\varepsilon_k^2\int_{\mathbb{R}^n}\sum_{i,j,\ell=1}^n(\partial_i(h_k)_{j\ell})^2 B_{k,t,z}^2\,dx \\ &= \varepsilon_k^2\mu_k^4\left(\frac{1}{4}\int_{\mathbb{R}^n}\sum_{i,j,p=1}^n h_{ip}\,h_{pj}\,\partial_i B_{t,z}\,\partial_j B_{t,z}\,dx - \frac{n-2}{32(n-1)}\int_{\mathbb{R}^n}\sum_{i,j,\ell=1}^n(\partial_i h_{j\ell})^2 B_{t,z}^2\,dx\right) \end{aligned}$$

and

$$\frac{1}{2}\int_{\mathbb{R}^n}|\nabla R_{k,t,z}|^2 dx = \frac{1}{2}\varepsilon_k^2\mu_k^4\int_{\mathbb{R}^n}|\nabla R_{t,z}|^2 dx,$$

where $R_{t,z}$ and $B_{t,z}$ are as in (3.14) and (3.16). Together with (3.1), (3.36), (3.37) and (3.39), this concludes the proof of (3.35). The expansions of the derivatives with respect to t and z follow from similar estimates. □

To conclude this section, we show that $I\left(u_{k,t,z}\right)$ expands at first-order as $I\left(W_{k,t,z}\right)$ and that this expansion is C^1 in t:

Proposition 3.5. *We have*

$$I\left(u_{k,t,z}\right) = I\left(W_{k,t,z}\right) + \mathrm{o}\left(\mu_k^{\frac{n-2}{2}}\right) \text{ and } \partial_t\left(I\left(u_{k,t,z}\right)\right) = \partial_t\left(I\left(W_{k,t,z}\right)\right) + \mathrm{o}\left(\mu_k^{\frac{n-2}{2}}\right)$$

as $k\to\infty$, uniformly with respect to $(t,z)\in[1/A,A]\times\overline{B(0,1)}$, where $u_{k,t,z}$ is as in Proposition 3.3.

C^1-expansions in the z variable also hold true, but we will not need them here.

Proof. By the mean value theorem, (3.21) and (3.29), we have

$$\begin{aligned} I\left(u_{k,t,z}\right) &= I\left(W_{k,t,z}\right) + DI\left(W_{k,t,z}\right)\left(\varphi_k(t,z)\right) + \mathrm{O}\left(\|\varphi_k(t,z)\|_{H^1(M)}^2\right) \\ &= I\left(W_{k,t,z}\right) + \int_M E_{k,t,z}\varphi_k(t,z)\,dv_{\breve{g}} + \mathrm{o}\left(\mu_k^{\frac{n-2}{2}}\right) = I\left(W_{k,t,z}\right) + \mathrm{o}\left(\mu_k^{\frac{n-2}{2}}\right), \end{aligned}$$

which proves the first equality. We now show the C^1–estimate in t. By Proposition 3.3, there exist, for any $(t,z)\in[1/A,A]\times B(0,1)$, real numbers $(\lambda_k^j(t,z))_{0\le j\le n}$ such that $u_{k,t,z}$ satisfies

$$\left(\triangle_{\breve{g}} + c_n\,\mathrm{Scal}_{\breve{g}}\right)u_{k,t,z} - f(u_{k,t,z}) = \sum_{j=0}^n \lambda_k^j(t,z)\left(\triangle_{\breve{g}} + c_n\,\mathrm{Scal}_{\breve{g}}\right)\hat{Z}_{j,k,t,z} \text{ in } M. \tag{3.46}$$

By definition of $u_{k,t,z}$, we have independently

$$\left(\triangle_{\breve g} + c_n \operatorname{Scal}_{\breve g}\right) u_{k,t,z} - f(u_{k,t,z}) = E_{k,t,z} + \left(\triangle_{\breve g} + c_n \operatorname{Scal}_{\breve g} - f'(W_{k,t,z})\right) \varphi_k(t,z) - \left(f(u_{k,t,z}) - f(W_{k,t,z}) - f'(W_{k,t,z})\varphi_k(t,z)\right), \quad (3.47)$$

where $E_{k,t,z}$ is as in (3.20). By (3.21) and (3.29), Theorem 4.3 of Premoselli [39] applies and shows the existence of a sequence σ_k of positive numbers with $\sigma_k \to 0$ as $k \to \infty$ such that for any $x \in M$ and $(t,z) \in [1/A, A] \times B(0,1)$,

$$|\varphi_k(t,z)(x)| \le \sigma_k \left(1 + U_{k,t,z}\left(\Phi(x)\right)\right) \quad (3.48)$$

holds. By (3.13) and (3.18), we can then let $R > 0$ be large enough so that

$$W_{k,t,z}\left(\Phi^{-1}(x)\right) \ge \frac{1}{2} B_{k,t,z}(x)$$

holds for any $x \in B(\xi_k(z), \sqrt{\mu_k}/R) \subset \mathbb{R}^n$ and for any $k \in \mathbb{N}$ and $(t,z) \in [1/A, A] \times B(0,1)$. In $\Phi^{-1}\left(B(\xi_k(z), \sqrt{\mu_k}/R)\right)$, we then have

$$|f(u_{k,t,z}) - f(W_{k,t,z}) - f'(W_{k,t,z})\varphi_k(t,z)| \le C\left(U_{k,t,z} \circ \Phi\right)^{2^*-3} \varphi_k(t,z)^2.$$

In $M \backslash \Phi^{-1}\left(B(\xi_k(z), \sqrt{\mu_k}/R)\right)$, (3.48) similarly shows that

$$|f(u_{k,t,z}) - f(W_{k,t,z}) - f'(W_{k,t,z})\varphi_k(t,z)| \le C|\varphi_k(t,z)|.$$

We let $j \in \{0, \dots, n\}$ and integrate (3.47) against $\hat Z_{j,k,t,z}$. Since $|\hat Z_{j,k,t,z}| \le C\left(U_{k,t,z} \circ \Phi\right)$ in M, by using the latter inequality together with straightforward computations, we obtain

$$\begin{aligned}
&\int_M \left(f(u_{k,t,z}) - f(W_{k,t,z}) - f'(W_{k,t,z})\varphi_k(t,z)\right) \hat Z_{j,k,t,z}\, dv_{\breve g} \\
&\quad = \mathrm{O}\left(\int_{\Phi^{-1}\left(B(\xi_k(z), \sqrt{\mu_k}/R)\right)} \left(U_{k,t,z} \circ \Phi\right)^{2^*-2} |\varphi_k(t,z)|^2 dv_{\breve g}\right) \\
&\qquad + \mathrm{O}\left(\int_{M \backslash \Phi^{-1}\left(B(\xi_k(z), \sqrt{\mu_k}/R)\right)} \left(U_{k,t,z} \circ \Phi\right) |\varphi_k(t,z)|\, dv_{\breve g}\right) \\
&\quad = \mathrm{O}\left(\|\varphi_k(t,z)\|^2_{H^1(M)} + \|\varphi_k(t,z)\|_{H^1(M)} \|B_{k,t,z}\|_{L^{\frac{2n}{n+2}}(\mathbb{R}^n \backslash B(\xi_k(z), \sqrt{\mu_k}/R))}\right) \\
&\quad = \mathrm{o}\left(\mu_k^{\frac{n-2}{2}}\right)
\end{aligned}$$

holds, where we have used (3.21) and (3.29) in the last line. With (3.33), (3.34), (3.46) and (3.47), the latter shows that

$$\begin{aligned}
(1 + \mathrm{o}(1)) \|\nabla V_j\|^2_{L^2(\mathbb{R}^n)} \lambda_k^j(t,z) &= \int_M E_{k,t,z} \hat Z_{j,k,t,z}\, dv_{\breve g} + \mathrm{o}\left(\mu_k^{\frac{n-2}{2}}\right) \\
&\quad + \int_M \varphi_k(t,z) \left(\triangle_{\breve g} + c_n \operatorname{Scal}_{\breve g} - f'(W_{k,t,z})\right) \hat Z_{j,k,t,z}\, dv_{\breve g}. \quad (3.49)
\end{aligned}$$

By (3.28) and Proposition 3.4, we have

$$\int_M E_{k,t,z} \hat Z_{0,k,t,z}\, dv_{\breve g} = \begin{cases} \dfrac{2}{n-2}\, t\, \partial_t \left(I\left(W_{k,t,z}\right)\right) = \mathrm{O}\left(\mu_k^{\frac{n-2}{2}}\right) & \text{if } j = 0 \\ -\, n\, t\, \partial_{z_j} \left(I\left(W_{k,t,z}\right)\right) = \mathrm{O}\left(\mu_k^{\frac{n-2}{2}}\right) & \text{if } 1 \le j \le n. \end{cases} \quad (3.50)$$

By (3.13), (3.18), (3.28) and (3.30), we have, for $0 \le j \le n$ and $x \in B(y_k, 2r_k)$,

$$\begin{aligned}&\left(\triangle_{\breve g} + c_n \operatorname{Scal}_{\breve g} - f'(W_{k,t,z})\right) \hat Z_{j,k,t,z}(\Phi^{-1}(x))\\ &\quad = \left(\triangle_{\tilde g} + c_n \operatorname{Scal}_{\tilde g} - f'(B_{k,t,z})\right)\left(Z_{j,k,t,z} + Q_{j,k,t,z}\right)(x) + \mathrm{O}\left(r_k^{-n}\mu_k^{\frac{n-2}{2}}\right)\\ &+ \mathrm{O}\left(\frac{\varepsilon_k \mu_k^{\frac{n+2}{2}}}{(\mu_k + |\xi_k(z) - x|)^n}\right) + \mathrm{O}\left(\begin{Bmatrix} B_{k,t,z}(x)^{2^*-2} & \text{if } |x - \xi_k(z)| \le \sqrt{\mu_k(t)}\\ B_{k,t,z}(x) & \text{otherwise}\end{Bmatrix}\right).\end{aligned}$$

Using then (3.24), (3.25), (3.26), (3.31) and (3.32), we find that, for $x \in B(y_k, 2r_k)$,

$$\left(\triangle_{\tilde g} + c_n \operatorname{Scal}_{\tilde g} - f'(B_{k,t,z})\right)\left(Z_{j,k,t,z} + Q_{j,k,t,z}\right)(x) = \mathrm{O}\left(\frac{\varepsilon_k \mu_k^{\frac{n-2}{2}}}{(\mu_k + |\xi_k(z) - x|)^{n-2}}\right)$$

holds. Thus,

$$\begin{aligned}&\left|\left(\triangle_{\breve g} + c_n \operatorname{Scal}_{\breve g} - f'(W_{k,t,z})\right) \hat Z_{j,k,t,z}\left(\Phi^{-1}(x)\right)\right|\\ &\quad \le C\left(r_k^{-n}\mu_k^{\frac{n-2}{2}} + \varepsilon_k B_{k,t,z}(x) + \begin{Bmatrix} B_{k,t,z}(x)^{2^*-2} & \text{if } |x - \xi_k(z)| \le \sqrt{\mu_k(t)}\\ B_{k,t,z}(x) & \text{otherwise}\end{Bmatrix}\right)\end{aligned} \tag{3.51}$$

holds for any $x \in B(y_k, 2r_k)$. As a consequence, with (3.1),

$$\begin{aligned}\left\|\Big(\triangle_{\breve g} + c_n \operatorname{Scal}_{\breve g} - f'(W_{k,t,z})\Big)\hat Z_{j,k,t,z}\right\|_{L^{\frac{2n}{n+2}}(M)} &\le C\left(r_k^{-n}\mu_k^{\frac{n-2}{2}} + \varepsilon_k \mu_k^2 + \mu_k^{\frac{n+2}{4}}\right)\\ &\le C\mu_k^{\frac{n-2}{4}}.\end{aligned}$$

Hence, with (3.21) and (3.29), Hölder's inequality shows that

$$\left|\int_M \varphi_k(t,z)\left(\triangle_{\breve g} + c_n \operatorname{Scal}_{\breve g} - f'(W_{k,t,z})\right)\hat Z_{j,k,t,z}\, dv_{\breve g}\right| = \mathrm{o}\left(\mu_k^{\frac{n-2}{2}}\right). \tag{3.52}$$

Together with (3.49) and (3.50), this shows that

$$\sum_{j=0}^n |\lambda_k^j(t,z)| = \mathrm{O}\left(\mu_k^{\frac{n-2}{2}}\right). \tag{3.53}$$

We can now use (3.46) to write that

$$\begin{aligned}\partial_t\left(I\left(u_{k,t,z}\right)\right) &= DI\left(u_{k,t,z}\right)\left(\partial_t u_{k,t,z}\right)\\ &= \sum_{j=0}^n \lambda_k^j(t,z)\big(\hat Z_{j,k,t,z}, \partial_t W_{k,t,z} + \partial_t \varphi_k(t,z)\big)_{H^1(M)},\end{aligned}$$

where $W_{k,t,z}$ is as in (3.18). On the one side, with (3.9), (3.10), (3.33) and (3.34), we have

$$\big(\hat Z_{j,k,t,z}, \partial_t W_{k,t,z}\big)_{H^1(M)} = \frac{n-2}{2t}\delta_{j0}\|\nabla V_0\|^2_{L^2(\mathbb{R}^n)} + \mathrm{o}(1)$$

as $k \to \infty$, where δ_{j0} is the Kronecker symbol. On the other side, since $\varphi_k(t,z) \in \hat K^\perp_{k,t,z}$, we have $\big(\varphi_k(t,z), \hat Z_{j,k,t,z}\big)_{H^1(M)} = 0$ for all $0 \le j \le n$, hence

$$\big(\hat Z_{j,k,t,z}, \partial_t \varphi_k(t,z)\big)_{H^1(M)} = -\big(\partial_t \hat Z_{j,k,t,z}, \varphi_k(t,z)\big)_{H^1(M)} = \mathrm{o}(1),$$

where we have used (3.21), (3.29), (3.33) and (3.34). Together with (3.53), we thus obtain that

$$\partial_t \left(I\left(u_{k,t,z}\right)\right) = \frac{n-2}{2t}\delta_{j0}\|\nabla V_0\|^2_{L^2(\mathbb{R}^n)}\lambda_k^0(t,z) + \mathrm{o}\left(\mu_k^{\frac{n-2}{2}}\right).$$

Together with (3.49), (3.50) and (3.52), this becomes

$$\partial_t \left(I\left(u_{k,t,z}\right)\right) = \partial_t \left(I\left(W_{k,t,z}\right)\right) + \mathrm{o}\left(\mu_k^{\frac{n-2}{2}}\right),$$

which concludes the proof of Proposition 3.5. □

4. Proof of Theorem 2.2 – Part 2: Finding critical points of the reduced energy

Let $n \geq 11$ and let $W : (\mathbb{R}^n)^4 \to \mathbb{R}$ be a four-linear form in $\mathbb{R}^n$ possessing the same symmetries as a Weyl tensor, that is $W_{ijk\ell} = -W_{jik\ell} = -W_{ij\ell k} = W_{k\ell ij}$, $\sum_{i=1}^n W_{iji\ell} = 0$ and

$$W_{ijk\ell} + W_{jki\ell} + W_{kij\ell} = 0$$

for any $i,j,k,\ell \in \{1,\dots,n\}$. Assume that W is not identically zero, that is

$$|W|^2 = \sum_{i,j,k,\ell=1}^n (W_{ijk\ell})^2 > 0.$$

For any $k,\ell \in \{1,\dots,n\}$, let

$$T_{k\ell} := \sum_{p,q,r=1}^n \left(W_{kpqr} + W_{krqp}\right)\left(W_{\ell pqr} + W_{\ell rqp}\right). \tag{4.1}$$

Straightforward computations using the symmetries of W show that

$$T_{k\ell} = 3\sum_{p,q,r=1}^n W_{kpqr}W_{\ell pqr} \quad \text{and} \quad \sum_{k=1}^n T_{kk} = 3|W|^2 > 0.$$

We let, for $x \in \mathbb{R}^n$ and $1 \leq i,j \leq n$,

$$h(x)_{ij} := \frac{1}{3}\sum_{p,q=1}^n W_{ipjq}x_p x_q. \tag{4.2}$$

Using the symmetries of W, it is easily seen that h satisfies (3.2). We define, for $t > 0$ and $z \in B(0,1)$,

$$\begin{aligned} F(t,z) := {} & \frac{1}{4}\int_{\mathbb{R}^n}\sum_{i,j,p=1}^n h_{ip}\,h_{pj}\,\partial_i B_{t,z}\,\partial_j B_{t,z}\,dx - \frac{n-2}{32(n-1)}\int_{\mathbb{R}^n}\sum_{i,j,\ell=1}^n (\partial_i h_{j\ell})^2 B_{t,z}^2\,dx \\ & - \frac{1}{2}\int_{\mathbb{R}^n}|\nabla R_{t,z}|^2 dx + \Lambda(n)\check{u}_0(x_0)t^{\frac{n-2}{2}} \\ =: {} & F_1(t,z) + F_2(t,z) + F_3(t,z) + \Lambda(n)\check{u}_0(x_0)t^{\frac{n-2}{2}}, \end{aligned} \tag{4.3}$$

where $\check{u}_0$ is as in (3.5) and x_0 is as in the beginning of Section 3, $R_{t,z}$ is defined by (3.14) for h given by (4.2), $B_{t,z}$ is as in (3.16) and $\Lambda(n)$ is the positive numerical constant given by (3.38). The function F is smooth in $(0,\infty)\times B(0,1)$. By (3.15), the uniqueness of $R_{t,z}$ and since $h(-x) = h(x)$ for all $x \in \mathbb{R}^n$, we have, for any

$t>0, z \in B(0,1)$ and $x \in \mathbb{R}^n$, $R_{t,-z}(x) = R_{t,z}(-x)$ and thus $F(t,-z) = F(t,z)$. This shows that, for all $t>0$ and $1 \le i,j,k \le n$,

$$\partial_t \partial_{z_i} F(t,z)_{|z=0} = 0 \quad \text{and} \quad \partial_t \partial^3_{z_i z_j z_k} F(t,z)_{|z=0} = 0. \tag{4.4}$$

For real numbers $p, q \ge 0$ with $p > q+2$, we define

$$I_p^q := \int_0^\infty \frac{r^q}{(1+r)^p} dr.$$

The following induction formulas are easily proven (see Aubin [3]):

$$I_{p+1}^q = \frac{p-q-1}{p} I_p^q \quad \text{and} \quad I_{p+1}^{q+1} = \frac{q+1}{p-q-1} I_{p+1}^q. \tag{4.5}$$

From (4.5) the following expressions are easily obtained:

$$\begin{aligned} I_n^{\frac{n}{2}} &= \frac{n}{n-2} I_n^{\frac{n-2}{2}}, \quad I_n^{\frac{n+2}{2}} = \frac{n+2}{n-4}\frac{n}{n-2} I_n^{\frac{n-2}{2}}, \, I_{n+2}^{\frac{n+4}{2}} = \frac{(n+4)(n+2)}{4(n-2)(n+1)} I_n^{\frac{n-2}{2}}, \\ I_{n+1}^{\frac{n+2}{2}} &= \frac{n+2}{2(n-2)} I_n^{\frac{n-2}{2}}, \, I_{n-2}^{\frac{n-2}{2}} = \frac{4(n-1)}{n-4} I_n^{\frac{n-2}{2}}, \, I_{n-2}^{\frac{n}{2}} = \frac{4n(n-1)}{(n-4)(n-6)} I_n^{\frac{n-2}{2}}. \end{aligned} \tag{4.6}$$

Recall that, for any $t>0$ and $z \in \mathbb{R}^n$, we have $\int_{\mathbb{R}^n} B_{t,z}^{2^*} dx = K_n^{-n}$. A simple change of variables then yields

$$I_n^{\frac{n-2}{2}} = \frac{2}{\omega_{n-1}} \left(n(n-2)\right)^{-\frac{n}{2}} K_n^{-n}, \tag{4.7}$$

where ω_{n-1} is the volume of the standard unit sphere of dimension $n-1$. This allows us to compute the integrals in (4.6).

We first prove the following lemma which shows the existence of a strict global minimum of $t \mapsto F(t,0)$:

Lemma 4.1. *Assume that $n \ge 11$. Then $t \in [0,\infty) \mapsto F(t,0)$ possesses a unique global minimum $t_0 > 0$ such that $F(t_0,0) < 0$. This minimum is also non-degenerate, meaning that $\partial_t^2 F(t_0,0) > 0$.*

Proof. First, by (3.15) and the uniqueness of $R_{t,z}$, we have $R_{t,0} = 0$, hence $F_3(t,0) = 0$ for all $t>0$. Let $t>0$ and $z \in B(0,1)$. Since $\sum_{j=1}^n x_j h(x)_{ij} = 0$ for all $1 \le i \le n$ and by definition of $B_{t,z}$, we have

$$\begin{aligned} &\frac{1}{4} \sum_{i,j,p=1}^n h(x)_{ip} h(x)_{pj} \, \partial_i B_{t,z}(x) \, \partial_j B_{t,z}(x) \\ &\qquad = \frac{1}{4n^2} \frac{t^{n-2}}{\left(t^2 + \frac{|x-z|^2}{n(n-2)}\right)^n} \sum_{i,j,p=1}^n h(x)_{ip} h(x)_{pj} z_i z_j. \end{aligned} \tag{4.8}$$

Thus $F_1(t,0) = 0$ and $F(t,0) = F_2(t,0) + \Lambda(n) \check{u}_0(x_0) t^{\frac{n-2}{2}}$ for any $t>0$. By (4.2), we have, for all $x \in \mathbb{R}^n$,

$$\partial_i h_{j\ell}(x) = \frac{1}{3} \sum_{p=1}^n (W_{ji\ell p} + W_{jp\ell i}) x_p. \tag{4.9}$$

Hence, by (4.3), we have

$$F_2(t,0) = -\frac{n-2}{32(n-1)} t^4 \int_{\mathbb{S}^{n-1}} \sum_{i,j,\ell=1}^n (\partial_i h_{j\ell}(x))^2 d\sigma(x) \cdot \int_0^\infty \frac{r^{n+1} dr}{\left(1 + \frac{r^2}{n(n-2)}\right)^{n-2}}.$$

Recall that
$$\int_{\mathbb{S}^{n-1}} x_i x_j \, d\sigma(x) = \frac{\omega_{n-1}}{n}\delta_{ij} \tag{4.10}$$
for all $i,j \in \{1,\dots,n\}$. As a consequence, we have
$$\int_{\mathbb{S}^{n-1}} \sum_{i,j,\ell=1}^n (\partial_i h_{j\ell}(x))^2 d\sigma(x) = \frac{\omega_{n-1}}{9n}\sum_{k=1}^n T_{kk},$$
where T_{kk} is as in (4.1). Independently, straightforward computations with (4.6) and (4.7) give that
$$\int_0^\infty \frac{r^{n+1}dr}{\left(1+\frac{r^2}{n(n-2)}\right)^{n-2}} = \frac{4n^2(n-1)(n-2)}{(n-4)(n-6)}\frac{K_n^{-n}}{\omega_{n-1}}.$$
This shows that
$$F(t,0) = -\left(\frac{n(n-2)^2}{72(n-4)(n-6)}K_n^{-n}\sum_{k=1}^n T_{kk}\right)t^4 + \Lambda(n)\check{u}_0(x_0)t^{\frac{n-2}{2}}. \tag{4.11}$$
Since $\Lambda(n) > 0$, $\check{u}_0 > 0$ in M, $\sum_{k=1}^n T_{kk} > 0$ and $n \geq 11$, Lemma 4.1 follows. $\square$

The value of t_0 only depends on n, W and $\check{u}_0$, that have been fixed from the beginning. From now on, we will let $A > 0$ be chosen large enough so that $t_0 \in [2/A, A/2]$. We now show that the second derivatives of F at $(t,0)$ in z vanish for every $t > 0$:

Proposition 4.2. *Let $n \geq 11$ and let $t > 0$ be fixed. Then, for any $1 \leq k,\ell \leq n$,*
$$\partial^2_{z_k z_\ell} F(t,z)_{|z=0} = 0.$$

Proof. Let $t > 0$ and $k,\ell \in \{1,\dots,n\}$ be fixed throughout this proof. First, by (3.15) and the uniqueness of $R_{t,z}$, we have $R_{t,0} = 0$ and $\partial_{z_i}(R_{t,z})_{|z=0} = 0$ for all $1 \leq i \leq n$. As a consequence, $\partial^2_{z_k z_\ell} F_3(t,z)_{|z=0} = 0$. On the one hand, with (4.8) and differentiating under the integral, we get that
$$\begin{aligned}\partial^2_{z_k z_\ell} F_1(t,z)_{|z=0} &= \frac{1}{2n^2}\int_{\mathbb{R}^n}\sum_{p=1}^n h(x)_{kp}\, h(x)_{p\ell}\frac{t^{n-2}dx}{\left(t^2+\frac{|x|^2}{n(n-2)}\right)^n}.\\ &= \frac{t^2}{2n^2}\int_{\mathbb{S}^{n-1}}\sum_{p=1}^n h(x)_{kp}\, h(x)_{p\ell}\, d\sigma(x)\cdot\int_0^\infty \frac{r^{n+3}dr}{\left(1+\frac{r^2}{n(n-2)}\right)^n}.\end{aligned}$$
By using the symmetries of W, it is easily seen that
$$\sum_{p,i,j=1}^n W_{kpij}\left(W_{\ell pij}+W_{\ell jip}\right) = \frac{1}{2}T_{k\ell},$$
where $T_{k\ell}$ is as in (4.1). Using (4.2) and Corollary 29 of Brendle [7], we thus obtain
$$\int_{\mathbb{S}^{n-1}}\sum_{p=1}^n h(x)_{kp}\, h(x)_{p\ell}\, d\sigma(x) = \frac{\omega_{n-1}}{18n(n+2)}T_{k\ell}.$$
Together with (4.6) and (4.7), we then obtain
$$\partial^2_{z_k z_\ell} F_1(t,z)_{|z=0} = \frac{t^2}{36}\frac{n-2}{n-4}K_n^{-n}T_{k\ell}. \tag{4.12}$$

By using (4.9) together with change of variables, we obtain

$$F_2(t,z) = -\frac{n-2}{288(n-1)} \int_{\mathbb{R}^n} \sum_{p,q=1}^n T_{pq}(x+z)_p(x+z)_q \frac{t^{n-2}dx}{\left(t^2 + \frac{|x|^2}{n(n-2)}\right)^{n-2}},$$

so that differentiating and using (4.6), (4.7) and (4.10) yields

$$\partial^2_{z_k z_\ell} F_2(t,z)_{|z=0} = -\frac{n-2}{144(n-1)} T_{k\ell} \int_{\mathbb{R}^n} \frac{t^2 dx}{\left(1 + \frac{|x|^2}{n(n-2)}\right)^{n-2}} = -\frac{t^2}{36} \frac{n-2}{n-4} K_n^{-n} T_{k\ell}. \tag{4.13}$$

Together with (4.12), this concludes the proof of Proposition 4.2. □

In order to prove that $(t_0, 0)$ is a non-degenerate critical point of F, we now expand, for a fixed value of t, $z \mapsto F(t,z)$ to the fourth order as $z \to 0$. A first, obvious, remark is that for $i,j,k,l \in \{1,\dots,n\}$, we have

$$\partial^4_{z_i z_j z_k z_l} F_2(t,z)_{|z=0} = 0. \tag{4.14}$$

This easily follows from the expression of F_2 given by (4.3). Indeed, by (4.2), $\sum_{i,j,\ell=1}^n (\partial_i h_{j\ell}(x))^2$ is a second-order polynomial in x, hence $F_2(t,z)$ is a second-order polynomial in z by a change of variables. We now expand $F_1 + F_2$:

Proposition 4.3. *Let $t > 0$. We have, as $z \to 0$,*

$$F_1(t,z) + F_2(t,z) = F_2(t,0) + \frac{K_n^{-n}}{4n} \sum_{p,q=1}^n \left(h_{pq}(z)\right)^2 + \mathrm{O}\left(|z|^5\right).$$

Proof. By definition, F_1 and F_2 are even in z. The expansions to second-order thus follow from $F_1(t,0) = 0$, (4.12) and (4.13). By (4.14), we only have to expand $F_1(t,z)$ to fourth order. By (4.3), (4.8) and a change of variables, we have

$$F_1(t,z) = \frac{1}{4n^2} \int_{\mathbb{R}^n} \frac{t^{n-2}}{\left(t^2 + \frac{|x|^2}{n(n-2)}\right)^n} \sum_{i,j,p=1}^n h(x+z)_{ip}\, h(x+z)_{pj} z_i z_j \, dx.$$

It is easily seen that, for any $x, z \in \mathbb{R}^n$ and $1 \le a,b,c,d \le n$,

$$\begin{aligned}(x+z)_a(x+z)_b(x+z)_c(x+z)_d &= x_a x_b x_c x_d + x_a x_b x_c z_d + x_a x_b x_d z_c + x_a x_c x_d z_b \\ + x_b x_c x_d z_a + x_a x_b z_c z_d &+ x_a x_c z_b z_d + x_a x_d z_b z_c + x_b x_c z_a z_d + x_b x_d z_a z_c + x_c x_d z_a z_b \\ &+ \mathrm{O}\left(|z|^3|x| + |z|^4\right).\end{aligned}$$

As a consequence and by (4.2), we have, as $z \to 0$,

$$F_1(t,z) + F_2(t,z) = F_2(t,0) + G(z) + \mathrm{O}\left(|z|^5\right),$$

where

$$\begin{aligned}G(z) := \frac{1}{36n^2} \int_{\mathbb{R}^n} \left(1 + \frac{|x|^2}{n(n-2)}\right)^{-n} &\sum_{i,j,p,a,b,c,d=1}^n W_{iapb} W_{pcjd} z_i z_j (x_a x_b z_c z_d \\ &+ x_a x_c z_b z_d + x_a x_d z_b z_c + x_b x_c z_a z_d + x_b x_d z_a z_c + x_c x_d z_a z_b)\, dx.\end{aligned}$$

By (4.6), (4.7) and (4.10), we have, for $1 \le a, b \le n$,

$$\int_{\mathbb{R}^n} \frac{x_a x_b \, dx}{\left(1 + \frac{|x|^2}{n(n-2)}\right)^n} = n K_n^{-n} \delta_{ab}.$$

Since W is totally traceless, we thus obtain

$$G(z) = \frac{1}{36n} K_n^{-n} \sum_{i,j,p,a,b,c=1}^{n} \big(W_{iapb} W_{pajc} z_b z_c z_i z_j + W_{iapb} W_{pcja} z_b z_c z_i z_j + W_{iapb} W_{pbjc} z_a z_c z_i z_j + W_{iapb} W_{pcjb} z_a z_c z_i z_j \big). \quad (4.15)$$

Since $W_{pajc} = -W_{pacj}$, we have $\sum_{i,j,p,a,b,c=1}^{n} W_{iapb} W_{pajc} z_b z_c z_i z_j = 0$. Similarly, we obtain that the last two terms in the right-hand side of (4.15) vanish. Since moreover $W_{pcja} = W_{japc}$, we then obtain

$$\begin{aligned} G(z) &= \frac{1}{36n} K_n^{-n} \sum_{i,j,p,a,b,c=1}^{n} W_{iapb} W_{japc} z_b z_c z_i z_j \\ &= \frac{1}{36n} K_n^{-n} \sum_{a,p=1}^{n} \left(- \sum_{i,b=1}^{n} W_{iabp} z_i z_b \right)^2 = \frac{1}{4n} K_n^{-n} \sum_{a,p=1}^{n} (h_{ap}(z))^2 \end{aligned}$$

by (4.2), which concludes the proof of Proposition 4.3. □

We next expand $F_3(t,z)$ at fourth-order in z at $(t,0)$ for some fixed $t > 0$:

Proposition 4.4. *Let $t > 0$. As $z \to 0$, we have*

$$F_3(t,z) = -\frac{n+4}{48(n+1)} K_n^{-n} \sum_{p,q=1}^{n} (h_{pq}(z))^2 + \mathrm{O}\left(|z|^5\right).$$

Proof. By (3.15), for $(t,z) \in (0,\infty) \times \mathbb{R}^n$, we have $R_{t,z} = \tilde{R}_{t,z}(x-z)$, where $\tilde{R}_{t,z}$ is the unique solution in $K_{t,0}^{\perp}$ of

$$\triangle_{\delta_0} \tilde{R}_{t,z}(x) - (2^*-1) B_{t,0}(x)^{2^*-2} \tilde{R}_{t,z}(x) = -\frac{t^{\frac{n-2}{2}}}{n(n-2)} \frac{\sum_{p,q=1}^{n} h_{pq}(x+z) z_p z_q}{\left(t^2 + \frac{|x|^2}{n(n-2)}\right)^{\frac{n+2}{2}}}, \quad (4.16)$$

where h is given by (4.2). Hence, for fixed $t > 0$, all the derivatives of $R_{t,z}$ with respect to z, taken at $z = 0$, belong to $K_{t,0}^{\perp}$. We let in what follows, for $a, b \in \{1, \ldots, n\}$,

$$L_{ab} := \partial^2_{z_a z_b} \left(R_{t,z}\right)_{|z=0} \in K_{t,0}^{\perp}.$$

Since $R_{t,0} = 0$ and $\partial_{z_i} \left(R_{t,z}\right)_{|z=0} = 0$ for all $1 \le i \le n$, it is easily seen that, for $1 \le a, b, c, d \le n$, we have

$$\partial_{z_a} F_3(t,z)_{|z=0} = 0, \quad \partial^2_{z_a z_b} F_3(t,z)_{|z=0} = 0, \quad \partial^3_{z_a z_b z_c} F_3(t,z)_{|z=0} = 0 \quad \text{and}$$
$$\partial^4_{z_a z_b z_c z_d} F_3(t,z)_{|z=0} = -\int_{\mathbb{R}^n} \left(\langle \nabla L_{ab}, \nabla L_{cd} \rangle + \langle \nabla L_{ac}, \nabla L_{bd} \rangle + \langle \nabla L_{ad}, \nabla L_{bd} \rangle \right) dx. \quad (4.17)$$

Differentiating (4.16) shows that L_{ab} satisfies

$$\triangle_{\delta_0} L_{ab}(x) - (2^*-1) B_{t,0}^{2^*-2} L_{ab}(x) = -\frac{2t^{\frac{n-2}{2}}}{n(n-2)} \frac{h_{ab}(x)}{\left(t^2 + \frac{|x|^2}{n(n-2)}\right)^{\frac{n+2}{2}}}.$$

In particular, for any $x \in \mathbb{R}^n$, we have $L_{ab}(x) = t^{1-\frac{n}{2}} L^0_{ab}(x/t)$, where $L^0_{ab} \in K^\perp_{1,0}$ satisfies

$$\triangle_{\delta_0} L^0_{ab}(x) - (2^* - 1) B_{1,0}^{2^*-2} L^0_{ab}(x) = -\frac{2}{n(n-2)} \frac{h_{ab}(x)}{\left(1 + \frac{|x|^2}{n(n-2)}\right)^{\frac{n+2}{2}}}. \tag{4.18}$$

Coming back to (4.17), we also have, with a change of variables,

$$\partial^4_{z_a z_b z_c z_d} F_3(t,z)_{|z=0} = -\int_{\mathbb{R}^n} \left(\langle \nabla L^0_{ab}, \nabla L^0_{cd}\rangle + \langle \nabla L^0_{ac}, \nabla L^0_{bd}\rangle + \langle \nabla L^0_{ad}, \nabla L^0_{bd}\rangle\right) dx. \tag{4.19}$$

We first find an explicit expression for L^0_{ab}. Since W is totally traceless, h is a harmonic polynomial in $\mathbb{R}^n$. As a consequence, we can look for a solution of (4.18) under the form

$$L^0_{ab}(x) = h_{ab}(x) f(|x|^2).$$

Simple computations show that $f(t) = -\frac{1}{n}\left(1 + \frac{t}{n(n-2)}\right)^{-n/2}$ satisfies (4.18) for all $t \in \mathbb{R}$. By using (4.2), we thus obtain

$$L^0_{ab}(x) = -\frac{1}{n} \frac{h_{ab}(x)}{\left(1 + \frac{|x|^2}{n(n-2)}\right)^{\frac{n}{2}}} = -\frac{1}{3n} \frac{\sum_{p,q=1}^n W_{apbq} x_p x_q}{\left(1 + \frac{|x|^2}{n(n-2)}\right)^{\frac{n}{2}}} \tag{4.20}$$

for $1 \le a, b \le n$ and all $x \in \mathbb{R}^n$. Note that L^0_{ab} given by (4.20) belongs to $K^\perp_{1,0}$, which follows again from the symmetries of W, and is thus the unique solution of (4.18) in $K^\perp_{1,0}$. Straightforward computations with (4.20) then show that

$$\int_{\mathbb{R}^n} \langle \nabla L^0_{ab}, \nabla L^0_{cd}\rangle \, dx = K_1 + K_2 + K_3, \tag{4.21}$$

where

$$A_1 := \frac{1}{9n^2} \int_{\mathbb{R}^n} \left(1 + \frac{|x|^2}{n(n-2)}\right)^{-n} \sum_{i,p,q=1}^n (W_{aibp} + W_{apbi})(W_{cidq} + W_{cqdi}) x_p x_q \, dx,$$

$$A_2 := \frac{1}{n^2(n-2)^2} \int_{\mathbb{R}^n} \left(1 + \frac{|x|^2}{n(n-2)}\right)^{-n-2} |x|^2 h_{ab}(x)\, h_{cd}(x)\, dx,$$

$$A_3 := -\frac{4}{n^2(n-2)} \int_{\mathbb{R}^n} \left(1 + \frac{|x|^2}{n(n-2)}\right)^{-n-1} h_{ab}(x) h_{cd}(x)\, dx.$$

By (4.6), (4.7) and (4.10), we have

$$\begin{aligned} A_1 &= \frac{1}{9n} K_n^{-n} \sum_{p,q=1}^n (W_{apbq} + W_{aqbp})(W_{cpdq} + W_{cqdp}) \\ &= \frac{2}{9n} K_n^{-n} \sum_{p,q=1}^n W_{apbq} (W_{cpdq} + W_{cqdp}). \end{aligned}$$

Using again Corollary 29 of Brendle [7], we have

$$\int_{\mathbb{S}^{n-1}} h_{ab}(x)\, h_{cd}(x)\, d\sigma(x) = \frac{1}{9n(n+2)} \omega_{n-1} \sum_{p,q=1}^n W_{apbq} (W_{cpdq} + W_{cqdp}).$$

Together with (4.6) and (4.7), we thus obtain

$$\begin{aligned}A_2 &= \frac{1}{n^2(n-2)^2}\int_0^\infty \frac{r^{n+5}}{(1+\frac{r^2}{n(n-2)})^{n+2}}dr\cdot\int_{\mathbb{S}^{n-1}} h_{ab}(x)\,h_{cd}(x)\,d\sigma(x)\\ &= \frac{n+4}{36(n+1)}K_n^{-n}\sum_{p,q=1}^n W_{apbq}\left(W_{cpdq}+W_{cqdp}\right).\end{aligned}$$

Similarly, we get

$$A_3 = -\frac{2}{9n}K_n^{-n}\sum_{p,q=1}^n W_{apbq}\left(W_{cpdq}+W_{cqdp}\right).$$

Combining the latter equalities with (4.21) finally shows that

$$\int_{\mathbb{R}^n}\langle\nabla L^0_{ab},\nabla L^0_{cd}\rangle\,dx = \frac{n+4}{36(n+1)}K_n^{-n}\sum_{p,q=1}^n W_{apbq}\left(W_{cpdq}+W_{cqdp}\right).$$

Going back to (4.19), we have thus proven that

$$\begin{aligned}\partial^4_{z_az_bz_cz_d}F_3(t,z)_{|z=0} = -\frac{n+4}{36(n+1)}K_n^{-n}\sum_{p,q=1}^n &(W_{apbq}(W_{cpdq}+W_{cqdp})\\ &+W_{apcq}(W_{bpdq}+W_{bqdp})+W_{apdq}(W_{bpcq}+W_{bqcp})).\end{aligned}\tag{4.22}$$

Using the symmetries of W, it is now easily seen that

$$\sum_{a,b,c,d,p,q=1}^n W_{apbq}\left(W_{cpdq}+W_{cqdp}\right)z_az_bz_cz_d = 18\sum_{p,q=1}^n\left(h_{pq}(z)\right)^2.$$

A Taylor formula for F_3 at fourth order in $(t,0)$ thus shows with (4.22) that

$$\begin{aligned}F_3(t,z) &= \frac{1}{24}\sum_{a,b,c,d=1}^n \partial^4_{z_az_bz_cz_d}F_3(t,z)_{|z=0}z_az_bz_cz_d + \mathrm{O}\left(|z|^5\right)\\ &= -\frac{n+4}{48(n+1)}K_n^{-n}\sum_{p,q=1}^n\left(h_{pq}(z)\right)^2+\mathrm{O}\left(|z|^5\right),\end{aligned}$$

which concludes the proof of Proposition 4.4. □

Combining Propositions 4.2, 4.3 and 4.4 and since F is even in z, we have therefore proven that, for any $t>0$ fixed,

$$F(t,z) = F(t,0) - \frac{n^2-8n-12}{48n(n+1)}K_n^{-n}\sum_{p,q=1}^n\left(h_{pq}(z)\right)^2+\mathrm{O}\left(|z|^5\right)\tag{4.23}$$

as $z\to 0$. It is easily checked that $n^2-8n-12>0$ for $n\geq 11$. We are now in position to conclude the proof of Theorem 2.2.

Proof of Theorem 2.2. Let $(M,\hat g)$ be a locally conformally flat manifold that is Y-non-degenerate in the sense of Definition 2.1. Let $W:(\mathbb{R}^n)^4\to\mathbb{R}$ be a four-linear form as in Section 4 and h be defined by (4.2). By the symmetries of W, h satisfies (3.2). We assume that W is chosen so that

$$\sum_{p,q=1}^n\left(h_{pq}(z)\right)^2\geq C|z|^4\quad\text{ for all } z\in\mathbb{R}^n\tag{4.24}$$

for some constant $C > 0$ independent of z. Assumption (4.24) is for example satisfied when W has the following diagonal form:

$$W_{ijk\ell} = \frac{A_{ij}}{2}\left(\delta_{ik}\delta_{j\ell} - \delta_{jk}\delta_{i\ell}\right),$$

where A is a nonzero symmetric matrix satisfying $A_{ii} = 0$ and $\sum_{j=1}^n A_{ij} = 0$ for any $1 \le i \le n$ and $A_{ij} \neq 0$ whenever $i \neq j$. Let $(\varepsilon_k)_k, (\mu_k)_k$ and $(r_k)_k$ be sequences as in Section 3, $\tilde{g}$ and g be defined as in (3.3) and $\check{g}$ and $\check{u}_0$ be defined as in (3.5). The analysis of Section 3 applies. For $(t,z) \in [1/A, A] \times B(0,1)$, we let $u_{k,t,z} := W_{k,t,z} + \varphi_k(t,z)$, where $W_{k,t,z}$ is as in (3.18) and $\varphi_k(t,z)$ is given by Proposition 3.3. For $k \in \mathbb{N}$, we let $F_k : [1/A, A] \times B(0,1) \to \mathbb{R}$ be defined by

$$F_k(t,z) := \mu_k^{1-\frac{n}{2}}\Big(I\left(u_{k,t,z}\right) - \frac{1}{n}K_n^{-n} - I(\check{u}_0)\Big).$$

By Propositions 3.4 and 3.5, we have

$$F_k(t,z) = F(t,z) + \Lambda_k^0(t,z) \quad \text{and} \quad \partial_t F_k(t,z) = \partial_t F(t,z) + \Lambda_k^1(t,z) \tag{4.25}$$

uniformly with respect to $(t,z) \in [1/A, A] \times B(0,1)$, where Λ_k^0 and Λ_k^1 converge uniformly to 0 in compact subsets of $[1/A, A] \times B(0,1)$ and F is as in (4.3). By Lemma 4.1, (4.23) and (4.24), F has a saddle-type geometry on $[1/A, A] \times B(0,1)$, we can let ε and η be small enough so that

$$\min_{t \in [t_0-\eta, t_0+\eta]} F(t,0) = F(t_0,0) > \max_{t \in [t_0-\eta, t_0+\eta], |z|=\varepsilon} F(t,z)$$

and $\partial_t F(t_0-\eta, z) < 0$ and $\partial_t F(t_0+\eta, z) > 0$ for any $z \in B(0,\varepsilon)$. Since $F(t_0,0) < 0$, by decreasing ε and η if necessary, we can also assume that

$$\max_{(t,z) \in [t_0-\eta, t_0+\eta] \times \overline{B(0,\varepsilon)}} F(t,z) < 0.$$

By (4.25), choosing k large enough, we again have

$$\min_{t \in [t_0-\eta, t_0+\eta]} F_k(t,0) > \max_{t \in [t_0-\eta, t_0+\eta], |z|=\varepsilon} F_k(t,z),$$

and $\partial_t F_k(t_0-\eta, z) < 0$ and $\partial_t F_k(t_0+\eta, z) > 0$ for any $z \in B(0,\varepsilon)$. Thus F_k also has a saddle-type geometry for k large enough. Lemma A.1 of Thizy–Vétois [55] then applies and shows that, for k large enough, F_k admits a critical point (t_k, z_k) in $(t_0-\eta, t_0+\eta) \times B(0,\varepsilon)$. By Proposition 3.3, u_{k,t_k,z_k} is then a solution of (3.6). Standard arguments show that u_k blows-up as $k \to \infty$. Finally, by Propositions 3.4 and 3.5 and since u_{k,t_k,z_k} solves (3.6), we have

$$\begin{aligned}\int_M |u_{k,t_k,z_k}|^{2^*} dv_{\tilde{g}} = nI\left(u_{k,t_k,z_k}\right) &= K_n^{-n} + \int_M \check{u}_0^{2^*} dv_{\check{g}} + n\mu_k^{\frac{n-2}{2}}\left(F(t_k,z_k) + \mathrm{o}(1)\right)\\ &< K_n^{-n} + \int_M \check{u}_0^{2^*} dv_{\check{g}}\end{aligned}$$

since $F(t_k,z_k) < 0$. Since $\int_M \check{u}_0^{2^*} dv_{\check{g}} = Y(M,[g])^{\frac{n}{2}}$ by the Y-non-degeneracy assumption, this concludes the proof of Theorem 2.2. ☐

Remark 4.5. *If $\tilde{g}$ is defined as in* (3.3)*, it follows from an expansion at first-order in $x - y_k$ that $\tilde{g}$ satisfies*

$$\mathrm{Weyl}_{\tilde{g}}(y_k) = -\varepsilon_k W$$

for any $k \in \mathbb{N}$. In view of (4.11)*, Proposition 3.4 then shows that*

$$I\left(W_{k,t,0}\right) = \frac{1}{n} K_n^{-n} + I(\check{u}_0) + \Lambda(n)\check{u}_0(x_0)\mu_k(t)^{\frac{n-2}{2}} - C(n)|\operatorname{Weyl}_{\tilde{g}}(y_k)|_{\tilde{g}}^2 \mu_k(t)^4 + \mathrm{o}\left(\mu_k^{\frac{n-2}{2}}\right),$$

an expansion that is reminiscent of those in Esposito–Pistoia–Vétois [21] *and Premoselli–Wei* [43]*.*

5. Asymptotic analysis at the lowest sign-changing energy level and proof of Theorem 1.2

We begin this section with a simple result showing that $Y\left(\mathbb{S}^n, [g_0]\right)^{\frac{n}{2}} + Y\left(M, [g]\right)^{\frac{n}{2}}$ is the minimal energy level for the blow-up of sign-changing solutions of (1.1):

Proposition 5.1. *Let (M, g) be a smooth, closed Riemannian manifold of dimension $n \geq 3$ and positive Yamabe type. Let $(u_k)_{k\in\mathbb{N}}$ be a blowing-up sequence of solutions to* (1.1)*. If u_k changes sign for large k and blows-up as $k \to \infty$, then*

$$E\left(u_k\right) \geq Y\left(M, [g]\right)^{\frac{n}{2}} + Y\left(\mathbb{S}^n, [g_{std}]\right)^{\frac{n}{2}} + \mathrm{o}\left(1\right) \tag{5.1}$$

as $k \to \infty$. Moreover, if equality holds true in (5.1) *and (M, g) is not conformally diffeomorphic to $(\mathbb{S}^n, g_{std})$, then up to a subsequence and replacing u_k by $-u_k$ if necessary, u_k is of the form*

$$u_k = u_0 - \left(\frac{\mu_k}{\mu_k^2 + \frac{d_g(\cdot,\xi_k)^2}{n(n-2)}}\right)^{\frac{n-2}{2}} + \mathrm{o}\left(1\right) \text{ in } H^1\left(M\right) \text{ as } k \to \infty, \tag{5.2}$$

where u_0 is a positive solution to the Yamabe equation (1.2) *that attains $Y\left(M, [g]\right)$, d_g is the geodesic distance with respect to the metric g and $\left(\mu_k\right)_k$ and $\left(\xi_k\right)_k$ are two families of positive numbers and points in M, respectively, such that $\mu_k \to 0$ and $\xi_k \to \xi_0$ as $k \to \infty$ for some point $\xi_0 \in M$.*

Proof of Proposition 5.1. We let

$$E := \liminf_{k\to\infty} E\left(u_k\right) \quad \text{and} \quad E_0 := Y\left(M, [g]\right)^{\frac{n}{2}} + Y\left(\mathbb{S}^n, [g_{std}]\right)^{\frac{n}{2}}.$$

On the one hand, $Y\left(M, [g]\right) < Y\left(\mathbb{S}^n, [g_{std}]\right)$ when (M, g) is not conformally diffeomorphic to $(\mathbb{S}^n, g_{std})$ by the results of Trudinger [56], Aubin [3] and Schoen [50]. On the other hand, it is easy to see that the energy of any sign-changing solutions to the Yamabe equation on $(\mathbb{S}^n, g_{std})$ is greater than $2Y\left(\mathbb{S}^n, [g_{std}]\right)^{\frac{n}{2}}$. Struwe's decomposition [53] then shows that up to a subsequence and replacing u_k by $-u_k$ if necessary,

$$\begin{cases} \text{(5.2) holds true with } u_0 = 0 \text{ in } M \text{ if } E < E_0, \\ \text{(5.2) holds true for some energy-minimizing solution } u_0 \text{ of (1.2) if } E = E_0 \text{ and} \\ (M, g) \text{ is not conformally diffeomorphic to } (\mathbb{S}^n, g_0). \end{cases}$$

We are left to show that (5.2) cannot hold true with $u_0 = 0$ in M. Assume by contradiction that this is the case. For large k, by testing (1.1) with $u_k^+ := \max\left(u_k, 0\right)$, we obtain

$$\int_M \left(\left|\nabla u_k^+\right|^2 + c_n \operatorname{Scal}_g \left(u_k^+\right)^2\right) dv_g = \int_M \left(u_k^+\right)^{2^*} dv_g. \tag{5.3}$$

Since u_k changes sign the right-hand side in (5.3) is nonzero. An easy density argument then gives

$$\frac{\int_M \left(\left|\nabla u_k^+\right|^2 + c_n \operatorname{Scal}_g \left(u_k^+\right)^2 \right) dv_g}{\left(\int_M \left(u_k^+\right)^{2^*} dv_g\right)^{2/2^*}} \geq \inf_{u\in C^\infty(M),\, u>0} \frac{\int_M \left(|\nabla u|^2 + c_n \operatorname{Scal}_g u^2\right) dv_g}{\left(\int_M u^{2^*} dv_g\right)^{2/2^*}} = c_n Y\left(M,[g]\right), \tag{5.4}$$

where $c_n := \frac{n-2}{4(n-1)}$. By using (5.3) and (5.4), we then obtain

$$c_n Y\left(M,[g]\right) \leq \left(\int_M \left(u_k^+\right)^{2^*} dv_g\right)^{2/n}. \tag{5.5}$$

Since $Y\left(M,[g]\right) > 0$, it then follows from (5.5) that $u_k^+ \not\to 0$ in $L^{2^*}(M)$ as $k\to\infty$. This is in contradiction with (5.2) when $u_0 = 0$ in M. $\square$

The rest of this subsection is devoted to the proof of Theorem 1.2. By Lee and Parker's construction of conformal normal coordinates [26] there exists a smooth family of metrics $(g_\xi)_{\xi\in M}$, $g_\xi = \Lambda_\xi^{2^*-2} g$, such that for every point $\xi \in M$, the function Λ_ξ is smooth, positive and satisfies

$$\Lambda_\xi(\xi) = 1, \quad \nabla\Lambda_\xi(\xi) = 0 \quad \text{and} \quad dv_{g_\xi}(x) = \left(1 + \mathrm{O}\left(|x|^N\right)\right) dv_{\delta_0}(x) \tag{5.6}$$

in geodesic normal coordinates, where dv_{g_ξ} and dv_{δ_0} are the volume elements of the metric g_ξ and the Euclidean metric δ_0, respectively, and $N\in\mathbb{N}$ can be chosen arbitrarily large. In particular (see [26]), it follows from (5.6) that

$$\operatorname{Scal}_{g_\xi}(\xi) = 0, \quad \nabla \operatorname{Scal}_{g_\xi}(\xi) = 0 \quad \text{and} \quad \Delta_g \operatorname{Scal}_{g_\xi}(\xi) = \frac{1}{6}|\operatorname{Weyl}_g(\xi)|_g^2. \tag{5.7}$$

Also, if g is locally conformally flat in a neighbourhood of $\xi\in M$, then g_ξ can be chosen flat in a neighbourhood of ξ. We first state some preliminary lemmas that apply to the more general case where the weak limit u_0 is a possibly sign-changing solution of

$$\Delta_g u_0 + c_n \operatorname{Scal}_g u_0 = |u_0|^{2^*-2} u_0 \quad \text{in } M. \tag{5.8}$$

The following lemma is essentially contained in Premoselli [39] (see also Druet–Hebey–Robert [20] in the case of positive solutions):

Lemma 5.2. *Let (M,g) be a smooth, closed Riemannian manifold of dimension $n\geq 3$ and u_0 be a solution to* (5.8). *Assume that there exists a sequence of solutions $(u_k)_{k\in\mathbb{N}}$ of type* (5.2) *to* (1.1). *Let $(g_\xi)_{\xi\in M}$, $g_\xi = \Lambda_\xi^{2^*-2} g$, be the Lee–Parker family of conformal metrics to g. Let $(\overline{\mu}_k)_{k\in\mathbb{N}}$ and $\left(\overline{\xi}_k\right)_{k\in\mathbb{N}}$ be families of positive numbers and points in M, respectively, such that*

$$u_k\left(\overline{\xi}_k\right) = \min_M u_k = -\overline{\mu}_k^{1-\frac{n}{2}}. \tag{5.9}$$

Then

$$u_k - u_0 + B_k = \mathrm{o}\left(B_k + 1\right) \tag{5.10}$$

as $k\to\infty$, uniformly in M, where B_k is defined as

$$B_k(x) := \Lambda_{\overline{\xi}_k}(x) \frac{\overline{\mu}_k^{\frac{n-2}{2}}}{\left(\overline{\mu}_k^2 + \frac{d_{g_{\overline{\xi}_k}}\left(x,\overline{\xi}_k\right)^2}{n(n-2)}\right)^{\frac{n-2}{2}}} \quad \forall x\in M.$$

Proof of Lemma 5.2. Since $(u_k)_k$ is of the form (5.2), the main result in Premoselli [39] shows that

$$u_k = u_0 - \frac{\mu_k^{\frac{n-2}{2}}}{\left(\mu_k^2 + \frac{d_g(x,\xi_k)^2}{n(n-2)}\right)^{\frac{n-2}{2}}} + \mathrm{o}\left(1 + \left(\frac{\mu_k}{\mu_k^2 + d_g\left(\cdot,\xi_k\right)^2}\right)^{\frac{n-2}{2}}\right) \tag{5.11}$$

as $k \to \infty$, uniformly in M, where $(\mu_k)_k$ and $(\xi_k)_k$ are as in (5.2). Let $(\overline{\mu}_k)_k$ and $\left(\overline{\xi}_k\right)_k$ be such that (5.9) holds true. It then follows from (5.11) that

$$\overline{\mu}_k \sim \mu_k \quad \text{ and } \quad d_g\left(\overline{\xi}_k, \xi_k\right) = \mathrm{o}\left(\mu_k\right) \tag{5.12}$$

as $k \to \infty$. Moreover, since $\Lambda_{\overline{\xi}_k}\left(\overline{\xi}_k\right) = 1$, we obtain

$$d_{g_{\overline{\xi}_k}}\left(x, \overline{\xi}_k\right)^2 = d_g\left(x, \overline{\xi}_k\right)^2 + \mathrm{O}\left(d_g\left(x, \overline{\xi}_k\right)^3\right) \tag{5.13}$$

uniformly with respect to $k \in \mathbb{N}$ and $x \in M$. By using again that $\Lambda_{\overline{\xi}_k}\left(\overline{\xi}_k\right) = 1$ together with (5.12) and (5.13), we obtain

$$\frac{\mu_k^{\frac{n-2}{2}}}{\left(\mu_k^2 + \frac{d_g(x,\xi_k)^2}{n(n-2)}\right)^{\frac{n-2}{2}}} = B_k + \mathrm{o}\left(B_k + 1\right) \tag{5.14}$$

as $k \to \infty$, uniformly in M, so that (5.10) follows from (5.11) and (5.14). □

We then state the following result from Premoselli–Vétois [42], which essentially follows from a Pohozaev-type identity together with the estimate (5.11):

Lemma 5.3. *(Lemma 3.2 in* [42]*) Let (M,g) be a smooth, closed Riemannian manifold of dimension $n \geq 3$ and u_0 be a solution to* (5.8)*. Assume that there exists a sequence of solutions $(u_k)_{k\in\mathbb{N}}$ to* (1.1) *satisfying* (5.2)*. Let $(\overline{\mu}_k)_k$, $\left(\overline{\xi}_k\right)_k$, $(B_k)_k$ and $(g_\xi)_\xi$ be as in Lemma 5.2. Then*

$$\begin{aligned}\int_{B\left(0,1/\sqrt{\overline{\mu}_k}\right)} \left(\langle \nabla \hat{u}_k, \cdot\rangle_{\delta_0} + \frac{n-2}{2}\hat{u}_k\right)\left(\left(\Delta_{\hat{g}_k} - \Delta_{\delta_0}\right)\hat{u}_k + \overline{\mu}_k^2 \hat{h}_k \hat{u}_k\right) dv_{\hat{g}_k} \\ = \frac{1}{2} n^{\frac{n-2}{2}} (n-2)^{\frac{n+2}{2}} \omega_{n-1} u_0\left(\xi_0\right) \overline{\mu}_k^{\frac{n-2}{2}} + \mathrm{o}\left(\overline{\mu}_k^{\frac{n-2}{2}}\right)\end{aligned} \tag{5.15}$$

as $k \to \infty$, where

$$\hat{u}_k\left(y\right) := \overline{\mu}_k^{\frac{n-2}{2}}\left(\Lambda_{\overline{\xi}_k}^{-1} u_k\right)\left(\exp_{\overline{\xi}_k}\left(\overline{\mu}_k y\right)\right), \quad \hat{g}_k\left(y\right) := \exp_{\overline{\xi}_k}^* g_{\overline{\xi}_k}\left(\overline{\mu}_k y\right) \tag{5.16}$$

and

$$\hat{h}_k\left(y\right) := c_n \operatorname{Scal}_{\hat{g}_k}\left(\exp_{\overline{\xi}_k}\left(\overline{\mu}_k y\right)\right) \tag{5.17}$$

for all points $y \in B\left(0, 1/\sqrt{\overline{\mu}_k}\right)$, ω_{n-1} is the volume of the standard unit sphere of dimension $n-1$, δ_0 is the Euclidean metric in $\mathbb{R}^n$, $\exp_{\overline{\xi}_k}$ is the exponential map at the point $\overline{\xi}_k$ with respect to the metric $g_{\overline{\xi}_k}$ and we identify $T_{\overline{\xi}_k}M$ with $\mathbb{R}^n$.

Finally, to prove Theorem 1.2 in dimensions $n \geq 7$, we need the following refinement of Lemma 5.2:

Lemma 5.4. *Let (M,g) be a smooth, closed Riemannian manifold of dimension $n \geq 7$ and u_0 be a solution to (5.8). Let $(u_k)_k$, $(\overline{\mu}_k)_k$, $\left(\overline{\xi}_k\right)_k$, $(B_k)_k$ and $(g_\xi)_\xi$ be as in Lemma 5.2. Then, for $i \in \{0,1,2\}$,*

$$
\begin{aligned}
&\left|\nabla^i \left(u_k - u_0 + B_k\right)(x)\right| \\
&= \mathrm{O}\left(\left\{\begin{matrix} \dfrac{1}{\left(\overline{\mu}_k + d_{g_{\overline{\xi}_k}}\left(x, \overline{\xi}_k\right)\right)^i} & \text{if } (M,g) \text{ is l.c.f.} \\ \dfrac{1}{\left(\overline{\mu}_k + d_{g_{\overline{\xi}_k}}\left(x, \overline{\xi}_k\right)\right)^i} + \dfrac{\overline{\mu}_k^{\frac{n-2}{2}}}{\left(\overline{\mu}_k + d_{g_{\overline{\xi}_k}}\left(x, \overline{\xi}_k\right)\right)^{n-6+i}} & \text{otherwise}\end{matrix}\right\}\right)
\end{aligned}
\tag{5.18}
$$

uniformly with respect to $k \in \mathbb{N}$ and $x \in M$, where l.c.f. stands for locally conformally flat.

Proof of Lemma 5.4. We only need to prove (5.18) when $i = 0$, since the cases $i = 1$ and $i = 2$ follow by standard interior estimates. Remark that when x satisfies $d_{g_{\overline{\xi}_k}}\left(x, \overline{\xi}_k\right) \geq \sqrt{\overline{\mu}_k}$, the estimate (5.18) when $i = 0$ follows from (5.10). Therefore, it suffices to consider the case where $d_{g_{\overline{\xi}_k}}\left(x, \overline{\xi}_k\right) < \sqrt{\overline{\mu}_k}$ as $k \to \infty$. In this case, we use an approach inspired from the work of Chen–Lin [10] in the case of positive solutions that was applied to the Yamabe equation by Marques [30]. By letting $x = \exp_{\overline{\xi}_k}\left(\overline{\mu}_k y\right)$ for $y \in B\left(0, 1/\sqrt{\overline{\mu}_k}\right)$, it is easily seen that (5.18) is implied by the following estimate:

$$
\left|\nabla^i \left(\hat{u}_k + B_0\right)(y)\right| = \mathrm{O}\left(\left\{\begin{matrix} \dfrac{\overline{\mu}_k^{\frac{n-2}{2}}}{\left(1+|y|\right)^i} & \text{if } (M,g) \text{ is l.c.f.} \\ \dfrac{\overline{\mu}_k^{\frac{n-2}{2}}}{\left(1+|y|\right)^i} + \dfrac{\overline{\mu}_k^{4}}{\left(1+|y|\right)^{n-6+i}} & \text{otherwise}\end{matrix}\right\}\right),
\tag{5.19}
$$

where $\hat{u}_k$ is as in (5.16) and we have let

$$
B_0(y) := \left(1 + \frac{|y|^2}{n(n-2)}\right)^{-\frac{n-2}{2}} \qquad \forall y \in \mathbb{R}^n. \tag{5.20}
$$

As mentioned before, we only have to prove (5.19) for $i = 0$. By conformal invariance of the conformal Laplacian, the equation (1.1) can be rewritten as

$$
\Delta_{\hat{g}_k} \hat{u}_k + \overline{\mu}_k^2 \hat{h}_k \hat{u}_k = |\hat{u}_k|^{2^*-2} \hat{u}_k \quad \text{in } B\big(0, 1/\sqrt{\overline{\mu}_k}\big), \tag{5.21}
$$

where $\hat{h}_k$ and $\hat{g}_k$ are as in (5.16) and (5.17). We let $y_k \in \overline{B\big(0, 1/\sqrt{\overline{\mu}_k}\big)}$ be such that

$$
\left|\left(\hat{u}_k + B_0\right)(y_k)\right| = \max_{\overline{B\left(0,1/\sqrt{\overline{\mu}_k}\right)}} \left|\left(\hat{u}_k + B_0\right)\right| =: \lambda_k, \tag{5.22}
$$

and we define

$$
\psi_k := \lambda_k^{-1}\left(\hat{u}_k + B_0\right).
$$

By (5.10), $\lambda_k = \mathrm{o}(1)$ as $k \to \infty$. By using (5.21) together with the equation $\Delta_{\delta_0} B_0 = B_0^{2^*-1}$, we obtain

$$
\Delta_{\hat{g}_k} \psi_k = \left(2^*-1\right) B_0^{2^*-2} \psi_k + \lambda_k^{-1} f_k \quad \text{in } B\big(0, 1/\sqrt{\overline{\mu}_k}\big), \tag{5.23}
$$

where

$$f_k := (\Delta_{\hat{g}_k} - \Delta_{\delta_0}) B_0 - \overline{\mu}_k^2 \hat{h}_k \hat{u}_k + |\hat{u}_k|^{2^*-2} \hat{u}_k + B_0^{2^*-1} - (2^*-1) \lambda_k B_0^{2^*-2} \psi_k. \quad (5.24)$$

We now estimate the terms in the right-hand side of (5.24). Since B_0 is radially symmetric, it follows from (5.6) that

$$(\Delta_{\hat{g}_k} - \Delta_{\delta_0}) B_0(y) = \mathrm{O}\left(\overline{\mu}_k^N |y|^{N-1} |\nabla B_0(y)|\right) = \mathrm{O}\left(\frac{\overline{\mu}_k^N |y|^N}{(1+|y|)^n}\right) \quad (5.25)$$

uniformly with respect to $k \in \mathbb{N}$ and $y \in B(0, 1/\sqrt{\overline{\mu}_k})$. By using (5.7) and (5.10), we obtain

$$\hat{h}_k(y) \hat{u}_k(y) = \begin{cases} 0 & \text{if } (M,g) \text{ is l.c.f.} \\ \mathrm{O}\left(\dfrac{\overline{\mu}_k^2 |y|^2}{(1+|y|)^{n-2}}\right) & \text{otherwise} \end{cases} \quad (5.26)$$

uniformly with respect to $k \in \mathbb{N}$ and $y \in B(0, 1/\sqrt{\overline{\mu}_k})$. Since $2^* - 2 < 2$ when $n \geq 7$, by using (5.10) together with straightforward estimates, we obtain

$$|\hat{u}_k|^{2^*-2} \hat{u}_k + B_0^{2^*-1} - (2^*-1) \lambda_k B_0^{2^*-2} \psi_k = \mathrm{o}\left(\lambda_k B_0^{2^*-2} |\psi_k|\right) \quad (5.27)$$

as $k \to \infty$, uniformly with respect to $y \in B(0, 1/\sqrt{\overline{\mu}_k})$. Let G_k be the Green's function of the operator $\Delta_{\hat{g}_k}$ in $B(0, 1/\sqrt{\overline{\mu}_k})$ with zero Dirichlet boundary condition. Then

$$\psi_k(y) = \int_{B\left(0, 1/\sqrt{\overline{\mu}_k}\right)} G_k(y, \cdot) \left((2^*-1) B_0^{2^*-2} \psi_k + \lambda_k^{-1} f_k\right) dv_{\hat{g}_k} - \int_{\partial B\left(0, 1/\sqrt{\overline{\mu}_k}\right)} \partial_\nu G_k(y, \cdot) \psi_k \, d\sigma_{\hat{g}_k} \quad (5.28)$$

for all points $x \in B(0, 1/\sqrt{\overline{\mu}_k})$. Standard estimates on Green's functions (see e.g. Robert [45]) give that for large k,

$$G_k(y, z) = \mathrm{O}\left(|y - z|^{2-n}\right) \quad \text{for } z \in B\left(0, 1/\sqrt{\overline{\mu}_k}\right) \quad (5.29)$$

and

$$\partial_\nu G_k(y, z) = \mathrm{O}\left(|y - z|^{1-n}\right) \quad \text{for } z \in \partial B\left(0, 1/\sqrt{\overline{\mu}_k}\right) \quad (5.30)$$

uniformly with respect to $k \in \mathbb{N}$ and $y \in B(0, 1/\sqrt{\overline{\mu}_k})$. Remark also that (5.10) gives

$$\psi_k = \mathrm{O}\left(\lambda_k^{-1} \overline{\mu}_k^{\frac{n-2}{2}}\right) \text{ as } k \to \infty, \text{ uniformly in } B\left(0, 1/\sqrt{\overline{\mu}_k}\right) \backslash B\left(0, 1/2\sqrt{\overline{\mu}_k}\right). \quad (5.31)$$

By using (5.25), (5.26), (5.27) and (5.28)–(5.31) together with straightforward integral estimates and the fact that $n \geq 7$, we obtain

$$\psi_k(y) = \mathrm{O}\left(\int_{B\left(0, 1/\sqrt{\overline{\mu}_k}\right)} \frac{|\psi_k(z)| \, dz}{|y - z|^{n-2} (1+|z|)^4} + \lambda_k^{-1} \overline{\mu}_k^{\frac{n-2}{2}} \left\{ + \frac{\lambda_k^{-1} \overline{\mu}_k^4}{(1+|y|)^{n-6}} \quad \text{if } (M,g) \text{ is not l.c.f.} \right\}\right) \quad (5.32)$$

uniformly with respect to $k\in\mathbb{N}$ and $y\in B\left(0,1/\sqrt{\overline{\mu}_k}\right)$. We now claim that

$$\lambda_k=\mathrm{O}\left(\overline{\mu}_k^{\frac{n-2}{2}}\ \left\{+\overline{\mu}_k^4\quad\text{if }\ (M,g)\ \text{ is not l.c.f.}\right\}\right)\tag{5.33}$$

uniformly with respect to $k\in\mathbb{N}$. Assume by contradiction that (5.33) does not hold true, i.e. there exists a subsequence $\left(k_j\right)_{j\in\mathbb{N}}$ of positive numbers such that

$$k_j\to 0\quad\text{and}\quad \overline{\mu}_{k_j}^{\frac{n-2}{2}}\left\{+\overline{\mu}_{k_j}^4\quad\text{if }\ (M,g)\ \text{ is not l.c.f.}\right\}=\mathrm{o}\left(\lambda_{k_j}\right)\tag{5.34}$$

as $j\to\infty$. Since $\left|\psi_{k_j}\right|\le 1$ in $B\big(0,1/\sqrt{\overline{\mu}_{k_j}}\big)$ and $\hat{g}_{k_j}\to\delta_0$ as $j\to\infty$, uniformly in compact subsets of $\mathbb{R}^n$, it follows from (5.23)–(5.27) together with standard elliptic estimates that up to a subsequence, $\left(\psi_{k_j}\right)_j$ converges in $C^1_{loc}\left(\mathbb{R}^n\right)$ as $j\to\infty$ to a solution $\psi_0\in C^2\left(\mathbb{R}^n\right)$ of the equation

$$\Delta_{\delta_0}\psi_0=\left(2^*-1\right)B_0^{2^*-2}\psi_0\quad\text{in }\mathbb{R}^n.\tag{5.35}$$

Independently, it follows from (5.32) and (5.34) that

$$\psi_{k_j}\left(y\right)=\mathrm{O}\left(\left(1+|y|\right)^{-2}\right)+\mathrm{o}\left(1\right)\tag{5.36}$$

as $j\to\infty$, uniformly with respect to $y\in B\big(0,1/\sqrt{\overline{\mu}_{k_j}}\big)$. Passing to the limit into (5.36) as $j\to\infty$, we then obtain

$$\psi_0\left(y\right)=\mathrm{O}\left(\left(1+|y|\right)^{-2}\right)\tag{5.37}$$

uniformly with respect to $y\in\mathbb{R}^n$. By applying Lemma 2.4 of Chen–Lin [9], it follows from (5.35) and (5.37) that

$$\psi_0\left(y\right)=c_0\frac{1-\frac{|y|^2}{n(n-2)}}{\left(1+\frac{|y|^2}{n(n-2)}\right)^{\frac{n}{2}}}+\sum_{i=1}^n c_i\frac{y_i}{\left(1+\frac{|y|^2}{n(n-2)}\right)^{\frac{n}{2}}}\quad\forall y\in\mathbb{R}^n$$

for some numbers $c_0,\dots,c_n\in\mathbb{R}$. On the other hand, it follows from (5.9) that $\psi_k\left(0\right)=\left|\nabla\psi_k\left(0\right)\right|=0$, which gives $\psi_0\left(0\right)=\left|\nabla\psi_0\left(0\right)\right|=0$. Therefore, we obtain $c_0=\dots=c_n=0$, and so $\psi_0=0$ in $\mathbb{R}^n$. Since $\psi_{k_j}\left(y_{k_j}\right)=1$ for all $j\in\mathbb{N}$, where y_{k_j} is as in (5.22), we then obtain that $\left|y_{k_j}\right|\to\infty$ as $j\to\infty$, which is in contradiction with (5.36). This proves that (5.33) holds true. Finally, by using (5.33) together with successive applications of (5.32), we obtain that

$$\psi_k\left(y\right)=\mathrm{O}\left(\frac{\ln\left(2+|y|\right)}{\left(1+|y|\right)^{n-2}}+\lambda_k^{-1}\overline{\mu}_k^{\frac{n-2}{2}}\left\{+\frac{\lambda_k^{-1}\overline{\mu}_k^4}{\left(1+|y|\right)^{n-6}}\quad\text{if }\ (M,g)\ \text{ is not l.c.f.}\right\}\right)\tag{5.38}$$

uniformly with respect to $k\in\mathbb{N}$ and $y\in B\big(0,1/\sqrt{\overline{\mu}_k}\big)$. The estimate (5.19) with $i=0$ then follows from (5.33) and (5.38). ∎

We can now use Lemmas 5.3 and 5.4 to prove Theorem 1.2.

Proof of Theorem 1.2. Assume by contradiction that

$$E(M,[g])\le Y(\mathbb{S}^n,[g_{std}])^{\frac{n}{2}}+Y(M,[g])^{\frac{n}{2}}.$$

By using the definition of $E(M,[g])$ together with a diagonal argument, it follows that there exists a sequence of blowing-up solutions $\left(u_k\right)_{k\in\mathbb{N}}$ to (1.1) such that

$$E\left(u_k\right)\le Y\left(M,[g]\right)^{\frac{n}{2}}+Y\left(\mathbb{S}^n,[g_{std}]\right)^{\frac{n}{2}}+\mathrm{o}\left(1\right)$$

as $k \to \infty$. In the case where the functions u_k do not change sign, we obtain a contradiction with the assumptions of Theorem 1.2 by applying the compactness results for positive solutions of the Yamabe equation (see Schoen [51, 52], Li–Zhu [29], Druet [19], Marques [30], Li–Zhang [27, 28] and Khuri–Marques–Schoen [25]). Therefore, in what follows, we assume that the functions u_k change sign. It then follows from Proposition 5.1 that up to a subsequence and replacing u_k by $-u_k$ if necessary, the functions u_k are of the form (5.2) for some energy-minimizing, positive solution u_0 to (1.2). In the case where $n \leq 6$, we can apply a more general compactness result that we obtained in Premoselli–Vétois [42] (Theorem 1.2 in [42]). Therefore, in what follows, we assume that $n \geq 7$. By using (5.18) (see also (5.19)) and (5.25), we then obtain

$$\begin{aligned}\left(\Delta_{\hat{g}_k} - \Delta_{\delta_0}\right) \hat{u}_k &= \mathrm{O}\Bigg(\frac{\overline{\mu}_k^N \left|y\right|^N}{\left(1+\left|y\right|\right)^n} + \left|\left(\hat{g}_k - \delta_0\right)(y)\right| \left|\nabla^2 \left(\hat{u}_k + B_0\right)(y)\right| \\ &\qquad + \left|\nabla\left(\hat{g}_k - \delta_0\right)(y)\right| \left|\nabla\left(\hat{u}_k + B_0\right)(y)\right| \Bigg) \\ &= \mathrm{O}\left(\overline{\mu}_k^{\frac{n+2}{2}} \left\{ + \frac{\overline{\mu}_k^6 \left|y\right|}{\left(1+\left|y\right|\right)^{n-4}} \quad \text{if } (M,g) \text{ is not l.c.f.} \right\} \right) \end{aligned} \tag{5.39}$$

uniformly with respect to $k \in \mathbb{N}$ and $y \in B\left(0, 1/\sqrt{\overline{\mu}_k}\right)$. It then follows from (5.18), (5.26) and (5.39) that

$$\begin{aligned}\int_{B\left(0,1/\sqrt{\overline{\mu}_k}\right)} \left(\langle \nabla \hat{u}_k, \cdot \rangle_{\delta_0} + \frac{n-2}{2} \hat{u}_k \right) \left(\left(\Delta_{\hat{g}_k} - \Delta_{\delta_0}\right) \hat{u}_k + \overline{\mu}_k^2 \hat{h}_k \hat{u}_k \right) dv_{\delta_0} \\ - \overline{\mu}_k^2 \int_{B\left(0,1/\sqrt{\overline{\mu}_k}\right)} \left(\langle \nabla B_0, \cdot \rangle_{\delta_0} + \frac{n-2}{2} B_0 \right) \hat{h}_k B_0 \, dv_{\delta_0} \\ = \mathrm{O}\left(\left\{ \begin{array}{ll} \overline{\mu}_k^{\frac{n}{2}} & \text{if } (M,g) \text{ is l.c.f.} \\ \overline{\mu}_k^{\frac{n}{2}} + \overline{\mu}_k^6 & \text{otherwise} \end{array} \right\} \right)\end{aligned} \tag{5.40}$$

for large k. On the other hand, by using (5.6) and (5.7) together with straightforward computations and symmetry arguments, we obtain

$$\begin{aligned}\int_{B\left(0,1/\sqrt{\overline{\mu}_k}\right)} \left(\langle \nabla B_0, \cdot \rangle_{\delta_0} + \frac{n-2}{2} B_0 \right) \hat{h}_k B_0 \, dv_{\delta_0} \\ = \begin{cases} 0 & \text{if } (M,g) \text{ is l.c.f.} \\ a_n \left|\mathrm{Weyl}_g\left(\xi_0\right)\right|_g^2 \overline{\mu}_k^2 + \mathrm{o}\left(\overline{\mu}_k^2\right) & \text{otherwise} \end{cases}\end{aligned} \tag{5.41}$$

as $k \to \infty$, where

$$a_n := \frac{c_n}{24}(n-2)^2 \int_{\mathbb{R}^n} \left(1 + \frac{\left|y\right|^2}{n(n-2)}\right)^{1-n} \left(\frac{\left|y\right|^2}{n(n-2)} - 1\right) \frac{\left|y\right|^2 dy}{n(n-2)}. \tag{5.42}$$

The constant a_n is computed by using (4.5) and (4.7), and we obtain

$$a_n = \frac{c_n}{6} \frac{n^{\frac{n}{2}}(n-2)^{\frac{n+4}{2}}}{n-6} \omega_{n-1} I_{n-1}^{\frac{n}{2}} = \frac{n^{\frac{n+2}{2}}(n-2)^{\frac{n+4}{2}}}{12(n-6)(n-4)} \omega_{n-1} I_n^{\frac{n-2}{2}}$$
$$= \frac{n(n-2)^2}{6(n-6)(n-4)} K_n^{-n}. \tag{5.43}$$

By putting together (5.15), (5.40) and (5.41), we then obtain

$$\frac{1}{2} n^{\frac{n-2}{2}} (n-2)^{\frac{n+2}{2}} \omega_{n-1} u_0(\xi_0) \overline{\mu}_k^{\frac{n-6}{2}} - a_n \left|\mathrm{Weyl}_g(\xi_0)\right|_g^2 \overline{\mu}_k^2$$
$$= \mathrm{o}\left(\begin{Bmatrix} \overline{\mu}_k^{\frac{n-6}{2}} & \text{if } (M,g) \text{ is l.c.f.} \\ \overline{\mu}_k^2 + \overline{\mu}_k^{\frac{n-6}{2}} & \text{otherwise} \end{Bmatrix}\right) \tag{5.44}$$

as $k \to \infty$. In the case where $n = 10$, we obtain

$$2 \cdot 10^{-4} 8^{-6} a_{10} = \frac{5}{567}\, \omega_9. \tag{5.45}$$

Finally, by using (5.44) and (5.45), we obtain a contradiction with the assumptions of Theorem 1.2. $\square$

Remark 5.5. *The approach used in the proof of Theorem 1.2 can be extended to the case of sequences of solutions $(u_k)_{k\in\mathbb{N}}$ of type (5.2) to linear perturbations of the Yamabe equation of the form*

$$\Delta_g u_k + (c_n \,\mathrm{Scal}_g + \varepsilon_k h)\, u_k = |u_k|^{2^*-2} u_k \quad \text{in } M, \tag{5.46}$$

where $h \in C^{0,\vartheta}(M)$, $\vartheta \in (0,1)$, and $(\varepsilon_k)_k$ is a sequence of positive real numbers such that $\varepsilon_k \to 0$ as $k \to \infty$. In the case of positive solutions, perturbed equations of the form (5.2) have been studied for example by Esposito–Pistoia–Vétois [21], Morabito–Pistoia–Vaira [34], Premoselli [40] and Robert–Vétois [49]. In this case, in place of (5.18), we obtain, for $i \in \{0,1,2\}$,

$$\left|\nabla^i (u_k - u_0 + B_k)(x)\right|$$
$$= \mathrm{O}\left(\begin{Bmatrix} \dfrac{1}{\left(\overline{\mu}_k + d_{g_{\overline{\xi}_k}}(x,\overline{\xi}_k)\right)^i} + \dfrac{\varepsilon_k \overline{\mu}_k^{\frac{n-2}{2}}}{\left(\overline{\mu}_k + d_{g_{\overline{\xi}_k}}(x,\overline{\xi}_k)\right)^{n-4+i}} & \textit{if } (M,g) \textit{ is l.c.f.} \\ \dfrac{1}{\left(\overline{\mu}_k + d_{g_{\overline{\xi}_k}}(x,\overline{\xi}_k)\right)^i} + \dfrac{\overline{\mu}_k^{\frac{n-2}{2}}\left(\varepsilon_k + \overline{\mu}_k^2 + d_{g_{\overline{\xi}_k}}(x,\overline{\xi}_k)^2\right)}{\left(\overline{\mu}_k + d_{g_{\overline{\xi}_k}}(x,\overline{\xi}_k)\right)^{n-4+i}} & \textit{otherwise} \end{Bmatrix}\right)$$

uniformly with respect to $k \in \mathbb{N}$ and $x \in M$, and in place of (5.44), we obtain

$$\frac{1}{2} n^{\frac{n-2}{2}} (n-2)^{\frac{n+2}{2}} \omega_{n-1} u_0(\xi_0) \overline{\mu}_k^{\frac{n-6}{2}} - a_n \left|\mathrm{Weyl}_g(\xi_0)\right|_g^2 \overline{\mu}_k^2 + b_n \varepsilon_k h(\xi_0)$$
$$= \mathrm{o}\left(\begin{Bmatrix} \varepsilon_k + \overline{\mu}_k^{\frac{n-6}{2}} & \textit{if } (M,g) \textit{ is l.c.f.} \\ \varepsilon_k + \overline{\mu}_k^2 + \overline{\mu}_k^{\frac{n-6}{2}} & \textit{otherwise} \end{Bmatrix}\right)$$

as $k \to \infty$, where a_n is as in (5.42) and b_n is another positive constant depending only on n. This yields that there does not exist any sign-changing blowing-up sequences of solutions of type (5.2) to (5.46) in each of the following situations:

- $3 \le n \le 6$,
- $h \ge 0$ *and* $7 \le n \le 9$,
- $h \ge 0$, $n = 10$ *and* $u_0 > \frac{5}{567} |\operatorname{Weyl}_g|_g^2$ *for all points in* M,
- $h \le 0$, $n = 10$ *and* $u_0 < \frac{5}{567} |\operatorname{Weyl}_g|_g^2$ *for all points in* M,
- $h \ge 0$, $n \ge 11$ *and* (M, g) *is locally conformally flat,*
- $h \le 0$, $n \ge 11$ *and* $\operatorname{Weyl}_g \ne 0$ *for all points in* M.

Conversely, it is not difficult to adapt the constructive proofs in Esposito–Pistoia–Vétois [21] *and Robert–Vétois* [49] *to prove that if* u_0 *is a nondegenerate positive solution to* (1.2)*, then there exists a blowing-up sequence of solutions of type* (5.2) *to* (5.46) *in each of the following situations:*

- $\min\limits_M h < 0$ *and* $7 \le n \le 9$,
- $\min\limits_M h < 0$, $n = 10$ *and* $u_0 > \frac{5}{567} |\operatorname{Weyl}_g|_g^2$ *for all points in* M,
- $\max\limits_M h > 0$, $n = 10$ *and* $u_0 < \frac{5}{567} |\operatorname{Weyl}_g|_g^2$ *for all points in* M,
- $\min\limits_M h < 0$, $n \ge 11$ *and* (M, g) *is locally conformally flat,*
- $\max\limits_M h > 0$, $n \ge 11$ *and* $\operatorname{Weyl}_g \ne 0$ *for all points in* M.

This is again in sharp contrast with the case of positive solutions (see Esposito–Pistoia–Vétois [21]*), where blowing-up sequences of solutions to* (5.46) *exist when* $\max\limits_M h > 0$ *in all dimensions* $n \ge 4$.

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