A PRIORI ESTIMATES FOR SOLUTIONS OF ANISOTROPIC ELLIPTIC EQUATIONS

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ABSTRACT. We prove universal, pointwise, *a priori* estimates for nonnegative solutions of anisotropic nonlinear elliptic equations.

1. Introduction

In dimension $n \geq 2$, given $\overrightarrow{p} = (p_1, \dots, p_n)$ with $p_i > 1$ for $i = 1, \dots, n$, the anisotropic Laplace operator $\Delta_{\overrightarrow{p}}$ is defined by

$$\Delta_{\overrightarrow{p}} u = \sum_{i=1}^{n} \frac{\partial}{\partial x_i} \nabla_{x_i}^{p_i} u\,, \tag{1.1}$$

where $\nabla_{x_i}^{p_i} u = |\partial u / \partial x_i|^{p_i - 2}\, \partial u / \partial x_i$. In this paper, we consider nonnegative solutions of anisotropic equations of the type

$$-\Delta_{\overrightarrow{p}} u = f(u) \tag{1.2}$$

in open subsets of $\mathbb{R}^n$, where f is continuous on $\mathbb{R}_+$. Anisotropic equations of type (1.2) have received much attention in recent years. They have been investigated by Alves–El Hamidi [1], Antontsev–Shmarev [3–7], Bendahmane–Karlsen [10–12], Bendahmane–Langlais–Saad [13], Cianchi [19], D'Ambrosio [21], Fragalà–Gazzola–Kawohl [25], Fragalà–Gazzola–Lieberman [26], El Hamidi–Rakotoson [22, 23], El Hamidi–Vétois [24], Li [33], Lieberman [34, 35], Mihăilescu–Pucci–Rădulescu [38, 39], Mihăilescu–Rădulescu–Tersian [40], and Vétois [51]. They have strong physical background. Time evolution versions of these equations emerge, for instance, from the mathematical description of the dynamics of fluids in anisotropic media when the conductivities of the media are different in different directions. We refer to the extensive books by Antontsev–Díaz–Shmarev [2] and Bear [9] for discussions in this direction. They also appear in biology as a model for the propagation of epidemic diseases in heterogeneous domains (see, for instance, Bendahmane–Karlsen [10] and Bendahmane–Langlais–Saad [13]).

In connection with the anisotropic Laplace operator (1.1), for any open subset Ω in $\mathbb{R}^n$, we define the Sobolev space

$$W^{1,\overrightarrow{p}}_{\mathrm{loc}}(\Omega) = \left\{ u \in L^1_{\mathrm{loc}}(\Omega)\,;\ \frac{\partial u}{\partial x_i} \in L^{p_i}_{\mathrm{loc}}(\Omega) \quad \forall i = 1, \dots, n \right\},$$

where for any real number $p \geq 1$, $L^p_{\mathrm{loc}}(\Omega)$ stands for the space of all measurable functions on Ω which belong to $L^p(\Omega')$ for all compact subsets Ω' of Ω. Possible references on anisotropic Sobolev spaces are Besov [14], Haškovec–Schmeiser [30], Kruzhkov–Kolodiĭ [31], Kruzhkov–Korolev [32], Lu [37], Nikol′skiĭ [45], Rákosník [47, 48], and Troisi [50]. We consider in this paper weak solutions in $W^{1,\overrightarrow{p}}_{\mathrm{loc}}(\Omega) \cap L^\infty_{\mathrm{loc}}(\Omega)$ of equations of type (1.2). In case $p_i \geq 2$ for all $i = 1, \dots, n$, we know by Lieberman [34, 35] that if f is continuous, then any weak solution in $W^{1,\overrightarrow{p}}_{\mathrm{loc}}(\Omega) \cap L^\infty_{\mathrm{loc}}(\Omega)$ of equation (1.2) belongs to $W^{1,\infty}_{\mathrm{loc}}(\Omega)$, and in particular, is continuous.

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In this paper, we aim to find universal, pointwise, *a priori* estimates for solutions of equations like (1.2). By universal, we mean that the estimate does not depend on the solution. In the classical case of the isotropic Laplace operator, it is well known since the work of Gidas–Spruck [28] that such estimates can be derived via rescaling arguments from a Liouville result. We state in Theorem 1.1 our *a priori* estimates in the anisotropic case. A large part of the paper relies on establishing Liouville results associated with the nonlinear anisotropic equation (1.2). Theorem 1.2, see Section 4, is actually a Liouville result of the type of Mitidieri–Pohožaev [41–44], where we prove nonexistence for inequalities. Theorem 1.3, see Section 5, is a Liouville result of the type of Gidas–Spruck [27] and Serrin–Zou [49], where we prove nonexistence for equations.

We define the critical exponent $p_{\text{cr}}(\overrightarrow{p})$ to be the supremum of the real numbers Q such that for any q in (p_+, Q), where $p_+ = \max(p_1, \dots, p_n)$, there does not exist any nontrivial, nonnegative solution in $W^{1,\overrightarrow{p}}_{\text{loc}}(\mathbb{R}^n) \cap L^\infty_{\text{loc}}(\mathbb{R}^n)$ of the equation

$$-\Delta_{\overrightarrow{p}} u = \lambda u^{q-1}, \tag{1.3}$$

where λ is any positive real number, with the convention that $p_{\text{cr}}(\overrightarrow{p}) = p_+$ in case such a real number Q does not exist. As a remark, by an easy change of variable, we can take $\lambda = 1$ in equation (1.3). In case $p_i = 2$ for $i = 1, \dots, n$, namely in the case of the isotropic Laplace operator, by Gidas–Spruck [27], we get $p_{\text{cr}} = +\infty$ in case $n = 2$ and $p_{\text{cr}} = 2n/(n-2)$ in case $n \geq 3$. In the anisotropic regime, our *a priori* estimate states as follows.

Theorem 1.1. *Let $n \geq 2$, $\overrightarrow{p} = (p_1, \dots, p_n)$, and q be such that $2 \leq p_i < q < p_{\text{cr}}(\overrightarrow{p})$ for $i = 1, \dots, n$. Let λ be a positive real number and f be a continuous function on $\mathbb{R}_+$ satisfying*

$$f(u) = u^{q-1}(\lambda + \text{o}(1)) \tag{1.4}$$

as $u \to +\infty$. Then there exist two positive constants $\Lambda_1 = \Lambda_1(n, \overrightarrow{p}, f)$ and $\Lambda_2 = \Lambda_2(n, \overrightarrow{p}, f)$ such that for any open subset Ω of $\mathbb{R}^n$ satisfying $\Omega \neq \mathbb{R}^n$, any nonnegative solution u in $W^{1,\overrightarrow{p}}_{\text{loc}}(\Omega) \cap L^\infty_{\text{loc}}(\Omega)$ of equation (1.2) satisfies

$$u(x) \leq \Lambda_1 + \Lambda_2 \left(\inf_{y \in \partial\Omega} \sum_{i=1}^n |x_i - y_i|^{\frac{p_i}{q-p_i}} \right)^{-1} \tag{1.5}$$

for all points x in Ω. Moreover, we can take $\Lambda_1 = 0$ in case $f(u) = \lambda u^{q-1}$.

When $p_{\text{cr}}(\overrightarrow{p}) > p_+$, Theorem 1.1 provides, in particular, universal, *a priori* bounds on compact subsets of Ω for nonnegative weak solutions of equation (1.2). Such nontrivial solutions are proved to exist by Fragalà–Gazzola–Kawohl [25] when $f(u) = \lambda u^{q-1}$.

We are now led to the difficult question of estimating the critical exponent $p_{\text{cr}}(\overrightarrow{p})$. As already mentioned, this question was solved by Gidas–Spruck [27] in the case of the classical Laplace operator. We also refer to Serrin–Zou [49] for an extension of this result in the context of the p-Laplace operator. In Theorems 1.2 and 1.3, we state our results concerning the anisotropic case. In case $\sum_{i=1}^n \frac{1}{p_i} > 1$, we let p_* be the exponent defined by

$$p_* = \frac{n-1}{\sum_{i=1}^n \frac{1}{p_i} - 1}, \tag{1.6}$$

and p^* be the anisotropic Sobolev critical exponent (see, for instance, Troisi [50]), namely

$$p^* = \frac{n}{\sum_{i=1}^n \frac{1}{p_i} - 1}. \tag{1.7}$$

We then get the following result.

Theorem 1.2. *Let $n \geq 2$ and $\overrightarrow{p} = (p_1, \dots, p_n)$ be such that $p_i > 1$ for $i = 1, \dots, n$, and let p_* and p^* be as in* (1.6) *and* (1.7)*. There hold $p_{\text{cr}}(\overrightarrow{p}) = +\infty$ in case $\sum_{i=1}^n \frac{1}{p_i} \leq 1$, $p_* \leq p_{\text{cr}}(\overrightarrow{p})$ in case $p_* > p_+$ and $\sum_{i=1}^n \frac{1}{p_i} > 1$, and finally, $p_{\text{cr}}(\overrightarrow{p}) \leq p^*$ in case $p^* > p_+$ and $\sum_{i=1}^n \frac{1}{p_i} > 1$.*

We can state another result as follows where we prove that $p_{\text{cr}}(\overrightarrow{p})$ is a small perturbation of the isotropic critical exponent $np/(n-p)$ as $\overrightarrow{p} \to (p, \dots, p)$ with $2 \leq p \leq (n+1)/2$. A more general nonexistence result is given and commented in Section 4.

Theorem 1.3. *Let $n \geq 3$. For any real number p in $[2, (n+1)/2]$, there holds*

$$p_{\text{cr}}(\overrightarrow{p}) \longrightarrow \frac{np}{n-p}$$

as $p_i \to p$ with $p_i \geq 2$ for $i = 1, \dots, n$, where $\overrightarrow{p} = (p_1, \dots, p_n)$.

2. Some comments

Under the notations in Theorem 1.1, increasing if necessary the constant Λ_2, in the isotropic case $p_i = p$ for $i = 1, \dots, n$, we can rewrite estimate (1.5) as

$$u(x) \leq \Lambda_1 + \Lambda_2 d(x, \partial\Omega)^{\frac{-p}{q-p}}, \tag{2.1}$$

where $d(x, \partial\Omega)$ is the distance from the point x to the boundary of the domain Ω. In case $p = 2$, namely in the case of the classical Laplace operator, some important references related to the *a priori* estimate (2.1) are Bidaut-Véron–Véron [17], Dancer [20], Gidas–Spruck [27], Poláčik–Quittner–Souplet [46], and Serrin–Zou [49] (the last two references are concerned with the p-Laplace operator, but in case $p = 2$, they both extend the results in [17, 20, 27] to more general nonlinearities). Our proof of Theorem 1.1 is inspired by the recent work of Poláčik–Quittner–Souplet [46] on the derivation of *a priori* estimates from Liouville results. This technique is based on rescaling arguments together with a so-called doubling property.

We observe that in the anisotropic case, the possible behaviors, allowed by our estimate (1.5), of the nonnegative weak solutions of equation (1.2) near a boundary depend on the geometry of this boundary, on its orientation, and not only on the distance to it as in (2.1).

As an interesting particular case, Theorem 1.1 provides *a priori* estimates near an isolated singularity. We point out that when no anisotropy is involved, namely when $p_i = p$ for $i = 1, \dots, n$, if $n > p$ and $p_* < q < p^*$, where $p_* = p(n-1)/(n-p)$ and $p^* = np/(n-p)$, then an explicit nonnegative weak solution of equation (1.3) in $\mathbb{R}^n \backslash \{0\}$ with $\lambda = 1$ is given by

$$u(x) = C_{n,p} \left(\sum_{i=1}^n |x_i|^{\frac{p}{p-1}} \right)^{\frac{1-p}{q-p}},$$

where

$$C_{n,p} = \frac{p^{\frac{p-1}{q-p}} \left(q(n-p) - p(n-1) \right)^{\frac{1}{q-p}}}{(q-p)^{\frac{p}{q-p}}}.$$

As is easily seen, in this case, the growth near the boundary in our estimate (1.5) is sharp. Whereas, these estimates are no more sharp in the case of the equation $-\Delta u = u^{q-1}$ when $2 < q \leq 2_*$. In this case, the local behavior near an isolated singularity was established by Lions [36] for q in $(2, 2_*)$ (see Bidaut-Véron [15] for an extension to the p-Laplace operator) and by Aviles [8] for $q = 2_*$.

Our last remark on Theorem 1.1 is that the nonexistence of nontrivial, nonnegative weak solution of equation (1.3) on the whole Euclidean space is a necessary condition. Indeed, if such a solution exists, then by rescaling, we can construct a family of solutions with arbitrarily large maximum values on a compact subset of a domain Ω, and this contradicts (1.5).

3. Proof of Theorem 1.1

In this section, we let $\overrightarrow{p} = (p_1, \dots, p_n)$ and q satisfy $1 < p_i < q < p_{\text{cr}}(\overrightarrow{p})$ for $i = 1, \dots, n$, and f be a continuous function on $\mathbb{R}_+$ satisfying (1.4) for some positive real number λ. Our proof of Theorem 1.1 is inspired by the recent work of Poláčik–Quittner–Souplet [46].

Proof of Theorem 1.1. We proceed by contradiction and assume that for any natural number α and any $\lambda_\alpha > 0$, there exist an open subset Ω_α of $\mathbb{R}^n$ such that $\Omega_\alpha \neq \mathbb{R}^n$, a point x_α in Ω_α, and a nonnegative solution u_α in $W^{1,\overrightarrow{p}}_{\text{loc}}(\Omega_\alpha) \cap L^\infty_{\text{loc}}(\Omega_\alpha)$ of equation (1.2) such that

$$u_\alpha(x_\alpha) > \lambda_\alpha \Bigg(1 + \bigg(\inf_{y \in \partial\Omega_\alpha} \sum_{i=1}^n |(x_\alpha - y)_i|^{\frac{p_i}{q-p_i}}\bigg)^{-1}\Bigg). \tag{3.1}$$

We define a distance function $d_{\overrightarrow{p},q}$ on $\mathbb{R}^n$ by

$$d_{\overrightarrow{p},q}(x,y) = \sum_{i=1}^n |x_i - y_i|^{\frac{p_i(q-p_+)}{p_+(q-p_i)}}.$$

For any point y in $\mathbb{R}^n$ and for any positive real number r, we let $B^{\overrightarrow{p},q}_y(r)$ be the ball of center y and radius r with respect to the metric $d_{\overrightarrow{p},q}$. For any α, letting $\lambda_\alpha = (2\alpha)^{p_+/(q-p_+)}$ in (3.1), it easily follows that

$$\overline{B^{\overrightarrow{p},q}_{x_\alpha}\Big(2\alpha u_\alpha(x_\alpha)^{\frac{p_+-q}{p_+}}\Big)} \subset \Omega_\alpha.$$

Moreover, since f is continuous, by Lieberman [34, 35], we get that the function u_α belongs to $W^{1,\infty}_{\text{loc}}(\Omega_\alpha)$, and in particular, is continuous. The doubling property (see Poláčik–Quittner–Souplet [46, Lemma 5.1]) then yields the existence of a point y_α in Ω_α such that there hold

$$\begin{aligned} &B^{\overrightarrow{p},q}_{y_\alpha}\Big(2\alpha u_\alpha(y_\alpha)^{\frac{p_+-q}{p_+}}\Big) \subset \Omega_\alpha, \quad u_\alpha(x_\alpha) \le u_\alpha(y_\alpha), \\ &\text{and} \quad u_\alpha(y) \le 2^{\frac{p_+}{q-p_+}} u_\alpha(y_\alpha) \quad \forall y \in B^{\overrightarrow{p},q}_{y_\alpha}\Big(\alpha u_\alpha(y_\alpha)^{\frac{p_+-q}{p_+}}\Big). \end{aligned} \tag{3.2}$$

For any α, we set

$$\mu_\alpha = \frac{1}{u_\alpha(y_\alpha)}. \tag{3.3}$$

By (3.1) and (3.2), since $\lambda_\alpha \to +\infty$, we get that there holds $\mu_\alpha \to 0$ as $\alpha \to +\infty$. We then define the anisotropic affine transformation $\tau_\alpha : \mathbb{R}^n \to \mathbb{R}^n$ by

$$\tau_\alpha(y) = \Big(\mu_\alpha^{\frac{p_1-q}{p_1}}(y - y_\alpha)_1, \dots, \mu_\alpha^{\frac{p_n-q}{p_n}}(y - y_\alpha)_n\Big).$$

We let $\widetilde{u}_\alpha$ be the function defined on $\widetilde{\Omega}_\alpha = \tau_\alpha(\Omega_\alpha)$ by

$$\widetilde{u}_\alpha = \mu_\alpha u_\alpha \circ \tau_\alpha^{-1}.$$

Since u_α is a weak solution of (1.2) on Ω_α, we get that $\widetilde{u}_\alpha$ is a weak solution of the equation

$$-\Delta_{\overrightarrow{p}} \widetilde{u}_\alpha = \mu_\alpha^{q-1} f\left(\mu_\alpha^{-1} \widetilde{u}_\alpha\right) \tag{3.4}$$

in $\widetilde{\Omega}_\alpha$. Moreover, by (3.2), we get

$$B^{\overrightarrow{p},q}_0(2\alpha) \subset \widetilde{\Omega}_\alpha, \quad \widetilde{u}_\alpha(0) = 1, \quad \text{and} \quad \widetilde{u}_\alpha(y) \le 2^{\frac{p_+}{q-p_+}} \quad \forall y \in B^{\overrightarrow{p},q}_0(\alpha). \tag{3.5}$$

By (1.4) and since the function f is continuous, we get that there exist two positive constants C_1 and C_2 such that for any nonnegative real number u, there holds

$$|f(u)| \le C_1 + C_2 u^{q-1}. \tag{3.6}$$

It follows from (3.5) and (3.6) that the right-hand side in (3.4) is uniformly bounded on $B_0^{\overrightarrow{p},q}(\alpha)$. By Lieberman [34, 35], we then get

$$\|\widetilde{u}_\alpha\|_{W^{1,\infty}(B_0^{\overrightarrow{p},q}(\alpha/2))} \le C \tag{3.7}$$

for all α, where C is a positive constant independent of α. Passing if necessary to a subsequence, we may assume that for any bounded subset Ω' of $\mathbb{R}^n$, the sequence $(\widetilde{u}_\alpha)_\alpha$ converges to a function $\widetilde{u}$ strongly in $C^0(\Omega')$ and weakly in $W^{1,s}(\Omega')$ for all real numbers $s \ge 1$. Since the functions $\widetilde{u}_\alpha$ are nonnegative, so is $\widetilde{u}$. We claim that $(\widetilde{u}_\alpha)_\alpha$ converges in fact strongly to the function $\widetilde{u}$ in $W^{1,s}(\Omega')$ for all real numbers $s \ge 1$. In order to prove this claim, we let φ be a nonnegative smooth function with compact support in $\mathbb{R}^n$, and for α large enough so that the support of φ is included in the set $\widetilde{\Omega}_\alpha$, we multiply equation (3.4) by the function $(\widetilde{u}_\alpha - \widetilde{u})\varphi$, and we integrate by parts. It follows that

$$\sum_{i=1}^n \int_{\widetilde{\Omega}_\alpha} \left|\frac{\partial \widetilde{u}_\alpha}{\partial x_i}\right|^{p_i-2} \frac{\partial \widetilde{u}_\alpha}{\partial x_i}\left(\frac{\partial \widetilde{u}_\alpha}{\partial x_i} - \frac{\partial \widetilde{u}}{\partial x_i}\right)\varphi dx = \int_{\widetilde{\Omega}_\alpha} \mu_\alpha^{q-1} f\left(\mu_\alpha^{-1}\widetilde{u}_\alpha\right)(\widetilde{u}_\alpha - \widetilde{u})\varphi dx - \sum_{i=1}^n \int_{\widetilde{\Omega}_\alpha} \left|\frac{\partial \widetilde{u}_\alpha}{\partial x_i}\right|^{p_i-2} \frac{\partial \widetilde{u}_\alpha}{\partial x_i}(\widetilde{u}_\alpha - \widetilde{u})\frac{\partial \varphi}{\partial x_i} dx\,. \tag{3.8}$$

By (3.6), we can write

$$\left|\int_{\widetilde{\Omega}_\alpha} \mu_\alpha^{q-1} f\left(\mu_\alpha^{-1}\widetilde{u}_\alpha\right)(\widetilde{u}_\alpha - \widetilde{u})\varphi dx\right| = \mathrm{O}\left(\left(\mu_\alpha^{q-1} + \|\widetilde{u}_\alpha\|^{q-1}_{C^0(\mathrm{Supp}(\varphi))}\right)\|\varphi\|_{C^0(\mathbb{R}^n)}\|\widetilde{u}_\alpha - \widetilde{u}\|_{L^1(\mathrm{Supp}(\varphi))}\right) \longrightarrow 0 \tag{3.9}$$

as $\alpha \to +\infty$. By Hölder's inequality, we get

$$\left|\int_{\widetilde{\Omega}_\alpha} \left|\frac{\partial \widetilde{u}_\alpha}{\partial x_i}\right|^{p_i-2} \frac{\partial \widetilde{u}_\alpha}{\partial x_i}(\widetilde{u}_\alpha - \widetilde{u})\frac{\partial \varphi}{\partial x_i} dx\right| \le \left\|\frac{\partial \widetilde{u}_\alpha}{\partial x_i}\right\|^{p_i-1}_{L^{p_i}(\mathrm{Supp}(\varphi))}\left\|\frac{\partial \varphi}{\partial x_i}\right\|_{L^{2p_i}(\mathbb{R}^n)}\|\widetilde{u}_\alpha - \widetilde{u}\|_{L^{2p_i}(\mathrm{Supp}(\varphi))} \longrightarrow 0 \tag{3.10}$$

as $\alpha \to +\infty$. By (3.8)–(3.10), we get

$$\sum_{i=1}^n \int_{\widetilde{\Omega}_\alpha} \left|\frac{\partial \widetilde{u}_\alpha}{\partial x_i}\right|^{p_i-2} \frac{\partial \widetilde{u}_\alpha}{\partial x_i}\left(\frac{\partial \widetilde{u}_\alpha}{\partial x_i} - \frac{\partial \widetilde{u}}{\partial x_i}\right)\varphi dx \longrightarrow 0 \tag{3.11}$$

as $\alpha \to +\infty$. Independently, since the sequence $(\widetilde{u}_\alpha)_\alpha$ converges weakly to the function $\widetilde{u}$ in $W^{1,\overrightarrow{p}}(\mathrm{Supp}(\varphi))$, there holds

$$\int_{\widetilde{\Omega}_\alpha} \left|\frac{\partial \widetilde{u}}{\partial x_i}\right|^{p_i-2} \frac{\partial \widetilde{u}}{\partial x_i}\frac{\partial \widetilde{u}_\alpha}{\partial x_i}\varphi dx \longrightarrow \int_{\widetilde{\Omega}_\alpha} \left|\frac{\partial \widetilde{u}}{\partial x_i}\right|^{p_i}\varphi dx \tag{3.12}$$

as $\alpha \to +\infty$ for $i = 1, \dots, n$. By (3.11) and (3.12), we get

$$\sum_{i=1}^n \int_{\widetilde{\Omega}_\alpha} \left(\left|\frac{\partial \widetilde{u}_\alpha}{\partial x_i}\right|^{p_i-2} \frac{\partial \widetilde{u}_\alpha}{\partial x_i} - \left|\frac{\partial \widetilde{u}}{\partial x_i}\right|^{p_i-2} \frac{\partial \widetilde{u}}{\partial x_i}\right)\left(\frac{\partial \widetilde{u}_\alpha}{\partial x_i} - \frac{\partial \widetilde{u}}{\partial x_i}\right)\varphi dx \longrightarrow 0$$

as $\alpha \to +\infty$. Since this estimate holds true for all nonnegative smooth functions φ with compact support in $\mathbb{R}^n$, it easily follows that

$$\int_{\Omega'} \left(\left| \frac{\partial \widetilde{u}_\alpha}{\partial x_i} \right|^{p_i-2} \frac{\partial \widetilde{u}_\alpha}{\partial x_i} - \left| \frac{\partial \widetilde{u}}{\partial x_i} \right|^{p_i-2} \frac{\partial \widetilde{u}}{\partial x_i} \right) \left(\frac{\partial \widetilde{u}_\alpha}{\partial x_i} - \frac{\partial \widetilde{u}}{\partial x_i} \right) dx \longrightarrow 0$$

as $\alpha \to +\infty$ for $i = 1, \dots, n$ and for all bounded subsets Ω' of $\mathbb{R}^n$. In particular, up to a subsequence, we get

$$\left(\left| \frac{\partial \widetilde{u}_\alpha}{\partial x_i} \right|^{p_i-2} \frac{\partial \widetilde{u}_\alpha}{\partial x_i} - \left| \frac{\partial \widetilde{u}}{\partial x_i} \right|^{p_i-2} \frac{\partial \widetilde{u}}{\partial x_i} \right) \left(\frac{\partial \widetilde{u}_\alpha}{\partial x_i} - \frac{\partial \widetilde{u}}{\partial x_i} \right) \longrightarrow 0 \quad \text{a.e. in } \mathbb{R}^n$$

as $\alpha \to +\infty$. As an easy consequence, for $i = 1, \dots, n$, the functions $\partial \widetilde{u}_\alpha / \partial x_i$ converge almost everywhere to $\partial \widetilde{u} / \partial x_i$ in $\mathbb{R}^n$ as $\alpha \to +\infty$. By (3.7), it follows that for any bounded subset Ω' of $\mathbb{R}^n$, the functions $\widetilde{u}_\alpha$ converge strongly to $\widetilde{u}$ in $W^{1,s}(\Omega')$ for all real numbers $s \ge 1$ as $\alpha \to +\infty$, and our claim is proved. For any smooth function φ with compact support in $\mathbb{R}^n$, we then get

$$\int_{\widetilde{\Omega}_\alpha} \left| \frac{\partial \widetilde{u}_\alpha}{\partial x_i} \right|^{p_i-2} \frac{\partial \widetilde{u}_\alpha}{\partial x_i} \varphi dx \longrightarrow \int_{\mathbb{R}^n} \left| \frac{\partial \widetilde{u}}{\partial x_i} \right|^{p_i-2} \frac{\partial \widetilde{u}}{\partial x_i} \varphi dx \tag{3.13}$$

as $\alpha \to +\infty$ for $i = 1, \dots, n$. By (1.4), (3.6), and since the sequence $(\widetilde{u}_\alpha)_\alpha$ converges to $\widetilde{u}$ in $C^0(\mathrm{Supp}(\varphi))$, we then get

$$\int_{\widetilde{\Omega}_\alpha} \mu_\alpha^{q-1} f\left(\mu_\alpha^{-1} \widetilde{u}_\alpha(x)\right) \varphi dx \longrightarrow \lambda \int_{\mathbb{R}^n} \widetilde{u}^{q-1} \varphi dx \,. \tag{3.14}$$

It follows from (3.4), (3.13), and (3.14) that the function $\widetilde{u}$ is a nonnegative solution in $W^{1,\overrightarrow{p}}_{\mathrm{loc}}(\mathbb{R}^n) \cap L^\infty_{\mathrm{loc}}(\mathbb{R}^n)$ of equation (1.3). By assumption, we then get that $\widetilde{u}$ is identically zero which is in contradiction with $\widetilde{u}(0) = 1$. This ends the proof of Theorem 1.1 in the general case where the function f satisfies (1.4). In case $f(u) = \lambda u^{q-1}$, in the same way, by contradiction and by the doubling property, we construct, for any α, a nonnegative weak solution u_α of equation (1.2) in an open set Ω_α of $\mathbb{R}^n$, and a point y_α in Ω_α such that (3.2) holds true. The difference here is that up to a subsequence, it occurs that $\mu_\alpha \ge C > 0$ for all α, where μ_α is as in (3.3). However, since we now get

$$\int_{\widetilde{\Omega}_\alpha} \mu_\alpha^{q-1} f\left(\mu_\alpha^{-1} \widetilde{u}_\alpha(x)\right) \varphi dx \longrightarrow \lambda \int_{\mathbb{R}^n} \widetilde{u}_\alpha^{q-1} \varphi dx \,,$$

the above proof carries over the same. □

4. Proof of Theorem 1.2

In this section, we let $n \ge 2$, $\overrightarrow{p} = (p_1, \dots, p_n)$ satisfy $p_i > 1$ for $i = 1, \dots, n$. More than Theorem 1.2, we show the nonexistence of solutions of inequalities of the type

$$-\Delta_{\overrightarrow{p}} u \ge \lambda u^{q-1} \text{ in } \mathbb{R}^n \,, \tag{4.1}$$

where λ is a positive real number. More precisely, we prove that inequality (4.1) does not admit any nontrivial nonnegative solution in $W^{1,\overrightarrow{p}}_{\mathrm{loc}}(\mathbb{R}^n)$ when there holds $p_+ < q < +\infty$ in case $\sum_{i=1}^n \frac{1}{p_i} \le 1$, and $p_+ < q \le p_*$ in case $p_* > p_+$ and $\sum_{i=1}^n \frac{1}{p_i} > 1$, where the exponent p_* is as in (1.6). In the context of the p-Laplace operator, this result is due to Mitidieri–Pohožaev [41, 42]. Extensions to more general classes of operators can also be found in Bidaut-Véron–Pohožaev [16], Birindelli–Demengel [18], D'Ambrosio [21], and Mitidieri–Pohožaev [41–44]. In particular, in D'Ambrosio [21], the case of the anisotropic Laplace operator is explicitly treated as a particular case among very general classes of operators.

Proof of Theorem 1.2. In case $p^* > p_+$ and $\sum_{i=1}^n \frac{1}{p_i} > 1$, we know by El Hamidi–Rakotoson [23] that equation (1.3) admits at least one nontrivial nonnegative weak solution in $\mathbb{R}^n$. It follows that in this case, there holds $p_{\text{cr}}(\overrightarrow{p}) \le p^*$. We now have to prove that any nonnegative solution in $W^{1,\overrightarrow{p}}_{\text{loc}}(\mathbb{R}^n)$ of equation (1.3) (or, more generally, of inequality (4.1)) is identically zero when there holds $p_+ < q < +\infty$ in case $\sum_{i=1}^n \frac{1}{p_i} \le 1$, and $p_+ < q \le p_*$ in case $p_* > p_+$ and $\sum_{i=1}^n \frac{1}{p_i} > 1$, where the exponent p_* is as in (1.6). We proceed by contradiction and assume that such a solution u is not identically zero. In case u is not positive, we take $u_\delta = u + \delta$ instead of u in the following arguments, and we pass to the limit as $\delta \to 0$. We let φ be a nonnegative smooth function with compact support in $\mathbb{R}^n$ to be chosen later on so that the integrals below are finite. By multiplying inequality (4.1) by $u^{-\varepsilon}\varphi$, where ε is a positive real number to be fixed small later on, and by integrating by parts, we get

$$\int_{\mathbb{R}^n} u^{q-\varepsilon-1}\varphi dx + \varepsilon \sum_{i=1}^n \int_{\mathbb{R}^n} u^{-\varepsilon-1} \left|\frac{\partial u}{\partial x_i}\right|^{p_i} \varphi dx \le \sum_{i=1}^n \int_{\mathbb{R}^n} u^{-\varepsilon} \left|\frac{\partial u}{\partial x_i}\right|^{p_i-2} \frac{\partial u}{\partial x_i}\frac{\partial \varphi}{\partial x_i} dx \le \sum_{i=1}^n \int_{\mathbb{R}^n} u^{-\varepsilon} \left|\frac{\partial u}{\partial x_i}\right|^{p_i-1} \left|\frac{\partial \varphi}{\partial x_i}\right| dx\,. \tag{4.2}$$

For $i = 1,\dots,n$ and $C > 0$, Young's inequality gives

$$\int_{\mathbb{R}^n} u^{-\varepsilon} \left|\frac{\partial u}{\partial x_i}\right|^{p_i-1} \left|\frac{\partial \varphi}{\partial x_i}\right| dx \le \frac{C}{p_i} \int_{\mathbb{R}^n} u^{p_i-\varepsilon-1} \left|\frac{\partial \varphi}{\partial x_i}\right|^{p_i} \varphi^{1-p_i} dx + \frac{p_i-1}{p_i} C^{\frac{-1}{p_i-1}} \int_{\mathbb{R}^n} u^{-\varepsilon-1} \left|\frac{\partial u}{\partial x_i}\right|^{p_i} \varphi dx \tag{4.3}$$

and

$$\int_{\mathbb{R}^n} u^{p_i-\varepsilon-1} \left|\frac{\partial \varphi}{\partial x_i}\right|^{p_i} \varphi^{1-p_i} dx \le \frac{p_i-\varepsilon-1}{q-\varepsilon-1} C \int_{\mathbb{R}^n} u^{q-\varepsilon-1}\varphi dx + \frac{q-p_i}{q-\varepsilon-1} C^{\frac{\varepsilon+1-p_i}{q-p_i}} \int_{\mathbb{R}^n} \left|\frac{\partial \varphi}{\partial x_i}\right|^{\frac{p_i(q-\varepsilon-1)}{q-p_i}} \varphi^{1-\frac{p_i(q-\varepsilon-1)}{q-p_i}} dx \tag{4.4}$$

for $\varepsilon < p_i - 1$. By (4.2)–(4.4), we get that there exists a positive constant C independent of u and φ such that there holds

$$\int_{\mathbb{R}^n} u^{q-\varepsilon-1}\varphi dx + \varepsilon \sum_{i=1}^n \int_{\mathbb{R}^n} u^{-\varepsilon-1} \left|\frac{\partial u}{\partial x_i}\right|^{p_i} \varphi dx \le C \sum_{i=1}^n \int_{\mathbb{R}^n} \left|\frac{\partial \varphi}{\partial x_i}\right|^{\frac{p_i(q-\varepsilon-1)}{q-p_i}} \varphi^{1-\frac{p_i(q-\varepsilon-1)}{q-p_i}} dx\,. \tag{4.5}$$

Independently, multiplying inequality (4.1) by φ and integrating by parts yield

$$\int_{\mathbb{R}^n} h u^{q-1}\varphi dx \le \sum_{i=1}^n \int_{\mathbb{R}^n} \left|\frac{\partial u}{\partial x_i}\right|^{p_i-2} \frac{\partial u}{\partial x_i}\frac{\partial \varphi}{\partial x_i} dx \le \sum_{i=1}^n \int_{\mathbb{R}^n} \left|\frac{\partial u}{\partial x_i}\right|^{p_i-1} \left|\frac{\partial \varphi}{\partial x_i}\right| dx\,. \tag{4.6}$$

For $i = 1,\dots,n$, Hölder's inequality gives

$$\int_{\mathbb{R}^n} \left|\frac{\partial u}{\partial x_i}\right|^{p_i-1} \left|\frac{\partial \varphi}{\partial x_i}\right| dx \le \left(\int_{\mathbb{R}^n} u^{-\varepsilon-1} \left|\frac{\partial u}{\partial x_i}\right|^{p_i} \varphi dx\right)^{\frac{p_i-1}{p_i}} \times \left(\int_{\mathbb{R}^n} u^{(p_i-1)(\varepsilon+1)} \left|\frac{\partial \varphi}{\partial x_i}\right|^{p_i} \varphi^{1-p_i} dx\right)^{\frac{1}{p_i}} \tag{4.7}$$

and

$$\int_{\mathbb{R}^n} u^{(p_i-1)(\varepsilon+1)} \left|\frac{\partial \varphi}{\partial x_i}\right|^{p_i} \varphi^{1-p_i} dx \le \left(\int_{\text{Supp}\,\frac{\partial\varphi}{\partial x_i}} u^{q-1}\varphi dx\right)^{\frac{(p_i-1)(\varepsilon+1)}{q-1}}$$
$$\times \left(\int_{\mathbb{R}^n} \left|\frac{\partial \varphi}{\partial x_i}\right|^{\frac{p_i(q-1)}{q-1-(p_i-1)(\varepsilon+1)}} \varphi^{1-\frac{p_i(q-1)}{q-1-(p_i-1)(\varepsilon+1)}} dx\right)^{\frac{q-1-(p_i-1)(\varepsilon+1)}{q-1}} \tag{4.8}$$

for $\varepsilon < \frac{q-p_+}{p_+-1}$. By (4.5)–(4.8), we get

$$\int_{\mathbb{R}^n} u^{q-1}\varphi dx \le C \sum_{i=1}^{n} \left(\sum_{j=1}^{n} \int_{\mathbb{R}^n} \left|\frac{\partial \varphi}{\partial x_j}\right|^{\frac{p_j(q-\varepsilon-1)}{q-p_j}} \varphi^{1-\frac{p_j(q-\varepsilon-1)}{q-p_j}} dx\right)^{\frac{p_i-1}{p_i}} \tag{4.9}$$
$$\times \left(\int_{\mathbb{R}^n} \left|\frac{\partial \varphi}{\partial x_i}\right|^{\frac{p_i(q-1)}{q-1-(p_i-1)(\varepsilon+1)}} \varphi^{1-\frac{p_i(q-1)}{q-1-(p_i-1)(\varepsilon+1)}} dx\right)^{\frac{q-1-(p_i-1)(\varepsilon+1)}{p_i(q-1)}} \left(\int_{\text{Supp}\,\frac{\partial\varphi}{\partial x_i}} u^{q-1}\varphi dx\right)^{\frac{(p_i-1)(\varepsilon+1)}{p_i(q-1)}}$$

for some positive constant C independent of u and φ. We then let η be a smooth cutoff function satisfying $\eta \equiv 1$ in $[0,1]$, $0 \le \eta \le 1$ in $[1,2]$, and $\eta \equiv 0$ in $[1,+\infty)$, and for any positive real number R, we let φ_R be the function defined on $\mathbb{R}^n$ by

$$\varphi_R(x) = \eta\left(\sqrt{\sum_{i=1}^{n}\left(R^{\frac{p_i-q}{p_i}} x_i\right)^2}\right)^{\kappa},$$

where κ is a positive real number large enough so that the integrals above are finite. By (4.9), we get that there exists a positive constant C independent of u and R such that there holds

$$\int_{\mathbb{R}^n} u^{q-1}\varphi_R dx$$
$$\le C \sum_{i=1}^{n} R^{\left(n-1-\left(\sum_{i=1}^{n}\frac{1}{p_i}-1\right)q\right)\frac{(p_i-1)(\varepsilon+1)-p_i(q-1)}{p_i(q-1)}} \left(\int_{\text{Supp}\,\frac{\partial\varphi_R}{\partial x_i}} u^{q-1}\varphi_R dx\right)^{\frac{(p_i-1)(\varepsilon+1)}{p_i(q-1)}}. \tag{4.10}$$

It follows that

$$\int_{\mathbb{R}^n} u^{q-1}\varphi_R dx \le C R^{\left(\sum_{i=1}^{n}\frac{1}{p_i}-1\right)q-n+1} \tag{4.11}$$

for some positive constant C independent of u and R. Since, by assumption,

$$\left(\sum_{i=1}^{n}\frac{1}{p_i}-1\right)q \le n-1\,, \tag{4.12}$$

passing to the limit as $R \to +\infty$ into (4.11) then gives

$$\int_{\mathbb{R}^n} u^{q-1} dx = 0 \tag{4.13}$$

in case inequality (4.12) is strict, and

$$\int_{\mathbb{R}^n} u^{q-1} dx < +\infty$$

in case equality holds in (4.12). In this last case, passing to the limit into (4.10) as $R \to +\infty$ also yields (4.13). It follows from (4.13) that the function u is identically zero. This ends the proof of Theorem 1.2. □

5. A nonexistence result for equation (1.3)

This section is devoted to the following result.

Theorem 5.1. *Let $n \geq 2$, $\overrightarrow{p} = (p_1, \dots, p_n)$ and q be such that $2 \leq p_i < q$ for $i = 1, \dots, n$. Assume that there exist some real numbers a, b_{ij}, c_{ij}, λ_{ij}, μ_{ij}, and ν_i satisfying*

$$(p_j - 1)\, b_{ij} = (p_i - 1)\, b_{ji}\,, \quad \lambda_{ij} = -\lambda_{ji}\,, \quad \sum_{k \neq i} \mu_{ik}^2 + \sum_{k \neq j} \mu_{jk}^2 - 2\mu_{ij}\mu_{ji} = 1\,, \tag{5.1}$$

and

$$\nu_i = \frac{1}{2}\big(p_i + p_j - 1 - 2p_i \sum_{k \neq i} \mu_{ik}^2\big) a + (p_j - 1)\, b_{ij} - 2p_i c_{ij} \mu_{ij} \mu_{ji} - p_i p_j \lambda_{ij} \tag{5.2}$$

for all distinct indices $i, j = 1, \dots, n$, and such that

$$a > \max\left(2 - 2p_-, 3 - 2q, 2 - n + \left(\sum_{i=1}^n \frac{1}{p_i} - 2\right) q\right), \tag{5.3}$$

$$A_i < 0\,, \quad B_{ij} < 0\,, \quad \text{and} \quad \nu_i > p_i\,(q-1) \sum_{k \neq i} \mu_{ik}^2\,, \tag{5.4}$$

where $p_- = \min(p_1, \dots, p_n)$,

$$A_i = \frac{p_i - 1}{2p_i - 1}\,(a-1)\left(\frac{\nu_i}{p_i} + \sum_{k \neq i} (a + 2c_{ik})\, \mu_{ik}^2\right) + \sum_{k \neq i} c_{ik}^2 \mu_{ik}^2\,, \tag{5.5}$$

and

$$B_{ij} = \frac{1}{2p_i p_j}\,(a-1)\left(\left(p_i\,(p_i - 1) + p_j\,(p_j - 1)\right) a + 2\,(p_i + p_j)\,(p_j - 1)\, b_{ij}\right) + (p_i - p_j)\,(a-1)\,\lambda_{ij} + b_{ij} b_{ji} + 2c_{ij} c_{ji} \mu_{ij} \mu_{ji} \tag{5.6}$$

for all distinct indices $i, j = 1, \dots, n$. Then equation (1.3) *does not admit any nontrivial, nonnegative solution in $W^{1,\overrightarrow{p}}_{\text{loc}}(\mathbb{R}^n) \cap L^\infty_{\text{loc}}(\mathbb{R}^n)$.*

Assuming Theorem 5.1, we can prove Theorem 1.3. The proof of Theorem 5.1 is left to Sections 5 and 6.

Proof of Theorem 1.3. We let $n \geq 3$ and p be a real number in $[2, (n+1)/2]$. By El Hamidi–Rakotoson [23], there exists at least one nontrivial, nonnegative weak solution of equation (1.3) when $\sum_{i=1}^n 1/p_i > 1$, $q = p^*$, and $p_+ < p^*$, where p^* is as in (1.7). It follows

$$\limsup p_{\text{cr}}(\overrightarrow{p}) \leq \frac{np}{n-p}$$

as $p_i \to p$ for $i = 1, \dots, n$. It remains to prove that for any real number q in (p, p^*), where $p^* = np/(n-p)$, if $\overrightarrow{p}$ is close enough to $(p, \dots, p)$ and satisfies $p_i \geq 2$ for $i = 1, \dots, n$, then there exist some real numbers a, b_{ij}, c_{ij}, λ_{ij}, μ_{ij}, and ν_i satisfying (5.1)–(5.4). In the isotropic case $\overrightarrow{p} = (p, \dots, p)$, we claim that, by setting $q_d = q + d$ for $d > 0$ small enough, (5.1)–(5.4) hold true when

$$\lambda_{ij} = 0\,, \quad \mu_{ij} = \begin{cases} 1/\sqrt{2n} & \text{if } i < j\,, \\ -1/\sqrt{2n} & \text{if } i > j\,, \end{cases} \quad \nu_i = \frac{p\,(n-1)\,(q_d - 1)}{2n}\,, \tag{5.7}$$

$$a = \frac{(2n - (n+1)\,p)\,(q_d - 1) + 2\sqrt{n\,(p-1)\,(q-1)\,(np - (n-p)\,q_d)}}{(n+1)\,p - n}\,, \tag{5.8}$$

and $b_{ij} = c_{ij} = b$, where

$$b = \frac{n\,(p-1)\,(q_d - 1) - \sqrt{n\,(p-1)\,(q-1)\,(np - (n-p)\,q_d)}}{(n+1)\,p - n}\,. \tag{5.9}$$

In this case, the equalities in (5.1) and (5.2) follow from straightforward computations. Moreover, for $d > 0$ small enough, we compute

$$\nu_i > \frac{p\,(q-1)\,(n-1)}{2n} = p\,(q-1)\sum_{k \neq i} \mu_{ik}^2$$

and

$$A_i = \frac{1}{2} B_{ij} = \frac{(n-1)\,(p-1)\,(n-p)\,(q_d - p^*)\,d}{2\,((n+1)\,p - n)^2} > 0\,.$$

Hence, (5.4) holds true. As a remark, when $d = 0$, we get the equalities in (5.4) instead of the inequalities. Still in the isotropic case, we now prove that the inequality in (5.3) holds true when $d = 0$, and thus when d is small. Taking into account that

$$a > \frac{(2n - (n+1)\,p)\,(q-1)}{(n+1)\,p - n}$$

and that $p < q < p^*$, one can easily see that it suffices to prove the extremal inequalities

$$\frac{(2n - (n+1)\,p)\,(p^* - 1)}{(n+1)\,p - n} \geq \max\left(2 - 2p, 2 - n + \frac{(n-2p)\,p^*}{p}\right) \tag{5.10}$$

and

$$\frac{(2n - (n+1)\,p)\,(p-1)}{(n+1)\,p - n} \geq 3 - 2p\,. \tag{5.11}$$

Since $2 \leq p \leq (n+1)/2$, we compute

$$\frac{(2n - (n+1)\,p)\,(p^* - 1)}{(n+1)\,p - n} - (2 - 2p) = \frac{n + 1 - 2p}{n - p} \geq 0\,. \tag{5.12}$$

We also compute

$$\frac{(2n - (n+1)\,p)\,(p^* - 1)}{(n+1)\,p - n} - \left(2 - n + \frac{(n-2p)\,p^*}{p}\right) = \frac{p}{n-p} > 0 \tag{5.13}$$

and

$$\frac{(2n - (n+1)\,p)\,(p-1)}{(n+1)\,p - n} - (3 - 2p) = \frac{\left(\sqrt{n+1}\,(p-1) - 1\right)\left(\sqrt{n+1}\,(p-1) + 1\right)}{(n+1)\,p - n} > 0\,. \tag{5.14}$$

Then (5.10) and (5.11) follow from (5.12)–(5.14). This ends the proof of our claim, namely that in the isotropic case $\overrightarrow{p} = (p, \dots, p)$, (5.1)–(5.4) hold true with the above definition of a, b_{ij}, c_{ij}, λ_{ij}, μ_{ij}, and ν_i. In the anisotropic case, we can choose the real numbers μ_{ij} and a as in (5.7) and (5.8), and take

$$b_{ij} = \frac{p-1}{p_j - 1} b\,, \quad c_{ij} = b\,, \quad \lambda_{ij} = \frac{p_j - p_i}{2 p_i p_j} a\,,$$

and

$$\nu_i = \frac{1}{2}\left(\frac{2n-1}{n} p_i - 1\right) a + \left(\frac{p_i}{n} + p - 1\right) b\,,$$

where a and b are as in (5.8) and (5.9). One then easily checks that (5.1) and (5.2) hold true, and if $\overrightarrow{p}$ is close enough to $(p,\dots,p)$, then we also get (5.3) and (5.4). This ends the proof of Theorem 1.3. □

In the context of the p-Laplace operator, the case where the exponent p is large ($p > n/2$ if $n \geq 3$ and $p > \left(1+\sqrt{17}\right)/4$ if $n = 2$) is treated in Serrin–Zou [49] by using a Harnack-type inequality, and the case $1 < p < 2$ is treated by using [49, Proposition 8.1]. We lack both of these two properties in the anisotropic case.

We are now led to consider the set

$$\mathcal{Q}\left(\overrightarrow{p}\right) = \{q \in (p_+,+\infty)\ ;\ \exists a, b_{ij}, c_{ij}, \lambda_{ij}, \mu_{ij}, \nu_i \in \mathbb{R} \text{ s.t. } (5.1)\text{–}(5.4)\}.$$

We define the exponent

$$p_{\max}\left(\overrightarrow{p}\right) = \sup\{q \in \mathcal{Q}\left(\overrightarrow{p}\right)\ ;\ (p_+,q) \subset \mathcal{Q}\left(\overrightarrow{p}\right)\}$$

in case the set $\mathcal{Q}\left(\overrightarrow{p}\right)$ is not empty and $p_{\max}\left(\overrightarrow{p}\right) = p_+$ otherwise. By Theorem 5.1, we get $p_{\max}\left(\overrightarrow{p}\right) \leq p_{\text{cr}}\left(\overrightarrow{p}\right)$ when $p_i \geq 2$ for $i = 1,\dots,n$. Due to the large number of nonlinear equations and inequalities, $p_{\max}\left(\overrightarrow{p}\right)$ is excessively hard to estimate. In the simpler case $p_2 = p_3 = \dots = p_n$, we can give numerical estimates. In this case, we reduce the number of unknowns by assuming that

$$b_{1i} = b_{1j}\,, \quad c_{1i} = c_{1j}\,, \quad c_{i1} = c_{j1}\,, \quad \lambda_{1i} = \lambda_{1j}\,, \quad \mu_{1i} = \mu_{1j}\,, \quad \mu_{i1} = \mu_{j1}\,, \tag{5.15}$$

$$\mu_{ij} = -\mu_{ji}\,, \quad \lambda_{ij} = 0 \tag{5.16}$$

for $i,j = 2,\dots,n$ satisfying $i \neq j$, and that

$$b_{ij} = b_{kl}\,, \quad c_{ij} = c_{kl}\,, \quad |\mu_{ij}| = |\mu_{kl}| \tag{5.17}$$

for $i,j,k,l = 2,\dots,n$ satisfying $i \neq j$ and $k \neq l$. Another reduction consists in replacing the inequalities (5.4) by the equations

$$A_i = -\varepsilon\,, \quad B_{ij} = -\varepsilon\,, \quad \text{and} \quad \nu_i = p_i\,(q-1)\sum_{k\neq i}\mu_{ik}^2 + \varepsilon \tag{5.18}$$

for $i,j = 1,\dots,n$ satisfying $i \neq j$, where ε is a positive parameter to be chosen small (we take $\varepsilon = 10^{-3}$ in our numerical estimates below). This way, we define the set

$$\widetilde{\mathcal{Q}}\left(\overrightarrow{p}\right) = \{q \in (p_+,+\infty)\ ;\ \exists a, b_{ij}, c_{ij}, \lambda_{ij}, \mu_{ij}, \nu_i \in \mathbb{R} \text{ s.t. } (5.1)\text{–}(5.3),\ (5.15)\text{–}(5.18)\}$$

and the exponent

$$\widetilde{p}_{\max}\left(\overrightarrow{p}\right) = \sup\{q \in \widetilde{\mathcal{Q}}\left(\overrightarrow{p}\right)\ ;\ (p_+,q) \subset \widetilde{\mathcal{Q}}\left(\overrightarrow{p}\right)\}$$

which is a lower bound for $p_{\max}\left(\overrightarrow{p}\right)$, and thus for $p_{\text{cr}}\left(\overrightarrow{p}\right)$, and which we can estimate numerically. In Figures 1 and 2 below, we plot our numerical estimates in case $n = 4$, $p_2 = p_3 = p_4 = 2$ and in case $n = 6$, $p_2 = p_3 = \dots = p_6 = 3$, the exponent p_1 being on the abscissa and taking values from $p_- = 2$ to $p_+ = 4$. For any q in the filled region $p_+ < q < \widetilde{p}_{\max}$, we get the nonexistence of nontrivial, nonnegative solution in $W^{1,\overrightarrow{p}}_{\text{loc}}\left(\mathbb{R}^n\right) \cap L^\infty_{\text{loc}}\left(\mathbb{R}^n\right)$ for equation (1.3). We also plot in Figures 1 and 2 the values of the Sobolev critical exponent $q = p^*$ given by (1.7) and for which such a nonexistence result is known to be false, see El Hamidi–Rakotoson [23]. In both Figures 1 and 2, one can observe in particular that $\widetilde{p}_{\max}\left(\overrightarrow{p}\right)$ converges to p^* as p_1 converges to $p_2 = p_3 = \dots = p_n$, namely as $\overrightarrow{p}$ converges to $(p_n,\dots,p_n)$. Indeed, as stated in Theorem 1.3, this holds true when $2 \leq p_n \leq (n+1)/2$, and in particular, in the cases illustrated in Figures 1 and 2.

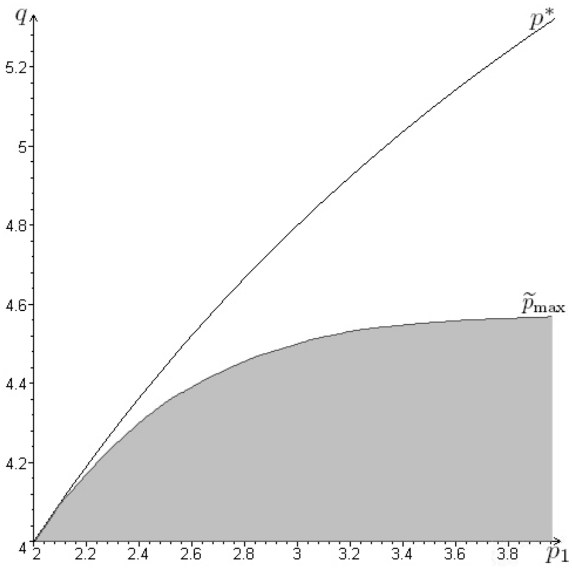

FIGURE 1. $\widetilde{p}_{\max}$ and p^* in case $n = 4$, $2 \le p_1 \le 4$, and $p_2 = p_3 = p_4 = 2$.

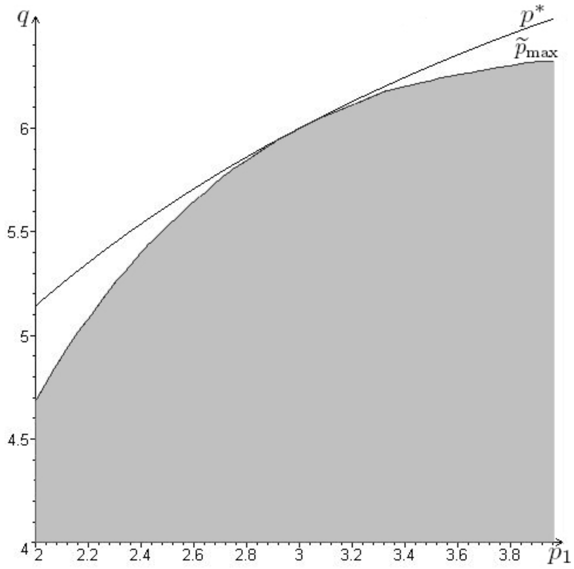

FIGURE 2. $\widetilde{p}_{\max}$ and p^* in case $n = 6$, $2 \le p_1 \le 4$, and $p_2 = p_3 = \cdots = p_6 = 3$.

6. The Key estimate

Proposition 6.1 below is a crucial step in the proof of Theorem 5.1. It generalizes an estimate of Gidas–Spruck [27] (see Serrin–Zou [49] for an extension to the p-Laplace operator).

Proposition 6.1. *Let $\overrightarrow{p} = (p_1, \dots, p_n)$ with $p_i \ge 2$ for $i = 1, \dots, n$. Assume that there exist some real numbers a, b_{ij}, c_{ij}, λ_{ij}, μ_{ij}, and ν_i satisfying* (5.1) *and* (5.2). *Let f be a C^1-function on $\mathbb{R}_+$ and u be a nonnegative solution in $W^{1,\overrightarrow{p}}_{\text{loc}}(\mathbb{R}^n) \cap L^\infty_{\text{loc}}(\mathbb{R}^n)$ of equation* (1.2). *Then, for any positive real number δ and any smooth function φ with compact support in $\mathbb{R}^n$, there holds*

$$\sum_{i=1}^n \int_{\mathbb{R}^n} u_\delta^{a-1} \Big(\frac{\nu_i}{p_i} f(u) - \sum_{k \ne i} \mu_{ik}^2 u_\delta f'(u) \Big) \left| \frac{\partial u}{\partial x_i} \right|^{p_i} \varphi dx$$
$$- \sum_{i=1}^n A_i \int_{\mathbb{R}^n} u_\delta^{a-2} \left| \frac{\partial u}{\partial x_i} \right|^{2p_i} \varphi dx - \sum_{i=1}^n \sum_{j=1}^{i-1} B_{ij} \int_{\mathbb{R}^n} u_\delta^{a-2} \left| \frac{\partial u}{\partial x_i} \right|^{p_i} \left| \frac{\partial u}{\partial x_j} \right|^{p_j} \varphi dx$$
$$\le \sum_{i=1}^n \Big(\sum_{k \ne i} \mu_{ik}^2 - 1 \Big) \int_{\mathbb{R}^n} u_\delta^a f(u) \left| \frac{\partial u}{\partial x_i} \right|^{p_i - 2} \frac{\partial u}{\partial x_i} \frac{\partial \varphi}{\partial x_i} dx$$
$$+ \sum_{i=1}^n C_i \int_{\mathbb{R}^n} u_\delta^{a-1} \left| \frac{\partial u}{\partial x_i} \right|^{2p_i - 2} \frac{\partial u}{\partial x_i} \frac{\partial \varphi}{\partial x_i} dx$$
$$+ \sum_{i=1}^n \sum_{j \ne i} D_{ij} \int_{\mathbb{R}^n} u_\delta^{a-1} \left| \frac{\partial u}{\partial x_j} \right|^{p_i - 2} \frac{\partial u}{\partial x_i} \left| \frac{\partial u}{\partial x_j} \right|^{p_j} \frac{\partial \varphi}{\partial x_i} dx$$
$$+ \sum_{i=1}^n \sum_{j=1}^{i-1} \int_{\mathbb{R}^n} u_\delta^a \left| \frac{\partial u}{\partial x_i} \right|^{p_i - 2} \frac{\partial u}{\partial x_i} \left| \frac{\partial u}{\partial x_j} \right|^{p_j - 2} \frac{\partial u}{\partial x_j} \frac{\partial^2 \varphi}{\partial x_i \partial x_j} dx$$
$$+ \frac{1}{2} \sum_{i=1}^n \int_{\mathbb{R}^n} u_\delta^a \left| \frac{\partial u}{\partial x_i} \right|^{2p_i - 2} \frac{\partial^2 \varphi}{\partial x_i^2} dx \,, \tag{6.1}$$

where $u_\delta = u + \delta$, *where* A_i *and* B_{ij} *are as in* (5.5) *and* (5.6), *and where*

$$C_i = \frac{a}{2} + \frac{p_i - 1}{2p_i - 1}\left(\frac{\nu_i}{p_i} + \sum_{k \neq i} (a + 2c_{ik})\,\mu_{ik}^2\right) \tag{6.2}$$

and

$$D_{ij} = \frac{1}{2p_j}\left((p_i + p_j - 1)\,a + 2\,(p_j - 1)\,b_{ij}\right) + p_i \lambda_{ij} \tag{6.3}$$

for all distinct indices $i, j = 1, \dots, n$.

In the proof of Proposition 6.1, we approximate the solution u of equation (1.2) by a family of solutions of regularized problems. In what follows, we fix a positive real number R, and we consider ε as a small positive parameter. Since f is a C^1-function on $\mathbb{R}_+$ and u belongs to $W^{1,\overrightarrow{p}}_{\text{loc}}(\mathbb{R}^n) \cap L^\infty_{\text{loc}}(\mathbb{R}^n)$, by Lieberman [34,35], we get that the functions u and $f(u)$ belong to $W^{1,\infty}_{\text{loc}}(\mathbb{R}^n)$. It easily follows that there exist two families of smooth functions v_ε and f_ε on $\overline{B_0(R)}$, uniformly bounded in $C^1(B_0(R))$ and converging respectively to u and $f(u)$ in $W^{1,r}(B_0(R))$ as $\varepsilon \to 0$ for all r in $[1, +\infty)$. We also approximate $\Delta_{\overrightarrow{p}} u$ by $\operatorname{div}(L_\varepsilon(\nabla u))$, where $L_\varepsilon(\nabla u_\varepsilon) = \left(L_i^\varepsilon(\partial u_\varepsilon/\partial x_i)\right)_{i=1,\dots,n}$ is defined by

$$L_i^\varepsilon(X) = \left(\varepsilon^2 + X^2\right)^{\frac{p_i-2}{2}} X$$

for $i = 1, \dots, n$. In particular, we compute

$$(L_i^\varepsilon)'(X) = \left(\varepsilon^2 + X^2\right)^{\frac{p_i-4}{2}}\left(\varepsilon^2 + (p_i - 1)\,X^2\right).$$

Aiming to prove Proposition 6.1, we shall state some preliminary steps. The first one is as follows.

Step 6.2. *There exists a unique smooth solution* u_ε *of the Dirichlet problem*

$$\begin{cases} -\operatorname{div}(L_\varepsilon(\nabla u_\varepsilon)) + u_\varepsilon = f_\varepsilon + v_\varepsilon & \textit{in } B_0(R)\,, \\ u_\varepsilon = v_\varepsilon & \textit{on } \partial B_0(R)\,. \end{cases} \tag{6.4}$$

Proof. We use here similar arguments as those in Fragalà–Gazzola–Lieberman [26] for another family of anisotropic elliptic problems. We fix a real number θ in $(0,1)$. By Gilbarg–Trudinger [29, Theorem 6.14], for any function v in $C^{1,\theta}(\overline{B_0(R)})$, there exists a unique solution $w = T(v)$ in $C^{2,\theta}(\overline{B_0(R)})$ of the problem

$$\begin{cases} -\,(L_i^\varepsilon)'\left(\dfrac{\partial v}{\partial x_i}\right)\dfrac{\partial^2 w}{\partial x_i^2} = f_\varepsilon + v_\varepsilon - v & \text{in } B_0(R)\,, \\ w = v_\varepsilon & \text{on } \partial B_0(R)\,. \end{cases} \tag{6.5}$$

By the compactness of the embedding of $C^{2,\theta}(\overline{B_0(R)})$ into $C^{1,\theta}(\overline{B_0(R)})$, we get that the operator $T : C^{1,\theta}(\overline{B_0(R)}) \to C^{1,\theta}(\overline{B_0(R)})$ is compact. We claim that there exists a uniform bound in $C^{1,\theta}(\overline{B_0(R)})$ on the set of all functions w satisfying $w = \lambda T(w)$ for some real number λ in $[0,1]$. In order to prove this claim, for such a function w, we set

$$W = \left\{x \in B_0(R)\;;\; |w(x)| > M\right\},$$

where M is a real number satisfying $|f_\varepsilon| + |v_\varepsilon| \le M$ on $B_0(R)$. Multiplying (6.5) by the function $\operatorname{sign}(w)\max(|w| - M, 0)$ and integrating by parts on W give

$$\sum_{i=1}^n \int_W \left|\frac{\partial w}{\partial x_i}\right|^{p_i} dx \le \sum_{i=1}^n \int_W L_i^\varepsilon\left(\frac{\partial w}{\partial x_i}\right)\frac{\partial w}{\partial x_i} dx$$
$$= \lambda \int_W (|w| - M)\left(\operatorname{sign}(w)(f_\varepsilon + v_\varepsilon) - |w|\right) dx \le 0\,.$$

It follows that the set W is empty, and thus that there holds $|w| \le M$ in $B_0(R)$. By Gilbarg–Trudinger [29, Theorems 13.2, 14.4, and 15.6], we then get the expected uniform bound in $C^{1,\theta}(\overline{B_0(R)})$. Applying Fragalà–Gazzola–Lieberman [26, Lemma 2] gives the existence of a solution in $C^{1,\theta}(\overline{B_0(R)})$ of problem (6.4). The smoothness of this solution follows from Gilbarg–Trudinger [29, Theorem 6.19] by bootstrap method, and its uniqueness follows from the strict convexity of the functional I defined by

$$I(w) = \sum_{i=1}^n \int_{B_0(R)} L_i^\varepsilon\left(\frac{\partial w}{\partial x_i}\right)\frac{\partial w}{\partial x_i} dx + \int_{B_0(R)} w^2 dx - \int_{B_0(R)} (f_\varepsilon + v_\varepsilon) w dx\,.$$

This ends the proof of Step 6.2. ☐

The second step states as follows.

Step 6.3. *The functions u_ε are uniformly bounded in $C^0(B_0(R))$, $W^{1,\vec{p}}(B_0(R))$, and $C^1(\Omega)$ for all compact subsets Ω of $B_0(R)$.*

Proof. We begin with proving that the functions u_ε are uniformly bounded in $C^0(B_0(R))$. As in Step 6.2, for any ε, we set

$$W_\varepsilon = \{x \in B_0(R)\ ;\ |u_\varepsilon(x)| > M\}\,,$$

where M is a real number satisfying $|f_\varepsilon| + |v_\varepsilon| \le M$ on $B_0(R)$ for all ε. Multiplying equation (6.4) by the function $\operatorname{sign}(u_\varepsilon)\max(|u_\varepsilon| - M, 0)$ and integrating by parts on W_ε give

$$\sum_{i=1}^n \int_{W_\varepsilon} \left|\frac{\partial u_\varepsilon}{\partial x_i}\right|^{p_i} dx \le \sum_{i=1}^n \int_{W_\varepsilon} L_i^\varepsilon\left(\frac{\partial u_\varepsilon}{\partial x_i}\right)\frac{\partial u_\varepsilon}{\partial x_i} dx$$
$$= \int_{W_\varepsilon} (|u_\varepsilon| - M)\left(\operatorname{sign}(u_\varepsilon)(f_\varepsilon + v_\varepsilon) - |u_\varepsilon|\right) dx \le 0\,.$$

It follows that the set W_ε is empty, and thus that there holds $|u_\varepsilon| \le M$ in $B_0(R)$ for all ε. By Lieberman [34, 35], we then get that the functions u_ε are uniformly bounded in $C^1(\Omega)$ for all compact subsets Ω of $B_0(R)$. We now prove that the functions u_ε are uniformly bounded in $W^{1,\vec{p}}(B_0(R))$. For any ε, multiplying equation (6.4) by the function $u_\varepsilon - v_\varepsilon$ and integrating by parts on $B_0(R)$ yield

$$\sum_{i=1}^n \int_{B_0(R)} L_i^\varepsilon\left(\frac{\partial u_\varepsilon}{\partial x_i}\right)\left(\frac{\partial u_\varepsilon}{\partial x_i} - \frac{\partial v_\varepsilon}{\partial x_i}\right) dx = \int_{B_0(R)} (f_\varepsilon + v_\varepsilon - u_\varepsilon)(u_\varepsilon - v_\varepsilon) dx\,. \tag{6.6}$$

On the one hand, there holds

$$\int_{B_0(R)} L_i^\varepsilon\left(\frac{\partial u_\varepsilon}{\partial x_i}\right)\frac{\partial u_\varepsilon}{\partial x_i} dx \ge \int_{B_0(R)} \left|\frac{\partial u_\varepsilon}{\partial x_i}\right|^{p_i} dx\,. \tag{6.7}$$

On the other hand, by Hölder's inequality and since the functions v_ε are uniformly bounded in $C^1(B_0(R))$, we get

$$\begin{aligned}\left|\int_{B_0(R)} L_i^\varepsilon\left(\frac{\partial u_\varepsilon}{\partial x_i}\right)\frac{\partial v_\varepsilon}{\partial x_i}dx\right| &\le C\int_{B_0(R)}\left(\varepsilon^{p_i-2}+\left|\frac{\partial u_\varepsilon}{\partial x_i}\right|^{p_i-2}\right)\left|\frac{\partial u_\varepsilon}{\partial x_i}\frac{\partial v_\varepsilon}{\partial x_i}\right|dx\\ &\le C'\left(\varepsilon^{p_i-2}\left\|\frac{\partial u_\varepsilon}{\partial x_i}\right\|_{L^{p_i}(B_0(R))}+\left\|\frac{\partial u_\varepsilon}{\partial x_i}\right\|_{L^{p_i}(B_0(R))}^{p_i-1}\right)\end{aligned} \tag{6.8}$$

for some positive constants C and C' independent of ε. Since the functions u_ε, v_ε, and f_ε are uniformly bounded in $C^0(B_0(R))$, it follows from (6.6)–(6.8) that the u_ε's are uniformly bounded in $W^{1,\overrightarrow{p}}(B_0(R))$. This ends the proof of Step 6.3. □

The third step in the proof of Proposition 6.1 is as follows.

Step 6.4. *For $i,j=1,\dots,n$, the functions $(\varepsilon^2+(\partial u_\varepsilon/\partial x_i)^2)^{(p_i-2)/4}\partial^2 u_\varepsilon/\partial x_i\partial x_j$ are uniformly bounded in $L^2(\Omega)$ for any compact subset Ω of $B_0(R)$. In particular, the functions $L_i^\varepsilon(\partial u_\varepsilon/\partial x_i)$ are uniformly bounded in $W^{1,2}(\Omega)$.*

Proof. For any real numbers $R'<R''$ in $(0,R)$, we let $\eta_{R'}$ be a smooth cutoff function on $\mathbb{R}^n$ satisfying $\eta_{R'}\equiv 1$ in $B_0(R')$, $0\le\eta_{R'}\le 1$ in $B_0(R'')\backslash B_0(R')$, and $\eta_{R'}\equiv 0$ out of $B_0(R'')$. For $j=1,\dots,n$ and for any ε, multiplying equation (6.4) by $\partial\left((\partial u_\varepsilon/\partial x_j)\,\eta_{R'}^2\right)/\partial x_j$ and integrating by parts, we get

$$\begin{aligned}\sum_{i=1}^n\left(\int_{\mathbb{R}^n}(L_i^\varepsilon)'\left(\frac{\partial u_\varepsilon}{\partial x_i}\right)\left(\frac{\partial^2 u_\varepsilon}{\partial x_i\partial x_j}\eta_{R'}\right)^2dx+2\int_{\mathbb{R}^n}(L_i^\varepsilon)'\left(\frac{\partial u_\varepsilon}{\partial x_i}\right)\frac{\partial u_\varepsilon}{\partial x_j}\frac{\partial^2 u_\varepsilon}{\partial x_i\partial x_j}\frac{\partial\eta_{R'}}{\partial x_i}\eta_{R'}dx\right)\\ =\int_{\mathbb{R}^n}\frac{\partial\left(f_\varepsilon+v_\varepsilon-u_\varepsilon\right)}{\partial x_j}\frac{\partial u_\varepsilon}{\partial x_j}\eta_{R'}^2dx\,.\end{aligned} \tag{6.9}$$

For $i,j=1,\dots,n$ and for any $C>0$, Young's inequality gives

$$\left|\frac{\partial u_\varepsilon}{\partial x_j}\frac{\partial^2 u_\varepsilon}{\partial x_i\partial x_j}\frac{\partial\eta_{R'}}{\partial x_i}\eta_{R'}\right|\le C\left(\frac{\partial^2 u_\varepsilon}{\partial x_i\partial x_j}\eta_{R'}\right)^2+\frac{1}{C}\left(\frac{\partial u_\varepsilon}{\partial x_j}\frac{\partial\eta_{R'}}{\partial x_i}\right)^2\quad\text{in } B_0(R)\,. \tag{6.10}$$

By (6.9) and (6.10), we get that there exists a positive constant C independent of ε such that, for $j=1,\dots,n$, there holds

$$\begin{aligned}&\sum_{i=1}^n\int_{B_0(R')}\left(\varepsilon^2+\left(\frac{\partial u_\varepsilon}{\partial x_i}\right)^2\right)^{\frac{p_i-2}{2}}\left(\frac{\partial^2 u_\varepsilon}{\partial x_i\partial x_j}\right)^2dx\\ &\le C\sum_{i=1}^n\left(\int_{\mathbb{R}^n}\left(\varepsilon^2+\left(\frac{\partial u_\varepsilon}{\partial x_i}\right)^2\right)^{\frac{p_i-2}{2}}\left(\frac{\partial u_\varepsilon}{\partial x_j}\frac{\partial\eta_{R'}}{\partial x_i}\right)^2dx+\int_{B_0(R'')}\left|\frac{\partial\left(f_\varepsilon+v_\varepsilon-u_\varepsilon\right)}{\partial x_j}\frac{\partial u_\varepsilon}{\partial x_j}\right|dx\right).\end{aligned}$$

Taking into account that there holds $p_i\ge 2$ for $i=1,\dots,n$ and that the functions u_ε, v_ε, and f_ε are uniformly bounded in $C^1(B_0(R''))$, it follows that for $i,j=1,\dots,n$ the functions $(\varepsilon^2+(\partial u_\varepsilon/\partial x_i)^2)^{(p_i-2)/4}\partial^2u_\varepsilon/\partial x_i\partial x_j$ are uniformly bounded in $L^2(B_0(R'))$. This ends the proof of Step 6.4. □

The fourth step in the proof of Proposition 6.1 states as follows.

Step 6.5. *For any compact subset Ω of $B_0(R)$, the functions u_ε converge to u in $W^{1,r}(\Omega)$ as $\varepsilon\to 0$ for all r in $[1,+\infty)$.*

Proof. By Step 6.3, for any sequence $(\varepsilon_\alpha)_\alpha$ of positive real numbers converging to 0, up to a subsequence, $(u_{\varepsilon_\alpha})_\alpha$ converges weakly to a function w in $W^{1,\overrightarrow{p}}(B_0(R))$. Moreover, by the compactness of the embedding of $W^{1,\overrightarrow{p}}(B_0(R))$ into $L^1(B_0(R))$ and since $(u_{\varepsilon_\alpha})_\alpha$ is bounded in $C^0(B_0(R))$, we get that $(u_{\varepsilon_\alpha})_\alpha$ converges to w in $L^r(B_0(R))$ as $\varepsilon \to 0$ for all r in $[1,+\infty)$. By Step 6.4 and by the compactness of the embedding of $W^{1,2}(\Omega)$ into $L^1(\Omega)$ for all compact subsets Ω of $B_0(R)$, we get that for $i=1,\dots,n$, up to a subsequence, $(L_i^{\varepsilon_\alpha}(\partial u_{\varepsilon_\alpha}/\partial x_i))_\alpha$ converges to a function Ψ_i in $L^1(\Omega)$, and thus almost everywhere in $B_0(R)$. It easily follows that there holds $\Psi_i = |\partial w/\partial x_i|^{p_i-2}\partial w/\partial x_i$ and that the functions $\partial u_{\varepsilon_\alpha}/\partial x_i$ converge almost everywhere to $\partial w/\partial x_i$ in $B_0(R)$ as $\alpha \to +\infty$ for $i=1,\dots,n$. By the compactness of the trace embedding of $W^{1,\overrightarrow{p}}(B_0(R))$ into $L^1(\partial B_0(R))$, since $u_{\varepsilon_\alpha} \equiv v_{\varepsilon_\alpha}$ on $\partial B_0(R)$ for all α, and since $(v_{\varepsilon_\alpha})_\alpha$ converges to u in $C^0(\partial B_0(R))$ as $\varepsilon \to 0$, we get $w \equiv u$ on $\partial B_0(R)$. Multiplying equations (1.2) and (6.4) by the function $w-u$ and integrating by parts on $B_0(R)$ give

$$\sum_{i=1}^n \int_{B_0(R)} \left(L_i^{\varepsilon_\alpha}\left(\frac{\partial u_{\varepsilon_\alpha}}{\partial x_i}\right) - \left|\frac{\partial u}{\partial x_i}\right|^{p_i-2}\frac{\partial u}{\partial x_i}\right)\left(\frac{\partial w}{\partial x_i} - \frac{\partial u}{\partial x_i}\right)dx = \int_{B_0(R)} \left(f_{\varepsilon_\alpha} - f(u) + v_{\varepsilon_\alpha} - u_{\varepsilon_\alpha}\right)(w-u)\,dx\,. \tag{6.11}$$

By Step 6.3, we get that the sequence $(L_i^{\varepsilon_\alpha}(\partial u_{\varepsilon_\alpha}/\partial x_i))_\alpha$ is bounded in $L^{p_i/(p_i-1)}(B_0(R))$ for $i=1,\dots,n$. On the other hand, $(L_i^{\varepsilon_\alpha}(\partial u_{\varepsilon_\alpha}/\partial x_i))_\alpha$ converges almost everywhere in $B_0(R)$ to $|\partial w/\partial x_i|^{p_i-2}\partial w/\partial x_i$ as $\alpha \to +\infty$. By standard integration theory, it follows that $(L_i^{\varepsilon_\alpha}(\partial u_{\varepsilon_\alpha}/\partial x_i))_\alpha$ converges weakly to $|\partial w/\partial x_i|^{p_i-2}\partial w/\partial x_i$ in $L^{p_i/(p_i-1)}(B_0(R))$. Taking into account that $(u_{\varepsilon_\alpha})_\alpha$, $(v_{\varepsilon_\alpha})_\alpha$, and $(f_{\varepsilon_\alpha})_\alpha$ converge respectively to w, u, and $f(u)$ in $L^2(B_0(R))$, passing to the limit as $\alpha \to +\infty$ into (6.11) then yields

$$\sum_{i=1}^n \int_{B_0(R)} \left(\left|\frac{\partial w}{\partial x_i}\right|^{p_i-2}\frac{\partial w}{\partial x_i} - \left|\frac{\partial u}{\partial x_i}\right|^{p_i-2}\frac{\partial u}{\partial x_i}\right)\left(\frac{\partial w}{\partial x_i} - \frac{\partial u}{\partial x_i}\right)dx = -\int_{B_0(R)} (w-u)^2\,dx \le 0\,. \tag{6.12}$$

For $i=1,\dots,n$, one can easily check

$$\left(\left|\frac{\partial w}{\partial x_i}\right|^{p_i-2}\frac{\partial w}{\partial x_i} - \left|\frac{\partial u}{\partial x_i}\right|^{p_i-2}\frac{\partial u}{\partial x_i}\right)\left(\frac{\partial w}{\partial x_i} - \frac{\partial u}{\partial x_i}\right) \ge 0 \quad \text{in } B_0(R)\,. \tag{6.13}$$

It follows from (6.12) and (6.13) that there hold $w=u$ and $\partial w/\partial x_i = \partial u/\partial x_i$ almost everywhere in $B_0(R)$ for $i=1,\dots,n$. The above holds true for all sequences $(\varepsilon_\alpha)_\alpha$ of positive real numbers converging to 0. Hence, we get that the functions u_ε and $\partial u_\varepsilon/\partial x_i$ converge almost everywhere respectively to u and $\partial u/\partial x_i$ in $B_0(R)$ as $\varepsilon \to 0$ for $i=1,\dots,n$. Since, by Step 6.3, the functions u_ε are uniformly bounded in $C^1(\Omega)$ for all compact subsets Ω of $B_0(R)$, it follows that they converge to u in $W^{1,r}(\Omega)$ as $\varepsilon \to 0$ for all r in $[1,+\infty)$. □

It follows from Step 6.5 that for any compact subset Ω of $B_0(R)$, the functions u_ε converge to u in $C^0(\Omega)$ as $\varepsilon \to 0$. In particular, since u is nonnegative, for any positive real number δ, the function $u_{\varepsilon,\delta} = u_\varepsilon + \delta$ is positive in Ω for ε small. In the following two steps, we enumerate several integral estimates.

Step 6.6. *Let a be a real number, δ be a positive real number, and φ be a smooth function with compact support in $B_0(R)$. For $i,j=1,\dots,n$ and for ε small, define $\mathcal{E}_{ij}^\varepsilon(u_{\varepsilon,\delta},\varphi)$ and $\mathcal{F}_{ij}^\varepsilon(u_{\varepsilon,\delta},\varphi)$ by*

$$\begin{aligned}\mathcal{E}_{ij}^\varepsilon(u_{\varepsilon,\delta},\varphi)=a&\int_{\mathbb{R}^n}u_{\varepsilon,\delta}^{a-1}L_i^\varepsilon\left(\frac{\partial u_\varepsilon}{\partial x_i}\right)\frac{\partial u_\varepsilon}{\partial x_i}\frac{\partial}{\partial x_j}\left(L_j^\varepsilon\left(\frac{\partial u_\varepsilon}{\partial x_j}\right)\right)\varphi dx\\&+\int_{\mathbb{R}^n}u_{\varepsilon,\delta}^{a}L_i^\varepsilon\left(\frac{\partial u_\varepsilon}{\partial x_i}\right)\frac{\partial^2}{\partial x_i\partial x_j}\left(L_j^\varepsilon\left(\frac{\partial u_\varepsilon}{\partial x_j}\right)\right)\varphi dx\\&+\int_{\mathbb{R}^n}u_{\varepsilon,\delta}^{a}L_i^\varepsilon\left(\frac{\partial u_\varepsilon}{\partial x_i}\right)\frac{\partial}{\partial x_j}\left(L_j^\varepsilon\left(\frac{\partial u_\varepsilon}{\partial x_j}\right)\right)\frac{\partial\varphi}{\partial x_i}dx\end{aligned}$$

and

$$\begin{aligned}\mathcal{F}_{ij}^\varepsilon(u_{\varepsilon,\delta},\varphi)=p_i(a-1)&\int_{\mathbb{R}^n}u_{\varepsilon,\delta}^{a-2}L_i^\varepsilon\left(\frac{\partial u_\varepsilon}{\partial x_i}\right)\frac{\partial u_\varepsilon}{\partial x_i}L_j^\varepsilon\left(\frac{\partial u_\varepsilon}{\partial x_j}\right)\frac{\partial u_\varepsilon}{\partial x_j}\varphi dx\\&+p_i\int_{\mathbb{R}^n}u_{\varepsilon,\delta}^{a-1}\frac{\partial}{\partial x_i}\left(L_i^\varepsilon\left(\frac{\partial u_\varepsilon}{\partial x_i}\right)\right)L_j^\varepsilon\left(\frac{\partial u_\varepsilon}{\partial x_j}\right)\frac{\partial u_\varepsilon}{\partial x_j}\varphi dx\\&+p_i\int_{\mathbb{R}^n}u_{\varepsilon,\delta}^{a-1}L_i^\varepsilon\left(\frac{\partial u_\varepsilon}{\partial x_i}\right)L_j^\varepsilon\left(\frac{\partial u_\varepsilon}{\partial x_j}\right)\frac{\partial u_\varepsilon}{\partial x_j}\frac{\partial\varphi}{\partial x_i}dx\,.\end{aligned}$$

For $i=1,\dots,n$ and for ε small, define also $\mathcal{G}_i^\varepsilon(u_{\varepsilon,\delta},\varphi)$, and $\mathcal{H}_i^\varepsilon(u_{\varepsilon,\delta},\varphi)$ by

$$\begin{aligned}\mathcal{G}_i^\varepsilon(u_{\varepsilon,\delta},\varphi)=&a\int_{\mathbb{R}^n}u_{\varepsilon,\delta}^{a-1}\left(L_i^\varepsilon\left(\frac{\partial u_\varepsilon}{\partial x_i}\right)\right)^2\frac{\partial u_\varepsilon}{\partial x_i}\frac{\partial\varphi}{\partial x_i}dx\\&+2\int_{\mathbb{R}^n}u_{\varepsilon,\delta}^{a}L_i^\varepsilon\left(\frac{\partial u_\varepsilon}{\partial x_i}\right)\frac{\partial}{\partial x_i}\left(L_i^\varepsilon\left(\frac{\partial u_\varepsilon}{\partial x_i}\right)\right)\frac{\partial\varphi}{\partial x_i}dx\\&+\int_{\mathbb{R}^n}u_{\varepsilon,\delta}^{a}\left(L_i^\varepsilon\left(\frac{\partial u_\varepsilon}{\partial x_i}\right)\right)^2\frac{\partial^2\varphi}{\partial x_i^2}dx\end{aligned}$$

and

$$\begin{aligned}\mathcal{H}_i^\varepsilon(u_{\varepsilon,\delta},\varphi)=(p_i-1)(a-1)&\int_{\mathbb{R}^n}u_{\varepsilon,\delta}^{a-2}\left(L_i^\varepsilon\left(\frac{\partial u_\varepsilon}{\partial x_i}\right)\right)^2\left(\frac{\partial u_\varepsilon}{\partial x_i}\right)^2\varphi dx\\&+(2p_i-1)\int_{\mathbb{R}^n}u_{\varepsilon,\delta}^{a-1}L_i^\varepsilon\left(\frac{\partial u_\varepsilon}{\partial x_i}\right)\frac{\partial u_\varepsilon}{\partial x_i}\frac{\partial}{\partial x_i}\left(L_i^\varepsilon\left(\frac{\partial u_\varepsilon}{\partial x_i}\right)\right)\varphi dx\\&+(p_i-1)\int_{\mathbb{R}^n}u_{\varepsilon,\delta}^{a-1}\left(L_i^\varepsilon\left(\frac{\partial u_\varepsilon}{\partial x_i}\right)\right)^2\frac{\partial u_\varepsilon}{\partial x_i}\frac{\partial\varphi}{\partial x_i}dx\,.\end{aligned}$$

Then there hold $\mathcal{E}_{ij}^\varepsilon(u_{\varepsilon,\delta},\varphi)=\mathcal{E}_{ji}^\varepsilon(u_{\varepsilon,\delta},\varphi)$, $\mathcal{F}_{ij}^\varepsilon(u_{\varepsilon,\delta},\varphi)=\mathcal{F}_{ji}^\varepsilon(u_{\varepsilon,\delta},\varphi)+\mathrm{O}(\varepsilon)$, $\mathcal{G}_i^\varepsilon(u_{\varepsilon,\delta},\varphi)=0$, and $\mathcal{H}_i^\varepsilon(u_{\varepsilon,\delta},\varphi)=\mathrm{O}(\varepsilon^2)$ as $\varepsilon\to0$, for $i,j=1,\dots,n$.

Proof. For $i,j=1,\dots,n$ and for ε small, an easy integration by parts gives

$$\mathcal{E}_{ij}^\varepsilon(u_{\varepsilon,\delta},\varphi)=-\int_{\mathbb{R}^n}u_{\varepsilon,\delta}^{a}\frac{\partial}{\partial x_i}\left(L_i^\varepsilon\left(\frac{\partial u_\varepsilon}{\partial x_i}\right)\right)\frac{\partial}{\partial x_j}\left(L_j^\varepsilon\left(\frac{\partial u_\varepsilon}{\partial x_j}\right)\right)\varphi dx\,. \tag{6.14}$$

In particular, we get $\mathcal{E}_{ij}^\varepsilon(u_{\varepsilon,\delta},\varphi)=\mathcal{E}_{ji}^\varepsilon(u_{\varepsilon,\delta},\varphi)$. Another integration by parts gives

$$\mathcal{F}_{ij}^\varepsilon(u_{\varepsilon,\delta},\varphi)=-p_i\int_{\mathbb{R}^n}u_{\varepsilon,\delta}^{a-1}L_i^\varepsilon\left(\frac{\partial u_\varepsilon}{\partial x_i}\right)\frac{\partial}{\partial x_i}\left(L_j^\varepsilon\left(\frac{\partial u_\varepsilon}{\partial x_j}\right)\frac{\partial u_\varepsilon}{\partial x_j}\right)\varphi dx\,. \tag{6.15}$$

We then compute

$$\mathcal{F}_{ij}^{\varepsilon}\left(u_{\varepsilon,\delta},\varphi\right)=-p_ip_j\int_{\mathbb{R}^n}u_{\varepsilon,\delta}^{a-1}L_i^{\varepsilon}\left(\frac{\partial u_\varepsilon}{\partial x_i}\right)L_j^{\varepsilon}\left(\frac{\partial u_\varepsilon}{\partial x_j}\right)\frac{\partial^2 u_\varepsilon}{\partial x_i\partial x_j}\varphi dx$$
$$+p_i\left(p_j-2\right)\varepsilon^2\int_{\mathbb{R}^n}u_{\varepsilon,\delta}^{a-1}L_i^{\varepsilon}\left(\frac{\partial u_\varepsilon}{\partial x_i}\right)\left(\varepsilon^2+\left(\frac{\partial u_\varepsilon}{\partial x_j}\right)^2\right)^{\frac{p_j-4}{2}}\frac{\partial u_\varepsilon}{\partial x_j}\frac{\partial^2 u_\varepsilon}{\partial x_i\partial x_j}\varphi dx\,.$$

It follows that

$$\left|\mathcal{F}_{ij}^{\varepsilon}\left(u_{\varepsilon,\delta},\varphi\right)+p_ip_j\int_{\mathbb{R}^n}u_{\varepsilon,\delta}^{a-1}L_i^{\varepsilon}\left(\frac{\partial u_\varepsilon}{\partial x_i}\right)L_j^{\varepsilon}\left(\frac{\partial u_\varepsilon}{\partial x_j}\right)\frac{\partial^2 u_\varepsilon}{\partial x_i\partial x_j}\varphi dx\right|$$
$$\le p_i\left(p_j-2\right)\varepsilon\int_{\mathbb{R}^n}u_{\varepsilon,\delta}^{a-1}\left(\varepsilon^2+\left(\frac{\partial u_\varepsilon}{\partial x_i}\right)^2\right)^{\frac{p_i-1}{2}}\left(\varepsilon^2+\left(\frac{\partial u_\varepsilon}{\partial x_j}\right)^2\right)^{\frac{p_j-2}{2}}\left|\frac{\partial^2 u_\varepsilon}{\partial x_i\partial x_j}\varphi\right|dx\,.$$

Since $u_{\varepsilon,\delta}\ge\delta/2$ on $\operatorname{Supp}(\varphi)$ for ε small, since $p_i\ge 2$ for $i=1,\dots,n$, by Steps 6.3 and 6.4, we then get

$$\mathcal{F}_{ij}^{\varepsilon}\left(u_{\varepsilon,\delta},\varphi\right)=-p_ip_j\int_{\mathbb{R}^n}u_{\varepsilon,\delta}^{a-1}L_i^{\varepsilon}\left(\frac{\partial u_\varepsilon}{\partial x_i}\right)L_j^{\varepsilon}\left(\frac{\partial u_\varepsilon}{\partial x_j}\right)\frac{\partial^2 u_\varepsilon}{\partial x_i\partial x_j}\varphi dx+\mathrm{O}\left(\varepsilon\right)\tag{6.16}$$

as $\varepsilon\to 0$ for $i,j=1,\dots,n$. In particular, we get $\mathcal{F}_{ij}^{\varepsilon}\left(u_{\varepsilon,\delta},\varphi\right)=\mathcal{F}_{ji}^{\varepsilon}\left(u_{\varepsilon,\delta},\varphi\right)+\mathrm{O}\left(\varepsilon\right)$ as $\varepsilon\to 0$. For $i=1,\dots,n$ and for ε small, the identity $\mathcal{G}_i^{\varepsilon}\left(u_{\varepsilon,\delta},\varphi\right)=0$ follows from a straightforward integration by parts. Another integration by parts gives

$$\mathcal{H}_i^{\varepsilon}\left(u_{\varepsilon,\delta},\varphi\right)=\int_{\mathbb{R}^n}u_{\varepsilon,\delta}^{a-1}L_i^{\varepsilon}\left(\frac{\partial u_\varepsilon}{\partial x_i}\right)\frac{\partial u_\varepsilon}{\partial x_i}\frac{\partial}{\partial x_i}\left(L_i^{\varepsilon}\left(\frac{\partial u_\varepsilon}{\partial x_i}\right)\right)\varphi dx$$
$$-\left(p_i-1\right)\int_{\mathbb{R}^n}u_{\varepsilon,\delta}^{a-1}\left(L_i^{\varepsilon}\left(\frac{\partial u_\varepsilon}{\partial x_i}\right)\right)^2\frac{\partial^2 u_\varepsilon}{\partial x_i^2}\varphi dx$$
$$=-\left(p_i-2\right)\varepsilon^2\int_{\mathbb{R}^n}u_{\varepsilon,\delta}^{a-1}\left(\varepsilon^2+\left(\frac{\partial u_\varepsilon}{\partial x_i}\right)^2\right)^{p_i-3}\left(\frac{\partial u_\varepsilon}{\partial x_i}\right)^2\frac{\partial^2 u_\varepsilon}{\partial x_i^2}\varphi dx\,.$$

Since $u_{\varepsilon,\delta}\ge\delta/2$ on $\operatorname{Supp}(\varphi)$ for ε small, since $p_i\ge 2$ for $i=1,\dots,n$, by Steps 6.3 and 6.4, we then get

$$\left|\mathcal{H}_i^{\varepsilon}\left(u_{\varepsilon,\delta},\varphi\right)\right|\le\left(p_i-2\right)\varepsilon^2\int_{\mathbb{R}^n}u_{\varepsilon,\delta}^{a-1}\left(\varepsilon^2+\left(\frac{\partial u_\varepsilon}{\partial x_i}\right)^2\right)^{p_i-2}\left|\frac{\partial^2 u_\varepsilon}{\partial x_i^2}\varphi\right|dx=\mathrm{O}\left(\varepsilon^2\right)$$

as $\varepsilon\to 0$. This ends the proof of Step 6.6. ◻

The next step in the proof of Proposition 6.1 is as follows.

Step 6.7. *Let a, b_{ij}, and c_{ij} be some real numbers, δ be a positive real number, and φ be a smooth function with compact support in $B_0(R)$. For $i,j=1,\dots,n$ and for ε small, define $\mathcal{P}_{ij}^{\varepsilon}\left(u_{\varepsilon,\delta},\varphi\right)$ and $\mathcal{Q}_{ij}^{\varepsilon}\left(u_{\varepsilon,\delta},\varphi\right)$ by*

$$\mathcal{P}_{ij}^{\varepsilon}\left(u_{\varepsilon,\delta},\varphi\right)=\int_{\mathbb{R}^n}u_{\varepsilon,\delta}^{a+c_{ij}+c_{ji}}\frac{\partial}{\partial x_i}\left(u_{\varepsilon,\delta}^{-c_{ij}}L_i^{\varepsilon}\left(\frac{\partial u_\varepsilon}{\partial x_i}\right)\right)\frac{\partial}{\partial x_j}\left(u_{\varepsilon,\delta}^{-c_{ji}}L_j^{\varepsilon}\left(\frac{\partial u_\varepsilon}{\partial x_j}\right)\right)\varphi dx$$

and

$$\mathcal{Q}_{ij}^{\varepsilon}\left(u_{\varepsilon,\delta},\varphi\right)=\int_{\mathbb{R}^n}u_{\varepsilon,\delta}^{a+b_{ij}+b_{ji}}\frac{\partial}{\partial x_j}\left(u_{\varepsilon,\delta}^{-b_{ij}}L_i^{\varepsilon}\left(\frac{\partial u_\varepsilon}{\partial x_i}\right)\right)\frac{\partial}{\partial x_i}\left(u_{\varepsilon,\delta}^{-b_{ji}}L_j^{\varepsilon}\left(\frac{\partial u_\varepsilon}{\partial x_j}\right)\right)\varphi dx\,.$$

For $i,j=1,\dots,n$, there hold

$$\begin{aligned}\mathcal{P}_{ij}^{\varepsilon}\left(u_{\varepsilon,\delta},\varphi\right)=&c_{ij}c_{ji}\int_{\mathbb{R}^n}u_{\varepsilon,\delta}^{a-2}L_i^{\varepsilon}\left(\frac{\partial u_\varepsilon}{\partial x_i}\right)\frac{\partial u_\varepsilon}{\partial x_i}L_j^{\varepsilon}\left(\frac{\partial u_\varepsilon}{\partial x_j}\right)\frac{\partial u_\varepsilon}{\partial x_j}\varphi dx\\&-c_{ji}\int_{\mathbb{R}^n}u_{\varepsilon,\delta}^{a-1}\frac{\partial}{\partial x_i}\left(L_i^{\varepsilon}\left(\frac{\partial u_\varepsilon}{\partial x_i}\right)\right)L_j^{\varepsilon}\left(\frac{\partial u_\varepsilon}{\partial x_j}\right)\frac{\partial u_\varepsilon}{\partial x_j}\varphi dx\\&-(a+c_{ij})\int_{\mathbb{R}^n}u_{\varepsilon,\delta}^{a-1}L_i^{\varepsilon}\left(\frac{\partial u_\varepsilon}{\partial x_i}\right)\frac{\partial u_\varepsilon}{\partial x_i}\frac{\partial}{\partial x_j}\left(L_j^{\varepsilon}\left(\frac{\partial u_\varepsilon}{\partial x_j}\right)\right)\varphi dx\\&-\int_{\mathbb{R}^n}u_{\varepsilon,\delta}^{a}L_i^{\varepsilon}\left(\frac{\partial u_\varepsilon}{\partial x_i}\right)\frac{\partial^2}{\partial x_i\partial x_j}\left(L_j^{\varepsilon}\left(\frac{\partial u_\varepsilon}{\partial x_j}\right)\right)\varphi dx\\&-\int_{\mathbb{R}^n}u_{\varepsilon,\delta}^{a}L_i^{\varepsilon}\left(\frac{\partial u_\varepsilon}{\partial x_i}\right)\frac{\partial}{\partial x_j}\left(L_j^{\varepsilon}\left(\frac{\partial u_\varepsilon}{\partial x_j}\right)\right)\frac{\partial\varphi}{\partial x_i}dx\end{aligned}\tag{6.17}$$

for ε small and

$$\begin{aligned}\mathcal{Q}_{ij}^{\varepsilon}\left(u_{\varepsilon,\delta},\varphi\right)=&E_{ij}\int_{\mathbb{R}^n}u_{\varepsilon,\delta}^{a-2}L_i^{\varepsilon}\left(\frac{\partial u_\varepsilon}{\partial x_i}\right)\frac{\partial u_\varepsilon}{\partial x_i}L_j^{\varepsilon}\left(\frac{\partial u_\varepsilon}{\partial x_j}\right)\frac{\partial u_\varepsilon}{\partial x_j}\varphi dx\\&+F_{ij}\int_{\mathbb{R}^n}u_{\varepsilon,\delta}^{a-1}\frac{\partial}{\partial x_i}\left(L_i^{\varepsilon}\left(\frac{\partial u_\varepsilon}{\partial x_i}\right)\right)L_j^{\varepsilon}\left(\frac{\partial u_\varepsilon}{\partial x_j}\right)\frac{\partial u_\varepsilon}{\partial x_j}\varphi dx\\&-\int_{\mathbb{R}^n}u_{\varepsilon,\delta}^{a}\frac{\partial^2}{\partial x_j\partial x_i}\left(L_i^{\varepsilon}\left(\frac{\partial u_\varepsilon}{\partial x_i}\right)\right)L_j^{\varepsilon}\left(\frac{\partial u_\varepsilon}{\partial x_j}\right)\varphi dx\\&+\int_{\mathbb{R}^n}u_{\varepsilon,\delta}^{a}L_i^{\varepsilon}\left(\frac{\partial u_\varepsilon}{\partial x_i}\right)\frac{\partial}{\partial x_j}\left(L_j^{\varepsilon}\left(\frac{\partial u_\varepsilon}{\partial x_j}\right)\right)\frac{\partial\varphi}{\partial x_i}dx\\&+G_{ij}\int_{\mathbb{R}^n}u_{\varepsilon,\delta}^{a-1}L_i^{\varepsilon}\left(\frac{\partial u_\varepsilon}{\partial x_i}\right)L_j^{\varepsilon}\left(\frac{\partial u_\varepsilon}{\partial x_j}\right)\frac{\partial u_\varepsilon}{\partial x_j}\frac{\partial\varphi}{\partial x_i}dx\\&+\int_{\mathbb{R}^n}u_{\varepsilon,\delta}^{a}L_i^{\varepsilon}\left(\frac{\partial u_\varepsilon}{\partial x_i}\right)L_j^{\varepsilon}\left(\frac{\partial u_\varepsilon}{\partial x_j}\right)\frac{\partial^2\varphi}{\partial x_i\partial x_j}dx+\mathrm{O}\left(\varepsilon\right)\end{aligned}\tag{6.18}$$

as $\varepsilon\to 0$, where

$$E_{ij}=\frac{1}{p_j}\left(a-1\right)\left(\left(p_i-1\right)a+\left(p_j-1\right)b_{ij}+\left(p_i-1\right)b_{ji}\right)+b_{ij}b_{ji}\,,\tag{6.19}$$

$$F_{ij}=\frac{1}{p_j}\left(\left(p_i-1\right)a+\left(p_j-1\right)b_{ij}+\left(p_i-1\right)b_{ji}\right)\,,\tag{6.20}$$

$$G_{ij}=\frac{1}{p_j}\left(\left(p_i+p_j-1\right)a+\left(p_j-1\right)b_{ij}+\left(p_i-1\right)b_{ji}\right)\,.\tag{6.21}$$

Proof. For $i,j=1,\dots,n$ and for ε small, a straightforward computation yields

$$\begin{aligned}\mathcal{P}_{ij}^{\varepsilon}\left(u_{\varepsilon,\delta},\varphi\right)=&c_{ij}c_{ji}\int_{\mathbb{R}^n}u_{\varepsilon,\delta}^{a-2}L_i^{\varepsilon}\left(\frac{\partial u_\varepsilon}{\partial x_i}\right)\frac{\partial u_\varepsilon}{\partial x_i}L_j^{\varepsilon}\left(\frac{\partial u_\varepsilon}{\partial x_j}\right)\frac{\partial u_\varepsilon}{\partial x_j}\varphi dx\\&-c_{ji}\int_{\mathbb{R}^n}u_{\varepsilon,\delta}^{a-1}\frac{\partial}{\partial x_i}\left(L_i^{\varepsilon}\left(\frac{\partial u_\varepsilon}{\partial x_i}\right)\right)L_j^{\varepsilon}\left(\frac{\partial u_\varepsilon}{\partial x_j}\right)\frac{\partial u_\varepsilon}{\partial x_j}\varphi dx\\&-c_{ij}\int_{\mathbb{R}^n}u_{\varepsilon,\delta}^{a-1}L_i^{\varepsilon}\left(\frac{\partial u_\varepsilon}{\partial x_i}\right)\frac{\partial u_\varepsilon}{\partial x_i}\frac{\partial}{\partial x_j}\left(L_j^{\varepsilon}\left(\frac{\partial u_\varepsilon}{\partial x_j}\right)\right)\varphi dx\\&+\int_{\mathbb{R}^n}u_{\varepsilon,\delta}^{a}\frac{\partial}{\partial x_i}\left(L_i^{\varepsilon}\left(\frac{\partial u_\varepsilon}{\partial x_i}\right)\right)\frac{\partial}{\partial x_j}\left(L_j^{\varepsilon}\left(\frac{\partial u_\varepsilon}{\partial x_j}\right)\right)\varphi dx\,.\end{aligned}\tag{6.22}$$

Then (6.17) follows from (6.14) and (6.22). We now prove (6.18). For $i, j = 1, \ldots, n$ and for ε small, an easy integration by parts gives

$$\begin{aligned}\mathcal{Q}^{\varepsilon}_{ij}\left(u_{\varepsilon,\delta},\varphi\right) = &-\int_{\mathbb{R}^n} u_{\varepsilon,\delta}^{a+b_{ij}}\frac{\partial}{\partial x_j}\left(u_{\varepsilon,\delta}^{-b_{ij}}L_i^{\varepsilon}\left(\frac{\partial u_\varepsilon}{\partial x_i}\right)\right)L_j^{\varepsilon}\left(\frac{\partial u_\varepsilon}{\partial x_j}\right)\frac{\partial\varphi}{\partial x_i}dx\\ &-\int_{\mathbb{R}^n} u_{\varepsilon,\delta}^{a+b_{ij}}\frac{\partial^2}{\partial x_j\partial x_i}\left(u_{\varepsilon,\delta}^{-b_{ij}}L_i^{\varepsilon}\left(\frac{\partial u_\varepsilon}{\partial x_i}\right)\right)L_j^{\varepsilon}\left(\frac{\partial u_\varepsilon}{\partial x_j}\right)\varphi dx\\ &-\left(a+b_{ij}+b_{ji}\right)\int_{\mathbb{R}^n} u_{\varepsilon,\delta}^{a+b_{ij}-1}\frac{\partial}{\partial x_j}\left(u_{\varepsilon,\delta}^{-b_{ij}}L_i^{\varepsilon}\left(\frac{\partial u_\varepsilon}{\partial x_i}\right)\right)\frac{\partial u_\varepsilon}{\partial x_i}L_j^{\varepsilon}\left(\frac{\partial u_\varepsilon}{\partial x_j}\right)\varphi dx\,. \qquad (6.23)\end{aligned}$$

Another integration by parts gives

$$\begin{aligned}&\int_{\mathbb{R}^n} u_{\varepsilon,\delta}^{a+b_{ij}}\frac{\partial}{\partial x_j}\left(u_{\varepsilon,\delta}^{-b_{ij}}L_i^{\varepsilon}\left(\frac{\partial u_\varepsilon}{\partial x_i}\right)\right)L_j^{\varepsilon}\left(\frac{\partial u_\varepsilon}{\partial x_j}\right)\frac{\partial\varphi}{\partial x_i}dx\\ &\quad= -\left(a+b_{ij}\right)\int_{\mathbb{R}^n} u_{\varepsilon,\delta}^{a-1}L_i^{\varepsilon}\left(\frac{\partial u_\varepsilon}{\partial x_i}\right)L_j^{\varepsilon}\left(\frac{\partial u_\varepsilon}{\partial x_j}\right)\frac{\partial u_\varepsilon}{\partial x_j}\frac{\partial\varphi}{\partial x_i}dx\\ &\quad\quad-\int_{\mathbb{R}^n} u_{\varepsilon,\delta}^{a}L_i^{\varepsilon}\left(\frac{\partial u_\varepsilon}{\partial x_i}\right)\frac{\partial}{\partial x_j}\left(L_j^{\varepsilon}\left(\frac{\partial u_\varepsilon}{\partial x_j}\right)\right)\frac{\partial\varphi}{\partial x_i}dx\\ &\quad\quad-\int_{\mathbb{R}^n} u_{\varepsilon,\delta}^{a}L_i^{\varepsilon}\left(\frac{\partial u_\varepsilon}{\partial x_i}\right)L_j^{\varepsilon}\left(\frac{\partial u_\varepsilon}{\partial x_j}\right)\frac{\partial^2\varphi}{\partial x_i\partial x_j}dx\,. \qquad (6.24)\end{aligned}$$

We compute

$$\begin{aligned}&\int_{\mathbb{R}^n} u_{\varepsilon,\delta}^{a+b_{ij}}\frac{\partial^2}{\partial x_j\partial x_i}\left(u_{\varepsilon,\delta}^{-b_{ij}}L_i^{\varepsilon}\left(\frac{\partial u_\varepsilon}{\partial x_i}\right)\right)L_j^{\varepsilon}\left(\frac{\partial u_\varepsilon}{\partial x_j}\right)\varphi dx\\ &= \int_{\mathbb{R}^n} u_{\varepsilon,\delta}^{a+b_{ij}}\frac{\partial}{\partial x_j}\left(u_{\varepsilon,\delta}^{-b_{ij}}\frac{\partial}{\partial x_i}\left(L_i^{\varepsilon}\left(\frac{\partial u_\varepsilon}{\partial x_i}\right)\right)\right)L_j^{\varepsilon}\left(\frac{\partial u_\varepsilon}{\partial x_j}\right)\varphi dx\\ &\quad- b_{ij}\int_{\mathbb{R}^n} u_{\varepsilon,\delta}^{a+b_{ij}}\frac{\partial}{\partial x_j}\left(u_{\varepsilon,\delta}^{-b_{ij}-1}L_i^{\varepsilon}\left(\frac{\partial u_\varepsilon}{\partial x_i}\right)\frac{\partial u_\varepsilon}{\partial x_i}\right)L_j^{\varepsilon}\left(\frac{\partial u_\varepsilon}{\partial x_j}\right)\varphi dx\\ &= \int_{\mathbb{R}^n} u_{\varepsilon,\delta}^{a}\frac{\partial^2}{\partial x_j\partial x_i}\left(L_i^{\varepsilon}\left(\frac{\partial u_\varepsilon}{\partial x_i}\right)\right)L_j^{\varepsilon}\left(\frac{\partial u_\varepsilon}{\partial x_j}\right)\varphi dx\\ &\quad- b_{ij}\int_{\mathbb{R}^n} u_{\varepsilon,\delta}^{a-1}\frac{\partial}{\partial x_i}\left(L_i^{\varepsilon}\left(\frac{\partial u_\varepsilon}{\partial x_i}\right)\right)L_j^{\varepsilon}\left(\frac{\partial u_\varepsilon}{\partial x_j}\right)\frac{\partial u_\varepsilon}{\partial x_j}\varphi dx\\ &\quad- b_{ij}\int_{\mathbb{R}^n} u_{\varepsilon,\delta}^{a-1}\frac{\partial}{\partial x_j}\left(L_i^{\varepsilon}\left(\frac{\partial u_\varepsilon}{\partial x_i}\right)\frac{\partial u_\varepsilon}{\partial x_i}\right)L_j^{\varepsilon}\left(\frac{\partial u_\varepsilon}{\partial x_j}\right)\varphi dx\\ &\quad+ b_{ij}\left(b_{ij}+1\right)\int_{\mathbb{R}^n} u_{\varepsilon,\delta}^{a-2}L_i^{\varepsilon}\left(\frac{\partial u_\varepsilon}{\partial x_i}\right)\frac{\partial u_\varepsilon}{\partial x_i}L_j^{\varepsilon}\left(\frac{\partial u_\varepsilon}{\partial x_j}\right)\frac{\partial u_\varepsilon}{\partial x_j}\varphi dx \qquad (6.25)\end{aligned}$$

and

$$\begin{aligned}
&\int_{\mathbb{R}^n} u_{\varepsilon,\delta}^{a+b_{ij}-1} \frac{\partial}{\partial x_j}\left(u_{\varepsilon,\delta}^{-b_{ij}} L_i^\varepsilon\left(\frac{\partial u_\varepsilon}{\partial x_i}\right)\right)\frac{\partial u_\varepsilon}{\partial x_i} L_j^\varepsilon\left(\frac{\partial u_\varepsilon}{\partial x_j}\right)\varphi dx\\
&= \int_{\mathbb{R}^n} u_{\varepsilon,\delta}^{a-1}\frac{\partial}{\partial x_j}\left(L_i^\varepsilon\left(\frac{\partial u_\varepsilon}{\partial x_i}\right)\frac{\partial u_\varepsilon}{\partial x_i}\right)L_j^\varepsilon\left(\frac{\partial u_\varepsilon}{\partial x_j}\right)\varphi dx\\
&\quad - \int_{\mathbb{R}^n} u_{\varepsilon,\delta}^{a-1} L_i^\varepsilon\left(\frac{\partial u_\varepsilon}{\partial x_i}\right) L_j^\varepsilon\left(\frac{\partial u_\varepsilon}{\partial x_j}\right)\frac{\partial^2 u}{\partial x_i \partial x_j}\varphi dx\\
&\quad - b_{ij}\int_{\mathbb{R}^n} u_{\varepsilon,\delta}^{a-2} L_i^\varepsilon\left(\frac{\partial u_\varepsilon}{\partial x_i}\right)\frac{\partial u_\varepsilon}{\partial x_i} L_j^\varepsilon\left(\frac{\partial u_\varepsilon}{\partial x_j}\right)\frac{\partial u_\varepsilon}{\partial x_j}\varphi dx\,.
\end{aligned} \tag{6.26}$$

Then (6.18) follows from (6.15), (6.16), and (6.23)–(6.26). □

The last step in the proof of Proposition 6.1 states as follows.

Step 6.8. *Let a and b_{ij} be some real numbers, δ be a positive real number, and φ be a smooth function with compact support in $B_0(R)$. Assume that there holds $(p_j-1)\,b_{ij} = (p_i-1)\,b_{ji}$ for $i,j=1,\dots,n$. Then, for $i,j=1,\dots,n$, there holds*

$$\liminf_{\varepsilon\to 0} \mathcal{Q}_{ij}^\varepsilon\left(u_{\varepsilon,\delta},\varphi\right) \ge 0\,. \tag{6.27}$$

Proof. For ε small and for $i,j=1,\dots,n$, we compute from the definition of $\mathcal{Q}_{ij}^\varepsilon(u_{\varepsilon,\delta},\varphi)$ that

$$\begin{aligned}
\mathcal{Q}_{ij}^\varepsilon\left(u_{\varepsilon,\delta},\varphi\right) = \int_{\mathbb{R}^n} u_{\varepsilon,\delta}^{a-2}&\left(\varepsilon^2+\left(\frac{\partial u_\varepsilon}{\partial x_i}\right)^2\right)^{\frac{p_i-4}{2}}\left(\varepsilon^2+\left(\frac{\partial u_\varepsilon}{\partial x_j}\right)^2\right)^{\frac{p_j-4}{2}}\\
&\times\left(X_{ij}^\varepsilon\left(u_{\varepsilon,\delta},\nabla u_\varepsilon\right)+Y_{ij}^\varepsilon\left(u_{\varepsilon,\delta},\nabla u_\varepsilon\right)+Z_{ij}^\varepsilon\left(u_{\varepsilon,\delta},\nabla u_\varepsilon\right)\right)\varphi dx\,,
\end{aligned} \tag{6.28}$$

where

$$\begin{aligned}
X_{ij}^\varepsilon\left(u_{\varepsilon,\delta},\nabla u_\varepsilon\right) = &\left(b_{ij}b_{ji}\left(\frac{\partial u_\varepsilon}{\partial x_i}\right)^2\left(\frac{\partial u_\varepsilon}{\partial x_j}\right)^2 + (p_i-1)(p_j-1)\,u_{\varepsilon,\delta}^2\left(\frac{\partial^2 u_\varepsilon}{\partial x_i\partial x_j}\right)^2\right.\\
&\left.-\left((p_j-1)\,b_{ij}+(p_i-1)\,b_{ji}\right)u_{\varepsilon,\delta}\frac{\partial u_\varepsilon}{\partial x_i}\frac{\partial u_\varepsilon}{\partial x_j}\frac{\partial^2 u_\varepsilon}{\partial x_i\partial x_j}\right)\left(\frac{\partial u_\varepsilon}{\partial x_i}\right)^2\left(\frac{\partial u_\varepsilon}{\partial x_j}\right)^2,
\end{aligned}$$

$$\begin{aligned}
Y_{ij}^\varepsilon\left(u_{\varepsilon,\delta},\nabla u_\varepsilon\right) = &\, b_{ij}b_{ji}\varepsilon^2\left(\varepsilon^2+\left(\frac{\partial u_\varepsilon}{\partial x_i}\right)^2+\left(\frac{\partial u_\varepsilon}{\partial x_j}\right)^2\right)\left(\frac{\partial u_\varepsilon}{\partial x_i}\right)^2\left(\frac{\partial u_\varepsilon}{\partial x_j}\right)^2\\
&+\varepsilon^2 u_{\varepsilon,\delta}^2\left(\varepsilon^2+(p_i-1)\left(\frac{\partial u_\varepsilon}{\partial x_i}\right)^2+(p_j-1)\left(\frac{\partial u_\varepsilon}{\partial x_j}\right)^2\right)\left(\frac{\partial^2 u_\varepsilon}{\partial x_i\partial x_j}\right)^2,
\end{aligned}$$

and

$$\begin{aligned}
Z_{ij}^\varepsilon\left(u_{\varepsilon,\delta},\nabla u_\varepsilon\right) = -\varepsilon^2 u_{\varepsilon,\delta}&\Bigg(\left(b_{ij}+b_{ji}\right)\varepsilon^2+\left(b_{ij}+(p_i-1)\,b_{ji}\right)\left(\frac{\partial u_\varepsilon}{\partial x_i}\right)^2\\
&+\left((p_j-1)\,b_{ij}+b_{ji}\right)\left(\frac{\partial u_\varepsilon}{\partial x_j}\right)^2\Bigg)\frac{\partial u_\varepsilon}{\partial x_i}\frac{\partial u_\varepsilon}{\partial x_j}\frac{\partial^2 u_\varepsilon}{\partial x_i\partial x_j}\,.
\end{aligned}$$

Since there holds $(p_j-1)\,b_{ij}=(p_i-1)\,b_{ji}$ for $i,j=1,\dots,n$, we get

$$X_{ij}^\varepsilon\left(u_{\varepsilon,\delta},\nabla u_\varepsilon\right)=\frac{p_i-1}{p_j-1}\left(b_{ji}\frac{\partial u_\varepsilon}{\partial x_i}\frac{\partial u_\varepsilon}{\partial x_j}-(p_j-1)\,u_{\varepsilon,\delta}\frac{\partial^2u_\varepsilon}{\partial x_i\partial x_j}\right)^2\times\left(\frac{\partial u_\varepsilon}{\partial x_i}\right)^2\left(\frac{\partial u_\varepsilon}{\partial x_j}\right)^2\ge 0\quad\text{in }\operatorname{Supp}(\varphi)\,. \tag{6.29}$$

We also get $Y_{ij}^\varepsilon\left(u_{\varepsilon,\delta},\nabla u_\varepsilon\right)\ge 0$ in $\operatorname{Supp}(\varphi)$. Since $u_{\varepsilon,\delta}\ge\delta/2$ in $\operatorname{Supp}(\varphi)$ for ε small, since $p_i\ge 2$ for $i=1,\dots,n$, by Steps 6.3 and 6.4, we finally compute

$$\begin{aligned}&\left|\int_{\mathbb{R}^n}u_{\varepsilon,\delta}^{a-2}\left(\varepsilon^2+\left(\frac{\partial u_\varepsilon}{\partial x_i}\right)^2\right)^{\frac{p_i-4}{2}}\left(\varepsilon^2+\left(\frac{\partial u_\varepsilon}{\partial x_j}\right)^2\right)^{\frac{p_j-4}{2}}Z_{ij}^\varepsilon\left(u_{\varepsilon,\delta},\nabla u_\varepsilon\right)\varphi dx\right|\\&\quad\le C\varepsilon\int_{\mathbb{R}^n}u_{\varepsilon,\delta}^{a-1}\left(\varepsilon^2+\left(\frac{\partial u_\varepsilon}{\partial x_i}\right)^2\right)^{\frac{p_i-2}{2}}\left(\varepsilon^2+\left(\frac{\partial u_\varepsilon}{\partial x_j}\right)^2\right)^{\frac{p_j-2}{2}}\\&\qquad\times\left(\varepsilon+\left|\frac{\partial u_\varepsilon}{\partial x_i}\right|+\left|\frac{\partial u_\varepsilon}{\partial x_j}\right|\right)\left|\frac{\partial^2u_\varepsilon}{\partial x_i\partial x_j}\varphi\right|dx=\mathrm{O}(\varepsilon)\end{aligned} \tag{6.30}$$

as $\varepsilon\to 0$. Then (6.27) follows from (6.28)–(6.30). □

We can now prove Proposition 6.1 by using the above preliminary steps.

End of proof of Proposition 6.1. For ε small, given some real numbers a, b_{ij}, c_{ij}, α_{ij}, β_{ij}, γ_i, δ_i, μ_{ij}, and σ_{ij} satisfying

$$(p_j-1)\,b_{ij}=(p_i-1)\,b_{ji}\,,\quad \alpha_{ij}=-\alpha_{ji}\,,\quad\text{and }\beta_{ij}=-\beta_{ji}\,, \tag{6.31}$$

we let

$$\begin{aligned}\Theta_\varepsilon\left(u_{\varepsilon,\delta},\varphi\right)=\sum_{i=1}^n\Bigg(\sum_{j\ne i}\Big(&\mu_{ij}^2\widetilde{\mathcal{P}}_{ij}^\varepsilon\left(u_{\varepsilon,\delta},\varphi\right)+\mu_{ij}\mu_{ji}\mathcal{P}_{ij}^\varepsilon\left(u_{\varepsilon,\delta},\varphi\right)+\sigma_{ij}^2\mathcal{Q}_{ij}^\varepsilon\left(u_{\varepsilon,\delta},\varphi\right)\\&+\alpha_{ij}\mathcal{E}_{ij}^\varepsilon\left(u_{\varepsilon,\delta},\varphi\right)+\beta_{ij}\mathcal{F}_{ij}^\varepsilon\left(u_{\varepsilon,\delta},\varphi\right)\Big)+\gamma_i\mathcal{G}_i^\varepsilon\left(u_{\varepsilon,\delta},\varphi\right)+\delta_i\mathcal{H}_i^\varepsilon\left(u_{\varepsilon,\delta},\varphi\right)\Bigg),\end{aligned} \tag{6.32}$$

where $\mathcal{E}_{ij}^\varepsilon\left(u_{\varepsilon,\delta},\varphi\right)$, $\mathcal{F}_{ij}^\varepsilon\left(u_{\varepsilon,\delta},\varphi\right)$, $\mathcal{G}_i^\varepsilon\left(u_{\varepsilon,\delta},\varphi\right)$, and $\mathcal{H}_i^\varepsilon\left(u_{\varepsilon,\delta},\varphi\right)$ are as in Step 6.6, where $\mathcal{P}_{ij}^\varepsilon\left(u_{\varepsilon,\delta},\varphi\right)$ and $\mathcal{Q}_{ij}^\varepsilon\left(u_{\varepsilon,\delta},\varphi\right)$ are as in Step 6.7, and where

$$\widetilde{\mathcal{P}}_{ij}^\varepsilon\left(u_{\varepsilon,\delta},\varphi\right)=\int_{\mathbb{R}^n}u_{\varepsilon,\delta}^{a+2c_{ij}}\frac{\partial}{\partial x_i}\left(u_{\varepsilon,\delta}^{-c_{ij}}L_i^\varepsilon\left(\frac{\partial u_\varepsilon}{\partial x_i}\right)\right)\frac{\partial}{\partial x_i}\left(u_{\varepsilon,\delta}^{-c_{ij}}L_i^\varepsilon\left(\frac{\partial u_\varepsilon}{\partial x_i}\right)\right)\varphi dx\,.$$

We note that

$$\begin{aligned}&\sum_{i=1}^n\sum_{j\ne i}\left(\mu_{ij}^2\widetilde{\mathcal{P}}_{ij}^\varepsilon\left(u_{\varepsilon,\delta},\varphi\right)+\mu_{ij}\mu_{ji}\mathcal{P}_{ij}^\varepsilon\left(u_{\varepsilon,\delta},\varphi\right)\right)\\&=\sum_{i=1}^n\sum_{j=1}^{i-1}\int_{\mathbb{R}^n}u_{\varepsilon,\delta}^a\left(\mu_{ij}u_{\varepsilon,\delta}^{c_{ij}}\frac{\partial}{\partial x_i}\left(u_{\varepsilon,\delta}^{-c_{ij}}L_i^\varepsilon\left(\frac{\partial u_\varepsilon}{\partial x_i}\right)\right)+\mu_{ji}u_{\varepsilon,\delta}^{c_{ji}}\frac{\partial}{\partial x_j}\left(u_{\varepsilon,\delta}^{-c_{ji}}L_j^\varepsilon\left(\frac{\partial u_\varepsilon}{\partial x_j}\right)\right)\right)^2\varphi dx\ge 0\,.\end{aligned} \tag{6.33}$$

It follows from (6.33) and from Steps 6.6 and 6.8 that

$$\liminf_{\varepsilon\to 0}\Theta_\varepsilon\left(u_{\varepsilon,\delta},\varphi\right)\ge 0\,. \tag{6.34}$$

We can develop (6.32) by using (6.17) and (6.18). We then get

$$\begin{aligned}
\Theta_\varepsilon\left(u_{\varepsilon,\delta},\varphi\right)=&\sum_{i=1}^n\int_{\mathbb{R}^n}u_{\varepsilon,\delta}^{a-1}\operatorname{div}\left(R_i^\varepsilon\left(\nabla u_\varepsilon\right)\right)L_i^\varepsilon\left(\frac{\partial u_\varepsilon}{\partial x_i}\right)\frac{\partial u_\varepsilon}{\partial x_i}\varphi dx\\
&-\sum_{i=1}^n\int_{\mathbb{R}^n}u_{\varepsilon,\delta}^{a}\frac{\partial\left(\operatorname{div}\left(S_i^\varepsilon\left(\nabla u_\varepsilon\right)\right)\right)}{\partial x_i}L_i^\varepsilon\left(\frac{\partial u_\varepsilon}{\partial x_i}\right)\varphi dx\\
&+\sum_{i=1}^n\int_{\mathbb{R}^n}u_{\varepsilon,\delta}^{a}\operatorname{div}\left(T_i^\varepsilon\left(\nabla u_\varepsilon\right)\right)L_i^\varepsilon\left(\frac{\partial u_\varepsilon}{\partial x_i}\right)\frac{\partial\varphi}{\partial x_i}dx\\
&+\sum_{i=1}^n M_i\int_{\mathbb{R}^n}u_{\varepsilon,\delta}^{a-2}\left(L_i^\varepsilon\left(\frac{\partial u_\varepsilon}{\partial x_i}\right)\right)^2\left(\frac{\partial u_\varepsilon}{\partial x_i}\right)^2\varphi dx\\
&+\sum_{i=1}^n\sum_{j=1}^{i-1}N_{ij}\int_{\mathbb{R}^n}u_{\varepsilon,\delta}^{a-2}L_i^\varepsilon\left(\frac{\partial u_\varepsilon}{\partial x_i}\right)\frac{\partial u_\varepsilon}{\partial x_i}L_j^\varepsilon\left(\frac{\partial u_\varepsilon}{\partial x_j}\right)\frac{\partial u_\varepsilon}{\partial x_j}\varphi dx\\
&+\sum_{i=1}^n\left(a\gamma_i+\left(p_i-1\right)\delta_i\right)\int_{\mathbb{R}^n}u_{\varepsilon,\delta}^{a-1}\left(L_i^\varepsilon\left(\frac{\partial u_\varepsilon}{\partial x_i}\right)\right)^2\frac{\partial u_\varepsilon}{\partial x_i}\frac{\partial\varphi}{\partial x_i}dx\\
&+\sum_{i=1}^n\sum_{j\neq i}\left(p_i\beta_{ij}+\sigma_{ij}^2G_{ij}\right)\int_{\mathbb{R}^n}u_{\varepsilon,\delta}^{a-1}L_i^\varepsilon\left(\frac{\partial u_\varepsilon}{\partial x_i}\right)L_j^\varepsilon\left(\frac{\partial u_\varepsilon}{\partial x_j}\right)\frac{\partial u_\varepsilon}{\partial x_j}\frac{\partial\varphi}{\partial x_i}dx\\
&+\sum_{i=1}^n\sum_{j=1}^{i-1}\left(\sigma_{ij}^2+\sigma_{ji}^2\right)\int_{\mathbb{R}^n}u_{\varepsilon,\delta}^{a}L_i^\varepsilon\left(\frac{\partial u_\varepsilon}{\partial x_i}\right)L_j^\varepsilon\left(\frac{\partial u_\varepsilon}{\partial x_j}\right)\frac{\partial^2\varphi}{\partial x_i\partial x_j}dx\\
&+\sum_{i=1}^n\gamma_i\int_{\mathbb{R}^n}u_{\varepsilon,\delta}^{a}\left(L_i^\varepsilon\left(\frac{\partial u_\varepsilon}{\partial x_i}\right)\right)^2\frac{\partial^2\varphi}{\partial x_i^2}dx+\mathrm{O}\left(\varepsilon\right)
\end{aligned}\tag{6.35}$$

as $\varepsilon\to 0$, where E_{ij}, F_{ij}, and G_{ij} are as in (6.19)–(6.21) and where

$$M_i=\left(p_i-1\right)\left(a-1\right)\delta_i+\sum_{k\neq i}c_{ik}^2\mu_{ik}^2$$

and

$$N_{ij}=\left(p_i-p_j\right)\left(a-1\right)\beta_{ij}+2c_{ij}c_{ji}\mu_{ij}\mu_{ji}+\sigma_{ij}^2E_{ij}+\sigma_{ji}^2E_{ji}$$

for all distinct $i,j=1,\dots,n$. In (6.35), $R_i^\varepsilon\left(\nabla u_\varepsilon\right)=\left(R_{ij}^\varepsilon\left(\partial u_\varepsilon/\partial x_j\right)\right)_{j=1,\dots,n}$ is given by

$$R_{ii}^\varepsilon\left(\partial u_\varepsilon/\partial x_i\right)=\Big(\left(2p_i-1\right)\delta_i-\sum_{k\neq i}\left(a+2c_{ik}\right)\mu_{ik}^2\Big)L_i^\varepsilon\left(\frac{\partial u_\varepsilon}{\partial x_i}\right)\tag{6.36}$$

and

$$R_{ij}^\varepsilon\left(\partial u_\varepsilon/\partial x_j\right)=\left(a\alpha_{ij}+p_j\beta_{ji}-\left(a+2c_{ij}\right)\mu_{ij}\mu_{ji}+\sigma_{ji}^2F_{ji}\right)L_j^\varepsilon\left(\frac{\partial u_\varepsilon}{\partial x_j}\right)\tag{6.37}$$

for all distinct $i,j=1,\dots,n$, and $S_i^\varepsilon\left(\nabla u_\varepsilon\right)=\left(S_{ij}^\varepsilon\left(\partial u_\varepsilon/\partial x_j\right)\right)_{j=1,\dots,n}$ is given by

$$S_{ii}^\varepsilon\left(\partial u_\varepsilon/\partial x_i\right)=-\sum_{k\neq i}\mu_{ik}^2L_i^\varepsilon\left(\frac{\partial u_\varepsilon}{\partial x_i}\right)\tag{6.38}$$

and

$$S_{ij}^\varepsilon\left(\partial u_\varepsilon/\partial x_j\right)=\left(\alpha_{ij}-\mu_{ij}\mu_{ji}-\sigma_{ji}^2\right)L_j^\varepsilon\left(\frac{\partial u_\varepsilon}{\partial x_j}\right)\tag{6.39}$$

for all distinct $i, j = 1, \ldots, n$. In a similar way, $T_i^\varepsilon(\nabla u_\varepsilon) = \big(T_{ij}^\varepsilon(\partial u_\varepsilon/\partial x_j)\big)_{j=1,\ldots,n}$ is given by

$$T_{ii}^\varepsilon(\partial u_\varepsilon/\partial x_i) = \big(2\gamma_i - \sum_{k\neq i}\mu_{ik}^2\big)L_i^\varepsilon\left(\frac{\partial u_\varepsilon}{\partial x_i}\right) \tag{6.40}$$

and

$$T_{ij}^\varepsilon(\partial u_\varepsilon/\partial x_j) = \big(\alpha_{ij} - \mu_{ij}\mu_{ji} + \sigma_{ij}^2\big)L_j^\varepsilon\left(\frac{\partial u_\varepsilon}{\partial x_j}\right) \tag{6.41}$$

for all distinct $i, j = 1, \ldots, n$. For any real numbers a, b_{ij}, c_{ij}, α_{ij}, β_{ij}, γ_i, δ_i, μ_{ij}, σ_{ij}, r_i, s_i, and t_i, we now consider the system consisting of equation (6.31) together with the equations $R_i^\varepsilon(\nabla u_\varepsilon) = r_i L_\varepsilon(\nabla u_\varepsilon)$, $S_i^\varepsilon(\nabla u_\varepsilon) = s_i L_\varepsilon(\nabla u_\varepsilon)$, and $T_i^\varepsilon(\nabla u_\varepsilon) = t_i L_\varepsilon(\nabla u_\varepsilon)$ for $i = 1, \ldots, n$. Using (6.36)–(6.41), we can eliminate the unknowns r_i, s_i, and t_i for $i = 1, \ldots, n$, and state our system as follows

$$\begin{cases} (p_j-1)\,b_{ij} = (p_i-1)\,b_{ji}\,, \quad \alpha_{ij} = -\alpha_{ji}\,, \quad \beta_{ij} = -\beta_{ji}\,, \\ a\alpha_{ij} + p_j\beta_{ji} - (a+2c_{ij})\,\mu_{ij}\mu_{ji} + \sigma_{ji}^2F_{ji} = (2p_i-1)\,\delta_i - \sum_{k\neq i}(a+2c_{ik})\,\mu_{ik}^2\,, \\ \alpha_{ij} - \mu_{ij}\mu_{ji} - \sigma_{ji}^2 = -\sum_{k\neq i}\mu_{ik}^2\,, \\ \alpha_{ij} - \mu_{ij}\mu_{ji} + \sigma_{ij}^2 = 2\gamma_i - \sum_{k\neq i}\mu_{ik}^2\,, \end{cases}$$

where the equations have to be satisfied for all distinct $i, j = 1, \ldots, n$. Easy manipulations lead to the following equivalent system

$$\begin{cases} (p_j-1)\,b_{ij} = (p_i-1)\,b_{ji}\,, \quad \beta_{ij} = -\beta_{ji}\,, \\ \sum_{k\neq i}\mu_{ik}^2 + \sum_{k\neq j}\mu_{jk}^2 - 2\mu_{ij}\mu_{ji} = \sigma_{ij}^2 + \sigma_{ji}^2 = 2\gamma_i\,, \\ \alpha_{ij} = \sum_{k\neq j}\mu_{jk}^2 - \mu_{ij}\mu_{ji} - \sigma_{ij}^2\,, \\ \delta_i = \dfrac{1}{2p_i-1}\big(p_j\beta_{ji} - 2c_{ij}\mu_{ij}\mu_{ji} + 2\sum_{k\neq i}c_{ik}\mu_{ik}^2 + (a+F_{ji})\,\sigma_{ji}^2\big). \end{cases} \tag{6.42}$$

In particular, the second line in (6.42) implies that there holds $\gamma_i = \gamma$, where γ does not depend on the index i. By the changes of unknowns

$$\lambda_{ij} = \beta_{ij} + \frac{1}{2p_ip_j}\left((p_i+p_j-1)\,a + 2\,(p_j-1)\,b_{ij}\right)\big(\sigma_{ij}^2 - \sigma_{ji}^2\big) \tag{6.43}$$

and

$$\nu_i = p_i\,(2p_i-1)\,\delta_i - p_i\sum_{k\neq i}(a+2c_{ik})\,\mu_{ik}^2\,, \tag{6.44}$$

and by eliminating α_{ij}, β_{ij}, δ_i, and σ_{ij} for $i, j = 1, \ldots, n$, the system (6.42) can be written as

$$\begin{cases} (p_j-1)\,b_{ij} = (p_i-1)\,b_{ji}\,, \quad \lambda_{ij} = -\lambda_{ji}\,, \quad \sum_{k\neq i}\mu_{ik}^2 + \sum_{k\neq j}\mu_{jk}^2 - 2\mu_{ij}\mu_{ji} = 2\gamma\,, \\ \nu_i = \big(\gamma\,(p_i+p_j-1) - p_i\sum_{k\neq i}\mu_{ik}^2\big)a + 2\gamma\,(p_j-1)\,b_{ij} - 2p_ic_{ij}\mu_{ij}\mu_{ji} - p_ip_j\lambda_{ij}\,. \end{cases}$$

Multiplying $\Theta_\varepsilon\left(u_{\varepsilon,\delta},\varphi\right)$ by $1/\gamma$ for ε small, we can fix $\gamma=1/2$, and we then recover equations (5.1) and (5.2). In particular, if the real numbers a, b_{ij}, c_{ij}, μ_{ij}, λ_{ij}, and ν_i satisfy (5.1) and (5.2), then for $i=1,\dots,n$, we get

$$R_i^\varepsilon\left(\nabla u_\varepsilon\right)=\frac{\nu_i}{p_i}L_\varepsilon\left(\nabla u_\varepsilon\right),\quad S_i^\varepsilon\left(\nabla u_\varepsilon\right)=-\sum_{k\neq i}\mu_{ik}^2L_\varepsilon\left(\nabla u_\varepsilon\right),$$

and

$$T_i^\varepsilon\left(\nabla u_\varepsilon\right)=\Big(1-\sum_{k\neq i}\mu_{ik}^2\Big)L_\varepsilon\left(\nabla u_\varepsilon\right).$$

Taking into account that u_ε satisfies equation (6.4), that $u_{\varepsilon,\delta}\ge\delta/2$ on $\operatorname{Supp}(\varphi)$ for ε small, and that $p_i\ge2$ for $i=1,\dots,n$, by Step 6.5, by (6.34), and by (6.42)–(6.44) passing to the limit into (6.35) as $\varepsilon\to0$ gives (6.1). ☐

7. Proof of Theorem 5.1

In this section, we let $\overrightarrow{p}=(p_1,\dots,p_n)$ and q be such that $2\le p_i<q$ for $i=1,\dots,n$, and we assume that there exist some real numbers a, b_{ij}, c_{ij}, λ_{ij}, μ_{ij}, and ν_i satisfying (5.1)–(5.4). We now prove Theorem 5.1 by using Proposition 6.1.

Proof of Theorem 5.1. We let u be a nonnegative solution in $W^{1,\overrightarrow{p}}_{\text{loc}}\left(\mathbb{R}^n\right)\cap L^\infty_{\text{loc}}\left(\mathbb{R}^n\right)$ of equation (1.3). Changing, if necessary, the variable, we may assume that the positive constant λ in (1.3) is equal to 1. For any positive real number δ, we set $u_\delta=u+\delta$. We let φ be a nonnegative smooth function with compact support in $\mathbb{R}^n$ to be chosen later on so that any of the integrals below are finite. We begin with applying Young's inequality in order to estimate some of the terms in equation (6.1). For $i,j=1,\dots,n$ and for any $C>0$, we get

$$\int_{\mathbb{R}^n}u_\delta^a u^{q-1}\left|\frac{\partial u}{\partial x_i}\right|^{p_i-1}\left|\frac{\partial\varphi}{\partial x_i}\right|dx\le\frac{C}{p_i}\int_{\mathbb{R}^n}u_\delta^{a+p_i-1}u^{q-1}\left|\frac{\partial\varphi}{\partial x_i}\right|^{p_i}\varphi^{1-p_i}dx+\frac{p_i-1}{p_i}C^{\frac{-1}{p_i-1}}\int_{\mathbb{R}^n}u_\delta^{a-1}u^{q-1}\left|\frac{\partial u}{\partial x_i}\right|^{p_i}\varphi dx\,,\quad(7.1)$$

$$\int_{\mathbb{R}^n}u_\delta^{a-1}\left|\frac{\partial u}{\partial x_i}\right|^{p_i-1}\left|\frac{\partial u}{\partial x_j}\right|^{p_j}\left|\frac{\partial\varphi}{\partial x_i}\right|dx\le\frac{C}{p_i}\int_{\mathbb{R}^n}u_\delta^{a+p_i-2}\left|\frac{\partial u}{\partial x_j}\right|^{p_j}\left|\frac{\partial\varphi}{\partial x_i}\right|^{p_i}\varphi^{1-p_i}dx+\frac{p_i-1}{p_i}C^{\frac{-1}{p_i-1}}\int_{\mathbb{R}^n}u_\delta^{a-2}\left|\frac{\partial u}{\partial x_i}\right|^{p_i}\left|\frac{\partial u}{\partial x_j}\right|^{p_j}\varphi dx\,,\quad(7.2)$$

$$\int_{\mathbb{R}^n}u_\delta^{a}\left|\frac{\partial u}{\partial x_i}\right|^{2p_i-2}\left|\frac{\partial^2\varphi}{\partial x_i^2}\right|dx\le\frac{C}{p_i}\int_{\mathbb{R}^n}u_\delta^{a+2p_i-2}\left|\frac{\partial^2\varphi}{\partial x_i^2}\right|^{p_i}\varphi^{1-p_i}dx+\frac{p_i-1}{p_i}C^{\frac{-1}{p_i-1}}\int_{\mathbb{R}^n}u_\delta^{a-2}\left|\frac{\partial u}{\partial x_i}\right|^{2p_i}\varphi dx\,,\quad(7.3)$$

and

$$\int_{\mathbb{R}^n}u_\delta^{a}\left|\frac{\partial u}{\partial x_i}\right|^{p_i-1}\left|\frac{\partial u}{\partial x_j}\right|^{p_j-1}\left|\frac{\partial^2\varphi}{\partial x_i\partial x_j}\right|dx\le\frac{C}{p_i}\int_{\mathbb{R}^n}u_\delta^{a+p_i-1}\left|\frac{\partial u}{\partial x_j}\right|^{p_j-1}\left|\frac{\partial^2\varphi}{\partial x_i\partial x_j}\right|^{p_i}\left|\frac{\partial\varphi}{\partial x_j}\right|^{1-p_i}dx+\frac{p_i-1}{p_i}C^{\frac{-1}{p_i-1}}\int_{\mathbb{R}^n}u_\delta^{a-1}\left|\frac{\partial u}{\partial x_i}\right|^{p_i}\left|\frac{\partial u}{\partial x_j}\right|^{p_j-1}\left|\frac{\partial\varphi}{\partial x_j}\right|dx\,.\quad(7.4)$$

Still applying Young's inequality, we estimate the first term on the right-hand side of (7.4). For $i,j=1,\dots,n$ and for any $C>0$, there holds

$$\begin{aligned}\int_{\mathbb{R}^n} u_\delta^{a+p_i-1}\left|\frac{\partial u}{\partial x_j}\right|^{p_j-1}\left|\frac{\partial^2\varphi}{\partial x_i\partial x_j}\right|^{p_i}\left|\frac{\partial\varphi}{\partial x_j}\right|^{1-p_i}dx\\ \le\frac{C}{p_j}\int_{\mathbb{R}^n} u_\delta^{a+p_i+p_j-2}\left|\frac{\partial^2\varphi}{\partial x_i\partial x_j}\right|^{p_ip_j}\left|\frac{\partial\varphi}{\partial x_i}\right|^{-p_i(p_j-1)}\left|\frac{\partial\varphi}{\partial x_j}\right|^{-p_j(p_i-1)}\varphi^{(p_i-1)(p_j-1)}dx\\ +\frac{p_j-1}{p_j}C^{\frac{-1}{p_j-1}}\int_{\mathbb{R}^n} u_\delta^{a+p_i-2}\left|\frac{\partial u}{\partial x_j}\right|^{p_j}\left|\frac{\partial\varphi}{\partial x_i}\right|^{p_i}\varphi^{1-p_i}dx\,. \quad (7.5)\end{aligned}$$

Moreover, for $i=1,\dots,n$, we get

$$\begin{aligned}\int_{\mathbb{R}^n} u_\delta^{a-1}\Big(\frac{\nu_i}{p_i}u^{q-1}-(q-1)\sum_{k\neq i}\mu_{ik}^2u_\delta u^{q-2}\Big)\left|\frac{\partial u}{\partial x_i}\right|^{p_i}\varphi dx \quad (7.6)\\ =\Big(\frac{\nu_i}{p_i}-(q-1)\sum_{k\neq i}\mu_{ik}^2\Big)\int_{\mathbb{R}^n} u_\delta^{a-1}u^{q-1}\left|\frac{\partial u}{\partial x_i}\right|^{p_i}\varphi dx-\delta\,(q-1)\sum_{k\neq i}\mu_{ik}^2\int_{\mathbb{R}^n} u_\delta^{a-1}u^{q-2}\left|\frac{\partial u}{\partial x_i}\right|^{p_i}\varphi dx.\end{aligned}$$

Multiplying (1.3) by $u_\delta^a u^{q-1}\varphi$ and integrating by parts yield

$$\begin{aligned}\int_{\mathbb{R}^n} u_\delta^a u^{2q-2}\varphi dx=\sum_{i=1}^n\Bigg(a\int_{\mathbb{R}^n} u_\delta^{a-1}u^{q-1}\left|\frac{\partial u}{\partial x_i}\right|^{p_i}\varphi dx\\ +(q-1)\int_{\mathbb{R}^n} u_\delta^{a-1}u^{q-2}\left|\frac{\partial u}{\partial x_i}\right|^{p_i}\varphi dx+\int_{\mathbb{R}^n} u_\delta^a u^{q-1}\left|\frac{\partial u}{\partial x_i}\right|^{p_i-2}\frac{\partial u}{\partial x_i}\frac{\partial\varphi}{\partial x_i}dx\Bigg). \quad (7.7)\end{aligned}$$

On the other hand, for $i=1,\dots,n$ and for any $C>0$, Young's inequality gives

$$\int_{\mathbb{R}^n} u_\delta^{a-1}u^{q-2}\left|\frac{\partial u}{\partial x_i}\right|^{p_i}\varphi dx\le\frac{C}{2}\int_{\mathbb{R}^n} u_\delta^a u^{2q-4}\varphi dx+\frac{1}{2C}\int_{\mathbb{R}^n} u_\delta^{a-2}\left|\frac{\partial u}{\partial x_i}\right|^{2p_i}\varphi dx\,. \quad (7.8)$$

By (5.4), (6.1), and (7.1)–(7.8), it follows that there exists a positive constant C independent of u, δ, and φ such that

$$\begin{aligned}\int_{\mathbb{R}^n} u_\delta^a u^{2q-2}\varphi dx\le C\sum_{i=1}^n\Bigg(\delta\int_{\mathbb{R}^n} u_\delta^a u^{2q-4}\varphi dx\\ +\int_{\mathbb{R}^n} u_\delta^{a+p_i-1}u^{q-1}\left|\frac{\partial\varphi}{\partial x_i}\right|^{p_i}\varphi^{1-p_i}dx+\sum_{j=1}^n\int_{\mathbb{R}^n} u_\delta^{a+p_i-2}\left|\frac{\partial u}{\partial x_j}\right|^{p_j}\left|\frac{\partial\varphi}{\partial x_i}\right|^{p_i}\varphi^{1-p_i}dx\\ +\sum_{j=1}^{i-1}\int_{\mathbb{R}^n} u_\delta^{a+p_i+p_j-2}\left|\frac{\partial^2\varphi}{\partial x_i\partial x_j}\right|^{p_ip_j}\left|\frac{\partial\varphi}{\partial x_i}\right|^{-p_i(p_j-1)}\left|\frac{\partial\varphi}{\partial x_j}\right|^{-p_j(p_i-1)}\varphi^{(p_i-1)(p_j-1)}dx\\ +\int_{\mathbb{R}^n} u_\delta^{a+2p_i-2}\left|\frac{\partial^2\varphi}{\partial x_i^2}\right|^{p_i}\varphi^{1-p_i}dx\Bigg). \quad (7.9)\end{aligned}$$

For $i=1,\dots,n$, we let g_i be the function defined on $\mathbb{R}_+$ by

$$g_i(s)=\begin{cases}\dfrac{s^{a+p_i-1}}{a+p_i-1} & \text{if } a+p_i-1\neq 0,\\ \ln s & \text{if } a+p_i-1=0.\end{cases}$$

Multiplying (1.3) by $g_i(u_\delta)\left|\partial\varphi/\partial x_i\right|^{p_i}\varphi^{1-p_i}$ and integrating by parts yield

$$\begin{aligned}\int_{\mathbb{R}^n} g_i(u_\delta)\,u^{q-1}\left|\frac{\partial\varphi}{\partial x_i}\right|^{p_i}\varphi^{1-p_i}dx &= \sum_{j=1}^{n}\int_{\mathbb{R}^n} u_\delta^{a+p_i-2}\left|\frac{\partial u}{\partial x_j}\right|^{p_j}\left|\frac{\partial\varphi}{\partial x_i}\right|^{p_i}\varphi^{1-p_i}dx\\ &\quad-(p_i-1)\sum_{j=1}^{n}\int_{\mathbb{R}^n} g_i(u_\delta)\left|\frac{\partial u}{\partial x_j}\right|^{p_j-2}\frac{\partial u}{\partial x_j}\left|\frac{\partial\varphi}{\partial x_i}\right|^{p_i}\frac{\partial\varphi}{\partial x_j}\varphi^{-p_i}dx\\ &\quad+p_i\sum_{j=1}^{i-1}\int_{\mathbb{R}^n} g_i(u_\delta)\left|\frac{\partial u}{\partial x_j}\right|^{p_j-2}\frac{\partial u}{\partial x_j}\frac{\partial^2\varphi}{\partial x_i\partial x_j}\left|\frac{\partial\varphi}{\partial x_i}\right|^{p_i-2}\frac{\partial\varphi}{\partial x_i}\varphi^{1-p_i}dx\,. \end{aligned}\tag{7.10}$$

For $i,j=1,\dots,n$ and for any $C>0$, Young's inequality gives

$$\begin{aligned}&\int_{\mathbb{R}^n}\left|g_i(u_\delta)\right|\left|\frac{\partial u}{\partial x_j}\right|^{p_j-1}\left|\frac{\partial\varphi}{\partial x_i}\right|^{p_i}\left|\frac{\partial\varphi}{\partial x_j}\right|\varphi^{-p_i}dx\\ &\quad\le\frac{C}{p_j}\int_{\mathbb{R}^n}\left|g_i(u_\delta)\right|^{p_j}u_\delta^{-(p_j-1)(a+p_i-2)}\left|\frac{\partial\varphi}{\partial x_i}\right|^{p_i}\left|\frac{\partial\varphi}{\partial x_j}\right|^{p_j}\varphi^{1-p_i-p_j}dx\\ &\qquad+\frac{p_j-1}{p_j}C^{\frac{-1}{p_j-1}}\int_{\mathbb{R}^n}u_\delta^{a+p_i-2}\left|\frac{\partial u}{\partial x_j}\right|^{p_j}\left|\frac{\partial\varphi}{\partial x_i}\right|^{p_i}\varphi^{1-p_i}dx\end{aligned}\tag{7.11}$$

and

$$\begin{aligned}&\int_{\mathbb{R}^n}\left|g_i(u_\delta)\right|\left|\frac{\partial u}{\partial x_j}\right|^{p_j-1}\left|\frac{\partial^2\varphi}{\partial x_i\partial x_j}\right|\left|\frac{\partial\varphi}{\partial x_i}\right|^{p_i-1}\varphi^{1-p_i}dx\\ &\quad\le\frac{C}{p_j}\int_{\mathbb{R}^n}\left|g_i(u_\delta)\right|^{p_j}u_\delta^{-(p_j-1)(a+p_i-2)}\left|\frac{\partial^2\varphi}{\partial x_i\partial x_j}\right|^{p_j}\left|\frac{\partial\varphi}{\partial x_i}\right|^{p_i-p_j}\varphi^{1-p_i}dx\\ &\qquad+\frac{p_j-1}{p_j}C^{\frac{-1}{p_j-1}}\int_{\mathbb{R}^n}u_\delta^{a+p_i-2}\left|\frac{\partial u}{\partial x_j}\right|^{p_j}\left|\frac{\partial\varphi}{\partial x_i}\right|^{p_i}\varphi^{1-p_i}dx\,.\end{aligned}\tag{7.12}$$

We let θ be a positive real number to be chosen small later on. We then get a positive constant C_θ such that for any $s>0$ and for $i,j=1,\dots,n$, there holds

$$s^{a+p_i-1}+\left|g_i(s)\right|+\left|g_i(s)\right|^{p_j}s^{-(p_j-1)(a+p_i-1)}\le C_\theta s^{a+p_i-1}h_\theta(s)\;,\tag{7.13}$$

where the function h_θ is defined by $h_\theta(s)=s^\theta+s^{-\theta}$. Increasing, if necessary, the constant C in (7.9), by (7.10)–(7.13), we get

$$\begin{aligned}\int_{\mathbb{R}^n}u_\delta^a u^{2q-2}\varphi dx&\le C\sum_{i=1}^{n}\Bigg(\delta\int_{\mathbb{R}^n}u_\delta^a u^{2q-4}\varphi dx+\int_{\mathbb{R}^n}h_\theta(u_\delta)\,u_\delta^{a+p_i-1}u^{q-1}\left|\frac{\partial\varphi}{\partial x_i}\right|^{p_i}\varphi^{1-p_i}dx\\ &\quad+\sum_{j=1}^{i-1}\int_{\mathbb{R}^n}u_\delta^{a+p_i+p_j-2}\left|\frac{\partial^2\varphi}{\partial x_i\partial x_j}\right|^{p_ip_j}\left|\frac{\partial\varphi}{\partial x_i}\right|^{-p_i(p_j-1)}\left|\frac{\partial\varphi}{\partial x_j}\right|^{-p_j(p_i-1)}\varphi^{(p_i-1)(p_j-1)}dx\\ &\quad+\sum_{j=1}^{n}\int_{\mathbb{R}^n}h_\theta(u_\delta)\,u_\delta^{a+p_i+p_j-2}\left(\left|\frac{\partial\varphi}{\partial x_i}\right|^{p_i}\left|\frac{\partial\varphi}{\partial x_j}\right|^{p_j}\varphi^{1-p_i-p_j}+\left|\frac{\partial^2\varphi}{\partial x_i\partial x_j}\right|^{p_j}\left|\frac{\partial\varphi}{\partial x_i}\right|^{p_i-p_j}\varphi^{1-p_i}\right)dx\Bigg),\end{aligned}\tag{7.14}$$

where h_θ is as in (7.13). For $i = 1, \dots, n$ and for $C > 0$, Young's inequality gives

$$\int_{\mathbb{R}^n} h_\theta(u_\delta)\, u_\delta^{a+p_i-1} u^{q-1} \left|\frac{\partial \varphi}{\partial x_i}\right|^{p_i} \varphi^{1-p_i} dx \\ \le \frac{C}{2} \int_{\mathbb{R}^n} h_\theta(u_\delta)^2\, u_\delta^{a+2p_i-2} \left|\frac{\partial \varphi}{\partial x_i}\right|^{2p_i} \varphi^{1-2p_i} dx + \frac{1}{2C} \int_{\mathbb{R}^n} u_\delta^a u^{2q-2} \varphi dx\,. \quad (7.15)$$

We note that there holds $h_{2\theta} \le C_\theta h_\theta^2$ for some positive constant C_θ. Increasing, if necessary, the constant C in (7.14), by (7.15), we then get

$$\int_{\mathbb{R}^n} u_\delta^a u^{2q-2} \varphi dx \le C \sum_{i=1}^n \Bigg(\delta \int_{\mathbb{R}^n} u_\delta^a u^{2q-4} \varphi dx \quad (7.16) \\ + \sum_{j=1}^{i-1} \int_{\mathbb{R}^n} u_\delta^{a+p_i+p_j-2} \left|\frac{\partial^2 \varphi}{\partial x_i \partial x_j}\right|^{p_i p_j} \left|\frac{\partial \varphi}{\partial x_i}\right|^{-p_i(p_j-1)} \left|\frac{\partial \varphi}{\partial x_j}\right|^{-p_j(p_i-1)} \varphi^{(p_i-1)(p_j-1)} dx \\ + \sum_{j=1}^n \int_{\mathbb{R}^n} h_{2\theta}(u_\delta)\, u_\delta^{a+p_i+p_j-2} \left(\left|\frac{\partial \varphi}{\partial x_i}\right|^{p_i} \left|\frac{\partial \varphi}{\partial x_j}\right|^{p_j} \varphi^{1-p_i-p_j} + \left|\frac{\partial^2 \varphi}{\partial x_i \partial x_j}\right|^{p_j} \left|\frac{\partial \varphi}{\partial x_i}\right|^{p_i-p_j} \varphi^{1-p_i} \right) dx \Bigg).$$

We let κ be a real number in $(0,1)$ to be chosen close to 1 later on. One easily constructs a smooth cutoff function η satisfying $\eta \equiv 1$ in $[0,1]$, $0 \le \eta \le 1$ in $[1,2]$, $\eta \equiv 0$ in $[2,+\infty)$, and such that for $i, j = 1, \dots, n$, the functions $|\eta'|^{p_i+p_j} \eta^{1-\kappa-p_i-p_j}$, $|\eta''|^{p_j} |\eta'|^{p_i-p_j} \eta^{1-\kappa-p_i}$, $|\eta''|^{p_i p_j} |\eta'|^{p_i+p_j-2p_i p_j} \eta^{1-\kappa+p_i p_j-p_i-p_j}$, $|\eta'|^{p_i} \eta^{1-\kappa-p_i}$, and $|\eta'|^{p_i+p_j-p_i p_j} \eta^{1-\kappa+p_i p_j-p_i-p_j}$ extended by 0 outside of their domain of definition, are continuous on $\mathbb{R}^+$. For any positive real number R, we let φ_R be the function defined on $\mathbb{R}^n$ by

$$\varphi_R(x) = \eta\left(\sqrt{\sum_{i=1}^n \left(R^{\frac{p_i-q}{p_i}} x_i \right)^2} \right).$$

By (7.16) we get that there exists a positive constant C independent of u, δ, and R such that

$$\int_{\mathbb{R}^n} u_\delta^a u^{2q-2} \varphi_R dx \le C \Bigg(\delta \int_{\mathbb{R}^n} u_\delta^a u^{2q-4} \varphi_R^\kappa dx \\ + \sum_{i=1}^n \sum_{j=1}^n R^{p_i+p_j-2q} \int_{\mathbb{R}^n} h_{2\theta}(u_\delta)\, u_\delta^{a+p_i+p_j-2} \varphi_R^\kappa dx \Bigg). \quad (7.17)$$

Since $q > 2$, we get

$$\int_{\mathbb{R}^n} u_\delta^a u^{2q-4} \varphi_R^\kappa dx \le \int_{\mathbb{R}^n} u_\delta^{a+2q-4} \varphi_R^\kappa dx \\ \le \begin{cases} \left(\|u\|_{C^0(\mathrm{Supp}(\varphi_R))} + \delta \right)^{a+2q-4} \displaystyle\int_{\mathbb{R}^n} \varphi_R^\kappa dx & \text{if } a \ge 4-2q\,, \\ \delta^{a+2q-4} \displaystyle\int_{\mathbb{R}^n} \varphi_R^\kappa dx & \text{if } a < 4-2q\,. \end{cases} \quad (7.18)$$

Increasing, if necessary, the constant C in (7.17), it follows from (7.18) that

$$\int_{\mathbb{R}^n} u_\delta^a u^{2q-2} \varphi_R dx \le C \Bigg(\left(C_R + \delta^{a+2q-4}\right) \delta \int_{\mathbb{R}^n} \varphi_R^\kappa dx + \sum_{i=1}^n \sum_{j=1}^n R^{p_i+p_j-2q} \int_{\mathbb{R}^n} h_{2\theta}\left(u_\delta\right) u_\delta^{a+p_i+p_j-2} \varphi_R^\kappa dx \Bigg) \quad (7.19)$$

for some positive constant C_R independent of δ. By (5.3), we can choose θ small enough so that $a + 2p_- - 2 \ge 2\theta$ and $q > p_+ + \theta$. Passing to the limit into (7.19) as $\delta \to 0$ then yields

$$\int_{\mathbb{R}^n} u^{a+2q-2} \varphi_R dx \le C \sum_{i=1}^n \sum_{j=1}^n R^{p_i+p_j-2q} \int_{\mathbb{R}^n} h_{2\theta}(u)\, u^{a+p_i+p_j-2} \varphi_R^\kappa dx\,. \quad (7.20)$$

Still by (5.3), Young's inequality gives that for any $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that for $i, j = 1, \dots, n$, there holds

$$R^{p_i+p_j-2q} \int_{\mathbb{R}^n} u^{a+p_i+p_j-2\pm 2\theta} \varphi_R^\kappa dx \le \varepsilon \int_{\mathbb{R}^n} u^{a+2q-2} \varphi_R dx + C_\varepsilon R^{\frac{(p_i+p_j-2q)(a+2q-2)}{2q-p_i-p_j \mp 2\theta}} \int_{\mathbb{R}^n} \varphi_R^{\frac{(a+2q-2)\kappa - (a+p_i+p_j-2\pm 2\theta)}{2q-p_i-p_j\mp 2\theta}} dx\,. \quad (7.21)$$

In order to get the finiteness of the right-hand side of (7.21), one has to choose the real number κ close enough to 1 so that

$$\kappa \ge \frac{a + 2p_+ - 2 + 2\theta}{a + 2q - 2}\,.$$

Increasing, if necessary, the constant C in (7.20), it follows from (7.21) that there holds

$$\int_{\mathbb{R}^n} u^{a+2q-2} \varphi_R dx \le C \sum_{i=1}^n \sum_{j=1}^n R^{\frac{(p_i+p_j-2q)(a+2q-2)}{2q-p_i-p_j+2\theta}} \int_{\mathbb{R}^n} \varphi_R^{\frac{(a+2q-2)\kappa-(a+p_i+p_j-2+2\theta)}{2q-p_i-p_j-2\theta}} dx \quad (7.22)$$

for R large. Moreover, for $i, j = 1, \dots, n$, we easily compute

$$\int_{\mathbb{R}^n} \varphi_R^{\frac{(a+2q-2)\kappa-(a+p_i+p_j-2+2\theta)}{2q-p_i-p_j-2\theta}} dx \le C R^{q \sum_{k=1}^n \frac{1}{p_k} - n} \quad (7.23)$$

for some positive constant C independent of i, j, and R. It follows from (7.22) and (7.23) that for R large, there holds

$$\int_{\mathbb{R}^n} u^{a+2q-2} \varphi_R dx \le C R^{2-n+\left(\sum_{i=1}^n \frac{1}{p_i} - 2\right) q - a + \frac{\theta}{q-p_-+\theta}}\,. \quad (7.24)$$

By (5.3), we can choose the real number θ small enough so that passing to the limit into (7.24) as $R \to +\infty$ yields

$$\int_{\mathbb{R}^n} u^{a+2q-2} dx = 0\,,$$

and thus the function u is identically zero. This ends the proof of Theorem 5.1. □

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