A PRIORI ESTIMATES AND APPLICATION TO THE SYMMETRY OF SOLUTIONS FOR CRITICAL p–LAPLACE EQUATIONS

JÉRÔME VÉTOIS

Abstract. We establish pointwise a priori estimates for solutions in $D^{1,p}(\mathbb{R}^n)$ of equations of type $-\Delta_p u = f(x,u)$, where $p \in (1,n)$, $\Delta_p := \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right)$ is the p–Laplace operator, and f is a Caratheodory function with critical Sobolev growth. In the case of positive solutions, our estimates allow us to extend previous radial symmetry results. In particular, by combining our results and a result of Damascelli–Ramaswamy [6], we are able to extend a recent result of Damascelli–Merchán–Montoro–Sciunzi [7] on the symmetry of positive solutions in $D^{1,p}(\mathbb{R}^n)$ of the equation $-\Delta_p u = u^{p^*-1}$, where $p^* := np/(n-p)$.

1. Introduction and main results

In this paper, we are interested in problems of the type

$$\begin{cases} -\Delta_p u = f(x,u) & \text{in } \mathbb{R}^n,\\ u \in D^{1,p}(\mathbb{R}^n), \end{cases} \tag{1.1}$$

where $p \in (1,n)$, $\Delta_p u := \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right)$, $D^{1,p}(\mathbb{R}^n)$ is the completion of $C_c^\infty(\mathbb{R}^n)$ with respect to the norm $\|u\|_{D^{1,p}(\mathbb{R}^n)} := \left(\int_{R^n}|\nabla u|^p\,dx\right)^{1/p}$, and $f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ is a Caratheodory function such that

$$|f(x,s)| \le \Lambda |s|^{p^*-1} \quad \text{for all } s \in \mathbb{R} \text{ and a.e. } x \in \mathbb{R}^n, \tag{1.2}$$

for some real number $\Lambda > 0$, with $p^* := np/(n-p)$.

Our main result is as follows.

Theorem 1.1. *Let $p \in (1,n)$, $f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ be a Caratheodory function such that (1.2) holds true and u be a solution of (1.1). Then there exists a constant $C_0 = C_0(n,p,\Lambda,u)$ such that*

$$|u(x)| \le C_0\big(1+|x|^{\frac{n-p}{p-1}}\big)^{-1} \quad \text{and} \quad |\nabla u(x)| \le C_0\big(1+|x|^{\frac{n-1}{p-1}}\big)^{-1} \tag{1.3}$$

for all $x \in \mathbb{R}^n$. If moreover $u \ge 0$ in $\mathbb{R}^n$ and $\int_{\mathbb{R}^n} f(x,u)\,dx > 0$, then we have

$$u(x) \ge C_1\big(1+|x|^{\frac{n-p}{p-1}}\big)^{-1} \tag{1.4}$$

for all $x \in \mathbb{R}^n$, for some constant $C_1 = C_1(n,p,\lambda,\Lambda,u) > 0$, where λ is a real number such that $0 < \lambda < \int_{\mathbb{R}^n} f(x,u)\,dx$.

Date: July 25, 2014. *Revised:* April 11, 2015.

Published in *Journal of Differential Equations* **260** (2016), no. 1, 149–161.

The dependence on u of the constants C_0 and C_1 will be made more precise in Remarks 4.1 and 4.3.

In the case of the Laplace operator ($p=2$), the upper bound estimates (1.3) have been established by Jannelli–Solimini [15] for nonlinearities of the form $f(x,u)=\sum_{i=1}^N a_i(x)\left|u\right|^{q_i^*-2}u$, where $q_i^*:=2^*(1-1/q_i)$, $q_i\in(n/2,\infty]$, $\left|a_i(x)\right|=\mathrm{O}\big(\left|x\right|^{-n/q_i}\big)$ for large $|x|$, and a_i belongs to the Marcinkiewicz space $M^{q_i}(\mathbb{R}^n)$ for all $i=1,\dots,N$. The case of unbounded domains $\Omega\neq\mathbb{R}^n$ is also treated in [15].

Since the pioneer work of Gidas–Ni–Nirenberg [12] and later extensions by Li [18] in case $p=2$ and Damascelli–Ramaswamy [6] in case $1<p<2$, decay estimates are known to be useful to derive radial symmetry results for C^1–solutions of problems of the type

$$\begin{cases} -\Delta_p u=f(u), \quad u>0 \quad \text{in } \mathbb{R}^n,\\ u(x)\longrightarrow 0 \quad \text{as } |x|\longrightarrow 0\,. \end{cases} \tag{1.5}$$

Here, we consider the following result of Damascelli–Ramaswamy [6] and Li [18]: if $1<p\le 2$, f is a locally Lipschitz continuous function in $(0,\infty)$ such that

$$\frac{f(v)-f(u)}{v-u}\le\Lambda\max(u^\alpha,v^\alpha)\quad\forall u,v\text{ such that }0<u<v<s_0 \tag{1.6}$$

for some real numbers $\Lambda,s_0>0$, and $\alpha>p-2$, and u is a C^1–solution of (1.5) such that

$$u(x)=\mathrm{O}\left(\left|x\right|^{-m}\right)\quad\text{and}\quad\left|\nabla u(x)\right|=\mathrm{O}\left(\left|x\right|^{-m-1}\right) \tag{1.7}$$

$$(\text{and } u(x)\ge C\left|x\right|^{-m}\ \text{ for large } |x| \text{ when } \alpha<0) \tag{1.8}$$

for some real numbers $C>0$ and $m>p/(\alpha+2-p)$, then u is radially symmetric and strictly radially decreasing about some point $x_0\in\mathbb{R}^n$, i.e. there exists $v\in C^1(0,\infty)$ such that $v'(r)<0$ for all $r>0$ and $u(x)=v(|x-x_0|)$ for all $x\in\mathbb{R}^n$. We also mention that other symmetry results for problems of type (1.5) have been established without any decay assumption in the case where f is nonincreasing near 0 (see Gidas–Ni–Nirenberg [12], Li [18], and Li–Ni [19] in case $p=2$, Damascelli–Pacella–Ramaswamy [5], Damascelli–Ramaswamy [6], and Serrin–Zou [26] in case $p\neq 2$).

In case $\alpha=p^*-2$, the conditions (1.7)–(1.8) follow from (1.3)–(1.4) with $m=(n-p)/(p-1)$ (which is greater than $p/(\alpha+2-p)=(n-p)/p$). Consequently, by combining Theorem 1.1, the results of Damascelli–Ramaswamy [6] and Li [18], and the regularity results that are referred to in Lemma 2.1 below, we obtain the following corollary.

Corollary 1.2. *Assume that* $1<p\le 2$. *Let* f *be a locally Lipschitz continuous function in* $(0,\infty)$ *such that* (1.2) *and* (1.6) *hold true with* $\alpha=p^*-2$. *Then any nonnegative solution of* (1.1) *is radially symmetric and strictly radially decreasing about some point* $x_0\in\mathbb{R}^n$.

Let us now comment on the positive solutions of the equation with pure power nonlinearity, namely

$$-\Delta_p u = u^{p^*-1}, \quad u > 0 \quad \text{in } \mathbb{R}^n. \tag{1.9}$$

Guedda–Véron [14] proved that the only positive, radially symmetric solutions of (1.9) are of the form

$$u_{\mu,x_0}(x) = (n\mu)^{\frac{n-p}{p^2}} \left(\frac{n-p}{p-1}\right)^{\frac{(n-p)(p-1)}{p^2}} \left(\mu + |x-x_0|^{\frac{p}{p-1}}\right)^{\frac{p-n}{p}} \tag{1.10}$$

for all $x \in \mathbb{R}^n$, for some real number $\mu > 0$ and point $x_0 \in \mathbb{R}^n$. In case $p = 2$, Caffarelli–Gidas–Spruck [2] (see also Chen–Li [3]) proved that the functions (1.10) are the only positive solutions of (1.9). In a recent paper, Damascelli–Merchán–Montoro–Sciunzi [7] proved that any solution in $D^{1,p}(\mathbb{R}^n)$ of (1.9) is radially symmetric provided that $2n/(n+2) \le p < 2$. The condition $p \ge 2n/(n+2)$ corresponds to the values of p for which the function $s \mapsto s^{p^*-1}$ is Lipschitz continuous near 0. With the above Corollary 1.2, we extend the result of Damascelli–Merchán–Montoro–Sciunzi [7] to the whole interval $1 < p < 2$. By combining the result of Guedda–Véron [14] and Corollary 1.2, we obtain the following corollary.

Corollary 1.3. *Assume that* $1 < p < 2$. *Then the functions* (1.10) *are the only positive solutions in* $D^{1,p}(\mathbb{R}^n)$ *of* (1.9).

As a motivation to our results, it is well known that the profile of solutions of the equation

$$-\Delta_p u = |u|^{p^*-2} u \quad \text{in } \mathbb{R}^n \tag{1.11}$$

plays a central role in the blow-up theories of critical equations. Possible references in book form on this subject and its applications in case $p = 2$ are Druet–Hebey–Robert [9], Ghoussoub [11], and Struwe [28]. In case $p \ne 2$, global compactness results in energy spaces in the spirit of Struwe [27] have been established in different contexts by Alves [1] for equations posed in the whole $\mathbb{R}^n$, Saintier [23] in the case of a smooth, compact manifold, and Mercuri–Willem [20] and Yan [34] in the case of a smooth, bounded domain. In view of these results, it is likely that the new information provided by Theorem 1.1 and Corollary 1.2 on the solutions of (1.11) will lead to new existence and multiplicity results as it is the case for $p = 2$.

The paper is organized as follows. Section 2 is concerned with global boundedness results. The key result in this section is a global bound in weak Lebesgue spaces which we obtain by arguments of measure theory. In Section 3, we establish a preliminary decay estimate which is not sharp but which turns out to be a crucial ingredient in what follows. To prove this estimate, we exploit the scaling law of the equation, and we apply a doubling property from Poláčik–Quittner–Souplet [22]. In Section 4, we conclude the proof of Theorem 1.1. The proof of the upper bound estimates (1.3) follows from the results of Sections 2 and 3 together with Harnack-type inequalities of Serrin [25] and Trudinger [30]. The proof of the lower bound estimate (1.4) relies on a Harnack inequality on annuli, which is inspired from similar results used in Friedman–Véron [10] and Véron [33] for the study of singular solutions of p–Laplace equations in pointed domains.

Note. Since this paper was written, the result of Corollary 1.3 has been extended to all $p \in (1,n)$ by Sciunzi [24]. The proof in [24] is based on the moving plane method. It uses the estimates of Theorem 1.1 together with a sharp lower bound estimate for the norm of the gradient of the solutions.

Acknowledgments. The author wishes to express his gratitude to Emmanuel Hebey for helpful comments on the manuscript.

2. Global boundedness results

The first result of this section refers to some known regularity results for critical equations.

Lemma 2.1. *Let* $f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ *be a Caratheodory function such that* (1.2) *holds true. Then any solution of* (1.1) *belongs to* $W^{1,\infty}(\mathbb{R}^n) \cap C^{1,\theta}_{\text{loc}}(\mathbb{R}^n)$ *for some* $\theta \in (0,1)$.

Proof of Lemma 2.1. A straightforward adaptation of Peral [21, Theorem E.0.20] (which in turn is adapted from Trudinger [31, Theorem 3]) yields that for any solution u of (1.1), there exist constants $C, R > 0$ and $\beta > 1$ such that $\|u\|_{L^{\beta p^*}(B(x,R))} \le C$ for all $x \in \mathbb{R}^n$, where $B(x,R)$ is the Euclidean ball of center x and radius R. We then obtain a global L^∞–bound by applying Serrin [25, Theorem 1].

Once we have the L^∞–boundedness of the solutions, the results of DiBenedetto [8] and Tolksdorf [29] provide global L^∞–bounds and local Hölder regularity for the derivatives. ☐

The next result is concerned with the boundedness of solutions of (1.1) in weak Lebesgue spaces. For any $s \in (0,\infty)$ and any domain $\Omega \subset \mathbb{R}^n$, we define $L^{s,\infty}(\Omega)$ as the set of all measurable functions $u : \Omega \to \mathbb{R}$ such that

$$\|u\|_{L^{s,\infty}(\Omega)} := \sup_{h>0} \left(h \cdot \operatorname{meas}(\{|u| > h\})^{1/s} \right) < \infty\,,$$

where $\operatorname{meas}(\{|u| > h\})$ is the measure of the set $\{x \in \Omega : |u(x)| > h\}$. The map $\|\cdot\|_{L^{s,\infty}(\Omega)}$ defines a quasi-norm on $L^{s,\infty}(\Omega)$ (see for instance Grafakos [13]).

Our result is as follows.

Lemma 2.2. *Let* $f:\mathbb{R}^n\times\mathbb{R}\to\mathbb{R}$ *be a Caratheodory function such that* (1.2) *holds true. Then any solution of* (1.1) *belongs to* $L^{p_*-1,\infty}(\mathbb{R}^n)$*, where* $p_*:=p(n-1)/(n-p)$*. Hence, by interpolation (see for instance Grafakos* [13, Proposition 1.1.14]*), since by Lemma 2.1 any solution of* (1.1) *belongs to* $L^\infty(\mathbb{R}^n)$*, we obtain that the solutions belong to* $L^s(\mathbb{R}^n)$ *for all* $s\in(p_*-1,\infty]$*.*

Proof of Lemma 2.2. We let u be a nontrivial solution of (1.1). For any $h>0$, by testing (1.1) with $T_h(u):=\operatorname{sgn}(u)\cdot\min(|u|,h)$, where $\operatorname{sgn}(u)$ denotes the sign of u, we obtain

$$\int_{|u|\le h}|\nabla u|^p\,dx=\int_{|u|\le h}f(x,u)\cdot u\,dx+h\int_{|u|>h}f(x,u)\cdot\operatorname{sgn}(u)\,dx\,. \tag{2.1}$$

It follows from (1.2) and (2.1)that

$$\int_{|u|\le h}|\nabla u|^p\,dx\le\Lambda\left(\int_{|u|\le h}|u|^{p^*}\,dx+h\int_{|u|>h}|u|^{p^*-1}\,dx\right). \tag{2.2}$$

We then write

$$\int_{|u|\le h}|u|^{p^*}\,dx=\int_{\mathbb{R}^n}|T_h(u)|^{p^*}\,dx-h^{p^*}\operatorname{meas}(\{|u|>h\}) \tag{2.3}$$

and

$$\begin{aligned}\int_{|u|>h}|u|^{p^*-1}\,dx&=(p^*-1)\int_0^\infty s^{p^*-2}\operatorname{meas}(\{|u|>\max(s,h)\})\,ds\\&=h^{p^*-1}\operatorname{meas}(\{|u|>h\})+(p^*-1)\int_h^\infty s^{p^*-2}\operatorname{meas}(\{|u|>s\})\,ds\,.\end{aligned} \tag{2.4}$$

It follows from (2.2)–(2.4) that

$$\begin{aligned}\int_{|u|\le h}|\nabla u|^p\,dx\le\Lambda\Bigg(&\int_{\mathbb{R}^n}|T_h(u)|^{p^*}\,dx\\&+(p^*-1)\,h\int_h^\infty s^{p^*-2}\operatorname{meas}(\{|u|>s\})\,ds\Bigg).\end{aligned} \tag{2.5}$$

Sobolev inequality gives

$$\int_{\mathbb{R}^n}|T_h(u)|^{p^*}\,dx\le K\left(\int_{|u|\le h}|\nabla u|^p\,dx\right)^{\frac{n}{n-p}} \tag{2.6}$$

for some constant $K=K(n,p)$. By (2.3), (2.5), (2.6), and since $\int_{\mathbb{R}^n}|T_h(u)|^{p^*}\,dx=\mathrm{o}(1)$ as $h\to0$, we obtain

$$\begin{aligned}h^{p^*}\operatorname{meas}(\{|u|>h\})&\le\int_{\mathbb{R}^n}|T_h(u)|^{p^*}\,dx\\&\le C\left(h\int_h^\infty s^{p^*-2}\operatorname{meas}(\{|u|>s\})\,ds\right)^{\frac{n}{n-p}}\end{aligned} \tag{2.7}$$

for small h, for some constant $C = C(n,p,\Lambda)$. We then define

$$G(h) := \left(\int_h^\infty g(s)\,ds\right)^{\frac{-p}{n-p}}, \quad \text{where } g(s) := s^{p^*-2}\operatorname{meas}(\{|u| > s\}).$$

Since the function $t \mapsto t^{-p/(n-p)}$ is locally Lipschitz in $(0,\infty)$ and $\int_h^\infty g(s)\,ds > 0$ for all $h < \|u\|_{L^\infty(\mathbb{R}^n)}$, we get that G is locally absolutely continuous in $\left(0, \|u\|_{L^\infty(\mathbb{R}^n)}\right)$ with derivative

$$G'(h) = \frac{p}{n-p}\left(\int_h^\infty g(s)\,ds\right)^{\frac{-n}{n-p}} g(h) \tag{2.8}$$

for a.e. $h \in \left(0, \|u\|_{L^\infty(\mathbb{R}^n)}\right)$ (see for instance Leoni [17, Theorem 3.68]). By (2.7) and (2.8), we obtain

$$G'(h) \le C \cdot \frac{p}{n-p} \cdot h^{\frac{2p-n}{n-p}} \tag{2.9}$$

for small h. Integrating (2.9) gives

$$G(h) - G(0) \le C \cdot h^{\frac{p}{n-p}} \tag{2.10}$$

for small h, where $G(0) := \lim_{h\to 0} G(h)$. On the other hand, by (2.4) and dominated convergence, we have

$$(p^*-1)\,hG(h)^{\frac{p-n}{p}} \le h\int_{|u|>h} |u|^{p^*-1}\,dx = \mathrm{o}(1) \tag{2.11}$$

as $h \to 0$. It follows from (2.10) and (2.11) that $G(0) > 0$, i.e. $\int_0^\infty g(s)\,ds < \infty$. By (2.7) and since $p^* - \frac{n}{n-p} = p_* - 1$ and G is nonincreasing, we then get

$$h^{p_*-1}\operatorname{meas}(\{|u| > h\}) \le C \cdot G(h)^{-n/p} \le C \cdot G(0)^{-n/p}$$

for small h, and hence we obtain $\|u\|_{L^{p_*-1,\infty}(\mathbb{R}^n)} < \infty$. □

By (1.2) and a weak version of Kato's inequality [16] (see Cuesta Leon [4, Proposition 3.2]), we obtain

$$-\Delta_p |u| \le |f(x,u)| \le \Lambda |u|^{p^*-1} \quad \text{in } \mathbb{R}^n, \tag{2.12}$$

where the inequality is in the sense that

$$\int_{\mathbb{R}^n} |\nabla |u||^{p-2}\,\nabla |u| \cdot \nabla\varphi\,dx \le \Lambda \int_{\mathbb{R}^n} |u|^{p^*-1}\varphi\,dx$$

for all nonnegative, smooth functions φ with compact support in $\mathbb{R}^n$.

Our last result in this section is as follows.

Lemma 2.3. *For any real number $\Lambda > 0$ and any nonnegative, nontrivial solution $v \in D^{1,p}(\mathbb{R}^n)$ of the inequality $-\Delta_p v \le \Lambda v^{p^*-1}$ in $\mathbb{R}^n$, we have $\|v\|_{L^{p^*}(\mathbb{R}^n)} \ge \kappa_0$ for some constant $\kappa_0 = \kappa_0(n,p,\Lambda) > 0$.*

Proof. By applying Sobolev inequality and testing $-\Delta_p v \le \Lambda v^{p^*-1}$ with the function v, we obtain

$$\int_{\mathbb{R}^n} v^{p^*} dx \le K \left(\int_{\mathbb{R}^n} |\nabla v|^p \, dx\right)^{\frac{n}{n-p}} \le K \left(\Lambda \int_{\mathbb{R}^n} v^{p^*} dx\right)^{\frac{n}{n-p}} \tag{2.13}$$

for some constant $K = K(n,p)$. The result then follows immediately from (2.13). ☐

3. A preliminary decay estimate

The following result provides a decay estimate which is not sharp but which will serve as a preliminary step in the proof of Theorem 1.1.

Lemma 3.1. *Let* κ_0 *be as in Lemma 2.3,* $f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ *be a Caratheodory function such that* (1.2) *holds true, and* u *be a solution of* (1.1). *For any* $\kappa > 0$, *we define*

$$r_\kappa(u) := \inf\left(\left\{r > 0 \,:\, \|u\|_{L^{p^*}(\mathbb{R}^n \backslash B(0,r))} < \kappa\right\}\right),$$

where $B(0,r)$ *is the Euclidean ball of center 0 and radius* r. *Then for any* $\kappa \in (0,\kappa_0)$ *and* $r > r_\kappa(u)$, *there exists a constant* $K_0 = K_0\big(n,p,\Lambda,\kappa,r,r_\kappa(u), \|u\|_{L^{p^*}(\mathbb{R}^n)}\big)$ *such that*

$$|u(x)| \le K_0 |x|^{\frac{p-n}{p}} \quad \text{for all } x \in \mathbb{R}^n \backslash B(0,r). \tag{3.1}$$

The proof of Lemma 3.1 relies on scaling arguments and the following doubling property from Poláčik–Quittner–Souplet [22].

Lemma 3.2. *Let* (X, dist) *be a complete metric space,* D *and* Σ *be two subsets of* X *such that* $D \ne \emptyset$, $D \subset \Sigma$, *and* Σ *is closed. Let* M *be a nonnegative function on* D *which is bounded on compact subsets of* D. *Then for any point* x_0 *in* D *and any positive real number* α_0 *such that*

$$\mathrm{dist}(x_0, \Sigma\backslash D)\, M(x_0) > 2\alpha_0,$$

there exists a point y_0 *in* D *such that*

$$\mathrm{dist}(y_0, \Sigma\backslash D)\, M(y_0) > 2\alpha_0, \quad M(x_0) \le M(y_0), \tag{3.2}$$

and

$$M(y) \le 2M(y_0) \quad \text{for all } y \in D \cap \overline{B_X(y_0, \alpha_0/M(y_0))}, \tag{3.3}$$

where $B_X(y_0, \alpha_0/M(y_0))$ *is the ball of center* y_0 *and radius* $\alpha_0/M(y_0)$ *with respect to the distance* dist. *In the case where* $X = \mathbb{R}^n$, dist *is the Euclidean distance,* D *is open, and* $\Sigma = \overline{D}$, *it follows from the first inequality in* (3.2) *that* $\overline{B_X(y_0, \alpha_0/M(y_0))} \subset D$, *and hence* (3.3) *holds true for all* $y \in \overline{B_X(y_0, \alpha_0/M(y_0))}$.

We refer to [22] for the proof of Lemma 3.2. Now we prove Lemma 3.1.

Proof of Lemma 3.1. We fix $\Lambda > 0$, $\kappa \in (0, \kappa_0)$, $\kappa' > \kappa_0$, $r > 0$, and $r' \in (0, r)$. As is easily seen, in order to prove Lemma 3.1, it is sufficient to prove that there exists a constant $K_1 = K_1 (n, p, \Lambda, \kappa, \kappa', r, r')$ such that for any solution u of (1.1) such that $r_\kappa (u) \leq r'$ and $\|u\|_{L^{p^*}(\mathbb{R}^n)} \leq \kappa'$, we have

$$\operatorname{dist}(x, B(0, r''))\, |u(x)|^{\frac{p}{n-p}} \leq K_1 \quad \text{for all } x \in \mathbb{R}^n \backslash B(0, r), \tag{3.4}$$

where $r'' := (r + r')/2$ and dist is the Euclidean distance function.

We prove (3.4) by contradiction. Suppose that for any $\alpha \in \mathbb{N}$, there exists a Caratheodory function $f_\alpha : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ such that (1.2) holds true, a solution u_α of (1.1) with $f = f_\alpha$ such that $r_\kappa (u_\alpha) \leq r'$ and $\|u_\alpha\|_{L^{p^*}(\mathbb{R}^n)} \leq \kappa'$, and a point $x_\alpha \in \mathbb{R}^n \backslash B(0, r)$ such that

$$\operatorname{dist}(x_\alpha, B(0, r''))\, |u_\alpha (x_\alpha)|^{\frac{p}{n-p}} > 2\alpha. \tag{3.5}$$

By (3.5) and Lemma 3.2, and since $B(0, r'') \subset B(0, r)$, we get that there exists a point $y_\alpha \in \mathbb{R}^n \backslash B(0, r'')$ such that

$$\operatorname{dist}(y_\alpha, B(0, r''))\, |u_\alpha (y_\alpha)|^{\frac{p}{n-p}} > 2\alpha, \quad |u_\alpha (x_\alpha)| \leq |u_\alpha (y_\alpha)|, \tag{3.6}$$

and

$$|u_\alpha (y)| \leq 2^{\frac{n-p}{p}} |u_\alpha (y_\alpha)| \quad \text{for all } y \in B\big(y_\alpha, \alpha\, |u_\alpha (y_\alpha)|^{\frac{-p}{n-p}}\big). \tag{3.7}$$

For any α and $y \in \mathbb{R}^n$, we define

$$\widetilde{u}_\alpha (y) := \mu_\alpha \cdot u_\alpha\big(\mu_\alpha^{\frac{p}{n-p}} \cdot y + y_\alpha\big), \tag{3.8}$$

where $\mu_\alpha := |u_\alpha (y_\alpha)|^{-1}$. By (1.1), we obtain

$$-\Delta_p \widetilde{u}_\alpha = \mu_\alpha^{p^*-1} \cdot f_\alpha\big(\mu_\alpha^{\frac{p}{n-p}} \cdot y + y_\alpha, \mu_\alpha^{-1} \cdot \widetilde{u}_\alpha\big) \quad \text{in } \mathbb{R}^n. \tag{3.9}$$

It follows from (1.2) that

$$\big|\mu_\alpha^{p^*-1} \cdot f_\alpha\big(\mu_\alpha^{\frac{p}{n-p}} \cdot y + y_\alpha, \mu_\alpha^{-1} \cdot \widetilde{u}_\alpha\big)\big| \leq \Lambda\, |\widetilde{u}_\alpha|^{p^*-1} \quad \text{in } \mathbb{R}^n. \tag{3.10}$$

Moreover, by (3.7) and (3.8), we obtain

$$|\widetilde{u}_\alpha (0)| = 1 \quad \text{and} \quad |\widetilde{u}_\alpha (y)| \leq 2^{\frac{n-p}{p}} \quad \text{for all } y \in B(0, \alpha). \tag{3.11}$$

By DiBenedetto [8] and Tolksdorf [29], it follows from (3.10) and (3.11) that there exists a constant $C > 0$ and a real number $\theta \in (0, 1)$ such that for point $x \in \mathbb{R}^n$, we have

$$\|\widetilde{u}_\alpha\|_{C^{1,\theta}(B(x,1))} \leq C \tag{3.12}$$

for large α. By compactness of $C^{1,\theta}(B(x, 1)) \hookrightarrow C^1(B(x, 1))$, it follows from (3.12) that $(\widetilde{u}_\alpha)_\alpha$ converges up to a subsequence in $C^1_{\text{loc}}(\mathbb{R}^n)$ to some function $\widetilde{u}_\infty$. By (3.11), we obtain $|\widetilde{u}_\infty (0)| = 1$. Moreover, by applying the inequality (2.12), we obtain

$$\int_{\mathbb{R}^n} |\nabla |\widetilde{u}_\alpha||^p \, dx = \int_{\mathbb{R}^n} |\nabla |u_\alpha||^p \, dx \leq \Lambda \int_{\mathbb{R}^n} |u_\alpha|^{p^*} dx \leq \Lambda (\kappa')^{p^*},$$

and hence $|\widetilde{u}_\infty| \in D^{1,p}(\mathbb{R}^n)$. By observing that the inequality (2.12) is invariant by the change of scale (3.8), we then get that $|\widetilde{u}_\infty|$ is a weak solution of

$$-\Delta_p |\widetilde{u}_\infty| \le \Lambda |\widetilde{u}_\infty|^{p^*-1} \quad \text{in } \mathbb{R}^n. \tag{3.13}$$

On the other hand, for any $R > 0$, we have

$$\|\widetilde{u}_\alpha\|_{L^{p^*}(B(0,R))} = \|u_\alpha\|_{L^{p^*}(B(y_\alpha, R\mu_\alpha^{\frac{p}{n-p}}))}. \tag{3.14}$$

By (3.6) and since $r_\kappa(u_\alpha) < r''$, we get

$$B\big(y_\alpha, R\mu_\alpha^{\frac{p}{n-p}}\big) \cap B(0, r_\kappa(u_\alpha)) = \emptyset \tag{3.15}$$

for large α. By (3.14), (3.15), and by definition of $r_\kappa(u_\alpha)$, we obtain

$$\|\widetilde{u}_\alpha\|_{L^{p^*}(B(0,R))} \le \kappa \tag{3.16}$$

for large α. Passing to the limit into (3.16) as $\alpha \to \infty$ and then as $R \to \infty$ yields

$$\|\widetilde{u}_\infty\|_{L^{p^*}(\mathbb{R}^n)} \le \kappa\,. \tag{3.17}$$

Since $\kappa < \kappa_0$, by Lemma 2.3, (3.13), and (3.17), we get that $\widetilde{u}_\infty \equiv 0$, which is in contradiction with $|\widetilde{u}_\infty(0)| = 1$. This ends the proof of Lemma 3.1. □

4. Proof of Theorem 1.1

We can now prove Theorem 1.1 by applying Lemmas 2.2, 3.1, and Harnack-type inequalities of Serrin [25] and Trudinger [30].

Proof of (1.3). We let u be a solution of (1.1). We let κ and r be as in Lemma 3.1. For any $R > 0$ and $y \in \mathbb{R}^n$, we define

$$u_R(y) := R^{\frac{n-p}{p-1}} \cdot u\big(R \cdot y\big). \tag{4.1}$$

By (1.1), we obtain

$$-\Delta_p u_R = R^n \cdot f\big(R \cdot y, R^{\frac{p-n}{p-1}} \cdot u_R\big) \quad \text{in } \mathbb{R}^n. \tag{4.2}$$

It follows from (1.2) that

$$\big|R^n \cdot f\big(R \cdot y, R^{\frac{p-n}{p-1}} \cdot u_R\big)\big| \le \Lambda \cdot R^{\frac{-p}{p-1}} \cdot |u_R|^{p^*-1} \quad \text{in } \mathbb{R}^n. \tag{4.3}$$

Moreover, similarly to (2.12), it follows from (4.2) and (4.3) that $|u_R|$ is a weak solution of

$$-\Delta_p |u_R| \le \Lambda \cdot R^{\frac{-p}{p-1}} \cdot |u_R|^{p^*-1} \quad \text{in } \mathbb{R}^n. \tag{4.4}$$

By writing $|u_R|^{p^*-1} = |u_R|^{p^*-p} \cdot |u_R|^{p-1}$ and applying Lemma 3.1, we obtain

$$R^{\frac{-p}{p-1}} \cdot |u_R|^{p^*-1} \le K_0^{p^*-p} |u_R|^{p-1} \quad \text{in } \mathbb{R}^n \backslash B(0,1) \tag{4.5}$$

provided that $R \ge r$. It follows from (4.4), (4.5), and Trudinger [30, Theorem 1.3] that for any $\varepsilon > 0$, we have

$$\|u_R\|_{L^\infty(B(0,2)\backslash B(0,4))} \le c_\varepsilon \|u_R\|_{L^{p-1+\varepsilon}(B(0,5)\backslash B(0,1))}\,. \tag{4.6}$$

for some constant $c_\varepsilon = c\left(n, p, \Lambda, K_0, \varepsilon\right)$. We fix $\varepsilon_0 = \varepsilon_0\left(n, p\right)$ such that $0 < \varepsilon_0 < p_* - p$, where p_* is as in Lemma 2.2. By a generalized version of Hölder's inequality (see for instance Grafakos [13, Exercise 1.1.11]), we obtain that there exists a constant $c_0 = c_0\left(n, p\right)$ such that

$$\left\|u_R\right\|_{L^{p-1+\varepsilon_0}(B(0,5)\backslash B(0,1))} \le c_0 \left\|u_R\right\|_{L^{p_*-1,\infty}(B(0,5)\backslash B(0,1))} . \tag{4.7}$$

By observing that the quasi-norm $\left\|\cdot\right\|_{L^{p_*-1,\infty}(R^n)}$ is left invariant by the change of scale (4.1), we deduce from (4.6), (4.7), and Lemma 2.2 that

$$\left\|u_R\right\|_{L^\infty(B(0,2)\backslash B(0,4))} \le c_1 \tag{4.8}$$

for some constant $c_1 = c_1\big(n, p, \Lambda, K_0, \left\|u\right\|_{L^{p_*-1,\infty}(\mathbb{R}^n)}\big)$. By (4.2)–(4.5), (4.8), and the estimates of DiBenedetto [8] and Tolksdorf [29], we get

$$\left\|\nabla u_R\right\|_{L^\infty(B(0,5/2)\backslash B(0,7/2))} \le c_2 . \tag{4.9}$$

for some constant $c_2 = c_2\big(n, p, \Lambda, K_0, \left\|u\right\|_{L^{p_*-1,\infty}(\mathbb{R}^n)}\big)$. Finally, for any $x \in \mathbb{R}^n \backslash B\left(0, 3r\right)$, by applying (4.8) and (4.9) with $R = \left|x\right|/3$, we obtain

$$\left|u\left(x\right)\right| \le c_3 \left|x\right|^{\frac{p-n}{p-1}} \quad \text{and} \quad \left|\nabla u\left(x\right)\right| \le c_3 \left|x\right|^{\frac{1-n}{p-1}} \tag{4.10}$$

for some constant $c_3 = c_3\big(n, p, \Lambda, K_0, \left\|u\right\|_{L^{p_*-1,\infty}(\mathbb{R}^n)}\big)$. Since on the other hand u and ∇u are uniformly bounded in $B\left(0, 3r\right)$, we can deduce (1.3) from (4.10). ☐

Remark 4.1. *As one can see from the above proof, the constant* C_0 *in* (1.3) *depends on* n, p, Λ, κ, r, $r_\kappa\left(u\right)$, $\left\|u\right\|_{L^{p_*-1,\infty}(\mathbb{R}^n)}$, $\left\|u\right\|_{L^{p^*}(\mathbb{R}^n)}$, *and* $\left\|u\right\|_{W^{1,\infty}(B(0,3r))}$.

In order to prove the lower bound estimate (1.4), we need the following Harnack inequality on annuli. This result is inspired from Friedman–Véron [10] and Véron [33] where similar results are used for the study of singular solutions of p–Laplace equations in pointed domains.

Lemma 4.2. *Let* $f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ *be a Caratheodory function such that* (1.2) *holds true,* u *be a nonnegative solution of* (1.1), κ *and* r *be as in Lemma 3.1, and* K_0 *be the constant given by Lemma 3.1. Then there exists a constant* $c_4 = c_4\left(n, p, \Lambda, K_0\right)$ *such that*

$$\sup_{2R<|x|<5R} \left(u\left(x\right)\right) \le c_4 \cdot \inf_{2R<|x|<5R} \left(u\left(x\right)\right) \tag{4.11}$$

for all $R \ge r$.

Proof of Lemma 4.2. For any $R > 0$, we define u_R as in (4.1). By (4.2), (4.3), (4.5), and Serrin [25, Theorem 5], we obtain that there exists a constant $c = c\left(n, p, \Lambda, K_0\right)$ such that

$$\sup_{z\in B(y,1/3)} \left(u_R\left(z\right)\right) \le c \cdot \inf_{z\in B(y,1/3)} \left(u_R\left(z\right)\right) \tag{4.12}$$

for all points y in the annulus $A := B(0,5)\backslash B(0,2)$. Moreover, we can join every two points in A by 17 connected balls of radius 1/3 and centers in A. Hence (4.11) follows from (4.12) with $c_4 := c^{17}$. ☐

We can now prove (1.4) by applying Lemma 4.2.

Proof of (1.4). We let u be a nonnegative solution of (1.1) such that $\int_{\mathbb{R}^n} f(x,u)\,dx > 0$. In particular, in view of (1.2), we have $u \not\equiv 0$, and hence $u > 0$ in $\mathbb{R}^n$ by the strong maximum principle of Vázquez [32]. By Lemma 4.2, we then get that in order to prove (1.4), it is sufficient to obtain a lower bound estimate of $\|u\|_{L^\infty(B(0,5R)\backslash B(0,2R))}$ for large R.

By (4.2), (4.3), (4.5), and Serrin [25, Theorem 1], we obtain

$$\begin{aligned}\|\nabla u_R\|_{L^p(B(0,4)\backslash B(0,3))} &\le c_5 \|u_R\|_{L^p(B(0,5)\backslash B(0,2))}\\ &\le c_5' \|u_R\|_{L^\infty(B(0,5)\backslash B(0,2))}\end{aligned} \tag{4.13}$$

for some constants c_5 and c_5' depending only on n, p, Λ, and K_0, where u_R is as in (4.1). By changing the scale of (4.13), we then get

$$\|\nabla u\|_{L^p(B(0,4R)\backslash B(0,3R))} \le c_5' R^{\frac{n-p}{p}} \|u\|_{L^\infty(B(0,5R)\backslash B(0,2R))}\,. \tag{4.14}$$

Next, we claim that if $\int_{\mathbb{R}^n} f(x,u)\,dx > \lambda$ for some real number $\lambda > 0$, then we have

$$\|\nabla u\|_{L^p(B(0,4R)\backslash B(0,3R))} \ge c_6 R^{\frac{p-n}{p(p-1)}} \tag{4.15}$$

for large R, for some constant $c_6 = c_6(n,p,\lambda) > 0$. For any $x \in \mathbb{R}^n$ and $R > 0$, we define $\chi_R(x) := \chi(|x|/R)$, where $\chi \in C^1(0,\infty)$ is a cutoff function such that $\chi \equiv 1$ on $[0,3]$, $\chi \equiv 0$ on $[4,\infty)$, $0 \le \eta \le 1$ and $|\eta'| \le 2$ on $(3,4)$. By testing (1.1) with χ_R and applying Hölder's inequality, we obtain

$$\begin{aligned}\int_{\mathbb{R}^n} f(x,u)\,\chi_R\,dx &= \int_{\mathbb{R}^n} |\nabla u|^{p-2}\,\nabla u \cdot \nabla \chi_R\,dx\\ &\le \|\nabla u\|^{p-1}_{L^p(\operatorname{supp}(\nabla\chi_R))} \cdot \|\nabla \chi_R\|_{L^p(\operatorname{supp}(\nabla\chi_R))}\,,\end{aligned} \tag{4.16}$$

where $\operatorname{supp}(\chi_R)$ denotes the support of χ_R. It follows from (4.16) and the definition of χ_R that

$$\int_{\mathbb{R}^n} f(x,u)\,\chi_R\,dx \le C R^{\frac{n-p}{p}} \|\nabla u\|^{p-1}_{L^p(B(0,4R)\backslash B(0,3R))} \tag{4.17}$$

for some constant $C = C(n,p) > 0$. Then (4.15) follows from (4.17) with $c_6 := (\lambda/C)^{\frac{1}{p-1}}$

Finally, we deduce (1.4) from (4.11), (4.14), and (4.15). ☐

Remark 4.3. *As one can see from the above proof, the constant* C_1 *in* (1.4) *depends on* n, p, λ, Λ, κ, r, $r_\kappa(u)$, $\|u\|_{L^{p^*}(\mathbb{R}^n)}$, *and a lower bound for* u *on the ball* $B(0, 2\max(r, R_{\lambda,f}(u)))$, *where*

$$R_{\lambda,f}(u) := \inf\left(\left\{R > 0 : \int_{\mathbb{R}^n} f(x,u)\,\chi_{R'}dx > \lambda\,, \quad \forall R' > R\right\}\right).$$

References

[1] C. O. Alves, *Existence of positive solutions for a problem with lack of compactness involving the p-Laplacian*, Nonlinear Anal. **51** (2002), no. 7, 1187–1206.

[2] L. A. Caffarelli, B. Gidas, and J. Spruck, *Asymptotic symmetry and local behavior of semilinear elliptic equations with critical Sobolev growth*, Comm. Pure Appl. Math. **42** (1989), no. 3, 271–297.

[3] W. Chen and C. Li, *Classification of solutions of some nonlinear elliptic equations*, Duke Math. J. **63** (1991), no. 3, 615–622.

[4] M. C. Cuesta Leon, *Existence results for quasilinear problems via ordered sub- and supersolutions*, Ann. Fac. Sci. Toulouse Math. (6) **6** (1997), no. 4, 591–608.

[5] L. Damascelli, F. Pacella, and M. Ramaswamy, *Symmetry of ground states of p-Laplace equations via the moving plane method*, Arch. Ration. Mech. Anal. **148** (1999), no. 4, 291–308.

[6] L. Damascelli and M. Ramaswamy, *Symmetry of C^1 solutions of p-Laplace equations in $\mathbb{R}^N$*, Adv. Nonlinear Stud. **1** (2001), no. 1, 40–64.

[7] L. Damascelli, S. Merchán, L. Montoro, and B. Sciunzi, *Radial symmetry and applications for a problem involving the $-\Delta_p(\cdot)$ operator and critical nonlinearity in $\mathbb{R}^n$*, Adv. Math. **265** (2014), no. 10, 313–335.

[8] E. DiBenedetto, *$C^{1+\alpha}$ local regularity of weak solutions of degenerate elliptic equations*, Nonlinear Anal. **7** (1983), no. 8, 827–850.

[9] O. Druet, E. Hebey, and F. Robert, *Blow-up theory for elliptic PDEs in Riemannian geometry*, Mathematical Notes, vol. 45, Princeton University Press, 2004.

[10] A. Friedman and L. Véron, *Singular solutions of some quasilinear elliptic equations*, Arch. Rational Mech. Anal. **96** (1986), no. 4, 359–387.

[11] N. Ghoussoub, *Duality and perturbation methods in critical point theory*, Cambridge Tracts in Mathematics, vol. 107, Cambridge University Press, 1993.

[12] B. Gidas, W. Ni, and L. Nirenberg, *Symmetry of positive solutions of nonlinear elliptic equations in $\mathbb{R}^n$*, Mathematical analysis and applications, Part A, Adv. in Math. Suppl. Stud., vol. 7, Academic Press, New York–London, 1981, pp. 369–402.

[13] L. Grafakos, *Classical Fourier Analysis*, 2nd ed., Graduate Texts in Mathematics, vol. 249, Springer, New York, 2008.

[14] M. Guedda and L. Véron, *Local and global properties of solutions of quasilinear elliptic equations*, J. Differential Equations **76** (1988), no. 1, 159–189.

[15] E. Jannelli and S. Solimini, *Concentration estimates for critical problems*, Ricerche Mat. **48** (1999), no. suppl., 233–257.

[16] T. Kato, *Schrödinger operators with singular potentials*, Proceedings of the International Symposium on partial Differential Equations and the Geometry of Normed Linear Spaces (Jerusalem, 1972), 1973, pp. 135–148.

[17] G. Leoni, *A first course in Sobolev spaces*, Graduate Studies in Mathematics, vol. 105, American Mathematical Society, Providence, RI, 2009.

[18] C. Li, *Monotonicity and symmetry of solutions of fully nonlinear elliptic equations on unbounded domains*, Comm. partial Differential Equations **16** (1991), no. 4-5, 585–615.

[19] Y. Li and W. Ni, *Radial symmetry of positive solutions of nonlinear elliptic equations in $\mathbb{R}^n$*, Comm. partial Differential Equations **18** (1993), no. 5-6, 1043–1054.

[20] C. Mercuri and M. Willem, *A global compactness result for the p–Laplacian involving critical nonlinearities*, Discrete Contin. Dyn. Syst. **28** (2010), no. 2, 469–493.

[21] I. Peral, *Multiplicity of Solutions for the p-Laplacian.* Lecture Notes at the Second School on Nonlinear Functional Analysis and Applications to Differential Equations, ICTP, Trieste, 1997.

[22] P. Poláčik, P. Quittner, and P. Souplet, *Singularity and decay estimates in superlinear problems via Liouville-type theorems. I. Elliptic equations and systems*, Duke Math. J. **139** (2007), no. 3, 555–579.

[23] N. Saintier, *Asymptotic estimates and blow-up theory for critical equations involving the p-Laplacian*, Calc. Var. partial Differential Equations **25** (2006), no. 3, 299–331.

[24] B. Sciunzi, *Classification of positive* $D^{1,p}\left(\mathbb{R}^N\right)$*–solutions to the critical p–Laplace equation in* $\mathbb{R}^N$. Preprint at arXiv:1506.03653.

[25] J. Serrin, *Local behavior of solutions of quasi-linear equations*, Acta Math. **111** (1964), no. 1, 247–302.

[26] J. Serrin and H. Zou, *Symmetry of ground states of quasilinear elliptic equations*, Arch. Ration. Mech. Anal. **148** (1999), no. 4, 265–290.

[27] M. Struwe, *A global compactness result for elliptic boundary value problems involving limiting nonlinearities*, Math. Z. **187** (1984), no. 4, 511–517.

[28] ———, *Variational methods: Applications to nonlinear partial differential equations and Hamiltonian systems*, Springer-Verlag, Berlin, 1990.

[29] P. Tolksdorf, *Regularity for a more general class of quasilinear elliptic equations*, J. Differential Equations **51** (1984), no. 1, 126–150.

[30] N. S. Trudinger, *On Harnack type inequalities and their application to quasilinear elliptic equations*, Comm. Pure Appl. Math. **20** (1967), 721–747.

[31] ———, *Remarks concerning the conformal deformation of Riemannian structures on compact manifolds*, Ann. Scuola Norm. Sup. Pisa (3) **22** (1968).

[32] J. L. Vázquez, *A strong maximum principle for some quasilinear elliptic equations*, Appl. Math. Optim. **12** (1984), no. 3, 191–202.

[33] L. Véron, *Singular solutions of some nonlinear elliptic equations*, Nonlinear Anal. **5** (1981), no. 3, 225–242.

[34] S. Yan, *A global compactness result for quasilinear elliptic equation involving critical Sobolev exponent*, Chinese J. Contemp. Math. **16** (1995), no. 3, 227–234.

Jérôme Vétois, McGill University Department of Mathematics and Statistics, 805 Sherbrooke Street West, Montreal, Quebec H3A 0B9, Canada.

E-mail address: `jerome.vetois@mcgill.ca`