SIGN-CHANGING BLOW-UP FOR THE MOSER–TRUDINGER EQUATION

LUCA MARTINAZZI, PIERRE-DAMIEN THIZY, AND JÉRÔME VÉTOIS

ABSTRACT. Given a sufficiently symmetric domain $\Omega \Subset \mathbb{R}^2$, for any $k \in \mathbb{N} \setminus \{0\}$ and $\beta > 4\pi k$ we construct blowing-up solutions $(u_\varepsilon) \subset H_0^1(\Omega)$ to the Moser–Trudinger equation such that as $\varepsilon \downarrow 0$, we have $\|\nabla u_\varepsilon\|_{L^2}^2 \to \beta$, $u_\varepsilon \rightharpoonup u_0$ in H_0^1 where u_0 is a sign-changing solution of the Moser–Trudinger equation and u_ε develops k positive spherical bubbles, all concentrating at $0 \in \Omega$. These 3 features (lack of quantization, non-zero weak limit and bubble clustering) stand in sharp contrast to the positive case ($u_\varepsilon > 0$) studied by the second author and Druet [8].

1. Introduction and main result

Given a smooth, bounded domain $\Omega \subset \mathbb{R}^2$ and a smooth, positive function h on $\overline{\Omega}$, we consider the Moser–Trudinger functional $I_h : H_0^1(\Omega) \to \mathbb{R}$ defined as

$$I_h(u) := \int_\Omega h \exp\left(u^2\right) dx \qquad \forall u \in H_0^1(\Omega).$$

For any $\beta > 0$, let $E_{h,\beta}$ be the set of all the critical points $u \in H_0^1(\Omega)$ of I_h under the constraint $\|\nabla u\|_{L^2}^2 = \beta$. Note that $u \in E_{h,\beta}$ if and only if u is a solution of the problem

$$\begin{cases} \Delta u = \lambda h f(u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{$\mathcal{E}_{h,\beta}$}$$

where we use the notation $\Delta := -\partial_{x_1}^2 - \partial_{x_2}^2$,

$$f(u) := u \exp\left(u^2\right) \quad \text{and} \quad \lambda := \frac{2\beta}{DI_h(u).u} = \frac{\beta}{\int_\Omega h u^2 \exp(u^2)\, dx}. \tag{1.1}$$

We first introduce the following definition in the spirit of [13, Chapter 5] (see also Remark 4.8):

Definition 1.1. *We say that $\beta > 0$ is a* stable energy level of I_h *if, for all (h_ε), (β_ε) and (λ_ε) such that $h_\varepsilon \to h$ in $C^2(\bar{\Omega})$ and $\beta_\varepsilon \to \beta$ with $\lambda_\varepsilon = O(1)$, any family (u_ε) such that u_ε solves $(\mathcal{E}_{h_\varepsilon,\beta_\varepsilon})$ with $\lambda = \lambda_\varepsilon$ for all ε converges in $C^2(\bar{\Omega})$ to some u solving $(\mathcal{E}_{h,\beta})$ as $\varepsilon \to 0$, up to a subsequence. We say that $\beta > 0$ is a* positively stable energy level of I_h *if the same holds true with $u_\varepsilon \geq 0$.*

Date: April 10, 2021.

Published in *Journal of Functional Analysis* **282** (2022), no. 2, 109288, 85 p.

The first author was supported by the Swiss National Foundation Grants PP00P2-144669 and PP00P2-170588/1. The second author was supported by the French ANR Grant BLADE-JC. The third author was supported by the Discovery Grant RGPIN-2016-04195 from the Natural Sciences and Engineering Research Council of Canada.

As a consequence of the Moser–Trudinger inequality [17,22], every $\beta \in (0, 4\pi)$ is a stable energy level of I_h. Druet–Thizy [8] obtained that every $\beta \in (0,\infty) \backslash 4\pi\mathbb{N}^\star$ ($\mathbb{N}^\star := \mathbb{N} \setminus \{0\}$) is a *positively* stable energy level of I_h. In contrast to this result, we obtain in this paper that every $\beta \geq 4\pi$ is an unstable energy level provided Ω and h are such that $0 \in \Omega$ and the following symmetric condition holds true for some even number $l \in 2\mathbb{N}^\star$:

(A) Ω is symmetric and h is even with respect to the lines

$$\ell_j := \left\{ \left(t \cos\left(\frac{j\pi}{2l}\right), t \sin\left(\frac{j\pi}{2l}\right) \right) : \, t \in \mathbb{R} \right\}, \quad 0 \leq j \leq 2l-1.$$

Under this assumption, we obtain the following:

Theorem 1.2. *Let $\Omega \subset \mathbb{R}^2$ be a smooth, bounded domain, $l \in 2\mathbb{N}^*$, $\alpha \in (0,1)$ and $h \in C^{l-2,\alpha}(\Omega) \cap C^2\left(\overline{\Omega}\right)$ be a positive function such that $0 \in \Omega$ and (A) holds true. Then every $\beta \geq 4\pi$ is an unstable energy level of I_h.*

In order to prove Theorem 1.2 we will construct a sign-changing weak limit w_0 with arbitrary energy $\beta_0 \in (0, 8\pi l)$ and use a Lyapunov-Schmidt procedure to glue to w_0 an arbitrary number $k \in \mathbb{N}^*$ of bubbles, all concentrating at the origin. This is in sharp contrast to the positive case studied by Druet–Thizy [8], in which blow-up can happen only at energy levels $\beta \in 4\pi\mathbb{N}^\star$, the weak limit vanishes and the bubbles blow up at distinct points. See also [5,6,9] for the constructive counterpart of [8].

To be more concrete, given $h \in C^{l-2,\alpha}(\Omega) \cap C^2\left(\overline{\Omega}\right)$ and $\beta_0 > 0$, using the symmetry of Ω and h, we will construct $w_0 \in E_{h,\beta_0}$ such that

$$w_0(x_1, 0) \sim a_0 x_1^l, \quad \text{as } x_1 \to 0, \quad \text{for some } a_0 > 0. \tag{1.2}$$

Up to a perturbation and a diagonal argument, we can assume that w_0 is non degenerate, and construct families $h_\varepsilon \to h$ in $C^2\left(\overline{\Omega}\right)$, $\beta_\varepsilon \to \beta_0$ and w_ε, $0 \leq \varepsilon \leq \varepsilon_0$, smooth with respect to ε such that $w_\varepsilon \in E_{h_\varepsilon,\beta_\varepsilon}$ and $0 > w_\varepsilon(0) \uparrow 0$ as $\varepsilon \to 0$. The behaviour (1.2) of the weak limit w_0 near the origin will be crucial to glue bubbles and the value of $w_\varepsilon(0) \uparrow 0$ will determine the parameter $\overline{\gamma}_\varepsilon \to \infty$ (see (3.6)), which is the approximate height of the bubbles.

In fact, if $\overline{B}_\gamma$ is the radial solution to $\Delta \overline{B}_\gamma = f\left(\overline{B}_\gamma\right)$ with $\overline{B}_\gamma(0) = \gamma$, we will attach to the function w_ε a fixed number k of perturbations of $\overline{B}_{\gamma_\varepsilon}$ along the x_1 axis, at points $(\tau_{\varepsilon,1}, 0), \ldots, (\tau_{\varepsilon,k}, 0)$. The centers $(\tau_{\varepsilon,i}, 0)$ of the bubbles will satisfy for some $\delta \in (0,1)$,

$$-\frac{kd_\varepsilon}{\delta} < \tau_{\varepsilon,1} < \cdots < \tau_{\varepsilon,k} < \frac{kd_\varepsilon}{\delta}, \quad |\tau_{\varepsilon,i} - \tau_{\varepsilon,j}| > \delta d_\varepsilon, \quad d_\varepsilon := \overline{\gamma}_\varepsilon^{-1/l} \to 0, \tag{1.3}$$

and, up to scaling, $(\tau_{\varepsilon,1}/d_\varepsilon, \ldots, \tau_{\varepsilon,k}/d_\varepsilon)$ will converge to a zero of $N = (N^1, \ldots, N^k)$, defined in a suitable convex subset of $\mathbb{R}^k$ as

$$N^i(y_1, \ldots, y_k) := a_0 l y_i^{l-1} - \sum_{j \neq i} \frac{2}{y_i - y_j}. \tag{1.4}$$

Note that, contrary to the case studied in [5,6,8], the function h (more specifically, its gradient) plays no role in (1.4), hence at main order it does not influence the location of the bubbles, which instead depends on $a_0 > 0$ and l as in (1.2) and on k.

A diagonal argument allows to treat the case $\beta_0 = 0$. Thus we finally obtain:

Theorem 1.3. *Given Ω, l, α and h as in Theorem 1.2 and $\beta \geq 4\pi$, $k \in \mathbb{N}^*$, $\beta_0 \geq 0$ such that $\beta = \beta_0 + 4\pi k$, there exist $w_0 \in E_{h,\beta_0}$ and $\varepsilon_0 > 0$ such that for every $\varepsilon \in (0, \varepsilon_0)$, we can find $h_\varepsilon \to h$ in $C^2(\overline{\Omega})$, $\beta'_\varepsilon \to \beta_0$, as $\varepsilon \to 0$, $w_\varepsilon \in E_{h_\varepsilon,\beta'_\varepsilon}$ as in (1.2), numbers $\beta_\varepsilon \to \beta$,*

$\overline{\gamma}_\varepsilon \to \infty$, γ_ε, τ_ε, $\theta_\varepsilon \in \mathbb{R}^k$, with $\gamma_{\varepsilon,i} \sim \overline{\gamma}_\varepsilon$, $\theta_{\varepsilon,i} \to 0$ as $\varepsilon \to 0$, $\tau_{\varepsilon,i}$ as in (1.3), and a function $u_\varepsilon \in E_{h_\varepsilon,\beta_\varepsilon}$ of the form

$$u_\varepsilon = w_\varepsilon + \sum_{i=1}^{k}(1+\theta_{\varepsilon,i})B_{\varepsilon,\gamma_{\varepsilon,i},\tau_{\varepsilon,i}} + \Psi_{\varepsilon,\gamma_\varepsilon,\tau_\varepsilon} + \Phi_{\varepsilon,\gamma_\varepsilon,\theta_\varepsilon,\tau_\varepsilon}\,, \tag{1.5}$$

where the approximate bubble $B_{\varepsilon,\gamma_{\varepsilon,i},\tau_{\varepsilon,i}} \in H_0^1(\Omega)$ is as in Section 3.1 and the remainder $H_0^1(\Omega)$-terms $\Psi_{\varepsilon,\gamma_\varepsilon,\tau_\varepsilon}$ and $\Phi_{\varepsilon,\gamma_\varepsilon,\theta_\varepsilon,\tau_\varepsilon}$ are given by Propositions 3.2 and 4.2. In particular,

$$\|\nabla\Psi_{\varepsilon,\gamma_\varepsilon,\tau_\varepsilon}\|_{L^2} = \mathrm{o}(1), \quad \|\nabla\Phi_{\varepsilon,\gamma_\varepsilon,\theta_\varepsilon,\tau_\varepsilon}\|_{L^2} = \mathrm{o}(1), \quad \text{as } \varepsilon \to 0.$$

In contrast with several other works constructing blowing-up solutions to the Moser–Trudinger equation, starting with del Pino–Musso–Ruf [5,6], our Lyapunov-Schmidt reduction will be performed in $H_0^1(\Omega)$, avoiding the use of weighted C^0-norms. While this is more in the spirit of the seminal work of Rey [19], the elegance of working with the Hilbert space $H_0^1(\Omega)$ requires a very precise ansatz (see Section 3), and a very sharp analysis of the radial bubble $\overline{B}_\gamma$, as obtained in [8,14,15] and further extended in our Section 6. In fact, we will construct the ansatz in two steps. First we construct the approximate solution

$$U_{\varepsilon,\gamma,\tau} = w_\varepsilon + \sum_{i=1}^{k} B_{\varepsilon,\gamma_i,\tau_i} + \Psi_{\varepsilon,\gamma,\tau},$$

for every τ as in (1.3) and γ in a fairly broad range (see (3.4) and Proposition 3.2). Then we shall strongly restrict the range of γ (Proposition 3.3 and (3.28)) and add the terms $\theta_i B_{\varepsilon,\gamma_i,\tau_i}$ to the ansatz. This will be crucial in the energy estimates of Section 4. In order to estimate the error terms near the bubbles, we shall use the spherical profile of the bubble to treat the blow-up regions as approximate spheres and apply Poincaré–Sobolev-type estimates, as given in Section 7. Finally we will perform the Lyapunov-Schmidt reduction to find the correct value $(\gamma_\varepsilon, \theta_\varepsilon, \tau_\varepsilon)$ and the correction term $\Phi_{\varepsilon,\gamma_\varepsilon,\theta_\varepsilon,\tau_\varepsilon}$, to finally obtain u_ε as in (1.5) (see also Remark 4.6).

Recently, Problem $(\mathcal{E}_{h,\beta})$ has received attention also when the nonlinearity f is suitably perturbed. Mancini and the second author [16] constructed radial (both positive and nodal) solutions u_γ to $(\mathcal{E}_{h,\beta})$ on the unit disk, blowing up at 0 and having non-zero weak limit as $\gamma \to \infty$, in the case $h \equiv 1$, $f_\gamma(u) = \lambda_\gamma u + \beta_\gamma u\exp(u^2)$ or $f_\gamma(u) = \beta_\gamma u \exp(u^2 - au)$ for suitable λ_γ, β_γ and $a > 0$. Grossi–Mancini–Naimen–Pistoia [12] constructed nodal solutions u_p to $(\mathcal{E}_{h,\beta})$ with $h \equiv 1$ and $f_p(u) = u\exp\left(u^2 + |u|^p\right)$, having *one* blow-up point as $p \downarrow 1$. Naimen [18] further gave a very detailed blow-up analysis of the blow-up of *radial* nodal solutions to $(\mathcal{E}_{h,\beta})$ when $h \equiv 1$ and $f(u) = u\exp\left(u^2 + \alpha|u|^\beta\right)$, $\alpha > 0$. To our knowledge, our work is the first one in which non-zero weak limits appear in the unperturbed case $f(u) = u\exp(u^2)$, and an arbitrary number of bubbles concentrates at the same point. Indeed, these two phenomena cannot occur in the unperturbed case without (radial) symmetry breaking of the solution.

To conclude, it would be very interesting to investigate whether our symmetry assumptions on the domain Ω can be relaxed or even removed. Indeed, we mainly use these assumptions to obtain the existence of a weak limit w_0 such that the point $\xi_0 \in \Omega$ at which the bubbles concentrate not only is a critical point of w_0 but also satisfies the closed condition $w_0(\xi_0) = 0$. This latter delicate condition is crucial for our construction in the unperturbed case $f(u) = u\exp(u^2)$. By contrast, in the strongly perturbed regime considered in [12], Grossi–Mancini–Naimen–Pistoia were able to construct solutions with one bubble which concentrates at a critical point ξ_0 of w_0 satisfying the open condition $w_0(\xi_0) > 1/2$. Then their assumptions not only cover the case of a symmetric and convex domain Ω but even a generic class of domains (see [12, Remark 1.2]).

2. Preliminary steps

This section is devoted to the construction of a smooth family of critical points satisfying some regularity, symmetry and asymptotic conditions which we will then use in the next sections to construct our blowing-up solutions.

Definition 2.1. *For every $l \in \mathbb{N}^*$, $p \in \mathbb{N}$ and $\alpha \in (0,1)$, we let $C^{p,\alpha}_{l,sym}(\Omega)$ be the vector space of all functions in $C^{p,\alpha}(\Omega)$ that are even with respect to the line ℓ_{2j} for all $j \in \{0,\dots,l-1\}$, where ℓ_{2j} is as in (A).*

Definition 2.2. *Let $\beta > 0$, h be a continuous, positive function on $\overline{\Omega}$ and $w \in E_{h,\beta}$. Then we say that w is non-degenerate if there does not exist any solution $v \neq 0$ to the problem*

$$\begin{cases} \Delta v = \lambda h f'(w)\, v & \text{in } \Omega \\ v = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.1}$$

where λ and f are as in (1.1). We let $E^{nd}_{h,\beta}$ be the set of all non-degenerate elements of $E_{h,\beta}$.

The main result of this section is the following:

Proposition 2.3. *Let Ω be a smooth, bounded domain, $l \in 2\mathbb{N}^\star$, $\beta_0 \in (0, 8l\pi)$, $\alpha \in (0,1)$ and $h \in C^{l-2,\alpha}(\Omega) \cap C^{0,\alpha}(\overline{\Omega})$ be a positive function such that $0 \in \Omega$ and (A) holds true. Then we have the following:*

- (i) *There exist $w_0 \in E_{h,\beta_0} \cap C^{l,\alpha}_{l,sym}(\Omega) \cap C^2(\overline{\Omega})$ and $a_0 > 0$ such that $w_0(x_1, 0) \sim a_0 x_1^l$ as $x_1 \to 0$.*
- (ii) *There exists $\kappa_0 > 0$ such that for every $\kappa \in (-\kappa_0, \kappa_0) \setminus \{0\}$, $w_\kappa := (1+\kappa) w_0 \in E^{nd}_{h_\kappa,\beta_\kappa} \cap C^{l,\alpha}_{l,sym}(\Omega) \cap C^2(\overline{\Omega})$, where $\beta_\kappa := (1+\kappa)^2 \beta_0$ and $h_\kappa := h \exp\left(-\kappa(\kappa+2) w_0^2\right)$.*
- (iii) *For every $\kappa \in (-\kappa_0, \kappa_0) \setminus \{0\}$, there exist $\hat{h}_\kappa \in C^{l-2,\alpha}_{l,sym}(\Omega) \cap C^2(\overline{\Omega})$ and $\varepsilon_0(\kappa) \in (0,1)$ such that for every $\varepsilon \in (0, \varepsilon_0(\kappa))$, there exist $\beta_{\kappa,\varepsilon} > 0$ and $w_{\kappa,\varepsilon} \in E_{h_{\kappa,\varepsilon},\beta_{\kappa,\varepsilon}} \cap C^{l,\alpha}_{l,sym}(\Omega) \cap C^2(\overline{\Omega})$, where $h_{\kappa,\varepsilon} := h_\kappa + \varepsilon \hat{h}_\kappa$, such that the families $(\beta_{\kappa,\varepsilon})_{0 \le \varepsilon \le \varepsilon_0(\kappa)}$ and $(w_{\kappa,\varepsilon})_{0 \le \varepsilon \le \varepsilon_0(\kappa)}$, where $\beta_{\kappa,0} := \beta_\kappa$ and $w_{\kappa,0} := w_\kappa$, are smooth in ε and moreover $\partial_\varepsilon [w_{\kappa,\varepsilon}(0)]_{\varepsilon=0} < 0$ and $w_{\kappa,\varepsilon}(0) < 0$ for all $\varepsilon \in (0, \varepsilon_0(\kappa))$.*

Proof of Proposition 2.3 (i). Define

$$\Omega_1 := \left\{ (x_1, x_2) \in \Omega :\ |x_2| < x_1 \tan\left(\frac{\pi}{2l}\right) \right\}.$$

Since Ω satisfies (A), we obtain that Ω_1 is symmetric with respect to the line ℓ_0. In particular, we can define the vector space $\mathcal{H}$ of all functions in $H^1_0(\Omega_1)$ that are even in x_2. Note that (A) also gives that $h|_{\Omega_1}$ is even in x_2. By applying standard variational arguments (see for instance Proposition 6 of Mancini–Martinazzi [15] in case $h \equiv 1$ and $\mathcal{H} = H^1_0(\Omega_1)$), we then obtain that for every $\beta_0 \in (0, 8l\pi)$, there exists a critical point w_0 of the functional $I_h|_{\mathcal{H}}$ under the constraint $\|\nabla w_0\|^2_{L^2(\Omega_1)} = \beta_0/2l$ such that $w_0 > 0$ in Ω_1. By using (A), we can then extend w_0 to the whole domain Ω as an odd function with respect to the line ℓ_{2j+1} for all $j \in \{0,\dots,l-1\}$. We claim that $w_0 \in E_{h,\beta_0}$. To see this, for every test function $v \in H^1_0(\Omega)$, we define

$$v_{\text{sym}} := \sum_{j=0}^{l-1} v \circ S_{2j+1} \circ S_1 - \sum_{j=0}^{l-1} v \circ S_{2j+1},$$

where $S_{2j+1}:\Omega\to\Omega$ is the symmetry operator with respect to the line ℓ_{2j+1}. By remarking that $v_{\mathrm{sym}}\in H^1_0(\Omega_1)$ and using v_{sym} as a test function for the Euler–Lagrange equation of u_0, we obtain

$$\int_{\Omega_1}\langle\nabla u_0,\nabla v_{\mathrm{sym}}\rangle\,dx=\frac{\beta_0}{l\int_{\Omega_1}hw_0^2\exp\left(w_0^2\right)dx}\int_{\Omega_1}hf\left(w_0\right)v_{\mathrm{sym}}dx. \tag{2.2}$$

By changes of variable and using the symmetry of w_0 and h, we obtain

$$\int_{\Omega_1}\langle\nabla w_0,\nabla v_{\mathrm{sym}}\rangle\,dx=\int_{\Omega}\langle\nabla w_0,\nabla v\rangle\,dx, \tag{2.3}$$

$$\int_{\Omega_1}hw_0^2\exp\left(w_0^2\right)dx=\frac{1}{2l}\int_{\Omega}hw_0^2\exp\left(w_0^2\right)dx=\frac{1}{2l}DI_h\left(w_0\right).w_0, \tag{2.4}$$

$$\int_{\Omega_1}hf\left(w_0\right)v_{\mathrm{sym}}dx=\int_{\Omega}hf\left(w_0\right)vdx. \tag{2.5}$$

By putting together (2.2)–(2.5), we obtain

$$\int_{\Omega}\langle\nabla w_0,\nabla v\rangle\,dx=\frac{2\beta_0}{DI_h\left(w_0\right).w_0}\int_{\Omega}hf\left(w_0\right)vdx$$

and so $w_0\in E_{h,\beta_0}$. Since $h\in C^{l-2,\alpha}(\Omega)\cap C^{0,\alpha}\left(\overline{\Omega}\right)$ and $\partial\Omega$ is smooth, by using the Moser–Trudinger inequality together with standard elliptic regularity theory, we then obtain that $w_0\in C^{l,\alpha}(\Omega)\cap C^{2,\alpha}\left(\overline{\Omega}\right)$. Since $w_0|_{\Omega_1}$ is even in x_2 and w_0 is odd with respect to the line ℓ_{2j+1} for all $j\in\{0,\dots,l-1\}$, we then obtain that w_0 is even with respect to ℓ_{2j} for all $j\in\{0,\dots,l-1\}$, i.e. $w_0\in C^{l,\alpha}_{l,\mathrm{sym}}(\Omega)$. Furthermore, since $w_0\in C^{l,\alpha}(\Omega)$ and $w_0=0$ on ℓ_{2j+1} for all $j\in\{0,\dots,l-1\}$, we obtain that $D^jw_0(0)=0$ for all $j\in\{0,\dots,l-1\}$ and so

$$w_0\left(x_1,0\right)=a_0x_1^l+\mathrm{O}\left(x_1^{l+\alpha}\right) \tag{2.6}$$

as $x_1\to0$ for some $a_0\in\mathbb{R}$. It remains to prove that $a_0>0$. Since $0\in\Omega$ and Ω is open, there exists $r_0>0$ such that $B(0,r_0)\subset\Omega$. For every $\varepsilon>0$, we define

$$S_{l,\varepsilon}\left(r_0\right):=\left\{\left(r\cos\theta,r\sin\theta\right):\,0<r<r_0\text{ and }|\theta|<\pi/\left(2\left(l+\varepsilon\right)\right)\right\}$$

and let $v_{l,\varepsilon}:S_{l,\varepsilon}(r_0)\to\mathbb{R}$ be the function defined as

$$v_{l,\varepsilon}\left(r\cos\theta,r\sin\theta\right):=r^{l+\varepsilon}\cos\left(\left(l+\varepsilon\right)\theta\right)$$

for all $(r\cos\theta,r\sin\theta)\in S_{l,\varepsilon}(r_0)$. It is easy to check that $v_{l,\varepsilon}$ is harmonic in $S_{l,\varepsilon}(r_0)$, continuous on $\overline{S_{l,\varepsilon}(r_0)}$ and $v_{l,\varepsilon}=0$ on $B(0,r_0)\cap\partial S_{l,\varepsilon}(r_0)$. On the other hand, since $S_{l,\varepsilon}(r_0)\subset\Omega_1$, we have that w_0 is continuous on $\overline{S_{l,\varepsilon}(r_0)}$ and positive on $\overline{S_{l,\varepsilon}(r_0)}\backslash\{0\}$. Furthermore, since $h,w_0>0$ in $S_{l,\varepsilon}(r_0)$, it follows from the Euler–Lagrange equation of w_0 that $\Delta w_0>0$ in $S_{l,\varepsilon}(r_0)$. It follows that there exists $\delta_{l,\varepsilon}>0$ such that $w_0\ge\delta_{l,\varepsilon}v_{l,\varepsilon}$ on $\partial S_{l,\varepsilon}(r_0)\cap\partial B(0,r_0)$. By comparison, we then obtain that $w_0\ge\delta_{l,\varepsilon}v_{l,\varepsilon}$ in $S_{l,\varepsilon}(r_0)$. Since $v_{l,\varepsilon}(r,0)=r^{l+\varepsilon}$, by taking $\varepsilon<\alpha$, we then obtain that the number a_0 in (2.6) is positive. This ends the proof of (i) in Proposition 2.3. ☐

Proof of Proposition 2.3 (ii). It is easy to check that $w_\kappa\in E_{h_\kappa,\beta_\kappa}\cap C^{2,\alpha}_{l,\mathrm{sym}}\left(\overline{\Omega}\right)$ for all $\kappa\in(-1,1)$. It remains to prove that $w_\kappa\in E^{nd}_{h_\kappa,\beta_\kappa}$ for $\kappa\in(-\kappa_0,\kappa_0)\setminus\{0\}$ with κ_0 small enough. Assume by contradiction that this is not the case, i.e. there exists a sequence of real numbers $(\kappa_j)_{j\in\mathbb{N}}$ such that w_{κ_j} is degenerate and $\kappa_j\to0$. Let v_j be a nonzero solution of the linearized equation

$$\begin{cases}\Delta v_j=\lambda_{\kappa_j}h_{\kappa_j}f'\left(w_{\kappa_j}\right)v_j & \text{in }\Omega\\ v_j=0 & \text{on }\partial\Omega,\end{cases}$$

with

$$\lambda_{\kappa_j} := \frac{2\beta_{\kappa_j}}{DI_{h_{\kappa_j}}(w_{\kappa_j}).w_{\kappa_j}} = \frac{2\beta_0}{DI_h(w_0).w_0} = \lambda.$$

By renormalizing and passing to a subsequences, we may assume without loss of generality that $\|\nabla v_j\|_{L^2} = 1$ and $(v_j)_{j\in\mathbb{N}}$ converges weakly to some function v_0 in $H_0^1(\Omega)$. By using the compactness of the embedding $H_0^1(\Omega) \hookrightarrow L^2(\Omega)$ and remarking that $\beta_\kappa \to \beta_0$ and $h_\kappa, w_\kappa \to h, w_0$ in $C^0(\overline{\Omega})$, we obtain that $(v_j)_{j\in\mathbb{N}}$ converges strongly to v_0 in $H_0^1(\Omega)$ and so $\|\nabla v_0\|_{L^2} = 1$. Furthermore, we obtain that v_0 is a solution of (2.1) with $\kappa = 0$. By using the definitions of h_κ, β_κ and w_κ, in particular noticing that $h_\kappa \exp(w_\kappa^2) = h \exp(w_0^2)$, and recalling the equation satisfied by v_j and v_0, it follows that

$$\int_\Omega h\left(1 + 2w_{\kappa_j}^2\right)\exp\left(w_0^2\right)v_jv_0dx = \frac{1}{2}DI_h(w_0).w_0\int_\Omega\langle\nabla v_j, \nabla v_0\rangle dx = \int_\Omega hf'(w_0)v_jv_0dx$$

and so

$$\int_\Omega hw_0^2\exp\left(w_0^2\right)v_jv_0dx = \int_\Omega h\frac{w_{\kappa_j}^2 - w_0^2}{\kappa_j(\kappa_j+2)}\exp\left(w_0^2\right)v_jv_0dx = 0. \tag{2.7}$$

By passing to the limit into (2.7), we obtain

$$\int_\Omega hw_0^2\exp\left(w_0^2\right)v_0^2dx = 0,$$

which gives $w_0v_0 = 0$ in Ω. Since $\|\nabla w_0\|_{L^2}^2 = \beta_0 \neq 0$, by unique continuation (see Aronszajn [1] and Cordes [4]), we obtain that $w_0 \neq 0$ in a dense subset D of Ω and so $v_0 = 0$ on D. By continuity of v_0, it follows that $v_0 = 0$ in Ω. This is in contradiction with $\|\nabla v_0\|_{L^2} = 1$. This ends the proof of (ii) in Proposition 2.3. □

The result of (iii) in Proposition 2.3 will follow from the following:

Proposition 2.4. *Let $l \in \mathbb{N}^\star$, $p \geq 2$, $\alpha \in (0,1)$ and Ω be a smooth, bounded domain such that $0 \in \Omega$ and Ω is symmetric with respect to the line ℓ_{2j} for all $j \in \{0,\dots,l-1\}$. Let $\overline{\beta}_0 > 0$, $\overline{h} \in C_{l,sym}^{p-2,\alpha}(\Omega) \cap C^2(\overline{\Omega})$ be positive in $\overline{\Omega}$ and $\overline{w}_0 \in E_{\overline{h},\overline{\beta}_0}^{nd} \cap C_{l,sym}^{p,\alpha}(\Omega) \cap C^2(\overline{\Omega})$ be such that $\overline{w}_0(0,0) = 0$ and $\overline{w}_0(r,0) > 0$ for small $r > 0$. Let D be the set of all functions $\hat{h} \in C_{l,sym}^{p-2,\alpha}(\Omega) \cap C^2(\overline{\Omega})$ such that*

$$\int_\Omega G_{\overline{h}}(\cdot,0)\,\hat{h}f(\overline{w}_0)\,dx < 0,$$

where $G_{\overline{h}}$ is the Green's function of the operator

$$\Delta - \frac{2\overline{\beta}_0\overline{h}f'(\overline{w}_0)}{DI_{\overline{h}}(\overline{w}_0).\overline{w}_0}$$

with boundary condition $G_{\overline{h}}(\cdot,0)\big|_{\partial\Omega} = 0$. Then D is a non-empty open subset of $C_{l,sym}^{p-2,\alpha}(\Omega) \cap C^2(\overline{\Omega})$ and for every $\hat{h} \in D$, there exists $\varepsilon_0 > 0$ such that for every $\varepsilon \in (0,\varepsilon_0)$, there exist $\overline{\beta}_\varepsilon > 0$ and $\overline{w}_\varepsilon \in E_{\overline{h}_\varepsilon,\overline{\beta}_\varepsilon} \cap C_{l,sym}^{p,\alpha}(\Omega) \cap C^2(\overline{\Omega})$, where $\overline{h}_\varepsilon := \overline{h} + \varepsilon\hat{h}$, such that $\left(\overline{\beta}_\varepsilon\right)_{0\leq\varepsilon\leq\varepsilon_0}$ and $(\overline{w}_\varepsilon)_{0\leq\varepsilon\leq\varepsilon_0}$ are smooth in ε and $\partial_\varepsilon[\overline{w}_\varepsilon(0)]_{\varepsilon=0} < 0$ and $\overline{w}_\varepsilon(0) < 0$ for all $\varepsilon \in (0,\varepsilon_0)$.

Proof. We begin with proving that D is not empty. Since $G_{\overline{h}}(\cdot,0) > 0$ near 0, $\overline{w}_0 \in C_{l,\mathrm{sym}}^{p,\alpha}(\Omega)$ and $\overline{w}_0(r,0) > 0$ for small $r > 0$, we obtain that there exists $x_0 \in \Omega$ and $r_0 > 0$ such that $G_{\overline{h}}(\cdot,0)\,\overline{w}_0 > 0$ in $B(x_0,r_0)$ and $B(x_0,r_0) \subset \Omega_0$, where

$$\Omega_0 := \{(x_1,x_2) \in \Omega :\ 0 < x_2 < x_1\tan(\pi/l)\}.$$

Let $\chi \in C^{\infty}(\Omega)$ be such that $\chi > 0$ in $B(x_0, r_0)$ and $\chi \equiv 0$ in $B(x_0, r_0)^c$. Let χ_{sym} be the unique function in $C^{p-2,\alpha}_{l,\text{sym}}(\Omega) \cap C^2(\overline{\Omega})$ such that $\chi_{\text{sym}} \equiv \chi$ in Ω_0. By symmetry and since $G_{\overline{h}}(\cdot, 0)\chi\overline{w}_0 > 0$ in $B(x_0, r_0)$ and $\chi = 0$ in $B(x_0, r_0)^c$, we obtain

$$\int_{\Omega} G_{\overline{h}}(\cdot, 0)\chi_{\text{sym}} f(\overline{w}_0)\, dx = 2l \int_{\Omega} G_{\overline{h}}(\cdot, 0)\chi f(\overline{w}_0)\, dx > 0,$$

i.e. $-\chi_{\text{sym}} \in D$. This proves that D is not empty. Now, we prove the second part of Proposition 2.4. Since $\overline{h} \in C^{p-2,\alpha}_{l,\text{sym}}(\Omega) \cap C^2(\overline{\Omega})$ and $\overline{w}_0 \in E^{nd}_{\overline{h},\overline{\beta}_0} \cap C^{p,\alpha}_{l,\text{sym}}(\Omega) \cap C^2(\overline{\Omega})$, it follows from the implicit function theorem together with standard elliptic regularity that there exist a neighborhood $\mathcal{N}$ of $\overline{h}$ in $C^{p-2,\alpha}_{l,\text{sym}}(\Omega) \cap C^2(\overline{\Omega})$ and a smooth mapping $\overline{w} : \mathcal{N} \to C^{p,\alpha}_{l,\text{sym}}(\Omega) \cap C^2(\overline{\Omega})$ such that $\overline{w}(\overline{h}) = \overline{w}_0$ and for every $\tilde{h} \in \mathcal{N}$, $\widetilde{U} = \overline{w}(\tilde{h})$ is a solution of the problem

$$\begin{cases} \Delta \widetilde{U} = \dfrac{2\overline{\beta}_0 \tilde{h} f(\widetilde{U})}{DI_{\overline{h}}(\overline{w}_0).\overline{w}_0} & \text{in } \Omega \\ \widetilde{U} = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.8}$$

Note that (2.8) is equivalent to $\widetilde{U} \in E_{\tilde{h},\overline{\beta}(\tilde{h})}$, where

$$\overline{\beta}(\tilde{h}) := \frac{\overline{\beta}_0 DI_{\tilde{h}}(\widetilde{U}).\widetilde{U}}{DI_{\overline{h}}(\overline{w}_0).\overline{w}_0}.$$

In particular, we obtain that for every $\hat{h} \in D$, there exists $\varepsilon_0 \in (0,1)$ such that for every $\varepsilon \in (0, \varepsilon_0)$, there exist $\overline{\beta}_\varepsilon = \overline{\beta}(\overline{h}_\varepsilon) > 0$ and $\overline{w}_\varepsilon = \overline{w}(\overline{h}_\varepsilon) \in E_{\overline{h}_\varepsilon, \overline{\beta}_\varepsilon} \cap C^{p,\alpha}_{l,\text{sym}}(\Omega) \cap C^2(\overline{\Omega})$, where $\overline{h}_\varepsilon := \overline{h} + \varepsilon\hat{h}$ such that $(\overline{\beta}_\varepsilon)_{0 \le \varepsilon \le \varepsilon_0}$ and $(\overline{w}_\varepsilon)_{0 \le \varepsilon \le \varepsilon_0}$ are smooth in ε. Furthermore, by differentiating (2.8), we obtain

$$\begin{cases} \left(\Delta - 2\overline{\beta}_0 \left(DI_{\overline{h}}(\overline{w}_0).\overline{w}_0\right)^{-1} \overline{h} f'(\overline{w}_0)\right) \partial_\varepsilon [\overline{w}_\varepsilon]_{\varepsilon=0} = \dfrac{2\overline{\beta}_0 \hat{h} f(\overline{w}_0)}{DI_{\overline{h}}(\overline{w}_0).\overline{w}_0} & \text{in } \Omega \\ \partial_\varepsilon [\overline{w}_\varepsilon]_{\varepsilon=0} = 0 & \text{on } \partial\Omega. \end{cases}$$

Since $\hat{h} \in D$, it follows that

$$\partial_\varepsilon [\overline{w}_\varepsilon(0)]_{\varepsilon=0} = \int_{\Omega} G_{\overline{h}}(\cdot, 0)\hat{h} f(\overline{w}_0)\, dx < 0.$$

Since $\overline{w}_0(0) = 0$, by taking ε_0 smaller if necessary, we then obtain that $\overline{w}_\varepsilon(0) < 0$ for all $\varepsilon \in (0, \varepsilon_0)$. This ends the proof of Proposition 2.4. □

Proof of Proposition 2.3 (iii). The result of (iii) in Proposition 2.3 is a direct consequence of Proposition 2.4 applied to $\overline{\beta}_0 := \beta_\kappa$, $\overline{h}_0 := h_\kappa$ and $\overline{w}_0 := w_\kappa$. □

3. Construction of the ansatz

This section is devoted to the construction of our ansatz. We let Ω, l, α and h be as in Theorem 1.2, fix $\beta > 4\pi$, $\beta_0 > 0$ and $k \in \mathbb{N}^*$ such that $\beta = \beta_0 + 4k\pi$ and let a_0, κ_0, ε_0, β_κ, w_κ, h_κ, $\beta_{\kappa,\varepsilon}$, $u_{\kappa,\varepsilon}$ and $h_{\kappa,\varepsilon}$ be as in Proposition 2.3. To prove that β is an unstable energy level of I_h, by using a diagonal argument, one can easily see that it suffices to show that for every $\kappa \in (-\kappa_0, \kappa_0) \setminus \{0\}$, the number $\beta_\kappa + 4k\pi$ is an unstable energy level of I_{h_κ}. In what follows, we fix $\kappa \in (-\kappa_0, \kappa_0) \setminus \{0\}$ and for the sake of simplicity, we drop the dependance in κ from our notations. More precisely, we denote $\varepsilon_0 := \varepsilon_0(\kappa)$, $\beta_0 := \beta_\kappa$, $h_0 := h_\kappa$, $w_0 := w_\kappa$,

$\beta_\varepsilon := \beta_{\kappa,\varepsilon}$, $h_\varepsilon := h_{\kappa,\varepsilon}$ and $w_\varepsilon := w_{\kappa,\varepsilon}$. Remark that the new function w_0 still satisfies the properties of (i) in Proposition 2.3 but now this function is moreover non-degenerate.

3.1. **The bubbles.** For every $\gamma_0 > 0$, we let $\overline{B}_{\gamma_0}$ be the unique radial solution to the problem

$$\begin{cases} \Delta \overline{B}_{\gamma_0} = f\left(\overline{B}_{\gamma_0}\right) & \text{in } \mathbb{R}^2 \\ \overline{B}_{\gamma_0}(0) = \gamma_0, \end{cases}$$

where $f(s) := s\exp\left(s^2\right)$ for all $s \in \mathbb{R}$. Note that by standard ordinary differential equations theory, $\overline{B}_{\gamma_0}$ is defined on $[0,\infty)$. For every $\varepsilon \in (0,\varepsilon_0)$, $\gamma_0 > 0$ and $x_0 \in \Omega$, we then define

$$\overline{B}_{\varepsilon,\gamma_0,x_0}(x) := \overline{B}_{\gamma_0}\left(\sqrt{\lambda_\varepsilon h_\varepsilon(x_0)}\,|x-x_0|\right) \qquad \forall x \in \mathbb{R}^2,$$

where

$$\lambda_\varepsilon := \frac{2\beta_\varepsilon}{DI_{h_\varepsilon}(w_\varepsilon).w_\varepsilon} \longrightarrow \frac{2\beta}{DI_h(w_0).w_0} =: \lambda > 0,$$

so that $\overline{B}_{\varepsilon,\gamma_0,x_0}$ solves the problem

$$\begin{cases} \Delta \overline{B}_{\varepsilon,\gamma_0,x_0} = \lambda_\varepsilon h_\varepsilon(x_0) f\left(\overline{B}_{\varepsilon,\gamma_0,x_0}\right) & \text{in } \mathbb{R}^2 \\ \overline{B}_{\varepsilon,\gamma_0,x_0}(x_0) = \gamma_0. \end{cases}$$

For every $r > 0$ such that $B(x_0,r) \subset \Omega$, we then let $B_{\varepsilon,\gamma_0,x_0,r} : \Omega \to \mathbb{R}$ be the function defined as

$$B_{\varepsilon,\gamma_0,x_0,r}(x) := \begin{cases} \overline{B}_{\varepsilon,\gamma_0,x_0}(x) - C_{\varepsilon,\gamma_0,x_0,r} + A_{\varepsilon,\gamma_0,x_0,r} H(x,x_0) & x \in B(x_0,r) \\ A_{\varepsilon,\gamma_0,x_0,r} G(x,x_0) & \text{otherwise} \end{cases} \tag{3.1}$$

for all $x \in \Omega$, where G is the Green's function of the Laplace operator in Ω with boundary condition $G(\cdot,x_0)|_{\partial\Omega} = 0$, H is the regular part of G, i.e.

$$G(x,x_0) = \frac{1}{2\pi}\ln\frac{1}{|x-x_0|} + H(x,x_0)$$

and $A_{\varepsilon,\gamma_0,x_0,r}$, $C_{\varepsilon,\gamma_0,x_0,r}$ are constants chosen so that $B_{\varepsilon,\gamma_0,x_0,r} \in C^1\left(\overline{\Omega}\right)$, i.e.

$$A_{\varepsilon,\gamma_0,x_0,r} := \int_{B(x_0,r)} \Delta \overline{B}_{\varepsilon,\gamma_0,x_0}\, dx, \tag{3.2}$$

$$C_{\varepsilon,\gamma_0,x_0,r} := \overline{B}_{\gamma_0}\left(\sqrt{\lambda_\varepsilon h_\varepsilon(x_0)}r\right) - \frac{A_{\varepsilon,\gamma_0,x_0,r}}{2\pi}\ln\frac{1}{r}\,. \tag{3.3}$$

3.2. **The primary ansatz.** For every $\varepsilon \in (0,\varepsilon_0)$ and $\delta \in (0,1)$, let Γ^k_ε and $T^k_\varepsilon(\delta)$ be the sets of parameters defined as

$$\Gamma^k_\varepsilon(\delta) := \left\{\gamma = (\gamma_1,\dots,\gamma_k) \in (0,\infty)^k : |\gamma_i - \overline{\gamma}_\varepsilon| < \delta\overline{\gamma}_\varepsilon,\ \forall i \in \{1,\dots,k\}\right\}, \tag{3.4}$$

$$T^k_\varepsilon(\delta) := \Big\{\tau = (\tau_1,\dots,\tau_k) \in \mathbb{R}^k : -\frac{kd_\varepsilon}{\delta} < \tau_1 < \dots < \tau_k < \frac{kd_\varepsilon}{\delta}$$
$$\text{and } |\tau_i - \tau_j| > \delta d_\varepsilon,\ \forall i,j \in \{1,\dots,k\}, i \neq j\Big\}, \tag{3.5}$$

where

$$\overline{\gamma}_\varepsilon := \frac{2(k+l-1)}{l\,|w_\varepsilon(0)|}\ln\frac{1}{|w_\varepsilon(0)|} \quad \text{and} \quad d_\varepsilon := \overline{\gamma}_\varepsilon^{-1/l}. \tag{3.6}$$

From (3.6), $w_0(0) = 0$ and $\partial_\varepsilon[w_\varepsilon(0)]_{\varepsilon=0} \neq 0$, we get

$$w_\varepsilon(0) \sim -\frac{2(k+l-1)}{l}\frac{\ln\overline{\gamma}_\varepsilon}{\overline{\gamma}_\varepsilon}, \quad \varepsilon \sim \frac{w_\varepsilon(0)}{\partial_\varepsilon[w_\varepsilon(0)]_{\varepsilon=0}} = \mathrm{O}\left(\frac{\ln\overline{\gamma}_\varepsilon}{\overline{\gamma}_\varepsilon}\right), \quad \text{as } \varepsilon \to 0, \tag{3.7}$$

and since $w_0(r,0) \sim a_0 r^l$ as $r \to 0$, using the continuity of $\partial_\varepsilon w_\varepsilon(x)$ jointly in ε and x, and (3.7) we get for some $\varepsilon_1 \in (0,\varepsilon)$

$$\begin{aligned} w_\varepsilon(\overline{\tau_i}) &= w_0(\overline{\tau_i}) + [w_\varepsilon(\overline{\tau_i}) - w_0(\overline{\tau_i})] = \mathrm{O}\left(d_\varepsilon^l\right) + \varepsilon\partial_\varepsilon[w_\varepsilon(\overline{\tau_i})]_{\varepsilon=\varepsilon_1} \\ &\sim \varepsilon\partial_\varepsilon[w_\varepsilon(0)]_{\varepsilon=0} \sim w_\varepsilon(0) \sim -\frac{2(k+l-1)\ln\overline{\gamma}_\varepsilon}{l\overline{\gamma}_\varepsilon}, \quad \text{as } \varepsilon \to 0, \end{aligned} \tag{3.8}$$

uniformly in $\tau \in T_\varepsilon^k(\delta)$. For every $(\gamma,\tau) \in \Gamma_\varepsilon^k(\delta) \times T_\varepsilon^k(\delta)$, we define

$$\widetilde{U}_{\varepsilon,\gamma,\tau} := w_\varepsilon + \sum_{i=1}^k B_{\varepsilon,\gamma_i,\tau_i},$$

where $B_{\varepsilon,\gamma_i,\tau_i} := B_{\varepsilon,\gamma_i,\overline{\tau_i},r_\varepsilon}$, $\overline{\tau_i} := (\tau_i, 0)$, and for $\delta_0 \in (0,1/2)$ to be fixed later,

$$r_\varepsilon := \overline{\mu}_\varepsilon^{\delta_0}, \quad \overline{\mu}_\varepsilon^2 := \exp\left(-\overline{\gamma}_\varepsilon^2\right). \tag{3.9}$$

Claim 3.1. *Set $A_{\varepsilon,\gamma_i,\tau_i} := A_{\varepsilon,\gamma_i,\overline{\tau_i},r_\varepsilon}$ and $C_{\varepsilon,\gamma_i,\tau_i} := C_{\varepsilon,\gamma_i,\overline{\tau_i},r_\varepsilon}$. For every $\delta \in (0,1)$ and $i \in \{1,\dots,k\}$, we have*

$$A_{\varepsilon,\gamma_i,\tau_i} = \frac{4\pi}{\gamma_i} + \mathrm{O}\left(\frac{1}{\overline{\gamma}_\varepsilon^3}\right), \quad C_{\varepsilon,\gamma_i,\tau_i} = -\frac{2\ln\overline{\gamma}_\varepsilon}{\gamma_i} + \mathrm{O}\left(\frac{1}{\overline{\gamma}_\varepsilon}\right), \tag{3.10}$$

$$\partial_{\gamma_i}\left[A_{\varepsilon,\gamma_i,\tau_i}\right] = -\frac{4\pi}{\gamma_i^2} + \mathrm{O}\left(\frac{1}{\overline{\gamma}_\varepsilon^4}\right) \quad \text{and} \quad \partial_{\gamma_i}\left[C_{\varepsilon,\gamma_i,\tau_i}\right] = \frac{2\ln\overline{\gamma}_\varepsilon}{\gamma_i^2} + \mathrm{O}\left(\frac{1}{\overline{\gamma}_\varepsilon^2}\right) \tag{3.11}$$

as $\varepsilon \to 0$, uniformly in $(\gamma,\tau) \in \Gamma_\varepsilon^k(\delta) \times T_\varepsilon^k(\delta)$. Furthermore, for every $a \geq 0$ and $\delta' \in \left(0, 1-\sqrt{\delta_0}\right)$ (i.e. such that $(1-\delta')^2 > \delta_0$), we have

$$\partial_{\tau_i}\left[A_{\varepsilon,\gamma_i,\tau_i}\right] = \mathrm{O}\left(\frac{1}{\overline{\gamma}_\varepsilon^a}\right) \quad \text{and} \quad \partial_{\tau_i}\left[C_{\varepsilon,\gamma_i,\tau_i}\right] \sim -\frac{\partial_{x_1}h_\varepsilon(\overline{\tau_i})}{h_\varepsilon(\overline{\tau_i})\gamma_i} = \mathrm{O}\left(\frac{1}{\overline{\gamma}_\varepsilon}\right) \tag{3.12}$$

as $\varepsilon \to 0$, uniformly in $(\gamma,\tau) \in \Gamma_\varepsilon^k(\delta') \times T_\varepsilon^k(\delta)$.

The proof of Claim 3.1 is based on a precise asymptotic study of the bubbles $\overline{B}_\gamma$ and is postponed to the Appendix.

3.3. **Correction of the error at the bottom of the bubbles.** In this section, we modify our ansatz so to correct the error made outside the balls $B(\overline{\tau_i}, 2r_\varepsilon)$. We prove the following:

Proposition 3.2. *Let Ω, l α and h be as in Theorem 1.2. Let k, ε_0, h_ε, w_ε, λ_ε, $\overline{\gamma}_\varepsilon$, $\overline{\tau_i}$, r_ε, δ_0, $\Gamma_\varepsilon^k(\delta)$, $T_\varepsilon^k(\delta)$ and $\widetilde{U}_{\varepsilon,\gamma,\tau}$ be as in Sections 3.1 and 3.2. Let $\chi \in C^\infty(\mathbb{R})$ be such that $0 \leq \chi \leq 1$ in $\mathbb{R}$, $\chi \equiv 1$ in $(-\infty,1]$ and $\chi \equiv 0$ in $[2,\infty)$. Define*

$$\chi_{\varepsilon,\tau}(x) := 1 - \sum_{i=1}^k \chi\left(\left(|x-\overline{\tau_i}| + r_\varepsilon^2 - r_\varepsilon\right)/r_\varepsilon^2\right) \qquad \forall x \in \mathbb{R}^2.$$

For every $\delta \in (0,1)$ and $\delta' \in (0, 1-\sqrt{2\delta_0})$, there exist $\varepsilon_1(\delta,\delta') \in (0,\varepsilon_0)$ and $C_1 = C_1(\delta,\delta') > 0$ such that for every $\varepsilon \in (0, \varepsilon_1(\delta,\delta'))$ and $(\gamma,\tau) \in \Gamma_\varepsilon^k(\delta') \times T_\varepsilon^k(\delta)$, there exists a unique solution $\Psi_{\varepsilon,\gamma,\tau} \in C^{l,\alpha}(\Omega) \cap C^2\left(\overline{\Omega}\right)$ to the problem

$$\begin{cases} \Delta\left(w_\varepsilon + \Psi_{\varepsilon,\gamma,\tau}\right) = \lambda_\varepsilon h_\varepsilon \chi_{\varepsilon,\tau} f\left(\widetilde{U}_{\varepsilon,\gamma,\tau} + \Psi_{\varepsilon,\gamma,\tau}\right) & \text{in } \Omega \\ \Psi_{\varepsilon,\gamma,\tau} = 0 & \text{on } \partial\Omega \end{cases} \tag{3.13}$$

such that $\Psi_{\varepsilon,\gamma,\tau}$ is even in x_2, continuously differentiable in (γ,τ) and

$$\left\|\Psi_{\varepsilon,\gamma,\tau}\right\|_{C^1} \leq \frac{C_1}{\overline{\gamma}_\varepsilon}, \quad \left\|D_\gamma\left[\Psi_{\varepsilon,\gamma,\tau}\right]\right\|_{C^1} \leq \frac{C_1}{\overline{\gamma}_\varepsilon^2}, \tag{3.14}$$

$$\left\|D_\tau\left[\Psi_{\varepsilon,\gamma,\tau}\right]\right\|_{H^1}+\left\|D_\tau\left[\Psi_{\varepsilon,\gamma,\tau}\right]\right\|_{C^0}\le\frac{C_1}{\overline{\gamma}_\varepsilon}\,. \tag{3.15}$$

Finally, setting $U_{\varepsilon,\gamma,\tau}:=\widetilde{U}_{\varepsilon,\gamma,\tau}+\Psi_{\varepsilon,\gamma,\tau}$, *there exists* $p_0=p_0\left(\delta_0,\delta'\right)$ *such that for every* $p\in[1,p_0]$, $a\ge 0$ *and* $i\in\{1,\dots,k\}$, *we have*

$$\left\|\exp\left(U_{\varepsilon,\gamma,\tau}^2\right)\mathbf{1}_{A(\overline{\tau_i},r_\varepsilon,R_\varepsilon)}\right\|_{L^p}=\mathrm{O}\left(\frac{1}{\overline{\gamma}_\varepsilon^a}\right),\ \left\|\exp\left(U_{\varepsilon,\gamma,\tau}^2\right)B_{\varepsilon,\gamma_i,\tau_i}^a\mathbf{1}_{\Omega_{R_\varepsilon,\tau}}\right\|_{L^p}=\mathrm{O}\left(\frac{1}{\overline{\gamma}_\varepsilon^a}\right), \tag{3.16}$$

$$\left\|\partial_{\tau_i}\left[\chi_{\varepsilon,\tau}\right]f\left(U_{\varepsilon,\gamma,\tau}\right)\right\|_{L^p}=\mathrm{O}\left(\frac{1}{\overline{\gamma}_\varepsilon^a}\right),\ \left\|f'\left(U_{\varepsilon,\gamma,\tau}\right)\partial_{\tau_i}\left[U_{\varepsilon,\gamma,\tau}\right]\mathbf{1}_{\Omega_{r_\varepsilon,\tau}}\right\|_{L^p}=\mathrm{O}\left(\overline{\gamma}_\varepsilon\right) \tag{3.17}$$

uniformly in $(\gamma,\tau)\in\Gamma_\varepsilon^k\left(\delta'\right)\times T_\varepsilon^k\left(\delta\right)$, *where* $R_\varepsilon:=\exp\left(-\overline{\gamma}_\varepsilon\right)$ *and*

$$A\left(\overline{\tau_i},r,R\right):=B\left(\overline{\tau_i},R\right)\backslash B\left(\overline{\tau_i},r\right)\quad\text{and}\quad\Omega_{r,\tau}:=\Omega\backslash\left(\bigcup_{i=1}^k B\left(\overline{\tau_i},r\right)\right) \tag{3.18}$$

for all $R>r>0$

In other words, the function

$$U_{\varepsilon,\gamma,\tau}:=\widetilde{U}_{\varepsilon,\gamma,\tau}+\Psi_{\varepsilon,\gamma,\tau}=w_\varepsilon+\sum_{i=1}^k B_{\varepsilon,\gamma_i,\tau_i}+\Psi_{\varepsilon,\gamma,\tau}, \tag{3.19}$$

where $\Psi_{\varepsilon,\gamma,\tau}$ is given by Proposition 3.2, is an exact solution outside the balls $B\left(\overline{\tau}_i,r_\varepsilon+r_\varepsilon^2\right)$ for all $i\in\{1,\dots,k\}$, and it satisfies

$$\Delta U_{\varepsilon,\gamma,\tau}=\begin{cases}\Delta B_{\varepsilon,\gamma_i,\tau_i}=\Delta\overline{B}_{\varepsilon,\gamma_i,\overline{\tau_i}}=\lambda_\varepsilon h_\varepsilon\left(\overline{\tau_i}\right)f\left(\overline{B}_{\varepsilon,\gamma_i,\overline{\tau_i}}\right) & \text{in } B\left(\overline{\tau_i},r_\varepsilon\right)\\ \lambda_\varepsilon h_\varepsilon\chi_{\varepsilon,\tau}f\left(U_{\varepsilon,\gamma,\tau}\right) & \text{in } \Omega_{r_\varepsilon,\tau}.\end{cases} \tag{3.20}$$

Since the proof of Proposition 3.2 is lenghty, but not necessary to understand the rest of the construction, it is postponed to Section 5.

For later use, we also observe that (3.1), (3.6), (3.9), (3.10) and (3.14) give $U_{\varepsilon,\gamma,\tau}=\delta_0\overline{\gamma}_\varepsilon(1+\mathrm{o}(1))$ in $\Omega_\varepsilon^i:=B(\overline{\tau_i},r_\varepsilon+r_\varepsilon^2)\setminus B(\overline{\tau_i},r_\varepsilon)$, hence

$$f(U_{\varepsilon,\gamma,\tau})=\mathrm{O}\left(\overline{\mu}_\varepsilon^{-2\delta_0^2+\mathrm{o}(1)}\right),\quad\text{in } \Omega_\varepsilon^i. \tag{3.21}$$

3.4. **Adjustment of the values at the centers of the bubbles.** In this section, we refine the range of the parameters γ_i so to optimize the error made in the regions $B\left(\overline{\tau_i},r_\varepsilon\right)$. Let us start by expanding

$$U_{\varepsilon,\gamma,\tau}\left(x\right)=\overline{B}_{\varepsilon,\gamma_i,\overline{\tau_i}}\left(x\right)+E_{\varepsilon,\gamma,\tau}^{(i)}+F_{\varepsilon,\gamma,\tau}^{(i)}\left(x\right) \tag{3.22}$$

for all $x\in B\left(\overline{\tau_i},r_\varepsilon\right)$, where

$$E_{\varepsilon,\gamma,\tau}^{(i)}:=w_\varepsilon\left(\overline{\tau_i}\right)-C_{\varepsilon,\gamma_i,\tau_i}+A_{\varepsilon,\gamma_i,\tau_i}H\left(\overline{\tau_i},\overline{\tau_i}\right)+\sum_{j\ne i}A_{\varepsilon,\gamma_j,\tau_j}G\left(\overline{\tau_i},\overline{\tau_j}\right)+\Psi_{\varepsilon,\gamma,\tau}\left(\overline{\tau_i}\right), \tag{3.23}$$

$$\begin{aligned}F_{\varepsilon,\gamma,\tau}^{(i)}\left(x\right):=&\,w_\varepsilon\left(x\right)-w_\varepsilon\left(\overline{\tau_i}\right)+A_{\varepsilon,\gamma_i,\tau_i}\left(H(x,\overline{\tau_i})-H\left(\overline{\tau_i},\overline{\tau_i}\right)\right)\\&+\sum_{j\ne i}A_{\varepsilon,\gamma_j,\tau_j}\left(G\left(x,\overline{\tau_j}\right)-G\left(\overline{\tau_i},\overline{\tau_j}\right)\right)+\Psi_{\varepsilon,\gamma,\tau}\left(x\right)-\Psi_{\varepsilon,\gamma,\tau}\left(\overline{\tau_i}\right).\end{aligned} \tag{3.24}$$

Note that $F_{\varepsilon,\gamma,\tau}^{(i)}\left(\overline{\tau_i}\right)=0$, so $F_{\varepsilon,\gamma,\tau}^{(i)}$ is small in $B\left(\overline{\tau}_i,r_\varepsilon\right)$. Instead the constant $E_{\varepsilon,\gamma,\tau}^{(i)}$ might be large depending on the choice of γ and τ. In the next proposition we show that we can choose $\overline{\gamma}_\varepsilon\left(\tau\right)\sim\overline{\gamma}_\varepsilon$ depending on τ and ε in such a way that $E_{\varepsilon,\overline{\gamma}_\varepsilon(\tau),\tau}^{(i)}=0$ for all $i\in\{1,\dots,k\}$.

Proposition 3.3. *Let δ_0, ε_1 and $\Psi_{\varepsilon,\gamma,\tau}$ be as in Proposition 3.2. Then for every $\delta \in (0,1)$ and $\delta' \in \left(0, 1-\sqrt{2\delta_0}\right)$, there exists $\varepsilon_2(\delta,\delta') \in (0, \varepsilon_1(\delta,\delta'))$ such that for every $\varepsilon \in (0, \varepsilon_2(\delta,\delta'))$ and $\tau \in T_\varepsilon^k(\delta)$, there exists a unique $\overline{\gamma}_\varepsilon(\tau) = \left(\overline{\gamma}_{1,\varepsilon}(\tau), \dots, \overline{\gamma}_{k,\varepsilon}(\tau)\right) \in \Gamma_\varepsilon^k(\delta')$ such that $\overline{\gamma}_{k,\varepsilon}(\tau)$ is continuous in τ and for every $i \in \{1,\dots,k\}$, we have*

$$U_{\varepsilon,\overline{\gamma}_\varepsilon(\tau),\tau}(\overline{\tau_i}) = \overline{\gamma}_{i,\varepsilon}(\tau) \quad \text{and} \quad \overline{\gamma}_{i,\varepsilon}(\tau) \sim \overline{\gamma}_\varepsilon \tag{3.25}$$

as $\varepsilon \to 0$, uniformly in $\tau \in T_\varepsilon^k(\delta)$.

Proof. For every $\gamma \in \Gamma_\varepsilon^k(\delta')$, we denote $\tilde{\gamma} := \gamma/\overline{\gamma}_\varepsilon$. We let $I := (1-\delta', 1+\delta')$ and $E_{\varepsilon,\tau} : I^k \to \mathbb{R}^k$, $E_{\varepsilon,\tau} = \left(E_{\varepsilon,\tau}^{(1)}, \dots, E_{\varepsilon,\tau}^{(k)}\right)$ be the function defined by

$$E_{\varepsilon,\tau}^{(i)}(\tilde{\gamma}) := \frac{\overline{\gamma}_\varepsilon}{\ln \overline{\gamma}_\varepsilon} E_{\varepsilon,\gamma,\tau}^{(i)} \quad \forall \gamma \in I^k,\ i \in \{1,\dots,k\}.$$

In particular, $E_{\varepsilon,\tau} \in C^1(I^k)$. By definition of d_ε, G and H, we obtain

$$G(\overline{\tau_i}, \overline{\tau_j}) \sim \frac{1}{2\pi} \ln \frac{1}{d_\varepsilon} \sim \frac{\ln \overline{\gamma}_\varepsilon}{2l\pi} \quad \text{and} \quad H(\overline{\tau_i}, \overline{\tau_i}) = \mathrm{O}(1) \tag{3.26}$$

as $\varepsilon \to 0$, uniformly in $\tau \in T_\varepsilon^k(\delta)$. It follows from (3.8), (3.10), (3.11), (3.14) and (3.26) that $E_{\varepsilon,\tau} \to E_0 = \left(E_0^{(1)}, \cdots, E_0^{(k)}\right)$ in $C^1(I^k)$ as $\varepsilon \to 0$, uniformly in $\tau \in T_\varepsilon^k(\delta)$, where

$$E_0^{(i)}(\tilde{\gamma}) := \frac{2}{\tilde{\gamma}_i} + \frac{2}{l} \sum_{j \neq i} \frac{1}{\tilde{\gamma}_j} - \frac{2(k+l-1)}{l}$$

for all $i \in \{1,\dots,k\}$ and $\tilde{\gamma} = (\tilde{\gamma}_1, \dots, \tilde{\gamma}_k) \in I^k$. In particular,

$$E_0(1,\dots,1) = 0 \quad \text{and} \quad \det(DE_0(1,\dots,1)) \neq 0. \tag{3.27}$$

By applying the implicit function theorem, it follows from (3.27) that there exists $\varepsilon_2(\delta,\delta') \in (0, \varepsilon_1(\delta,\delta'))$ such that for every $\varepsilon \in (0, \varepsilon_2(\delta,\delta'))$ and $\tau \in T_\varepsilon^k(\delta)$, there exists a unique $\tilde{\gamma}_\varepsilon(\tau) \in I^k$ such that $\tilde{\gamma}_\varepsilon(\tau)$ is continuous in τ, $E_{\varepsilon,\tau}(\tilde{\gamma}_\varepsilon(\tau)) = 0$ and $\tilde{\gamma}_\varepsilon(\tau) \to (1,\dots,1)$ as $\varepsilon \to 0$, uniformly in $\tau \in T_\varepsilon^k(\delta)$, i.e. there exists a unique $\overline{\gamma}_\varepsilon(\tau) = \overline{\gamma}_\varepsilon \tilde{\gamma}_\varepsilon(\tau) \in \Gamma_\varepsilon^k(\delta')$ such that $\overline{\gamma}_\varepsilon(\tau)$ is continuous in τ and (3.25) holds true. This ends the proof of Proposition 3.3. ☐

Now, we refine the set $\Gamma_\varepsilon^k(\delta')$ by defining

$$\overline{\Gamma}_\varepsilon^k(\tau) := \left\{\gamma = (\gamma_1, \dots, \gamma_k) \in (0,\infty)^k : |\gamma_i - \overline{\gamma}_{i,\varepsilon}(\tau)| < \frac{\delta_\varepsilon}{\overline{\gamma}_\varepsilon},\ \forall i \in \{1,\dots,k\}\right\}, \tag{3.28}$$

where $\overline{\gamma}_{1,\varepsilon}(\tau), \dots, \overline{\gamma}_{k,\varepsilon}(\tau)$ are the numbers obtained in Proposition 3.3 and

$$\delta_\varepsilon := \overline{\mu}_\varepsilon^{\delta_1 + 1/2}, \tag{3.29}$$

where $\overline{\mu}_\varepsilon$ is as in (3.9) and $\delta_1 \in (0, 1/2)$ is a number that we shall fix later.

Note that for every $\delta, \delta' \in (0,1)$, we have

$$\overline{\Gamma}_\varepsilon^k(\tau) \subset \Gamma_\varepsilon^k(\delta') \tag{3.30}$$

for small $\varepsilon > 0$, uniformly in $\tau \in T_\varepsilon^k(\delta)$. Therefore, we can fix

$$\delta' := \frac{1 - \sqrt{2\delta_0}}{2}$$

in what follows and let $\varepsilon_3(\delta) \in (0, \varepsilon_2(\delta,\delta'))$ be such that (3.30) holds true together with the results of Propositions 3.2 and 3.3 for all $\varepsilon \in (0, \varepsilon_3(\delta))$ and $\tau \in T_\varepsilon^k(\delta)$.

3.5. **An additional variation in the directions of the bubbles.** We now introduce an additional family of parameters $\theta = (\theta_1, \dots, \theta_k) \in \mathbb{R}^k$ and define our final ansatz as

$$U_{\varepsilon,\gamma,\tau,\theta} := U_{\varepsilon,\gamma,\tau} + \sum_{i=1}^{k} \theta_i B_{\varepsilon,\gamma_i,\tau_i} = w_\varepsilon + \sum_{i=1}^{k} (1+\theta_i)\, B_{\varepsilon,\gamma_i,\tau_i} + \Psi_{\varepsilon,\gamma,\tau},$$

for

$$\theta \in \Theta_\varepsilon^k(\delta) := \left\{ \theta = (\theta_1, \dots, \theta_k) \in \mathbb{R}^k : |\theta_i| < \frac{\delta_\varepsilon \ln \overline{\gamma}_\varepsilon}{\overline{\gamma}_\varepsilon^4},\ \forall i \in \{1, \dots, k\} \right\}, \tag{3.31}$$

where $\overline{\gamma}_\varepsilon$ and δ_ε are as in (3.6) and (3.29). Finally, we define

$$P_\varepsilon^k(\delta) := \left\{ (\gamma, \tau, \theta) \in (0,\infty)^k \times T_\varepsilon^k(\delta) \times \Theta_\varepsilon^k : \gamma \in \overline{\Gamma}_\varepsilon^k(\tau) \right\},$$

where $T_\varepsilon^k(\delta)$, $\overline{\Gamma}_\varepsilon^k(\tau)$ and Θ_ε^k are defined as in (3.5), (3.28) and (3.31), respectively.

3.6. **Pointwise estimates near the centers of the bubbles.** We can now prove the following:

Proposition 3.4. *Let $\overline{\gamma}_\varepsilon(\tau)$ be as in Proposition 3.3. Then for every $i \in \{1, \dots, k\}$ and $\delta \in (0,1)$ we have*

$$\partial_{\gamma_i}\left[E_{\varepsilon,\gamma,\tau}^{(i)}\right] = -\frac{2\ln\overline{\gamma}_\varepsilon}{\overline{\gamma}_\varepsilon^2} + \mathrm{o}\left(\frac{\ln\overline{\gamma}_\varepsilon}{\overline{\gamma}_\varepsilon^2}\right), \quad \partial_{\gamma_j}\left[E_{\varepsilon,\gamma,\tau}^{(i)}\right] = -\frac{2\ln\overline{\gamma}_\varepsilon}{l\overline{\gamma}_\varepsilon^2} + \mathrm{o}\left(\frac{\ln\overline{\gamma}_\varepsilon}{\overline{\gamma}_\varepsilon^2}\right), \quad \textit{for } j \neq i,$$

$$E_{\varepsilon,\gamma,\tau}^{(i)} = -\frac{2\ln\overline{\gamma}_\varepsilon}{\overline{\gamma}_\varepsilon^2}\left((\gamma_i - \overline{\gamma}_{i,\varepsilon}(\tau)) + \sum_{j\neq i}\frac{\gamma_j - \overline{\gamma}_{j,\varepsilon}(\tau)}{l}\right) + \mathrm{o}\left(|\gamma - \overline{\gamma}_\varepsilon(\tau)|\frac{\ln\overline{\gamma}_\varepsilon}{\overline{\gamma}_\varepsilon^2}\right)$$

as $\varepsilon \to 0$, uniformly in $\tau \in T_\varepsilon^k(\delta)$ and $\gamma \in \overline{\Gamma}_\varepsilon^k(\tau)$.

Proof. Using (3.11), (3.23), (3.14) and noticing that for $(\gamma_1, \dots, \gamma_k) \in \overline{\Gamma}_\varepsilon^k(\tau)$ we have $\gamma_j \sim \overline{\gamma}_\varepsilon$ for $j = 1, \dots, k$, we get

$$\begin{aligned}\partial_{\gamma_i}\left[E_{\varepsilon,\gamma,\tau}^{(i)}\right] &= -\partial_{\gamma_i}\left[C_{\varepsilon,\gamma_i,\tau_i}\right] + \partial_{\gamma_i}\left[A_{\varepsilon,\gamma_i,\tau_i}\right] H(\overline{\tau_i}, \overline{\tau_i}) + \partial_{\gamma_i}\left[\Psi_{\varepsilon,\gamma,\tau}\right] \\ &= -\frac{2\ln\overline{\gamma}_\varepsilon}{\gamma_i^2}(1 + \mathrm{o}(1)) - \frac{4\pi}{\gamma_i^2}\mathrm{O}(1) + \mathrm{O}\left(\frac{1}{\overline{\gamma}_\varepsilon^2}\right) = -\frac{2\ln\overline{\gamma}_\varepsilon}{\overline{\gamma}_\varepsilon^2} + \mathrm{o}\left(\frac{\ln\overline{\gamma}_\varepsilon^2}{\overline{\gamma}_\varepsilon^2}\right)\end{aligned}$$

as $\varepsilon \to 0$. For the case $j \neq i$, using (3.6), we estimate

$$G(\overline{\tau_i}, \overline{\tau_j}) = \frac{1}{2\pi}\ln\frac{1}{d_\varepsilon} + \mathrm{O}(1) = \frac{1}{2\pi l}\ln\overline{\gamma}_\varepsilon + \mathrm{O}(1),$$

uniformly in $\tau \in T_\varepsilon^k(\delta)$, hence

$$\begin{aligned}\partial_{\gamma_j}\left[E_{\varepsilon,\gamma,\tau}^{(i)}\right] &= \partial_{\gamma_j}\left[A_{\varepsilon,\gamma_j,\tau_j}\right] G(\overline{\tau_i}, \overline{\tau_j}) + \partial_{\gamma_j}\left[\Psi_{\varepsilon,\gamma,\tau}\right] \\ &= -\frac{4\pi}{\gamma_j^2}\left(\frac{1}{2\pi l}\ln\overline{\gamma}_\varepsilon + \mathrm{O}(1)\right) + \mathrm{O}\left(\frac{1}{\overline{\gamma}_\varepsilon^2}\right) = -\frac{2\ln\overline{\gamma}_\varepsilon}{l\overline{\gamma}_\varepsilon^2} + \mathrm{o}\left(\frac{\ln\overline{\gamma}_\varepsilon^2}{\overline{\gamma}_\varepsilon^2}\right).\end{aligned}$$

Now, since $E_{\varepsilon,\overline{\gamma}_\varepsilon(\tau),\tau}^{(i)} = 0$, integrating the gradient of $E_{\varepsilon,\gamma,\tau}^{(i)}$ with respect to γ from $\overline{\gamma}_\varepsilon(\tau)$ to a generic $\gamma \in \overline{\Gamma}_\varepsilon^k(\tau)$, the last identity follows at once. □

Proposition 3.5. *For every $i \in \{1, \dots, k\}$ and $\delta \in (0,1)$, we have*

$$F^{(i)}_{\varepsilon,\gamma,\tau}(x) = \left(a_0 l \tau_i^{l-1} - \frac{2}{\overline{\gamma}_\varepsilon} \sum_{j \neq i} \frac{1}{\tau_i - \tau_j} \right) (x_1 - \tau_i) + \mathrm{o}\left(\frac{|x - \overline{\tau}_i|}{\overline{\gamma}_\varepsilon d_\varepsilon} \right), \tag{3.32}$$

and for every $i, j \in \{1, \dots, k\}$,

$$\partial_{\gamma_j} \left[F^{(i)}_{\varepsilon,\gamma,\tau} \right](x) = \mathrm{O}\left(\frac{|x - \overline{\tau}_i|}{\overline{\gamma}_\varepsilon^2 d_\varepsilon} \right) \tag{3.33}$$

as $\varepsilon \to 0$, uniformly in $x = (x_1, x_2) \in B(\overline{\tau}_i, r_\varepsilon)$ and $(\gamma, \tau, \theta) \in P^k_\varepsilon(\delta)$.

Proof. Note that $F^{(i)}_{\varepsilon,\gamma,\tau}(\overline{\tau}_i) = 0$. Then, by using (3.7), (3.10) and (3.15) and since $w_\varepsilon = w_0 + \mathrm{O}(\varepsilon)$ in $C^1(\Omega)$, $w_0(r, 0) \sim a_0 r^l$ as $r \to 0$ and $\partial_{x_2} w_0(0,0) = 0$, we obtain

$$\begin{aligned}
F^{(i)}_{\varepsilon,\gamma,\tau}(x) &= \int_0^1 \left\langle \nabla F^{(i)}_{\varepsilon,\gamma,\tau}((1-t)\overline{\tau}_i + tx), x - \overline{\tau}_i \right\rangle dt \\
&= \int_0^1 \left(\langle \nabla w_0((1-t)\overline{\tau}_i + tx), x - \overline{\tau}_i \rangle - \sum_{j \neq i} \frac{A_{\varepsilon,\gamma_j,\tau_j} \langle (1-t)\overline{\tau}_i + tx - \overline{\tau}_j, x - \overline{\tau}_i \rangle}{2\pi |(1-t)\overline{\tau}_i + tx - \overline{\tau}_j|^2} \right) dt \\
&\quad + \mathrm{O}(\varepsilon |x - \overline{\tau}_i|) = \left(a_0 l \tau_i^{l-1} - \frac{2}{\overline{\gamma}_\varepsilon} \sum_{j \neq i} \frac{1}{\tau_i - \tau_j} \right) (x_1 - \tau_i) + \mathrm{o}\left(\frac{|x - \overline{\tau}_i|}{\overline{\gamma}_\varepsilon^{1-1/l}} \right)
\end{aligned}$$

as $\varepsilon \to 0$, uniformly in $x = (x_1, x_2) \in B(\overline{\tau}_i, r_\varepsilon)$, $\tau \in T^k_\varepsilon(\delta)$ and $\gamma \in \overline{\Gamma}^k_\varepsilon(\tau)$, hence proving (3.32). Differentiating (3.24) and using Claim 3.1, (3.33) also follows at once. □

Proposition 3.6. *For every $i \in \{1, \dots, k\}$ and $\delta \in (0,1)$, we have*

$$\begin{aligned}
U_{\varepsilon,\gamma,\tau,\theta}(x) = \overline{B}_{\varepsilon,\gamma_i,\tau_i}(x) &+ \left(a_0 l \tau_i^{l-1} - \frac{2}{\overline{\gamma}_\varepsilon} \sum_{j \neq i} \frac{1}{\tau_i - \tau_j} \right) (x_1 - \tau_i) \\
&+ \mathrm{o}\left(\frac{|x - \overline{\tau}_i|}{\overline{\gamma}_\varepsilon^{1-1/l}} \right) + \mathrm{O}\left(\frac{\delta_\varepsilon \ln \overline{\gamma}_\varepsilon}{\overline{\gamma}_\varepsilon^3} \right)
\end{aligned} \tag{3.34}$$

as $\varepsilon \to 0$, uniformly in $x = (x_1, x_2) \in B(\overline{\tau}_i, r_\varepsilon)$ and $(\gamma, \tau, \theta) \in P^k_\varepsilon(\delta)$. In particular, for every $\delta \in (0,1)$, there exists $\varepsilon_4(\delta) \in (0, \varepsilon_3(\delta))$, where $\varepsilon_3(\delta)$ is as in Section 3.5, such that

$$\overline{B}_{\varepsilon,\gamma_i,\tau_i}(x) > 0 \text{ and } U_{\varepsilon,\gamma,\tau,\theta} > 0 \text{ in } B(\overline{\tau}_i, r_\varepsilon) \tag{3.35}$$

for all $\varepsilon(\delta) \in (0, \varepsilon_4(\delta))$, $(\gamma, \tau, \theta) \in P^k_\varepsilon(\delta)$ and $i \in \{1, \dots, k\}$.

Proof. In order to prove (3.34), it suffices to write

$$U_{\varepsilon,\gamma,\tau,\theta}(x) = \overline{B}_{\varepsilon,\gamma_i,\tau_i}(x) + E^{(i)}_{\varepsilon,\gamma,\tau} + F^{(i)}_{\varepsilon,\gamma,\tau}(x) + \sum_{j=1}^k \theta_j B_{\varepsilon,\gamma_j,\tau_j}(x) \quad \text{in } B(\overline{\tau}_i, r_\varepsilon)$$

and apply Proposition 3.4 to bound

$$E^{(i)}_{\varepsilon,\gamma,\tau} + \sum_{j=1}^k \theta_i B_{\varepsilon,\gamma_j,\tau_j} = \mathrm{O}\left(\frac{\delta_\varepsilon \ln \overline{\gamma}_\varepsilon}{\overline{\gamma}_\varepsilon^3} \right)$$

and Proposition 3.5 to estimate $F^{(i)}_{\varepsilon,\gamma,\tau}(x)$. It then follows from (3.34) and (6.2) that (3.35) holds true for small $\varepsilon > 0$, uniformly in $(\gamma, \tau, \theta) \in P^k_\varepsilon(\delta)$. □

4. Proof of Theorems 1.2 and 1.3

This section is devoted to the proof of Theorems 1.2 and 1.3. We let Ω, l, α and h be as in Theorem 1.2, fix $\beta > 4\pi$, $\beta_0 > 0$ and $k \in \mathbb{N}^*$ such that $\beta = \beta_0 + 4k\pi$ and let β_ε, h_ε, w_ε, λ_ε, $\overline{\gamma}_\varepsilon$, $\overline{\mu}_\varepsilon$, d_ε, r_ε, δ_ε, δ_0, δ_1, $\overline{\gamma}_{i,\varepsilon}(\tau)$, $\overline{B}_{\varepsilon,\gamma_0,x_0}$, $A_{\varepsilon,\gamma,x,r}$, $C_{\varepsilon,\gamma,x,r}$, G, H, $B_{\varepsilon,\gamma_i,\tau_i}$, $\widetilde{U}_{\varepsilon,\gamma,\tau}$, $\chi_{\varepsilon,\tau}$, $\Psi_{\varepsilon,\gamma,\tau}$, $U_{\varepsilon,\gamma,\tau,\theta}$, $\Gamma^k_\varepsilon(\delta)$, $\overline{\Gamma}^k_\varepsilon(\tau)$, $T^k_\varepsilon(\delta)$, Θ^k_ε and $P^k_\varepsilon(\delta)$ be as in Section 3. We define

$$R_{\varepsilon,\gamma,\tau,\theta} := U_{\varepsilon,\gamma,\tau,\theta} - \Delta^{-1}\left[\lambda_\varepsilon h_\varepsilon f\left(U_{\varepsilon,\gamma,\tau,\theta}\right)\right]. \tag{4.1}$$

As a first step, we obtain the following:

Proposition 4.1. *Let ε_4 be as in Proposition 3.6. Assume that*

$$\tfrac{3-\sqrt{5}}{4} < \delta_0 < \tfrac{1}{2} \quad \text{and} \quad 0 < \delta_1 < 3\delta_0 - 2\delta_0^2 - \tfrac{1}{2}. \tag{4.2}$$

Then for every $\delta \in (0,1)$, there exist $\varepsilon_5(\delta) \in (0, \varepsilon_4(\delta))$ and $C_5 = C_5(\delta) > 0$ such that

$$\left\|R_{\varepsilon,\gamma,\tau,\theta}\right\|_{H^1_0} \le C_5 \frac{\delta_\varepsilon \ln \overline{\gamma}_\varepsilon}{\overline{\gamma}_\varepsilon^2} \tag{4.3}$$

for all $\varepsilon \in (0, \varepsilon_5(\delta))$ and $(\gamma,\tau,\theta) \in P^k_\varepsilon(\delta)$.

Proof. For every $\psi \in H^1_0(\Omega)$, using that $B_{\varepsilon,\gamma_i,\tau_i} \in V_{\varepsilon,\gamma,\tau}$, integrating by parts and using (3.20), we obtain

$$\begin{aligned}
\left\langle R_{\varepsilon,\gamma,\tau,\theta}, \psi\right\rangle_{H^1_0} &= \left\langle U_{\varepsilon,\gamma,\tau,\theta} - \Delta^{-1}\left[\lambda_\varepsilon h_\varepsilon f\left(U_{\varepsilon,\gamma,\tau,\theta}\right)\right], \psi\right\rangle_{H^1_0}\\
&= \int_\Omega \big(\Delta U_{\varepsilon,\gamma,\tau,\theta} - \lambda_\varepsilon h_\varepsilon f\left(U_{\varepsilon,\gamma,\tau,\theta}\right)\big)\psi dx\\
&= \lambda_\varepsilon \int_\Omega \Bigg(\sum_{i=1}^k (1+\theta_i) h_\varepsilon(\overline{\tau_i}) f\left(\overline{B}_{\varepsilon,\gamma_i,\tau_i}\right)\mathbf{1}_{B(\overline{\tau_i}, r_\varepsilon)} + h_\varepsilon \chi_{\varepsilon,\tau} f\big(U_{\varepsilon,\gamma,\tau}\big)\\
&\quad - h_\varepsilon f\left(U_{\varepsilon,\gamma,\tau,\theta}\right)\Bigg)\psi dx + \sum_{i=1}^k \theta_i \int_\Omega \psi \Delta B_{\varepsilon,\gamma_i,\tau_i} dx.
\end{aligned} \tag{4.4}$$

By using the definition of $\chi_{\varepsilon,\tau}$ together with the mean value theorem, we obtain

$$\begin{aligned}
&\left|\sum_{i=1}^k (1+\theta_i) h_\varepsilon(\overline{\tau_i}) f\left(\overline{B}_{\varepsilon,\gamma_i,\tau_i}\right)\mathbf{1}_{B(\overline{\tau_i}, r_\varepsilon)} + h_\varepsilon \chi_{\varepsilon,\tau} f\big(U_{\varepsilon,\gamma,\tau}\big) - h_\varepsilon f\left(U_{\varepsilon,\gamma,\tau,\theta}\right)\right|\\
&\quad \le \sum_{i=1}^k \left(\left|h_\varepsilon(\overline{\tau_i}) f\left(\overline{B}_{\varepsilon,\gamma_i,\tau_i}\right) - h_\varepsilon f\left(U_{\varepsilon,\gamma,\tau,\theta}\right)\right| + |\theta_i| h_\varepsilon(\overline{\tau_i}) f\left(\overline{B}_{\varepsilon,\gamma_i,\tau_i}\right)\right)\mathbf{1}_{B(\overline{\tau_i}, r_\varepsilon)}\\
&\qquad + h_\varepsilon \sum_{i=1}^k \left|f\big(U_{\varepsilon,\gamma,\tau}\big)\right| \mathbf{1}_{A(\overline{\tau_i}, r_\varepsilon, r_\varepsilon + r_\varepsilon^2)} + h_\varepsilon \left|f\big(U_{\varepsilon,\gamma,\tau}\big) - f\left(U_{\varepsilon,\gamma,\tau,\theta}\right)\right| \mathbf{1}_{\Omega_{r_\varepsilon,\tau}}\\
&\quad \le \sum_{i=1}^k \Big(h_\varepsilon f'\left((1-t_1)\overline{B}_{\varepsilon,\gamma_i,\tau_i} + t_1 U_{\varepsilon,\gamma,\tau,\theta}\right)\left|U_{\varepsilon,\gamma,\tau,\theta} - \overline{B}_{\varepsilon,\gamma_i,\tau_i}\right|\\
&\qquad\quad + \left|\nabla h_\varepsilon\left((1-t_2)\overline{\tau_i} + t_2 x\right)\right| \left|x - \overline{\tau_i}\right| f\left(\overline{B}_{\varepsilon,\gamma_i,\tau_i}\right)\Big)\mathbf{1}_{B(\overline{\tau_i}, r_\varepsilon)}
\end{aligned}$$

$$+ h_\varepsilon \sum_{i=1}^{k} \left|f\left(U_{\varepsilon,\gamma,\tau}\right)\right| \mathbf{1}_{A(\overline{\tau_i}, r_\varepsilon, r_\varepsilon + r_\varepsilon^2)} + \bigg(h_\varepsilon\left(\overline{\tau_i}\right) f\left(\overline{B}_{\varepsilon,\gamma_i,\tau_i}\right)$$

$$+ h_\varepsilon f'\bigg(U_{\varepsilon,\gamma,\tau} + t_3 \sum_{i=1}^{k} \theta_i B_{\varepsilon,\gamma_i,\tau_i} \bigg)\bigg) \sum_{i=1}^{k} |\theta_i|\, B_{\varepsilon,\gamma_i,\tau_i} \mathbf{1}_{\Omega_{r_\varepsilon,\tau}} \tag{4.5}$$

for some functions $t_1, t_2, t_3 : \Omega \to [0,1]$, where $A\left(\overline{\tau_i}, r_\varepsilon, r_\varepsilon + r_\varepsilon^2\right)$ and $\Omega_{r_\varepsilon,\tau}$ are as in (3.18). Since $\lambda_\varepsilon \to \lambda_0$ and $h_\varepsilon \to h_0$ in $C^1\left(\overline{\Omega}\right)$, it follows from (4.4) and (4.5) that

$$\langle R_{\varepsilon,\gamma,\tau,\theta}, \psi \rangle_{H_0^1} = \mathrm{O}\Bigg(\sum_{i=1}^{k} \int_\Omega \bigg(\Big(f'\left(\left(1-t_1\right)\overline{B}_{\varepsilon,\gamma_i,\tau_i} + t_1 U_{\varepsilon,\gamma,\tau,\theta}\right) \left|U_{\varepsilon,\gamma,\tau,\theta} - \overline{B}_{\varepsilon,\gamma_i,\tau_i}\right|$$

$$+ |x - \overline{\tau_i}|\, f\left(\overline{B}_{\varepsilon,\gamma_i,\tau_i}\right) \Big) \mathbf{1}_{B(\overline{\tau_i}, r_\varepsilon)} + \left|f\left(U_{\varepsilon,\gamma,\tau}\right)\right| \mathbf{1}_{A(\overline{\tau_i}, r_\varepsilon, r_\varepsilon + r_\varepsilon^2)}$$

$$+ f'\bigg(U_{\varepsilon,\gamma,\tau} + t_4 \sum_{j=1}^{k} \theta_j B_{\varepsilon,\gamma_j,\tau_j} \bigg) |\theta_i|\, B_{\varepsilon,\gamma_i,\tau_i} \mathbf{1}_{\Omega_{r_\varepsilon,\tau}} \bigg) |\psi| dx \Bigg). \tag{4.6}$$

For every $i \in \{1, \dots, k\}$, by using (3.34) and remarking that $f'(u) \le 3u f(u)$ for all $u \ge 1$, we obtain

$$\int_{B(\overline{\tau_i}, r_\varepsilon)} \left(f'\left(\left(1-t_1\right)\overline{B}_{\varepsilon,\gamma_i,\tau_i} + t_1 U_{\varepsilon,\gamma,\tau,\theta}\right) \left|U_{\varepsilon,\gamma,\tau,\theta} - \overline{B}_{\varepsilon,\gamma_i,\tau_i}\right| + |x - \overline{\tau_i}|\, f\left(\overline{B}_{\varepsilon,\gamma_i,\tau_i}\right) \right)$$

$$\times |\psi| dx = \mathrm{O}\Bigg(\int_{B(\overline{\tau_i}, r_\varepsilon)} f\left(\overline{B}_{\varepsilon,\gamma_i,\tau_i}\right) \bigg(\frac{\delta_\varepsilon \ln \overline{\gamma}_\varepsilon}{\overline{\gamma}_\varepsilon^2} + \overline{\gamma}_\varepsilon^{1/l} |x - \overline{\tau_i}| \bigg) |\psi| dx \Bigg). \tag{4.7}$$

By integrating by parts, we obtain

$$\int_{B(\overline{\tau_i}, r_\varepsilon)} f\left(\overline{B}_{\varepsilon,\gamma_i,\tau_i}\right) |\psi|\, dx = \left(\lambda_\varepsilon h_\varepsilon\left(\overline{\tau_i}\right)\right)^{-1} \left\langle B_{\varepsilon,\gamma_i,\tau_i}, |\psi| \right\rangle_{H_0^1}$$

$$\le \left(\lambda_\varepsilon h_\varepsilon\left(\overline{\tau_i}\right)\right)^{-1} \left\|B_{\varepsilon,\gamma_i,\tau_i}\right\|_{H_0^1} \|\psi\|_{H_0^1} = \mathrm{O}\left(\|\psi\|_{H_0^1} \right). \tag{4.8}$$

On the other hand, for every $p > 1$, by using Hölder's inequality together with the continuity of the embedding $H_0^1(\Omega) \hookrightarrow L^{p'}(\Omega)$, where p' is the conjugate exponent of p, we obtain

$$\int_{B(\overline{\tau_i}, r_\varepsilon)} f\left(\overline{B}_{\varepsilon,\gamma_i,\tau_i}\right) |x - \overline{\tau_i}|\, |\psi|\, dx = \mathrm{O}\left(\left\| f\left(\overline{B}_{\varepsilon,\gamma_i,\tau_i}\right) |x - \overline{\tau_i}|\, \mathbf{1}_{B(\overline{\tau_i}, r_\varepsilon)} \right\|_{L^p} \|\psi\|_{H_0^1} \right), \tag{4.9}$$

$$\int_{A(\overline{\tau_i}, r_\varepsilon, r_\varepsilon + r_\varepsilon^2)} \left|f\left(U_{\varepsilon,\gamma,\tau}\right)\right| |\psi|\, dx = \mathrm{O}\left(\left\| f\left(U_{\varepsilon,\gamma,\tau}\right) \mathbf{1}_{A(\overline{\tau_i}, r_\varepsilon, r_\varepsilon + r_\varepsilon^2)} \right\|_{L^p} \|\psi\|_{H_0^1} \right), \tag{4.10}$$

$$\int_{\Omega_{r_\varepsilon,\tau}} \bigg(f\left(\overline{B}_{\varepsilon,\gamma_i,\tau_i}\right) + f'\bigg(U_{\varepsilon,\gamma,\tau} + t_4 \sum_{j=1}^{k} \theta_j B_{\varepsilon,\gamma_j,\tau_j} \bigg) B_{\varepsilon,\gamma_i,\tau_i} \bigg) |\psi|\, dx$$

$$= \mathrm{O}\Bigg(\bigg\| \bigg(f\left(\overline{B}_{\varepsilon,\gamma_i,\tau_i}\right) + f'\bigg(U_{\varepsilon,\gamma,\tau} + t_4 \sum_{j=1}^{k} \theta_j B_{\varepsilon,\gamma_j,\tau_j} \bigg) B_{\varepsilon,\gamma_i,\tau_i} \bigg) \mathbf{1}_{\Omega_{r_\varepsilon,\tau}} \bigg\|_{L^p} \|\psi\|_{H_0^1} \Bigg). \tag{4.11}$$

By rescaling, we obtain

$$\left\| f\left(\overline{B}_{\varepsilon,\gamma_i,\tau_i}\right) |x - \overline{\tau_i}|\, \mathbf{1}_{B(\overline{\tau_i}, r_\varepsilon)} \right\|_{L^p}^p = \mu_{i,\varepsilon}^{p+2} \int_{B(0, r_\varepsilon/\mu_{i,\varepsilon})} f\left(\overline{B}_{\varepsilon,\gamma_i,\tau_i}\left(\overline{\tau_i} + \mu_{i,\varepsilon} x\right)\right)^p |x|^p\, dx,$$

where $\mu_{i,\varepsilon}$ is defined by $\mu_{i,\varepsilon}^2 := 4\gamma_{i,\varepsilon}^{-2}\exp\left(-\gamma_{i,\varepsilon}^2\right)$. By using (3.10) and (6.2), it follows that

$$\begin{aligned}&\left\|f\left(\overline{B}_{\varepsilon,\gamma_i,\tau_i}\right)\left|x-\overline{\tau_i}\right|\mathbf{1}_{B(\overline{\tau_i},r_\varepsilon)}\right\|_{L^p}^p\\&=\mathrm{O}\left(\mu_{i,\varepsilon}^{p+2}\int_{B(0,r_\varepsilon/\mu_{i,\varepsilon})}f\left(\gamma_{i,\varepsilon}-\frac{1}{\gamma_{i,\varepsilon}}\ln\frac{1}{1+\lambda_\varepsilon h_\varepsilon\left(\overline{\tau_i}\right)|x|^2}\right)^p|x|^p\,dx\right)\\&=\mathrm{O}\left(\frac{\mu_{i,\varepsilon}^{2-p}}{\overline{\gamma}_\varepsilon^p}\int_{B(0,r_\varepsilon/\mu_{i,\varepsilon})}\frac{|x|^p\,dx}{\left(1+\lambda_\varepsilon h_\varepsilon\left(\overline{\tau_i}\right)|x|^2\right)^{2p}}\right)=\mathrm{O}\left(\frac{\mu_{i,\varepsilon}^{2-p}}{\overline{\gamma}_\varepsilon^p}\right)=\mathrm{o}\left(\left(\frac{\delta_\varepsilon\ln\overline{\gamma}_\varepsilon}{\overline{\gamma}_\varepsilon^{2+1/l}}\right)^p\right)\end{aligned}\tag{4.12}$$

provided we choose p such that $2-p>p\left(\delta_1+1/2\right)$, i.e. $1<p<4/\left(2\delta_1+3\right)$, which is possible since $\delta_1<3\delta_0-2\delta_0^2-1/2<1/2$. As regards the terms in the right-hand sides of (4.10) and (4.11), by using (3.10) and (5.11) and proceeding as in (5.12)–(5.15) and (5.29), we obtain

$$\begin{aligned}\left\|f\left(U_{\varepsilon,\gamma,\tau}\right)\mathbf{1}_{A(\overline{\tau_i},r_\varepsilon,r_\varepsilon+r_\varepsilon^2)}\right\|_{L^p}^p&=\mathrm{O}\left(\overline{\gamma}_\varepsilon^p\ln\left(1+r_\varepsilon\right)\exp\left(\left(p\delta_0-1\right)\delta_0\overline{\gamma}_\varepsilon^2+\mathrm{o}\left(\overline{\gamma}_\varepsilon\right)\right)\right)\\&=\mathrm{O}\left(\overline{\gamma}_\varepsilon^p\exp\left(\left(p\delta_0-\frac{3}{2}\right)\delta_0\overline{\gamma}_\varepsilon^2+\mathrm{o}\left(\overline{\gamma}_\varepsilon^2\right)\right)\right)=\mathrm{o}\left(\left(\frac{\delta_\varepsilon\ln\overline{\gamma}_\varepsilon}{\overline{\gamma}_\varepsilon^2}\right)^p\right),\end{aligned}\tag{4.13}$$

$$\begin{aligned}&\left\|\left(f\left(\overline{B}_{\varepsilon,\gamma_i,\tau_i}\right)+f'\left(U_{\varepsilon,\gamma,\tau}+t_4\sum_{j=1}^k\theta_jB_{\varepsilon,\gamma_j,\tau_j}\right)B_{\varepsilon,\gamma_i,\tau_i}\right)\mathbf{1}_{\Omega_{r_\varepsilon,\tau}}\right\|_{L^p}^p\\&\quad=\mathrm{O}\left(\overline{\gamma}_\varepsilon^{3p+2}\exp\left(\left(p\delta_0-1\right)\delta_0\overline{\gamma}_\varepsilon^2+\mathrm{o}\left(\overline{\gamma}_\varepsilon^2\right)\right)+\frac{1}{\overline{\gamma}_\varepsilon^p}\int_{\Omega_{R_\varepsilon,\tau}}\left|\ln\left|x-\overline{\tau_i}\right|+\mathrm{O}\left(1\right)\right|^p dx\right)\\&\quad=\mathrm{o}\left(1\right)\end{aligned}\tag{4.14}$$

as $\varepsilon\to0$, uniformly in $(\gamma,\tau,\theta)\in P_\varepsilon^k\left(\delta\right)$, provided we choose p such that

$$\left(p\delta_0-\frac{3}{2}\right)\delta_0<-\frac{p}{2}\left(\delta_1+\frac{1}{2}\right)\quad\text{and}\quad p\delta_0-1<0,$$

$$\text{i.e. }1<p<\min\left(\frac{3\delta_0}{2\delta_0^2+\delta_1+1/2},\frac{1}{\delta_0}\right)=\frac{3\delta_0}{2\delta_0^2+\delta_1+1/2},$$

which is possible when assuming (4.2). By putting together (4.6)–(4.14) and using the fact that $|\theta_i|<\delta_\varepsilon\overline{\gamma}_\varepsilon^{-4}\ln\overline{\gamma}_\varepsilon$, we obtain (4.3). This ends the proof of Proposition 4.1. ☐

We let $\mathcal{H}$ be the vector space of all functions in $H_0^1\left(\Omega\right)$ that are even in x_2. For every $\tau\in T_\varepsilon^k\left(\delta\right)$ and $\gamma\in\overline{\Gamma}_\varepsilon^k\left(\tau\right)$, we define

$$V_{\varepsilon,\gamma,\tau}:=\operatorname{span}\left\{Z_{0,i,\varepsilon,\gamma,\tau},Z_{1,i,\varepsilon,\gamma,\tau},B_{\varepsilon,\gamma_i,\tau_i}\right\}_{1\le i\le k},$$

where

$$Z_{0,i,\varepsilon,\gamma,\tau}:=\partial_{\gamma_i}\left[U_{\varepsilon,\gamma,\tau,0}\right]\quad\text{and}\quad Z_{1,i,\varepsilon,\gamma,\tau}:=\partial_{\tau_i}\left[U_{\varepsilon,\gamma,\tau,0}\right]\quad\forall i\in\left\{1,\dots,k\right\}.$$

Note that $U_{\varepsilon,\gamma,\tau,0}\in\mathcal{H}$ and $V_{\varepsilon,\gamma,\tau}\subset\mathcal{H}$. We let $\Pi_{\varepsilon,\gamma,\tau}$ and $\Pi_{\varepsilon,\gamma,\tau}^\perp$ be the orthogonal projection of $\mathcal{H}$ onto $V_{\varepsilon,\gamma,\tau}$ and $V_{\varepsilon,\gamma,\tau}^\perp$, respectively. We obtain the following:

Proposition 4.2. *Assume that (4.2) holds true. Let ε_5 be as in Proposition 4.1. Then for every $\delta\in(0,1)$, there exist $\varepsilon_6\left(\delta\right)\in\left(0,\varepsilon_5\left(\delta\right)\right)$ and $C_6=C_6\left(\delta\right)>0$ such that for every $\varepsilon\in\left(0,\varepsilon_6\left(\delta\right)\right)$ and $(\gamma,\tau,\theta)\in P_\varepsilon^k\left(\delta\right)$, there exists a unique solution $\Phi_{\varepsilon,\gamma,\tau,\theta}\in V_{\varepsilon,\gamma,\tau}^\perp$ to the equation*

$$\Pi_{\varepsilon,\gamma,\tau}^\perp\left(U_{\varepsilon,\gamma,\tau,\theta}+\Phi_{\varepsilon,\gamma,\tau,\theta}-\Delta^{-1}\left[\lambda_\varepsilon h_\varepsilon f\left(U_{\varepsilon,\gamma,\tau,\theta}+\Phi_{\varepsilon,\gamma,\tau,\theta}\right)\right]\right)=0\tag{4.15}$$

such that

$$\left\|\Phi_{\varepsilon,\gamma,\tau,\theta}\right\|_{H_0^1}\le C_6\frac{\delta_\varepsilon\ln\overline{\gamma}_\varepsilon}{\overline{\gamma}_\varepsilon^2}. \tag{4.16}$$

Furthermore, $\Phi_{\varepsilon,\gamma,\tau,\theta}$ *is continuous in* (γ,τ,θ)*.*

The proof of Proposition 4.2 relies on the following:

Lemma 4.3. *Let* ε_5 *be as in Proposition 4.1. For every* $\delta\in(0,1)$*, there exist* $\varepsilon_5'(\delta)\in(0,\varepsilon_5(\delta))$ *and* $C_5'=C_5'(\delta)>0$ *such that for every* $\varepsilon\in(0,\varepsilon_5'(\delta))$ *and* $(\gamma,\tau,\theta)\in P_\varepsilon^k(\delta)$*, the operator* $L_{\varepsilon,\gamma,\tau,\theta}:V_{\varepsilon,\gamma,\tau}^\perp\to V_{\varepsilon,\gamma,\tau}^\perp$ *defined by*

$$L_{\varepsilon,\gamma,\tau,\theta}(\Phi)=\Pi_{\varepsilon,\gamma,\tau}^\perp\left(\Phi-\Delta^{-1}\left[\lambda_\varepsilon h_\varepsilon f'\left(U_{\varepsilon,\gamma,\tau,\theta}\right)\Phi\right]\right)\quad\forall\Phi\in V_{\varepsilon,\gamma,\tau}^\perp \tag{4.17}$$

satisfies

$$\|\Phi\|_{H_0^1}\le C_5'\left\|L_{\varepsilon,\gamma,\tau,\theta}(\Phi)\right\|_{H_0^1}. \tag{4.18}$$

In particular, $L_{\varepsilon,\gamma,\tau,\theta}$ *is an isomorphism.*

Proposition 4.2 and Lemma 4.3 (together with Proposition 4.4 and Lemma 4.5), are the heart of the Lyapunov–Schmidt procedure. We prove them by using a similar approach as in the case of higher dimensions (see for instance Deng–Musso–Wei [7] and Robert–Vétois [20,21]). Aside from the usual differences in the computations due to the exponential term, the main difference here lies in the use of the Poincaré–Sobolev inequalities (7.2) and (7.7), which take advantage of the additional dimensions of the kernel $V_{\varepsilon,\gamma,\tau}$ given by the directions of the bubbles.

Proof of Lemma 4.3. We proceed by contradiction. We assume that there exist sequences $(\varepsilon_n,\gamma_n,\tau_n,\theta_n,\Phi_n)_{n\in\mathbb{N}^*}$ such that $\varepsilon_n\to0$, $(\gamma_n,\tau_n,\theta_n)\in P_\varepsilon^k(\delta)$ and

$$\Phi_n\in V_{\varepsilon_n,\gamma_n,\tau_n}^\perp,\quad\|\Phi_n\|_{H_0^1}=1\quad\text{and}\quad\left\|L_{\varepsilon_n,\gamma_n,\tau_n,\theta_n}(\Phi_n)\right\|_{H_0^1}=\mathrm{o}(1) \tag{4.19}$$

as $n\to\infty$. For simplicity of notations, we denote $\overline{\gamma}_n:=\overline{\gamma}_{\varepsilon_n}$, $r_n:=r_{\varepsilon_n}$, $d_n:=d_{\varepsilon_n}$, $\lambda_n:=\lambda_{\varepsilon_n}$, $h_n:=h_{\varepsilon_n}$, $w_n:=w_{\varepsilon_n}$, $\Psi_n:=\Psi_{\varepsilon_n,\gamma_n,\tau_n}$, $U_n:=U_{\varepsilon_n,\gamma_n,\tau_n,\theta_n}$, $\overline{B}_{i,n}:=\overline{B}_{\varepsilon_n,\gamma_{i,n},\overline{\tau_{i,n}}}$, $B_{i,n}:=B_{\varepsilon_n,\gamma_{i,n},\tau_{i,n}}$, $L_n:=L_{\varepsilon_n,\gamma_n,\tau_n,\theta_n}$, $V_n^\perp:=V_{\varepsilon_n,\gamma_n,\tau_n}^\perp$ and $Z_{j,i,n}:=Z_{j,i,\varepsilon_n,\gamma_n,\tau_n}$ for all $i\in\{1,\dots,k\}$ and $j\in\{0,1\}$, where $\gamma_n:=(\gamma_{1,n},\dots,\gamma_{k,n})$, $\tau_n:=(\tau_{1,n},\dots,\tau_{k,n})$, $\overline{\tau_{i,n}}:=(\tau_{i,n},0)$ and $\theta_n:=(\theta_{1,n},\dots,\theta_{k,n})$. It follows from (4.19) that

$$\lambda_n\int_\Omega h_nf'(U_n)\,\Phi_n^2dx=\|\Phi_n\|_{H_0^1}^2-\langle\Phi_n,L_n(\Phi_n)\rangle_{H_0^1}=1+\mathrm{o}(1) \tag{4.20}$$

as $n\to\infty$. On the other hand, since $f'>0$, $\lambda_n\to\lambda_0$ and $h_n\to h_0$ in $C^0\left(\overline{\Omega}\right)$, we obtain

$$\lambda_n\int_\Omega h_nf'(U_n)\,\Phi_n^2dx=\mathrm{O}(I_n),\quad\text{where }I_n:=\int_\Omega f'(U_n)\,\Phi_n^2dx. \tag{4.21}$$

In what follows, we will prove that $I_n\to0$ as $n\to\infty$, thus contradicting (4.20) and (4.21).

Estimation of I_n *in the balls* $B\left(\overline{\tau_{i,n}},r_n\right)$*.* For $i\in\{1,\dots,k\}$, by rescaling and using (4.19), we obtain

$$\int_{B(\overline{\tau_{i,n}},r_n)}f'(U_n)\,\Phi_n^2dx=\mu_{i,n}^2\int_{B(0,r_n/\mu_{i,n})}f'\big(\gamma_{i,n}^{-1}\widehat{U}_n+\gamma_{i,n}\big)\widehat{\Phi}_n^2dx, \tag{4.22}$$

$$\int_{(\Omega-\overline{\tau_{i,n}})/\mu_{i,n}}\langle\nabla\widehat{\Phi}_n,\nabla\psi\rangle dx-\lambda_n\mu_{i,n}^2\int_{(\Omega-\overline{\tau_{i,n}})/\mu_{i,n}}\hat{h}_nf'\big(\gamma_{i,n}^{-1}\widehat{U}_n+\gamma_{i,n}\big)\widehat{\Phi}_n\psi dx$$
$$=\mathrm{o}\left(\|\nabla\psi\|_{L^2}\right)\qquad\forall\psi\in C_c^\infty\left(\mathbb{R}^2\right) \tag{4.23}$$

as $n\to\infty$, where $\mu_{i,n}$, $\hat{h}_n$, $\widehat{\Phi}_n$ and $\widehat{U}_n$ are defined by

$$\mu_{i,n}^2:=4\gamma_{i,n}^{-2}\exp\left(-\gamma_{i,n}^2\right),\quad \hat{h}_n\left(x\right):=h_n\left(\overline{\tau_{i,n}}+\mu_{i,n}x\right),$$
$$\widehat{\Phi}_n\left(x\right):=\Phi_n\left(\overline{\tau_{i,n}}+\mu_{i,n}x\right)\quad\text{and}\quad \widehat{U}_n\left(x\right):=\gamma_{i,n}\left(U_n\left(\overline{\tau_{i,n}}+\mu_{i,n}x\right)-\gamma_{i,n}\right)$$

for all $x\in(\Omega-\overline{\tau_{i,n}})/n$. By using (3.10), (3.34) and (6.2), we obtain

$$\widehat{U}_n\left(x\right)\sim\gamma_{i,n}\left(\overline{B}_{i,n}\left(\overline{\tau_{i,n}}+\mu_{i,n}x\right)-\gamma_{i,n}\right)\sim\ln\frac{1}{1+\lambda_nh_n\left(\overline{\tau_{i,n}}\right)\left|x\right|^2}\tag{4.24}$$

as $n\to\infty$, uniformly in $x\in B\left(0,r_n/\mu_{i,n}\right)$. By using (4.24) together with the definition of $\mu_{i,n}$, we obtain

$$\mu_{i,n}^2f'\big(\gamma_{i,n}^{-1}\widehat{U}_n+\gamma_{i,n}\big)\sim\frac{8}{\left(1+\lambda_nh_n\left(\overline{\tau_{i,n}}\right)\left|x\right|^2\right)^2}\tag{4.25}$$

as $n\to\infty$, uniformly in $x\in B\left(0,r_n/\mu_{i,n}\right)$. By remarking that

$$\big\|\nabla\widehat{\Phi}_n\big\|_{L^2}=\left\|\nabla\Phi_n\right\|_{L^2}=1$$

and using (4.23) and (4.25), we obtain that $\big(\widehat{\Phi}_n\big)_n$ converges, up to a subsequence, weakly in $D^{1,2}\left(\mathbb{R}^2\right)$, strongly in $L^p_{\text{loc}}\left(\mathbb{R}^2\right)$ for all $p\ge1$ and pointwise almost everywhere in $\mathbb{R}^2$ to a solution $\widehat{\Phi}_0$ of the equation

$$\Delta\widehat{\Phi}_0=\frac{8\lambda_0h_0\left(0\right)\widehat{\Phi}_0}{\left(1+\lambda_0h_0\left(0\right)\left|x\right|^2\right)^2}\qquad\text{in }\mathbb{R}^2.\tag{4.26}$$

Furthermore, since $\Phi_n\in\mathcal{H}$, we obtain that $\widehat{\Phi}_0$ is even in x_2. By using a result of Baraket–Pacard [2], it follows that $\widehat{\Phi}_0\in\operatorname{span}\left\{Z_0,Z_1\right\}$, where

$$Z_0\left(x\right):=\frac{1-\lambda_0h_0\left(0\right)\left|x\right|^2}{1+\lambda_0h_0\left(0\right)\left|x\right|^2}\quad\text{and}\quad Z_1\left(x\right):=\frac{2\lambda_0h_0\left(0\right)x_1}{1+\lambda_0h_0\left(0\right)\left|x\right|^2}\quad\forall x\in\mathbb{R}^2\,.$$

In particular, note that the Poincaré–Sobolev inequality (7.2) applies to $\widehat{\Phi}_0$. On the other hand, for every $i\in\{1,\dots,k\}$, since $\Phi_n\in E^\perp_{\varepsilon_n,\gamma_n,\tau_n}$, we get $\left\langle B_{i,n},\Phi_n\right\rangle_{H^1_0}=\left\langle Z_{0,i,n},\Phi_n\right\rangle_{H^1_0}=\left\langle Z_{1,i,n},\Phi_n\right\rangle_{H^1_0}=0$, which, by integrating by parts and using the equations satisfied by $B_{i,n}$, $Z_{0,i,n}$ and $Z_{1,i,n}$, gives

$$\int_{B(\overline{\tau_{i,n}},r_n)}f\left(\overline{B}_{i,n}\right)\Phi_ndx=0,\tag{4.27}$$

$$\lambda_nh_n\left(\overline{\tau_{i,n}}\right)\int_{B(\overline{\tau_{i,n}},r_n)}f'\left(\overline{B}_{i,n}\right)\partial_{\gamma_i}\left[\overline{B}_{\varepsilon_n,\gamma,\overline{\tau_{i,n}}}\right]_{\gamma=\gamma_{i,n}}\Phi_ndx+\Big\langle\Phi_n,\partial_{\gamma_i}\left[\Psi_{\varepsilon_n,\gamma,\tau_n,0}\right]_{\gamma=\gamma_n}\Big\rangle_{H^1_0}=0\tag{4.28}$$

together with an analogous estimate for the derivative in τ_i. It follows from (4.19), (4.27) and the Poincaré–Sobolev inequality (7.7) that

$$\int_{B(\overline{\tau_{i,n}},r_n)}f'\left(\overline{B}_{i,n}\right)\Phi_n^2dx=\mathrm{O}\left(\left\|\nabla\Phi_n\right\|^2_{L^2}\right)=\mathrm{O}\left(1\right).\tag{4.29}$$

On the other hand, by using Cauchy–Schwartz' inequality together with (3.14), (3.15) and (4.19), we obtain

$$\Big\langle\Phi_n,\partial_{\gamma_i}\left[\Psi_{\varepsilon_n,\gamma,\tau_n,0}\right]_{\gamma=\gamma_n}\Big\rangle_{H^1_0}=\mathrm{o}\left(1\right)\ \text{ and }\ \Big\langle\Phi_n,\partial_{\tau_i}\left[\Psi_{\varepsilon_n,\gamma_n,\tau,0}\right]_{\tau=\tau_n}\Big\rangle_{H^1_0}=\mathrm{o}\left(1\right)\tag{4.30}$$

as $n\to\infty$. By rescaling, it follows from (4.28) and (4.29) that

$$\mu_{i,n}^2\int_{B(0,r_n/\mu_{i,n})} f'\left(\overline{B}_{i,n}\left(\overline{\tau_{i,n}}+\mu_{i,n}x\right)\right)\widehat{\Phi}_n\left(x\right)^2dx=\mathrm{O}\left(1\right),\tag{4.31}$$

$$\left\langle\Phi_n,\partial_{\gamma_i}\left[\Psi_{\varepsilon_n,\gamma,\tau_n,0}\right]_{\gamma=\gamma_n}\right\rangle_{H_0^1}=-\lambda_nh_n\left(\overline{\tau_{i,n}}\right)\mu_{i,n}^2\int_{B(0,r_n/\mu_{i,n})}f'\left(\overline{B}_{i,n}\left(\overline{\tau_{i,n}}+\mu_{i,n}x\right)\right)\times\partial_{\gamma_i}\left[\overline{B}_{\varepsilon_n,\gamma,\overline{\tau_{i,n}}}\left(\overline{\tau_{i,n}}+\mu_{i,n}x\right)\right]_{\gamma=\gamma_n}\widehat{\Phi}_n\left(x\right)dx.\tag{4.32}$$

Here again, we obtain an analogous estimate to (4.32) for the derivative in τ_i. By using (6.2) and (6.3) together with the definition of $\mu_{i,n}$, we obtain

$$\partial_{\gamma_i}\left[\overline{B}_{\varepsilon_n,\gamma,\tau_n}\left(\overline{\tau_{i,n}}+\mu_{i,n}x\right)\right]_{\gamma=\gamma_n}\longrightarrow Z_0\left(x\right)\ \text{ for a.e. } x\in\mathbb{R}^2,\tag{4.33}$$

$$\partial_{\gamma_i}\left[\overline{B}_{\varepsilon_n,\gamma,\tau_n}\left(\overline{\tau_{i,n}}+\mu_{i,n}x\right)\right]_{\gamma=\gamma_n}=\mathrm{O}\left(1\right),\tag{4.34}$$

$$\partial_{\tau_i}\left[\overline{B}_{\varepsilon_n,\gamma_{i,n},(\tau,0)}\left(\overline{\tau_{i,n}}+\mu_{i,n}x\right)\right]_{\tau=\tau_{i,n}}=\frac{Z_1\left(x\right)}{\mu_{i,n}\gamma_{i,n}}+\mathrm{o}\left(\frac{1}{\mu_{i,n}\gamma_{i,n}}\right)\tag{4.35}$$

as $n\to\infty$, uniformly in $x\in B\left(0,r_n/\mu_{i,n}\right)$. For every $R>0$, since $\left(\widehat{\Phi}_n\right)_n$ converges strongly to $\widehat{\Phi}_0$ in $L^1_{\text{loc}}\left(\mathbb{R}^2\right)$, it follows from (4.24), (4.25) and (4.33) that

$$\mu_{i,n}^2\int_{B(0,R)}f'\left(\overline{B}_{i,n}\left(\overline{\tau_{i,n}}+\mu_{i,n}x\right)\right)\partial_{\gamma_i}\left[\overline{B}_{\varepsilon_n,\gamma,\overline{\tau_{i,n}}}\left(\overline{\tau_{i,n}}+\mu_{i,n}x\right)\right]_{\gamma=\gamma_n}\widehat{\Phi}_n\left(x\right)dx\longrightarrow 8\int_{B(0,R)}\frac{8Z_0\left(x\right)\widehat{\Phi}_0\left(x\right)dx}{\left(1+\lambda_0h_0\left(0\right)\left|x\right|^2\right)^2}\tag{4.36}$$

as $n\to\infty$. On the other hand, by using Hölder's inequality together with (4.24), (4.25), (4.31), (4.34) and (7.2), we obtain

$$\begin{aligned}&\left|\mu_{i,n}^2\int_{A(0,R,r_n/\mu_{i,n})}f'\left(\overline{B}_{i,n}\left(\overline{\tau_{i,n}}+\mu_{i,n}x\right)\right)\partial_{\gamma_i}\left[\overline{B}_{\varepsilon_n,\gamma,\overline{\tau_{i,n}}}\left(\overline{\tau_{i,n}}+\mu_{i,n}x\right)\right]_{\gamma=\gamma_n}\widehat{\Phi}_n\left(x\right)dx\right|\\&\quad=\mathrm{O}\left(\left(\mu_{i,n}^2\int_{A(0,R,r_n/\mu_{i,n})}f'\left(\overline{B}_{i,n}\left(\overline{\tau_{i,n}}+\mu_{i,n}x\right)\right)dx\right)^{1/2}\right)\\&\quad=\mathrm{O}\left(\left(\int_{B(0,R)^c}\frac{dx}{\left(1+\lambda_nh_n\left(\overline{\tau_{i,n}}\right)\left|x\right|^2\right)^2}\right)^{1/2}\right)=\mathrm{o}_R\left(1\right),\end{aligned}\tag{4.37}$$

$$\left|\int_{B(0,R)^c}\frac{Z_0\left(x\right)\widehat{\Phi}_0\left(x\right)dx}{\left(1+\lambda_0h_0\left(0\right)\left|x\right|^2\right)^2}\right|=\mathrm{O}\left(\left(\int_{B(0,R)^c}\frac{dx}{\left(1+\lambda_0h_0\left(0\right)\left|x\right|^2\right)^2}\right)^{1/2}\right)=\mathrm{o}_R\left(1\right),\tag{4.38}$$

where $\mathrm{o}_R\left(1\right)\to 0$ as $R\to\infty$, uniformly in $n\in\mathbb{N}^*$, where $A\left(0,R,r_n/\mu_{i,n}\right)$ is as in (3.18). It follows from (4.30), (4.32) and (4.36)–(4.38) that

$$\int_{\mathbb{R}^2}\frac{Z_0\left(x\right)\widehat{\Phi}_0\left(x\right)dx}{\left(1+\lambda_0h_0\left(0\right)\left|x\right|^2\right)^2}=0.\tag{4.39}$$

By proceeding in the same way but using (4.35) instead of (4.33)–(4.34), we obtain

$$\int_{\mathbb{R}^2}\frac{Z_1\left(x\right)\widehat{\Phi}_0\left(x\right)dx}{\left(1+\lambda_0h_0\left(0\right)\left|x\right|^2\right)^2}=0.\tag{4.40}$$

Since $\widehat{\Phi}_0 \in \operatorname{span}\{Z_0, Z_1\}$, it follows from (4.26), (4.39) and (4.40) that $\widehat{\Phi}_0 \equiv 0$. For every $R > 0$, by using (4.25) and since $\big(\widehat{\Phi}_n\big)_n$ converges strongly to $\widehat{\Phi}_0$ in $L^2_{\text{loc}}\left(\mathbb{R}^2\right)$, we then obtain

$$\mu_{i,n}^2 \int_{B(0,R)} f'\big(\gamma_{i,n}^{-1}\widehat{U}_n + \gamma_{i,n}\big)\Phi_n^2 dx = \mathrm{o}\left(1\right) \tag{4.41}$$

as $n \to \infty$. On the other hand, by proceeding as in (4.37), we obtain

$$\mu_{i,n}^2 \int_{A(0,R,r_n/\mu_{i,n})} f'\big(\gamma_{i,n}^{-1}\widehat{U}_n + \gamma_{i,n}\big)\Phi_n^2 dx = \mathrm{o}_R\left(1\right), \tag{4.42}$$

where $\mathrm{o}_R(1) \to 0$ as $R \to \infty$, uniformly in $n \in \mathbb{N}^*$. It follows from (4.22), (4.41) and (4.42) that

$$\int_{B(\overline{\tau_{i,n}},r_n)} f'\left(U_n\right)\Phi_n^2 dx = \mathrm{o}\left(1\right) \tag{4.43}$$

as $n \to \infty$.

Estimation of I_n in the annuli $A\left(\overline{\tau_{i,n}}, r_n, R_n\right)$, where $R_n := \exp\left(-\overline{\gamma}_n\right)$. For every $i \in \{1,\dots,k\}$, for small $p > 1$, by using Hölder's inequality, (3.16) and (4.19) together with the continuity of the embedding $H^1_0(\Omega) \hookrightarrow L^{2p'}(\Omega)$, we obtain

$$\begin{aligned}\int_{A(\overline{\tau_{i,n}},r_n,R_n)} f'\left(U_n\right)\Phi_n^2 dx &= \mathrm{O}\left(\left\|f'\left(U_n\right)\mathbf{1}_{A(\overline{\tau_{i,n}},r_n,R_n)}\right\|_{L^p}\left\|\Phi_n\right\|^2_{H^1_0}\right)\\ &= \mathrm{O}\left(\left\|f'\left(U_n\right)\mathbf{1}_{A(\overline{\tau_{i,n}},r_n,R_n)}\right\|_{L^p}\right) = \mathrm{o}\left(1\right)\end{aligned} \tag{4.44}$$

as $n \to \infty$.

Estimation of I_n in Ω_{R_n,τ_n}. By using (4.19), we obtain that $\left(\Phi_n\right)_n$ converges, up to a subsequence, weakly in $H^1_0(\Omega)$ and pointwise almost everywhere in Ω to a function Φ_0. Furthermore, (4.19) gives that

$$\int_\Omega \langle\nabla\Phi_n, \nabla\psi\rangle\, dx - \lambda_n \int_\Omega h_n f'\left(U_n\right)\Phi_n\psi dx = \mathrm{o}\left(1\right) \tag{4.45}$$

as $n \to \infty$, for all $\psi \in C^\infty_c(\Omega)$. By rescaling as in (4.22) and using (4.25) together with the fact that $\widehat{\Phi}_n \rightharpoonup 0$ in $D^{1,2}\left(\mathbb{R}^2\right)$, we obtain that

$$\sum_{i=1}^k \int_{B(\overline{\tau_{i,n}},r_n)} h_n f'\left(U_n\right)\Phi_n\psi dx = \mathrm{o}\left(1\right) \tag{4.46}$$

as $n \to \infty$. By using similar estimates as in (4.44), we obtain

$$\sum_{i=1}^k \int_{A(\overline{\tau_{i,n}},r_n,R_n)} h_n f'\left(U_n\right)\Phi_n\psi dx = \mathrm{o}\left(1\right) \tag{4.47}$$

as $n \to \infty$. By using (3.10), (3.14) and since $w_n \to w_0$ in $C^0\left(\overline{\Omega}\right)$, we obtain that $U_n\mathbf{1}_{\Omega_{R_n,\tau_n}}$ is uniformly bounded and converges pointwise to u_0 in Ω. Since moreover $\Phi_n \rightharpoonup \Phi_0$ in $H^1_0(\Omega)$, $\lambda_n \to \lambda_0$ and $h_n \to h_0$ in $C^0\left(\overline{\Omega}\right)$, it follows from (4.45)–(4.47) that Φ_0 is a solution of the equation

$$\Delta\Phi_0 = \lambda_0 h_0 f'\left(w_0\right)\Phi_0 \qquad \text{in } \mathbb{R}^n.$$

Since w_0 is non-degenerate, we then obtain that $\Phi_0 \equiv 0$. It then follows from standard integration theory that

$$\int_{\Omega_{R_n,\tau_n}} f'\left(U_n\right)\Phi_n^2 dx = \mathrm{o}\left(1\right) \tag{4.48}$$

as $n \to \infty$.

Finally, by combining (4.43), (4.44) and (4.48), we obtain a contradiction with (4.20) and (4.21). This ends the proof of Lemma 4.3. □

Proof of Proposition 4.2. We let $N_{\varepsilon,\gamma,\tau,\theta}: V^{\perp}_{\varepsilon,\gamma,\tau} \to V^{\perp}_{\varepsilon,\gamma,\tau}$ and $T_{\varepsilon,\gamma,\tau,\theta}: V^{\perp}_{\varepsilon,\gamma,\tau} \to V^{\perp}_{\varepsilon,\gamma,\tau}$ be the operators defined as

$$N_{\varepsilon,\gamma,\tau,\theta}(\Phi) := \Pi^{\perp}_{\varepsilon,\gamma,\tau}\big(\Delta^{-1}[\lambda_\varepsilon h_\varepsilon (f(U_{\varepsilon,\gamma,\tau,\theta}+\Phi) - f(U_{\varepsilon,\gamma,\tau,\theta}) - f'(U_{\varepsilon,\gamma,\tau,\theta})\Phi)]\big),$$
$$T_{\varepsilon,\gamma,\tau,\theta}(\Phi) := L^{-1}_{\varepsilon,\gamma,\tau,\theta}\big(N_{\varepsilon,\gamma,\tau,\theta}(\Phi) - \Pi^{\perp}_{\varepsilon,\gamma,\tau}(R_{\varepsilon,\gamma,\tau,\theta})\big)$$

for all $\Phi \in V^{\perp}_{\varepsilon,\gamma,\tau}$, where $R_{\varepsilon,\gamma,\tau,\theta}$ and $L_{\varepsilon,\gamma,\tau,\theta}$ are as in (4.1) and (4.17). Remark that the equation (4.15) can be rewritten as the fixed point equation $T_{\varepsilon,\gamma,\tau,\theta}(\Phi) = \Phi$. For every $C > 0$, $\varepsilon \in (0, \varepsilon'_5)$ and $(\gamma, \tau, \theta) \in P^k_\varepsilon(\delta)$, we define

$$V_{\varepsilon,\gamma,\tau,\theta}(C) := \left\{\Phi \in V^{\perp}_{\varepsilon,\gamma,\tau} : \|\Phi\|_{H^1_0} \le C \frac{\delta_\varepsilon \ln \overline{\gamma}_\varepsilon}{\overline{\gamma}_\varepsilon^2}\right\}.$$

We will prove that if C is chosen large enough, then $T_{\varepsilon,\gamma,\tau,\theta}$ has a fixed point in $V_{\gamma,\tau,\theta}(C)$. By using (4.18), we obtain

$$\|T_{\varepsilon,\gamma,\tau,\theta}(\Phi)\|_{H^1_0} \le C'_5\big(\|N_{\varepsilon,\gamma,\tau,\theta}(\Phi)\|_{H^1_0} + \|R_{\varepsilon,\gamma,\tau,\theta}\|_{H^1_0}\big). \tag{4.49}$$

For every $\Phi_1, \Phi_2 \in V_{\varepsilon,\gamma,\tau,\theta}(C)$ and $\psi \in V^{\perp}_{\varepsilon,\gamma,\tau}$, by integrating by parts and applying the mean value theorem, we obtain

$$\begin{aligned}\langle N_{\varepsilon,\gamma,\tau,\theta}(\Phi_1) - N_{\varepsilon,\gamma,\tau,\theta}(\Phi_2), \psi\rangle_{H^1_0} &\\ = \lambda_\varepsilon \int_\Omega h_\varepsilon \left(f'(U_{\varepsilon,\gamma,\tau,\theta} + t\Phi_1 + (1-t)\Phi_2) - f'(U_{\varepsilon,\gamma,\tau,\theta})\right)(\Phi_1 - \Phi_2)\psi dx &\\ = \lambda_\varepsilon \int_\Omega h_\varepsilon f''(U_{\varepsilon,\gamma,\tau,\theta} + st\Phi_1 + s(1-t)\Phi_2)(t\Phi_1 + (1-t)\Phi_2)(\Phi_1 - \Phi_2)\psi dx & \quad (4.50)\end{aligned}$$

for some functions $s, t : \Omega \to [0,1]$. Since $\lambda_\varepsilon \to \lambda_0$, $h_\varepsilon \to h_0$ in $C^0\left(\overline{\Omega}\right)$ and f'' is increasing, it follows from (4.50) that

$$\begin{aligned}\langle N_{\varepsilon,\gamma,\tau,\theta}(\Phi_1) - N_{\varepsilon,\gamma,\tau,\theta}(\Phi_2), \psi\rangle_{H^1_0} &\\ = \mathrm{O}\left(\int_\Omega f''(|U_{\varepsilon,\gamma,\tau,\theta}| + |\Phi_1| + |\Phi_2|)(|\Phi_1| + |\Phi_2|)|\Phi_1 - \Phi_2||\psi| dx\right). & \quad (4.51)\end{aligned}$$

For every $p > 1$, by using Hölder's inequality together with the continuity of the embedding $H^1_0(\Omega) \hookrightarrow L^{3p'}(\Omega)$, we obtain

$$\begin{aligned}\int_\Omega f''(|U_{\varepsilon,\gamma,\tau,\theta}| + |\Phi_1| + |\Phi_2|)(|\Phi_1| + |\Phi_2|)|\Phi_1 - \Phi_2||\psi| dx &\\ = \mathrm{O}\left(\|f''(|U_{\varepsilon,\gamma,\tau,\theta}| + |\Phi_1| + |\Phi_2|)\|_{L^p} \||\Phi_1| + |\Phi_2|\|_{H^1_0} \|\Phi_1 - \Phi_2\|_{H^1_0} \|\psi\|_{H^1_0}\right). & \quad (4.52)\end{aligned}$$

Since f'' is increasing, we obtain

$$f''(|U_{\varepsilon,\gamma,\tau,\theta}| + |\Phi_1| + |\Phi_2|) \le f''\big(\widetilde{U}_{\varepsilon,\gamma,\tau,\theta}\big) + f''\big(\widetilde{\Phi}_\varepsilon\big), \tag{4.53}$$

where

$$\widetilde{U}_{\varepsilon,\gamma,\tau,\theta} := (1+\delta_\varepsilon)|U_{\varepsilon,\gamma,\tau,\theta}| \quad \text{and} \quad \widetilde{\Phi}_\varepsilon := \left(1+\delta_\varepsilon^{-1}\right)(|\Phi_1| + |\Phi_2|).$$

Remark that $\widetilde{\Phi}_\varepsilon \to 0$ in $H^1_0(\Omega)$ as $\varepsilon \to 0$ since $\Phi_1, \Phi_2 \in V_{\varepsilon,\gamma,\tau,\theta}(C)$. By using Hölder's inequality together with the Moser–Trudinger's inequality and the continuity of the embedding

$H_0^1(\Omega)\hookrightarrow L^{6p}(\Omega)$, we then obtain

$$\begin{aligned}\big\|f''\big(\widetilde{\Phi}_\varepsilon\big)\big\|_{L^p}&=2\big\|\widetilde{\Phi}_\varepsilon\big(3+2\widetilde{\Phi}_\varepsilon^2\big)\exp\big(\widetilde{\Phi}_\varepsilon^2\big)\big\|_{L^{2p}}\\&\le 2\big\|\widetilde{\Phi}_\varepsilon\big\|_{H_0^1}\big(3+2\big\|\widetilde{\Phi}_\varepsilon\big\|_{H_0^1}^2\big)\big\|\exp\big(\widetilde{\Phi}_\varepsilon^2\big)\big\|_{L^{2p}}=\mathrm{o}(1)\end{aligned}\tag{4.54}$$

as $\varepsilon\to 0$. For every $i\in\{1,\dots,k\}$, by remarking that $f''(s)\le 6sf'(s)$ for all $s\ge 0$ and using similar estimates as in (4.24) and (4.25), we obtain

$$f''\big(\widetilde{U}_{\varepsilon,\gamma,\tau,\theta}(x)\big)=\mathrm{O}\left(\frac{\overline{\gamma}_i(\tau)\,\overline{\mu}_i(\tau)^2}{\Big(\overline{\mu}_i(\tau)^2+|x-\overline{\tau_i}|^2\Big)^2}\right)\tag{4.55}$$

uniformly in $x\in B(\overline{\tau_i},r_\varepsilon)$, where $\overline{\mu}_i(\tau)$ is defined by

$$\overline{\mu}_i(\tau)^2:=4\overline{\gamma}_i(\tau)^{-2}\exp\big(-\overline{\gamma}_i(\tau)^2\big).$$

It follows from (4.55) that

$$\big\|f''\big(\widetilde{U}_{\varepsilon,\gamma,\tau,\theta}\big)\mathbf{1}_{B(\overline{\tau_i},r_\varepsilon)}\big\|_{L^p}=\mathrm{O}\Big(\overline{\gamma}_i(\tau)\,\overline{\mu}_i(\tau)^{-2/p'}\Big),\tag{4.56}$$

where p' is the conjugate exponent of p. By using (3.16), we obtain

$$\big\|f''\big(\widetilde{U}_{\varepsilon,\gamma,\tau,\theta}\big)\mathbf{1}_{A(\overline{\tau_i},r_\varepsilon,R_\varepsilon)}\big\|_{L^p}=\mathrm{o}(1)\tag{4.57}$$

as $\varepsilon\to 0$, where $R_\varepsilon:=\exp(-\overline{\gamma}_\varepsilon)$, provided we choose p such that $p<1/\delta_0$. Furthermore, since $U_{\varepsilon,\gamma,\tau,\theta}$ is uniformly bounded in $\Omega_{R_\varepsilon,\tau}$, we obtain

$$\big\|f''\big(\widetilde{U}_{\varepsilon,\gamma,\tau,\theta}\big)\mathbf{1}_{\Omega_{R_\varepsilon,\tau}}\big\|_{L^p}=\mathrm{O}(1)\,.\tag{4.58}$$

By putting together (4.51)–(4.53), (4.54) and (4.24) and (4.58), we obtain

$$\|N_{\varepsilon,\gamma,\tau,\theta}(\Phi_1)-N_{\varepsilon,\gamma,\tau,\theta}(\Phi_2)\|_{H_0^1}=\mathrm{O}\Big(\overline{\gamma}_i(\tau)\,\overline{\mu}_i(\tau)^{-2/p'}\,\||\Phi_1|+|\Phi_2|\|_{H_0^1}\,\|\Phi_1-\Phi_2\|_{H_0^1}\Big)\,.\tag{4.59}$$

Remark that since $\Phi_1,\Phi_2\in V_{\varepsilon,\gamma,\tau,\theta}(C)$, we obtain

$$\overline{\gamma}_i(\tau)\,\overline{\mu}_i(\tau)^{-2/p'}\,\||\Phi_1|+|\Phi_2|\|_{H_0^1}=\mathrm{o}(1)\tag{4.60}$$

as $\varepsilon\to 0$, provided we choose p such that $2/p'<\delta_1+1/2$, i.e. $p<4/(3-2\delta_1)$. It follows from (4.59) and (4.60) that

$$\|N_{\varepsilon,\gamma,\tau,\theta}(\Phi_1)-N_{\varepsilon,\gamma,\tau,\theta}(\Phi_2)\|_{H_0^1}=\mathrm{o}\big(\|\Phi_1-\Phi_2\|_{H_0^1}\big)\tag{4.61}$$

as $\varepsilon\to 0$. By using (4.3), (4.49), (4.61) and since $N_{\varepsilon,\gamma,\tau,\theta}(0)=0$, we obtain that there exist $\varepsilon_6(\delta)\in(0,\varepsilon_5(\delta))$ and $C_6=C_6(\delta)>0$ such that for every $\varepsilon\in(0,\varepsilon_6(\delta))$ and $(\gamma,\tau,\theta)\in P_\varepsilon^k(\delta)$, $T_{\varepsilon,\gamma,\tau,\theta}$ is a contraction mapping on $V_{\varepsilon,\gamma,\tau,\theta}(C_6)$. We can then apply the fixed point theorem, which gives that there exists a unique solution $\Phi_{\varepsilon,\gamma,\tau,\theta}\in V_{\varepsilon,\gamma,\tau,\theta}(C_6)$ to the equation (4.15). The continuity of $\Phi_{\varepsilon,\gamma,\tau,\theta}$ in (γ,τ,θ) follows from the continuity of $U_{\varepsilon,\gamma,\tau,\theta}$, $Z_{0,i,\varepsilon,\gamma,\tau,\theta}$ and $Z_{1,i,\varepsilon,\gamma,\tau,\theta}$ in (γ,τ,θ). This ends the proof of Proposition 4.2. □

As a last step, we prove the following:

Proposition 4.4. *Let ε_6 and $\Phi_{\varepsilon,\gamma,\tau,\theta}$ be as in Proposition 4.2. Then there exists $\delta_7\in(0,1)$ such that for every $\delta\in(0,\delta_7)$, there exists $\varepsilon_7(\delta)\in(0,\varepsilon_7(\delta))$ such that for every $\varepsilon\in(0,\varepsilon_7(\delta))$, there exists $(\gamma_\varepsilon,\tau_\varepsilon,\theta_\varepsilon)\in P_\varepsilon^k(\delta)$ such that*

$$U_{\varepsilon,\gamma_\varepsilon,\tau_\varepsilon,\theta_\varepsilon}+\Phi_{\varepsilon,\gamma_\varepsilon,\tau_\varepsilon,\theta_\varepsilon}=\Delta^{-1}\left[\lambda_\varepsilon h_\varepsilon f\left(U_{\varepsilon,\gamma_\varepsilon,\tau_\varepsilon,\theta_\varepsilon}+\Phi_{\varepsilon,\gamma_\varepsilon,\tau_\varepsilon,\theta_\varepsilon}\right)\right].\tag{4.62}$$

The proof of Proposition 4.4 relies on the following:

Lemma 4.5. *Set*

$$\widetilde{R}_{\varepsilon,\gamma,\tau,\theta} := U_{\varepsilon,\gamma,\tau,\theta} + \Phi_{\varepsilon,\gamma,\tau,\theta} - \Delta^{-1}\left[\lambda_\varepsilon h_\varepsilon f\left(U_{\varepsilon,\gamma,\tau,\theta} + \Phi_{\varepsilon,\gamma,\tau,\theta}\right)\right].$$

Then for every $i \in \{1,\dots,k\}$ *and* $\delta \in (0,1)$*, we have*

$$\left\langle \widetilde{R}_{\varepsilon,\gamma,\tau,\theta}, Z_{0,i,\varepsilon,\gamma,\tau}\right\rangle_{H_0^1} = -8\pi \sum_{j=1}^k \partial_{\gamma_i}\left[E^{(j)}_{\varepsilon,\gamma,\tau}\right]\left(E^{(j)}_{\varepsilon,\gamma,\tau} + \theta_j \overline{\gamma}_\varepsilon\right) + \frac{4\pi}{\overline{\gamma}_\varepsilon^2} E^{(i)}_{\varepsilon,\gamma,\tau} + \mathrm{o}\left(\frac{\delta_\varepsilon \ln \overline{\gamma}_\varepsilon}{\overline{\gamma}_\varepsilon^5}\right), \tag{4.63}$$

$$\left\langle \widetilde{R}_{\varepsilon,\gamma,\tau,\theta}, B_{\varepsilon,\gamma_i,\tau_i}\right\rangle_{H_0^1} = -8\pi\overline{\gamma}_\varepsilon\left(E^{(i)}_{\varepsilon,\gamma,\tau} + \theta_i\overline{\gamma}_\varepsilon\right) + \mathrm{o}\left(\frac{\delta_\varepsilon}{\overline{\gamma}_\varepsilon^2}\right), \tag{4.64}$$

$$\left\langle \widetilde{R}_{\varepsilon,\gamma,\tau,\theta}, Z_{1,i,\varepsilon,\gamma,\tau}\right\rangle_{H_0^1} = -\frac{4\pi}{\overline{\gamma}_\varepsilon}\left(a_0 l \tau_i^{l-1} - \frac{2}{\overline{\gamma}_\varepsilon}\sum_{j\neq i}\frac{1}{\tau_i - \tau_j}\right) + \mathrm{o}\left(\frac{1}{\overline{\gamma}_\varepsilon^2 d_\varepsilon}\right) \tag{4.65}$$

as $\varepsilon \to 0$*, uniformly in* $(\gamma,\tau,\theta) \in P^k_\varepsilon(\delta)$.

Remark 4.6. *As an evidence of the* strong *interaction generated by the Moser–Trudinger critical nonlinearity, we stress that the variables* θ *and* γ *are intricately coupled in the expansions* (4.63)–(4.65) *used to determine* $(\gamma_\varepsilon, \theta_\varepsilon, \tau_\varepsilon)$*. This is not the case for 2-dimensional Liouville-type equations (see for instance* [3]*), for which it is possible to construct blowing-up solutions without introducing neither the parameter* θ *nor the bubbles* $B_{\varepsilon,\gamma_i,\tau_i}$ *in* $V_{\varepsilon,\gamma,\tau}$ *(see for instance* [10] *working also in the* $H^1_0(\Omega)$*-framework). Finally, even not facing a situation with clustering or nonzero weak limit like ours, it is delicate to get a clean energy expansion in the Moser–Trudinger critical case (see* [6]*). In particular, this expansion has to eventually fit with the cancellation pointed out by* [15] *for the blow-up solutions.*

Proof of (4.63). We start with computations that will be used also in the proofs of (4.64)-(4.64). Given $Z \in H^1_0(\Omega)$, integration by parts yields

$$\left\langle \widetilde{R}_{\varepsilon,\gamma,\tau,\theta}, Z\right\rangle_{H_0^1} = \int_\Omega [\Delta(U_{\varepsilon,\gamma,\tau,\theta} + \Phi_{\varepsilon,\gamma,\tau,\theta}) - f_\varepsilon(U_{\varepsilon,\gamma,\tau,\theta} + \Phi_{\varepsilon,\gamma,\tau,\theta})] Z dx,$$

where we use the notation $f_\varepsilon = \lambda_\varepsilon h_\varepsilon f$.[1] We now expand for real numbers U and R,

$$\exp[(U+R)^2] = \exp(U^2)\exp(2UR + R^2) = \exp(U^2)[1 + 2UR + \mathrm{O}(U^2R^2)] \tag{4.66}$$

uniformly for $|UR| \le 1$ and $|R| \le 1 \le |U|$, so that, recalling that $f'(t) = \left(1 + 2t^2\right)\exp\left(t^2\right)$,

$$f(U+R) = f(U) + f'(U)R + \mathrm{O}\left(U^3R^2\exp\left(U^2\right)\right), \tag{4.67}$$

and similarly for f_ε since $\lambda_\varepsilon h_\varepsilon = \mathrm{O}(1)$. We apply this to

$$U = U_{\varepsilon,\gamma,\tau} = u_\varepsilon + \sum_{i=1}^k B_{\varepsilon,\gamma_i,\tau_i} + \Psi_{\varepsilon,\gamma,\tau}, \quad R = \widetilde{\Phi}_{\varepsilon,\gamma,\tau,\theta} := \sum_{i=1}^k \theta_i B_{\varepsilon,\gamma_i,\tau_i} + \Phi_{\varepsilon,\gamma,\tau,\theta} \tag{4.68}$$

to obtain

$$f_\varepsilon\left(U_{\varepsilon,\gamma,\tau,\theta} + \Phi_{\varepsilon,\gamma,\tau,\theta}\right) = f_\varepsilon\left(U_{\varepsilon,\gamma,\tau}\right) + f'_\varepsilon\left(U_{\varepsilon,\gamma,\tau}\right)\left(\sum_{i=1}^k \theta_i B_{\varepsilon,\gamma_i,\tau_i} + \Phi_{\varepsilon,\gamma,\tau,\theta}\right) + \mathrm{O}\left(\exp\left(U^2_{\varepsilon,\gamma,\tau}\right)U^3_{\varepsilon,\gamma,\tau}\widetilde{\Phi}^2_{\varepsilon,\gamma,\tau,\theta}\right).$$

[1] We shall always write $f_\varepsilon(U)$ instead of $f_\varepsilon(x,U)$, ignoring the dependence on x.

Recalling Proposition 3.2, and in particular that $U_{\varepsilon,\gamma,\tau}$ is an exact solution outside the balls $B\left(\overline{\tau_j}, r_\varepsilon + r_\varepsilon^2\right)$, we get

$$\begin{aligned}\left\langle \widetilde{R}_{\varepsilon,\gamma,\tau,\theta}, Z\right\rangle_{H_0^1} &= \sum_{j=1}^k \int_{B(\overline{\tau_j}, r_\varepsilon)} \left[\Delta U_{\varepsilon,\gamma,\tau} - f_\varepsilon\left(U_{\varepsilon,\gamma,\tau}\right)\right] Z dx \\ &\quad + \sum_{j=1}^k \int_{\Omega_\varepsilon^j} \left[\Delta U_{\varepsilon,\gamma,\tau} - f_\varepsilon\left(U_{\varepsilon,\gamma,\tau}\right)\right] Z dx \\ &\quad + \sum_{j=1}^k \theta_j \int_\Omega \left[\Delta B_{\varepsilon,\gamma_j,\tau_j} - f_\varepsilon'\left(U_{\varepsilon,\gamma,\tau}\right) B_{\varepsilon,\gamma_j,\tau_j}\right] Z dx \\ &\quad + \int_\Omega \left[\Delta \Phi_{\varepsilon,\gamma,\tau,\theta} - f_\varepsilon'\left(U_{\varepsilon,\gamma,\tau}\right) \Phi_{\varepsilon,\gamma,\tau,\theta}\right] Z dx \\ &\quad + \mathrm{O}\left(\int_\Omega \left|U_{\varepsilon,\gamma,\tau}\right|^3 \exp\left(U_{\varepsilon,\gamma,\tau}^2\right) \widetilde{\Phi}_{\varepsilon,\gamma,\tau,\theta}^2 \left|Z\right| dx\right) \\ &=: \sum_{j=1}^k \left[(A)_j + (A')_j + (B)_j\right] + (C) + (D), \end{aligned} \tag{4.69}$$

where $\Omega_\varepsilon^j := B\left(\overline{\tau_i}, r_\varepsilon + r_\varepsilon^2\right) \backslash B\left(\overline{\tau_i}, r_\varepsilon\right)$. We now set $Z = Z_{0,i,\varepsilon,\gamma,\tau}$ in (4.69) and estimate the various terms.

In order to evaluate $(A) := \sum_{j=1}^k (A)_j$, expand as in (3.22)

$$U_{\varepsilon,\gamma,\tau} = \overline{B}_{\varepsilon,\gamma_j,\overline{\tau_j}} + E_{\varepsilon,\gamma,\tau}^{(j)} + F_{\varepsilon,\gamma,\tau}^{(j)}, \quad \text{in } B\left(\overline{\tau_j}, r_\varepsilon\right). \tag{4.70}$$

Using Proposition 3.5 and omitting some indices, we get

$$R_j(x) := E_{\varepsilon,\gamma,\tau}^{(j)} + F_{\varepsilon,\gamma,\tau}^{(j)}(x) = E_{\varepsilon,\gamma,\tau}^{(j)} + \mathrm{O}\left(\frac{|x - \overline{\tau_j}|}{\overline{\gamma}_\varepsilon d_\varepsilon}\right) =: R_j^s(x) + R_j^r, \tag{4.71}$$

for all $x \in B\left(\overline{\tau_j}, r_\varepsilon\right)$, where the letters s and r stand for "symmetric" and "remainder", respectively. Using (4.70) and (3.33), we get

$$\begin{aligned} Z_{0,i} := Z_{0,i,\varepsilon,\gamma,\tau} &= \partial_{\gamma_i}\left[\overline{B}_{\varepsilon,\gamma_j,\overline{\tau_j}} + R_j\right] \\ &= \partial_{\gamma_i}\left[\overline{B}_{\varepsilon,\gamma_j,\overline{\tau_j}}(x) + E_{\varepsilon,\gamma,\tau}^{(j)}\right] + \mathrm{O}\left(|x - \overline{\tau_j}|\right), \quad \text{in } B\left(\overline{\tau_j}, r_\varepsilon\right), \end{aligned} \tag{4.72}$$

where we also replaced $\mathrm{O}\left(|x - \overline{\tau_j}| / \left(\overline{\gamma}_\varepsilon^2 d_\varepsilon\right)\right)$ by $\mathrm{O}\left(|x - \overline{\tau_j}|\right)$ for simplicity. Using Proposition 3.2 and (3.20), i.e. $\Delta B_{\varepsilon,\gamma_j,\tau_j} = \Delta \overline{B}_{\varepsilon,\gamma_j,\overline{\tau_j}}$, in $B\left(\overline{\tau_j}, r_\varepsilon\right)$, we can write

$$(A)_j = \int_{B(\overline{\tau_j}, r_\varepsilon)} \left[\Delta \overline{B}_{\varepsilon,\gamma_j,\overline{\tau_j}} - f_\varepsilon\left(U_{\varepsilon,\gamma,\tau}\right)\right] Z_{0,i} dx.$$

We now Taylor expand as in (4.67) with

$$U = \overline{B}_{\varepsilon,\gamma_j,\overline{\tau_j}}, \quad R = R_j = E_{\varepsilon,\gamma,\tau}^{(j)} + F_{\varepsilon,\gamma,\tau}^{(j)},$$

and since $\overline{B}_{\varepsilon,\gamma_i,\overline{\tau_i}}$ is an exact solution in $B\left(\overline{\tau_i}, r_\varepsilon\right)$, we estimate

$$\begin{aligned}(A)_j = &\int_{B(\overline{\tau_j},r_\varepsilon)} \underbrace{\left[\Delta \overline{B}_{\varepsilon,\gamma_j,\overline{\tau_j}} - \lambda_\varepsilon h_\varepsilon\left(\overline{\tau_j}\right) f\left(\overline{B}_{\varepsilon,\gamma_j,\overline{\tau_j}}\right)\right]}_{=0} Z_{0,i} dx \\ &- \int_{B(\overline{\tau_j},r_\varepsilon)} \lambda_\varepsilon\left(h_\varepsilon - h_\varepsilon\left(\overline{\tau_j}\right)\right) f\left(\overline{B}_{\varepsilon,\gamma_j,\overline{\tau_j}}\right) Z_{0,i} dx - \int_{B(\overline{\tau_j},r_\varepsilon)} \lambda_\varepsilon h_\varepsilon f'\left(\overline{B}_{\varepsilon,\gamma_j,\overline{\tau_j}}\right) R_j Z_{0,i} dx \\ &+ \mathrm{O}\left(\int_{B(\overline{\tau_j},r_\varepsilon)} \overline{\gamma}_\varepsilon^3 \exp\left(\overline{B}^2_{\varepsilon,\gamma_j,\overline{\tau_j}}\right) R_j^2 \left|Z_{0,i}\right| dx\right). \end{aligned} \tag{4.73}$$

Observing that $h_\varepsilon - h_\varepsilon\left(\overline{\tau_j}\right) = \mathrm{O}\left(|x-\overline{\tau_j}|\right)$, using (4.71) to bound $F^{(j)}_{\varepsilon,\gamma,\tau}$, writing

$$f'\left(\overline{B}_{\varepsilon,\gamma_j,\overline{\tau_j}}\right) = \mathrm{O}\left(\overline{\gamma}_\varepsilon f\left(\overline{B}_{\varepsilon,\gamma_j,\overline{\tau_j}}\right)\right) = \mathrm{O}\left(\overline{\gamma}_\varepsilon^2 \exp\left(\overline{B}^2_{\varepsilon,\gamma_j,\overline{\tau_j}}\right)\right), \quad \text{in } B\left(\overline{\tau_j}, r_\varepsilon\right),$$

and using $|Z_{0,i}| = \mathrm{O}(1)$, we simplify to

$$\begin{aligned}(A)_j = &- \int_{B(\overline{\tau_j},r_\varepsilon)} \lambda_\varepsilon h_\varepsilon\left(\overline{\tau_j}\right) f'\left(\overline{B}_{\varepsilon,\gamma_j,\overline{\tau_j}}\right) E^{(j)}_{\varepsilon,\gamma,\tau} \partial_{\gamma_i}\left[\overline{B}_{\varepsilon,\gamma_j,\overline{\tau_j}} + E^{(j)}_{\varepsilon,\gamma,\tau}\right] dx \\ &+ \mathrm{O}\left(\int_{B(\overline{\tau_j},r_\varepsilon)} \overline{\gamma}_\varepsilon^3 \exp\left(\overline{B}^2_{\varepsilon,\gamma_j,\overline{\tau_j}}\right)\left(|E^{(j)}_{\varepsilon,\gamma,\tau}|^2 + |x-\overline{\tau_j}|\right) dx\right). \end{aligned} \tag{4.74}$$

Now write

$$\partial_{\gamma_i}\left[\overline{B}_{\varepsilon,\gamma_j,\overline{\tau_j}} + E^{(j)}_{\varepsilon,\gamma,\tau}\right] = \delta_{ij}\partial_{\gamma_i}\left[\overline{B}_{\varepsilon,\gamma_i,\overline{\tau_i}}\right] + \partial_{\gamma_i}\left[E^{(j)}_{\varepsilon,\gamma,\tau}\right], \quad \text{on } B\left(\overline{\tau_j}, r_\varepsilon\right),$$

where δ_{ij} is the Kronecker symbol. Observing that

$$\overline{B}_{\varepsilon,\gamma_j,\overline{\tau_j}}(x) = \overline{B}_{\gamma_j}\left(\sqrt{\lambda_{\varepsilon,j}}\left(x-\overline{\tau_j}\right)\right), \quad \text{where } \lambda_{\varepsilon,j} := \lambda_\varepsilon h_\varepsilon\left(\overline{\tau_j}\right), \tag{4.75}$$

with the change of variables $\sqrt{\lambda_{\varepsilon,j}}\left(x-\overline{\tau_j}\right) = y$ and Proposition 6.3, we get

$$\int_{B(\overline{\tau_j},r_\varepsilon)} \lambda_{\varepsilon,j} f'\left(\overline{B}_{\varepsilon,\gamma_j,\overline{\tau_j}}\right) dx = \int_{B(0,\sqrt{\lambda_{\varepsilon,j}}r_\varepsilon)} f'\left(\overline{B}_{\gamma_j}\right) dy = 8\pi + \mathrm{O}\left(\frac{1}{\overline{\gamma}_\varepsilon^2}\right), \tag{4.76}$$

where $\overline{B}_{\gamma_j}$ is as in Proposition 6.1. With the same change of variables and Proposition 6.3, we also get

$$\int_{B(\overline{\tau_i},r_\varepsilon)} \lambda_{\varepsilon,i} f'(\overline{B}_{\varepsilon,\gamma_i,\overline{\tau_i}}) \partial_{\gamma_i}\left[\overline{B}_{\varepsilon,\gamma_i,\overline{\tau_i}}\right] dx = \int_{B(0,\sqrt{\lambda_{\varepsilon,i}}r_\varepsilon)} f'\left(\overline{B}_{\gamma_i}\right) Z_{0,\gamma_i} dy = -\frac{4\pi + \mathrm{o}(1)}{\overline{\gamma}_\varepsilon^2},$$

where Z_{0,γ_i} is as in Proposition 6.2. Now, using Proposition 3.4, the dominant term in $(A)_j$ becomes

$$\begin{aligned} &- E^{(j)}_{\varepsilon,\gamma,\tau} \int_{B(\overline{\tau_j},r_\varepsilon)} \lambda_{\varepsilon,j} f'\left(\overline{B}_{\varepsilon,\gamma_j,\overline{\tau_j}}\right)\left(\partial_{\gamma_i}\left[E^{(j)}_{\varepsilon,\gamma,\tau}\right] + \partial_{\gamma_i}\left[\overline{B}_{\varepsilon,\gamma_j,\overline{\tau_j}}\right]\right) dy \\ &= -E^{(j)}_{\varepsilon,\gamma,\tau}\left(\left(8\pi + \mathrm{O}\left(\frac{1}{\overline{\gamma}_\varepsilon^2}\right)\right)\partial_{\gamma_i}\left[E^{(j)}_{\varepsilon,\gamma,\tau}\right] - \delta_{ij}\frac{4\pi + \mathrm{o}(1)}{\overline{\gamma}_\varepsilon^2}\right) \\ &= -E^{(j)}_{\varepsilon,\gamma,\tau}\left(8\pi\partial_{\gamma_i}\left[E^{(j)}_{\varepsilon,\gamma,\tau}\right] - \delta_{ij}\frac{4\pi}{\overline{\gamma}_\varepsilon^2}\right) + \mathrm{o}\left(\frac{\delta_\varepsilon \ln \overline{\gamma}_\varepsilon}{\overline{\gamma}_\varepsilon^5}\right) \end{aligned}$$

Concerning the remainder term in (4.74), again using Proposition 6.3, we have

$$\int_{B(\overline{\tau_j},r_\varepsilon)} \exp\left(\overline{B}^2_{\varepsilon,\gamma_j,\overline{\tau_j}}\right) \overline{\gamma}_\varepsilon^3 \delta_\varepsilon^2 dx = \mathrm{O}\left(\overline{\gamma}_\varepsilon \delta_\varepsilon^2\right) = \mathrm{o}\left(\frac{\delta_\varepsilon \ln \overline{\gamma}_\varepsilon}{\overline{\gamma}_\varepsilon^5}\right),$$

and, with the usual change of variables and Proposition 6.4, we obtain

$$\int_{B(\overline{\tau_j},r_\varepsilon)} \exp\left(\overline{B}^2_{\varepsilon,\gamma_j,\overline{\tau_j}}\right)\overline{\gamma}^3_\varepsilon \left|x-\overline{\tau_j}\right| dx = \mathrm{O}\left(\overline{\gamma}^3_\varepsilon \mu_{\gamma_j}^{3\delta_0-2\delta_0^2+\mathrm{o}(1)}\right) = \mathrm{O}\left(\mu_\varepsilon^{3\delta_0-2\delta_0^2+\mathrm{o}(1)}\right)$$
$$= \mathrm{o}\left(\frac{\delta_\varepsilon \ln \overline{\gamma}_\varepsilon}{\overline{\gamma}^5_\varepsilon}\right),$$

where in the last identity, we used that $\delta_\varepsilon = \overline{\mu}_\varepsilon^{\delta_1+1/2}$ and $3\delta_0 - 2\delta_0^2 > \delta_1 + \frac{1}{2}$ thanks to (4.2), so that

$$\mu_\varepsilon^{3\delta_0-2\delta_0^2+\mathrm{o}(1)} = \mathrm{O}\left(\frac{\delta_\varepsilon}{\overline{\gamma}^a_\varepsilon}\right), \quad \text{for any } a\in\mathbb{R}.$$

We therefore get

$$(A)_j = -E^{(j)}_{\varepsilon,\gamma,\tau}\left(8\pi\partial_{\gamma_i}\left[E^{(j)}_{\varepsilon,\gamma,\tau}\right] - \delta_{ij}\frac{4\pi}{\overline{\gamma}^2_\varepsilon}\right) + \mathrm{o}\left(\frac{\delta_\varepsilon \ln\overline{\gamma}_\varepsilon}{\overline{\gamma}^5_\varepsilon}\right)$$

Summing over j, we then obtain

$$(A) = \sum_{j=1}^k (A)_j = -8\pi \sum_{j=1}^k E^{(j)}_{\varepsilon,\gamma,\tau}\partial_{\gamma_i}\left[E^{(j)}_{\varepsilon,\gamma,\tau}\right] + \frac{4\pi}{\overline{\gamma}^2_\varepsilon}E^{(i)}_{\varepsilon,\gamma,\tau} + \mathrm{o}\left(\frac{\delta_\varepsilon\ln\overline{\gamma}_\varepsilon}{\overline{\gamma}^5_\varepsilon}\right). \tag{4.77}$$

As for the error term in the annuli, we have from Proposition 3.2,

$$(A')_j = \int_{\Omega^j_\varepsilon}\left(\chi_{\varepsilon,\tau}-1\right) f_\varepsilon\left(U_{\varepsilon,\gamma,\tau}\right) Z_{0,i,\varepsilon,\gamma,\tau}dx,$$

hence, from (3.21),

$$(A') = \sum_{j=1}^k \mathrm{O}\left(|\Omega^j_\varepsilon|\overline{\mu}_\varepsilon^{-2\delta_0^2+\mathrm{o}(1)}\right) = \mathrm{O}\left(\overline{\mu}_\varepsilon^{3\delta_0-2\delta_0^2+\mathrm{o}(1)}\right) = \mathrm{o}\left(\frac{\delta_\varepsilon\ln\overline{\gamma}_\varepsilon}{\overline{\gamma}^5_\varepsilon}\right), \tag{4.78}$$

where in the last line, we used that $\delta_\varepsilon = \overline{\mu}_\varepsilon^{\delta_1+1/2}$ and $3\delta_0 - 2\delta_0^2 > \delta_1 + 1/2$.

We now move on to the estimate of (B). Integration by parts and using that $U_{\varepsilon,\gamma,\tau}$ is an exact solution outside the balls $B\left(\overline{\tau_m}, r_\varepsilon + r_\varepsilon^2\right)$ give

$$\begin{aligned}
(B)_j &= \theta_j\int_\Omega \left[\Delta Z_{0,i} - f'_\varepsilon\left(U_{\varepsilon,\gamma,\theta}\right)Z_{0,i}\right] B_{\varepsilon,\gamma_j,\tau_j}dx\\
&= \theta_j\int_\Omega \partial_{\gamma_i}\left[\Delta U_{\varepsilon,\gamma,\tau} - f_\varepsilon\left(U_{\varepsilon,\gamma,\theta}\right)\right] B_{\varepsilon,\gamma_j,\tau_j}dx\\
&= \theta_j\sum_{m=1}^k\int_{B(\overline{\tau_m},r_\varepsilon)} \partial_{\gamma_i}\left[\Delta U_{\varepsilon,\gamma,\tau} - f_\varepsilon\left(U_{\varepsilon,\gamma,\theta}\right)\right] B_{\varepsilon,\gamma_j,\tau_j}dx\\
&\quad + \theta_j\sum_{m=1}^k\int_{\Omega^m_\varepsilon} \partial_{\gamma_i}\left[\Delta U_{\varepsilon,\gamma,\tau} - f_\varepsilon\left(U_{\varepsilon,\gamma,\theta}\right)\right] B_{\varepsilon,\gamma_j,\tau_j}dx\\
&=: \sum_{m=1}^k\left[(B)_{jm} + (B')_{jm}\right].
\end{aligned} \tag{4.79}$$

Using the same notations as in (4.70)–(4.71) and using (4.66), which gives

$$f'\left(B+R\right) = f'\left(B\right) + \mathrm{O}\left(B^3\left|R\right|\exp\left(B^2\right)\right) \tag{4.80}$$

with

$$B = \overline{B}_m = \overline{B}_{\varepsilon,\gamma_m,\tau_m}, \quad R = R_m = E^{(m)}_{\varepsilon,\gamma,\tau} + F^{(m)}_{\varepsilon,\gamma,\tau},$$

on $B\left(\overline{\tau_m}, r_\varepsilon\right)$, we can now write

$$\begin{aligned}
\partial_{\gamma_i}\left[\Delta U_{\varepsilon,\gamma,\tau} - f_\varepsilon\left(U_{\varepsilon,\gamma,\tau}\right)\right] &= \partial_{\gamma_i}\left[\Delta \overline{B}_m\right] - f_\varepsilon'\left(U_{\varepsilon,\gamma,\tau}\right)\partial_{\gamma_i}\left[U_{\varepsilon,\gamma,\tau}\right]\\
&= \partial_{\gamma_i}\left[\Delta \overline{B}_m\right] - \left[f_\varepsilon'\left(\overline{B}_m\right) + \mathrm{O}\left(\overline{\gamma}_\varepsilon^3 \exp\left(\overline{B}_m^2\right)\left|R_m\right|\right)\right]\partial_{\gamma_i}\left[\overline{B}_m + R_m\right]\\
&= \partial_{\gamma_i}\underbrace{\left[\Delta \overline{B}_m - \lambda_\varepsilon h_\varepsilon\left(\overline{\tau_m}\right) f\left(\overline{B}_m\right)\right]}_{=0} + \mathrm{O}\left(\overline{\gamma}_\varepsilon^3 \exp\left(\overline{B}_m^2\right)\left(\left|x-\overline{\tau_m}\right| + \left|R_m\right|\right)\partial_{\gamma_i}\left[B_m\right]\right)\\
&\qquad - \left[f_\varepsilon'\left(\overline{B}_m\right) + \mathrm{O}\left(\overline{\gamma}_\varepsilon^3 \exp\left(\overline{B}_m^2\right)\left|R_m\right|\right)\right]\partial_{\gamma_i}\left[R_m\right], \quad (4.81)
\end{aligned}$$

where we have also bound $h_\varepsilon - h_\varepsilon\left(\overline{\tau_m}\right) = \mathrm{O}\left(\left|x-\overline{\tau_m}\right|\right)$. Expanding $\partial_{\gamma_i}\left[R_m\right]$ as in (4.72), we then get

$$\begin{aligned}
(B)_{jm} = -\theta_j \int_{B\left(\overline{\tau_m}, r_\varepsilon\right)} f_\varepsilon'\left(\overline{B}_{\varepsilon,\gamma_m,\overline{\tau_m}}\right) B_{\varepsilon,\gamma_j,\tau_j}\partial_{\gamma_i}\left[R_m\right] dx&\\
+ \mathrm{O}\left(\left|\theta_j\right| \overline{\gamma}_\varepsilon^4 \int_{B\left(\overline{\tau_m}, r_\varepsilon\right)} \exp\left(\overline{B}_{\varepsilon,\gamma_m,\overline{\tau_m}}^2\right)\left(\delta_\varepsilon + \left|x-\overline{\tau_m}\right|\right) dx\right)&,
\end{aligned}$$

hence

$$(B)_{jm} = -\theta_j \partial_{\gamma_i}\left[E_{\varepsilon,\gamma,\tau}^{(m)}\right] \int_{B\left(\overline{\tau_m}, r_\varepsilon\right)} f_\varepsilon'\left(\overline{B}_{\varepsilon,\gamma_m,\overline{\tau_m}}\right) B_{\varepsilon,\gamma_j,\tau_j} dx + \mathrm{o}\left(\frac{\delta_\varepsilon \ln \overline{\gamma}_\varepsilon}{\overline{\gamma}_\varepsilon^5}\right).$$

Together with (6.11), for $j = m$, we obtain

$$(B)_{jj} = -8\pi \overline{\gamma}_\varepsilon \theta_j \partial_{\gamma_i}\left[E_{\varepsilon,\gamma,\tau}^{(j)}\right] + \mathrm{o}\left(\frac{\delta_\varepsilon \ln \overline{\gamma}_\varepsilon}{\overline{\gamma}_\varepsilon^5}\right),$$

while, observing that $B_{\varepsilon,\gamma_j,\tau_j} = \mathrm{O}(1)$ on $B\left(\overline{\tau_m}, r_\varepsilon\right)$ if $j \neq m$, we get

$$(B)_{jm} = \mathrm{o}\left(\frac{\delta_\varepsilon \ln \overline{\gamma}_\varepsilon}{\overline{\gamma}_\varepsilon^5}\right), \quad \text{for } j \neq m.$$

As for $(B')_{jm}$, similarly as in (4.78), we can bound with (3.21) and Proposition 6.4

$$\begin{aligned}
(B')_{jm} &= \int_{\Omega_\varepsilon^m} \left(\chi_{\varepsilon,\tau} - 1\right)\partial_{\gamma_i}\left[f_\varepsilon\left(U_{\varepsilon,\gamma,\tau}\right)\right] B_{\varepsilon,\gamma_j,\tau_j} dx\\
&= \mathrm{O}\left(\overline{\gamma}_\varepsilon \int_{\Omega_\varepsilon^m} \left|f_\varepsilon'\left(U_{\varepsilon,\gamma,\tau}\right)\right|\left|Z_{0,j,\varepsilon,\gamma,\tau}\right| dx\right) = \mathrm{O}\left(\overline{\mu}_\varepsilon^{3\delta_0 - 2\delta_0^2 + \mathrm{o}(1)}\right) = \mathrm{o}\left(\frac{\delta_\varepsilon \ln \overline{\gamma}_\varepsilon}{\overline{\gamma}_\varepsilon^5}\right). \quad (4.82)
\end{aligned}$$

Hence, finally, summing over m and j, we obtain

$$(B) = \sum_{j=1}^{k} (B)_j = -8\pi \overline{\gamma}_\varepsilon \sum_{j=1}^{k} \theta_j \partial_{\gamma_i}\left[E_{\varepsilon,\gamma,\tau}^{(j)}\right] + \mathrm{o}\left(\frac{\delta_\varepsilon \ln \overline{\gamma}_\varepsilon}{\overline{\gamma}_\varepsilon^5}\right). \quad (4.83)$$

We now estimate the term (C). Similar to (4.79), integration by parts and Proposition 3.2 give

$$(C) = \int_\Omega \left[\Delta Z_{0,i} - f_\varepsilon'\left(U_{\varepsilon,\gamma,\theta}\right) Z_{0,i}\right] \Phi_{\varepsilon,\gamma,\tau,\theta} dx$$

$$= \sum_{j=1}^{k} \int_{B(\overline{\tau_j}, r_\varepsilon)} \partial_{\gamma_i} \left[\Delta U_{\varepsilon,\gamma,\tau} - f_\varepsilon \left(U_{\varepsilon,\gamma,\tau} \right) \right] \Phi_{\varepsilon,\gamma,\tau,\theta} dx$$
$$+ \sum_{j=1}^{k} \int_{\Omega_\varepsilon^j} \partial_{\gamma_i} \left[\Delta U_{\varepsilon,\gamma,\tau} - f_\varepsilon \left(U_{\varepsilon,\gamma,\tau} \right) \right] \Phi_{\varepsilon,\gamma,\tau,\theta} dx =: \sum_{j=1}^{k} \left[(C)_j + (C')_j \right].$$

We can now use (4.81), and with the same notations, we write

$$(C)_j = - \int_{B(\overline{\tau_j}, r_\varepsilon)} f'_\varepsilon \left(\overline{B}_{\varepsilon,\gamma_j,\overline{\tau_j}} \right) \partial_{\gamma_i} \left[R_j \right] \Phi_{\varepsilon,\gamma,\tau,\theta} dx + \mathrm{O} \Bigg(\int_{B(\overline{\tau_j}, r_\varepsilon)} \overline{\gamma}_\varepsilon^3 \exp \left(\overline{B}^2_{\varepsilon,\gamma_j,\overline{\tau_j}} \right)$$
$$\times \left(\left(|x - \overline{\tau_j}| + |R_j| \right) \left| \partial_{\gamma_i} \left[\overline{B}_j \right] \right| + |R_j| \left| \partial_{\gamma_i} \left[R_j \right] \right| \right) |\Phi_{\varepsilon,\gamma,\tau,\theta}| \, dx \Bigg).$$

The main term in $(C)_j$ will be

$$(C_1)_j = -\partial_{\gamma_i} \left[E^{(j)}_{\varepsilon,\gamma,\tau} \right] \int_{B(\overline{\tau_j}, r_\varepsilon)} f'_\varepsilon \left(\overline{B}_{\varepsilon,\gamma_j,\overline{\tau_j}} \right) \Phi_{\varepsilon,\gamma,\tau,\theta} dx$$
$$= -\partial_{\gamma_i} \left[E^{(j)}_{\varepsilon,\gamma,\tau} \right] \int_{B(\overline{\tau_j}, r_\varepsilon)} 2\gamma_j f_\varepsilon \left(\overline{B}_{\varepsilon,\gamma_j,\overline{\tau_j}} \right) \Phi_{\varepsilon,\gamma,\tau,\theta} dx$$
$$+ \mathrm{O} \left(\frac{\ln \overline{\gamma}_\varepsilon}{\overline{\gamma}_\varepsilon^2} \int_{B(\overline{\tau_j}, r_\varepsilon)} \left| 2\gamma_j f_\varepsilon \left(\overline{B}_{\varepsilon,\gamma_j,\overline{\tau_j}} \right) - f'_\varepsilon \left(\overline{B}_{\varepsilon,\gamma_j,\overline{\tau_j}} \right) \right| |\Phi_{\varepsilon,\gamma,\tau,\theta}| \, dx \right)$$
$$= -2\gamma_j \partial_{\gamma_i} \left[E^{(j)}_{\varepsilon,\gamma,\tau} \right] \int_{B(\overline{\tau_j}, r_\varepsilon)} \Delta B_{\varepsilon,\gamma_j,\tau_j} \Phi_{\varepsilon,\gamma,\tau,\theta} dx$$
$$+ \mathrm{O} \left(\overline{\gamma}_\varepsilon \int_{B(\overline{\tau_j}, r_\varepsilon)} |x - \overline{\tau}_j| \, f \left(\overline{B}_{\varepsilon,\gamma_j,\overline{\tau_j}} \right) |\Phi_{\varepsilon,\gamma,\tau,\theta}| \, dx \right)$$
$$+ \mathrm{O} \left(\frac{\ln \overline{\gamma}_\varepsilon}{\overline{\gamma}_\varepsilon^2} \int_{B(\overline{\tau_j}, r_\varepsilon)} \exp \left(\overline{B}^2_{\varepsilon,\gamma_j,\overline{\tau_j}} \right) \left| 2\gamma_j \overline{B}_{\varepsilon,\gamma_j,\overline{\tau_j}} - 2\overline{B}^2_{\varepsilon,\gamma_j,\overline{\tau_j}} - 1 \right| |\Phi_{\varepsilon,\gamma,\tau,\theta}| \, dx \right),$$

where we used that

$$\lambda_{\varepsilon,j} f \left(\overline{B}_{\varepsilon,\gamma_j,\overline{\tau_j}} \right) = \Delta \overline{B}_{\varepsilon,\gamma_j,\overline{\tau_j}} = \Delta B_{\varepsilon,\gamma_j,\tau_j}, \quad \text{in } B \left(\overline{\tau_j}, r_\varepsilon \right).$$

Since $\Phi_{\varepsilon,\gamma,\tau,\theta} \perp B_{\varepsilon,\gamma_j,\tau_j}$ in $H^1_0(\Omega)$ and $\Delta B_{\varepsilon,\gamma_j,\tau_j} = 0$ in $\Omega \setminus B(\overline{\tau_j}, r_\varepsilon)$, we have

$$\int_{B(\overline{\tau_j}, r_\varepsilon)} \Delta B_{\varepsilon,\gamma_j,\tau_j} \Phi_{\varepsilon,\gamma,\tau,\theta} dx = \int_\Omega \left\langle \nabla B_{\varepsilon,\gamma_j,\tau_j}, \nabla \Phi_{\varepsilon,\gamma,\tau,\theta} \right\rangle dx = 0, \tag{4.84}$$

and by Proposition 6.1,

$$\gamma_j \overline{B}_{\varepsilon,\gamma_j,\overline{\tau_j}} - \overline{B}^2_{\varepsilon,\gamma_j,\overline{\tau_j}} = \mathrm{O} \left(1 + t_{\gamma_j} \left(\cdot - \overline{\tau_j} \right) \right), \quad \text{in } B \left(\overline{\tau_j}, r_\varepsilon \right).$$

so that with a change of variables and Propositions 4.2 and 7.2, we get

$$(C_1)_j = \mathrm{O} \left(\overline{\gamma}_\varepsilon^2 r_\varepsilon \int_{B(\overline{\tau_j}, r_\varepsilon)} \exp \left(\overline{B}^2_{\varepsilon,\gamma_j,\overline{\tau_j}} \right) |\Phi_{\varepsilon,\gamma,\tau,\theta}| \, dx \right)$$
$$+ \mathrm{O} \left(\frac{\ln \overline{\gamma}_\varepsilon}{\overline{\gamma}_\varepsilon^2} \int_{B(\overline{\tau_j}, r_\varepsilon)} \exp \left(\overline{B}^2_{\varepsilon,\gamma_j,\overline{\tau_j}} \right) \left(1 + t_{\gamma_j} \left(x - \overline{\tau_j} \right) \right) |\Phi_{\varepsilon,\gamma,\tau,\theta}| \, dx \right)$$

$$= \mathrm{O}\left(r_\varepsilon \|\nabla \Phi_{\varepsilon,\gamma,\tau,\theta}\|_{L^2}\right) + \mathrm{O}\left(\frac{\ln \overline{\gamma}_\varepsilon}{\overline{\gamma}_\varepsilon^4} \|\nabla \Phi_{\varepsilon,\gamma,\tau\theta}\|_{L^2}\right) = \mathrm{o}\left(\frac{\delta_\varepsilon \ln \overline{\gamma}_\varepsilon}{\overline{\gamma}_\varepsilon^5}\right).$$

Note that we crucially used the orthogonality condition (4.84) to gain a factor $\overline{\gamma}_\varepsilon^{-2}$. Again, with a change of variables and Proposition 7.2, we bound

$$\begin{aligned}
(C_2)_j &= -\int_{B(\overline{\tau_j}, r_\varepsilon)} f_\varepsilon'\left(\overline{B}_{\varepsilon,\gamma_j,\overline{\tau_j}}\right) \partial_{\gamma_i}\left[F^{(j)}_{\varepsilon,\gamma,\tau}\right] \Phi_{\varepsilon,\gamma,\tau,\theta} dx \\
&= \mathrm{O}\left(\int_{B(\overline{\tau_j}, r_\varepsilon)} f_\varepsilon'\left(\overline{B}_{\varepsilon,\gamma_j,\overline{\tau_j}}\right) |x - \overline{\tau_j}| \, |\Phi_{\varepsilon,\gamma,\tau,\theta}| \, dx\right) \\
&= \mathrm{O}\left(r_\varepsilon \int_{B(\overline{\tau_j}, r_\varepsilon)} f_\varepsilon'\left(\overline{B}_{\varepsilon,\gamma_j,\overline{\tau_j}}\right) |\Phi_{\varepsilon,\gamma,\tau,\theta}| \, dx\right) = \mathrm{O}\left(r_\varepsilon \|\nabla \Phi_{\varepsilon,\gamma,\tau,\theta}\|_{L^2}\right) = \mathrm{o}\left(\frac{\delta_\varepsilon \ln \overline{\gamma}_\varepsilon}{\overline{\gamma}_\varepsilon^5}\right).
\end{aligned}$$

Similarly, for some exponent $a > 0$ (which plays no role),

$$\begin{aligned}
(C_3)_j &= \mathrm{O}\Bigg(\int_{B(\overline{\tau_j}, r_\varepsilon)} \overline{\gamma}_\varepsilon^3 \exp\left(\overline{B}^2_{\varepsilon,\gamma_j,\overline{\tau_j}}\right) \left((|x - \overline{\tau_j}| + |R_j|) \left|\partial_{\gamma_i}\left[\overline{B}_j\right]\right| + |R_j| \, |\partial_{\gamma_i}[R_j]|\right) \\
&\qquad \times |\Phi_{\varepsilon,\gamma,\tau,\theta}| \, dx\Bigg) = \mathrm{O}\left((\delta_\varepsilon + r_\varepsilon)\, \overline{\gamma}_\varepsilon^a \int_{B(\overline{\tau_j}, r_\varepsilon)} \exp\left(\overline{B}^2_{\varepsilon,\gamma_j,\overline{\tau_j}}\right) |\Phi_{\varepsilon,\gamma,\tau,\theta}| \, dx\right) \\
&= \mathrm{O}\left((\delta_\varepsilon + r_\varepsilon)\, \overline{\gamma}_\varepsilon^{a-2} \|\nabla \Phi_{\varepsilon,\gamma,\tau,\theta}\|\|_{L^2}\right) = \mathrm{o}\left(\frac{\delta_\varepsilon \ln \overline{\gamma}_\varepsilon}{\overline{\gamma}_\varepsilon^5}\right).
\end{aligned} \tag{4.85}$$

As for $(C')_j$, in analogy with (4.82) (with $\Phi_{\varepsilon,\gamma,\tau}$ instead of $B_{\varepsilon,\gamma_j,\tau_j}$), using (3.21) together with the Hölder and Poincaré inequalities, we bound

$$\begin{aligned}
(C')_j &= \mathrm{O}\left(\int_{\Omega_\varepsilon^j} f'\left(U_{\varepsilon,\gamma,\tau}\right) |\Phi_{\varepsilon,\gamma,\tau,\theta}| \, dx\right) = \mathrm{O}\left(\|f'\left(U_{\varepsilon,\gamma,\tau}\right)\|_{L^2(\Omega_\varepsilon^j)} \|\Phi_{\varepsilon,\gamma,\tau,\theta}\|_{L^2(\Omega)}\right) \\
&= \mathrm{O}\left(\overline{\mu}_\varepsilon^{\frac{1}{2}(3\delta_0 - 2\delta_0^2) + \mathrm{o}(1)} \|\nabla \Phi_{\varepsilon,\gamma,\tau,\theta}\|_{L^2(\Omega)}\right) = \mathrm{o}\left(\frac{\delta_\varepsilon \ln \overline{\gamma}_\varepsilon}{\overline{\gamma}_\varepsilon^5}\right).
\end{aligned}$$

Summing over j, we arrive at

$$(C) = \sum_{j=1}^k \left[(C_1)_j + (C_2)_j + (C_3)_j + (C')_j\right] = \mathrm{o}\left(\frac{\delta_\varepsilon \ln \overline{\gamma}_\varepsilon}{\overline{\gamma}_\varepsilon^5}\right). \tag{4.86}$$

As for (D), recalling that $|Z_{0,i}| = \mathrm{O}(1)$, we bound

$$\begin{aligned}
(D) &= \mathrm{O}\left(\int_\Omega |U_{\varepsilon,\gamma,\tau}|^3 \exp\left(U^2_{\varepsilon,\gamma,\tau}\right) \left(\Phi^2_{\varepsilon,\gamma,\tau,\theta} + \sum_{i=1}^k \theta_i^2 \overline{\gamma}_\varepsilon^2\right) dx\right) \\
&= \mathrm{O}\left(\int_\Omega \overline{\gamma}_\varepsilon^3 \exp\left(U^2_{\varepsilon,\gamma,\tau}\right) \Phi^2_{\varepsilon,\gamma,\tau,\theta} dx\right) + \sum_{i=1}^k \mathrm{O}\left(\theta_i^2 \overline{\gamma}_\varepsilon^2 \int_\Omega |U_{\varepsilon,\gamma,\tau}|^3 \exp\left(U^2_{\varepsilon,\gamma,\tau}\right) dx\right) \\
&=: (D_1) + (D_2).
\end{aligned}$$

We first claim that

$$\int_\Omega \exp\left(U^2_{\varepsilon,\gamma,\tau}\right) dx = \mathrm{O}(1). \tag{4.87}$$

Indeed, with the usual decomposition given by (4.70), we get

$$\int_{B(\overline{\tau_j}, r_\varepsilon)} \exp\left(U^2_{\varepsilon,\gamma,\tau}\right) dx = \mathrm{O}\left(\int_{B(\overline{\tau_j}, r_\varepsilon)} \exp\left(\overline{B}^2_{\varepsilon,\gamma_j,\overline{\tau_j}}\right) dx\right) = \mathrm{O}\left(\frac{1}{\overline{\gamma}_\varepsilon^2}\right),$$

thanks to the usual change of variables and Proposition 6.3. Then, summing over $j=1,\dots,k$ and also using (3.16), we obtain (4.87). Then we immediately estimate

$$(D_2)=\mathrm{O}\left(\theta_i^2\overline{\gamma}_\varepsilon^5\right)=\mathrm{o}\left(\frac{\delta_\varepsilon\ln\overline{\gamma}_\varepsilon}{\overline{\gamma}_\varepsilon^5}\right).$$

As for (D_1), from Hölder's inequality and (3.16), we have

$$\begin{aligned}(D_1'):=\int_{\Omega_{r_\varepsilon,\tau}}\overline{\gamma}_\varepsilon^3\exp\left(U_{\varepsilon,\gamma,\tau}^2\right)\Phi_{\varepsilon,\gamma,\tau,\theta}^2dx&=\mathrm{O}\left(\overline{\gamma}_\varepsilon^3\left\|\exp\left(U_{\varepsilon,\gamma,\tau}^2\right)\mathbf{1}_{\Omega_{r_\varepsilon,\tau}}\right\|_{L^p}\left\|\Phi_{\varepsilon,\gamma,\tau,\theta}\right\|_{L^{2p'}}^2\right)\\&=\mathrm{O}\left(\overline{\gamma}_\varepsilon^3\left\|\nabla\Phi_{\varepsilon,\gamma,\tau,\theta}\right\|_{L^2}^2\right)=\mathrm{o}\left(\frac{\delta_\varepsilon\ln\overline{\gamma}_\varepsilon}{\overline{\gamma}_\varepsilon^5}\right),\end{aligned}$$

where p is sufficiently small and p' is the conjugate exponent of p. Moreover, with Proposition 7.2, and the same change of variables used to estimate $(C_1)_j$, we obtain

$$\begin{aligned}(D_1)_j:=\int_{B(\overline{\tau_j},r_\varepsilon)}\overline{\gamma}_\varepsilon^3\exp\left(U_{\varepsilon,\gamma,\tau}^2\right)\Phi_{\varepsilon,\gamma,\tau,\theta}^2dx&=\mathrm{O}\left(\overline{\gamma}_\varepsilon^3\int_{B(\overline{\tau_j},r_\varepsilon)}\exp\left(\overline{B}_{\varepsilon,\gamma_j,\overline{\tau_j}}^2\right)\Phi_{\varepsilon,\gamma,\tau,\theta}^2dx\right)\\&=\mathrm{O}\left(\overline{\gamma}_\varepsilon\left\|\nabla\Phi_{\varepsilon,\gamma,\tau,\theta}\right\|_{L^2}^2\right)=\mathrm{o}\left(\frac{\delta_\varepsilon\ln\overline{\gamma}_\varepsilon}{\overline{\gamma}_\varepsilon^5}\right).\end{aligned}\tag{4.88}$$

Summing up, we conclude

$$(D)=\sum_{j=1}^k(D_1)_j+(D_1')+(D_2)=\mathrm{o}\left(\frac{\delta_\varepsilon\ln\overline{\gamma}_\varepsilon}{\overline{\gamma}_\varepsilon^5}\right).\tag{4.89}$$

Now, putting together (4.77), (4.78), (4.83), (4.86) and (4.89), we conclude. □

Proof of (4.64). We consider now (4.69) with $Z=B_{\varepsilon,\gamma_i,\tau_i}$ and estimate the terms from (A) to (D). Using (4.67), we get

$$\begin{aligned}(A)_j=&\int_{B(\overline{\tau_j},r_\varepsilon)}\underbrace{\left[\Delta\overline{B}_{\varepsilon,\gamma_j,\overline{\tau_j}}-\lambda_{\varepsilon,j}f\left(\overline{B}_{\varepsilon,\gamma_j,\overline{\tau_j}}\right)\right]}_{=0}B_{\varepsilon,\gamma_i,\tau_i}dx\\&-\int_{B(\overline{\tau_j},r_\varepsilon)}\lambda_\varepsilon\left(h_\varepsilon-h_\varepsilon\left(\overline{\tau_j}\right)\right)f\left(\overline{B}_{\varepsilon,\gamma_j,\overline{\tau_j}}\right)B_{\varepsilon,\gamma_i,\tau_i}dx\\&-\int_{B(\overline{\tau_j},r_\varepsilon)}\lambda_\varepsilon h_\varepsilon f'\left(\overline{B}_{\varepsilon,\gamma_j,\overline{\tau_j}}\right)R_jB_{\varepsilon,\gamma_i,\tau_i}dx+\mathrm{O}\left(\int_{B(\overline{\tau_j},r_\varepsilon)}\overline{\gamma}_\varepsilon^4\exp\left(\overline{B}_{\varepsilon,\gamma_j,\overline{\tau_j}}^2\right)R_j^2\right),\end{aligned}$$

where R_j is as in (4.71). Similarly as in (4.74), we reduce to

$$\begin{aligned}(A)_j=&-\int_{B(\overline{\tau_j},r_\varepsilon)}\lambda_{\varepsilon,j}f'\left(\overline{B}_{\varepsilon,\gamma_j,\overline{\tau_j}}\right)E_{\varepsilon,\gamma,\tau}^{(j)}B_{\varepsilon,\gamma_i,\tau_i}dx\\&+\mathrm{O}\left(\int_{B(\overline{\tau_j},r_\varepsilon)}\overline{\gamma}_\varepsilon^4\exp\left(\overline{B}_{\varepsilon,\gamma_i,\overline{\tau_i}}^2\right)\left(|x-\overline{\tau_j}|+\delta_\varepsilon^2\right)dx\right).\end{aligned}$$

In the case $j=i$ we use that

$$B_{\varepsilon,\gamma_i,\tau_i}=\overline{B}_{\varepsilon,\gamma_i,\tau_i}\left(1+\mathrm{O}\left(\frac{\ln\overline{\gamma}_\varepsilon}{\overline{\gamma}_\varepsilon^2}\right)\right),\quad\text{in }B\left(\overline{\tau_i},r_\varepsilon\right),\tag{4.90}$$

(see Claim 3.1) and with the usual change of variables, taking (4.75) and Propositions 6.3 and 6.4 into account, we obtain

$$\begin{aligned}(A)_i &= -\left(1+\mathrm{O}\left(\frac{\ln\overline{\gamma}_\varepsilon}{\overline{\gamma}_\varepsilon^2}\right)\right)E^{(i)}_{\varepsilon,\gamma,\tau}\int_{B(\overline{\tau_i},r_\varepsilon)}\lambda_{\varepsilon,j}f'\left(\overline{B}_{\varepsilon,\gamma_i,\overline{\tau_i}}\right)\overline{B}_{\varepsilon,\gamma_i,\tau_i}dx+\mathrm{o}\left(\frac{\delta_\varepsilon}{\overline{\gamma}_\varepsilon^2}\right)\\ &= -\left(1+\mathrm{O}\left(\frac{\ln\overline{\gamma}_\varepsilon}{\overline{\gamma}_\varepsilon^2}\right)\right)E^{(i)}_{\varepsilon,\gamma,\tau}\int_{B(0,\sqrt{\lambda_{\varepsilon,i}}r_\varepsilon)}f'\left(\overline{B}_{\gamma_i}\right)\overline{B}_{\gamma_i}dx+\mathrm{o}\left(\frac{\delta_\varepsilon}{\overline{\gamma}_\varepsilon^2}\right)\\ &= -\left(8\pi+\mathrm{O}\left(\frac{\ln\overline{\gamma}_\varepsilon}{\overline{\gamma}_\varepsilon^2}\right)\right)\overline{\gamma}_\varepsilon E^{(i)}_{\varepsilon,\gamma,\tau}+\mathrm{o}\left(\frac{\delta_\varepsilon}{\overline{\gamma}_\varepsilon^2}\right)=-8\pi\overline{\gamma}_\varepsilon E^{(i)}_{\varepsilon,\gamma,\tau}+\mathrm{o}\left(\frac{\delta_\varepsilon}{\overline{\gamma}_\varepsilon^2}\right).\end{aligned}$$

For the case $j\neq i$, we use that $B_{\varepsilon,\gamma_i,\tau_i}=\mathrm{O}\left((\ln\overline{\gamma}_\varepsilon)/\overline{\gamma}_\varepsilon\right)$ in $B(\overline{\tau_j},r_\varepsilon)$ and with the same computations, we obtain

$$(A)_j=\mathrm{O}\left(\frac{\ln\overline{\gamma}_\varepsilon}{\overline{\gamma}_\varepsilon^2}\right)|E^{(j)}_{\varepsilon,\gamma,\tau}|\int_{B(0,\sqrt{\lambda_{\varepsilon,j}}r_\varepsilon)}f'\left(\overline{B}_{\gamma_j}\right)dx+\mathrm{o}\left(\frac{\delta_\varepsilon}{\overline{\gamma}_\varepsilon^2}\right)=\mathrm{o}\left(\frac{\delta_\varepsilon}{\overline{\gamma}_\varepsilon^2}\right),$$

so that summing up we conclude

$$(A)=\sum_{j=1}^k(A)_j=-8\pi\overline{\gamma}_\varepsilon E^{(i)}_{\varepsilon,\gamma,\tau}+\mathrm{o}\left(\frac{\delta_\varepsilon}{\overline{\gamma}_\varepsilon^2}\right). \tag{4.91}$$

As for the annuli, similarly as in (4.78), we bound

$$(A')=\int_{\cup_{j=1}^k\Omega_\varepsilon^j}\left(\chi_{\varepsilon,\tau}-1\right)f_\varepsilon\left(U_{\varepsilon,\gamma,\tau}\right)B_{\varepsilon,\gamma,\tau}dx=\mathrm{O}\left(\overline{\mu}_\varepsilon^{3\delta_0-2\delta_0^2+\mathrm{o}(1)}\right)=\mathrm{o}\left(\frac{\delta_\varepsilon}{\overline{\gamma}_\varepsilon^2}\right). \tag{4.92}$$

We now turn to the estimate of (B). Using a Taylor expansion, together with (4.66), (4.70) and (4.71), we write

$$\begin{aligned}(B)_{jm} &:= \theta_j\int_{B(\overline{\tau_m},r_\varepsilon)}\left[\Delta B_{\varepsilon,\gamma_j,\tau_j}-f'_\varepsilon\left(U_{\varepsilon,\gamma,\tau}\right)B_{\varepsilon,\gamma_j,\tau_j}\right]B_{\varepsilon,\gamma_i,\tau_i}dx\\ &= \theta_j\int_{B(\overline{\tau_m},r_\varepsilon)}\left[\delta_{jm}\Delta\overline{B}_{\varepsilon,\gamma_j,\overline{\tau_j}}-f'_\varepsilon\left(\overline{B}_{\varepsilon,\gamma_m,\overline{\tau_m}}\right)B_{\varepsilon,\gamma_j,\tau_j}\right]B_{\varepsilon,\gamma_i,\tau_i}dx\\ &\quad+\mathrm{O}\left(|\theta_j|\int_{B(\overline{\tau_m},r_\varepsilon)}\overline{\gamma}_\varepsilon^3\exp\left(\overline{B}^2_{\varepsilon,\gamma_m,\overline{\tau_m}}\right)R_mB_{\varepsilon,\gamma_i,\tau_i}dx\right).\end{aligned}$$

With Propositions 6.3 and 6.4, we estimate the last term as $\mathrm{o}\left(\delta_\varepsilon/\overline{\gamma}_\varepsilon^2\right)$. For $j=m=i$, still with Proposition 6.3, we compute, keeping (4.90) in mind

$$\begin{aligned}&\theta_i\int_{B(\overline{\tau_i},r_\varepsilon)}\left[\lambda_{\varepsilon,i}f\left(\overline{B}_{\varepsilon,\gamma_i,\overline{\tau_i}}\right)-f'_\varepsilon\left(\overline{B}_{\varepsilon,\gamma_i,\overline{\tau_i}}\right)B_{\varepsilon,\gamma_i,\tau_i}\right]B_{\varepsilon,\gamma_i,\tau_i}dx\\ &\qquad\qquad=-8\pi\theta_i\overline{\gamma}_\varepsilon^2+\mathrm{O}\left(|\theta_i|\,\overline{\gamma}_\varepsilon\right)=-8\pi\theta_i\overline{\gamma}_\varepsilon^2+\mathrm{o}\left(\frac{\delta_\varepsilon}{\overline{\gamma}_\varepsilon^2}\right),\end{aligned}$$

while for $j\neq m$, or $j\neq i$ a similar computation based on Proposition 6.3 and (4.90) gives $(B)_{jm}=\mathrm{o}\left(\delta_\varepsilon/\overline{\gamma}_\varepsilon^2\right)$. Considering the integral in $\Omega_{r_\varepsilon,\tau}$, where $\Delta B_{\varepsilon,\gamma_j,\tau_j}=0$ for every $j=1,\ldots,k$, we estimate with the help of (3.16),

$$\begin{aligned}(B')_{jm} &:= \theta_j\int_{B(\overline{\tau_m},R_\varepsilon)\backslash B(\overline{\tau_m},r_\varepsilon)}f'_\varepsilon\left(U_{\varepsilon,\gamma,\tau}\right)B_{\varepsilon,\gamma_j,\tau_j}B_{\varepsilon,\gamma_i,\tau_i}dx\\ &= \mathrm{O}\left(|\theta_j|\,\overline{\gamma}_\varepsilon^3\int_{B(\overline{\tau_m},R_\varepsilon)\backslash B(\overline{\tau_m},r_\varepsilon)}\exp\left(U^2_{\varepsilon,\gamma,\tau}\right)dx\right)=\mathrm{o}\left(\frac{\delta_\varepsilon}{\overline{\gamma}_\varepsilon^2}\right),\end{aligned}$$

where $R_\varepsilon = \exp(-\overline{\gamma}_\varepsilon)$ and, since $B_{\varepsilon,\gamma_j,\tau_j} = \mathrm{O}(1)$ in $\Omega \setminus B(\overline{\tau_j}, r_\varepsilon)$, still with (3.16), we get

$$(B'')_{jm} := \theta_j \int_{\Omega_{R_\varepsilon,\tau}} f'_\varepsilon(U_{\varepsilon,\gamma,\tau}) B_{\varepsilon,\gamma_j,\tau_j} B_{\varepsilon,\gamma_i,\tau_i} dx = \mathrm{O}(|\theta_j|) = \mathrm{o}\left(\frac{\delta_\varepsilon}{\overline{\gamma}_\varepsilon^2}\right),$$

In conclusion, we have proven that

$$(B) = \sum_{j=1}^{k} [(B)_{jm} + (B')_{jm} + (B'')_{jm}] = -8\pi\theta_i \overline{\gamma}_\varepsilon^2 + \mathrm{o}\left(\frac{\delta_\varepsilon}{\overline{\gamma}_\varepsilon^2}\right). \tag{4.93}$$

To estimate (C), we integrate by parts to obtain

$$(C) = \int_\Omega [\Delta B_{\varepsilon,\gamma_i,\tau_i} - f'_\varepsilon(U_{\varepsilon,\gamma,\tau}) B_{\varepsilon,\gamma_i,\tau_i}] \Phi_{\varepsilon,\gamma,\tau,\theta} dx = -\int_\Omega f'_\varepsilon(U_{\varepsilon,\gamma,\tau}) B_{\varepsilon,\gamma_i,\tau_i} \Phi_{\varepsilon,\gamma,\tau,\theta} dx,$$

where we also used that $\Phi_{\varepsilon,\gamma,\tau,\theta} \perp B_{\varepsilon,\gamma_i,\tau_i}$ in H_0^1. Using (4.80), we write

$$\begin{aligned}(C_1)_j &:= -\int_{B(\overline{\tau_j},r_\varepsilon)} f'_\varepsilon(U_{\varepsilon,\gamma,\tau}) B_{\varepsilon,\gamma_i,\tau_i} \Phi_{\varepsilon,\gamma,\tau,\theta} dx \\ &= -\int_{B(\overline{\tau_j},r_\varepsilon)} f'_\varepsilon\left(\overline{B}_{\varepsilon,\gamma_j,\overline{\tau_j}}\right) B_{\varepsilon,\gamma_i,\tau_i} \Phi_{\varepsilon,\gamma,\tau,\theta} dx \\ &\quad + \underbrace{\mathrm{O}\left(\int_{B(\overline{\tau_j},r_\varepsilon)} \overline{\gamma}_\varepsilon^3 \exp\left(\overline{B}_{\varepsilon,\gamma_j,\overline{\tau_j}}^2\right) |R_j B_{\varepsilon,\gamma_i,\tau_i} \Phi_{\varepsilon,\gamma,\tau,\theta}|\right)}_{=\mathrm{o}(\delta_\varepsilon/\overline{\gamma}_\varepsilon^2) \text{ as in (4.85)}}.\end{aligned}$$

Then, recalling that $f'(s) = (1+2s^2)\exp(s^2) = 2sf(s) + \exp(s^2)$, which gives

$$\begin{aligned}(C_1)_j &= -2\int_{B(\overline{\tau_j},r_\varepsilon)} \overline{B}_{\varepsilon,\gamma_j,\overline{\tau_j}} f_\varepsilon\left(\overline{B}_{\varepsilon,\gamma_j,\overline{\tau_j}}\right) B_{\varepsilon,\gamma_i,\tau_i} \Phi_{\varepsilon,\gamma,\tau,\theta} dx \\ &\quad + \mathrm{O}\left(\overline{\gamma}_\varepsilon \int_{B(\overline{\tau_j},r_\varepsilon)} \exp\left(\overline{B}_{\varepsilon,\gamma_j,\overline{\tau_j}}^2\right) |\Phi_{\varepsilon,\gamma,\tau,\theta}| dx\right) + \mathrm{o}\left(\frac{\delta_\varepsilon}{\overline{\gamma}_\varepsilon^2}\right),\end{aligned}$$

and using Proposition 7.2, we obtain

$$\overline{\gamma}_\varepsilon \int_{B(\overline{\tau_j},r_\varepsilon)} \exp\left(\overline{B}_{\varepsilon,\gamma_j,\overline{\tau_j}}^2\right) |\Phi_{\varepsilon,\gamma,\tau,\theta}| dx = \mathrm{O}\left(\frac{\|\nabla\Phi_{\varepsilon,\gamma,\tau,\theta}\|_{L^2}}{\overline{\gamma}_\varepsilon}\right) = \mathrm{o}\left(\frac{\delta_\varepsilon}{\overline{\gamma}_\varepsilon^2}\right).$$

Using Proposition 7.2 again, we simplify

$$\begin{aligned}(C_1)_i &= 2\gamma_i^2 \int_{B(\overline{\tau_i},r_\varepsilon)} \lambda_\varepsilon h_\varepsilon(\overline{\tau_i}) f\left(\overline{B}_{\varepsilon,\gamma_i,\overline{\tau_i}}\right) \Phi_{\varepsilon,\gamma,\tau,\theta} dx \\ &\quad + \mathrm{O}\left(\overline{\gamma}_\varepsilon^2 \int_{B(\overline{\tau_i},r_\varepsilon)} |x-\overline{\tau_i}| f\left(\overline{B}_{\varepsilon,\gamma_i,\overline{\tau_i}}\right) |\Phi_{\varepsilon,\gamma,\tau,\theta}| dx\right) \\ &\quad + \mathrm{O}\left(\int_{B(\overline{\tau_i},r_\varepsilon)} (1+t_{\gamma_i}(x-\overline{\tau_i})) f\left(\overline{B}_{\varepsilon,\gamma_i,\overline{\tau_i}}\right) |\Phi_{\varepsilon,\gamma,\tau,\theta}| dx\right) + \mathrm{o}\left(\frac{\delta_\varepsilon}{\overline{\gamma}_\varepsilon^2}\right) \\ &= 2\gamma_i^2 \int_{B(\overline{\tau_i},r_\varepsilon)} \Delta\overline{B}_{\varepsilon,\gamma_i,\overline{\tau_i}} \Phi_{\varepsilon,\gamma,\tau,\theta} dx + \mathrm{O}\left(\left(r_\varepsilon + \frac{1}{\overline{\gamma}_\varepsilon^2}\right) \|\nabla\Phi_{\varepsilon,\gamma,\tau,\theta}\|_{L^2}\right) + \mathrm{o}\left(\frac{\delta_\varepsilon}{\overline{\gamma}_\varepsilon^2}\right),\end{aligned}$$

where the last integral vanishes thanks to $\Delta\overline{B}_{\varepsilon,\gamma_i,\overline{\tau_i}} = \Delta B_{\varepsilon,\gamma_i,\tau_i}$ and to the condition $B_{\varepsilon,\gamma_i,\tau_i} \perp \Phi_{\varepsilon,\gamma,\tau,\theta}$. A similar computation holds on $B(\overline{\tau_j}, r_\varepsilon)$, where we can use that

$B_{\varepsilon,\gamma_i,\tau_i} = \mathrm{O}(1)$ if $j \neq i$. Hence

$$(C_1) = \sum_{j=1}^{k} (C_1)_j = \mathrm{o}\left(\frac{\delta_\varepsilon}{\overline{\gamma}_\varepsilon^2}\right).$$

With (3.16) and the Hölder and Poincaré inequalities, we now bound

$$\begin{aligned}(C_2)_j &:= -\sum_{j=1}^{k} \int_{B(\overline{\tau_j},R_\varepsilon)\setminus B(\overline{\tau_j},r_\varepsilon)} f_\varepsilon'(U_{\varepsilon,\gamma,\tau}) B_{\varepsilon,\gamma_i,\tau_i} \Phi_{\varepsilon,\gamma,\tau,\theta} dx \\ &= \sum_{j=1}^{k} \mathrm{O}\left(\overline{\gamma}_\varepsilon^3 \left\| \exp\left(U_{\varepsilon,\gamma,\tau}^2\right) \mathbf{1}_{B(\overline{\tau_j},R_\varepsilon)\setminus B(\overline{\tau_j},r_\varepsilon)} \right\|_{L^p} \left\|\Phi_{\varepsilon,\gamma,\tau,\theta}\right\|_{L^{p'}}\right) \\ &= \mathrm{O}\left(\frac{\left\|\nabla \Phi_{\varepsilon,\gamma,\tau,\theta}\right\|_{L^2}}{\overline{\gamma}_\varepsilon^{a-3}}\right) = \mathrm{o}\left(\frac{\delta_\varepsilon}{\overline{\gamma}_\varepsilon^2}\right),\end{aligned}$$

upon choosing $a > 3$. Again with (3.16) and the Hölder and Poincaré inequalities, and observing that $U_{\varepsilon,\gamma,\tau} = \mathrm{O}(1)$ in $\Omega_{R_\varepsilon,\tau}$, we get

$$\begin{aligned}(C_3) &:= -\int_{\Omega_{R_\varepsilon,\tau}} f_\varepsilon'(U_{\varepsilon,\gamma,\tau}) B_{\varepsilon,\gamma_i,\tau_i} \Phi_{\varepsilon,\gamma,\tau,\theta} dx \\ &= \mathrm{O}\left(\left\| \exp\left(U_{\varepsilon,\gamma,\tau}^2\right) B_{\varepsilon,\gamma_i,\tau_i} \mathbf{1}_{\Omega_{R_\varepsilon,\tau}} \right\|_{L^p} \left\|\Phi_{\varepsilon,\gamma,\tau,\theta}\right\|_{L^{p'}}\right) = \mathrm{O}\left(\frac{\left\|\nabla \Phi_{\varepsilon,\gamma,\tau,\theta}\right\|_{L^2}}{\overline{\gamma}_\varepsilon}\right) = \mathrm{o}\left(\frac{\delta_\varepsilon}{\overline{\gamma}_\varepsilon^2}\right),\end{aligned}$$

where p is sufficiently small and p' is the conjugate exponent of p. Adding up, we conclude

$$(C) = (C_1) + (C_2) + (C_3) = \mathrm{o}\left(\frac{\delta_\varepsilon}{\overline{\gamma}_\varepsilon^2}\right). \tag{4.94}$$

Finally the estimate

$$(D) = \mathrm{o}\left(\frac{\delta_\varepsilon}{\overline{\gamma}_\varepsilon^2}\right) \tag{4.95}$$

follows exactly as the analog estimate in the proof of (4.63) (replacing $Z_{0,i,\varepsilon,\gamma,\tau}$ by $B_{\varepsilon,\gamma_i,\tau_i}$), since all the terms contain θ_i^2 or $\left\|\nabla \Phi_{\varepsilon,\gamma,\tau,\theta}\right\|_{L^2}^2$, which actually allows an estimate of the form $(D) = \mathrm{O}(\delta_\varepsilon / \overline{\gamma}_\varepsilon^a)$ for every $a \geq 0$. Now, putting together (4.91), (4.92), (4.93), (4.94) and (4.95), we conclude. □

Proof of (4.65). We now use (4.69) with $Z = Z_{1,i,\varepsilon,\gamma,\tau}$, and again we need to estimate the terms from (A) to (D).

We start with some estimates of $Z_{1,i} := Z_{1,i,\varepsilon,\gamma,\tau}$. From Claim 3.1, we have

$$\begin{aligned}\partial_{\tau_i}\left[B_{\varepsilon,\gamma_i,\tau_i}\right] &= \partial_{\tau_i}\left[\overline{B}_{\varepsilon,\gamma_i,\overline{\tau_i}}\right] - \partial_{\tau_i}\left[C_{\varepsilon,\gamma_i,\tau_i}\right] + \partial_{\tau_i}\left[A_{\varepsilon,\gamma_i,\tau_i} H(\cdot,\overline{\tau_i})\right] \\ &= \partial_{\tau_i}\left[\overline{B}_{\varepsilon,\gamma_i,\overline{\tau_i}}\right] + \mathrm{O}\left(\frac{1}{\overline{\gamma}_\varepsilon}\right), \quad \text{in } B(\overline{\tau}_i, r_\varepsilon).\end{aligned}$$

Now, recalling (3.19) and using (3.15), we write

$$Z_{1,i} = \partial_{\tau_i}\left[B_{\varepsilon,\gamma_i,\tau_i}\right] + \partial_{\tau_i}\left[\Psi_{\varepsilon,\gamma,\tau}\right] = \partial_{\tau_i}\left[\overline{B}_{\varepsilon,\gamma_i,\overline{\tau_i}}\right] + \mathrm{O}\left(\frac{1}{\overline{\gamma}_\varepsilon}\right) =: Z_{1,i}^a + Z_{1,i}^r, \tag{4.96}$$

in $B(\overline{\tau_i}, r_\varepsilon)$, and with (4.75) and Proposition 6.1, we estimate

$$Z_{1,i}^a = \frac{2}{\gamma_i} \frac{\lambda_{\varepsilon,i}(x_1 - \tau_i)}{\mu_i^2 + \lambda_{\varepsilon,i}|x - \overline{\tau_i}|^2} + \mathrm{O}\left(\frac{1}{\overline{\gamma}_\varepsilon^3} \frac{1}{\mu_i + |x - \overline{\tau_i}|}\right), \quad \text{in } B(\overline{\tau_i}, r_\varepsilon), \tag{4.97}$$

where $\mu_i := \mu_{\gamma_i} = \overline{\mu}_\varepsilon^{1+o(1)}$ is given by (6.1), while directly from the definition of $B_{\varepsilon,\gamma_i,\tau_i}$, Claim 3.1 and (3.15), we also obtain

$$Z_{1,i} = \frac{(2+o(1))\,(x_1-\tau_i)}{\overline{\gamma}_\varepsilon\,|x-\overline{\tau_i}|^2} + O\left(\frac{1}{\overline{\gamma}_\varepsilon}\right), \quad \text{in } \Omega\setminus B\left(\overline{\tau_i}, r_\varepsilon\right), \tag{4.98}$$

which can be specialized to

$$Z_{1,i} = O\left(\frac{1}{\overline{\gamma}_\varepsilon d_\varepsilon}\right), \quad \text{in } B\left(\overline{\tau_j}, r_\varepsilon\right), \quad \text{for } j\neq i. \tag{4.99}$$

Let us also write from (3.32),

$$F^{(i)}_{\varepsilon,\gamma,\tau}(x) = \Lambda^{(i)}_{\varepsilon,\tau}(x_1-\tau_i) + o\left(\frac{|x-\overline{\tau_i}|}{\overline{\gamma}_\varepsilon d_\varepsilon}\right) \quad \text{in } B\left(\overline{\tau_i}, r_\varepsilon\right), \tag{4.100}$$

where

$$\Lambda^{(i)}_{\varepsilon,\tau} := a_0 l \tau_i^{l-1} - \frac{2}{\overline{\gamma}_\varepsilon}\sum_{j\neq i}\frac{1}{\tau_i-\tau_j} = O\left(\frac{1}{\overline{\gamma}_\varepsilon d_\varepsilon}\right).$$

With the help of (4.70), as in (4.73), we can write

$$\begin{aligned}(A)_j &= -\int_{B(\overline{\tau_j},r_\varepsilon)} \lambda_\varepsilon\left(h_\varepsilon - h_\varepsilon\left(\overline{\tau_j}\right)\right) f\left(\overline{B}_{\varepsilon,\gamma_j,\overline{\tau_j}}\right) Z_{1,i}dx \\ &- \int_{B(\overline{\tau_j},r_\varepsilon)} \lambda_{\varepsilon,j} f'\left(\overline{B}_{\varepsilon,\gamma_j,\overline{\tau_j}}\right) R_j Z_{1,i}dx - \int_{B(\overline{\tau_j},r_\varepsilon)} \lambda_\varepsilon\left(h_\varepsilon - h_\varepsilon\left(\overline{\tau_j}\right)\right) f'\left(\overline{B}_{\varepsilon,\gamma_j,\overline{\tau_j}}\right) R_j Z_{1,i}dx \\ &+ O\left(\int_{B(\overline{\tau_j},r_\varepsilon)} \overline{\gamma}_\varepsilon^3 \exp\left(\overline{B}^2_{\varepsilon,\gamma_j,\overline{\tau_j}}\right) R_j^2\,|Z_{1,i}|\,dx\right) =: (A_1)_j + (A_2)_j + (A_3)_j + (A_4),\end{aligned}$$

where R_j is as in (4.71).

We start with the main order term, which turns out to be the one involving $F^{(i)}_{\varepsilon,\gamma,\tau}$ and which we write, using (4.97) and (4.100), as

$$\begin{aligned}(A_2^F)_i &:= -\int_{B(\overline{\tau_i},r_\varepsilon)} f'_\varepsilon\big(\overline{B}_{\varepsilon,\gamma_i,\overline{\tau_i}}\big) F^{(i)}_{\varepsilon,\gamma,\tau} Z_{1,i}dx \\ &= -\frac{\Lambda^{(i)}_{\varepsilon,\tau}}{\gamma_i}\int_{B(\overline{\tau_i},r_\varepsilon)} \lambda_{\varepsilon,i} f'\big(\overline{B}_{\varepsilon,\gamma_i,\overline{\tau_i}}\big)\left(\frac{2\lambda_{\varepsilon,i}\,(x_1-\tau_i)^2}{\mu_i^2+\lambda_{\varepsilon,i}\,|x-\overline{\tau_i}|^2} + O\left(\frac{1}{\overline{\gamma}_\varepsilon}\right)\right)dx \\ &\quad + o\left(\frac{1}{\overline{\gamma}_\varepsilon^2 d_\varepsilon}\int_{B(\overline{\tau_i},r_\varepsilon)} \lambda_{\varepsilon,i} f'\big(\overline{B}_{\varepsilon,\gamma_i,\overline{\tau_i}}\big)dx\right).\end{aligned}$$

With the usual change of variables $\sqrt{\lambda_{\varepsilon,i}}\,(x-\overline{\tau_i}) = y$ and using (4.75) and Proposition 6.3 together with $\gamma_i = \overline{\gamma}_\varepsilon\,(1+o\,(1))$ and $\Lambda^{(i)}_{\varepsilon,\tau} = O\left(1/\left(\overline{\gamma}_\varepsilon d_\varepsilon\right)\right)$, we get

$$\begin{aligned}(A_2^F)_i &= -\frac{\Lambda^{(i)}_{\varepsilon,\tau}}{\gamma_i}\int_{B(0,\sqrt{\lambda_{\varepsilon,i}}r_\varepsilon)} f'\left(\overline{B}_{\gamma_i}\right)\left(\frac{2y_1^2}{\mu_i^2+|y|^2} + o\,(1)\right)dy + o\left(\frac{1}{\overline{\gamma}_\varepsilon^2 d_\varepsilon}\right) \\ &= -\frac{(4\pi+o(1))\Lambda^{(i)}_{\varepsilon,\tau}}{\gamma_i} + o\left(\frac{1}{\overline{\gamma}_\varepsilon^2 d_\varepsilon}\right) = -\frac{4\pi\Lambda^{(i)}_{\varepsilon,\tau}}{\overline{\gamma}_\varepsilon} + o\left(\frac{1}{\overline{\gamma}_\varepsilon^2 d_\varepsilon}\right),\end{aligned}$$

For $j\neq i$, using (4.99) and Proposition 6.4, we get

$$(A_2^F)_j = O\left(\int_{B(\overline{\tau_j},r_\varepsilon)} f'\left(\overline{B}_{\varepsilon,\gamma_j,\overline{\tau_j}}\right)\left|F^{(j)}_{\varepsilon,\gamma,\tau}\right|\,|Z_{1,i}|\,dx\right)$$

$$
\begin{aligned}
&= \mathrm{O}\left(\int_{B(\overline{\tau_j},r_\varepsilon)} f'\left(\overline{B}_{\varepsilon,\gamma_j,\overline{\tau_j}}\right)\frac{|x-\overline{\tau_j}|}{\overline{\gamma}_\varepsilon d_\varepsilon}\frac{1}{\overline{\gamma}_\varepsilon d_\varepsilon}dx\right)\\
&= \mathrm{O}\left(\frac{1}{\overline{\gamma}_\varepsilon^2 d_\varepsilon^2}\int_{B(0,\sqrt{\lambda_{\varepsilon,j}}r_\varepsilon)} f'\left(\overline{B}_{\gamma_j}\right)|y|\,dy\right) = \mathrm{o}\left(\frac{1}{\overline{\gamma}_\varepsilon^2 d_\varepsilon}\right).
\end{aligned}
$$

Using (4.96), canceling the integral of the anti-symmetric term and using Proposition 6.3, we get

$$
\begin{aligned}
(A_2^E)_i &:= -\int_{B(\overline{\tau_i},r_\varepsilon)} f_\varepsilon'(\overline{B}_{\varepsilon,\gamma_i,\overline{\tau_i}})E^{(i)}_{\varepsilon,\gamma,\tau}(Z^a_{1,i}+Z^r_{1,i})dx\\
&= \mathrm{O}\left(\frac{\left|E^{(i)}_{\varepsilon,\gamma,\tau}\right|}{\overline{\gamma}_\varepsilon}\int_{B(\overline{\tau_i},r_\varepsilon)} f_\varepsilon'(\overline{B}_{\varepsilon,\gamma_i,\overline{\tau_i}})dx\right)\\
&= \mathrm{O}\left(\frac{\delta_\varepsilon \ln\overline{\gamma}_\varepsilon}{\overline{\gamma}_\varepsilon^4}\int_{B(0,\sqrt{\lambda_{\varepsilon,i}}r_\varepsilon)} f_\varepsilon'\left(\overline{B}_{\gamma_i}\right)dx\right) = \mathrm{o}\left(\frac{1}{\overline{\gamma}_\varepsilon^2 d_\varepsilon}\right).
\end{aligned}
$$

When $j\neq i$, we have thanks to (4.99),

$$
\begin{aligned}
(A_2^E)_j &= \mathrm{O}\left(\int_{B(\overline{\tau_j},r_\varepsilon)} f_\varepsilon'\left(\overline{B}_{\varepsilon,\gamma_j,\overline{\tau_j}}\right)|E^{(j)}_{\varepsilon,\gamma,\tau}|\,|Z_{1,i}|\,dx\right)\\
&= \mathrm{O}\left(\frac{\delta_\varepsilon \ln\overline{\gamma}_\varepsilon}{\overline{\gamma}_\varepsilon^3}\frac{1}{\overline{\gamma}_\varepsilon d_\varepsilon}\int_{B(0,\sqrt{\lambda_{\varepsilon,j}}r_\varepsilon)} f_\varepsilon'\left(\overline{B}_{\gamma_j}\right)dx\right) = \mathrm{o}\left(\frac{1}{\overline{\gamma}_\varepsilon^2 d_\varepsilon}\right).
\end{aligned}
$$

We now estimate (A_1). Using that $h_\varepsilon - h_\varepsilon(\overline{\tau_i}) = \mathrm{O}(|x-\overline{\tau_i}|)$ in $B(\overline{\tau_i},r_\varepsilon)$, by (4.97), we have

$$
|(h_\varepsilon - h_\varepsilon(\overline{\tau_i}))Z_{1,i}| = \mathrm{O}\left(\frac{1}{\overline{\gamma}_\varepsilon}\right),\quad \text{in } B(\overline{\tau_i},r_\varepsilon), \tag{4.101}
$$

and with Proposition 6.3, we estimate

$$
\begin{aligned}
(A_1)_i = \mathrm{O}\left(\frac{1}{\overline{\gamma}_\varepsilon}\int_{B(\overline{\tau_i},r_\varepsilon)} f(\overline{B}_{\varepsilon,\gamma_i,\overline{\tau_i}})dx\right) &= \mathrm{O}\left(\frac{1}{\overline{\gamma}_\varepsilon}\int_{B(0,\sqrt{\lambda_{\varepsilon,i}}r_\varepsilon)} f\left(\overline{B}_{\gamma_i}\right)dy\right)\\
&= \mathrm{O}\left(\frac{1}{\overline{\gamma}_\varepsilon^2}\right) = \mathrm{o}\left(\frac{1}{\overline{\gamma}_\varepsilon^2 d_\varepsilon}\right).
\end{aligned}
$$

Observe that this says that thanks to $d_\varepsilon = \mathrm{o}(1)$, the term $\nabla h_\varepsilon(\overline{\tau}_i)$ does not play a role, contrary to what happens when the blow-up points are separated by a finite distance. For $j\neq i$, with (4.99), Proposition 6.4 and the usual change of variables we obtain

$$
(A_1)_j = \mathrm{O}\left(\frac{1}{\overline{\gamma}_\varepsilon d_\varepsilon}\int_{B(\overline{\tau_j},r_\varepsilon)} f\left(\overline{B}_{\varepsilon,\gamma_j,\overline{\tau_j}}\right)|x-\overline{\tau_j}|\,dx\right) = \mathrm{o}\left(\frac{1}{\overline{\gamma}_\varepsilon^2 d_\varepsilon}\right).
$$

Similarly, one can bound with (4.97),

$$
\begin{aligned}
(A_3)_i &= \mathrm{O}\left(\int_{B(\overline{\tau_i},r_\varepsilon)} |x-\overline{\tau_i}|\,f'\left(\overline{B}_{\varepsilon,\gamma_i,\overline{\tau_i}}\right)|R_i|\frac{1}{\overline{\gamma}_\varepsilon|x-\overline{\tau_i}|}dx\right)\\
&= \mathrm{O}\left(\frac{1}{\overline{\gamma}_\varepsilon}\int_{B(\overline{\tau_i},r_\varepsilon)} f'\left(\overline{B}_{\varepsilon,\gamma_i,\overline{\tau_i}}\right)\left(\frac{\delta_\varepsilon\ln\overline{\gamma}_\varepsilon}{\overline{\gamma}_\varepsilon^3}+\frac{|x-\overline{\tau_i}|}{\overline{\gamma}_\varepsilon d_\varepsilon}\right)dx\right) = \mathrm{o}\left(\frac{1}{\overline{\gamma}_\varepsilon^2 d_\varepsilon}\right),
\end{aligned}
$$

where we also used Propositions 6.3 and 6.4. For $j \neq i$, an easier estimate holds, using (4.99) instead of (4.97), and $h_\varepsilon - h_\varepsilon(\tau_i) = \mathrm{O}(r_\varepsilon)$, so that

$$(A_3)_j = \mathrm{O}\left(\frac{r_\varepsilon}{\overline{\gamma}_\varepsilon d_\varepsilon}\int_{B(\overline{\tau_j}, r_\varepsilon)} f'\left(\overline{B}_{\varepsilon,\gamma_j,\overline{\tau_j}}\right)\left(\frac{\delta_\varepsilon \ln \overline{\gamma}_\varepsilon}{\overline{\gamma}_\varepsilon^3} + \frac{|x-\overline{\tau_j}|}{\overline{\gamma}_\varepsilon d_\varepsilon}\right) dx\right) = \mathrm{o}\left(\frac{1}{\overline{\gamma}_\varepsilon^2 d_\varepsilon}\right),$$

As for $(A_4)_i$, using that $\left|E^{(i)}_{\varepsilon,\gamma,\tau}\right| = \mathrm{o}\left(\delta_\varepsilon^2 (\ln \overline{\gamma}_\varepsilon)^2 / \overline{\gamma}_\varepsilon^6\right) = \mathrm{O}\left(\overline{\mu}_\varepsilon^{2\delta_1+1+\mathrm{o}(1)}\right)$, and $\overline{\gamma}_\varepsilon^a = \mathrm{O}\left(\overline{\mu}_\varepsilon^{\mathrm{o}(1)}\right)$ for every $a \in \mathbb{R}$, we bound

$$\begin{aligned}(A_4^E)_i &= \mathrm{O}\left(\overline{\mu}_\varepsilon^{2\delta_1+1+\mathrm{o}(1)}\int_{B(\overline{\tau_i}, r_\varepsilon)} \exp\left(\overline{B}^2_{\varepsilon,\gamma_i,\overline{\tau_i}}\right)\frac{1}{\mu_i + |x-\overline{\tau_i}|} dx\right) \\ &= \mathrm{O}\left(\overline{\mu}_\varepsilon^{2\delta_1+\mathrm{o}(1)}\int_{B(0,\sqrt{\lambda_{\varepsilon,i}} r_\varepsilon)} \exp\left(\overline{B}^2_{\gamma_i}\right) dx\right) = \mathrm{O}\left(\overline{\mu}_\varepsilon^{2\delta_1+\mathrm{o}(1)}\right) = \mathrm{o}\left(\frac{1}{\overline{\gamma}_\varepsilon^2 d_\varepsilon}\right),\end{aligned}$$

and, similarly, for $j \neq i$,

$$(A_4^E)_j = \mathrm{O}\left(\overline{\mu}_\varepsilon^{2\delta_1+1+\mathrm{o}(1)}\int_{B(\overline{\tau_j}, r_\varepsilon)} \exp\left(\overline{B}^2_{\varepsilon,\gamma_i,\overline{\tau_i}}\right)\frac{1}{\overline{\gamma}_\varepsilon d_\varepsilon} dx\right) = \mathrm{o}\left(\frac{1}{\overline{\gamma}_\varepsilon^2 d_\varepsilon}\right).$$

Using that $F^{(j)}_{\varepsilon,\gamma,\tau} = \mathrm{O}\left(r_\varepsilon / (\overline{\gamma}_\varepsilon d_\varepsilon)\right)$, similarly as in the case of $(A_4^E)_j$, we obtain $(A_4^F)_j = \mathrm{o}\left(\frac{1}{\overline{\gamma}_\varepsilon^2 d_\varepsilon}\right)$, including the case $j = i$. Summing over j, we obtain

$$(A) = \sum_{j=1}^k (A)_j = -\frac{4\pi\Lambda^{(i)}_{\varepsilon,\tau}}{\overline{\gamma}_\varepsilon} + \mathrm{o}\left(\frac{1}{\overline{\gamma}_\varepsilon^2 d_\varepsilon}\right). \tag{4.102}$$

It remains to show that all the remaining terms are $\mathrm{o}\left(1/(\overline{\gamma}_\varepsilon^2 d_\varepsilon)\right)$.

Let us now estimate (A'). By (3.20), (3.21), (4.98) and (4.99), we have

$$\begin{aligned}(A') &= \mathrm{O}\left(\sum_{j=1}^k \int_{\Omega_\varepsilon^j} |f_\varepsilon(U_{\varepsilon,\gamma,\tau})|\,|Z_{1,i}|\,dx\right) = \mathrm{O}\left(\frac{\overline{\mu}_\varepsilon^{-2\delta_0^2+\mathrm{o}(1)} r_\varepsilon^2}{\overline{\gamma}_\varepsilon}\right) \\ &= \mathrm{O}\left(\overline{\mu}_\varepsilon^{2\delta_0-2\delta_0^2+\mathrm{o}(1)}\right) = \mathrm{o}\left(\frac{1}{\overline{\gamma}_\varepsilon^2 d_\varepsilon}\right),\end{aligned} \tag{4.103}$$

absorbing powers of $\overline{\gamma}_\varepsilon$ in the term $\overline{\mu}_\varepsilon^{\mathrm{o}(1)}$ and using that $2\delta_0 - 2\delta_0^2 > 0$.

We now estimate (B). Since $\Delta B_{\varepsilon,\gamma_j,\tau_j} = \Delta \overline{B}_{\varepsilon,\gamma_j,\overline{\tau_j}} \mathbf{1}_{B(\overline{\tau_j}, r_\varepsilon)}$, we have

$$(B^\dagger)_j := \theta_j \int_\Omega \Delta B_{\varepsilon,\gamma_j,\tau_j} Z_{1,i} dx = \theta_j \int_{B(\overline{\tau_j}, r_\varepsilon)} \Delta \overline{B}_{\varepsilon,\gamma_j,\overline{\tau_j}} Z_{1,i} dx.$$

Then for $j \neq i$, together with (4.99) we obtain

$$\theta_j \int_{B(\overline{\tau_j}, r_\varepsilon)} \Delta \overline{B}_{\varepsilon,\gamma_j,\overline{\tau_j}} Z_{1,i} dx = \mathrm{O}\left(\frac{|\theta_j|}{\overline{\gamma}_\varepsilon d_\varepsilon}\int_{B(\overline{\tau_j}, r_\varepsilon)} f\left(\overline{B}_{\varepsilon,\gamma_j,\overline{\tau_j}}\right) dx\right) = \mathrm{o}\left(\frac{1}{\overline{\gamma}_\varepsilon^2 d_\varepsilon}\right).$$

For $j = i$, we use the anti-symmetry to obtain

$$\begin{aligned}\theta_i \int_{B(\overline{\tau_i}, r_\varepsilon)} \Delta \overline{B}_{\varepsilon,\gamma_i,\overline{\tau_i}} Z_{1,i} dx &= \theta_i \int_{B(\overline{\tau_i}, r_\varepsilon)} \lambda_{\varepsilon,i} f\left(\overline{B}_{\varepsilon,\gamma_i,\overline{\tau_i}}\right) \overline{B}_{\varepsilon,\gamma_i,\overline{\tau_i}} \left(Z^a_{1,i} + Z^r_{1,i}\right) dx \\ &= \theta_i \int_{B(\overline{\tau_i}, r_\varepsilon)} \lambda_{\varepsilon,i} f\left(\overline{B}_{\varepsilon,\gamma_i,\overline{\tau_i}}\right) \overline{B}_{\varepsilon,\gamma_i,\overline{\tau_i}} D_{\tau_i} \Psi_{\varepsilon,\gamma,\tau} dx = \mathrm{o}\left(\frac{1}{\overline{\gamma}_\varepsilon^2 d_\varepsilon}\right).\end{aligned}$$

In order to estimate the second term in the integral in $(B)_j$, we start with the integral away from the blow-up points, and using (3.17), the definition of $B_{\varepsilon,\gamma_j,\tau_j}$ and (3.10), we get

$$(B'_j) = \theta_j \int_{\Omega_{r_\varepsilon,\tau}} f'_\varepsilon\left(U_{\varepsilon,\gamma,\tau}\right) B_{\varepsilon,\gamma_j,\tau_j} Z_{1,i} dx = \mathrm{O}\left(|\theta_j| \left\| f'_\varepsilon\left(U_{\varepsilon,\gamma,\tau}\right) Z_{1,i} \mathbf{1}_{\Omega_{r_\varepsilon,\tau}} \right\|_{L^p} \right.$$
$$\left. \times \left\| B_{\varepsilon,\gamma_j,\tau_j} \mathbf{1}_{\Omega \setminus B(\overline{\tau_j},r_\varepsilon)} \right\|_{L^{p'}} \right) = \mathrm{O}\left(|\theta_j| \,\overline{\gamma}_\varepsilon \left\| \frac{1}{\overline{\gamma}_\varepsilon} \ln \frac{C}{|\cdot - \overline{\tau_j}|} \right\|_{L^{p'}} \right) = \mathrm{o}\left(\frac{1}{\overline{\gamma}_\varepsilon^2 d_\varepsilon} \right),$$

where p is sufficiently small and p' is the conjugate exponent of p. It remains to estimate

$$(B)_{jm} := \theta_j \int_{B(\overline{\tau_m},r_\varepsilon)} B_{\varepsilon,\gamma_j,\tau_j} f'_\varepsilon\left(U_{\varepsilon,\gamma,\tau}\right) Z_{1,i} dx.$$

For $m \neq i$, it easily follows from (4.99) and Proposition 6.3 that

$$(B)_{jm} = \mathrm{O}\left(\frac{|\theta_j|}{d_\varepsilon} \int_{B(\overline{\tau_m},r_\varepsilon)} f'_\varepsilon\left(U_{\varepsilon,\gamma,\tau}\right) dx \right) = \mathrm{o}\left(\frac{1}{\overline{\gamma}_\varepsilon^2 d_\varepsilon} \right).$$

The case $m = i$ is more subtle. Using (4.80) to split

$$(B)_{ji} = \theta_j \int_{B(\overline{\tau_i},r_\varepsilon)} B_{\varepsilon,\gamma_j,\tau_j} f'_\varepsilon\big(\overline{B}_{\varepsilon,\gamma_i,\overline{\tau_i}}\big) Z_{1,i} dx$$
$$+ \mathrm{O}\left(|\theta_j|\, \overline{\gamma}_\varepsilon^3 \int_{B(\overline{\tau_i},r_\varepsilon)} \exp\big(\overline{B}^2_{\varepsilon,\gamma_i,\overline{\tau_i}}\big) |R_i|\, |Z_{1,i}|\, dx \right) =: (B_1)_{ji} + (B_2)_{ji}. \quad (4.104)$$

Now, writing

$$B_{\varepsilon,\gamma_i,\tau_i} = B^s_{\varepsilon,\gamma_i,\tau_i} + B^r_{\varepsilon,\gamma_i,\tau_i},$$

where

$$B^s_{\varepsilon,\gamma_i,\tau_i} = \overline{B}_{\varepsilon,\gamma_i,\overline{\tau_i}} - C_{\varepsilon,\gamma_i,\tau_i} + A_{\varepsilon,\gamma_i,\tau_i} H\left(\overline{\tau_i},\overline{\tau_i}\right),$$
$$B^r_{\varepsilon,\gamma_i,\tau_i} = A_{\varepsilon,\gamma_i,\tau_i}\left(H\left(\cdot,\overline{\tau_i}\right) - H\left(\overline{\tau_i},\overline{\tau_i}\right)\right) = \mathrm{O}\left(\frac{|\cdot - \overline{\tau_i}|}{\overline{\gamma}_\varepsilon} \right), \quad \text{in } B\left(\overline{\tau_i}, r_\varepsilon\right),$$

and also using (4.96) and (4.101), we get

$$(B_1)_{ii} = \theta_i \int_{B(\overline{\tau_i},r_\varepsilon)} B^s_{\varepsilon,\gamma_i,\tau_i} \lambda_{\varepsilon,i} f'\big(\overline{B}_{\varepsilon,\gamma_i,\overline{\tau_i}}\big) Z^a_{1,i} dx$$
$$+ \mathrm{O}\left(|\theta_i| \int_{B(\overline{\tau_i},r_\varepsilon)} f'\big(\overline{B}_{\varepsilon,\gamma_i,\overline{\tau_i}}\big)\big(1 + \overline{\gamma}_\varepsilon \underbrace{|x - \overline{\tau_i}|\, |Z_{1,i}|}_{=\mathrm{O}(1/\overline{\gamma}_\varepsilon)}\big) dx \right) = \mathrm{o}\left(\frac{1}{\overline{\gamma}_\varepsilon^2 d_\varepsilon} \right), \quad (4.105)$$

where we used skew-symmetry to cancel the first integral, and the usual change of variables to estimate the second one with Proposition 6.3. When $j \neq i$, a similar approach gives analog results, with the splitting of $B_{\varepsilon,\gamma_j,\tau_j} = B^s_{\varepsilon,\gamma_j,\tau_j} + B^r_{\varepsilon,\gamma_j,\tau_j}(x)$, where

$$B^s_{\varepsilon,\gamma_j,\tau_j} = A_{\varepsilon,\gamma_j,\tau_j} G\left(\overline{\tau_j},\overline{\tau_i}\right),$$
$$B^r_{\varepsilon,\gamma_j,\tau_j} = A_{\varepsilon,\gamma_j,\tau_j}\left(G\left(\overline{\tau_j},x\right) - G\left(\overline{\tau_j},\overline{\tau_i}\right)\right) = \mathrm{O}\left(\frac{|x - \overline{\tau_i}|}{\overline{\gamma}_\varepsilon d_\varepsilon} \right), \quad \text{in } B\left(\overline{\tau_i}, r_\varepsilon\right),$$

which allows to cancel the symmetric term and obtain

$$(B_1)_{ji} = \mathrm{o}\left(\frac{1}{\overline{\gamma}_\varepsilon^2 d_\varepsilon} \right)$$

As for $(B_2)_{ji}$, the term involving $F^{(i)}_{\varepsilon,\gamma,\tau}$ can be estimated using a similar approach as in (4.105), since $F^{(i)}_{\varepsilon,\gamma,\tau}(x) = \mathrm{O}\left(|x-\overline{\tau_i}|/(\overline{\gamma}_\varepsilon d_\varepsilon)\right)$ in $B(\overline{\tau_i}, r_\varepsilon)$. In the term involving $E^{(i)}_{\varepsilon,\gamma,\tau} = \mathrm{O}\left(\delta_\varepsilon \ln \overline{\gamma}_\varepsilon / \overline{\gamma}_\varepsilon^3\right)$, we use the estimate

$$|Z_{1,i}| = \mathrm{O}\left(\frac{1}{\overline{\gamma}_\varepsilon\left(\mu_{\gamma_i} + |x-\overline{\tau_i}|\right)}\right) = \mathrm{O}\left(\frac{1}{\overline{\mu}_\varepsilon^{1+\mathrm{o}(1)}}\right) \tag{4.106}$$

to finally obtain

$$(B_2)_{ji} = \mathrm{O}\left(\frac{|\theta_j|\,\delta_\varepsilon}{\overline{\mu}_\varepsilon^{1+\mathrm{o}(1)}}\right) = \mathrm{o}\left(\frac{1}{\overline{\gamma}_\varepsilon^2 d_\varepsilon}\right).$$

Summing up, we conclude

$$(B) = \sum_{j=1}^{k}\left[(B^\dagger)_j + (B')_j + \sum_{m=1}^{k}(B)_{jm}\right] = \mathrm{o}\left(\frac{1}{\overline{\gamma}_\varepsilon^2 d_\varepsilon}\right). \tag{4.107}$$

To bound the term (C), let us start by observing that $\Phi_{\varepsilon,\gamma,\tau,\theta} \perp Z_{1,i}$ implies

$$(C_1) := \int_\Omega \Delta\Phi_{\varepsilon,\gamma,\tau,\theta} Z_{1,i}\, dx = 0,$$

so that it remains to bound

$$(C_2) := -\int_\Omega f'_\varepsilon\left(U_{\varepsilon,\gamma,\tau}\right)\Phi_{\varepsilon,\gamma,\tau,\theta} Z_{1,i}\, dx.$$

Observe that a rough estimate on $B(\overline{\tau_i}, r_\varepsilon)$ using $|Z_{1,i}| = \mathrm{O}\left(1/\overline{\mu}_\varepsilon\right)$ would lead to an exponentially large error term. Therefore we have to be more subtle and use again the Sobolev–Poincaré estimates which follow from $\Phi_{\varepsilon,\gamma,\tau,\theta} \perp B_{\varepsilon,\gamma_i,\tau_i}$. We start by noticing that by (3.17) and the Sobolev embedding, we have

$$\begin{aligned}(C_2^*) := -\int_{\Omega_{r_\varepsilon,\tau}} f'_\varepsilon\left(U_{\varepsilon,\gamma,\tau}\right)\Phi_{\varepsilon,\gamma,\tau,\theta} Z_{1,i}\, dx = \mathrm{O}\Big(&\left\|f'\left(U_{\varepsilon,\gamma,\tau}\right) Z_{1,i}\mathbf{1}_{\Omega_{r_\varepsilon,\tau}}\right\|_{L^p} \\ &\times \left\|\Phi_{\varepsilon,\gamma,\tau,\theta}\right\|_{L^{p'}}\Big) = \mathrm{O}\left(\overline{\gamma}_\varepsilon \left\|\nabla\Phi_{\varepsilon,\gamma,\tau,\theta}\right\|_{L^2}\right) = \mathrm{O}\left(\frac{\delta_\varepsilon \ln\overline{\gamma}_\varepsilon}{\overline{\gamma}_\varepsilon}\right) = \mathrm{o}\left(\frac{1}{\overline{\gamma}_\varepsilon^2 d_\varepsilon}\right).\end{aligned}$$

For $j \neq i$, we bound with (4.99) and Proposition 7.2 (which we can use thanks to (4.84)),

$$\begin{aligned}(C_2^\dagger)_j &:= -\int_{B(\overline{\tau_j}, r_\varepsilon)} f'_\varepsilon\left(U_{\varepsilon,\gamma,\tau}\right) Z_{1,i}\Phi_{\varepsilon,\gamma,\tau,\theta}\, dx \\ &= \mathrm{O}\left(\frac{1}{\overline{\gamma}_\varepsilon d_\varepsilon}\int_{B(\overline{\tau_j}, r_\varepsilon)} f'_\varepsilon\left(U_{\varepsilon,\gamma,\tau}\right)\left|\Phi_{\varepsilon,\gamma,\tau,\theta}\right| dx\right) \\ &= \mathrm{O}\left(\frac{\left\|\nabla\Phi_{\varepsilon,\gamma,\tau,\theta}\right\|_{L^2}}{\overline{\gamma}_\varepsilon d_\varepsilon}\right) = \mathrm{O}\left(\frac{\delta_\varepsilon \ln\overline{\gamma}_\varepsilon}{\overline{\gamma}_\varepsilon^3 d_\varepsilon}\right) = \mathrm{o}\left(\frac{1}{\overline{\gamma}_\varepsilon^2 d_\varepsilon}\right).\end{aligned} \tag{4.108}$$

We are left with $(C_2^\dagger)_i$ which we expand as in (4.80), giving

$$\begin{aligned}(C_2^\dagger)_i = &-\int_{B(\overline{\tau_i}, r_\varepsilon)} f'_\varepsilon\big(\overline{B}_{\varepsilon,\gamma_i,\overline{\tau_i}}\big) Z_{1,i}\Phi_{\varepsilon,\gamma,\tau,\theta}\, dx \\ &+ \mathrm{O}\left(\overline{\gamma}_\varepsilon^3\int_{B(\overline{\tau_i}, r_\varepsilon)} \exp\big(\overline{B}^2_{\varepsilon,\gamma_i,\overline{\tau_i}}\big)\left|\big(E^{(i)}_{\varepsilon,\gamma,\tau} + F^{(i)}_{\varepsilon,\gamma,\tau}\big) Z_{1,i}\Phi_{\varepsilon,\gamma,\tau,\theta}\right| dx\right)\end{aligned} \tag{4.109}$$

The remainder term in (4.109) can be estimated as follows. By (4.97) and (4.100), we get $|F^{(i)}_{\varepsilon,\gamma,\tau}Z_{1,i}| = \mathrm{O}\left(1/\left(\overline{\gamma}_\varepsilon^2 d_\varepsilon\right)\right)$, and we use Proposition 7.2 to obtain an error term of order

$$\mathrm{O}\left(\frac{\|\nabla\Phi_{\varepsilon,\gamma,\tau,\theta}\|_{L^2}}{\overline{\gamma}_\varepsilon d_\varepsilon}\right) = \mathrm{o}\left(\frac{1}{\overline{\gamma}_\varepsilon^2 d_\varepsilon}\right).$$

As regards the term involving $E^{(i)}_{\varepsilon,\gamma,\tau}$, using (4.106) and Proposition 7.2, we obtain an error term of order

$$\mathrm{O}\left(\frac{\delta_\varepsilon\|\nabla\Phi_{\varepsilon,\gamma,\tau,\theta}\|_{L^2}}{\overline{\mu}_\varepsilon^{1+\mathrm{o}(1)}}\right) = \mathrm{O}\left(\overline{\mu}_\varepsilon^{2\delta_1+\mathrm{o}(1)}\right) = \mathrm{o}\left(\frac{1}{\overline{\gamma}_\varepsilon^2 d_\varepsilon}\right).$$

The first integral in (4.109) can be estimated by using (4.96) together with the estimate $\big(h_\varepsilon - h_\varepsilon\left(\overline{\tau_i}\right)\big)\partial_{\tau_i}\left[\overline{B}_{\varepsilon,\gamma_i,\overline{\tau_i}}\right] = \mathrm{O}\left(1/\gamma_\varepsilon\right)$ to obtain

$$\begin{aligned}\int_{B(\overline{\tau_i},r_\varepsilon)} f_\varepsilon'\big(\overline{B}_{\varepsilon,\gamma_i,\overline{\tau_i}}\big)Z_{1,i}\Phi_{\varepsilon,\gamma,\tau,\theta}dx &= \int_{B(\overline{\tau_i},r_\varepsilon)} \lambda_{\varepsilon,i}f'\left(\overline{B}_{\varepsilon,\gamma_i,\overline{\tau_i}}\right)\partial_{\tau_i}\left[\overline{B}_{\varepsilon,\gamma_i,\overline{\tau_i}}\right]\Phi_{\varepsilon,\gamma,\tau,\theta}dx\\ &+\mathrm{O}\left(\frac{1}{\gamma_\varepsilon}\int_{B(\overline{\tau_i},r_\varepsilon)} f'\left(\overline{B}_{\varepsilon,\gamma_i,\overline{\tau_i}}\right)|\Phi_{\varepsilon,\gamma,\tau,\theta}|\,dx\right) =: -(C_2^\ddagger)_i + (C_2^r)_i.\end{aligned}$$

The remainder term $(C_2^r)_i$ can be handled as in (4.108), giving

$$(C_2^r)_i = \mathrm{O}\left(\frac{\|\nabla\Phi_{\varepsilon,\gamma,\tau,\theta}\|_{L^2}}{\overline{\gamma}_\varepsilon}\right) = \mathrm{o}\left(\frac{1}{\overline{\gamma}_\varepsilon^2 d_\varepsilon}\right).$$

Then we are left with the term $(C_2^\ddagger)_i$, which is actually more subtle to bound. Let us first rewrite it as

$$(C_2^\ddagger)_i = -\int_{B(\overline{\tau_i},r_\varepsilon)} \lambda_{\varepsilon,i}\Delta Z_{1,i}\Phi_{\varepsilon,\gamma,\tau,\theta}dx,$$

using that

$$\Delta Z_{1,i} = \partial_{\tau_i}\left[\Delta U_{\varepsilon,\gamma,\tau}\right] = \partial_{\tau_i}\left[\Delta\overline{B}_{\varepsilon,\gamma_i,\overline{\tau_i}}\right] = \lambda_{\varepsilon,i}\partial_{\tau_i}\left[f\left(\overline{B}_{\varepsilon,\gamma,\overline{\tau_i}}\right)\right], \quad \text{in } B\left(\overline{\tau_i},r_\varepsilon\right).$$

In order to estimate $(C_2^\ddagger)_i$ we start by observing that the orthogonality condition $Z_{1,i}\perp\Phi_{\varepsilon,\gamma,\tau,\theta}$ and integration by parts imply

$$\begin{aligned}0 = \int_\Omega\langle\nabla Z_{1,i},\nabla\Phi_{\varepsilon,\gamma,\tau,\theta}\rangle\,dx &= \int_{B(\overline{\tau_i},r_\varepsilon)}\Delta Z_{1,i}\,\Phi_{\varepsilon,\gamma,\tau,\theta}dx\\ &+\int_{\Omega\setminus B(\overline{\tau_i},r_\varepsilon)}\Delta Z_{1,i}\,\Phi_{\varepsilon,\gamma,\tau,\theta}dx + \int_{\partial B(\overline{\tau_i},r_\varepsilon)}\left(\partial_\nu Z_{1,i}^{int} - \partial_\nu Z_{1,i}^{ext}\right)\Phi_{\varepsilon,\gamma,\tau,\theta}d\sigma. \quad (4.110)\end{aligned}$$

Here ν denotes the exterior normal to $\partial B\left(\overline{\tau_i},r_\varepsilon\right)$ and

$$Z_{1,i}^{int} := Z_{1,i}|_{\overline{B(\overline{\tau_i},r_\varepsilon)}},\quad Z_{1,i}^{ext} := Z_{1,i}|_{\Omega\setminus B(\overline{\tau_i},r_\varepsilon)}.$$

Note that the boundary integral in (4.110) is in general non-zero because $B_{\varepsilon,\gamma_i,\tau_i}$ is C^1 but not smooth across $\partial B\left(\overline{\tau_i},r_\varepsilon\right)$. Now we reduced the estimate of $(C_2^\ddagger)_i$ to

$$(C_2^\ddagger)_i = \int_{\Omega\setminus B(\overline{\tau_i},r_\varepsilon)}\Delta Z_{1,i}\,\Phi_{\varepsilon,\gamma,\tau,\theta}dx + \int_{\partial B(\overline{\tau_i},r_\varepsilon)}\left(\partial_\nu Z_{1,i}^{int} - \partial_\nu Z_{1,i}^{ext}\right)\Phi_{\varepsilon,\gamma,\tau,\theta}d\sigma.$$

Now using that

$$\begin{aligned}\Delta Z_{1,i} &= \partial_{\tau_i}\left[\Delta U_{\varepsilon,\gamma,\tau}\right] = \partial_{\tau_i}\left[\chi_{\varepsilon,\tau}f_\varepsilon\left(U_{\varepsilon,\gamma,\tau}\right)\right]\\ &= \partial_{\tau_i}\left[\chi_{\varepsilon,\tau}\right]f_\varepsilon\left(U_{\varepsilon,\gamma,\tau}\right) + \chi_{\varepsilon,\tau}f_\varepsilon'\left(U_{\varepsilon,\gamma,\tau}\right)\partial_{\tau_i}\left[U_{\varepsilon,\gamma,\tau}\right], \quad \text{in } B\left(\overline{\tau_i},r_\varepsilon\right).\end{aligned}$$

from (3.17), we obtain

$$\left\|\Delta Z_{1,i}\mathbf{1}_{\Omega\setminus B(\overline{\tau_i},r_\varepsilon)}\right\|_{L^p}=\mathrm{O}\left(\overline{\gamma}_\varepsilon\right)$$

for some $p>1$, hence with the Hölder and Sobolev inequalities

$$\begin{aligned}\int_{\Omega\setminus B(\overline{\tau_i},r_\varepsilon)}\Delta Z_{1,i}\,\Phi_{\varepsilon,\gamma,\tau,\theta}dx&=\mathrm{O}\left(\left\|\Delta Z_{1,i}\mathbf{1}_{\Omega\setminus B(\overline{\tau_i},r_\varepsilon)}\right\|_{L^p}\left\|\Phi_{\varepsilon,\gamma,\tau,\theta}\right\|_{L^{p'}}\right)\\&=\mathrm{O}\left(\overline{\gamma}_\varepsilon\left\|\nabla\Phi_{\varepsilon,\gamma,\tau,\theta}\right\|_{L^2}\right)=\mathrm{o}\left(\frac{1}{\overline{\gamma}_\varepsilon d_\varepsilon}\right).\end{aligned}$$

Observe that $D_\tau\left[\Psi_{\varepsilon,\gamma,\tau}\right]\in C^1\left(\overline{\Omega}\right)$, by elliptic estimates (the function $\chi_{\varepsilon,\tau}$ in Proposition 3.2 is smooth), hence we get

$$\partial_\nu Z_{1,i}^{int}-\partial_\nu Z_{1,i}^{ext}=\partial_\nu\partial_{\tau_i}\left[B_{\varepsilon,\gamma_i,\tau_i}^{int}\right]-\partial_\nu\partial_{\tau_i}\left[B_{\varepsilon,\gamma_i,\tau_i}^{ext}\right] \tag{4.111}$$

Using the definition of $B_{\varepsilon,\gamma_i,\tau_i}$ and (3.12), we compute

$$\begin{aligned}\partial_\nu\partial_{\tau_i}\left[B_{\varepsilon,\gamma_i,\tau_i}^{int}\right]&=\partial_{\tau_i}\left[\partial_\nu B_{\varepsilon,\gamma_i,\tau_i}^{int}\right]=\partial_{\tau_i}\left[\partial_\nu\overline{B}_{\varepsilon,\gamma_i,\overline{\tau_i}}\right]+\partial_{\tau_i}\left[\partial_\nu(A_{\varepsilon,\gamma_i,\tau_i}H\left(\cdot,\overline{\tau_i}\right))\right]\\&=\partial_{\tau_i}\left[\partial_\nu\overline{B}_{\varepsilon,\gamma_i,\overline{\tau_i}}\right]+\mathrm{O}\left(\frac{1}{\overline{\gamma}_\varepsilon}\right).\end{aligned}$$

Similarly,

$$\begin{aligned}\partial_\nu\partial_{\tau_i}\left[B_{\varepsilon,\gamma_i,\tau_i}^{ext}\right]&=\partial_{\tau_i}\left[\frac{A_{\varepsilon,\gamma_i,\tau_i}}{2\pi}\partial_\nu\left(\ln\frac{1}{|x-\overline{\tau_i}|}\right)\right]+\partial_{\tau_i}\left[\partial_\nu\left(A_{\varepsilon,\gamma_i,\tau_i}H\left(\cdot,\overline{\tau_i}\right)\right)\right]\\&=-\partial_{\tau_i}\left[\frac{A_{\varepsilon,\gamma_i,\tau_i}}{2\pi\left|x-\overline{\tau_i}\right|}\right]+\mathrm{O}\left(\frac{1}{\overline{\gamma}_\varepsilon}\right).\end{aligned}$$

Now in order to compute the difference of the two terms in (4.111), set

$$v_{\varepsilon,\gamma_i,\tau_i}\left(x\right):=\frac{A_{\varepsilon,\gamma_i,\tau_i}}{2\pi}\ln\frac{1}{|x-\overline{\tau_i}|}$$

and Note that $v'_{\varepsilon,\gamma_i,\tau_i}\left(r_\varepsilon\right)=\overline{B}'_{\varepsilon,\gamma_i,\overline{\tau_i}}\left(r_\varepsilon\right)$ by the definitions in Section 3.1, where with a little abuse of notation, we use the prime to denote the radial derivative from $\overline{\tau_i}$. Then, with a similar abuse of notation

$$\begin{aligned}-\overline{B}''_{\varepsilon,\gamma_i,\overline{\tau_i}}\left(r_\varepsilon\right)-\frac{\overline{B}'_{\varepsilon,\gamma_i,\overline{\tau_i}}\left(r_\varepsilon\right)}{r_\varepsilon}&=\Delta\overline{B}_{\varepsilon,\gamma_i,\overline{\tau_i}}\left(r_\varepsilon\right)=\lambda_{\varepsilon,i}f\left(\overline{B}_{\varepsilon,\gamma_i,\overline{\tau_i}}\left(r_\varepsilon\right)\right),\\-v''_{\varepsilon,\gamma_i,\tau_i}\left(r_\varepsilon\right)-\frac{v'_{\varepsilon,\gamma_i,\tau_i}\left(r_\varepsilon\right)}{r_\varepsilon}&=\Delta v_{\varepsilon,\gamma_i,\tau_i}\left(r_\varepsilon\right)=0,\end{aligned}$$

and subtracting we finally estimate

$$\begin{aligned}\left|\partial_\nu Z_{1,i}^{int}-\partial_\nu Z_{1,i}^{ext}\right|&=\mathrm{O}\left(\left|\overline{B}''_{\varepsilon,\gamma_i,\overline{\tau_i}}\left(r_\varepsilon\right)-v''_{\varepsilon,\gamma_i,\tau_i}\left(r_\varepsilon\right)\right|\right)+\mathrm{O}\left(\frac{1}{\overline{\gamma}_\varepsilon}\right)\\&=\mathrm{O}\left(f\left(\overline{B}_{\varepsilon,\gamma_i,\overline{\tau_i}}\left(r_\varepsilon\right)\right)\right)+\mathrm{O}\left(\frac{1}{\overline{\gamma}_\varepsilon}\right)=\mathrm{O}\left(\frac{1}{\overline{\mu}_\varepsilon^{2\delta_0^2+\mathrm{o}(1)}}\right).\end{aligned} \tag{4.112}$$

We now claim that

$$\left\|\Phi_{\varepsilon,\gamma,\tau,\theta}\right\|_{L^1(\partial B(\overline{\tau_i},r_\varepsilon))}=\mathrm{O}\left(r_\varepsilon\sqrt{\ln\frac{1}{r_\varepsilon}}+\left\|\nabla\Phi_{\varepsilon,\gamma,\tau,\theta}\right\|_{L^2}\right)=\mathrm{O}\left(\overline{\mu}_\varepsilon^{\delta_0+\mathrm{o}(1)}\right). \tag{4.113}$$

This, together with (4.112) allows to bound

$$\int_{\partial B(\overline{\tau_i},r_\varepsilon)} \left(\partial_\nu Z_{1,i}^{int} - \partial_\nu Z_{1,i}^{ext}\right) \Phi_{\varepsilon,\gamma,\tau,\theta} d\sigma = \mathrm{O}\left(\frac{\|\Phi_{\varepsilon,\gamma,\tau,\theta}\|_{L^1(\partial B(\overline{\tau_i},r_\varepsilon))}}{\overline{\mu_\varepsilon}^{2\delta_0^2+\mathrm{o}(1)}}\right)$$
$$= \mathrm{O}\left(\overline{\mu_\varepsilon}^{\delta_0-2\delta_0^2+\mathrm{o}(1)}\right) = \mathrm{o}\left(\frac{1}{\overline{\gamma}_\varepsilon^2 d_\varepsilon}\right).$$

This completes the estimates of $(C_2^\ddagger)_i$, hence

$$(C) = (C_1) + (C_2^*) + \sum_{j=1}^k \left[(C_2^\dagger)_j + (C_2^\ddagger)_j\right] = \mathrm{o}\left(\frac{1}{\overline{\gamma}_\varepsilon^2 d_\varepsilon}\right). \tag{4.114}$$

In order to prove (4.113), set $\widetilde{\Phi}(y) := \Phi_{\varepsilon,\gamma,\tau,\theta}(\overline{\tau_i} + r_\varepsilon y)$. We then have

$$\left\|\nabla\widetilde{\Phi}\right\|_{L^2(B(0,1))} = \|\nabla\Phi_{\varepsilon,\gamma,\tau,\theta}\|_{L^2(B(\overline{\tau_i},r_\varepsilon))}.$$

By the trace inequality, we get

$$\frac{\|\Phi_{\varepsilon,\gamma,\tau,\theta}\|_{L^1(B(\overline{\tau_i},r_\varepsilon))}}{r_\varepsilon} = \left\|\widetilde{\Phi}\right\|_{L^1(B(0,1))} = \mathrm{O}\left(\left\|\nabla\widetilde{\Phi}\right\|_{L^2(B(0,1))} + \left|\frac{1}{|B(0,1)|}\int_{B(0,1)}\widetilde{\Phi}dy\right|\right), \tag{4.115}$$

since the right-hand side contains a norm equivalent to the H^1-norm. Now, by the Jensen and Moser–Trudinger inequalities, we have

$$\exp\left(\left(\frac{1}{|B(0,1)|}\int_{B(0,1)}\widetilde{\Phi}dy\right)^2\right) \le \frac{1}{|B(0,1)|}\int_{B(0,1)}\exp\left(\widetilde{\Phi}^2\right)dy$$
$$= \frac{1}{\pi r_\varepsilon}\int_{B(\overline{\tau_i},r_\varepsilon)}\exp\left(\Phi_{\varepsilon,\gamma,\tau,\theta}^2\right)dx \le \frac{1}{\pi r_\varepsilon}$$

It follows that

$$\left|\frac{1}{|B(0,1)|}\int_{B(0,1)}\widetilde{\Phi}dy\right| \le \sqrt{\ln\frac{1}{\pi r_\varepsilon}}$$

and (4.113) follows at once from (4.115). This completes the proof of (4.114).

We finally estimate

$$(D) = \mathrm{O}\left(\int_\Omega |U_{\varepsilon,\gamma,\tau}|^3 \exp\left(U_{\varepsilon,\gamma,\tau}^2\right)\left(\Phi_{\varepsilon,\gamma,\tau,\theta}^2 + \sum_{j=1}^k \theta_j^2 B_{\varepsilon,\gamma_j,\tau_j}^2\right)|Z_{1,i}|\,dx\right)$$
$$= \mathrm{O}\left(\int_\Omega |U_{\varepsilon,\gamma,\tau}|^3 \exp\left(U_{\varepsilon,\gamma,\tau}^2\right)\Phi_{\varepsilon,\gamma,\tau,\theta}^2\,|Z_{1,i}|\,dx\right)$$
$$+ \sum_{j=1}^k \mathrm{O}\left(\theta_j^2\overline{\gamma}_\varepsilon^2\int_\Omega |U_{\varepsilon,\gamma,\tau}|^3\exp\left(U_{\varepsilon,\gamma,\tau}^2\right)|Z_{1,i}|\,dx\right) =: (D_1) + \sum_{j=1}^k (D_2)_j.$$

For every $j \in \{1,\dots,k\}$, with the rough estimate

$$Z_{1,i} = \mathrm{O}\left(\mu_{\gamma_i}^{-1}\right) = \mathrm{O}\left(\overline{\mu_\varepsilon}^{-1+\mathrm{o}(1)}\right), \quad \text{in } B\left(\overline{\tau_j}, r_\varepsilon\right),$$

we obtain as in (4.88)

$$(D_1)_j := \int_{B(\overline{\tau_j},r_\varepsilon)} |U_{\varepsilon,\gamma,\tau}|^3 \exp\left(U_{\varepsilon,\gamma,\tau}^2\right)\Phi_{\varepsilon,\gamma,\tau,\theta}^2\,|Z_{1,i}|\,dx$$

$$= \mathrm{O}\left(\frac{\overline{\gamma}_\varepsilon^3}{\overline{\mu}_\varepsilon^{1+\mathrm{o}(1)}}\left(\int_{B(\overline{\tau_j},r_\varepsilon)}\exp\left(\overline{B}_{\varepsilon,\gamma_j,\tau_j}^2\right)\Phi_{\varepsilon,\gamma,\tau,\theta}^2 dx\right)\right)$$
$$= \mathrm{O}\left(\frac{\left\|\nabla\Phi_{\varepsilon,\gamma,\tau,\theta}\right\|_{L^2}^2}{\overline{\mu}_\varepsilon^{1+\mathrm{o}(1)}}\right) = \mathrm{O}\left(\frac{\delta_\varepsilon^2}{\overline{\mu}_\varepsilon^{1+\mathrm{o}(1)}}\right) = \mathrm{O}\left(\overline{\mu}_\varepsilon^{2\delta_1+\mathrm{o}(1)}\right) = \mathrm{o}\left(\frac{1}{\overline{\gamma}_\varepsilon^2 d_\varepsilon}\right).$$

Similarly, with (3.17),

$$(D_1') := \int_{\Omega_{r_\varepsilon,\tau}}\left|U_{\varepsilon,\gamma,\tau}\right|^3\exp\left(U_{\varepsilon,\gamma,\tau}^2\right)\Phi_{\varepsilon,\gamma,\tau,\theta}^2\left|Z_{1,i}\right|dx = \mathrm{O}\Big(\overline{\gamma}_\varepsilon\left\|f'\left(U_{\varepsilon,\gamma,\tau}\right)Z_{1,i}\mathbf{1}_{\Omega_{r_\varepsilon,\tau}}\right\|_{L^p}$$
$$\times\left\|\Phi_{\varepsilon,\gamma,\tau,\theta}\right\|_{L^{2p'}}^2\Big) = \mathrm{O}\left(\overline{\gamma}_\varepsilon^2\left\|\nabla\Phi_{\varepsilon,\gamma,\tau,\theta}\right\|_{L^2}^2\right) = \mathrm{o}\left(\frac{1}{\overline{\gamma}_\varepsilon^2 d_\varepsilon}\right).$$

As for the terms involving θ_j, we can use the rough estimate $Z_{1,i} = \mathrm{O}\left(\overline{\mu}_\varepsilon^{-1+\mathrm{o}(1)}\right)$ to get

$$(D_2)_j := \theta_j^2\overline{\gamma}_\varepsilon^2\int_{B(\overline{\tau_m},r_\varepsilon)}\left|U_{\varepsilon,\gamma,\tau}\right|^3\exp\left(U_{\varepsilon,\gamma,\tau}^2\right)\left|Z_{1,i}\right|dx$$
$$= \mathrm{O}\left(\frac{\theta_j^2\overline{\gamma}_\varepsilon^5}{\overline{\mu}_\varepsilon^{1+\mathrm{o}(1)}}\left(\int_{B(\overline{\tau_m},r_\varepsilon)}\exp\left(\overline{B}_{\varepsilon,\gamma_m,\tau_m}^2\right)dx\right)\right) = \mathrm{O}\left(\overline{\mu}_\varepsilon^{2\delta_1+\mathrm{o}(1)}\right) = \mathrm{o}\left(\frac{1}{\overline{\gamma}_\varepsilon^2 d_\varepsilon}\right),$$

while in $\Omega_{r_\varepsilon,\tau}$, we can use (3.17) with $p=1$ to obtain

$$(D_2')_j := \theta_j^2\overline{\gamma}_\varepsilon^2\int_{\Omega_{r_\varepsilon,\tau}}\left|U_{\varepsilon,\gamma,\tau}\right|^3\exp\left(U_{\varepsilon,\gamma,\tau}^2\right)\left|Z_{1,i}\right|dx = \mathrm{O}(\theta_j^2\overline{\gamma}_\varepsilon^4) = \mathrm{o}\left(\frac{1}{\overline{\gamma}_\varepsilon^2 d_\varepsilon}\right).$$

Summing up, we obtain

$$(D) = \sum_{j=1}^k (D_1)_j + (D_1') + \sum_{j=1}^k\left[(D_2)_j + (D_2')_j\right] = \mathrm{o}\left(\frac{1}{\overline{\gamma}_\varepsilon^2 d_\varepsilon}\right). \tag{4.116}$$

Now, (4.102), (4.103), (4.107), (4.114) and (4.116) allow us to conclude. □

Proof of Proposition 4.4. We claim that for δ and ε small enough, we can find $(\gamma_\varepsilon,\theta_\varepsilon,\tau_\varepsilon)\in P_\varepsilon^k(\delta)$ such that

$$\left\langle\widetilde{R}_{\varepsilon,\gamma_\varepsilon,\tau_\varepsilon,\theta_\varepsilon}, Z_{0,i,\varepsilon,\gamma_\varepsilon,\tau_\varepsilon}\right\rangle_{H_0^1} = \left\langle\widetilde{R}_{\varepsilon,\gamma_\varepsilon,\tau_\varepsilon,\theta_\varepsilon}, B_{\varepsilon,\gamma_\varepsilon,\tau_\varepsilon}\right\rangle_{H_0^1} = \left\langle\widetilde{R}_{\varepsilon,\gamma_\varepsilon,\tau_\varepsilon,\theta_\varepsilon}, Z_{1,i,\varepsilon,\gamma_\varepsilon,\tau_\varepsilon}\right\rangle_{H_0^1} = 0, \tag{4.117}$$

for $i=1,\dots,k$, so that

$$\Pi_{\varepsilon,\gamma_\varepsilon,\tau_\varepsilon}\left(U_{\varepsilon,\gamma_\varepsilon,\tau_\varepsilon,\theta_\varepsilon} + \Phi_{\varepsilon,\gamma_\varepsilon,\tau_\varepsilon,\theta_\varepsilon} - \Delta^{-1}\left(\lambda_\varepsilon h_\varepsilon f\left(U_{\varepsilon,\gamma_\varepsilon,\tau_\varepsilon,\theta_\varepsilon} + \Phi_{\varepsilon,\gamma_\varepsilon,\tau_\varepsilon,\theta_\varepsilon}\right)\right)\right) = 0,$$

hence, together with Proposition 4.2, $U_{\varepsilon,\gamma_\varepsilon,\tau_\varepsilon,\theta_\varepsilon} + \Phi_{\varepsilon,\gamma_\varepsilon,\tau_\varepsilon,\theta_\varepsilon}$ is a solution to (4.62). For every $\tau\in T_\varepsilon^k(\delta)$ and $\gamma\in\widehat{\Gamma}_\varepsilon^k(\tau)$, let us set $\hat{\gamma} := \gamma - \overline{\gamma}_\varepsilon(\tau)$ and

$$\widehat{\Gamma}_\varepsilon^k := \left\{\hat{\gamma} = (\hat{\gamma}_1,\dots,\hat{\gamma}_k)\in(0,\infty)^k : \left|\hat{\gamma}_i\right| < \frac{\delta_\varepsilon}{\overline{\gamma}_\varepsilon}, \forall i\in\{i,\dots,k\}\right\},$$

and define $(L_\varepsilon, M_\varepsilon, N_\varepsilon) : \widehat{P}_\varepsilon^k(\delta) := \widehat{\Gamma}_\varepsilon^k\times\Theta_\varepsilon^k\times T_\varepsilon^k(\delta)\to\mathbb{R}^{3k}$ as

$$L_\varepsilon^i(\hat{\gamma},\tau,\theta) := -\frac{1}{8\pi}\left\langle\widetilde{R}_{\varepsilon,\gamma,\tau,\theta}, Z_{0,i,\varepsilon,\gamma,\tau}\right\rangle_{H_0^1}$$
$$= \sum_{j=1}^k \partial_{\gamma_i}\left[E_{\varepsilon,\gamma,\tau}^{(j)}\right]\left(E_{\varepsilon,\gamma,\tau}^{(j)} + \theta_j\overline{\gamma}_\varepsilon\right) - \frac{E_{\varepsilon,\gamma,\tau}^{(i)}}{2\overline{\gamma}_\varepsilon^2} + \mathrm{o}\left(\frac{\delta_\varepsilon\ln\overline{\gamma}_\varepsilon}{\overline{\gamma}_\varepsilon^5}\right),$$

$$M_{\varepsilon}^{i}\left(\hat{\gamma},\tau,\theta\right):=-\frac{1}{8\pi\overline{\gamma}_{\varepsilon}}\left\langle \widetilde{R}_{\varepsilon,\gamma,\tau,\theta},B_{\varepsilon,\gamma_{i},\tau_{i}}\right\rangle_{H_{0}^{1}}=E_{\varepsilon,\gamma,\tau}^{(i)}+\theta_{i}\overline{\gamma}_{\varepsilon}+\mathrm{o}\left(\frac{\delta_{\varepsilon}}{\overline{\gamma}_{\varepsilon}^{3}}\right),$$
$$N_{\varepsilon}^{i}\left(\hat{\gamma},\tau,\theta\right):=-\frac{\overline{\gamma}_{\varepsilon}^{2}d_{\varepsilon}}{4\pi}\left\langle \widetilde{R}_{\varepsilon,\gamma,\tau,\theta},Z_{1,i,\varepsilon,\gamma,\tau}\right\rangle_{H_{0}^{1}}=a_{0}l\left(\frac{\tau_{i}}{d_{\varepsilon}}\right)^{l-1}-\sum_{j\neq i}\frac{2d_{\varepsilon}}{\tau_{i}-\tau_{j}}+\mathrm{o}\left(1\right)$$

for $i=1,\dots k$, where the error terms in the right-hand sides are uniform for $(\hat{\gamma},\tau,\theta)\in\widehat{P}_{\varepsilon}^{k}\left(\delta\right)$. (Note that we wrote $\hat{\gamma}$ in the left-hand side and γ instead of $\hat{\gamma}+\overline{\gamma}_{\varepsilon}\left(\tau\right)$ in the right-hand side for simplicity, so that for instance the terms $E_{\varepsilon,\gamma,\tau}^{(j)}$ should be read as $E_{\varepsilon,\hat{\gamma}+\overline{\gamma}_{\varepsilon}(\tau),\tau}^{(j)}$)

We claim that

$$\deg\left(\left(L_{\varepsilon},M_{\varepsilon},N_{\varepsilon}\right),\widehat{P}_{\varepsilon}^{k}\left(\delta\right),0\right)\neq0 \tag{4.118}$$

for δ and ε small (to be fixed), where deg denotes the Brouwer degree. Let us consider the homotopy $(L_{\varepsilon}^{t},M_{\varepsilon}^{t},N_{\varepsilon}^{t}):\widehat{P}_{\varepsilon}^{k}\left(\delta\right)\to\mathbb{R}^{3k}$ with $L_{\varepsilon}^{t}=\left(L_{\varepsilon}^{t,1},\dots,L_{\varepsilon}^{t,k}\right)$, etc. defined by

$$L_{\varepsilon}^{t,i}=\left(1-t\right)\overline{L}_{\varepsilon}^{i}+tL_{\varepsilon}^{i},\quad \overline{L}_{\varepsilon}^{i}:=\sum_{j=1}^{k}\partial_{\gamma_{i}}\left[E_{\varepsilon,\gamma,\tau}^{(j)}\right]\left(E_{\varepsilon,\gamma,\tau}^{(j)}+\theta_{j}\overline{\gamma}_{\varepsilon}\right)-\frac{E_{\varepsilon,\gamma,\tau}^{(i)}}{2\overline{\gamma}_{\varepsilon}^{2}},$$
$$M_{\varepsilon}^{t,i}=\left(1-t\right)\overline{M}_{\varepsilon}^{i}+tM_{\varepsilon}^{i},\quad \overline{M}_{\varepsilon}^{i}:=E_{\varepsilon,\gamma,\tau}^{(i)}+\theta_{i}\overline{\gamma}_{\varepsilon},$$
$$N_{\varepsilon}^{t,i}=\left(1-t\right)\overline{N}_{\varepsilon}^{i}+tN_{\varepsilon}^{i},\quad \overline{N}_{\varepsilon}^{i}:=a_{0}l\left(\frac{\tau_{i}}{d_{\varepsilon}}\right)^{l-1}-\sum_{j\neq i}\frac{2d_{\varepsilon}}{\tau_{i}-\tau_{j}}.$$

for $i=1,\dots k$ and $t\in[0,1]$. We first show that $(L_{\varepsilon}^{t},M_{\varepsilon}^{t},N_{\varepsilon}^{t})\neq0$ on $\partial\widehat{P}_{\varepsilon}^{k}\left(\delta\right)$ for any $t\in[0,1]$ if $\varepsilon>0$ is sufficiently small. Otherwise there would be a sequence $\varepsilon_{n}\downarrow0$ (which we still denote by ε), $t_{\varepsilon}\in[0,1]$ and

$$(\hat{\gamma}_{\varepsilon},\theta_{\varepsilon},\tau_{\varepsilon})\in\partial\widehat{P}_{\varepsilon}^{k},\quad\text{i.e. }\hat{\gamma}_{\varepsilon}\in\partial\widehat{\Gamma}_{\varepsilon}^{k},\text{ or }\theta_{\varepsilon}\in\partial\Theta_{\varepsilon}^{k},\text{ or }\tau_{\varepsilon}\in\partial T_{\varepsilon}^{k}\left(\delta\right), \tag{4.119}$$

such that

$$\left(L_{\varepsilon}^{t_{\varepsilon}}\left(\hat{\gamma}_{\varepsilon},\theta_{\varepsilon},\tau_{\varepsilon}\right),M_{\varepsilon}^{t_{\varepsilon}}\left(\hat{\gamma}_{\varepsilon},\theta_{\varepsilon},\tau_{\varepsilon}\right),N^{t_{\varepsilon}}\left(\hat{\gamma}_{\varepsilon},\theta_{\varepsilon},\tau_{\varepsilon}\right)\right)=0.$$

Then, multiplying $M_{\varepsilon}^{t_{\varepsilon},j}$ by $\partial_{\gamma_{i}}E_{\varepsilon,\gamma_{\varepsilon},\tau_{\varepsilon}}^{(j)}$, subtracting it from $L_{\varepsilon}^{t_{\varepsilon},i}$ for $j=1,\dots,k$ and using Proposition 3.4, we obtain (upon multiplication by $2\overline{\gamma}_{\varepsilon}^{2}$)

$$E_{\varepsilon,\gamma_{\varepsilon},\tau_{\varepsilon}}^{(i)}=\mathrm{o}\left(\frac{\delta_{\varepsilon}\ln\overline{\gamma}_{\varepsilon}}{\overline{\gamma}_{\varepsilon}^{3}}\right),\quad\text{for }i=1,\dots,k. \tag{4.120}$$

Plugging (4.120) into the equation for $M_{\varepsilon}^{t_{\varepsilon},i}$, we then obtain

$$\theta_{\varepsilon}=\mathrm{o}\left(\frac{\delta_{\varepsilon}\ln\overline{\gamma}_{\varepsilon}}{\overline{\gamma}_{\varepsilon}^{4}}\right), \tag{4.121}$$

hence $\theta_{\varepsilon}\notin\partial\Theta_{\varepsilon}^{k}$. Now (4.121), the equation for $L_{\varepsilon}^{t_{\varepsilon},i}$ and Proposition 3.4 yield

$$\sum_{j=1}^{k}\partial_{\gamma_{i}}\left[E_{\varepsilon,\gamma_{\varepsilon},\tau_{\varepsilon}}^{(j)}\right]E_{\varepsilon,\gamma_{\varepsilon},\tau_{\varepsilon}}^{(j)}=\mathrm{o}\left(\frac{\delta_{\varepsilon}\ln^{2}\overline{\gamma}_{\varepsilon}}{\overline{\gamma}_{\varepsilon}^{5}}\right),\quad\text{for }i=1,\dots,k. \tag{4.122}$$

We can rewrite (4.122) as

$$Q_{\varepsilon}E_{\varepsilon,\gamma_{\varepsilon},\tau_{\varepsilon}}=\mathrm{o}\left(\frac{\delta_{\varepsilon}\ln\overline{\gamma}_{\varepsilon}}{\overline{\gamma}_{\varepsilon}^{3}}\right), \tag{4.123}$$

where, taking Proposition 3.4 into account, $Q_\varepsilon = (Q_{\varepsilon,ij})_{1\le i,j\le k}$ is a $k\times k$ matrix with $Q_{\varepsilon,ij} = \mathcal{Q}_{ij} + \mathrm{o}(1)$ as $\varepsilon\to 0$ and

$$\mathcal{Q} = (\mathcal{Q}_{ij})_{1\le i,j\le k} := \begin{pmatrix} 1 & 1/l & \dots & 1/l \\ 1/l & 1 & \dots & 1/l \\ \vdots & & \ddots & \vdots \\ 1/l & 1/l & \dots & 1 \end{pmatrix}, \quad E_{\varepsilon,\gamma_\varepsilon,\tau_\varepsilon} = \begin{pmatrix} E^{(1)}_{\varepsilon,\gamma_\varepsilon,\tau_\varepsilon} \\ \vdots \\ E^{(k)}_{\varepsilon,\gamma_\varepsilon,\tau_\varepsilon} \end{pmatrix} \tag{4.124}$$

Now, since

$$\det Q_\varepsilon = \det\mathcal{Q} + \mathrm{o}(1) = \left(1+\frac{k-1}{l}\right)\left(1-\frac{1}{l}\right)^{k-1} + \mathrm{o}(1) > 0$$

for $\varepsilon>0$ sufficiently small, uniformly with respect to $(\gamma,\theta,\tau)\in P^k_\varepsilon(\delta)$, we can invert Q_ε in (4.123) and get

$$E^{(i)}_{\varepsilon,\gamma_\varepsilon,\tau_\varepsilon} = \mathrm{o}\left(\frac{\delta_\varepsilon\ln\overline{\gamma}_\varepsilon}{\overline{\gamma}_\varepsilon^3}\right), \quad \text{for } i=1,\dots,k. \tag{4.125}$$

On the other hand, still by Proposition 3.4, we have

$$E_{\varepsilon,\gamma_\varepsilon,\tau_\varepsilon} = -2\frac{\ln\overline{\gamma}_\varepsilon}{\overline{\gamma}_\varepsilon^2}Q_\varepsilon\hat{\gamma}_\varepsilon + \mathrm{o}\left(|\hat{\gamma}_\varepsilon|\,\frac{\ln\overline{\gamma}_\varepsilon}{\overline{\gamma}_\varepsilon^2}\right),$$

where we recall that $\hat{\gamma} = \gamma - \overline{\gamma}_\varepsilon(\tau)$. Then, inverting Q_ε and using (4.125) we end up with $\hat{\gamma}_\varepsilon = \mathrm{o}(\delta_\varepsilon/\overline{\gamma}_\varepsilon)$, which, for $\varepsilon>0$ sufficiently small implies that $\hat{\gamma}_\varepsilon\notin\partial\overline{\Gamma}^k_\varepsilon$.

Finally, writing $\hat{\tau}_\varepsilon := \tau_\varepsilon/d_\varepsilon$, we have $N^i(\hat{\tau}_\varepsilon) = \mathrm{o}(1)$, where $N = (N^1,\dots,N^k)$ is as in (4.130). On the other hand, $\tau_\varepsilon\in\partial T^k_\varepsilon(\delta)$ implies $\hat{\tau}_\varepsilon\in\partial\widehat{T}^k(\delta)$, where

$$\widehat{T}^k(\delta) := \Big\{y=(y_1,\dots,y_k)\in\mathbb{R}^k:\ -\frac{k}{\delta}<y_1<y_2<\dots<y_k<\frac{k}{\delta} \\ \text{and } |y_i-y_j|>\delta,\ \forall i,j\in\{1,\dots,k\},\ i\ne j\Big\}, \tag{4.126}$$

which is compact and this contradicts Lemma 4.7 for $\delta=\delta(a_0,l,k)>0$ sufficiently small such that $y^*\in\widehat{T}^k(\delta)$. Then we also have $\tau_\varepsilon\notin\partial T^k_\varepsilon(\delta)$, which contradicts (4.119).

We have therefore proven that $(L^t_\varepsilon, M^t_\varepsilon, N^t_\varepsilon)\ne 0$ on $\partial\widehat{P}^k_\varepsilon(\delta)$, for $\varepsilon>0$ sufficiently small, hence by homotopy invariance of the degree

$$\deg\big((L_\varepsilon,M_\varepsilon,N_\varepsilon),\widehat{P}^k_\varepsilon(\delta),0\big) = \deg\big(\big(\overline{L}_\varepsilon,\overline{M}_\varepsilon,\overline{N}_\varepsilon\big),\widehat{P}^k_\varepsilon(\delta),0\big). \tag{4.127}$$

The degree of $\big(\overline{L}_\varepsilon,\overline{M}_\varepsilon,\overline{N}_\varepsilon\big)$ does not change upon multiplication by an invertible matrix with determinant 1, namely if we consider

$$\begin{pmatrix}\widetilde{L}_\varepsilon\\ \overline{M}_\varepsilon\\ \overline{N}_\varepsilon\end{pmatrix} := \begin{pmatrix} I_k & -D_\gamma[E_{\varepsilon,\gamma,\tau}] & 0\\ 0 & I_k & 0\\ 0 & 0 & I_k\end{pmatrix}\begin{pmatrix}\overline{L}_\varepsilon\\ \overline{M}_\varepsilon\\ \overline{N}_\varepsilon\end{pmatrix},$$

where $D_\gamma[E_{\varepsilon,\gamma,\tau}] = \big(\partial_{\gamma_j}\big[E^{(i)}_{\varepsilon,\gamma,\tau}\big]\big)_{1\le i,j\le k}$, I_k is the $k\times k$ identity matrix and $\widetilde{L}_\varepsilon:\widehat{P}^k_\varepsilon(\delta)\to\mathbb{R}^k$, is defined by $\widetilde{L}^i_\varepsilon = -E^{(i)}_{\varepsilon,\gamma,\tau}$, for $i=1,\dots k$, we get

$$\deg\big(\big(\overline{L}_\varepsilon,\overline{M}_\varepsilon,\overline{N}_\varepsilon\big),\widehat{P}^k_\varepsilon(\delta),0\big) = \deg\big(\big(\widetilde{L}_\varepsilon,\overline{M}_\varepsilon,\overline{N}_\varepsilon\big),\widehat{P}^k_\varepsilon(\delta),0\big).$$

Expanding $E^{(i)}_{\varepsilon,\gamma,\tau}$ as in Proposition 3.4, we do a final homotopy between $\left(\widetilde{L}_\varepsilon, \overline{M}_\varepsilon\right)$ and $\left(L^*_\varepsilon, M^*_\varepsilon\right) : \widehat{P}^k_\varepsilon(\delta) \to \mathbb{R}^{2k}$, where

$$L^{*i}_\varepsilon = -\frac{2\ln\overline{\gamma}_\varepsilon}{\overline{\gamma}_\varepsilon^2}\left(\hat{\gamma}_i + \sum_{j\neq i}\frac{\hat{\gamma}_j}{l}\right), \quad M^{*i}_\varepsilon = L^{*i}_\varepsilon + \theta_i\overline{\gamma}_\varepsilon$$

for $i = 1, \dots k$ (with the same method as above to prevent zeroes on $\partial\widehat{P}^k_\varepsilon(\delta)$), so that

$$\deg\left(\left(\widetilde{L}_\varepsilon, \overline{M}_\varepsilon, \overline{N}_\varepsilon\right), \widehat{P}^k_\varepsilon(\delta), 0\right) = \deg\left(\left(L^*_\varepsilon, M^*_\varepsilon, \overline{N}_\varepsilon\right), \widehat{P}^k_\varepsilon(\delta), 0\right)$$

Using the matrix $\mathcal{Q}$ defined in (4.124), we see that

$$\begin{pmatrix} L^*_\varepsilon \\ M^*_\varepsilon \\ \overline{N}_\varepsilon \end{pmatrix} = \begin{pmatrix} -2\frac{\ln\overline{\gamma}_\varepsilon}{\overline{\gamma}_\varepsilon^2}\mathcal{Q} & 0 & 0 \\ -2\frac{\ln\overline{\gamma}_\varepsilon}{\overline{\gamma}_\varepsilon^2}\mathcal{Q} & \overline{\gamma}_\varepsilon I_k & 0 \\ 0 & 0 & I_k \end{pmatrix}\begin{pmatrix} \hat{\gamma} \\ \theta \\ \overline{N}_\varepsilon \end{pmatrix}. \tag{4.128}$$

Since $\mathcal{Q}$ has positive determinant, if we call $\mathcal{A}$ the $3k \times 3k$ matrix on the right-hand side of (4.128) we have $\operatorname{sign}(\det\mathcal{A}) = (-1)^k$, and noticing that $\overline{N}_\varepsilon$ only depends on τ, we obtain

$$\mathcal{A}^{-1}\begin{pmatrix} L^*_\varepsilon \\ M^*_\varepsilon \\ \overline{N}_\varepsilon \end{pmatrix} = \mathrm{Id} \times \mathrm{Id} \times \overline{N}_\varepsilon : \widehat{\Gamma}^k_\varepsilon \times \Theta^k_\varepsilon \times T^k_\varepsilon(\delta) \to \mathbb{R}^k \times \mathbb{R}^k \times \mathbb{R}^k,$$

and using the product formula for the degree, we finally obtain

$$\begin{aligned} \deg\left(\left(L_\varepsilon, M_\varepsilon, N_\varepsilon\right), \widehat{P}^k_\varepsilon(\delta), 0\right) &= \deg\left(\left(L^*_{\varepsilon,}, M^*_\varepsilon, \overline{N}_\varepsilon\right), \widehat{P}^k_\varepsilon(\delta), 0\right) \\ &= (-1)^k \deg\left(\mathrm{Id}, \widehat{\Gamma}^k_\varepsilon, 0\right)\deg\left(\mathrm{Id}, \Theta^k_\varepsilon, 0\right)\deg\left(\overline{N}_\varepsilon, T^k_\varepsilon(\delta), 0\right) \\ &= (-1)^k \deg\left(\overline{N}_\varepsilon, T^k_\varepsilon(\delta), 0\right). \end{aligned} \tag{4.129}$$

In order to compute the degree of $\overline{N}_\varepsilon$, observe that $\overline{N}^i_\varepsilon(\tau) = N^i(\hat{\tau}_\varepsilon)$, where $\hat{\tau}_\varepsilon = \tau/d_\varepsilon$ and $N = \left(N^1, \dots, N^k\right)$ is as in (4.130). Moreover, since $\delta \in (0,1)$ was chosen such that $y^* \in \widehat{T}^k(\delta)$, with y^* as in Lemma 4.7, it follows that

$$\deg\left(\overline{N}_\varepsilon, T^k_\varepsilon(\delta), 0\right) = \deg\left(N, \widehat{T}^k(\delta), 0\right) = 1.$$

We then conclude with (4.129) that there exists $(\hat{\gamma}_\varepsilon, \theta_\varepsilon, \tau_\varepsilon) \in \widehat{P}^k_\varepsilon(\delta)$ such that $(\gamma_\varepsilon, \theta_\varepsilon, \tau_\varepsilon) = (\hat{\gamma}_\varepsilon + \overline{\gamma}_\varepsilon(\tau), \theta_\varepsilon, \tau_\varepsilon) \in P^k_\varepsilon(\delta)$ solves (4.117). □

Lemma 4.7. *The function*

$$N : \widehat{T}^k(0) := \left\{y = (y_1, \dots, y_k) \in \mathbb{R}^k : y_1 < y_2 < \dots < y_k\right\} \to \mathbb{R}^k$$

given by

$$N^i(y_1, \dots, y_k) := a_0 l y_i^{l-1} - 2\sum_{j\neq i}\frac{1}{y_i - y_j}, \quad \textit{for } i = 1, \dots, k, \tag{4.130}$$

has exactly one zero, which we call y^*. Moreover $\deg(H, \widehat{T}^k(0), 0) = 1$.

Proof. We have that $N = \nabla J$, with

$$J(y) = a_0\sum_{i=1}^k y_i^l + \frac{1}{2}\sum_{i\neq j}\ln\frac{1}{(y_i - y_j)^2}, \quad \forall y = (y_1, \dots, y_k) \in \widehat{T}^k(0).$$

The Hessian $\nabla^2 J$ is positive definite on $\widehat{T}^k(0)$, since

$$\partial^2_{y_i} J = \partial_{y_i} N^i = a_0 l(l-1) y_i^{l-2} + \sum_{j\neq i} \frac{2}{\left(y_i - y_j\right)^2},$$
$$\partial_{y_i}\partial_{y_j} J = \partial_{y_i} N^j = -\frac{2}{\left(y_i - y_j\right)^2}, \quad \text{for } i \neq j,$$

so that for every $\xi \in \mathbb{R}^k \setminus \{0\}$, using that $\xi_i^2 + \xi_j^2 \geq 2\xi_i \xi_j$, we get

$$\begin{aligned}\xi^T \nabla^2 J \xi &= \sum_{i=1}^k \xi_i^2 \left(a_0 l\left(l-1\right) y_i^{l-2} + \sum_{j\neq i} \frac{2}{\left(y_i - y_j\right)^2}\right) - \sum_{i=1}^k \sum_{j \neq i} \frac{2\xi_j \xi_i}{\left(y_i - y_j\right)^2} \\ &\geq \sum_{i=1}^k \xi_i^2 a_0 l\left(l-1\right) y_i^{l-2},\end{aligned}$$

and, using that $y \in \widehat{T}^k(0)$ and $l \in 2\mathbb{N}^*$, the right-hand side is positive, unless $\xi = (0,\dots,\xi_{i_0},\dots 0)$ and $y_{i_0} = 0$ for some $i_0 \in \{1,\dots,k\}$, in which case

$$\xi^T \nabla^2 J \xi = \sum_{j \neq i_0} \frac{2\xi_{i_0}^2}{(y_{i_0} - y_j)^2} > 0.$$

Then J is strictly convex in $\widehat{T}^k(0)$ and since $|J(y)| \to \infty$ as $y \to \partial \widehat{T}^k(0)$ or $|y| \to \infty$, J has a minimum y^*, which is its only critical point and the only zero of N. Moreover $\det(\nabla N(y^*)) = \det(\nabla^2 J(y^*)) > 0$, hence $\deg(N, \widehat{T}^k(0), 0) = 1$. □

Finally, we can now conclude the proof of Theorems 1.2 and 1.3.

End of proof of Theorems 1.2 and 1.3. It follows from Proposition 4.4 that for small $\varepsilon > 0$, $u_\varepsilon := U_{\varepsilon,\gamma_\varepsilon,\tau_\varepsilon,\theta_\varepsilon} + \Phi_{\varepsilon,\gamma_\varepsilon,\tau_\varepsilon,\theta_\varepsilon} \in E_{h_\varepsilon,\beta_\varepsilon}$, where $\beta_\varepsilon := \left\|\nabla u_\varepsilon\right\|^2_{L^2}$. We denote $\gamma_\varepsilon = (\gamma_{1,\varepsilon},\dots,\gamma_{k,\varepsilon})$, $\tau_\varepsilon = (\tau_{1,\varepsilon},\dots,\tau_{k,\varepsilon})$, $\theta_\varepsilon = (\theta_{1,\varepsilon},\dots,\theta_{k,\varepsilon})$. By using (3.14) and (4.16), we obtain that $\Psi_{\varepsilon,\gamma_\varepsilon,\tau_\varepsilon}, \Phi_{\varepsilon,\gamma_\varepsilon,\tau_\varepsilon,\theta_\varepsilon} \to 0$ in $H^1_0(\Omega)$ as $\varepsilon \to 0$. Since moreover $w_\varepsilon \to w_0$ in $C^1\left(\overline{\Omega}\right)$, $H \in C^1\left(\overline{\Omega} \times \overline{\Omega}\right)$, $\theta_{i,\varepsilon} \to 0$ and $A_{\varepsilon,\gamma_{i,\varepsilon},\tau_{i,\varepsilon}} \to 0$ for all $i \in \{1,\dots,k\}$, we obtain

$$\begin{aligned}\left\|\nabla u_\varepsilon\right\|_{L^2} = \Big\| \nabla w_0 + \sum_{i=1}^k \left(1+\theta_{i,\varepsilon}\right) \big(\nabla \overline{B}_{\varepsilon,\gamma_{i,\varepsilon},\overline{\tau_{i,\varepsilon}}} \mathbf{1}_{B(\overline{\tau_{i,\varepsilon}},r_\varepsilon)} \\ + A_{\varepsilon,\gamma_{i,\varepsilon},\tau_{i,\varepsilon}} \nabla G\left(\cdot,\overline{\tau_{i,\varepsilon}}\right) \mathbf{1}_{\Omega\setminus B(\overline{\tau_{i,\varepsilon}},r_\varepsilon)}\big)\Big\|_{L^2} + \mathrm{o}\left(1\right) \quad (4.131)\end{aligned}$$

as $\varepsilon \to 0$. By integrating by parts, we obtain

$$\left\|\nabla G\left(\cdot,\overline{\tau_{i,\varepsilon}}\right) \mathbf{1}_{\Omega\setminus B(\overline{\tau_{i,\varepsilon}},r_\varepsilon)}\right\|^2_{L^2} = -\int_{\partial B(\overline{\tau_{i,\varepsilon}},r_\varepsilon)} G\left(\cdot,\overline{\tau_{i,\varepsilon}}\right) \partial_\nu G\left(\cdot,\overline{\tau_{i,\varepsilon}}\right) d\sigma \sim \frac{1}{2\pi} \ln \frac{1}{r_\varepsilon} \sim \frac{\delta_0 \overline{\gamma}_\varepsilon^2}{4\pi}, \quad (4.132)$$

$$\begin{aligned}\left\langle \nabla G\left(\cdot,\overline{\tau_{i,\varepsilon}}\right) \mathbf{1}_{\Omega\setminus B(\overline{\tau_{i,\varepsilon}},r_\varepsilon)}, \nabla w_0 \right\rangle_{L^2} &= \int_{\partial B(\overline{\tau_{i,\varepsilon}},r_\varepsilon)} w_0 \partial_\nu G\left(\cdot,\overline{\tau_{i,\varepsilon}}\right) d\sigma = \mathrm{O}\left(\left\|w_0\right\|_{C^0(\partial B(\overline{\tau_{i,\varepsilon}},r_\varepsilon))}\right) \\ &= \mathrm{o}\left(1\right) \quad (4.133)\end{aligned}$$

and

$$\left\langle \nabla G\left(\cdot,\overline{\tau_{i,\varepsilon}}\right)\mathbf{1}_{\Omega\setminus B\left(\overline{\tau_{i,\varepsilon}},r_\varepsilon\right)},\nabla G\left(\cdot,\overline{\tau_{j,\varepsilon}}\right)\mathbf{1}_{\Omega\setminus B\left(\overline{\tau_{j,\varepsilon}},r_\varepsilon\right)}\right\rangle_{L^2}$$
$$=\int_{\partial B\left(\overline{\tau_{i,\varepsilon}},r_\varepsilon\right)\cup\partial B\left(\overline{\tau_{j,\varepsilon}},r_\varepsilon\right)} G\left(\cdot,\overline{\tau_{i,\varepsilon}}\right)\partial_\nu G\left(\cdot,\overline{\tau_{j,\varepsilon}}\right)d\sigma=\mathrm{O}\left(\ln\frac{1}{d_\varepsilon}\right)=\mathrm{O}\left(\ln\overline{\gamma}_\varepsilon\right) \quad (4.134)$$

as $\varepsilon\to 0$ for all $i,j\in\{1,\dots,k\}$, $i\neq j$, where ν and $d\sigma$ are the outward unit normal vector and volume element of $\partial B\left(\overline{\tau_{i,\varepsilon}},r_\varepsilon\right)\cup\partial B\left(\overline{\tau_{j,\varepsilon}},r_\varepsilon\right)$, respectively. On the other hand, since $\gamma_{i,\varepsilon}\sim\overline{\gamma}_\varepsilon$, by using (6.2), we obtain

$$\left\|\nabla\overline{B}_{\varepsilon,\gamma_{i,\varepsilon},\overline{\tau_{i,\varepsilon}}}\mathbf{1}_{B\left(\overline{\tau_{i,\varepsilon}},r_\varepsilon\right)}\right\|_{L^2}^2=2\pi\int_0^{\sqrt{\lambda_\varepsilon h_\varepsilon\left(\overline{\tau_{i,\varepsilon}}\right)}r_\varepsilon}\left(\overline{B}'_{\gamma_{i,\varepsilon}}(r)\right)^2 r\,dr$$
$$\sim\frac{8\pi}{\gamma_{i,\varepsilon}^2}\ln\left(\frac{\sqrt{\lambda_\varepsilon h_\varepsilon\left(\overline{\tau_{i,\varepsilon}}\right)}r_\varepsilon}{\mu_{i,\varepsilon}}\right)\sim 4\pi\left(1-\delta_0\right) \quad (4.135)$$

for all $i\in\{1,\dots,k\}$, where $\mu_{i,\varepsilon}$ is defined by $\mu_{i,\varepsilon}^2:=4\gamma_{i,\varepsilon}^{-2}\exp\left(-\gamma_{i,\varepsilon}^2\right)$. For every $j\neq i$, by remarking that $B\left(\overline{\tau_{i,\varepsilon}},r_\varepsilon\right)\cap B\left(\overline{\tau_{j,\varepsilon}},r_\varepsilon\right)=\emptyset$ for small $\varepsilon>0$ and

$$\left\|A_{\varepsilon,\gamma_{j,\varepsilon},\tau_{j,\varepsilon}}\nabla G\left(\cdot,\overline{\tau_{j,\varepsilon}}\right)\mathbf{1}_{B\left(\overline{\tau_{i,\varepsilon}},r_\varepsilon\right)}\right\|_{C^0}=\mathrm{O}\left(\frac{1}{\overline{\gamma}_\varepsilon d_\varepsilon}\right)=\mathrm{o}\left(1\right),$$

we obtain

$$\Big\langle\nabla\overline{B}_{\varepsilon,\gamma_{i,\varepsilon},\overline{\tau_{i,\varepsilon}}}\mathbf{1}_{B\left(\overline{\tau_{i,\varepsilon}},r_\varepsilon\right)},\nabla w_0+\sum_{j\neq i}\left(1+\theta_{i,\varepsilon}\right)\Big(\nabla\overline{B}_{\varepsilon,\gamma_{i,\varepsilon},\overline{\tau_{i,\varepsilon}}}\mathbf{1}_{B\left(\overline{\tau_{i,\varepsilon}},r_\varepsilon\right)}+A_{\varepsilon,\gamma_{i,\varepsilon},\tau_{i,\varepsilon}}\nabla G\left(\cdot,\overline{\tau_{i,\varepsilon}}\right)$$
$$\times\mathbf{1}_{\Omega\setminus B\left(\overline{\tau_{i,\varepsilon}},r_\varepsilon\right)}\Big)\Big\rangle_{L^2}=\mathrm{O}\left(\left\|\nabla\overline{B}_{\varepsilon,\gamma_{i,\varepsilon},\overline{\tau_{i,\varepsilon}}}\mathbf{1}_{B\left(\overline{\tau_{i,\varepsilon}},r_\varepsilon\right)}\right\|_{L^1}\right)=\mathrm{O}\left(\frac{r_\varepsilon}{\overline{\gamma}_\varepsilon}\right)=\mathrm{o}\left(1\right) \quad (4.136)$$

as $\varepsilon\to 0$. Since moreover $\theta_\varepsilon\to 0$, it follows from (3.10) and (4.131)–(4.136) that

$$\left\|\nabla u_\varepsilon\right\|_{L^2}^2\longrightarrow\left\|\nabla w_0\right\|_{L^2}^2+4k\pi=\beta_0+4k\pi$$

as $\varepsilon\to 0$. Standard elliptic theory gives that $\left\|u_\varepsilon\right\|_{L^\infty}\to\infty$ as $\varepsilon\to 0$. Since $h_\varepsilon\to h_0$ in $C^2\left(\overline{\Omega}\right)$, we then obtain that $\beta_0+4k\pi$ is an unstable energy level of I_{h_0}. This ends the proof of Theorem 1.2.

The above construction also proves Theorem 1.3 if w_0 is non-degenerate. Otherwise we apply a diagonal procedure. More precisely, thanks to Proposition 2.3, for $\kappa\in(0,\kappa_0)$ and $\varepsilon=\varepsilon(\kappa)$ sufficiently small we construct $w_{\kappa,\varepsilon(\kappa)}\in E_{\beta_{\kappa,\varepsilon(\kappa)},h_{\kappa,\varepsilon(\kappa)}}$, with $w_{\kappa,\varepsilon(\kappa)}\to w_0$, $h_{\kappa,\varepsilon(\kappa)}\to h$ in $C^2(\overline{\Omega})$ and $\beta'_\kappa:=\beta_{\kappa,\varepsilon(\kappa)}\to\beta_0$ as $\kappa\to 0$; we further construct

$$u_\kappa=w_\kappa+\sum_{i=1}^k(1+\theta_{\kappa,i})B_{\kappa,\gamma_{\kappa,i},\tau_{\kappa,i}}+\Psi_{\kappa,\gamma_\kappa,\tau_\kappa}+\Phi_{\kappa,\gamma_\kappa,\theta_\kappa,\tau_\kappa}\in E_{h_\kappa,\beta_\kappa},$$

where each subscript κ on the right-hand side actually means $(\kappa,\varepsilon(\kappa))$, with $\varepsilon(\kappa)>0$ sufficiently small so that

$$\|\nabla\Psi_{\kappa,\varepsilon(\kappa),\gamma_{\kappa,\varepsilon(\kappa)},\tau_{\kappa,\varepsilon(\kappa)}}\|_{L^2}+\|\nabla\Phi_{\kappa,\varepsilon(\kappa),\gamma_{\kappa,\varepsilon(\kappa)},\theta_{\kappa,\varepsilon(\kappa)},\tau_{\kappa,\varepsilon(\kappa)}}\|_{L^2}=\mathrm{o}(1)\quad\text{as }\kappa\to 0.$$

Up to renaming the indices, we conclude. □

Remark 4.8 (Stable vs. positively stable energy levels)**.** *As in Definition 1.1, let (u_ε) be a family of functions such that $u_\varepsilon\in E_{h_\varepsilon,\beta_\varepsilon}$ with $h_\varepsilon\to h>0$ in $C^2(\bar{\Omega})$ and $\beta_\varepsilon\to\beta>0$. In particular, u_ε solves $(\mathcal{E}_{h_\varepsilon,\beta_\varepsilon})$ with $\lambda=\lambda_\varepsilon>0$ obtained from h_ε, β_ε and u_ε thanks to* (1.1)*. As a simple claim, testing $(\mathcal{E}_{h_\varepsilon,\beta_\varepsilon})$ against $v>0$, first eigenfuntion of Δ with zero Dirichlet* *condition on $\partial\Omega$, the bound $\lambda_\varepsilon = O(1)$ is automatic when defining a* positively *stable energy level in Definition 1.1. In the sign changing case, however, let us consider the following unstable situation: u_ε goes uniformly to $0 \notin E_{h,\beta}$, while looking like a (h-weighted) Dirichlet eigenfunction associated to some large eigenvalue $\bar{\lambda}_\varepsilon \sim \lambda_\varepsilon \to +\infty$, but still having the given energy $\beta_\varepsilon \sim \beta > 0$ as $\varepsilon \to 0$. Then, in order not to have an empty notion of* stable energy level*, we further assume the bound $\lambda_\varepsilon = O(1)$ in Definition 1.1.*

5. Proof of Proposition 3.2

We fix $\varepsilon \in (0,1)$ and $\delta' \in \left(0, 1-\sqrt{2\delta_0}\right)$. For every $p > 1$, we define $\mathcal{W}_0^{2,p}(\Omega) := W^{2,p}(\Omega) \cap H_0^1(\Omega)$. Note that we have a compact embedding of $W^{2,p}(\Omega)$ into $H^1(\Omega)$ and $C^0\left(\overline{\Omega}\right)$ when $p > 1$ and into $C^1\left(\overline{\Omega}\right)$ when $p > 2$. For every $\varepsilon \in (0, \varepsilon_0)$ and $\tau \in T_\varepsilon^k(\delta)$, we let $L_{\varepsilon,\tau} : \mathcal{W}_0^{2,p}(\Omega) \to \mathcal{W}_0^{2,p}(\Omega)$ be the operator defined as

$$L_{\varepsilon,\tau}(\Psi) = \Psi - \Delta^{-1}\left[\lambda_\varepsilon h_\varepsilon \chi_{\varepsilon,\tau} f'(w_\varepsilon)\Psi\right] \qquad \forall \Psi \in \mathcal{W}_0^{2,p}(\Omega). \tag{5.1}$$

As a first step, we prove that there exists a constant $C = C(p,\delta) > 0$ such that

$$\|\Psi\|_{W^{2,p}} \le C \left\|L_{\varepsilon,\tau}(\Psi)\right\|_{W^{2,p}} \qquad \forall \Psi \in \mathcal{W}_0^{2,p}(\Omega) \tag{5.2}$$

so that in particular $L_{\varepsilon,\tau}$ is an isomorphism. We assume by contradiction that there exist sequences $(\varepsilon_n, \tau_n, \Psi_n)_n$ such that $\varepsilon_n \to 0$, $\tau_n \in T_{\varepsilon_n}^k(\delta)$, $\Psi_n \in \mathcal{W}_0^{2,p}(\Omega)$ and

$$\|\Psi_n\|_{W^{2,p}} = 1 \quad \text{and} \quad \left\|L_{\varepsilon_n,\tau_n}(\Psi_n)\right\|_{W^{2,p}} \to 0 \tag{5.3}$$

as $n \to \infty$. In particular, we obtain that $(\Psi_n)_n$ converges, up to a subsequence, weakly in $W^{2,p}(\Omega)$ and strongly in $H_0^1(\Omega)$ and $C^0\left(\overline{\Omega}\right)$ to a function Ψ_0. By using the second part of (5.3), we obtain

$$\int_\Omega \langle \nabla\Psi_n, \nabla\phi\rangle\, dx - \lambda_{\varepsilon_n} \int_\Omega h_{\varepsilon_n} \chi_{\varepsilon_n,\tau_n} f'(u_{\varepsilon_n}) \Psi_n \phi\, dx = \mathrm{o}(1) \tag{5.4}$$

for all $\phi \in C_c^\infty(\Omega)$. Since $\lambda_{\varepsilon_n} h_{\varepsilon_n} \chi_{\varepsilon_n,\tau_n} f'(u_{\varepsilon_n})$ is uniformly bounded and converges pointwise to $\lambda_0 h_0 f'(u_0)$ in Ω and $\Psi_n \to \Psi_0$ in $H_0^1(\Omega)$ and $C^0\left(\overline{\Omega}\right)$, by passing to the limit into (5.4), we obtain that Ψ_0 is a solution of the problem

$$\begin{cases} \Delta\Psi_0 = \lambda_0 h_0 f'(u_0)\Psi_0 & \text{in } \Omega \\ \Psi_0 = 0 & \text{on } \partial\Omega. \end{cases}$$

Since u_0 is non-degenerate, it follows that $\Psi_0 \equiv 0$. By using (5.3) together with standard L^p-estimates for the Dirichlet problem (see Lemma 9.17 of Gilbarg–Trudinger [11]), we then obtain

$$\begin{aligned} \|\Psi_n\|_{W^{2,p}} &\le \left\|L_{\varepsilon_n,\tau_n}(\Psi_n)\right\|_{W^{2,p}} + \left\|\Delta^{-1}\left[\lambda_{\varepsilon_n} h_{\varepsilon_n} \chi_{\varepsilon_n,\tau_n} f'(u_{\varepsilon_n})\Psi_n\right]\right\|_{W^{2,p}} \\ &= \mathrm{o}(1) + \mathrm{O}\left(\left\|\lambda_{\varepsilon_n} h_{\varepsilon_n} \chi_{\varepsilon_n,\tau_n} f'(u_{\varepsilon_n})\Psi_n\right\|_{L^p}\right) = \mathrm{o}(1) \end{aligned}$$

as $n \to \infty$, which is in contradiction with (5.3). This ends the proof of (5.2).

Now, for every $\varepsilon \in (0, \varepsilon_0)$ and $(\gamma, \tau) \in T_\varepsilon^k(\delta) \times \Gamma_\varepsilon^k(\delta')$, we let $N_{\varepsilon,\gamma,\tau}, T_{\varepsilon,\gamma,\tau} : \mathcal{W}_0^{2,p}(\Omega) \to \mathcal{W}_0^{2,p}(\Omega)$ be the operators defined as

$$\begin{aligned} N_{\varepsilon,\gamma,\tau}(\Psi) &:= \Delta^{-1}\left[\lambda_\varepsilon h_\varepsilon \chi_{\varepsilon,\tau}\left(f\left(\widetilde{U}_{\varepsilon,\gamma,\tau} + \Psi\right) - f(w_\varepsilon) - f'(w_\varepsilon)\Psi\right)\right], \\ T_{\varepsilon,\gamma,\tau}(\Psi) &:= L_{\varepsilon,\tau}^{-1}\left(N_{\varepsilon,\gamma,\tau}(\Psi) - R_{\varepsilon,\tau}\right) \end{aligned}$$

for all $\Psi \in \mathcal{W}_0^{2,p}(\Omega)$, where

$$R_{\varepsilon,\tau} := w_\varepsilon - \Delta^{-1}\left[\lambda_\varepsilon h_\varepsilon \chi_{\varepsilon,\tau} f(w_\varepsilon)\right] = \Delta^{-1}\left[\lambda_\varepsilon h_\varepsilon (1 - \chi_{\varepsilon,\tau}) f(w_\varepsilon)\right].$$

Note that the problem (3.13) can be rewritten as the fixed point equation $T_{\varepsilon,\gamma,\tau}(\Psi) = \Psi$. For every $C > 0$ and $\varepsilon \in (0, \varepsilon_0)$, we define

$$V_\varepsilon(C) := \left\{ \Psi \in \mathcal{W}_0^{2,p}(\Omega) : \|\Psi\|_{W^{2,p}} \le C/\overline{\gamma}_\varepsilon \right\}.$$

We will prove that if C is chosen large enough, then $T_{\varepsilon,\gamma,\tau}$ has a fixed point in $V_\varepsilon(C)$ for small $\varepsilon > 0$. By using a standard L^p-estimate and since $\lambda_\varepsilon \to \lambda_0$, $h_\varepsilon \to h_0$ and $w_\varepsilon \to w_0$ in $C^0(\overline{\Omega})$, we obtain

$$\|R_{\varepsilon,\tau}\|_{W^{2,p}} = \mathrm{O}\left(\|\lambda_\varepsilon h_\varepsilon (1 - \chi_{\varepsilon,\tau}) f(w_\varepsilon)\|_{L^p}\right) = \mathrm{O}\left(\|1 - \chi_{\varepsilon,\tau}\|_{L^p}\right) = \mathrm{o}(1/\overline{\gamma}_\varepsilon) \tag{5.5}$$

as $\varepsilon \to 0$, uniformly in $\tau \in T_\varepsilon^k(\delta)$. Similarly, for every $\Psi, \Psi_1, \Psi_2 \in V_\varepsilon(C)$, we obtain

$$\|N_{\varepsilon,\gamma,\tau}(\Psi)\|_{W^{2,p}} = \mathrm{O}\left(\left\|\chi_{\varepsilon,\tau}\big(f\big(\widetilde{U}_{\varepsilon,\gamma,\tau} + \Psi\big) - f(w_\varepsilon) - f'(w_\varepsilon)\Psi\big)\right\|_{L^p}\right), \tag{5.6}$$

$$\begin{aligned}\|N_{\varepsilon,\gamma,\tau}(\Psi_1) - N_{\varepsilon,\gamma,\tau}(\Psi_2)\|_{W^{2,p}} = \mathrm{O}\Big(\Big\|\chi_{\varepsilon,\tau}\big(f\big(\widetilde{U}_{\varepsilon,\gamma,\tau} + \Psi_1\big) - f\big(\widetilde{U}_{\varepsilon,\gamma,\tau} + \Psi_2\big) \\ - f'(w_\varepsilon)(\Psi_1 - \Psi_2)\big)\Big\|_{L^p}\Big).\end{aligned} \tag{5.7}$$

By applying the mean value theorem together with Hölder's inequality, it follows from (5.6) and (5.7) that

$$\begin{aligned}\|N_{\varepsilon,\gamma,\tau}(\Psi)\|_{W^{2,p}} = \mathrm{O}\Bigg(\Bigg\|\chi_{\varepsilon,\tau} f'\Big(w_\varepsilon + t_1 \sum_{i=1}^k B_{\varepsilon,\gamma_i,\tau_i} + \Psi\Big) \sum_{i=1}^k B_{\varepsilon,\gamma_i,\tau_i}\Bigg\|_{L^p} \\ + \|\chi_{\varepsilon,\tau} f''(w_\varepsilon + s_1\Psi)\|_{L^p} \|\Psi\|_{C^0}^2\Bigg),\end{aligned} \tag{5.8}$$

$$\begin{aligned}&\|N_{\varepsilon,\gamma,\tau}(\Psi_1) - N_{\varepsilon,\gamma,\tau}(\Psi_2)\|_{W^{2,p}} \\ &= \mathrm{O}\left(\left\|\chi_{\varepsilon,\tau}\big(f'\big(\widetilde{U}_{\varepsilon,\gamma,\tau} + (1 - s_2)\Psi_1 + s_2\Psi_2\big) - f'(w_\varepsilon)\big)\right\|_{L^p} \|\Psi_1 - \Psi_2\|_{C^0}\right) \\ &= \mathrm{O}\Bigg(\Bigg\|\chi_{\varepsilon,\tau} f''\Big(w_\varepsilon + t_2 \sum_{i=1}^k B_{\varepsilon,\gamma_i,\tau_i} + t_2(1 - s_2)\Psi_1 + t_2 s_2 \Psi_2\Big) \\ &\qquad \times \Big(\sum_{i=1}^k B_{\varepsilon,\gamma_i,\tau_i} + (1 - s_2)\Psi_1 + s_2\Psi_2\Big)\Bigg\|_{L^p} \|\Psi_1 - \Psi_2\|_{C^0}\Bigg)\end{aligned} \tag{5.9}$$

for some functions $s_1, s_2, t_1, t_2 : \Omega \to [0, 1]$. Since $0 \le \chi_{\varepsilon,\tau} \le 1$ in Ω, $w_\varepsilon \to w_0$ in $C^0(\overline{\Omega})$ and $\Psi \in V_\varepsilon(C)$, we obtain

$$\|\chi_{\varepsilon,\tau} f''(w_\varepsilon + s_1\Psi)\|_{L^p} = \mathrm{O}(1). \tag{5.10}$$

For every $j \in \{1, \dots, k\}$, by using (3.10), we obtain

$$B_{\varepsilon,\gamma_j,\tau_j}(x) = \frac{2}{\gamma_j}\left(\ln \frac{1}{|x - \overline{\tau_j}|} + \mathrm{O}(1)\right) \tag{5.11}$$

uniformly in $x \in \Omega \backslash B(\overline{\tau_j}, r_\varepsilon)$. We let $R_\varepsilon := \exp(-\overline{\gamma}_\varepsilon) \gg r_\varepsilon$. Since $\chi_{\varepsilon,\tau} \equiv 0$ in $B(\overline{\tau_j}, r_\varepsilon)$, $0 \le \chi_{\varepsilon,\tau} \le 1$ in Ω, $w_\varepsilon \to w_0$ in $C^0(\overline{\Omega})$, $u_0(0) = 0$, $\Psi, \Psi_1, \Psi_2 \in V_\varepsilon(C)$ and $0 \le s_1, s_2, t_1, t_2 \le 1$, it follows from (5.11) that

$$\begin{aligned}&\left\|\chi_{\varepsilon,\tau} f'\Big(w_\varepsilon + t_1 \sum_{i=1}^k B_{\varepsilon,\gamma_i,\tau_i} + \Psi\Big) \sum_{i=1}^k B_{\varepsilon,\gamma_i,\tau_i} \mathbf{1}_{B(\overline{\tau_j}, R_\varepsilon)}\right\|_{L^p}^p \\ &\qquad = \mathrm{O}\left(\overline{\gamma}_\varepsilon^p \int_{r_\varepsilon}^{R_\varepsilon} f'\left(\frac{2t_1}{\gamma_j} \ln \frac{1}{r} + \mathrm{o}(1)\right)^p r dr\right)\end{aligned}$$

$$
\begin{aligned}
&= \mathrm{O}\left(\overline{\gamma}_\varepsilon^{p+2}\int_{2\overline{\gamma}_\varepsilon/\gamma_j^2}^{\delta_0\overline{\gamma}_\varepsilon^2/\gamma_j^2} f'\left(t_1\gamma_j s+\mathrm{o}\left(1\right)\right)^p\exp\left(-\gamma_j^2 s\right)ds\right)\\
&= \mathrm{O}\left(\overline{\gamma}_\varepsilon^{3p+2}\int_{2\overline{\gamma}_\varepsilon/\gamma_j^2}^{\delta_0\overline{\gamma}_\varepsilon^2/\gamma_j^2} \exp\left(\left(pt_1^2 s-1\right)s\gamma_j^2+\mathrm{o}\left(\gamma_j\right)\right)ds\right)\\
&= \mathrm{O}\left(\overline{\gamma}_\varepsilon^{3p+2}\int_{\delta_0/[\overline{\gamma}_\varepsilon(1+\delta')^2]}^{\delta_0/\left(1-\delta'\right)^2} \exp\left(\left(pt_1^2 s-1\right)s\gamma_j^2+\mathrm{o}\left(\gamma_j\right)\right)ds\right)=\mathrm{o}\left(1\right), \qquad (5.12)
\end{aligned}
$$

$$
\begin{aligned}
&\Bigg\|\chi_{\varepsilon,\tau}f''\Big(w_\varepsilon+t_2\sum_{i=1}^k B_{\varepsilon,\gamma_i,\tau_i}+t_2\left(1-s_2\right)\Psi_1+t_2 s_2\Psi_2\Big)\\
&\qquad\times\Big(\sum_{i=1}^k B_{\varepsilon,\gamma_i,\tau_i}+\left(1-s_2\right)\Psi_1+s_2\Psi_2\Big)\mathbf{1}_{B(\overline{\tau_j},R_\varepsilon)}\Bigg\|_{L^p}^p\\
&\qquad= \mathrm{O}\left(\overline{\gamma}_\varepsilon^{p}\int_{r_\varepsilon}^{R_\varepsilon} f''\left(\frac{2t_2}{\gamma_j}\ln\frac{1}{r}+\mathrm{o}\left(1\right)\right)^p rdr\right)\\
&\qquad= \mathrm{O}\left(\overline{\gamma}_\varepsilon^{p+2}\int_{2\overline{\gamma}_\varepsilon/\gamma_j^2}^{\delta_0\overline{\gamma}_\varepsilon^2/\gamma_j^2} f''\left(t_2\gamma_j s+\mathrm{o}\left(1\right)\right)^p\exp\left(-\gamma_j^2 s\right)ds\right)\\
&\qquad= \mathrm{O}\left(\overline{\gamma}_\varepsilon^{4p+2}\int_{2\overline{\gamma}_\varepsilon/\gamma_j^2}^{\delta_0\overline{\gamma}_\varepsilon^2/\gamma_j^2} \exp\left(\left(pt_2^2 s-1\right)s\gamma_j^2+\mathrm{o}\left(\gamma_j\right)\right)ds\right)\\
&\qquad= \mathrm{O}\left(\overline{\gamma}_\varepsilon^{4p+2}\int_{\delta_0/[\overline{\gamma}_\varepsilon(1+\delta')^2]}^{\delta_0/\left(1-\delta'\right)^2} \exp\left(\left(pt_2^2 s-1\right)s\gamma_j^2+\mathrm{o}\left(\gamma_j\right)\right)ds\right)=\mathrm{o}\left(1\right) \qquad (5.13)
\end{aligned}
$$

as $\varepsilon\to 0$, uniformly in $(\gamma,\tau)\in\Gamma_\varepsilon^k\left(\delta'\right)\times T_\varepsilon^k\left(\delta\right)$ and $\Psi,\Psi_1,\Psi_2\in V_\varepsilon(C)$, provided we choose p such that $p\delta_0/\left(1-\delta'\right)^2-1<0$, i.e. $p<\left(1-\delta'\right)^2/\delta_0$. By using (5.11), we obtain

$$
\begin{aligned}
&\Bigg\|\chi_{\varepsilon,\tau}f'\Big(w_\varepsilon+t_1\sum_{i=1}^k B_{\varepsilon,\gamma_i,\tau_i}+\Psi\Big)\sum_{i=1}^k B_{\varepsilon,\gamma_i,\tau_i}\mathbf{1}_{\Omega_{R_\varepsilon,\tau}}\Bigg\|_{L^p}^p\\
&\qquad\qquad= \mathrm{O}\left(\frac{1}{\overline{\gamma}_\varepsilon^p}\sum_{i=1}^k\int_{\Omega_{R_\varepsilon,\tau}}\left|\ln\left|x-\overline{\tau_i}\right|+\mathrm{O}\left(1\right)\right|^p dx\right)=\mathrm{o}\left(1\right) \qquad (5.14)
\end{aligned}
$$

and, similarly,

$$
\begin{aligned}
&\Bigg\|\chi_{\varepsilon,\tau}f''\Big(w_\varepsilon+t_2\sum_{i=1}^k B_{\varepsilon,\gamma_i,\tau_i}+t_2\left(1-s_2\right)\Psi_1+t_2 s_2\Psi_2\Big)\\
&\qquad\qquad\times\Big(\sum_{i=1}^k B_{\varepsilon,\gamma_i,\tau_i}+\left(1-s_2\right)\Psi_1+s_2\Psi_2\Big)\mathbf{1}_{\Omega_{R_\varepsilon,\tau}}\Bigg\|_{L^p}=\mathrm{o}\left(1\right) \qquad (5.15)
\end{aligned}
$$

as $\varepsilon\to 0$, uniformly in $(\gamma,\tau)\in\Gamma_\varepsilon^k\left(\delta'\right)\times T_\varepsilon^k\left(\delta\right)$ and $\Psi,\Psi_1,\Psi_2\in V_\varepsilon(C)$.

Note that similar estimates as in (5.12)–(5.14) yield (3.16).

By putting together (5.9)–(5.15) and using the continuity of the embedding $W^{2,p}\left(\Omega\right)\hookrightarrow C^0\left(\overline{\Omega}\right)$, we obtain

$$
\left\|N_{\varepsilon,\gamma,\tau}\left(\Psi\right)\right\|_{W^{2,p}}=\mathrm{o}\left(\left\|\Psi\right\|_{W^{2,p}}^2\right), \qquad (5.16)
$$

$$\|N_{\varepsilon,\gamma,\tau}(\Psi_1) - N_{\varepsilon,\gamma,\tau}(\Psi_2)\|_{W^{2,p}} = \mathrm{o}\left(\|\Psi_1 - \Psi_2\|_{W^{2,p}}\right) \tag{5.17}$$

as $\varepsilon \to 0$, uniformly in $(\gamma,\tau) \in \Gamma_\varepsilon^k(\delta') \times T_\varepsilon^k(\delta)$ and $\Psi, \Psi_1, \Psi_2 \in V_\varepsilon(C)$. It follows from (5.2), (5.5), (5.16) and (5.17) that there exist $\varepsilon_1(p,\delta,\delta') \in (0,\varepsilon_0)$ and $C = C(p,\delta,\delta') > 0$ (here we do not specify the dependence in δ_0 as this number is considered to be fixed) such that for every $\varepsilon \in (0,\varepsilon_1(\delta))$ and $(\gamma,\tau) \in \Gamma_\varepsilon^k(\delta') \times T_\varepsilon^k(\delta)$, $T_{\varepsilon,\gamma,\tau}$ is a contraction mapping on $V_\varepsilon(C)$. By the fixed point theorem, we then obtain that there exists a unique solution $\Psi_{\varepsilon,\gamma,\tau} \in V_\varepsilon(C)$ to the problem (3.13). By fixing a number p such that $2 < p < (1-\delta_0)^2/\delta_0$, the first inequality in (3.14) then follows from the continuity of the embedding $W^{2,p}(\Omega) \hookrightarrow C^1\left(\overline{\Omega}\right)$. By using the Moser–Trudinger inequality together with standard elliptic regularity theory, we obtain that $\Psi_{\varepsilon,\gamma,\tau} \in C^{l,\alpha}(\Omega) \cap C^2\left(\overline{\Omega}\right)$. Furthermore, by symmetry of Ω, w_ε, h_ε, $\chi_{\varepsilon,\tau}$ and $\widetilde{U}_{\varepsilon,\gamma,\tau}$, we obtain that $\Psi_{\varepsilon,\gamma,\tau}$ is even in x_2 and by using the continuous differentiability of $\widetilde{U}_{\varepsilon,\gamma,\tau}$ and $\chi_{\varepsilon,\tau}$ in (γ,τ), we obtain that $\Psi_{\varepsilon,\gamma,\tau}$ is continuously differentiable in (γ,τ).

Now, we prove the second inequality in (3.14). For $i \in \{1,\dots,k\}$, by differentiating (3.13) in γ_i, we obtain

$$\begin{aligned}\Delta\left[L_{\varepsilon,\tau}\left(\partial_{\gamma_i}\left[\Psi_{\varepsilon,\gamma,\tau}\right]\right)\right] &= \lambda_\varepsilon h_\varepsilon \chi_{\varepsilon,\tau} f'\big(\widetilde{U}_{\varepsilon,\gamma,\tau} + \Psi_{\varepsilon,\gamma,\tau}\big)\partial_{\gamma_i}\big[\widetilde{U}_{\varepsilon,\gamma,\tau}\big] \\ &\quad + \lambda_\varepsilon h_\varepsilon \chi_{\varepsilon,\tau}\big(f'\big(\widetilde{U}_{\varepsilon,\gamma,\tau} + \Psi_{\varepsilon,\gamma,\tau}\big) - f'(w_\varepsilon)\big)\partial_{\gamma_i}\left[\Psi_{\varepsilon,\gamma,\tau}\right],\end{aligned} \tag{5.18}$$

where $L_{\varepsilon,\tau}$ is as in (5.1). By using (5.2) and (5.18) together with a standard L^p-estimate and since $\lambda_\varepsilon \to \lambda_0$ and $h_\varepsilon \to h_0$ in $C^0\left(\overline{\Omega}\right)$, we then obtain

$$\begin{aligned}\left\|\partial_{\gamma_i}\left[\Psi_{\varepsilon,\gamma,\tau}\right]\right\|_{W^{2,p}} = \mathrm{O}\Bigg(&\left\|\chi_{\varepsilon,\tau} f'\big(\widetilde{U}_{\varepsilon,\gamma,\tau} + \Psi_{\varepsilon,\gamma,\tau}\big)\partial_{\gamma_i}\big[\widetilde{U}_{\varepsilon,\gamma,\tau}\big]\right\|_{L^p} \\ &+ \left\|\chi_{\varepsilon,\tau}\big(f'\big(\widetilde{U}_{\varepsilon,\gamma,\tau} + \Psi_{\varepsilon,\gamma,\tau}\big) - f'(w_\varepsilon)\big)\partial_{\gamma_i}\left[\Psi_{\varepsilon,\gamma,\tau}\right]\right\|_{L^p}\Bigg).\end{aligned} \tag{5.19}$$

By using (3.11), we obtain

$$\partial_{\gamma_i}\big[\widetilde{U}_{\varepsilon,\gamma,\tau}\big] = \frac{2}{\gamma_i^2}\left(\ln|x - \overline{\tau_i}| + \mathrm{O}(1)\right) \tag{5.20}$$

uniformly in $x \in \Omega\backslash B(\overline{\tau_i}, r_\varepsilon)$. By using (5.11) and (5.20) and proceeding as in (5.12)–(5.15), we obtain

$$\begin{aligned}\left\|\chi_{\varepsilon,\tau} f'\big(\widetilde{U}_{\varepsilon,\gamma,\tau} + \Psi_{\varepsilon,\gamma,\tau}\big)\partial_{\gamma_i}\big[\widetilde{U}_{\varepsilon,\gamma,\tau}\big]\right\|_{L^p}^p &= \mathrm{O}\left(\int_{r_\varepsilon}^{R_\varepsilon} f'\left(\frac{2}{\gamma_i}\ln\frac{1}{r} + \mathrm{o}(1)\right)^p r\,dr\right. \\ &\quad \left. + \frac{1}{\gamma_i^{2p}}\int_{\Omega_{R_\varepsilon,\tau}} \left|\ln|x - \overline{\tau_i}| + \mathrm{O}(1)\right|^p dx\right) = \mathrm{O}\left(\frac{1}{\overline{\gamma}_\varepsilon^{2p}}\right)\end{aligned} \tag{5.21}$$

uniformly in $(\gamma,\tau) \in \Gamma_\varepsilon^k(\delta') \times T_\varepsilon^k(\delta)$, provided p is chosen so that $p < (1-\delta')^2/\delta_0$. On the other hand, by applying the mean value theorem together with Hölder's inequality, we obtain

$$\begin{aligned}\left\|\chi_{\varepsilon,\tau}\big(f'\big(\widetilde{U}_{\varepsilon,\gamma,\tau} + \Psi_{\varepsilon,\gamma,\tau}\big) - f'(w_\varepsilon)\big)\partial_{\gamma_i}\left[\Psi_{\varepsilon,\gamma,\tau}\right]\right\|_{L^p} &\le \left\|\partial_{\gamma_i}\left[\Psi_{\varepsilon,\gamma,\tau}\right]\right\|_{C^0} \\ \times \left\|\chi_{\varepsilon,\tau} f''\Big(w_\varepsilon + t\sum_{i=1}^k B_{\varepsilon,\gamma_i,\tau_i} + t\Psi_{\varepsilon,\gamma,\tau}\Big)\Big(\sum_{i=1}^k B_{\varepsilon,\gamma_i,\tau_i} + \Psi_{\varepsilon,\gamma,\tau}\Big)\right\|_{L^p}\end{aligned} \tag{5.22}$$

for some function $t:\Omega\to[0,1]$. By using (5.11) and proceeding as in (5.12)–(5.15), we obtain

$$\left\|\chi_{\varepsilon,\tau}f''\Big(w_\varepsilon+t\sum_{i=1}^k B_{\varepsilon,\gamma_i,\tau_i}+t\Psi_{\varepsilon,\gamma,\tau}\Big)\Big(\sum_{i=1}^k B_{\varepsilon,\gamma_i,\tau_i}+\Psi_{\varepsilon,\gamma,\tau}\Big)\right\|_{L^p}^p=\mathrm{o}\left(1\right) \tag{5.23}$$

as $\varepsilon\to0$, uniformly in $(\gamma,\tau)\in\Gamma_\varepsilon^k(\delta')\times T_\varepsilon^k(\delta)$, provided we choose p such that $p<(1-\delta')^2/\delta_0$. By putting together (5.19) and (5.21)–(5.23), we obtain

$$\left\|\partial_{\gamma_i}\left[\Psi_{\varepsilon,\gamma,\tau}\right]\right\|_{W^{2,p}}=\mathrm{O}\left(\frac{1}{\overline{\gamma}_\varepsilon^2}\right)+\mathrm{o}\left(\left\|\partial_{\gamma_i}\left[\Psi_{\varepsilon,\gamma,\tau}\right]\right\|_{C^0}\right) \tag{5.24}$$

as $\varepsilon\to0$, uniformly in $(\gamma,\tau)\in\Gamma_\varepsilon^k(\delta')\times T_\varepsilon^k(\delta)$. By choosing p such that

$$2<p<\frac{(1-\delta_0)^2}{\delta_0}$$

and using the continuity of the embedding $W^{2,p}(\Omega)\hookrightarrow C^1\left(\overline{\Omega}\right)$, the second inequality in (3.14) then follows from (5.24).

Now, we prove (3.15). For every $i\in\{1,\dots,k\}$, by differentiating (3.13) in τ_i, we obtain

$$\begin{aligned}\Delta\left[L_{\varepsilon,\tau}\left(\partial_{\tau_i}\left[\Psi_{\varepsilon,\gamma,\tau}\right]\right)\right]&=\lambda_\varepsilon h_\varepsilon f\big(\widetilde{U}_{\varepsilon,\gamma,\tau}+\Psi_{\varepsilon,\gamma,\tau}\big)\partial_{\tau_i}\left[\chi_{\varepsilon,\tau}\right]+\lambda_\varepsilon h_\varepsilon\chi_{\varepsilon,\tau}f'\big(\widetilde{U}_{\varepsilon,\gamma,\tau}+\Psi_{\varepsilon,\gamma,\tau}\big)\\&\quad\times\partial_{\tau_i}\big[\widetilde{U}_{\varepsilon,\gamma,\tau}\big]+\lambda_\varepsilon h_\varepsilon\chi_{\varepsilon,\tau}\big(f'\big(\widetilde{U}_{\varepsilon,\gamma,\tau}+\Psi_{\varepsilon,\gamma,\tau}\big)-f'\left(w_\varepsilon\right)\big)\partial_{\tau_i}\left[\Psi_{\varepsilon,\gamma,\tau}\right],\end{aligned} \tag{5.25}$$

where $L_{\varepsilon,\tau}$ is as in (5.1). By using (5.2) and (5.25) together with a standard L^p–estimate and since $\lambda_\varepsilon\to\lambda_0$ and $h_\varepsilon\to h_0$ in $C^0\left(\overline{\Omega}\right)$, we obtain

$$\begin{aligned}\left\|\partial_{\tau_i}\left[\Psi_{\varepsilon,\gamma,\tau}\right]\right\|_{W^{2,p}}&=\mathrm{O}\Bigg(\left\|f\big(\widetilde{U}_{\varepsilon,\gamma,\tau}+\Psi_{\varepsilon,\gamma,\tau}\big)\partial_{\tau_i}\left[\chi_{\varepsilon,\tau}\right]\right\|_{L^p}+\Big\|\chi_{\varepsilon,\tau}f'\big(\widetilde{U}_{\varepsilon,\gamma,\tau}+\Psi_{\varepsilon,\gamma,\tau}\big)\\&\quad\times\partial_{\tau_i}\big[\widetilde{U}_{\varepsilon,\gamma,\tau}\big]\Big\|_{L^p}+\left\|\chi_{\varepsilon,\tau}\big(f'\big(\widetilde{U}_{\varepsilon,\gamma,\tau}+\Psi_{\varepsilon,\gamma,\tau}\big)-f'\left(w_\varepsilon\right)\big)\partial_{\tau_i}\left[\Psi_{\varepsilon,\gamma,\tau}\right]\right\|_{L^p}\Bigg).\end{aligned} \tag{5.26}$$

It is easy to see that

$$\partial_{\tau_i}\left[\chi_{\varepsilon,\tau}\right]=\mathrm{O}\left(\frac{1}{r_\varepsilon^2}\mathbf{1}_{A(\overline{\tau_i},r_\varepsilon,r_\varepsilon+r_\varepsilon^2)}\right) \tag{5.27}$$

uniformly in Ω. By using (3.10) and (3.12) and since $\delta'<1-\sqrt{2\delta_0}$, we obtain

$$\partial_{\tau_i}\big[\widetilde{U}_{\varepsilon,\gamma,\tau}\big]=\frac{2}{\gamma_i}\left(1+\mathrm{o}\left(1\right)\right)\frac{x_1-\tau_i}{\left|x-\overline{\tau_i}\right|^2}+\mathrm{O}\left(\frac{1}{\overline{\gamma}_\varepsilon}\right) \tag{5.28}$$

uniformly in $x=(x_1,x_2)\in\Omega\backslash B\left(\overline{\tau_i},r_\varepsilon\right)$. By using (5.11), (5.27), (5.28) and proceeding as in (5.12)–(5.15), we obtain

$$\begin{aligned}&\left\|f\big(\widetilde{U}_{\varepsilon,\gamma,\tau}+\Psi_{\varepsilon,\gamma,\tau}\big)\partial_{\tau_i}\left[\chi_{\varepsilon,\tau}\right]\right\|_{L^p}^p\\&\quad=\mathrm{O}\left(\frac{1}{r_\varepsilon^{2p}}\int_{r_\varepsilon}^{r_\varepsilon+r_\varepsilon^2}f\left(\frac{2}{\gamma_i}\ln\frac{1}{r}\left(1+\mathrm{o}\left(1\right)\right)+\mathrm{O}\left(\frac{1}{\overline{\gamma}_\varepsilon}\right)\right)^p rdr\right)\\&\quad=\mathrm{O}\left(\frac{\overline{\gamma}_\varepsilon^{p+2}}{r_\varepsilon^{2p}}\int_{\delta_0\overline{\gamma}_\varepsilon^2/\gamma_i^2-\left(2/\gamma_i^2\right)\ln(1+r_\varepsilon)}^{\delta_0\overline{\gamma}_\varepsilon^2/\gamma_i^2}\exp\left(\left(ps-1\right)s\gamma_i^2\left(1+\mathrm{o}\left(1\right)\right)\right)ds\right)\\&\quad=\mathrm{O}\left(\frac{\overline{\gamma}_\varepsilon^{p}}{r_\varepsilon^{2p}}\ln\left(1+r_\varepsilon\right)\exp\left(\left(p\delta_0\frac{\overline{\gamma}_\varepsilon^2}{\gamma_i^2}-1\right)\delta_0\overline{\gamma}_\varepsilon^2+\mathrm{o}\left(\overline{\gamma}_\varepsilon^2\right)\right)\right)\end{aligned}$$

$$= \mathrm{O}\left(\overline{\gamma}_\varepsilon^p \exp\left(p\delta_0^2 \frac{\overline{\gamma}_\varepsilon^4}{\gamma_i^2} + \left(p - \frac{3}{2}\right)\delta_0 \overline{\gamma}_\varepsilon^2 + \mathrm{o}\left(\overline{\gamma}_\varepsilon^2\right)\right)\right)$$

$$= \mathrm{O}\left(\overline{\gamma}_\varepsilon^p \exp\left(\left(\frac{p\delta_0}{\left(1-\delta'\right)^2} + p - \frac{3}{2}\right)\delta_0 \overline{\gamma}_\varepsilon^2 + \mathrm{o}\left(\overline{\gamma}_\varepsilon^2\right)\right)\right) = \mathrm{o}\left(\frac{1}{\overline{\gamma}_\varepsilon^p}\right), \tag{5.29}$$

$$\left\|\chi_{\varepsilon,\tau} f'\big(\widetilde{U}_{\varepsilon,\gamma,\tau} + \Psi_{\varepsilon,\gamma,\tau}\big)\partial_{\tau_i}\left[\widetilde{U}_{\varepsilon,\gamma,\tau}\right]\right\|_{L^p}^p = \mathrm{O}\left(\frac{1}{\gamma_i^p}\int_{\Omega_{R_\varepsilon,\tau}}\left(\frac{1}{|x-\overline{\tau}_i|}+1\right)^p dx\right.$$

$$\left. + \frac{1}{\gamma_i^p}\int_{r_\varepsilon}^{R_\varepsilon} f'\left(\frac{2}{\gamma_i}\ln\frac{1}{r}\left(1+\mathrm{o}\left(1\right)\right) + \mathrm{O}\left(\frac{1}{\overline{\gamma}_\varepsilon}\right)\right)^p r^{1-p} dr\right)$$

$$= \mathrm{O}\left(\frac{1}{\overline{\gamma}_\varepsilon^p} + \gamma_i^{p+2}\int_{\delta_0/[\overline{\gamma}_\varepsilon(1+\delta')^2]}^{\delta_0/\left(1-\delta'\right)^2} \exp\left(\left(ps + \frac{p}{2} - 1\right)s\gamma_i^2 + \mathrm{o}\left(\gamma_i^2\right)\right)ds\right) = \mathrm{O}\left(\frac{1}{\overline{\gamma}_\varepsilon^p}\right) \tag{5.30}$$

as $\varepsilon \to 0$, uniformly in $(\gamma,\tau) \in \Gamma_\varepsilon^k(\delta') \times T_\varepsilon^k(\delta)$, provided we choose p such that

$$\frac{p\delta_0}{\left(1-\delta'\right)^2} + p - \frac{3}{2} < 0 \quad \text{and} \quad \frac{p\delta_0}{\left(1-\delta'\right)^2} + \frac{p}{2} - 1 < 0,$$

$$\text{i.e. } \max\left(\frac{1}{2} + \frac{\delta_0}{\left(1-\delta'\right)^2}, \frac{2}{3}\left(1 + \frac{\delta_0}{\left(1-\delta'\right)^2}\right)\right) < \frac{1}{p} < 1,$$

which is possible since $\delta' < 1 - \sqrt{2\delta_0}$. Note that in this case, we cannot choose $p > 2$ and so $W^{2,p}(\Omega)$ does not embed into $C^1\left(\overline{\Omega}\right)$. Furthermore, by proceeding as in (5.22)–(5.23), we obtain

$$\left\|\chi_{\varepsilon,\tau}\left(f'\big(\widetilde{U}_{\varepsilon,\gamma,\tau} + \Psi_{\varepsilon,\gamma,\tau}\big) - f'\left(w_\varepsilon\right)\right)\partial_{\tau_i}\left[\Psi_{\varepsilon,\gamma,\tau}\right]\right\|_{L^p} = \mathrm{o}\left(\left\|\partial_{\tau_i}\left[\Psi_{\varepsilon,\gamma,\tau}\right]\right\|_{C^0}\right) \tag{5.31}$$

as $\varepsilon \to 0$, uniformly in $(\gamma,\tau) \in \Gamma_\varepsilon^k(\delta') \times T_\varepsilon^k(\delta)$. By putting together (5.26), (5.29), (5.30) and (5.31), we obtain

$$\left\|\partial_{\tau_i}\left[\Psi_{\varepsilon,\gamma,\tau}\right]\right\|_{W^{2,p}} = \mathrm{O}\left(\frac{1}{\overline{\gamma}_\varepsilon}\right) + \mathrm{o}\left(\left\|\partial_{\tau_i}\left[\Psi_{\varepsilon,\gamma,\tau}\right]\right\|_{C^0}\right) \tag{5.32}$$

as $\varepsilon \to 0$, uniformly in $(\gamma,\tau) \in \Gamma_\varepsilon^k(\delta') \times T_\varepsilon^k(\delta)$. By using the continuity of the embeddings of $W^{2,p}(\Omega)$ into $C^0\left(\overline{\Omega}\right)$ and $H^1(\Omega)$, (3.15) then follows from (5.32).

Note that (5.29) corresponds to the first identity in (3.17), while the second one follows from (5.30) together with the already proven (3.15) and (3.16), which yield

$$\left\|f'\left(U_{\varepsilon,\gamma,\tau}\right)D_{\tau_i}\Psi_{\varepsilon,\gamma,\tau}\mathbf{1}_{\Omega_{r_\varepsilon,\tau}}\right\|_{L^p} = \mathrm{O}\left(\overline{\gamma}_\varepsilon^2\left\|\exp\left(U_{\varepsilon,\gamma,\tau}^2\right)\mathbf{1}_{\Omega_{r_\varepsilon,\tau}}\right\|_{L^p}\left\|D_{\tau_i}\Psi_{\varepsilon,\gamma,\tau}\right\|_{C^0}\right) = \mathrm{O}\left(\overline{\gamma}_\varepsilon\right).$$

This ends the proof of Proposition 3.2.

6. Expansions of the bubble and its derivatives

In this section we give a precise asymptotic analysis of spherical solutions, and prove some useful consequences.

Proposition 6.1. *For every $\gamma > 0$, let $\overline{B}_\gamma$ be the unique radial solution to the problem*

$$\begin{cases} \Delta\overline{B}_\gamma = f\left(\overline{B}_\gamma\right) & \text{in } \mathbb{R}^2 \\ \overline{B}_\gamma(0) = \gamma, \end{cases}$$

where $f(s) := s\exp\left(s^2\right)$ *for all* $s \in \mathbb{R}$. *Set*

$$\mu_\gamma^2 := 4\gamma^{-2}\exp\left(-\gamma^2\right) \quad \text{and} \quad t(r) := \ln\left(1+r^2\right) \quad \forall r \geq 0 \tag{6.1}$$

and let φ *be the unique radial solution to the problem*

$$\begin{cases} \Delta\varphi = 4e^{-2t}\left(t^2 - t + 2\varphi\right) & \text{in } \mathbb{R}^2 \\ \varphi(0) = 0. \end{cases}$$

Then

$$\overline{B}_\gamma(r) = \gamma - \frac{t(r/\mu_\gamma)}{\gamma} + \frac{\varphi(r/\mu_\gamma)}{\gamma^3} + D_\gamma(r/\mu_\gamma),$$

where

$$D_\gamma(r) = \mathrm{O}\left(\frac{t(r)}{\gamma^5}\right) \quad \text{and} \quad D'_\gamma(r) = \mathrm{O}\left(\frac{1}{\gamma^5 r}\right) \tag{6.2}$$

as $\gamma \to \infty$, *uniformly in* $r \in \left(0, \mu_\gamma^{\delta-1}\right)$, $\delta \in (0,1)$ *fixed. Furthermore,* $\varphi(r) \sim -t(r)$ *and* $\varphi'(r) \sim -t'(r)$ *as* $r \to \infty$.

Proof. This was originally proven in [8], see Claim 5.1 and estimates (5.8) and (5.9) in particular (note that the function B_γ in [8] corresponds to the function $\overline{B}_\gamma$ via the relation $B_\gamma(r) = \overline{B}_\gamma(r/2)$). The estimates (5.8)–(5.9) in [8] are valid as long as $0 \leq t(r/\mu_\gamma) \leq \gamma^2 - T_\gamma$, where T_γ is chosen so that $\gamma^k e^{-T_\gamma} = \mathrm{o}(1)$ as $\gamma \to \infty$ for every $k \geq 0$. It is not difficult to see that this condition is satisfied uniformly for $0 \leq r \leq \mu_\gamma^\delta$, for any fixed $\delta > 0$. ☐

With regard to the derivative of $\overline{B}_\gamma$ with respect to γ, we obtain the following:

Proposition 6.2. *Let* $\overline{B}_\gamma$, μ_γ, t *and* φ *be as in Proposition 6.1. Set* $\overline{Z}_0(r) := \frac{1-r^2}{1+r^2}$ *and let* ψ *be the unique radial solution to the problem*

$$\begin{cases} \Delta\psi = 4e^{-2t}\left(\overline{Z}_0\left(1 - 4t + 2t^2 + 4\varphi\right) + 2\psi\right) & \text{in } \mathbb{R}^2 \\ \psi(0) = 0. \end{cases}$$

Then

$$Z_{0,\gamma}(r) := \partial_\gamma\left[\overline{B}_\gamma(r)\right] = \overline{Z}_0(r/\mu_\gamma) + \frac{\psi(r/\mu_\gamma)}{\gamma^2} + E_\gamma(r/\mu_\gamma),$$

where

$$E_\gamma(r) = \mathrm{O}\left(\frac{1 + t(r)}{\gamma^4}\right) \quad \text{and} \quad E'_\gamma(r) = \mathrm{O}\left(\frac{1}{\gamma^4 r}\right) \tag{6.3}$$

as $\gamma \to \infty$, *uniformly in* $r \in \left(0, \mu_\gamma^{\delta-1}\right)$, $\delta \in (0,1)$ *fixed. Furthermore,* $\psi(r) \sim t(r)$ *and* $\psi'(r) \sim t'(r)$ *as* $r \to \infty$.

Proof. We easily see that

$$\begin{cases} \Delta Z_{0,\gamma} = f'\left(\overline{B}_\gamma\right) Z_{0,\gamma} & \text{in } B(0, \mu_\gamma^\delta) \\ Z_{0,\gamma}(0) = 1, \end{cases}$$

with $f(s) = se^{s^2}$. Set

$$E_\gamma(r) := Z_{0,\gamma}(\mu_\gamma r) - \overline{Z}_0(r) - \frac{\psi(r)}{\gamma^2}$$

and observe that

$$\Delta\overline{Z}_0 = 8e^{-2t}\overline{Z}_0,$$

so that

$$\begin{cases}\Delta E_\gamma = \mu_\gamma^2 f'\left(\overline{B}_\gamma\left(\mu_\gamma\cdot\right)\right) Z_{0,\gamma}\left(\mu_\gamma\cdot\right) - 8e^{-2t}\overline{Z}_0 - \dfrac{\Delta\psi}{\gamma^2} & \text{in } B\left(0,\mu_\gamma^{\delta-1}\right)\\ E_\gamma\left(0\right)=0. & \end{cases} \tag{6.4}$$

In order to expand the right-hand side of (6.4) we use (6.2), $\varphi = \mathrm{O}\left(1+t\right)$ and recalling that $\mu_\gamma^2\gamma^2 e^{\gamma^2}=4$, we find

$$\begin{aligned} f'\left(\overline{B}_\gamma\left(\mu_\gamma\cdot\right)\right) &= \left(1+2\overline{B}_\gamma^2\left(\mu_\gamma\cdot\right)\right)\exp\left(\overline{B}_\gamma^2\left(\mu_\gamma\cdot\right)\right)\\ &= \left[1+2\left(\gamma-\frac{t}{\gamma}+\frac{\varphi}{\gamma^3}+\mathrm{O}\left(\frac{t}{\gamma^5}\right)\right)^2\right] e^{\left(\gamma-\frac{t}{\gamma}+\frac{\varphi}{\gamma^3}+\mathrm{O}\left(\frac{t}{\gamma^5}\right)\right)^2}\\ &= \frac{4e^{-2t}}{\mu_\gamma^2}\left[\frac{1}{\gamma^2}+2-\frac{4t}{\gamma^2}+\mathrm{O}\left(\frac{1+t^2}{\gamma^4}\right)\right] e^{\frac{t^2}{\gamma^2}} e^{\frac{2\varphi}{\gamma^2}+\mathrm{O}\left(\frac{1+t^2}{\gamma^4}\right)}. \end{aligned} \tag{6.5}$$

Using that $e^s = 1+s+\mathrm{O}(s^2)e^s$ for $s>0$, we write

$$e^{\frac{t^2}{\gamma^2}} = 1+\frac{t^2}{\gamma^2}+\mathrm{O}\left(\frac{t^4}{\gamma^4}\right)e^{\frac{t^2}{\gamma^2}},$$

and using that $t=\mathrm{O}\left(\gamma^2\right)$ uniformly on $\left(0,\mu_\gamma^{\delta-1}\right)$,

$$e^{\frac{2\varphi}{\gamma^2}+\mathrm{O}\left(\frac{1+t^2}{\gamma^4}\right)} = 1+\frac{2\varphi}{\gamma^2}+\mathrm{O}\left(\frac{1+t^2}{\gamma^4}\right).$$

We now multiply and reorder, using that $\exp\left(t^2/\gamma^2\right)\geq 1$, to obtain

$$\begin{aligned} f'\left(\overline{B}_\gamma\left(\mu_\gamma\cdot\right)\right) &= \frac{4e^{-2t}}{\mu_\gamma^2}\left(2+\frac{1}{\gamma^2}\left(1-4t+4\varphi\right)+\mathrm{O}\left(\frac{1+t^4}{\gamma^4}\right)\right)e^{\frac{t^2}{\gamma^2}}\\ &= \frac{4e^{-2t}}{\mu_\gamma^2}\left(2+\frac{1}{\gamma^2}\left(1-4t+2t^2+4\varphi\right)\right)+\frac{e^{-2t+\frac{t^2}{\gamma^2}}}{\mu_\gamma^2}\mathrm{O}\left(\frac{1+t^4}{\gamma^4}\right). \end{aligned}$$

Together with (6.4) and using that $\psi = \mathrm{O}\left(1+t\right)$ (as we shall prove later), we now estimate

$$\begin{aligned} \Delta E_\gamma &= \mu_\gamma^2\left(f'\left(\overline{B}_\gamma\left(\mu_\gamma\cdot\right)\right)\overline{Z}_0+\frac{f'\left(\overline{B}_\gamma\left(\mu_\gamma\cdot\right)\right)\psi}{\gamma^2}+f'\left(\overline{B}_\gamma\left(\mu_\gamma\cdot\right)\right)E_\gamma\right)\\ &\quad -8e^{-2t}\overline{Z}_0-4e^{-2t}\left(\frac{\overline{Z}_0}{\gamma^2}\left(1-4t+2t^2+4\varphi\right)+\frac{2\psi}{\gamma^2}\right)\\ &= \mu_\gamma^2 f'\left(\overline{B}_\gamma\left(\mu_\gamma\cdot\right)\right)E_\gamma+e^{-2t+\frac{t^2}{\gamma^2}}\mathrm{O}\left(\frac{1+t^4}{\gamma^4}\right). \end{aligned}$$

We now go back to (6.5) and, still using that $t=\mathrm{O}\left(\gamma^2\right)$ on $B\left(0,\mu_\gamma^{\delta-1}\right)$, we bound

$$f'\left(\overline{B}_\gamma\left(\mu_\gamma\cdot\right)\right)=\mathrm{O}\left(\frac{1}{\mu_\gamma^2}e^{-2t+\frac{t^2}{\gamma^2}}\right),$$

so that

$$\Delta E_\gamma = e^{-2t+\frac{t^2}{\gamma^2}}\left(\mathrm{O}\left(|E_\gamma|\right)+\mathrm{O}\left(\frac{1+t^4}{\gamma^4}\right)\right). \tag{6.6}$$

Multiplying by γ^4 and using ODE theory, we see that

$$\gamma^4 E_\gamma \longrightarrow \widetilde{E}_\infty \quad \text{in } C^1_{\mathrm{loc}}\left(\mathbb{R}^2\right).$$

In particular, for any fixed $T>0$ and for γ large ($\gamma \geq \gamma_0(T)$), we have

$$|E_\gamma| \leq \frac{C(T)}{\gamma^4} \quad \text{and} \quad \left|E'_\gamma\right| \leq \frac{C'(T)}{\gamma^4} \quad \text{on } [0,T]. \tag{6.7}$$

From now on, it is understood that $\gamma \geq \gamma_0(T)$, so that (6.7) holds. In order to prove (6.3), observe that the first identity in (6.3) follows from the second one and (6.7) by integration over $[T,r]$. Then, for $T, M>0$ to be chosen later, set

$$R_\gamma := \sup\left\{ r \in \left(T, \mu_\gamma^{\delta-1}\right] : \left|E'_\gamma(\rho)\right| \leq \frac{M}{\gamma^4 \rho}, \forall \rho \in [T,r] \right\}.$$

We shall prove that for T and M suitable, we have $R_\gamma = \mu_\gamma^{\delta-1}$ for every γ sufficiently large.

Arguing by contradiction, assume that $R_\gamma < \mu_\gamma^{\delta-1}$, so that in particular

$$\left|E'_\gamma(R_\gamma)\right| = \frac{M}{R_\gamma \gamma^4}. \tag{6.8}$$

By definition of R_γ, using (6.7) and integrating, we get

$$|E_\gamma(r)| \leq |E_\gamma(T)| + \int_T^r \frac{M}{\gamma^4 \rho} d\rho \leq \frac{C(T)}{\gamma^4} + \frac{M t(r)}{2\gamma^4} \quad \text{on } [T, R_\gamma]. \tag{6.9}$$

With the divergence theorem, (6.6) and (6.9), we now bound for $t \in [T, R_\gamma]$,

$$\begin{aligned}
\left|2\pi r E'_\gamma(r)\right| &\leq \left|2\pi T E'_\gamma(T)\right| + \int_{B(0,r)\setminus B(0,T)} |\Delta E_\gamma(x)\, dx| \\
&\leq \frac{2\pi T C'(T)}{\gamma^4} + \int_{B(0,r)\setminus B(0,T)} e^{-2t+\frac{t^2}{\gamma^2}} \left(\widetilde{C}|E_\gamma| + \widetilde{C}\left(\frac{1+t^4}{\gamma^4}\right)\right) dx \\
&\leq \frac{2\pi T C'(T)}{\gamma^4} + \frac{\widetilde{C} M}{2\gamma^4} \int_{B(0,r)\setminus B(0,T)} e^{-2t+\frac{t^2}{\gamma^2}} t dx \\
&\quad + \frac{1}{\gamma^4} \int_{B(0,r)\setminus B(0,T)} e^{-2t+\frac{t^2}{\gamma^2}} \widetilde{C}\left(C(T)+1+t^4\right) dx \\
&=: \frac{2\pi T C'(T)}{\gamma^4} + \frac{(I_\gamma)}{\gamma^4} + \frac{(II_\gamma)}{\gamma^4}.
\end{aligned} \tag{6.10}$$

Observing that

$$-2t + \frac{t^2}{\gamma^2} \leq -(1+\delta)t \quad \text{and} \quad e^{-\frac{\delta}{2}t} t^k = \mathrm{O}(1) \quad \text{on } B\left(0, \mu_\gamma^{\delta-1}\right), \quad \forall k \geq 0,$$

we bound

$$\int_{B\left(0,\mu_\gamma^{\delta-1}\right)\setminus B(0,T)} e^{-2t+\frac{t^2}{\gamma^2}} t^k dx = \mathrm{O}\left(\int_{B(0,T)^c} e^{-\left(1+\frac{\delta}{2}\right)t} dx\right) = \mathrm{o}_T(1),$$

with $\mathrm{o}_T(1) \to 0$ as $T \to \infty$. We can therefore choose T sufficiently large (independent of M) so that

$$(I_\gamma) \leq \frac{\pi M}{2}.$$

Then, choosing M sufficiently large (depending on T), so that

$$2\pi T C'(T) + (II_\gamma) \leq \frac{\pi M}{2}$$

and dividing by 2π in (6.10), we finally obtain

$$r\left|E'(r)\right| \le \frac{M}{2\gamma^4}, \quad \forall r \in [T, R_\gamma],$$

which for $r = R_\gamma$ is a contradiction to (6.8). Therefore $R_\gamma = \mu_\gamma^{\delta-1}$.

To prove that $\psi(r) \sim t(r)$ and $\psi'(r) \sim t'(r)$ as $r \to \infty$, we recall from [15, Lemmas 15 and 16] (see also [8, Lemma 5.1]) that if ψ is radially symmetric and solves

$$\Delta\psi = 4e^{-2t}(g + 2\psi),$$

with $g(r) = \mathrm{O}\left((\ln r)^k\right)$ as $r \to \infty$ for some $k \ge 1$, then

$$\psi(r) = \beta \ln r + \mathrm{O}(r), \quad \psi'(r) = \frac{\beta}{r} + \mathrm{O}\left(\frac{(\ln r)^k}{r^3}\right), \quad \beta := \frac{2}{\pi}\int_{\mathbb{R}^2} \overline{Z}_0 e^{-2t} g dx,$$

as $r \to \infty$. With $g = \overline{Z}_0\left(1 - 4t + 2t^2 + 4\varphi\right)$ we compute

$$\int_{\mathbb{R}^2} \overline{Z}_0^2 e^{-2t} dx = \frac{\pi}{3}, \quad \int_{\mathbb{R}^2} \overline{Z}_0^2 e^{-2t} 4t dx = \frac{16\pi}{9},$$
$$\int_{\mathbb{R}^2} \overline{Z}_0^2 e^{-2t} 2t^2 dx = \frac{70\pi}{27} \quad \text{and} \quad \int_{\mathbb{R}^2} \overline{Z}_0^2 e^{-2t} 4\varphi dx = -\frac{4\pi}{27},$$

so that $\beta = 2$. □

Let us see a few consequences of the above estimates.

Proposition 6.3. *Let $\overline{B}_\gamma$, μ_γ, t, $\overline{Z}_0$ and $Z_{0,\gamma}$ be as in Propositions 6.1 and 6.2. Given $\delta \in (0,1)$, $a, b \ge 0$ and $t_\gamma(r) = t(r/\mu_\gamma)$, we have*

$$\int_{B(0,r)} \exp\left(\overline{B}_\gamma^2\right) \overline{B}_\gamma^b \left(1 + \mathrm{O}\left(\frac{t_\gamma}{\gamma^2}\right)\right)^a dx = 4\pi\gamma^{b-2} + \mathrm{O}\left(\gamma^{b-4}\right), \tag{6.11}$$

as $\gamma \to \infty$, uniformly for $\gamma\mu_\gamma \le r \le \mu_\gamma^\delta$. Moreover,

$$\int_{B(0,r)} f'\left(\overline{B}_\gamma\right) Z_{0,\gamma} dx = \frac{-4\pi + \mathrm{o}(1)}{\gamma^2} \tag{6.12}$$

as $\gamma \to \infty$, uniformly for $\gamma\mu_\gamma = \mathrm{o}(r)$ and $r \le \mu_\gamma^\delta$, and

$$\int_{B(0,r)} f'\left(\overline{B}_\gamma(x)\right) \frac{2x_1^2}{\mu_\gamma^2 + |x|^2} dx = 4\pi + \mathrm{O}\left(\frac{1}{\gamma^2}\right) \tag{6.13}$$

as $\gamma \to \infty$, uniformly for $\gamma\mu_\gamma \le r \le \mu_\gamma^\delta$.

Proof. Using Proposition 6.1 and noticing that $\varphi = \mathrm{O}(1+t)$, $t_\gamma = \mathrm{O}\left(\gamma^2\right)$ in $B(0,r)$ for $r \le \mu_\gamma^\delta$, we write

$$\exp\left(\overline{B}_\gamma^2\right) = e^{\left[\gamma - \frac{t_\gamma}{\gamma} + \mathrm{O}\left(\frac{1+t_\gamma}{\gamma^3}\right)\right]^2} = e^{\gamma^2} e^{-2t_\gamma} e^{\frac{t_\gamma^2}{\gamma^2} + \mathrm{O}\left(\frac{1+t_\gamma}{\gamma^2}\right)} = \frac{4e^{-2t_\gamma}}{\mu_\gamma^2\gamma^2}\left(1 + \mathrm{O}\left(\frac{t_\gamma^2}{\gamma^2}\right) e^{\frac{t_\gamma^2}{\gamma^2}}\right), \tag{6.14}$$

where we used the inequality $|e^x - 1| \le |x| e^{|x|}$ to estimate

$$\left|e^{\mathrm{O}\left(\frac{t_\gamma^2}{\gamma^2}\right)} - 1\right| = \mathrm{O}\left(\frac{t_\gamma^2}{\gamma^2}\right) e^{\frac{t_\gamma^2}{\gamma^2}}.$$

Further, we use Proposition 6.1 together with $(1+x)^a = 1 + \mathrm{O}(x)$ uniformly for $x = \mathrm{O}(1)$, to bound

$$\overline{B}_\gamma^b \left(1 + \mathrm{O}\left(\frac{t_\gamma}{\gamma^2}\right)\right)^a = \left(\gamma + \mathrm{O}\left(\frac{1+t_\gamma}{\gamma}\right)\right)^b \left(1 + \mathrm{O}\left(\frac{t_\gamma}{\gamma^2}\right)\right)^a$$
$$= \gamma^b \left(1 + \mathrm{O}\left(\frac{1+t_\gamma}{\gamma}\right)\right)^{a+b} = \gamma^b \left(1 + \mathrm{O}\left(\frac{1+t_\gamma^2}{\gamma^2}\right)\right). \tag{6.15}$$

We can then estimate the left-hand side of (6.11) as

$$\int_{B(0,r)} \frac{4e^{-2t_\gamma}}{\mu_\gamma^2 \gamma^{2-b}} \left(1 + \mathrm{O}\left(\frac{1+t_\gamma^2}{\gamma^2} e^{\frac{t_\gamma^2}{\gamma^2}}\right)\right) dx$$
$$= \int_{B(0,r/\mu_\gamma)} \frac{4e^{-2t}}{\gamma^{2-b}} \left(1 + \mathrm{O}\left(\frac{1+t^2}{\gamma^2} e^{\frac{t^2}{\gamma^2}}\right)\right) dx = \frac{4\pi}{\gamma^{2-b}} \int_0^{r/\mu_\gamma} \frac{2\rho}{\left(1+\rho^2\right)^2} d\rho$$
$$+ \frac{1}{\gamma^{4-b}} \int_{B(0,r/\mu_\gamma)} \mathrm{O}\left(\left(1+t^2\right) e^{-t\left(2 - \frac{t}{\gamma^2}\right)}\right) dx =: (I)_\gamma + (II)_\gamma.$$

Using that $r \geq \gamma \mu_\gamma$, one computes

$$(I)_\gamma = \frac{4\pi}{\gamma^{2-b}} \left(1 + \mathrm{O}\left(\gamma^{-2}\right)\right),$$

and using that $0 \leq t/\gamma^2 \leq (1 - \delta + \mathrm{o}(1))$ in $B(0, r/\mu_\gamma)$ as $\gamma \to \infty$, uniformly for $r \leq \mu_\gamma^\delta$, and observing that $\left(1+t^2\right) e^{-\left(1+\delta'\right)t} \in L^1\left(\mathbb{R}^2\right)$ for every $\delta' > 0$, one has

$$(II)_\gamma = \mathrm{O}\left(\frac{1}{\gamma^{4-b}} \int_{B(0,r/\mu_\gamma)} \left(1+t^2\right) e^{-t(1+\delta+\mathrm{o}(1))}\right) = \mathrm{O}\left(\frac{1}{\gamma^{4-b}}\right) \tag{6.16}$$

as $\gamma \to \infty$, uniformly for $r \leq \mu_\gamma^\delta$, so that (6.11) is proven.

In order to prove (6.12) we use Proposition 6.2 to expand $Z_{0,\gamma}$ and compute

$$\int_{B(0,r)} f'\left(\overline{B}_\gamma\right) Z_{0,\gamma} dx = \int_{B(0,r)} \partial_\gamma \left[f\left(\overline{B}_\gamma\right)\right] dx = \int_{B(0,r)} \Delta Z_{0,\gamma} dx$$
$$= \int_{B(0,r)} \Delta \left(\overline{Z}_0\left(\frac{x}{\mu_\gamma}\right) + \frac{1}{\gamma^2} \psi\left(\frac{x}{\mu_\gamma}\right) + E_\gamma\left(\frac{x}{\mu_\gamma}\right)\right) dx$$
$$= -\frac{2\pi r}{\mu_\gamma} \left(\overline{Z}_0'\left(\frac{r}{\mu_\gamma}\right) + \frac{1}{\gamma^2} \psi'\left(\frac{r}{\mu_\gamma}\right) + E_\gamma'\left(\frac{r}{\mu_\gamma}\right)\right).$$

A direct computation shows

$$\frac{r}{\mu_\gamma} \overline{Z}_0'\left(\frac{r}{\mu_\gamma}\right) = \mathrm{O}\left(\frac{\mu_\gamma^2}{r^2}\right) = \mathrm{o}\left(\frac{1}{\gamma^2}\right)$$

as $\gamma \to \infty$, uniformly for $\gamma \mu_\gamma = \mathrm{o}(r)$. Using that

$$\psi'(s) = t'(s)(1 + \mathrm{o}(1)) = \frac{2s}{1+s^s} = \frac{2}{s}(1 + \mathrm{o}(1)) \quad \text{as } s \to \infty,$$

we obtain

$$\frac{r}{\mu_\gamma} \psi'\left(\frac{r}{\mu_\gamma}\right) = 2 + \mathrm{o}(1).$$

Finally, from the second part of (6.3), we infer

$$\frac{r}{\mu_\gamma} E'_\gamma\left(\frac{r}{\mu_\gamma}\right) = \mathrm{O}\left(\frac{1}{\gamma^4}\right) = \mathrm{o}\left(\frac{1}{\gamma^2}\right).$$

Summing up, (6.12) follows at once.

It remains to prove (6.13). Using (6.14) and (6.15), we write

$$\begin{aligned}
&\int_{B(0,r)} f'\left(\overline{B}_\gamma(x)\right) \frac{2x_1^2}{\mu_\gamma^2 + |x|^2} dx \\
&= 4\gamma^2 e^{\gamma^2} \int_{B(0,r)} \left(1 + \mathrm{O}\left(\frac{1+t_\gamma}{\gamma^2}\right)\right) e^{-3t_\gamma} \left(1 + \mathrm{O}\left(\frac{t_\gamma^2}{\gamma^2}\right) e^{\frac{t_\gamma^2}{\gamma^2}}\right) \left(\frac{y_1}{\mu_\gamma}\right)^2 dy \\
&= \int_{B(0,r/\mu_\gamma)} 16 e^{-3t} y_1^2 dy + \mathrm{O}\left(\int_{B(0,r/\mu_\gamma)} \frac{1+t^2}{\gamma^2} e^{-t\left(3-t/\gamma^2\right)} y_1^2 dy\right) =: (I)_\gamma + (II)_\gamma
\end{aligned}$$

To compute $(I)_\gamma$, we observe that its value does not change if we replace y_1 with y_2, so that

$$(I)_\gamma = \frac{1}{2}\int_{B(0,r/\mu_\gamma)} 16 e^{-3t} |x|^2 dy = 16\pi\left[-\frac{1+2\rho^2}{4\left(1+\rho^2\right)^2}\right]_{\rho=0}^{r/\mu_\gamma} = 4\pi + \mathrm{o}(1)$$

as $\gamma \to \infty$, uniformly for $r \geq \gamma\mu_\gamma$. The term $(II)_\gamma$ can be estimated as in (6.16) since $y_1^2 \leq e^t$, so that

$$(II)_\gamma = \mathrm{O}\left(\int_{B(0,r/\mu_\gamma)} \frac{t^2}{\gamma^2} e^{-t(1+\delta+\mathrm{o}(1))} dx\right) = \mathrm{O}\left(\frac{1}{\gamma^2}\right)$$

as $\gamma \to \infty$, uniformly for $r \leq \mu_\gamma^\delta$. □

Proposition 6.4. *Let $\overline{B}_\gamma$ and μ_γ be as in Proposition 6.1. Given $\delta_0 \in (0, 1/2)$, we have*

$$\int_{B(0,r)} \exp\left(\overline{B}_\gamma(x)^2\right) |x|\, dx = \mathrm{O}\left(\mu_\gamma^{3\delta_0 - 2\delta_0^2 + \mathrm{o}(1)}\right) \tag{6.17}$$

as $\gamma \to \infty$, uniformly for $r = \mathrm{O}\left(\mu_\gamma^{\delta_0}\right)$.

Proof. Let t_γ be as in Proposition 6.2. Using Proposition 6.1, we write

$$\begin{aligned}
\exp\left(\overline{B}_\gamma^2\right) &= \exp\left(\left[\gamma - \frac{t_\gamma}{\gamma} + \mathrm{O}\left(\frac{1+t_\gamma}{\gamma^3}\right)\right]^2\right) = e^{\gamma^2} e^{-2t_\gamma + \frac{t_\gamma^2}{\gamma^2}} e^{\mathrm{O}\left(\frac{1+t_\gamma}{\gamma^2}\right)} \\
&= \mathrm{O}\left(\frac{e^{-t_\gamma\left(2-t_\gamma/\gamma^2\right)}}{\mu_\gamma^2\gamma^2}\right), \quad \text{for } r = \mathrm{O}\left(\mu_\gamma^{\delta_0}\right),
\end{aligned} \tag{6.18}$$

Then, using that

$$t_\gamma(r) \leq \left(1 - \delta_0 + \mathrm{o}(1)\right)\gamma^2, \quad \text{for } r = \mathrm{O}\left(\mu_\gamma^{\delta_0}\right), \tag{6.19}$$

together with a change of variables, we get

$$\int_{B(0,r)} \exp\left(\overline{B}_\gamma(x)^2\right) |x|\, dx = \mathrm{O}\left(\int_{B(0,r)} \frac{e^{-t_\gamma\left(2-t_\gamma/\gamma^2\right)}}{\mu_\gamma^2\gamma^2} |x|\, dx\right)$$

$$= \mathrm{O}\left(\mu_\gamma \int_{B(0,r/\mu_\gamma)} \frac{|y|\,dy}{\left(1+|y|^2\right)^{2-t/\gamma^2}}\right) = \mathrm{O}\left(\mu_\gamma \int_{B(0,r/\mu_\gamma)} \frac{|y|\,dy}{\left(1+|y|^2\right)^{1+\delta_0+\mathrm{o}(1)}}\right)$$
$$= \mathrm{O}\left(\mu_\gamma \left(\frac{r}{\mu_\gamma}\right)^{1-2\delta_0+\mathrm{o}(1)}\right) = \mathrm{O}\left(\mu_\gamma^{3\delta_0-2\delta_0^2+\mathrm{o}(1)}\right)$$

as $\gamma \to \infty$, uniformly for $r = \mathrm{O}\left(\mu_\gamma^{\delta_0}\right)$, which proves (6.17). □

7. Poincaré–Sobolev inequalities

The standard Poincaré–Sobolev inequality on $\mathbb{S}^2$ says that for every $p \in [1,\infty)$ there exists $C_p > 0$ such that for every $\phi \in H^1(\mathbb{S}^2)$ with $\int_{\mathbb{S}^2} \phi dv_{\mathbb{S}^2} = 0$, we have

$$\int_{\mathbb{S}^2} |\phi|^p \, dv_{\mathbb{S}^2} \le C_p \left(\int_{\mathbb{S}^2} |\nabla \phi|^2 \, dv_{\mathbb{S}^2}\right)^{\frac{p}{2}}. \tag{7.1}$$

Pulling back the spherical metric onto $\mathbb{R}^2$, we can also rewrite (7.1) as

$$\int_{\mathbb{R}^2} |\phi|^p \, e^{-2t} dx \le C_p \left(\int_{\mathbb{R}^2} |\nabla \phi|^2 \, dx\right)^{\frac{p}{2}}, \tag{7.2}$$

for every $\phi \in D^{1,2}\left(\mathbb{R}^2\right)$ such that $\int_{\mathbb{R}^2} \phi e^{-2t} dx = 0$, where $t(x) := \ln\left(1+|x|^2\right)$, so that $4e^{-2t(x)} = 4\left(1+|x|^2\right)^{-2}$ is the conformal factor of the pull-back metric.

We will need a perturbed version of (7.2), where we replace e^{-2t} with suitable scaled versions of $\exp\left(\overline{B}_\gamma^2\right)$.

Lemma 7.1. *Let $(\chi_\varepsilon)_{\varepsilon>0}$ be a sequence of functions in $\mathbb{R}^2$ such that for every $q > 1$, we have $\chi_\varepsilon \to \chi_0$ as $\varepsilon \to 0$ in $L^q\left(\mathbb{R}^2, e^{-2t}dx\right)$, i.e.*

$$\int_{\mathbb{R}^2} |\chi_\varepsilon - \chi_0|^q \, e^{-2t} dx \longrightarrow 0$$

for some function χ_0 in $\mathbb{R}^2$ and further assume that

$$\int_{\mathbb{R}^2} \chi_0 e^{-2t} dx \neq 0. \tag{7.3}$$

Then, for every $p \in [1,\infty)$, there exists a constant $C > 0$ (depending on p and (χ_ε)) such that for $\varepsilon > 0$ small enough, the following holds:

$$\int_{\mathbb{R}^2} |\phi|^p \, e^{-2t} dx \le C \left(\int_{\mathbb{R}^2} |\nabla \phi|^2 \, dx\right)^{\frac{p}{2}} \tag{7.4}$$

for every $\phi \in D^{1,2}\left(\mathbb{R}^2\right)$ such that $\int_{\mathbb{R}^2} \phi \chi_\varepsilon e^{-2t} dx = 0$.

Proof. Assume by contradiction that there exists a sequence $(\phi_\varepsilon)_\varepsilon$ in $D^{1,2}\left(\mathbb{R}^2\right)$ such that

$$\int_{\mathbb{R}^2} |\phi_\varepsilon|^p e^{-2t} dx = 1, \quad \int_{\mathbb{R}^2} \phi_\varepsilon \chi_\varepsilon e^{-2t} dx = 0, \quad \lim_{\varepsilon \to 0} \int_{\mathbb{R}^2} |\nabla \phi_\varepsilon|^2 dx = 0. \tag{7.5}$$

Let $\Pi : \mathbb{S}^2 \to \mathbb{R}^2$ be the stereographic projection. By the first equation in (7.5), the average of $\phi_\varepsilon \circ \Pi$ on $\mathbb{S}^2$ is bounded, so by the Sobolev–Poincaré inequality and weak compactness, up to a subsequence, $\phi_\varepsilon \circ \Pi \to \phi_0 \circ \Pi$ strongly in $L^{q'}(\mathbb{S}^2)$, in $L^p(\mathbb{S}^2)$, and weakly in $H^1(\mathbb{S}^2)$, for some function $\phi_0 \in L^p(\mathbb{R}^2, e^{-2t}dx)$. By lower-semicontinuity of the Dirichlet integral we get

$\|\nabla(\phi_0\circ\Pi)\|_{L^2(\mathbb{S}^2)} = \|\nabla\phi_0\|_{L^2(\mathbb{R}^2)} = 0$, so that ϕ_0 is constant, non-zero since $\|\phi_0\circ\Pi\|_{L^p(\mathbb{S}^2)} = 1$. Then, we obtain

$$0 = \int_{\mathbb{R}^2} \phi_\varepsilon\chi_\varepsilon e^{-2t}dx \to \int_{\mathbb{R}^2} \phi_0\chi_0 e^{-2t}dx \quad \Rightarrow \quad \int_{\mathbb{R}^2} \chi_0 e^{-2t}dx = 0,$$

contradicting our assumption. □

Proposition 7.2. *Let $\overline{B}_\gamma$, μ_γ and t_γ be as in Propositions 6.1 and 6.2. Let $\phi \in D^{1,2}\left(\mathbb{R}^2\right)$ be such that*

$$\int_{B_r} f\left(\overline{B}_\gamma\right)\phi dx = 0 \tag{7.6}$$

for r such that $\mu_\gamma = \mathrm{o}\,(r)$ and $r = \mathrm{O}\left(\mu_\gamma^{\delta_0}\right)$ for some $\delta_0 \in (0,1)$. Then for every $p \in [1,\infty)$, we have

$$\int_{B(0,r)} \exp\left(\overline{B}_\gamma^2\right)(1+t_\gamma)\,|\phi|^p\,dx = \mathrm{O}\left(\frac{1}{\gamma^2}\left(\int_{\mathbb{R}^2}|\nabla\phi|^2\,dx\right)^{\frac{p}{2}}\right). \tag{7.7}$$

Proof. With Proposition 6.1 we can rewrite condition (7.6) as

$$\begin{aligned}0 &= \gamma\int_{B(0,r)}\left(1+\mathrm{O}\left(\frac{1+t_\gamma}{\gamma^2}\right)\right)e^{\gamma^2}e^{-2t_\gamma+\frac{t_\gamma^2}{\gamma^2}+\mathrm{O}\left(\frac{1+t_\gamma}{\gamma^2}\right)}\phi dx\\ &= \frac{4}{\gamma}\int_{B(0,r/\mu_\gamma)}\left(1+\mathrm{O}\left(\frac{1+t}{\gamma^2}\right)\right)e^{-2t+\frac{t^2}{\gamma^2}+\mathrm{O}\left(\frac{1+t}{\gamma^2}\right)}\phi(\mu_\gamma\cdot)dx = \frac{4}{\gamma}\int_{\mathbb{R}^2}e^{-2t}\chi_\gamma\tilde{\phi}dy,\end{aligned}$$

where $\widetilde{\Phi}\,(y) = \phi(\mu_\gamma y)$, and we claim that

$$\chi_\gamma := \mathbf{1}_{B(0,r/\mu_\gamma)}\left(1+\mathrm{O}\left(\frac{1+t}{\gamma^2}\right)\right)e^{\frac{t^2}{\gamma^2}+\mathrm{O}\left(\frac{1+t}{\gamma^2}\right)} \longrightarrow \chi_0 \equiv 1$$
$$\text{in } L^q(\mathbb{R}^2, e^{-2t}dx) \text{ for } 1 \le q < \frac{1}{1-\delta_0}. \tag{7.8}$$

Indeed, it is clear that $\chi_\gamma \to \chi_0$ pointwise, while we can uniformly bound χ_γ by a function in $L^q\left(\mathbb{R}^2, e^{-2t}dx\right)$ as follows. By using (6.19), we obtain

$$\chi_\gamma = \mathrm{O}\left(e^{\frac{t^2}{\gamma^2}}\right) = \mathrm{O}\left(e^{t(1-\delta_0+\mathrm{o}(1))}\right), \quad \text{so that } \chi_\gamma^q = \mathrm{O}\left(e^{tq(1-\delta_0+\mathrm{o}(1))}\right).$$

On the other hand,

$$\begin{aligned}\int_{\mathbb{R}^2} e^{tq(1-\delta_0+\mathrm{o}(1))}e^{-2t}dx &= \int_{\mathbb{R}^2} e^{-t(2-q+q\delta_0+\mathrm{o}(1))}dx = \int_{\mathbb{R}^2}\mathrm{O}\left(\frac{1}{1+|x|^{4-2q+2q\delta_0+\mathrm{o}(1)}}\right)dx\\ &= \mathrm{O}\,(1) \quad \text{for } 4-2q+2q\delta_0 > 2, \quad \text{i.e. } 1 \le q < \frac{1}{1-\delta_0},\end{aligned}$$

so that (7.8) follows by dominated convergence.

We can then apply Lemma 7.1 to $\widetilde{\Phi}$, so that (7.4) holds. On the other hand, for any $r \in [1,\infty)$,

$$\begin{aligned}&\int_{B(0,r)} \exp\left(\overline{B}_\gamma^2\right)(1+t_\gamma)\,|\phi|^p\,dx\\ &\qquad = \frac{1}{\gamma^2}\int_{B(0,r/\mu_\gamma)} e^{-2t+\frac{t^2}{\gamma^2}+\mathrm{O}\left(\frac{1+t}{\gamma^2}\right)}(1+t)\,|\tilde{\phi}|^p dx = \frac{1}{\gamma^2}\int_{\mathbb{R}^2}\tilde{\chi}_\gamma|\tilde{\phi}|^p e^{-2t}dx,\end{aligned} \tag{7.9}$$

where, as in (7.8) we have

$$\tilde{\chi}_\gamma := \mathbf{1}_{B(0,r/\mu_\gamma)}\,(1+t)\,e^{\frac{t^2}{\gamma^2}+\mathrm{O}\left(\frac{1+t}{\gamma^2}\right)} \longrightarrow 1+t \quad \text{in } L^q(\mathbb{R}^2, e^{-2t}dx)$$

for $q < 1/(1-\delta_0)$, and with Hölder's inequality, we obtain

$$\begin{aligned}\int_{\mathbb{R}^2} \tilde{\chi}_\gamma |\tilde{\phi}|^p e^{-2t}dx \le \left(\int_{\mathbb{R}^2} \tilde{\chi}_\gamma^q e^{-2t}dx\right)^{\frac{1}{q}} \left(\int_{\mathbb{R}^2} |\tilde{\phi}|^{pq'} e^{-2t}dx\right)^{\frac{1}{q'}} &= \mathrm{O}\left(\left(\int_{\mathbb{R}^2} |\nabla\tilde{\phi}|^2\right)^{\frac{p}{2}}\right)\\ &= \mathrm{O}\left(\left(\int_{\mathbb{R}^2} |\nabla\phi|^2\right)^{\frac{p}{2}}\right).\end{aligned}$$

Substituting into (7.9), we then obtain (7.7). □

8. Proof of Claim 3.1

From (3.2), (6.2) and the divergence theorem, we get

$$A_{\varepsilon,\gamma_i,\tau_i} = -2\pi r_\varepsilon \overline{B}'_{\varepsilon,\gamma_i,\overline{\tau_i}}(r_\varepsilon) = -2\pi r_\varepsilon \sqrt{\lambda_\varepsilon h_\varepsilon(\overline{\tau}_i)}\,\overline{B}'_{\gamma_i}\big(\sqrt{\lambda_\varepsilon h_\varepsilon(\overline{\tau}_i)}r_\varepsilon\big) = \frac{4\pi}{\gamma_i} + \mathrm{O}\left(\frac{1}{\overline{\gamma}_\varepsilon^3}\right). \tag{8.1}$$

Considering that

$$\ln\lambda_\varepsilon = \mathrm{O}(1) \quad \text{and} \quad \ln\mu_{\gamma_i} = -\frac{1}{2}\gamma_i^2 - \ln\gamma_i + \mathrm{O}(1),$$

from (6.2), we infer

$$\begin{aligned}\overline{B}_{\gamma_i}\big(\sqrt{\lambda_\varepsilon h_\varepsilon(\overline{\tau}_i)}r_\varepsilon\big) &= \gamma_i - \frac{2\ln(r_\varepsilon/\mu_{\gamma_i}) + \ln(\lambda_\varepsilon h_\varepsilon(\overline{\tau}_i))}{\gamma_i} + \mathrm{O}\left(\frac{1}{\gamma_i}\right)\\ &= -\frac{2\ln r_\varepsilon}{\gamma_i} - \frac{2\ln\gamma_i}{\gamma_i} + \mathrm{O}\left(\frac{1}{\overline{\gamma}_\varepsilon}\right),\end{aligned} \tag{8.2}$$

which together with (3.3) and (8.1) gives

$$C_{\varepsilon,\gamma_i,\tau_i} = -\frac{2\ln\gamma_i}{\gamma_i} + \mathrm{O}\left(\frac{1}{\overline{\gamma}_\varepsilon}\right) = -\frac{2\ln\overline{\gamma}_\varepsilon}{\gamma_i} + \mathrm{O}\left(\frac{1}{\overline{\gamma}_\varepsilon}\right).$$

This proves (3.10). Further, Proposition 6.2 gives

$$\begin{aligned}\partial_{\gamma_i}\left[A_{\varepsilon,\gamma_i,\tau_i}\right] &= -2\pi r_\varepsilon\sqrt{\lambda_\varepsilon h_\varepsilon(\overline{\tau}_i)}\,\partial_{\gamma_i}\left[\overline{B}'_{\gamma_i}\big(\sqrt{\lambda_\varepsilon h_\varepsilon(\overline{\tau}_i)}r_\varepsilon\big)\right]\\ &= -\frac{4\pi}{\gamma_i^2} + \mathrm{O}\left(\frac{\mu_{\gamma_i}^2}{r_\varepsilon^3}\right) + \mathrm{O}\left(\frac{1}{\overline{\gamma}_\varepsilon^4}\right) = -\frac{4\pi}{\gamma_i^2} + \mathrm{O}\left(\frac{1}{\overline{\gamma}_\varepsilon^4}\right).\end{aligned}$$

Similarly,

$$\partial_{\gamma_i}[\overline{B}_{\gamma_i}\big(\sqrt{\lambda_\varepsilon h_\varepsilon(\overline{\tau}_i)}r_\varepsilon\big)] = \frac{2\ln r_\varepsilon}{\gamma_i^2} + \frac{2\ln\gamma_i}{\gamma_i^2} + \mathrm{O}\left(\frac{1}{\overline{\gamma}_\varepsilon^2}\right),$$

so that

$$\begin{aligned}\partial_{\gamma_i}\left[C_{\varepsilon,\gamma_i,\tau_i}\right] &= \partial_{\gamma_i}[\overline{B}_{\gamma_i}\big(\sqrt{\lambda_\varepsilon h_\varepsilon(\overline{\tau}_i)}r_\varepsilon\big)] + \frac{1}{2\pi}\partial_{\gamma_i}[A_{\varepsilon,\gamma_i,\tau_i}]\ln r_\varepsilon\\ &= \frac{2\ln\gamma_i}{\gamma_i^2} + \mathrm{O}\left(\frac{1}{\overline{\gamma}_\varepsilon^2}\right) = \frac{2\ln\overline{\gamma}_\varepsilon}{\gamma_i^2} + \mathrm{O}\left(\frac{1}{\overline{\gamma}_\varepsilon^2}\right),\end{aligned}$$

which proves (3.11). To prove (3.12), we observe that

$$A_{\varepsilon,\gamma_i,\tau_i} = \lambda_\varepsilon h_\varepsilon\left(\overline{\tau_i}\right)\int_{B(\overline{\tau_i},r_\varepsilon)} f\left(\overline{B}_{\varepsilon,\gamma_i,\overline{\tau_i}}\right)dx = 2\pi\lambda_\varepsilon h_\varepsilon\left(\overline{\tau_i}\right)\int_0^{r_\varepsilon} f\Big(\overline{B}_{\gamma_i}\big(\sqrt{\lambda_\varepsilon h_\varepsilon\left(\overline{\tau_i}\right)}r\big)\Big)rdr$$
$$= 2\pi\int_0^{\sqrt{\lambda_\varepsilon h_\varepsilon(\overline{\tau_i})}r_\varepsilon} f\big(\overline{B}_{\gamma_i}\left(r\right)\big)rdr. \qquad (8.3)$$

By differentiating (8.3) in τ_i, we obtain

$$\partial_{\tau_i}\left[A_{\varepsilon,\gamma_i,\tau_i}\right] = \pi\lambda_\varepsilon\partial_{x_1}h_\varepsilon\left(\overline{\tau_i}\right)r_\varepsilon^2 f\Big(\overline{B}_{\gamma_i}\big(\sqrt{\lambda_\varepsilon h_\varepsilon\left(\overline{\tau_i}\right)}r_\varepsilon\big)\Big). \qquad (8.4)$$

By using (8.2) together with the definition of r_ε, and using that $\gamma_i \geq (1-\delta')\,\overline{\gamma}_\varepsilon$, we obtain

$$r_\varepsilon^2 f\big(\overline{B}_{\gamma_i}\big(\sqrt{\lambda_\varepsilon h_\varepsilon\left(\overline{\tau_i}\right)}r_\varepsilon\big) = \mathrm{O}\left(r_\varepsilon^2\overline{\gamma}_\varepsilon\exp\left(\frac{4}{\gamma_i^2}\left(\ln r_\varepsilon + \ln\gamma_i + \mathrm{O}\left(1\right)\right)^2\right)\right)$$
$$= \mathrm{O}\left(\overline{\gamma}_\varepsilon\exp\left(2\ln r_\varepsilon\left(1 + 2\frac{\ln r_\varepsilon}{\gamma_i^2} + 4\frac{\ln\gamma_i}{\gamma_i^2}\right)\right)\right)$$
$$= \mathrm{O}\left(\exp\left(-\delta_0\overline{\gamma}_\varepsilon^2\left(1 - \delta_0\frac{\overline{\gamma}_\varepsilon^2}{\gamma_i^2} + \mathrm{o}\left(1\right)\right) + \ln\overline{\gamma}_\varepsilon\right)\right)$$
$$= \mathrm{O}\left(\exp\left(-\delta_0\overline{\gamma}_\varepsilon^2\left(1 - \frac{\delta_0}{\left(1-\delta'\right)^2} + \mathrm{o}\left(1\right)\right)\right)\right) = \mathrm{o}\left(\frac{1}{\overline{\gamma}_\varepsilon^a}\right) \qquad (8.5)$$

uniformly in $(\gamma,\tau) \in \Gamma_\varepsilon^k(\delta') \times T_\varepsilon^k(\delta)$ for all $a \geq 0$, provided $\delta' < 1 - \sqrt{\delta_0}$. By using (8.4) and (8.5) and since $\lambda_\varepsilon \to \lambda_0$ and $h_\varepsilon \to h_0$ in $C^1\left(\overline{\Omega}\right)$, we obtain the first part of (3.12). By differentiating $C_{\varepsilon,\gamma_i,\tau_i}$ in τ_i and using (6.2), (8.4) and (8.5), we then obtain

$$\partial_{\tau_i}\left[C_{\varepsilon,\gamma_i,\tau_i}\right] = \frac{\sqrt{\lambda_\varepsilon}\partial_{x_1}h_\varepsilon\left(\overline{\tau}_i\right)r_\varepsilon}{2\sqrt{h_\varepsilon\left(\overline{\tau}_i\right)}}\overline{B}'_{\gamma_i}\big(\sqrt{\lambda_\varepsilon h_\varepsilon\left(\overline{\tau}_i\right)}r_\varepsilon\big)$$
$$-\frac{1}{2\pi}\partial_{\tau_i}\left[A_{\varepsilon,\gamma_i,\tau_i}\right]\ln\frac{1}{r_\varepsilon} = -\frac{\partial_{x_1}h_\varepsilon\left(\overline{\tau_i}\right)}{h_\varepsilon\left(\overline{\tau_i}\right)\gamma_i} + \mathrm{O}\left(\frac{\left|\partial_{x_1}h_\varepsilon\left(\overline{\tau_i}\right)\right|}{\overline{\gamma}_\varepsilon^3}\right),$$

which gives the second part of (3.12). This ends the proof of Claim 3.1.

References

[1] N. Aronszajn, *A unique continuation theorem for solutions of elliptic partial differential equations or inequalities of second order*, J. Math. Pures Appl. (9) **36** (1957), 235–249.

[2] S. Baraket and F. Pacard, *Construction of singular limits for a semilinear elliptic equation in dimension* 2, Calc. Var. Partial Differential Equations **6** (1998), no. 1, 1–38.

[3] C.-C. Chen and C.-S. Lin, *Topological degree for a mean field equation on Riemann surfaces*, Comm. Pure Appl. Math. **56** (2003), no. 12, 1667–1727.

[4] H. O. Cordes, *Über die eindeutige Bestimmtheit der Lösungen elliptischer Differentialgleichungen durch Anfangsvorgaben*, Nachr. Akad. Wiss. Göttingen. Math.-Phys. Kl. IIa. **1956** (1956), 239–258 (German).

[5] M. del Pino, M. Musso, and B. Ruf, *New solutions for Trudinger-Moser critical equations in* $\mathbb{R}^2$, J. Funct. Anal. **258** (2010), no. 2, 421–457.

[6] ———, *Beyond the Trudinger-Moser supremum*, Calc. Var. Partial Differential Equations **44** (2012), no. 3-4, 543–576.

[7] S. Deng, M. Musso, and J.-C. Wei, *Sign-changing blow-up solutions for Yamabe problem*, Int. Math. Res. Not. **13** (2019), 4159–4197.

[8] O. Druet and P.-D. Thizy, *Multi-bumps analysis for Trudinger-Moser nonlinearities I - quantification and location of concentration points*, J. Eur. Math. Soc. (JEMS) **22** (2020), no. 12, 4025–4096.

[9] O. Druet, A. Malchiodi, L. Martinazzi, and P.-D. Thizy, *Multi-bumps analysis for Trudinger-Moser nonlinearities II - Existence of solutions of high energies.* In preparation.

[10] P. Esposito, M. Grossi, and A. Pistoia, *On the existence of blowing-up solutions for a mean field equation*, Ann. Inst. H. Poincaré Anal. Non Linéaire **22** (2005), no. 2, 227–257.
[11] D. Gilbarg and N. S. Trudinger, *Elliptic partial differential equations of second order*, 2nd ed., Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 224, Springer-Verlag, Berlin, 1983.
[12] M. Grossi, G. Mancini, D. Naimen, and A. Pistoia, *Bubbling nodal solutions for a large perturbation of the Moser–Trudinger equation on planar domains*, Math. Ann. (2020). To appear.
[13] E. Hebey, *Compactness and stability for nonlinear elliptic equations*, Zurich Lectures in Advanced Mathematics, European Mathematical Society (EMS), Zürich, 2014.
[14] A. Malchiodi and L. Martinazzi, *Critical points of the Moser–Trudinger functional on a disk*, J. Eur. Math. Soc. (JEMS) **16** (2014), no. 5, 893–908.
[15] G. Mancini and L. Martinazzi, *The Moser-Trudinger inequality and its extremals on a disk via energy estimates*, Calc. Var. Partial Differential Equations **56** (2017), no. 4, Art. 94, 26 pp.
[16] G. Mancini and P.-D. Thizy, *Glueing a peak to a non-zero limiting profile for a critical Moser-Trudinger equation*, J. Math. Anal. Appl. **472** (2019), no. 15, 1430–1457.
[17] J. Moser, *A sharp form of an inequality by N. Trudinger*, Indiana Univ. Math. J. **20** (1970/71), 1077–1092.
[18] D. Naimen, *Concentration profile, energy, and weak limits of radial solutions to semilinear elliptic equations with Trudinger-Moser critical nonlinearities*, Calc. Var. Partial Differential Equations **60** (2021), Art. 66, 54 pp.
[19] O. Rey, *The role of the Green's function in a nonlinear elliptic equation involving the critical Sobolev exponent*, J. Funct. Anal. **89** (1990), no. 1, 1–52.
[20] F. Robert and J. Vétois, *Sign-Changing Blow-Up for Scalar Curvature Type Equations*, Comm. Partial Differential Equations **38** (2013), no. 8, 1437–1465.
[21] ———, *A general theorem for the construction of blowing-up solutions to some elliptic nonlinear equations via Lyapunov-Schmidt's reduction*, Concentration Analysis and Applications to PDE (ICTS Workshop, Bangalore, 2012), Trends in Mathematics, Springer, Basel, 2013, pp. 85–116.
[22] N. S. Trudinger, *On imbeddings into Orlicz spaces and some applications*, J. Math. Mech. **17** (1967), 473–483.

Luca Martinazzi, Università degli Studi di Padova, Dipartimento di Matematica Tullio Levi-Civita, Via Trieste, 63, 35121 Padova, Italy
Email address: luca.martinazzi@math.unipd.it

Pierre-Damien Thizy, Institut Camille Jordan, Université Claude Bernard Lyon 1, 21 avenue Claude Bernard, 69622 Villeurbanne Cedex, France
Email address: pierre-damien.thizy@univ-lyon1.fr

Jérôme Vétois, McGill University, Department of Mathematics and Statistics, 805 Sherbrooke Street West, Montreal, Quebec H3A 0B9, Canada
Email address: jerome.vetois@mcgill.ca