

SHARP POINTWISE ESTIMATES FOR WEIGHTED CRITICAL p -LAPLACE EQUATIONS

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ABSTRACT. We investigate the asymptotic behavior of solutions to a class of weighted quasilinear elliptic equations which arise from the Euler–Lagrange equation associated with the Caffarelli–Kohn–Nirenberg inequality. We obtain sharp pointwise estimates which extend and improve previous results obtained in the unweighted case. In particular, we show that we can refine the asymptotic expansion at infinity by using a Kelvin-type transformation, which reduces the problem to another elliptic-type problem near the origin. The application of this transformation is straightforward in the linear case but more delicate in the quasilinear case. In particular, it is necessary in this case to establish some preliminary estimates before being able to apply the transformation.

1. INTRODUCTION AND MAIN RESULTS

In this paper, we are interested in the elliptic problem

$$\begin{cases} -\operatorname{div}(|x|^{-ap} |\nabla u|^{p-2} \nabla u) = f(x, u) & \text{in } \mathbb{R}^n \\ u \in D^{1,p}(\mathbb{R}^n, |x|^{-ap}), \end{cases} \quad (1.1)$$

where $f : \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R}$ is a Caratheodory function satisfying

$$|f(x, s)| \leq \Lambda |x|^{-bq} |s|^{q-1} \quad \text{for all } s \in \mathbb{R} \text{ and a.e. } x \in \mathbb{R}^n \quad (1.2)$$

for some constant $\Lambda > 0$, the numbers p , a , b and q are such that

$$p > 1, \quad a < \frac{n-p}{p}, \quad a \leq b < a+1, \quad q = \frac{np}{n-p(1+a-b)} \quad (1.3)$$

and $D^{1,p}(\mathbb{R}^n, |x|^{-ap})$ is the completion of $C_c^\infty(\mathbb{R}^n)$ with respect to the norm

$$\|u\|_{D^{1,p}(\mathbb{R}^n, |x|^{-ap})} := \left(\int_{\mathbb{R}^n} |x|^{-ap} |\nabla u|^p dx \right)^{\frac{1}{p}}.$$

When $f(x, u) = \Lambda |x|^{-bq} |u|^{q-2} u$, (1.1) is the Euler–Lagrange equation associated with the Caffarelli–Kohn–Nirenberg inequality [3], which

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states, under conditions (1.3), that there exists a positive constant $C = C(n, a, b, p) > 0$ such that

$$\left(\int_{\mathbb{R}^n} |x|^{-bq} |u|^q dx \right)^{\frac{1}{q}} \leq C \left(\int_{\mathbb{R}^n} |x|^{-ap} |\nabla u|^p dx \right)^{\frac{1}{p}} \quad (1.4)$$

for all $u \in C_c^\infty(\mathbb{R}^n)$.

Our main result is the following:

Theorem 1.1. *Let n, p, a, b and q be such that (1.3) holds true, $f : \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R}$ be a Caratheodory function satisfying (1.2) and u be a solution of (1.1). Then there exists a constant $C_0 > 0$ such that*

$$|x|^\mu |u(x)| + |x|^{\mu+1} |\nabla u(x)| \leq C_0 \quad \forall x \in \mathbb{R}^n \setminus B(0, 1), \quad (1.5)$$

where $\mu := (n - p(1 + a)) / (p - 1)$. If moreover $u > 0$ and $f(x, u) \geq 0$ in \mathbb{R}^n , then there exist constants $\alpha, \delta, C_1 > 0$ such that

$$||x|^\mu u(x) - \alpha| \leq C_1 |x|^{-\delta} \quad \forall x \in \mathbb{R}^n \setminus B(0, 1) \quad (1.6)$$

and

$$|x|^{\mu+1} \nabla u(x) + \alpha \mu |x|^{-1} x \rightarrow 0 \quad \text{as } |x| \rightarrow \infty. \quad (1.7)$$

This theorem extends previous results obtained by Sciunzi [26] and Vétois [36] in the case where $a = b = 0$. In fact, for positive solutions, (1.6) and (1.7) improve the estimates obtained in [26, 36], where it was proven that $1/C \leq |x|^\mu u(x), |x|^{\mu+1} |\nabla u(x)| \leq C$ for some constant $C > 0$ independent of $x \in \mathbb{R}^n \setminus B(0, 1)$.

The proof of (1.6) relies in particular on the use of a Kelvin-type transformation of the form $u_*(y) := |x|^\mu u(x)$, where $x := |y|^{-2} y$. This transformation is well-known in the case $p = 2$ but, as far as the authors know, it has never been used in the case where $p \neq 2$. In contrast with the case where $p = 2$, the equation is not invariant under this transformation when $p \neq 2$. In this paper, we show that we can still apply this transformation in the case where $p \neq 2$, however it is necessary in this case to establish some preliminary estimates (see (1.8) below) and then the transformation can be used to improve these estimates and obtain for instance the Hölder-type estimate in (1.6). We believe that even sharper results could be achieved by this method if new regularity results were obtained for the weighted elliptic-type problem that results from this transformation.

We point out that in contrast with the case where $a = b = 0$, the solutions of (1.1) are not always radial in the presence of weights. Indeed, Horiuchi [20] obtained the existence of extremals when $a < 0$ and $a < b < a + 1$ (see also Catrina and Wang [5]). However, these extremals turn out to be non-radial when $a < b < h(a)$ for some function h such that $a < h(a) < a + 1$ (see Catrina and Wang [5] and

Felli and Schneider [17] for $p = 2$ and Byeon and Wang [1], Caldiroli and Musina [4] and Smets and Willem [29] for $p \neq 2$). Caldiroli and Musina also observed in [4, Section 4] that the Kelvin-type transformation $u_*(y) := u(x)$, where $x := |y|^{-2}y$, transforms the problem into a similar weighted elliptic-type problem, where the only difference with the original problem lies in the exponents of the weights. We note that this transformation is however different from the one we use in this paper since we multiply here by the weight $|x|^\mu$ in order to remove the singularity at 0 in the transformed function.

When $p = 2$ and $0 \leq a \leq b < a + 1$, Chou and Chu [6] obtained that the positive solutions of (1.1) are always radial, thus extending the well-known result obtained by Caffarelli, Gidas and Spruck [2] for the classical Laplace operator i.e. when $a = b = 0$. The optimal symmetry result in the case where $a < 0$ and $p = 2$ has recently been established by Dolbeault, Esteban and Loss [13]. We also refer on this topic to the recent survey article [14] by the same authors.

In the case where $a = b = 0$ and $p \neq 2$, the pointwise estimates obtained by Sciunzi [26] and Vétois [36] have found application for establishing symmetry results for the solutions of (1.1); see Damascelli and Ramaswamy [10] and Vétois [36] for $p < 2$ and Sciunzi [26] for $p > 2$. These results have recently been extended by Oliva, Sciunzi and Vaira [23] to a class of p -Laplace equations with Hardy potential, using pointwise estimates established by Xiang [37, 38] in this case. These results have also been extended, under a different method, still relying on pointwise estimates, by Ciruolo, Figalli and Roncoroni [7] to a class of p -Laplace-type equations in an anisotropic setting. We also refer to the work by Esposito [15] which treats the limit case where $p = n$ and the nonlinearity is exponential, and where, again, pointwise estimates play a crucial role.

The interest of this paper is therefore twofold. On the one hand, we believe that like in the aforementioned papers, our results will be useful to establish new symmetry results. On the other hand, since, as mentioned above, there exist situations where non-radial solutions exist, we are interested in developing a method to obtain sharp pointwise estimates for the solutions in this case.

We prove Theorem 1.1 through several steps. In Section 2, we obtain global boundedness results in $L^\infty(\mathbb{R}^n)$ and weak Lebesgue spaces. In Section 3, we establish (1.5) by using suitable changes of scales and Harnack-type inequalities. In Section 4, we prove that

$$||x|^\mu u(x) - \alpha| + ||x|^{\mu+1} \nabla u(x) + \alpha \mu |x|^{-1} x| \rightarrow 0 \quad \text{as } |x| \rightarrow \infty \quad (1.8)$$

by using an approach based on comparison arguments. In Section 5, we then reduce the problem at infinity to another elliptic-type problem

near the origin by using a Kelvin-type transformation. More precisely, we prove the following:

Proposition 1.1. *Let n, p, a, b, q, μ and f be as in Theorem 1.1, $R_0 > 0$ and u be a weak solution of the equation*

$$-\operatorname{div}(|x|^{-ap} |\nabla u|^{p-2} \nabla u) = f(x, u) \quad \text{in } \mathbb{R}^n \setminus B(0, R_0). \quad (1.9)$$

Assume that there exists a constant $\alpha > 0$ such that (1.8) holds true. Let $r_0 := 1/R_0$ and $u_ : B(0, r_0) \setminus \{0\} \rightarrow \mathbb{R}$ be the function defined by*

$$u_*(y) := |y|^{-\mu} u(|y|^{-2} y) \quad \forall y \in B(0, r_0). \quad (1.10)$$

Then there exists $r \in (0, r_0)$ such that u_ is a weak solution (see Definition 1.1 below) to an equation of the form*

$$-\operatorname{div}(A(y, u_*, \nabla u_*)) = B(y, u_*, \nabla u_*) \quad \text{in } B(0, r) \quad (1.11)$$

for some Caratheodory functions $A : B(0, r) \times \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ and $B : B(0, r) \times \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}$ satisfying

$$\begin{cases} |A(y, z, \xi)| \leq C |y|^\gamma |\xi| \\ A(y, z, \xi) \cdot \xi \geq C^{-1} |y|^\gamma |\xi|^2 \\ |B(y, z, \xi)| \leq C (|y|^\gamma |\xi|^2 + |y|^{\gamma'}), \end{cases} \quad (1.12)$$

where

$$\gamma := \mu + 2 - n, \quad \gamma' := \frac{\mu q}{p} - n \quad (1.13)$$

and C is a positive constant independent of $(y, z, \xi) \in B(0, r) \times \mathbb{R} \times \mathbb{R}^n$.

The definition of weak solution of (1.11) is as follows:

Definition 1.1. *For every $r > 0$ and $\gamma, \gamma' > -n$, we define the space $H^{1,2}(B(0, r), |y|^\gamma)$ as the completion of $C^\infty(B(0, r))$ with respect to the norm*

$$\|u\|_{H^{1,2}(B(0,r),|y|^\gamma)} = \left(\int_{B(0,r)} |y|^\gamma |u|^2 dy + \int_{B(0,r)} |y|^\gamma |\nabla u|^2 dy \right)^{\frac{1}{2}}.$$

Given two Caratheodory functions $A : B(0, r) \times \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ and $B : B(0, r) \times \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}$ satisfying (1.12), we say that u is a weak solution of the equation

$$-\operatorname{div}(A(y, u, \nabla u)) = B(y, u, \nabla u) \quad \text{in } B(0, r)$$

if $u \in H^{1,2}(B(0, r), |y|^\gamma)$ and

$$\int_{B(0,r)} (A(y, u, \nabla u) \cdot \nabla \varphi - B(y, u, \nabla u) \varphi) dy = 0 \quad \forall \varphi \in C_0^\infty(B(0, r)).$$

It is well known that in the case where $f(x, u) = \Lambda |x|^{-bq} |u|^{q-2} u$ and $p = 2$, the problem (1.1) is invariant under the transformation $u \mapsto u_*$. With Proposition 1.1, we show that a similar transformation can still be applied in the general case provided we first establish (1.8). It is interesting to remark that even in the unweighted case where $a = b = 0$, the problem that we obtain after transformation is actually of weighted-type when $p \neq 2$. At the end of Section 5, we then combine Proposition 1.1 with an Hölder continuity result of Stredulinski [30] to obtain that (1.6) holds true, thus completing the proof of Theorem 1.1. For other references on the Hölder continuity of solutions to weighted elliptic equations, let us mention for instance the works of Colorado and Peral [8], Di Fazio and Zamboni [12], Felli and Schneider [18] and Monticelli, Rodney and Wheeden [24].

Remark 1.1. *It is possible to prove a more general version of Proposition 1.1 in the case where u_* is defined as*

$$u_*(y) := |y|^{-\mu} u(|y|^{-\sigma} y) \quad \forall y \in B(0, r_0)$$

for some constant $\sigma > 1$. In this case, we obtain that u_* solves an equation of the form (1.11) for some Caratheodory functions $A : B(0, r) \times \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ and $B : B(0, r) \times \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}$ satisfying (1.12) with

$$\gamma := (\sigma - 1)\mu + 2 - n \quad \text{and} \quad \gamma' := (\sigma - 1)\mu \frac{q}{p} - n.$$

In particular, when $n \geq 3$, we can choose

$$\sigma := 1 + \frac{n-2}{\mu},$$

which gives $\gamma = 0$. Furthermore, in this case, a straightforward computation gives

$$\gamma' = (n-2) \frac{q}{p} - n,$$

which is greater than -2 since $q > p$. In particular, this allows to apply the Hölder continuity result of Trudinger [32, Theorem 5.2] to obtain that (1.6) holds true.

2. REGULARITY AND BOUNDEDNESS RESULTS

We start by introducing some suitable function spaces.

Definition 2.1. *For every $s > 0$, $\gamma \in \mathbb{R}$ and measurable set $\Omega \subseteq \mathbb{R}^n$, we define $L^s(\Omega, |x|^\gamma)$ as the set of all measurable functions $u : \Omega \rightarrow \mathbb{R}$ such that*

$$\|u\|_{L^s(\Omega, |x|^\gamma)} := \left(\int_{\Omega} |x|^\gamma |u|^s dx \right)^{\frac{1}{s}} < \infty.$$

Furthermore, we denote $L^s(\Omega) := L^s(\Omega, 1)$.

Definition 2.2. For every $s > 0$, $\gamma \in \mathbb{R}$ and measurable set $\Omega \subseteq \mathbb{R}^n$, we define $L^{s,\infty}(\Omega, |x|^\gamma)$ as the set of all measurable functions $u : \Omega \rightarrow \mathbb{R}$ such that

$$\|u\|_{L^{s,\infty}(\Omega, |x|^\gamma)} := \sup_{h>0} \left(h \left(\int_{W_h} |x|^\gamma dx \right)^{\frac{1}{s}} \right) < \infty,$$

where

$$W_h := \{x \in \mathbb{R}^n : |u(x)| > h\}. \quad (2.1)$$

Furthermore, we denote $L^{s,\infty}(\Omega) := L^{s,\infty}(\Omega, 1)$.

As a first step in the proof of Theorem 1.1, we prove the following:

Step 2.1. Let n, p, a, b, q and f be as in Theorem 1.1. Then every solution of (1.1) belongs to $L^\infty(\mathbb{R}^n)$.

Proof. Here we adapt some ideas which originates from Trudinger [33]. Let u be a solution of (1.1). We begin with proving that $|u|^{\beta/p} \in L^q(\mathbb{R}^n, |x|^{-bq})$ for all $\beta > p$. For every $h > 0$, we define

$$\varphi_h(u) := \min(|u|, h)^{\frac{\beta-p}{p}}.$$

By using $\varphi_h(u)^p u$ as a test function, we obtain

$$\begin{aligned} \int_{\mathbb{R}^n} |x|^{-ap} |\nabla u|^p (h^{\beta-p} \chi_{W_h} + (\beta - p + 1) |u|^{\beta-p} \chi_{\mathbb{R}^n \setminus W_h}) dx \\ = \int_{\mathbb{R}^n} f(x, u) \varphi_h(u)^p u dx, \end{aligned} \quad (2.2)$$

where W_h is as in (2.1) and χ_{W_h} and $\chi_{\mathbb{R}^n \setminus W_h}$ are the characteristic functions of the sets W_h and $\mathbb{R}^n \setminus W_h$, respectively. Since $\beta > p$, it follows from (1.2) and (2.2) that

$$\int_{\mathbb{R}^n} |x|^{-ap} |\nabla u|^p \varphi_h(u)^p dx \leq \Lambda \int_{\mathbb{R}^n} |x|^{-bq} |u|^q \varphi_h(u)^p dx. \quad (2.3)$$

On the other hand, for every $k \in (0, h)$, we have

$$\begin{aligned} \int_{\mathbb{R}^n} |x|^{-bq} |u|^q \varphi_h(u)^p dx \\ \leq k^{\beta-p} \int_{\mathbb{R}^n} |x|^{-bq} |u|^q dx + \int_{W_k} |x|^{-bq} |u|^q \varphi_h(u)^p dx. \end{aligned} \quad (2.4)$$

Since $q > p$, by using Hölder's inequality, we obtain

$$\begin{aligned} \int_{W_k} |x|^{-bq} |u|^q \varphi_h(u)^p dx \\ \leq \left(\int_{W_k} |x|^{-bq} |u|^q dx \right)^{\frac{q-p}{q}} \left(\int_{W_k} |x|^{-bq} |u|^q \varphi_h(u)^q dx \right)^{\frac{p}{q}}. \end{aligned} \quad (2.5)$$

By applying the Caffarelli–Kohn–Nirenberg inequality to the function $\varphi_h(u)u$, we obtain

$$\begin{aligned} & \left(\int_{\mathbb{R}^n} |x|^{-bq} |u|^q \varphi_h(u)^q dx \right)^{\frac{p}{q}} \\ & \leq K \int_{\mathbb{R}^n} |x|^{-ap} |\nabla u|^p (h^{\beta-p} \chi_{W_h} + (\beta/p)^p |u|^{\beta-p} \chi_{\mathbb{R}^n \setminus W_h}) dx \\ & \leq K (\beta/p)^p \int_{\mathbb{R}^n} |x|^{-ap} |\nabla u|^p \varphi_h(u)^p dx \end{aligned} \quad (2.6)$$

for some constant $K = K(n, a, b, p) > 0$. It follows from (2.3)–(2.6) that

$$\begin{aligned} & \int_{\mathbb{R}^n} |x|^{-ap} |\nabla u|^p \varphi_h(u)^p dx \leq \Lambda \left(k^{\beta-p} \int_{\mathbb{R}^n} |x|^{-bq} |u|^q dx \right. \\ & \left. + K (\beta/p)^p \left(\int_{W_k} |x|^{-bq} |u|^q dx \right)^{\frac{q-p}{q}} \int_{\mathbb{R}^n} |x|^{-ap} |\nabla u|^p \varphi_h(u)^p dx \right). \end{aligned} \quad (2.7)$$

Since $u \in L^q(\mathbb{R}^n, |x|^{-bq})$ and $q > p$, we have

$$\lim_{k \rightarrow +\infty} \left(\int_{W_k} |x|^{-bq} |u|^q dx \right)^{\frac{q-p}{q}} = 0.$$

Therefore, by choosing k sufficiently large (depending on u), it follows from (2.7) that

$$\int_{\mathbb{R}^n} |x|^{-ap} |\nabla u|^p \varphi_h(u)^p dx \leq 2\Lambda k^{\beta-p} \int_{\mathbb{R}^n} |x|^{-bq} |u|^q dx,$$

which together with (2.6) gives

$$\left(\int_{\mathbb{R}^n} |x|^{-bq} |u|^q \varphi_h(u)^q dx \right)^{\frac{p}{q}} \leq 2\Lambda k^{\beta-p} K (\beta/p)^p \int_{\mathbb{R}^n} |x|^{-bq} |u|^q dx. \quad (2.8)$$

Since $u \in L^q(\mathbb{R}^n, |x|^{-bq})$, by passing to the limit as $h \rightarrow +\infty$ into (2.8), we then obtain that $|u|^{\beta/p} \in L^q(\mathbb{R}^n, |x|^{-bq})$. Now we prove that $u \in L^\infty(\mathbb{R}^n)$. For every $h > 0$, we define

$$\psi_h(u) := \operatorname{sgn}(u) \max(|u| - h, 0),$$

where $\operatorname{sgn}(u)$ denotes the sign of u . By using $\psi_h(u)$ as a test function, we obtain

$$\int_{W_h} |x|^{-ap} |\nabla u|^p dx = \int_{W_h} f(x, u) \psi_h(u) dx, \quad (2.9)$$

where W_h is as in (2.1). For every $\beta > p$, it follows from (1.2), (2.9) and Hölder's inequality that

$$\begin{aligned} \int_{W_h} |x|^{-ap} |\nabla u|^p dx &\leq \Lambda \left(\int_{W_h} |x|^{-bq} dx \right)^{1 - \frac{1}{q} - \frac{p(q-1)}{\beta q}} \\ &\quad \times \left(\int_{W_h} |x|^{-bq} |u|^{\frac{\beta q}{p}} dx \right)^{\frac{p(q-1)}{\beta q}} \left(\int_{W_h} |x|^{-bq} |\psi_h(u)|^q dx \right)^{\frac{1}{q}}. \end{aligned} \quad (2.10)$$

By applying the Caffarelli–Kohn–Nirenberg inequality, we obtain

$$\left(\int_{W_h} |x|^{-bq} |\psi_h(u)|^q dx \right)^{\frac{p}{q}} \leq K \int_{W_h} |x|^{-ap} |\nabla u|^p dx. \quad (2.11)$$

Since $|u|^{\beta/p} \in L^q(\mathbb{R}^n, |x|^{-bq})$, it follows from (2.10) and (2.11) that

$$\int_{W_h} |x|^{-bq} |\psi_h(u)|^q dx \leq C \left(\int_{W_h} |x|^{-bq} dx \right)^{\frac{q}{p-1} \left(1 - \frac{1}{q} - \frac{p(q-1)}{\beta q} \right)} \quad (2.12)$$

for some constant $C > 0$ independent of h . It follows from Hölder's inequality and (2.12) that

$$\begin{aligned} &\int_{W_h} |x|^{-bq} |\psi_h(u)| dx \\ &\leq \left(\int_{W_h} |x|^{-bq} dx \right)^{\frac{q-1}{q}} \left(\int_{W_h} |x|^{-bq} |\psi_h(u)|^q dx \right)^{\frac{1}{q}} \\ &\leq C^{\frac{1}{q}} \left(\int_{W_h} |x|^{-bq} dx \right)^{\frac{q-1}{q} + \frac{1}{p-1} \left(1 - \frac{1}{q} - \frac{p(q-1)}{\beta q} \right)}. \end{aligned} \quad (2.13)$$

On the other hand, by applying Tonelli's theorem, we obtain

$$\int_{W_h} |x|^{-bq} |\psi_h(u)| dx = \int_h^\infty \int_{W_s} |x|^{-bq} dx ds. \quad (2.14)$$

Furthermore, by choosing β large enough so that $\beta > p(q-1)/(q-p)$, we obtain

$$\frac{q-1}{q} + \frac{1}{p-1} \left(1 - \frac{1}{q} - \frac{p(q-1)}{\beta q} \right) > 1. \quad (2.15)$$

Therefore, since $u \in L^q(\mathbb{R}^n, |x|^{-bq})$, it follows from (2.13)–(2.15) that

$$\int_{W_h} |x|^{-bq} dx = 0 \quad \text{for large } h > 0$$

and so $u \in L^\infty(\mathbb{R}^n)$. This ends the proof of Step 2.1. \square

We then obtain the following:

Step 2.2. *Let n, p, a, b, q and f be as in Theorem 1.1. Then every solution of (1.1) belongs to $C^1(\mathbb{R}^n \setminus \{0\})$.*

Proof. This step follows directly from Step 2.1 together with the regularity results of DiBenedetto [11] and Tolksdorf [31] (see also Evans [16], Lewis [21], Uhlenbeck [34] and Ural'ceva [35] for previous results on this question). \square

The next result is concerned with the boundedness of solutions of (1.1) in weak Lebesgue spaces.

Step 2.3. *Let n, p, a, b, q and f be as in Theorem 1.1. Then every solution of (1.1) belongs to $L^{q-a/p, \infty}(\mathbb{R}^n, |x|^{-bq})$.*

Proof. Let u be a non-trivial solution of (1.1). For every $h > 0$, we define

$$\tau_h(u) := \operatorname{sgn}(u) \min(|u|, h),$$

where $\operatorname{sgn}(u)$ denotes the sign of u . By testing (1.1) with $\tau_h(u)$ and using (1.2), we obtain

$$\int_{\mathbb{R}^n \setminus W_h} |x|^{-ap} |\nabla u|^p dx \leq \Lambda \left(\int_{\mathbb{R}^n \setminus W_h} |x|^{-bq} |u|^q dx + h \int_{W_h} |x|^{-bq} |u|^{q-1} dx \right), \quad (2.16)$$

where W_h is as in (2.1). On the other hand, straightforward computations give

$$\int_{\mathbb{R}^n \setminus W_h} |x|^{-bq} |u|^q dx = \int_{\mathbb{R}^n} |x|^{-bq} |\tau_h(u)|^q dx - h^q \int_{W_h} |x|^{-bq} dx \quad (2.17)$$

and

$$\int_{W_h} |x|^{-bq} |u|^{q-1} dx = (q-1) \int_h^\infty \left(\int_{W_s} |x|^{-bq} dx \right) s^{q-2} ds + h^{q-1} \int_{W_h} |x|^{-bq} dx. \quad (2.18)$$

Plugging (2.17) and (2.18) into (2.16), we obtain

$$\int_{\mathbb{R}^n \setminus W_h} |x|^{-ap} |\nabla u|^p dx \leq \Lambda \left(\int_{\mathbb{R}^n} |x|^{-bq} |\tau_h(u)|^q dx + (q-1) h \int_h^\infty \left(\int_{W_s} |x|^{-bq} dx \right) s^{q-2} ds \right). \quad (2.19)$$

The Caffarelli–Kohn–Nirenberg inequality yields

$$\int_{\mathbb{R}^n} |x|^{-bq} |\tau_h(u)|^q dx \leq K \left(\int_{\mathbb{R}^n \setminus W_h} |x|^{-ap} |\nabla u|^p dx \right)^{\frac{q}{p}} \quad (2.20)$$

for some constant $K = K(n, a, b, p) > 0$. Since $u \in D^{1,p}(\mathbb{R}^n, |x|^{-ap})$, it follows from (2.17), (2.19) and (2.20) that there exists a constant $C > 0$ such that

$$\begin{aligned} h^q \int_{W_h} |x|^{-bq} dx &\leq \int_{\mathbb{R}^n} |x|^{-bq} |\tau_h(u)|^q dx \\ &\leq C \left(h \int_h^\infty \left(\int_{W_s} |x|^{-bq} dx \right) s^{q-2} ds \right)^{\frac{q}{p}} \quad \text{for small } h > 0. \end{aligned} \quad (2.21)$$

We now define

$$G(h) := \left(\int_h^\infty g(s) ds \right)^{\frac{p-q}{p}}, \quad \text{where } g(s) := s^{q-2} \int_{W_s} |x|^{-bq} dx.$$

In particular, G is positive, non-decreasing and locally absolutely continuous in $(0, \|u\|_{L^\infty(\mathbb{R}^n)})$ with derivative

$$G'(h) = \frac{q-p}{p} \left(\int_h^\infty g(s) ds \right)^{-\frac{q}{p}} g(h) \quad \text{for a.e. } h \in (0, \|u\|_{L^\infty(\mathbb{R}^n)}).$$

By using (2.21), we then obtain

$$G'(h) \leq \frac{q-p}{p} C h^{\frac{q-2p}{p}} \quad \text{for small } h > 0. \quad (2.22)$$

By integrating (2.22), we obtain

$$G(h) - G(0) \leq C h^{\frac{q-p}{p}} \quad \text{for small } h > 0, \quad (2.23)$$

where $G(0)$ stands for the limit of $G(h)$ as $h \rightarrow 0$. On the other hand, by using (2.18) together with dominated convergence, we obtain

$$(q-1) h G(h)^{\frac{-p}{q-p}} \leq h \int_{W_h} |x|^{-bq} |u|^{q-1} dx = o(1) \quad \text{as } h \rightarrow 0.$$

This coupled with (2.23) yields that $G(0) > 0$. By using (2.21) and since G is non-decreasing, we then obtain

$$h^{\frac{q(p-1)}{p}} \int_{W_h} |x|^{-bq} dx \leq C G(h)^{-\frac{q}{q-p}} \leq C G(0)^{-\frac{q}{q-p}} \quad \text{for small } h > 0,$$

which implies that $u \in L^{q-q/p, \infty}(\mathbb{R}^n, |x|^{-bq})$. This ends the proof of Step 2.3. \square

3. THE UPPER BOUND ESTIMATES

This section is devoted to the proof of (1.5). We begin with establishing a decay estimate, which is weaker than (1.5), but which will serve as a preliminary step in the proof of (1.5).

Step 3.1. Let n, p, a, b, q and f be as in Theorem 1.1. Let u be a solution of (1.1). Then there exists a constant $K_0 > 0$ such that

$$|x|^\nu |u(x)| + |x|^{\nu+1} |\nabla u(x)| \leq K_0 \quad \forall x \in \mathbb{R}^n \setminus B(0, 1/2), \quad (3.1)$$

where $\nu := (n - p(1 + a)) / p$.

Proof. For every $R > 0$, we define

$$u_R(x) := R^\nu u(Rx) \quad \forall x \in \mathbb{R}^n.$$

As is easily seen, in order to prove (3.1), it suffices to show that

$$|u_R(x)| + |\nabla u_R(x)| \leq K_0 \quad \forall R > 1/2, x \in \mathbb{S}^n \quad (3.2)$$

for some constant $K_0 > 0$. By using (1.1), we obtain

$$\begin{aligned} -\Delta_p u_R &= R^{\nu(p-1)+p} (|Rx|^{ap} f(Rx, u(Rx)) \\ &\quad - ap |Rx|^{-2} |\nabla u(Rx)|^{p-2} \langle \nabla u(Rx), Rx \rangle) \\ &= R^{\nu(p-1)+(a+1)p} |x|^{ap} f(Rx, R^{-\nu} u_R) \\ &\quad - ap |x|^{-2} |\nabla u_R|^{p-2} \langle \nabla u_R, x \rangle =: g_R(x, u_R, \nabla u_R) \quad \text{in } \mathbb{R}^n \setminus \{0\}. \end{aligned} \quad (3.3)$$

By using (1.2) together with straightforward computations, we obtain

$$\begin{aligned} |g_R(x, u_R, \nabla u_R)| &\leq \Lambda R^{\nu(p-q)+(a+1)p-bq} |x|^{ap-bq} |u_R|^{q-1} \\ &\quad + |a| p |x|^{-1} |\nabla u_R|^{p-1} = \Lambda |x|^{ap-bq} |u_R|^{q-1} + |a| p |x|^{-1} |\nabla u_R|^{p-1} \end{aligned} \quad (3.4)$$

for a.e. $x \in \mathbb{R}^n$. Assuming by contradiction that (3.2) does not hold true, we obtain that there exist sequences $(R_k)_{k \in \mathbb{N}}$ in $(1/2, \infty)$ and $(y_k)_{k \in \mathbb{N}}$ in \mathbb{S}^n such that

$$|u_{R_k}(y_k)| + |\nabla u_{R_k}(y_k)| \geq k \quad \forall k \in \mathbb{N}. \quad (3.5)$$

By using a doubling property (see Poláčik, Quittner and Souplet [25, Lemma 5.1]), we then obtain that there exists a sequence $(x_k)_{k \in \mathbb{N}}$ in $B(0, 2) \setminus B(0, 1/2)$ such that

$$\lambda_k := \left(|u_{R_k}(x_k)|^{\frac{q}{n}} + |\nabla u_{R_k}(x_k)|^{\frac{q}{n+q}} \right)^{-1} \rightarrow 0 \quad \text{as } k \rightarrow \infty, \quad (3.6)$$

$$B(x_k, 2k\lambda_k) \subset B(0, 2) \setminus B(0, 1/2) \quad \forall k \in \mathbb{N} \quad (3.7)$$

and

$$|\tilde{u}_k(y)|^{\frac{q}{n}} + |\nabla \tilde{u}_k(y)|^{\frac{q}{n+q}} \leq 2 \quad \forall k \in \mathbb{N}, y \in B(0, k), \quad (3.8)$$

where

$$\tilde{u}_k(y) := \lambda_k^{\frac{n}{q}} u_{R_k}(x_k + \lambda_k y).$$

By using (3.3), we obtain

$$\begin{aligned} -\Delta_p \tilde{u}_k &= \lambda_k^{\frac{n}{q}(p-1)+p} g_{R_k}(x_k + \lambda_k y, u_{R_k}(x_k + \lambda_k y), \nabla u_{R_k}(x_k + \lambda_k y)) \\ &= \lambda_k^{\frac{n}{q}(p-1)+p} g_{R_k}\left(x_k + \lambda_k y, \lambda_k^{-\frac{n}{q}} \tilde{u}_k, \lambda_k^{-\frac{n+q}{q}} \nabla \tilde{u}_k\right) \end{aligned}$$

in $\mathbb{R}^n \setminus \{-x_k/\lambda_k\}$. Furthermore, by using (3.4)–(3.8) and observing that $\frac{n}{q}(p-q) + p = p(b-a) \geq 0$, we obtain

$$\begin{aligned} & \left| \lambda_k^{\frac{n}{q}(p-1)+p} g_{R_k} \left(x_k + \lambda_k y, \lambda_k^{-\frac{n}{q}} \tilde{u}_k, \lambda_k^{-\frac{n+q}{q}} \nabla \tilde{u}_k \right) \right| \\ & \leq \Lambda \lambda_k^{p(b-a)} |x_k + \lambda_k y|^{ap-bq} |\tilde{u}_k|^{q-1} + \lambda_k |a| p |x_k + \lambda_k y|^{-1} |\nabla \tilde{u}_k|^{p-1} \leq C \end{aligned}$$

for a.e. $y \in B(0, k)$, for some constant $C > 0$ independent of k . Since $|x_k|/\lambda_k \rightarrow \infty$ as $k \rightarrow \infty$, by applying the results of DiBenedetto [11] and Tolksdorf [31], we then obtain that $(\tilde{u}_k)_{k \in \mathbb{N}}$ is bounded in $C_{\text{loc}}^{1,\theta}(\mathbb{R}^n)$ for some $\theta \in (0, 1)$ and so $(\tilde{u}_k)_{k \in \mathbb{N}}$ converges up to a subsequence in $C_{\text{loc}}^1(\mathbb{R}^n)$ to some function $\tilde{u}_\infty \in C^1(\mathbb{R}^n)$. Remark that by definition of \tilde{u}_k , we have

$$|\tilde{u}_k(0)|^{\frac{q}{n}} + |\nabla \tilde{u}_k(0)|^{\frac{q}{n+q}} = 1. \quad (3.9)$$

By passing to the limit as $k \rightarrow \infty$ into (3.9), we then obtain

$$|\tilde{u}_\infty(0)|^{\frac{q}{n}} + |\nabla \tilde{u}_\infty(0)|^{\frac{q}{n+q}} = 1. \quad (3.10)$$

On the other hand, by using (3.7) together with straightforward computations, we obtain

$$\begin{aligned} \int_{B(0,k)} |\tilde{u}_k|^q dy &= \int_{B(x_k, k\lambda_k)} |u_{R_k}|^q dx \leq \int_{B(0,2) \setminus B(0,1/2)} |u_{R_k}|^q dx \\ &\leq 2^{|b|q} \int_{B(0,2) \setminus B(0,1/2)} |x|^{-bq} |u_{R_k}|^q dx \\ &\leq 2^{|b|q} \int_{B(0,2R_k) \setminus B(0,R_k/2)} |x|^{-bq} |u|^q dx. \end{aligned} \quad (3.11)$$

Since $R_k > 1/2$ and $u \in C^1(\mathbb{R}^n \setminus \{0\})$, it follows from (3.6) and (3.7) that $R_k \rightarrow \infty$ as $k \rightarrow \infty$. Since $u \in L^q(\mathbb{R}^n, |x|^{-bq})$, it then follows from (3.11) that $\tilde{u}_k \rightarrow 0$ in $L_{\text{loc}}^q(\mathbb{R}^n)$ and so $\tilde{u}_\infty \equiv 0$ in \mathbb{R}^n . This is in contradiction with (3.10). This ends the proof of Step 3.1. \square

The next step is as follows:

Step 3.2. *Let n, p, a, b, q and f be as in Theorem 1.1. Let u be a solution of (1.1). Let K_0 be as in Step 3.1. Then $v := |u|$ satisfies the inequality*

$$-\Delta_p v \leq \bar{g}(x, v, \nabla v) \quad \text{in } \mathbb{R}^n \setminus B(0, 1/2), \quad (3.12)$$

where

$$\bar{g}(x, v, \nabla v) := \Lambda K_0^{q-p} |x|^{-p} v^{p-1} + |a| p |x|^{-1} |\nabla v|^{p-1}. \quad (3.13)$$

The inequality (3.12) is to be understood in the sense that

$$\int_{\mathbb{R}^n} |\nabla v|^{p-2} \langle \nabla v, \nabla \varphi \rangle dx \leq \int_{\mathbb{R}^n} \bar{g}(x, v, \nabla v) \varphi dx$$

for all $\varphi \in C_c^\infty(\mathbb{R}^n \setminus B(0, 1/2))$ such that $\varphi \geq 0$ in $\mathbb{R}^n \setminus B(0, 1/2)$.

Proof. As is easily seen, the equation satisfied by u can be rewritten as

$$-\Delta_p u = |x|^{ap} f(x, u) - ap|x|^{-2} |\nabla u|^{p-2} \langle \nabla u, x \rangle \quad \text{in } \mathbb{R}^n \setminus \{0\}. \quad (3.14)$$

By using (1.2) and (3.1) together with straightforward computations, we obtain

$$\left| |x|^{ap} f(x, u) - ap|x|^{-2} |\nabla u|^{p-2} \langle \nabla u, x \rangle \right| \leq \bar{g}(x, v, \nabla v) \quad (3.15)$$

for a.e. $x \in B(0, 1/2)$, where $\bar{g}(x, v, \nabla v)$ is as in (3.13). The inequality (3.12) then follows from (3.14) and (3.15) by applying an extended version of Kato's inequality [22] for the p -Laplace operator (see Cuesta Leon [9, Proposition 3.2]). This ends the proof of Step 3.2. \square

We can now prove (1.5) by using Steps 3.1 and 3.2.

Proof of (1.5). Let u be a solution of (1.1) and $v := |u|$. For every $R > 0$, we define

$$u_R(x) := R^\mu u(Rx) \quad \text{and} \quad v_R(x) := R^\mu v(Rx) \quad \forall x \in \mathbb{R}^n, \quad (3.16)$$

where μ is as in (1.6). As is easily seen, in order to prove (1.5), it suffices to show that

$$v_R(x) + |\nabla u_R(x)| \leq C_0 \quad \forall R > 1, x \in \mathbb{S}^n \quad (3.17)$$

for some constant $C_0 > 0$. By using (3.12) and remarking that

$$R^{n-pa} \bar{g}(Rx, v(Rx), \nabla v(Rx)) = \bar{g}(x, v_R(x), \nabla v_R(x)),$$

we obtain that v_R satisfies the inequality

$$-\Delta_p v_R \leq \bar{g}(x, v_R, \nabla v_R) \quad \text{in } \mathbb{R}^n \setminus B(0, 1/(2R)).$$

We can then apply a weak Harnack inequality (see Trudinger [32, Theorem 1.3]), which gives that for every $\epsilon > 0$, there exists a constant $c_\epsilon > 0$ such that

$$\|v_R\|_{L^\infty(B(0,2) \setminus B(0,1/2))} \leq c_\epsilon \|v_R\|_{L^{p-1+\epsilon}(B(0,3) \setminus B(0,1/3))} \quad (3.18)$$

for all $R > 1$. Since $q - q/p > p - 1$, we can choose ϵ so that $p - 1 + \epsilon < q - q/p$. We then obtain

$$\|v_R\|_{L^{p-1+\epsilon}(B(0,3) \setminus B(0,1/3))} \leq c \|v_R\|_{L^{q-q/p, \infty}(B(0,3) \setminus B(0,1/3))} \quad (3.19)$$

for some constant $c = c(n, p, a, b) > 0$. Furthermore, straightforward computations give

$$\begin{aligned} \|v_R\|_{L^{q-q/p, \infty}(B(0,3) \setminus B(0,1/3))}^{q-q/p} &= R^{-bq} \|u\|_{L^{q-q/p, \infty}(B(0,3R) \setminus B(0,R/3))}^{q-q/p} \\ &\leq 3^{|b|q} \|u\|_{L^{q-q/p, \infty}(\mathbb{R}^n, |x|^{-bq})}^{q-q/p}. \end{aligned} \quad (3.20)$$

Since $u \in L^{q-q/p, \infty}(\mathbb{R}^n, |x|^{-bq})$, it follows from (3.18)–(3.20) that

$$\|v_R\|_{L^\infty(B(0,2) \setminus B(0,1/2))} \leq C \quad (3.21)$$

for some constant $C > 0$ independent of $R > 1$. We then infer (3.17) from (3.21) by applying the gradient estimates of DiBenedetto [11] and Tolksdorf [31]. This ends the proof of (1.5). \square

4. THE FIRST-ORDER TERM

This section is devoted to the proof of (1.8). We begin with proving the following:

Step 4.1. *Let n, p, a, b, q, μ and f be as in Theorem 1.1. Let u be a solution of (1.1) such that $u > 0$ and $f(x, u) \geq 0$ in \mathbb{R}^n . Then there exists a constant $c_0 > 0$ such that*

$$u(x) \geq c_0 |x|^{-\mu} \quad \forall x \in \mathbb{R}^n \setminus B(0, 1). \quad (4.1)$$

Proof. For every $R > 0$, we let u_R be the function defined as in (3.16). As is easily seen, in order to prove (4.1), it suffices to show that

$$u_R(x) \geq c_0 \quad \forall R > 1, x \in \mathbb{S}^n \quad (4.2)$$

for some constant $c_0 > 0$. By using (1.1), we obtain

$$-\operatorname{div}(|x|^{-ap} |\nabla u_R|^{p-2} \nabla u_R) = R^n f(Rx, R^{-\mu} u_R) \quad \text{in } \mathbb{R}^n. \quad (4.3)$$

Furthermore, by using (1.2) and (1.5), we obtain

$$|x|^\mu u_R(x) + |x|^{\mu+1} |\nabla u_R(x)| \leq C_0 \quad (4.4)$$

and

$$|R^n f(Rx, R^{-\mu} u_R)| \leq \Lambda C_0^{q-p} R^{n-bq-\mu(q-1)} |x|^{-bq-\mu(q-p)} u_R^{p-1} \quad (4.5)$$

for a.e. $x \in \mathbb{R}^n \setminus B(0, 1/R)$. By remarking that

$$n - bq - \mu(q-1) = -\frac{\mu(q-p)}{p} < 0 \quad (4.6)$$

and applying the Harnack inequality (see Serrin [27]), it then follows from (4.3)–(4.6) that

$$\sup_{B(0,4) \setminus B(0,1/4)} u_R \leq C \inf_{B(0,4) \setminus B(0,1/4)} u_R \quad (4.7)$$

for some constant $C > 0$ independent of $R > 1$. Now we assume by contradiction that (4.2) does not hold true. Since u is positive and continuous in $\mathbb{R}^n \setminus \{0\}$, it then follows from (4.7) that there exists a sequence $(R_k)_{k \in \mathbb{N}}$ such that

$$R_k \rightarrow \infty \quad \text{and} \quad \sup_{B(0,4) \setminus B(0,1/4)} u_{R_k} \rightarrow 0 \quad \text{as } k \rightarrow \infty. \quad (4.8)$$

Since u_{R_k} satisfies (4.3)–(4.6), by applying the Hölder estimates of DiBenedetto [11] and Tolksdorf [31], we obtain that $(u_{R_k})_{k \in \mathbb{N}}$ is bounded in $C^{1,\theta}(B(0,3) \setminus B(0,1/3))$ for some $\theta \in (0,1)$. It then follows from (4.8) that up to a subsequence $u_{R_k} \rightarrow 0$ in $C^1(B(0,2) \setminus B(0,1/2))$. Let

$\eta \in C^1(\mathbb{R}^n)$ be a cutoff function such that $\eta \equiv 1$ in $B(0, 1/2)$, $\eta \equiv 0$ in $\mathbb{R}^n \setminus B(0, 2)$ and $0 \leq \eta \leq 1$ in $B(0, 2) \setminus B(0, 1/2)$. By testing (1.1) with $\eta_k(x) := \eta(x/R_k)$ and using Hölder's inequality, we obtain

$$\begin{aligned} \int_{\mathbb{R}^n} f(x, u) \eta_k dx &= \int_{\mathbb{R}^n} |x|^{-ap} |\nabla u|^{p-2} \langle \nabla u, \nabla \eta_k \rangle dx \\ &= \int_{\mathbb{R}^n} |x|^{-ap} |\nabla u_{R_k}|^{p-2} \langle \nabla u_{R_k}, \nabla \eta \rangle dx \\ &\leq 2^{|a|p} \|\nabla u_{R_k}\|_{L^p(B(0,2) \setminus B(0,1/2))}^{p-1} \|\nabla \eta\|_{L^p(B(0,2) \setminus B(0,1/2))}. \end{aligned} \quad (4.9)$$

Since up to a subsequence $u_{R_k} \rightarrow 0$ in $C^1(B(0, 2) \setminus B(0, 1/2))$, it follows from (4.9) that

$$\int_{\mathbb{R}^n} f(x, u) dx = \lim_{k \rightarrow \infty} \int_{\mathbb{R}^n} f(x, u) \eta_k dx = 0. \quad (4.10)$$

Since $f(x, u) \geq 0$ in \mathbb{R}^n , it follows from (4.10) that $f(x, u) \equiv 0$ in \mathbb{R}^n . Since u is bounded in \mathbb{R}^n , by applying a weighted version of Liouville's theorem (see Heinonen, Kilpeläinen and Martio [19, Theorem 6.10]), we then obtain that u is constant, which is in contradiction with (1.5) and the fact that u is positive in \mathbb{R}^n . This ends the proof of Step 4.1. \square

The next step is as follows:

Step 4.2. *Let n, p, a, b, q, μ and f be as in Theorem 1.1. Let u be a solution of (1.1) such that $u > 0$ and $f(x, u) \geq 0$ in \mathbb{R}^n . Then there exists a constant $\alpha > 0$ such that*

$$\lim_{R \rightarrow \infty} \Gamma_R(u) = \alpha, \quad \text{where } \Gamma_R(u) := \min_{x \in \mathbb{S}^n} (R^\mu u(Rx)). \quad (4.11)$$

Proof. By applying Step 4.1, we obtain that

$$\alpha := \liminf_{R \rightarrow \infty} \Gamma_R(u) > 0.$$

Assume by contradiction that (4.11) is not true, namely that

$$\limsup_{R \rightarrow \infty} \Gamma_R(u) > \alpha.$$

We then obtain that there exist $R_1, R_2 > 0$ such that $R_1 < R_2$ and

$$\beta := \min_{R_1 < R < R_2} (\Gamma_R(u)) < \min(\Gamma_{R_1}(u), \Gamma_{R_2}(u)).$$

It follows that

$$\min_{x \in A} (u(x) - w_\beta(x)) = 0 < \min_{x \in \partial A} (u(x) - w_\beta(x)), \quad (4.12)$$

where $A := B(0, R_2) \setminus B(0, R_1)$ and

$$w_\beta(x) := \beta |x|^{-\mu} \quad \forall x \in \mathbb{R}^n \setminus \{0\}.$$

By observing that

$$\operatorname{div}(|x|^{-ap} |\nabla u|^{p-2} \nabla u) \leq 0 = \operatorname{div}(|x|^{-ap} |\nabla w_\beta|^{p-2} \nabla w_\beta) \quad \text{in } \mathbb{R}^n \setminus \{0\},$$

we then obtain that (4.12) contradicts the strict comparison principle of Serrin [28, Theorem 1]. This ends the proof of Step 4.2. \square

We can now end the proof of (1.8) by using Steps 4.1 and 4.2.

Proof of (1.8). Let u be a solution of (1.1) such that $u > 0$ and $f(x, u) \geq 0$ in \mathbb{R}^n . By applying Step 4.2, we obtain that (4.11) holds true. We will prove that (1.8) holds true with α given by (4.11). Assume by contradiction that (1.8) is not true. Then there exists a sequence $(R_k)_{k \in \mathbb{N}}$ such that $R_k \rightarrow \infty$ and

$$\limsup_{k \rightarrow \infty} \sup_{x \in \mathbb{S}^n} (|u_{R_k}(x) - \alpha| + |\nabla u_{R_k}(x) + \alpha \mu x|) > 0, \quad (4.13)$$

where u_{R_k} is as in (3.16). We recall that u_{R_k} satisfies (4.3)–(4.6). By applying the Hölder estimates of DiBenedetto [11] and Tolksdorf [31], we then obtain that for every compact set $\Omega \subset \mathbb{R}^n \setminus \{0\}$, there exists $\theta_\Omega \in (0, 1)$ such that $(u_{R_k})_{k \in \mathbb{N}}$ is bounded in $C^{1, \theta_\Omega}(\Omega)$ and so there exists a subsequence of $(u_{R_k})_{k \in \mathbb{N}}$ which converges in $C^1(\Omega)$ to some function $u_\infty \in C^1(\mathbb{R}^n \setminus \{0\})$. By passing to the limit as $k \rightarrow \infty$ into (4.3) and using (4.4)–(4.6), we obtain that u_∞ satisfies the equation

$$\operatorname{div}(|x|^{-ap} |\nabla u_\infty|^{p-2} \nabla u_\infty) = 0 \quad \text{in } \mathbb{R}^n \setminus \{0\}.$$

By using (4.11) and observing that $\Gamma_r(u_{R_k}) = \Gamma_{R_k r}(u)$, we obtain

$$\Gamma_r(u_\infty) = \lim_{k \rightarrow \infty} \Gamma_r(u_{R_k}) = \alpha \quad \forall r > 0.$$

By another application of the strict comparison principle [28, Theorem 1], we then obtain

$$u_\infty(x) = \alpha |x|^{-\mu} \quad \forall x \in \mathbb{R}^n \setminus \{0\},$$

which contradicts (4.13). This ends the proof of (1.8). \square

5. THE KELVIN-TYPE TRANSFORMATION

This section is devoted to the proof of Proposition 1.1 and its application to the proof of (1.6).

Proof of Proposition 1.1. Let n, p, a, b, q, μ and f be as in Theorem 1.1 and u be a weak solution of (1.9). Assume that there exists a constant $\alpha > 0$ such that (1.8) holds true. It follows from (1.8) that if R is chosen large enough, then $u > 0$ and $|\nabla u| > 0$ in $\mathbb{R}^n \setminus B(0, R)$. Let $r := 1/R$ and u_* be the function defined as in (1.10). Since $u \in C^1(\mathbb{R}^n \setminus B(0, R))$, we have $u_* \in C^1(\overline{B(0, r)} \setminus \{0\})$. In what follows, we will use the notations $\nabla_x, \operatorname{div}_x$ and $\nabla_y, \operatorname{div}_y$ for the gradient and

divergence with respect to x and y , respectively. It follows from (1.8) that u_* can be extended to a continuous function in $B(0, r)$ and

$$\begin{aligned} |\nabla_y u_*| &= |-\mu |y|^{-\mu-2} u y + |y|^{-\mu-2} \nabla_x u - 2 |y|^{-\mu-4} \langle \nabla_x u, y \rangle y| \\ &= |-\mu |x|^\mu u x + |x|^{\mu+2} \nabla_x u - 2 |x|^\mu \langle \nabla_x u, x \rangle x| \\ &= o(|x|) = o(|y|^{-1}) \end{aligned} \quad (5.1)$$

as $|y| \rightarrow 0$, where $x := |y|^{-2} y$. By letting γ be as in (1.13) and remarking that $\gamma - 2 > -n$, it follows from (5.1) that $u_* \in H^{1,2}(B(0, r), |y|^\gamma)$. Furthermore, for every $\varphi \in C_c^\infty(B(0, r) \setminus \{0\})$ straightforward computations give

$$\begin{aligned} \int_{B(0,r)} |x|^{-ap} |\nabla_x u|^{p-2} \langle \nabla_x u, \nabla_x (|x|^{-\mu} \varphi) \rangle dx &= \sum_{i,j=1}^n \int_{B(0,r)} |x|^{2n-2-ap} \\ &\quad \times |\nabla_x u|^{p-2} \partial_{x_i} u (\delta_{ij} - 2|x|^{-2} x_i x_j) \partial_{y_j} (|x|^{-\mu} \varphi) dy \\ &= \int_{B(0,r)} |x|^{2n-2-ap-\mu} |\nabla_x u|^{p-2} \langle \nabla_x u - 2|x|^{-2} \langle \nabla_x u, x \rangle x, \nabla_y \varphi \rangle dy \\ &\quad - \mu \int_{B(0,r)} |x|^{2n-2-ap-\mu} |\nabla_x u|^{p-2} \langle \nabla_x u, x \rangle \varphi dy. \end{aligned} \quad (5.2)$$

We now compute

$$\begin{aligned} \nabla_x u &= -\mu |x|^{-\mu-2} u_* x + |x|^{-\mu-2} \nabla_y u_* - 2 |x|^{-\mu-4} \langle \nabla_y u_*, x \rangle x \\ &= -\mu |y|^\mu u_* y + |y|^{\mu+2} \nabla_y u_* - 2 |y|^\mu \langle \nabla_y u_*, y \rangle y \end{aligned} \quad (5.3)$$

and

$$\langle \nabla_x u, x \rangle = -\mu |y|^\mu u_* - |y|^\mu \langle \nabla_y u_*, y \rangle. \quad (5.4)$$

It follows from (5.3) and (5.4) that

$$\nabla_x u - 2 |x|^{-2} \langle \nabla_x u, x \rangle x = \mu |y|^\mu u_* y + |y|^{\mu+2} \nabla_y u_* \quad (5.5)$$

and

$$|\nabla_x u|^2 = \mu^2 |y|^{2\mu+2} u_*^2 + 2\mu |y|^{2\mu+2} u_* \langle \nabla_y u_*, y \rangle + |y|^{2\mu+4} |\nabla_y u_*|^2. \quad (5.6)$$

By using (5.4)–(5.6) together with the fact that $\mu(p-1) + (a+1)p = n$, we obtain

$$\begin{aligned} |x|^{2n-2-ap-\mu} |\nabla_x u|^{p-2} (\nabla_x u - 2|x|^{-2} \langle \nabla_x u, x \rangle x) \\ = |y|^\gamma (\mu^2 u_*^2 + 2\mu u_* \langle \nabla_y u_*, y \rangle + |\nabla_y u_*|^2 |y|^2)^{\frac{p-2}{2}} \\ \quad \times (\mu |y|^{-2} u_* y + \nabla_y u_*) \end{aligned} \quad (5.7)$$

and

$$\begin{aligned} & |x|^{2n-2-ap-\mu} |\nabla_x u|^{p-2} \langle \nabla_x u, x \rangle \\ &= -|y|^{\gamma-2} \left(\mu^2 u_*^2 + 2\mu u_* \langle \nabla_y u_*, y \rangle + |\nabla_y u_*|^2 |y|^2 \right)^{\frac{p-2}{2}} \\ & \quad \times (\mu u_* + \langle \nabla_y u_*, y \rangle). \end{aligned} \quad (5.8)$$

Let $\tilde{A} : B(0, r) \times (0, \infty) \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ and $\tilde{B} : B(0, r) \times (0, \infty) \times \mathbb{R}^n \rightarrow \mathbb{R}$ be the Caratheodory functions defined by

$$\tilde{A}(y, z, \xi) := |y|^\gamma E(y, z, \xi)^{\frac{p-2}{2}} (\mu |y|^{-2} zy + \xi) - \mu^{p-1} |y|^{\gamma-2} z^{p-1} y$$

and

$$\begin{aligned} \tilde{B}(y, s, \xi) &:= -\mu |y|^{\gamma-2} E(y, z, \xi)^{\frac{p-2}{2}} (\mu z + \langle \xi, y \rangle) \\ & \quad + (p-1) \mu^{p-1} |y|^{\gamma-2} z^{p-2} \langle \xi, y \rangle + \mu^p |y|^{\gamma-2} z^{p-1} \\ & \quad + |y|^{\mu-2n} f(|y|^{-2} y, |y|^\mu z) \end{aligned}$$

for all $(y, z, \xi) \in B(0, r) \times (0, \infty) \times \mathbb{R}^n$, where

$$E(y, z, \xi) := \mu^2 z^2 + 2\mu z \langle \xi, y \rangle + |\xi|^2 |y|^2.$$

We claim that u_* solves the equation

$$-\operatorname{div}_y \tilde{A}(y, u_*, \nabla_y u_*) = \tilde{B}(y, u_*, \nabla_y u_*) \quad \text{in } B(0, r) \setminus \{0\}.$$

Indeed, by using (5.2), (5.7) and (5.8) together with the fact that u solves (1.1), we obtain

$$\operatorname{div}_y \tilde{A}(y, u_*, \nabla_y u_*) + \tilde{B}(y, u_*, \nabla_y u_*) = H(y, u_*, \nabla_y u_*) \quad \text{in } B(0, r) \setminus \{0\},$$

where

$$\begin{aligned} H(y, u_*, \nabla_y u_*) &= -\operatorname{div}_y (\mu^{p-1} |y|^{\gamma-2} u_*^{p-1} y) \\ & \quad + (p-1) \mu^{p-1} |y|^{\gamma-2} u_*^{p-2} \langle \nabla_y u_*, y \rangle + \mu^p |y|^{\gamma-2} u_*^{p-1}. \end{aligned}$$

Straightforward computations then give $H(y, u_*, \nabla_y u_*) = 0$, thus proving our claim. We now let $A : B(0, r) \times \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ and $B : B(0, r) \times \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}$ be the Caratheodory functions defined by

$$\begin{aligned} A(y, z, \xi) &:= \tilde{A}(y, g_0(z), g(y, \xi)) \\ & \quad + \mu^{p-2} |y|^\gamma g_0(z)^{p-2} ((p-2) |y|^{-2} \langle \xi - g(y, \xi), y \rangle y + \xi - g(y, \xi)) \end{aligned}$$

and

$$B(y, z, \xi) := \tilde{B}(y, g_0(z), g(y, \xi))$$

for all $(y, z, \xi) \in B(0, r) \times \mathbb{R} \times \mathbb{R}^n$, where

$$g_0(z) := \begin{cases} \alpha/2 & \text{if } z < \alpha/2 \\ z & \text{if } \alpha/2 \leq z \leq 3\alpha/2 \\ 3\alpha/2 & \text{if } z > 3\alpha/2 \end{cases}$$

and

$$g(y, \xi) := \min\left(1, \frac{\delta}{|y||\xi|}\right) \xi.$$

Here δ is a positive constant that will be chosen later. It follows from (1.8) and (5.1) that if R is chosen large enough i.e. r is chosen small enough, then $g_0(u_*) = u_*$ and $g(y, \nabla_y u_*) = \nabla_y u_*$ in $B(0, r) \setminus \{0\}$ and so u_* solves the equation

$$-\operatorname{div}_y A(y, u_*, \nabla_y u_*) = B(y, u_*, \nabla_y u_*) \quad \text{in } B(0, r) \setminus \{0\}. \quad (5.9)$$

Furthermore, since $\alpha/2 \leq g_0(z) \leq 3\alpha/2$ and $|g(y, \xi)| \leq \delta/|y|$, straightforward estimates give

$$\begin{aligned} E(y, g_0(z), g(y, \xi))^{\frac{p-2}{2}} &= \mu^{p-2} g_0(z)^{p-2} \\ &+ (p-2) \mu^{p-3} g_0(z)^{p-3} \langle g(y, \xi), y \rangle + O(|y|^2 |g(y, \xi)|^2), \end{aligned}$$

which then yields

$$\begin{aligned} A(y, z, \xi) &= \mu^{p-2} |y|^\gamma g_0(z)^{p-2} ((p-2) |y|^{-2} \langle \xi, y \rangle y + \xi) \\ &+ O(|y|^{\gamma+1} |g(y, \xi)|^2) \quad (5.10) \end{aligned}$$

and

$$B(y, z, \xi) = O(|y|^\gamma |g(y, \xi)|^2) + |y|^{\mu-2n} f(|y|^{-2} y, |y|^\mu g_0(z)) \quad (5.11)$$

uniformly in $(y, z, \xi) \in B(0, r) \times \mathbb{R} \times \mathbb{R}^n$. By using (5.11) together with (1.2) and remarking that $(b + \mu)q = \mu q/p + n$ and $|g(y, \xi)| \leq |\xi|$, we then obtain

$$B(y, z, \xi) = O(|y|^\gamma |g(y, \xi)|^2 + |y|^{\gamma'}) = O(|y|^\gamma |\xi|^2 + |y|^{\gamma'}),$$

where γ and γ' are as in (1.13). On the other hand, by using (5.10), we obtain

$$|A(y, z, \xi)| = O(|y|^\gamma |\xi| + |y|^{\gamma+1} |g(y, \xi)|^2) = O(|y|^\gamma |\xi|)$$

and

$$\begin{aligned} A(y, z, \xi) \cdot \xi &= \mu^{p-2} |y|^\gamma g_0(z)^{p-2} ((p-2) |y|^{-2} \langle \xi, y \rangle^2 + |\xi|^2) \\ &+ O(|y|^{\gamma+1} |g(y, \xi)|^2 |\xi|) \\ &\geq \mu^{p-2} |y|^\gamma g_0(z)^{p-2} (\min(p-1, 1) + O(\delta)) |\xi|^2 \\ &\geq C^{-1} |y|^\gamma |\xi|^2 \end{aligned}$$

for some constant $C > 0$ independent of $(y, z, \xi) \in B(0, r) \times \mathbb{R} \times \mathbb{R}^n$, provided we choose δ small enough. This proves that (1.12) holds true. It remains to show that u_* is a weak solution of (1.11). Let $\eta \in C^1(\mathbb{R}^n)$ be a cutoff function such that $\eta \equiv 1$ in $B(0, 1/2)$, $\eta \equiv 0$ in $\mathbb{R}^n \setminus B(0, 1)$ and $0 \leq \eta \leq 1$ in $B(0, 1) \setminus B(0, 1/2)$. For every $\varepsilon > 0$, let $\eta_\varepsilon \in C^1(\mathbb{R}^n)$

be the function defined by $\eta_\varepsilon(y) = \eta(y/\varepsilon)$ for all $y \in \mathbb{R}^n$. For every $\varphi \in C_0^\infty(B(0, r))$, by using $(1 - \eta_\varepsilon)\varphi$ as a test function, we obtain

$$\begin{aligned} \int_{B(0, r)} (1 - \eta_\varepsilon) (A(y, u_*, \nabla u_*) \cdot \nabla \varphi - B(y, u_*, \nabla u_*) \varphi) dy \\ = \int_{B(0, r)} (A(y, u_*, \nabla u_*) \cdot \nabla \eta_\varepsilon) \varphi dy. \end{aligned} \quad (5.12)$$

On the other hand, by using (1.12) and (5.1) together with our definition of η_ε , we obtain

$$\begin{aligned} \int_{B(0, r)} \eta_\varepsilon A(y, u_*, \nabla u_*) \cdot \nabla \varphi dy &= O\left(\int_{B(0, \varepsilon)} |y|^{\gamma-1} dy\right) \\ &= O(\varepsilon^{\gamma+n-1}) = o(1), \end{aligned} \quad (5.13)$$

$$\begin{aligned} \int_{B(0, r)} \eta_\varepsilon B(y, u_*, \nabla u_*) \varphi dy &= O\left(\int_{B(0, \varepsilon)} (|y|^{\gamma-2} + |y|^{\gamma'}) dy\right) \\ &= O(\varepsilon^{\gamma+n-2} + \varepsilon^{\gamma'+n}) = o(1) \end{aligned} \quad (5.14)$$

and

$$\begin{aligned} \int_{B(0, r)} (A(y, u_*, \nabla u_*) \cdot \nabla \eta_\varepsilon) \varphi dy &= O\left(\frac{1}{\varepsilon} \int_{B(0, \varepsilon)} |y|^{\gamma-1} dy\right) \\ &= O(\varepsilon^{\gamma+n-2}) = o(1) \end{aligned} \quad (5.15)$$

as $\varepsilon \rightarrow 0$. It follows from (5.12)–(5.15) that

$$\int_{B(0, r)} (A(y, u_*, \nabla u_*) \cdot \nabla \varphi - B(y, u_*, \nabla u_*) \varphi) dy = 0.$$

This proves that u_* is a weak solution of (1.11). This ends the proof of Proposition 1.1. \square

We can now end the proof of (1.6) and therefore of Theorem 1.1 by putting together (1.8), Proposition 1.1 and a result of Stredulinski [30].

End of proof of (1.6) and of Theorem 1.1. Let u be a solution of (1.1) such that $u > 0$ and $f(x, u) \geq 0$ in \mathbb{R}^n . Let u_* be as in (1.10) and α be given by (1.8). By applying Proposition 1.1 together with the Hölder continuity result of Stredulinski [30, Theorem 3.1.15] (combined with [30, Theorem 2.2.56 and Lemma 3.1.7]), we then obtain that u_* is Hölder continuous in $B(0, r)$ provided there exist constants $C, \sigma > 0$ and $s \in [1, 2)$ such that

$$\int_{B(y_0, \rho)} |y|^{\gamma'} |\varphi|^s dy \leq C \rho^{s+\sigma-2} \int_{B(y_0, \rho)} |y|^\gamma |\nabla \varphi|^s dy \quad (5.16)$$

for all $\varphi \in C_0^\infty(B(y_0, \rho))$, $y_0 \in \mathbb{R}^n$ and $\rho > 0$ such that $\overline{B(y_0, \rho)} \subset B(0, r)$, where γ and γ' are as in (5.1). By remarking that $\gamma' > \gamma - 2$ and

applying a weighted version of the Hardy inequality, which corresponds to the limit case of the Caffarelli–Kohn–Nirenberg inequality [3], we obtain that (5.16) holds true with $s = 2 - \sigma$ provided we choose σ small enough. Therefore, we obtain that u_* is Hölder continuous in $B(0, r)$. In particular, we obtain that there exist constants $C, \delta > 0$ such that

$$|u_*(y) - \alpha| = |u_*(y) - u_*(0)| \leq C|y|^\delta \quad \forall y \in B(0, r). \quad (5.17)$$

By putting together (5.17) with the definition of u_* , we then obtain that (1.6) holds true. This ends the proof of Theorem 1.1. \square

REFERENCES

- [1] J. Byeon and Z.-Q. Wang, *Symmetry breaking of extremal functions for the Caffarelli-Kohn-Nirenberg inequalities*, Commun. Contemp. Math. **4** (2002), no. 3, 457–465.
- [2] L. A. Caffarelli, B. Gidas, and J. Spruck, *Asymptotic symmetry and local behavior of semilinear elliptic equations with critical Sobolev growth*, Comm. Pure Appl. Math. **42** (1989), no. 3, 271–297.
- [3] L. Caffarelli, R. Kohn, and L. Nirenberg, *First order interpolation inequalities with weights*, Compositio Math. **53** (1984), no. 3, 259–275.
- [4] P. Caldiroli and R. Musina, *Symmetry breaking of extremals for the Caffarelli-Kohn-Nirenberg inequalities in a non-Hilbertian setting*, Milan J. Math. **81** (2013), no. 2, 421–430.
- [5] F. Catrina and Z.-Q. Wang, *On the Caffarelli-Kohn-Nirenberg inequalities: sharp constants, existence (and nonexistence), and symmetry of extremal functions*, Comm. Pure Appl. Math. **54** (2001), no. 2, 229–258.
- [6] K. S. Chou and C. W. Chu, *On the best constant for a weighted Sobolev-Hardy inequality*, J. London Math. Soc. (2) **48** (1993), no. 1, 137–151.
- [7] G. Ciraolo, A. Figalli, and A. Roncoroni, *Symmetry results for critical anisotropic p -Laplacian equations in convex cones*, Geom. Funct. Anal. **30** (2020), no. 3, 770–803.
- [8] E. Colorado and I. Peral, *Eigenvalues and bifurcation for elliptic equations with mixed Dirichlet-Neumann boundary conditions related to Caffarelli-Kohn-Nirenberg inequalities*, Topol. Methods Nonlinear Anal. **23** (2004), no. 2, 239–273.
- [9] M. C. Cuesta Leon, *Existence results for quasilinear problems via ordered sub- and supersolutions*, Ann. Fac. Sci. Toulouse Math. (6) **6** (1997), no. 4, 591–608.
- [10] L. Damascelli and M. Ramaswamy, *Symmetry of C^1 solutions of p -Laplace equations in \mathbb{R}^N* , Adv. Nonlinear Stud. **1** (2001), no. 1, 40–64.
- [11] E. DiBenedetto, *$C^{1+\alpha}$ local regularity of weak solutions of degenerate elliptic equations*, Nonlinear Anal. **7** (1983), no. 8, 827–850.
- [12] G. Di Fazio and P. Zanboni, *Regularity for quasilinear degenerate elliptic equations*, Math. Z. **253** (2006), no. 4, 787–803.
- [13] J. Dolbeault, M. J. Esteban, and M. Loss, *Rigidity versus symmetry breaking via nonlinear flows on cylinders and Euclidean spaces*, Invent. Math. **206** (2016), no. 2, 397–440.

- [14] ———, *Symmetry and symmetry breaking: rigidity and flows in elliptic PDEs*, Proceedings of the International Congress of Mathematicians, Vol. 3 (Rio de Janeiro, 2018), World Scientific Publishing Co., Inc. River Edge, NJ, USA, 2018, pp. 2279–2304.
- [15] P. Esposito, *A classification result for the quasi-linear Liouville equation*, Ann. Inst. H. Poincaré Anal. Non Linéaire **35** (2018), no. 3, 781–801.
- [16] L. C. Evans, *A new proof of local $C^{1,\alpha}$ regularity for solutions of certain degenerate elliptic p.d.e.*, J. Differential Equations **45** (1982), no. 3, 356–373.
- [17] V. Felli and M. Schneider, *Perturbation results of critical elliptic equations of Caffarelli-Kohn-Nirenberg type*, J. Differential Equations **191** (2003), no. 1, 121–142.
- [18] ———, *A note on regularity of solutions to degenerate elliptic equations of Caffarelli-Kohn-Nirenberg type*, Adv. Nonlinear Stud. **3** (2003), no. 4, 431–443.
- [19] J. Heinonen, T. Kilpeläinen, and O. Martio, *Nonlinear potential theory of degenerate elliptic equations*, Dover Publications, Inc., Mineola, NY, 2006.
- [20] T. Horiuchi, *Best constant in weighted Sobolev inequality with weights being powers of distance from the origin*, J. Inequal. Appl. **1** (1997), no. 3, 275–292.
- [21] J. L. Lewis, *Regularity of the derivatives of solutions to certain degenerate elliptic equations*, Indiana Univ. Math. J. **32** (1983), no. 6, 849–858.
- [22] T. Kato, *Schrödinger operators with singular potentials*, Israel J. Math. **13** (1973), 135–148.
- [23] F. Oliva, B. Sciunzi, and G. Vaira, *Radial symmetry for a quasilinear elliptic equation with a critical Sobolev growth and Hardy potential*, J. Math. Pures Appl. (9) **140** (2020), 89–109.
- [24] D. D. Monticelli, S. Rodney, and R. L. Wheeden, *Harnack’s inequality and Hölder continuity for weak solutions of degenerate quasilinear equations with rough coefficients*, Nonlinear Anal. **126** (2015), 69–114.
- [25] P. Poláčik, P. Quittner, and P. Souplet, *Singularity and decay estimates in superlinear problems via Liouville-type theorems. I. Elliptic equations and systems*, Duke Math. J. **139** (2007), no. 3, 555–579.
- [26] B. Sciunzi, *Classification of positive $D^{1,p}(\mathbb{R}^N)$ -solutions to the critical p -Laplace equation in \mathbb{R}^N* , Adv. Math. **291** (2016), no. 1, 12–23.
- [27] J. Serrin, *Local behavior of solutions of quasi-linear equations*, Acta Math. **111** (1964), no. 1, 247–302.
- [28] James Serrin, *On the strong maximum principle for quasilinear second order differential inequalities*, J. Functional Analysis **5** (1970).
- [29] D. Smets and M. Willem, *Partial symmetry and asymptotic behavior for some elliptic variational problems*, Calc. Var. Partial Differential Equations **18** (2003), no. 1, 57–75.
- [30] E. W. Stredulinsky, *Weighted inequalities and degenerate elliptic partial differential equations*, Lecture Notes in Mathematics, vol. 1074, Springer-Verlag, Berlin, 1984.
- [31] P. Tolksdorf, *Regularity for a more general class of quasilinear elliptic equations*, J. Differential Equations **51** (1984), no. 1, 126–150.
- [32] N. S. Trudinger, *On Harnack type inequalities and their application to quasilinear elliptic equations*, Comm. Pure Appl. Math. **20** (1967), 721–747.
- [33] ———, *Remarks concerning the conformal deformation of Riemannian structures on compact manifolds*, Ann. Scuola Norm. Sup. Pisa (3) **22** (1968).
- [34] K. Uhlenbeck, *Regularity for a class of non-linear elliptic systems*, Acta Math. **138** (1977), no. 3-4, 219–240.

- [35] N. N. Ural'ceva, *Degenerate quasilinear elliptic systems*, Zap. Naučn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. **7** (1968) (Russian).
- [36] J. Vétois, *A priori estimates and application to the symmetry of solutions for critical p -Laplace equations*, J. Differential Equations **260** (2016), no. 1, 149–161.
- [37] C.-L. Xiang, *Asymptotic behaviors of solutions to quasilinear elliptic equations with critical Sobolev growth and Hardy potential*, J. Differential Equations **259** (2015), no. 8, 3929–3954.
- [38] ———, *Gradient estimates for solutions to quasilinear elliptic equations with critical Sobolev growth and hardy potential*, Acta Math. Sci. Ser. B (Engl. Ed.) **37** (2017), no. 1, 58–68.

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