SIGN-CHANGING BUBBLE TOWERS FOR ASYMPTOTICALLY CRITICAL ELLIPTIC EQUATIONS ON RIEMANNIAN MANIFOLDS

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Abstract. Given a smooth compact Riemannian n–manifold (M, g), we consider the equation $\Delta_g u + hu = |u|^{2^*-2-\varepsilon} u$, where h is a C^1–function on M, the exponent $2^* := 2n/(n-2)$ is the critical Sobolev exponent, and ε is a small positive real parameter such that $\varepsilon \to 0$. We prove the existence of blowing-up families of sign-changing solutions which develop bubble towers at some point where the function h is greater than the Yamabe potential $\frac{n-2}{4(n-1)} \operatorname{Scal}_g$.

1. Introduction

We let (M, g) be a smooth compact Riemannian n–manifold. We consider the asymptotically critical equation

$$\Delta_g u + hu = |u|^{2^*-2-\varepsilon} u \quad \text{in } M, \tag{1.1}$$

where $\Delta_g := -\operatorname{div}_g \nabla$ is the Laplace–Beltrami operator, h is a C^1–function on M, ε is a small positive real parameter such that $\varepsilon \to 0$, and $2^* := \frac{2n}{n-2}$ is the critical exponent for the embeddings of $H_1^2(M)$ into Lebesgue spaces. Here, $H_1^2(M)$ is the Riemannian Sobolev space defined as the completion of $C^\infty(M)$ for the norm $\|u\|_{1,2} := \left(\|\nabla u\|_2^2 + \|u\|_2^2\right)^{1/2}$. We assume that the operator $\Delta_g + h$ is coercive in $H_1^2(M)$, i.e. the energy associated to the operator controls the H_1^2–norm.

We say that a family of solutions $(u_\varepsilon)_\varepsilon$ to equation (1.1) *blows up* if there exists a family of points $(\xi_\varepsilon)_\varepsilon$ in M such that $|u_\varepsilon(\xi_\varepsilon)| \to +\infty$ as $\varepsilon \to 0$. In this paper, we are interested in the question of existence of blowing-up families of sign-changing solutions to equation (1.1). It is proved in Vétois [39] that families of solutions to equation (1.1), if bounded in $H_1^2(M)$, do not blow up as $\varepsilon \to 0$ in case the manifold is conformally flat of dimension $n \geq 7$ and

$$h < \alpha_n \operatorname{Scal}_g \quad \text{in } M, \tag{1.2}$$

where $\alpha_n := (n-2)/(4(n-1))$ and Scal_g is the scalar curvature of the manifold. In Theorem 1.1 below, we prove that in dimensions $n \geq 4$, if the reverse inequality (1.2) holds at some point ξ_0 of the manifold together with a nondegeneracy assumption at ξ_0, then there exist blowing-up families of sign-changing solutions to equation (1.1).

Previous results of compactness and noncompactness have been established for positive solutions to equation (1.1). Compactness of positive solutions has been proved to be true by Druet [14] (see also Druet–Hebey–Vétois [15]) under the hypothesis (1.2) for a general manifold of dimension $n \geq 3$. In case of the Yamabe potential $h \equiv \alpha_n \operatorname{Scal}_g$, compactness of positive solutions has been proved to be true in the aspherical conformally flat case, see Schoen [36], and for a general aspherical manifold of dimension $n \leq 24$, see Khuri–Marques–Schoen [22]. Previous contributions on this question in lower dimensions are by Li–Zhu [27]

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($n = 3$), Druet [14] ($n \le 5$), Marques [28] ($n \le 7$), and Li–Zhang [24–26] ($n \le 11$). The condition $n \le 24$ in the result by Khuri–Marques–Schoen [22] is sharp. Indeed, compactness of positive solutions to the Yamabe equation has been proved not to hold in general in higher dimensions by Brendle [4] ($n \ge 52$) and Brendle–Marques [5] ($n \ge 25$). We also refer to Esposito–Pistoia–Vétois [16] for a recent result on the instability of positive solutions to the Yamabe equation under perturbation of the potential.

When the reverse inequality (1.2) holds at some point ξ_0 of the manifold, it is proved in Micheletti–Pistoia–Vétois [29] that equation (1.1) admits at least one blowing-up family of positive solutions. This result is proved in [29] under the assumption that $n \ge 6$ together with a nondegeneracy assumption at ξ_0. As a by-product of our paper, Theorem 1.1 below extends the result in [29] to dimensions $n = 4, 5$. In dimension $n = 3$, compactness of positive solutions to equation (1.1) is established under a more refined condition than (1.2) which involves a mass term, see Li–Zhu [27]. In case where (M, g) is the standard sphere and h is a constant greater than the Yamabe potential, we also refer to Chen–Wei–Yan [6] for an existence result of positive blowing-up solutions with unbounded energy.

As for the blow-up of sign-changing solutions, an historical contribution is by Ding [13] proving that on the standard sphere $(\mathbb{S}^n, \mathrm{std})$, the Yamabe equation $\Delta_{\mathrm{std}} u + \alpha_n \operatorname{Scal}_{\mathrm{std}} u = |u|^{2^*-2} u$ admits a blowing-up family of sign-changing solutions which are not conformally equivalent to each others. In this case, we also refer to the recent work by del Pino–Musso–Pacard–Pistoia [10, 11] where the authors construct families of sign-changing solutions to the Yamabe equation on $(\mathbb{S}^n, \mathrm{std})$ which concentrate along some special submanifolds (see also Guo–Li–Wei [21] for a similar result for Yamabe-type problems with polyharmonic operators).

The expression of the solutions we get in Theorem 1.1 below is said to be a *bubble tower*. We call *bubble* a family of functions $\left(B_{\delta_\varepsilon,\xi_\varepsilon}\right)_\varepsilon$ defined by

$$B_{\delta_\varepsilon,\xi_\varepsilon}(x) := \left(\frac{\delta_\varepsilon \sqrt{n(n-2)}}{\delta_\varepsilon^2 + d_g(x, \xi_\varepsilon)^2} \right)^{\frac{n-2}{2}} \tag{1.3}$$

for all points x in M, where d_g is the geodesic distance on M with respect to the metric g, $\xi_\varepsilon \in M$, $\delta_\varepsilon > 0$, $\delta_\varepsilon \to 0$ as $\varepsilon \to 0$. In particular, in case (M, g) is the standard sphere, the bubbles are the exact positive solutions to the Yamabe equation, see Lelong-Ferrand [23] and Obata [31]. In the general case, it is well known since Struwe [37] that the blow-up of solutions to equations like (1.1) is due to the presence of bubbles. The solutions we get in Theorem 1.1 below, see (1.5), consist in a finite sum of an arbitrary number k of bubbles, with alternating signs, and a remainder $R_\varepsilon \to 0$ in $H_1^2(M)$. The bubbles are all centered at the same points ξ_ε. Their weights $\delta_{j,\varepsilon}(t_j)$ have different rates of convergence as $\varepsilon \to 0$. Moreover, the alternating signs in (1.5) make the solutions to be sign-changing when $k \ge 2$.

Given a C^1–function φ, we say that a critical point ξ_0 of φ is *C^1-stable* if there exists an open neighborhood Ω of ξ_0 such that for any point ξ in $\overline{\Omega}$, there holds $\nabla\varphi(\xi) = 0 \Leftrightarrow \xi = \xi_0$ and such that $\deg\left(\nabla(\varphi \circ \psi), \psi^{-1}(\Omega), 0\right) \ne 0$, where deg is the Brouwer degree and (ψ, Ω'), $\Omega \subset \Omega'$, is a given chart of M at the point ξ_0. This definition does not depend on the chart (ψ, Ω'). If φ is a C^2–function, then any nondegenerate critical point of φ is C^1–stable. We state our result as follows.

Theorem 1.1. *Let (M, g) be a smooth compact Riemannian manifold of dimension $n \ge 4$, $k \ge 1$ be a natural number, ξ_0 be a point in M, and h be a C^1–function on M such that the operator $\Delta_g + h$ is coercive. Assume that ξ_0 is a C^1–stable critical point of the function $h - \alpha_n \operatorname{Scal}_g$ and that*

$$h(\xi_0) > \alpha_n \operatorname{Scal}_g(\xi_0). \tag{1.4}$$

Then for $\varepsilon > 0$ *small, equation* (1.1) *admits a solution* u_ε *of the form*

$$u_\varepsilon = \sum_{j=1}^{k} (-1)^{j-1} B_{\delta_{j,\varepsilon}(t_j),\xi_\varepsilon} + R_\varepsilon \,, \tag{1.5}$$

where $B_{\delta_{j,\varepsilon}(t_j),\xi_\varepsilon}$ *is as in* (1.3), $\delta_{j,\varepsilon}(t_j) := t_j \mu_\varepsilon \varepsilon^{p_j}$, $\mu_\varepsilon > 0$, $\mu_\varepsilon \to 0$ *if* $n = 4$, $\mu_\varepsilon = 1$ *if* $n \geq 5$, $p_j = \frac{n+4j-6}{2(n-2)}$, $t_j > 0$, $\xi_\varepsilon \to \xi_0$ *in* M, *and* $R_\varepsilon \to 0$ *in* $H_1^2(M)$ *as* $\varepsilon \to 0$. *The functions* u_ε *are positive in case* $k = 1$, *sign-changing in case* $k \geq 2$.

As discussed above, due to Vétois [39], under assumption of conformal flatness, we know that blowing-up families of sign-changing solutions do not exist in dimensions $n \geq 7$ when $h < \alpha_n \operatorname{Scal}_g$.

The proof of Theorem 1.1 relies on a Lyapunov–Schmidt reduction. Over the past two decades, there has been intensive developments on Lyapunov–Schmidt reductions applied to semilinear elliptic problems. A possible reference in book form on the topic is by Ambrosetti–Malchiodi [1]. In addition to the above mentioned references in the geometric context, an early reference for solutions to critical equations with a single peak is by Rey [33]. Concerning bubble towers, without pretending to exhaustivity, previous constructions in the Euclidean space are by Contreras–del Pino [7], del Pino–Dolbeault–Musso [8, 9], del Pino–Musso–Pistoia [12], Pistoia–Weth [32] in case of balls or symmetric domains, and Ge–Jing–Pacard [18], Ge–Jing–Zhou [19], Ge–Musso–Pistoia [20], Musso–Pistoia [30] in case of a general domain.

The proof consists in reducing the problem to finding a C^1–stable critical point of a function $\mathcal{J}_\varepsilon$ posed on a $(k \times (n+1))$–dimensional domain, k being the number of bubbles. To this aim, we need to derive a C^1–uniform expansion of the energy functional as $\varepsilon \to 0$. Because of the contributions in energy due to the interaction between the bubbles (and also even in case of one bubble in dimensions $n = 4, 5$), the approximation rate (see (2.15)) is not as small as the one in Micheletti–Pistoia–Vétois [29] which treats the case of one bubble, and this does not allow us to derive C^1–estimates in the same way as in [29]. To overcome this issue, we exploit the symmetry between the derivatives of the bubbles (1.3) with respect to the weights ξ_ε and to the variable x, an idea which goes back to Rey [33], with the difficulty here that we have to add a corrective term which is due to the derivatives of the geodesic distance.

Our construction fails in dimension $n = 3$ due to the presence of a mass term in the asymptotic expansion of the reduced energy (see Li–Zhu [27]). Note that in case $k = 1$ (positive blow-up with one peak), the result in Theorem 1.1 would not be true due to the compactness result by Li–Zhu [27]. However, it is proved in the recent work by Robert–Vétois [34] that blowing-up families of sign-changing solutions can still be constructed in dimension $n = 3$ in the form $u_\varepsilon = u_0 - B_\varepsilon + R_\varepsilon$, where u_0 is a nondegenerate solution to equation (1.1) with $\varepsilon = 0$, B_ε is a bubble, and $R_\varepsilon \to 0$ in $H_1^2(M)$ as $\varepsilon \to 0$. The construction in [34] holds more generally in dimensions $3 \leq n \leq 6$ for a general potential h and also in higher dimensions for the geometric potential $h \equiv \alpha_n \operatorname{Scal}_g$.

A natural guess is that the method should also apply to prove the existence of bubble towers with positive sign in the slightly supercritical case $\varepsilon < 0$. This problem is usually the dual of the problem of sign-changing bubble towers in the slightly subcritical case $\varepsilon > 0$ (see, for instance, Musso–Pistoia [30]).

Our paper is organized as follows. We describe the proof of Theorem 1.1 in Section 2. We prove the asymptotic expansion of the energy in Section 3. We prove the first derivatives estimates in Section 4 and the error estimates in Section 5.

2. Scheme of the proof of Theorem 1.1

First, we set some notations. Assuming that the operator $\Delta_g + h$ is coercive, we can provide the Sobolev space $H_1^2(M)$ with the scalar product $\langle \cdot, \cdot \rangle_h$ defined by

$$\langle u, v \rangle_h := \int_M \langle \nabla u, \nabla v \rangle_g \, dv_g + \int_M huv dv_g \,, \tag{2.1}$$

where dv_g is the volume element of the manifold. We let $\| \cdot \|_h$ be the norm induced by $\langle \cdot, \cdot \rangle_h$. Moreover, for any u in $L^q(M)$, we denote the L^q–norm of u by $\|u\|_q := \left(\int_M |u|^q dv_g \right)^{1/q}$. We let $i^* : L^{\frac{2n}{n+2}}(M) \to H_1^2(M)$ be the adjoint operator to the embedding $i : H_1^2(M) \hookrightarrow L^{2^*}(M)$, i.e. for any w in $L^{\frac{2n}{n+2}}(M)$, the function $u = i^*(w)$ in $H_1^2(M)$ is the unique solution to the equation $\Delta_g u + hu = w$ in M. Equation (1.1) rewrites

$$u = i^*\left(f_\varepsilon(u)\right), \quad u \in H_1^2(M) \,, \tag{2.2}$$

where $f_\varepsilon(u) := |u|^{2^*-2-\varepsilon} u$ in case $k \geq 2$ and $f_\varepsilon(u) := u_+^{2^*-1-\varepsilon}$ in case $k = 1$, where $u_+ := \max(u, 0)$ (since we intend to construct positive solutions in this case).

By compactness of M, we get that the injectivity radius i_g of the manifold is nonzero. We let r_0 be a positive real number such that $r_0 < i_g$. We let χ be a smooth cutoff function such that $0 \leq \chi \leq 1$ in $\mathbb{R}_+$, $\chi = 1$ in $[0, r_0/2]$, and $\chi = 0$ in $[r_0, +\infty)$. We let N be an open subset of M on which there exists a smooth orthonormal frame with respect to the metric g. N is to be fixed later on. Thanks to the frame on N, we identify the tangent space $T_\xi M$ with $\mathbb{R}^n$ for all points ξ in N so that $\exp_\xi$ is in fact the composition of the standard exponential map with a linear isometry $\Psi_\xi : \mathbb{R}^n \to T_\xi M$ which is smooth with respect to ξ. For any point ξ in N, any positive real number δ, and any point σ in $\mathbb{R}^n$, we define our test function $W_{\delta,\sigma,\xi}$ by

$$W_{\delta,\sigma,\xi}(x) := \chi\left(d_g(x,\xi)\right) \delta^{\frac{2-n}{2}} U\left(\delta^{-1} \exp_\xi^{-1}(x) - \sigma\right) \tag{2.3}$$

for all points x in M, where d_g is the geodesic distance on M with respect to the metric g and

$$U(y) := \left(\frac{\sqrt{n(n-2)}}{1 + |y|^2} \right)^{\frac{n-2}{2}} \tag{2.4}$$

for all points y in $\mathbb{R}^n$. In particular, we get that $W_{\delta,\sigma,\xi}(x) = \chi(d_g(x,\xi)) B_{\delta,\xi}(x)$ for all points x in M, where $B_{\delta,\xi}$ is as in (1.3). The function U is a solution to the equation $\Delta_{\text{Eucl}} U = |U|^{2^*-2} U$ in $\mathbb{R}^n$, where $\Delta_{\text{Eucl}} := -\operatorname{div}_{\text{Eucl}} \nabla$ is the Laplace operator with respect to the Euclidean metric. Associated to this nonlinear equation is the linear equation $\Delta_{\text{Eucl}} v = (2^* - 1) U^{2^*-2} v$ in $\mathbb{R}^n$. By Bianchi–Egnell [3], any solution in $D^{1,2}(\mathbb{R}^n)$ of the equation $\Delta_{\text{Eucl}} v = (2^* - 1) U^{2^*-2} v$ is a linear combination of the functions

$$V_0(y) := \frac{d}{d\delta}\left(\delta^{\frac{2-n}{2}} U\left(\delta^{-1} y\right)\right)\Big|_{\delta=1} = \frac{1}{2} n^{\frac{n-2}{4}} (n-2)^{\frac{n+2}{4}} \frac{|y|^2 - 1}{\left(1 + |y|^2\right)^{\frac{n}{2}}} \tag{2.5}$$

and

$$V_i(y) := -\frac{\partial U}{\partial y_i}(y) = n^{\frac{n-2}{4}} (n-2)^{\frac{n+2}{4}} \frac{y_i}{\left(1 + |y|^2\right)^{\frac{n}{2}}} \tag{2.6}$$

for all points y in $\mathbb{R}^n$ and all $i = 1, \ldots, n$. For any $i = 0, \ldots, n$, any point ξ in N, any positive real number δ, and any point σ in $\mathbb{R}^n$, we define the function $Z_{i,\delta,\sigma,\xi}$ by

$$Z_{i,\delta,\sigma,\xi}(x) := \chi\left(d_g(x,\xi)\right) \delta^{\frac{2-n}{2}} V_i\left(\delta^{-1} \exp_\xi^{-1}(x) - \sigma\right) \tag{2.7}$$

for all points x in M.

We fix a natural number $k \geq 2$. For any point ξ in N, any $\delta = (\delta_1, \dots, \delta_k)$ in $(\mathbb{R}_+^*)^k$ and any $\sigma = (\sigma_1, \dots, \sigma_{k-1})$ in $(\mathbb{R}^n)^{k-1}$, letting $\sigma_k = 0$, we define the projections $\Pi_{\delta,\sigma,\xi}$ and $\Pi_{\delta,\sigma,\xi}^{\perp}$ of the Sobolev space $H_1^2(M)$ onto the respective subspaces

$$K_{\delta,\sigma,\xi} := \operatorname{Span}\left\{Z_{0,\delta_1,\sigma_1,\xi}, \dots, Z_{n,\delta_1,\sigma_1,\xi}, \dots, Z_{0,\delta_k,\sigma_k,\xi}, \dots, Z_{n,\delta_k,\sigma_k,\xi}\right\}, \tag{2.8}$$

$$K_{\delta,\sigma,\xi}^{\perp} := \left\{\phi \in H_1^2(M);\ \left\langle \phi, Z_{i,\delta_j,\sigma_j,\xi}\right\rangle_h = 0 \quad \forall i = 0, \dots, n \quad \forall j = 1, \dots, k\right\}, \tag{2.9}$$

where $\langle \cdot, \cdot \rangle_h$ is as in (2.1). We intend to construct solutions to equation (2.2) of the form

$$u_\varepsilon := \operatorname{Tower}_{\delta_\varepsilon(t_\varepsilon),\sigma_\varepsilon,\xi_\varepsilon} + \phi_{\delta_\varepsilon(t_\varepsilon),\sigma_\varepsilon,\xi_\varepsilon}, \text{ with } \operatorname{Tower}_{\delta_\varepsilon(t_\varepsilon),\sigma_\varepsilon,\xi_\varepsilon} := \sum_{j=1}^{k} (-1)^{j-1} W_{\delta_{j,\varepsilon}(t_{j,\varepsilon}),\sigma_{j,\varepsilon},\xi_\varepsilon}, \tag{2.10}$$

where $W_{\delta_{j,\varepsilon}(t_{j,\varepsilon}),\sigma_{j,\varepsilon},\xi_\varepsilon}$ is as in (2.3), $\phi_{\delta_\varepsilon(t_\varepsilon),\sigma_\varepsilon,\xi_\varepsilon} \in K_{\delta_\varepsilon(t_\varepsilon),\sigma_\varepsilon,\xi_\varepsilon}^{\perp}$, $\phi_{\delta_\varepsilon(t_\varepsilon),\sigma_\varepsilon,\xi_\varepsilon} \to 0$ in $H_1^2(M)$, $\xi_\varepsilon \to \xi_0 \in N$, $t_\varepsilon := (t_{1,\varepsilon}, \dots, t_{k,\varepsilon})$, $t_{j,\varepsilon} \to t_j > 0$ for all $j = 1, \dots, k$, $\sigma_\varepsilon := (\sigma_{1,\varepsilon}, \dots, \sigma_{k-1,\varepsilon})$, $\sigma_{k,\varepsilon} := 0$, $\sigma_{j,\varepsilon} \to \sigma_j \in \mathbb{R}^n$ for all $j = 1, \dots, k-1$, and

$$\delta_\varepsilon(t_\varepsilon) := \left(\delta_{1,\varepsilon}(t_{1,\varepsilon}), \delta_{2,\varepsilon}(t_{2,\varepsilon}), \dots, \delta_{k,\varepsilon}(t_{k,\varepsilon})\right), \quad \delta_{j,\varepsilon}(t_{j,\varepsilon}) := t_{j,\varepsilon}\mu_\varepsilon \varepsilon^{\frac{n+4j-6}{2(n-2)}} \tag{2.11}$$

for all $j = 1, \dots, k$. Here, $\mu_\varepsilon := 1$ if $n \geq 5$, and $\mu_\varepsilon := \ell^{-1}(\varepsilon)/\sqrt{\varepsilon}$ if $n = 4$, where $\ell : \left(0, e^{-1/2}\right) \to (0, e^{-1}/2)$, $\ell : \mu \mapsto -\mu^2 \ln \mu$. As is easily checked, if $n = 4$, then $\mu_\varepsilon \sim \sqrt{2/|\ln \varepsilon|}$ as $\varepsilon \to 0$. Since $t_{j,\varepsilon} \to t_j > 0$, $\sigma_{j,\varepsilon} \to 0$, and $\left\|(1 - \chi(d_g(\cdot, \xi_\varepsilon)))\, B_{\delta_{j,\varepsilon}(t_j),\xi_\varepsilon}\right\|_h \to 0$ as $\varepsilon \to 0$ for all $j = 1, \dots, k$, we find

$$\left\|W_{\delta_{j,\varepsilon}(t_{j,\varepsilon}),\sigma_{j,\varepsilon},\xi_\varepsilon} - B_{\delta_{j,\varepsilon}(t_j),\xi_\varepsilon}\right\|_h \longrightarrow 0 \tag{2.12}$$

as $\varepsilon \to 0$. In particular (1.5) follows from (2.10) and (2.12).

Equation (2.2) rewrites as the couple of equations

$$\Pi_{\delta_\varepsilon(t),\sigma,\xi}\left(\operatorname{Tower}_{\delta_\varepsilon(t),\sigma,\xi} + \phi_{\delta_\varepsilon(t),\sigma,\xi} - i^*\left(f_\varepsilon\left(\operatorname{Tower}_{\delta_\varepsilon(t),\sigma,\xi} + \phi_{\delta_\varepsilon(t),\sigma,\xi}\right)\right)\right) = 0, \tag{2.13}$$

$$\Pi_{\delta_\varepsilon(t),\sigma,\xi}^{\perp}\left(\operatorname{Tower}_{\delta_\varepsilon(t),\sigma,\xi} + \phi_{\delta_\varepsilon(t),\sigma,\xi} - i^*\left(f_\varepsilon\left(\operatorname{Tower}_{\delta_\varepsilon(t),\sigma,\xi} + \phi_{\delta_\varepsilon(t),\sigma,\xi}\right)\right)\right) = 0, \tag{2.14}$$

where $\operatorname{Tower}_{\delta_\varepsilon(t),\sigma,\xi}$ and $\delta_\varepsilon(t)$ are as in (2.10) and (2.11). The first step in the proof consists in solving equation (2.14). This is done in Proposition 2.1 below. We skip the proof of this result which is rather standard in the literature on Lyapunov–Schmidt reductions (see, for instance, Musso–Pistoia [30]). The right-hand side in (2.15) is estimated in Section 5.

Proposition 2.1. *For any compact subset A of $(\mathbb{R}_+^*)^k \times (\mathbb{R}^n)^{k-1} \times N$, there exists a positive constant C_A such that for ε small, for any (t, σ, ξ) in A, there exists a unique function $\phi_{\delta_\varepsilon(t),\sigma,\xi}$ in $K_{\delta_\varepsilon(t),\sigma,\xi}^{\perp}$ which solves equation* (2.14) *and satisfies*

$$\left\|\phi_{\delta_\varepsilon(t),\sigma,\xi}\right\|_h \leq C_A \left\|i^*\left(f_\varepsilon\left(\operatorname{Tower}_{\delta_\varepsilon(t),\sigma,\xi}\right)\right) - \operatorname{Tower}_{\delta_\varepsilon(t),\sigma,\xi}\right\|_h. \tag{2.15}$$

Moreover, $\phi_{\delta_\varepsilon(t),\sigma,\xi}$ is continuously differentiable with respect to (t, σ, ξ).

For ε small, we let J_ε be the functional in $H_1^2(M)$ defined by

$$J_\varepsilon(u) := \frac{1}{2}\int_M |\nabla u|_g^2\, dv_g + \frac{1}{2}\int_M h u^2 dv_g - \int_M F_\varepsilon(u)\, dv_g, \tag{2.16}$$

where $F_\varepsilon(u) = \int_0^u f_\varepsilon(s)\, ds$. The critical points of J_ε are the solutions to equation (2.2). For any point ξ in N, any $t = (t_1, \dots, t_k)$ in $(\mathbb{R}_+^*)^k$, and $\sigma = (\sigma_1, \dots, \sigma_{k-1})$ in $(\mathbb{R}^n)^{k-1}$, we define

$$\mathcal{J}_\varepsilon(t, \sigma, \xi) := J_\varepsilon\left(\operatorname{Tower}_{\delta_\varepsilon(t),\sigma,\xi} + \phi_{\delta_\varepsilon(t),\sigma,\xi}\right), \tag{2.17}$$

where $\operatorname{Tower}_{\delta_\varepsilon(t),\sigma,\xi}$ is as in (2.10) and $\phi_{\delta_\varepsilon(t),\sigma,\xi}$ is given by Proposition 2.1. We solve equation (2.13) in Proposition 2.2 below. Given some C^1–functions φ_ε, we say that the estimate $\varphi_\varepsilon = \mathrm{o}(\varepsilon)$ is C^1–uniform if there hold both $\varphi_\varepsilon = \mathrm{o}(\varepsilon)$ and $\nabla\varphi_\varepsilon = \mathrm{o}(\varepsilon)$ as $\varepsilon \to 0$.

Proposition 2.2. *If $n \geq 4$, then*

$$\mathcal{J}_\varepsilon(t,\sigma,\xi) = c_1 - c_2\varepsilon\ln\mu_\varepsilon - c_3\varepsilon\ln\varepsilon - c_4\varepsilon + c_5\varepsilon t_1^2\Big(h(\xi) - \frac{n-2}{4(n-1)}\operatorname{Scal}_g(\xi)\Big) - c_6\varepsilon\sum_{j=1}^{k}\ln t_j + c_7\varepsilon\sum_{j=1}^{k-1}\left(\frac{t_{j+1}}{t_j}\right)^{\frac{n-2}{2}}\left(1+|\sigma_j|^2\right)^{\frac{2-n}{2}} + \mathrm{o}(\varepsilon) \quad (2.18)$$

as $\varepsilon \to 0$, C^1–uniformly with respect to (t,σ,ξ) in compact subsets of $(\mathbb{R}_+^*)^k \times (\mathbb{R}^n)^{k-1} \times N$, where the c_i's are positive constants depending only on k and n, Scal_g is the scalar curvature, and μ_ε is as in (2.11). Moreover, given a compact subset A of $(\mathbb{R}_+^*)^k \times (\mathbb{R}^n)^{k-1} \times N$, for ε small, if $(t_\varepsilon,\sigma_\varepsilon,\xi_\varepsilon) \in A$ is a critical point of $\mathcal{J}_\varepsilon$, then the function $\operatorname{Tower}_{\delta_\varepsilon(t_\varepsilon),\sigma_\varepsilon,\xi_\varepsilon} + \phi_{\delta_\varepsilon(t_\varepsilon),\sigma_\varepsilon,\xi_\varepsilon}$ is a solution to equation (2.2).

The proof of the asymptotic expansion (2.18) is postponed to the next section. The fact that critical points of $\mathcal{J}_\varepsilon$ provide solutions to equation (2.2) is again rather standard (see Musso–Pistoia [30]). We skip the proof of this part here. Now, we prove Theorem 1.1 by using Propositions 2.1 and 2.2.

Proof of Theorem 1.1. We let $\mathcal{G}$ be the function defined in $(\mathbb{R}_+^*)^k \times (\mathbb{R}^n)^{k-1} \times M$ by

$$\mathcal{G}(t,\sigma,\xi) := c_5 t_1^2\Big(h(\xi) - \frac{n-2}{4(n-1)}\operatorname{Scal}_g(\xi)\Big) - c_6\sum_{j=1}^{k}\ln t_j + c_7\sum_{j=1}^{k-1}\left(\frac{t_{j+1}}{t_j}\right)^{\frac{n-2}{2}}\left(1+|\sigma_j|^2\right)^{\frac{2-n}{2}},$$

where c_5, c_6, and c_7 are as in (2.18). We change variables by setting $s = \Theta(t)$, where

$$\Theta(t) := \left(t_1, \frac{t_2}{t_1}, \frac{t_3}{t_2}, \dots, \frac{t_k}{t_{k-1}}\right).$$

We then get

$$\mathcal{G}\left(\Theta^{-1}(s),\sigma,\xi\right) = c_5 s_1^2\Big(h(\xi) - \frac{n-2}{4(n-1)}\operatorname{Scal}_g(\xi)\Big) - c_6\sum_{j=1}^{k}(k-j+1)\ln s_j + c_7\sum_{j=1}^{k-1} s_{j+1}^{\frac{n-2}{2}}\left(1+|\sigma_j|^2\right)^{\frac{2-n}{2}}.$$

By assumption, we get the existence of a C^1–stable critical point ξ_0 of the function $h - \frac{n-2}{4(n-1)}\operatorname{Scal}_g$ satisfying $h(\xi_0) > \frac{n-2}{4(n-1)}\operatorname{Scal}_g(\xi_0)$. We then define $s_0 := (s_{0,1},\dots,s_{0,k})$, where

$$s_{0,1} := \sqrt{\frac{kc_6}{2c_5\big(h(\xi_0) - \frac{n-2}{4(n-1)}\operatorname{Scal}_g(\xi_0)\big)}} \quad \text{and} \quad s_{0,j} := \left(\frac{2(k-j+1)c_6}{(n-2)c_7}\right)^{\frac{2}{n-2}} \quad \forall j = 2,\dots,k.$$

We claim that the point $(\Theta^{-1}(s_0), 0, \xi_0)$ is a C^1–stable critical point of the function $\mathcal{G}$. In order to prove this claim, it suffices to prove that the point $(s_0,0,0)$ is a C^1–stable critical point of the function $\mathcal{H}$ defined by

$$\mathcal{H}(s,\sigma,y) := \mathcal{G}\left(\Theta^{-1}(s),\sigma,\exp_{\xi_0} y\right)$$

for all (s,σ,y) in $(\mathbb{R}_+^*)^k \times (\mathbb{R}^n)^{k-1} \times B_0(i_g)$, where i_g is the injectivity radius of the manifold. We find

$$\frac{\partial\mathcal{H}}{\partial s_1}(s,\sigma,y) = 2c_5 s_1\Big(h\left(\exp_{\xi_0} y\right) - \frac{n-2}{4(n-1)}\operatorname{Scal}_g\left(\exp_{\xi_0} y\right)\Big) - \frac{kc_6}{s_1},$$

$$\frac{\partial \mathcal{H}}{\partial s_j}(s,\sigma,y) = -(k-j+1)\frac{c_6}{s_j} + \frac{n-2}{2}c_7 s_j^{\frac{n-4}{2}}\left(1+|\sigma_{j-1}|^2\right)^{\frac{2-n}{2}} \quad \forall j=2,\dots,k\,,$$

$$\nabla_{\sigma_j}\mathcal{H}(s,\sigma,y) = -(n-2)\,c_7 s_{j+1}^{\frac{n-2}{2}}\left(1+|\sigma_j|^2\right)^{-\frac{n}{2}}\sigma_j \quad \forall j=1,\dots,k-1\,,$$

$$\nabla_y\mathcal{H}(s,\sigma,y) = c_5 s_1^2 \nabla_y\Big(h\left(\exp_{\xi_0} y\right) - \frac{n-2}{4(n-1)}\operatorname{Scal}_g\left(\exp_{\xi_0} y\right)\Big).$$

One easily checks that there hold $\nabla_s\nabla_y\mathcal{H}(s_0,0,0) = \nabla_y\nabla_s\mathcal{H}(s_0,0,0) = 0$, $\nabla_\sigma\nabla_y\mathcal{H}(s_0,0,0) = \nabla_y\nabla_\sigma\mathcal{H}(s_0,0,0) = 0$, $\nabla_s\nabla_\sigma\mathcal{H}(s_0,0,0) = \nabla_\sigma\nabla_s\mathcal{H}(s_0,0,0) = 0$, and that $\nabla_s^2\mathcal{H}(s_0,0,0)$ and $\nabla_\sigma^2\mathcal{H}(s_0,0,0)$ are nondegenerate. Moreover, by assumption, the point 0 is a C^1–stable critical point of the function $y\mapsto h\left(\exp_{\xi_0} y\right) - \frac{n-2}{4(n-1)}\operatorname{Scal}_g\left(\exp_{\xi_0} y\right)$, and thus of the function $y\mapsto \mathcal{H}(s_0,0,y)$. By standard properties of the Brouwer degree, see for instance [17], we then get that the point $(s_0,0,0)$ is a C^1–stable critical point of the function $\mathcal{H}$. It follows that the point $\left(\Theta^{-1}(s_0),0,\xi_0\right)$ is a C^1–stable critical point of the function $\mathcal{G}$. Proposition 2.2 yields

$$\left|\nabla\big(\varepsilon^{-1}\mathcal{J}_\varepsilon(t,\sigma,\xi) - \mathcal{G}(t,\sigma,\xi)\big)\right| \longrightarrow 0$$

as $\varepsilon\to 0$, uniformly with respect to (t,σ,ξ) in compact subsets of $(\mathbb{R}_+^*)^k\times(\mathbb{R}^n)^{k-1}\times N_0$, where $\mathcal{J}_\varepsilon$ is as in (2.17) and N_0 is some open neighborhood of the point ξ_0 on which there exists a smooth orthonormal frame with respect to the metric g. By standard properties of the Brouwer degree, we then get the existence of a family of critical points $(t_\varepsilon,\sigma_\varepsilon,\xi_\varepsilon)$ of $\mathcal{J}_\varepsilon$ converging to $\left(\Theta^{-1}(s_0),0,\xi_0\right)$ as $\varepsilon\to 0$. By Proposition 2.2, it follows that the function u_ε defined in (2.10), with $\phi_{\delta_\varepsilon(t_\varepsilon),\sigma_\varepsilon,\xi_\varepsilon}$ as in Proposition 2.1, is a solution to equation (2.2) for ε small. In particular, (1.5) follows from (2.12), Proposition 2.1, and Lemma 5.1. It remains to prove that for ε small, the function u_ε is positive in case $k=1$, sign-changing in case $k\ge 2$. The positivity of the function u_ε in case $k=1$ follows from the coercivity of the operator Δ_g+h and the fact that $f_\varepsilon(u_\varepsilon)\ge 0$ in this case. In case $k\ge 2$, we claim that for any $j=1,\dots,k$, given two real numbers a and b such that $a<b$, the function u_ε is negative (resp. positive) at some point in the annulus $A_{j,\varepsilon}(a,b) := B_{\xi_\varepsilon}(b\mu_\varepsilon\varepsilon^{p_j})\backslash B_{\xi_\varepsilon}(a\mu_\varepsilon\varepsilon^{p_j})$ if j is even (resp. odd) for ε small, where $B_\xi(r)$ is the geodesic ball of center ξ and radius r with respect to the metric g. In order to prove this claim, we proceed by contradiction and assume that u_ε is nonnegative (resp. nonpositive) everywhere in $A_{j,\varepsilon}(a,b)$. By straightforward computations, it follows from (1.5) that $R_\varepsilon \ge C\left(\mu_\varepsilon\varepsilon^{p_j}\right)^{(2-n)/2}$ (resp. $R_\varepsilon\le -C\left(\mu_\varepsilon\varepsilon^{p_j}\right)^{(2-n)/2}$) on $A_{j,\varepsilon}(a,b)$ for some positive constant C independent of ε. In particular, we get $R_\varepsilon\not\to 0$ as $\varepsilon\to 0$ in $L^{2^*}(M)$, and thus in $H_1^2(M)$. There is a contradiction, and this proves our claim, namely that in case $k\ge 2$, the function u_ε changes sign for ε small. This ends the proof of Theorem 1.1. ☐

3. The reduced energy

This section is devoted to the proof of the asymptotic expansion (2.18) in Proposition 2.2. We use the first derivatives estimates which are left to Section 4 and the error estimates which are left to Section 5. We also repeatedly use in our estimates the easy fact that given a compact subset A of $(\mathbb{R}_+^*)^k\times(\mathbb{R}^n)^{k-1}\times N$, for ε small, there exists a positive constant C_A such that for any (t,σ,ξ) in A, any point x in M, and any $j=1,\dots,k-1$, there holds

$$\frac{1}{C_A}\left(\delta_{j,\varepsilon}(t_j)^2 + d_g(x,\xi)^2\right) \le \delta_{j,\varepsilon}(t_j)^2 + \left|\exp_\xi^{-1}(x) - \delta_{j,\varepsilon}(t_j)\,\sigma_j\right|^2 \le C_A\left(\delta_{j,\varepsilon}(t_j)^2 + d_g(x,\xi)^2\right). \tag{3.1}$$

As a first step, we give the asymptotic expansion of $J_\varepsilon\left(\mathrm{Tower}_{\delta_\varepsilon(t),\sigma,\xi}\right)$ as $\varepsilon\to 0$, where J_ε is as in (2.16). We let K_n be the sharp constant for the embedding of $D^{1,2}(\mathbb{R}^n)$ into $L^{2^*}(\mathbb{R}^n)$. As computed independently by Rodemich [35], Aubin [2], and Talenti [38], there holds

$$K_n = \sqrt{\frac{4}{n\left(n-2\right)\omega_n^{2/n}}}\,, \tag{3.2}$$

where ω_n is the volume of the unit n–sphere.

Lemma 3.1. *If $n \geq 4$, then there holds*

$$\begin{aligned} J_\varepsilon\left(\mathrm{Tower}_{\delta_\varepsilon(t),\sigma,\xi}\right) = \frac{K_n^{-n}}{n}\Bigg(& k - \frac{k}{4}\left(n-2\right)^2\varepsilon\ln\mu_\varepsilon - \frac{k}{8}\left(n-2\right)\left(n+2k-4\right)\varepsilon\ln\varepsilon - k\beta_n\varepsilon \\ &+ \gamma_n\varepsilon t_1^2\Big(h\left(\xi\right) - \frac{n-2}{4(n-1)}\,\mathrm{Scal}_g\left(\xi\right)\Big) - \frac{\left(n-2\right)^2}{4}\varepsilon\sum_{j=1}^k\ln t_j \\ &+ 2^n\frac{\omega_{n-1}}{\omega_n}\varepsilon\sum_{j=1}^{k-1}\left(\frac{t_{j+1}}{t_j}\right)^{\frac{n-2}{2}}\left(1+\left|\sigma_j\right|^2\right)^{\frac{2-n}{2}} + \mathrm{o}\left(\varepsilon\right)\Bigg) \end{aligned} \tag{3.3}$$

as $\varepsilon \to 0$, C^1–uniformly with respect to (t,σ,ξ) in compact subsets of $(\mathbb{R}_+^*)^k \times (\mathbb{R}^n)^{k-1} \times N$, where Scal_g is the scalar curvature, μ_ε and $\delta_\varepsilon(t)$ are as in (2.11), ω_n (resp. ω_{n-1}) is the volume of the unit n–sphere (resp. $(n-1)$–sphere), K_n is as in (3.2), $\gamma_n := 3$ if $n = 4$, $\gamma_n := 2\left(n-1\right)/\left(\left(n-2\right)\left(n-4\right)\right)$ if $n \geq 5$, and

$$\beta_n := 2^{n-3}\left(n-2\right)^2\frac{\omega_{n-1}}{\omega_n}\int_0^{+\infty}\frac{r^{\frac{n-2}{2}}\ln\left(1+r\right)}{\left(1+r\right)^n}dr + \frac{\left(n-2\right)^2}{4n}\Big(1 - n\ln\sqrt{n\left(n-2\right)}\Big)\,. \tag{3.4}$$

Proof of Lemma 3.1. All our estimates in this proof are uniform with respect to (t,σ,ξ) in compact subsets of $(\mathbb{R}_+^*)^k \times (\mathbb{R}^n)^{k-1} \times N$ and with respect to ε in $(0,\varepsilon_0)$ for some fixed positive real number ε_0. We prove the C^0–expansion of (3.3). The C^1–expansions follow from similar estimates for the derivatives with respect to t, σ, and ξ. We get

$$\begin{aligned} J_\varepsilon\left(\mathrm{Tower}_{\delta_\varepsilon(t),\sigma,\xi}\right) = &\sum_{j=1}^k\left(\frac{1}{2}\int_M\left|\nabla W_{\delta_{j,\varepsilon}(t_j),\sigma_j,\xi}\right|_g^2 dv_g + \frac{1}{2}\int_M hW_{\delta_{j,\varepsilon}(t_j),\sigma_j,\xi}^2 dv_g\right. \\ &- \frac{1}{2^*-\varepsilon}\int_M W_{\delta_{j,\varepsilon}(t_j),\sigma_j,\xi}^{2^*-\varepsilon}dv_g + \sum_{l\neq j}\left(-1\right)^{j+l}\left(\frac{1}{2}\int_M\left\langle\nabla W_{\delta_{j,\varepsilon}(t_j),\sigma_j,\xi},\nabla W_{\delta_{l,\varepsilon}(t_l),\sigma_l,\xi}\right\rangle_g dv_g\right. \\ &\left.\left. + \frac{1}{2}\int_M hW_{\delta_{j,\varepsilon}(t_j),\sigma_j,\xi}W_{\delta_{l,\varepsilon}(t_l),\sigma_l,\xi}dv_g - \int_M W_{\delta_{j,\varepsilon}(t_j),\sigma_j,\xi}^{2^*-1-\varepsilon}W_{\delta_{l,\varepsilon}(t_l),\sigma_l,\xi}dv_g\right)\right) \\ &- \int_M\Bigg(F_\varepsilon\left(\mathrm{Tower}_{\delta_\varepsilon(t),\sigma,\xi}\right) - \sum_{j=1}^k F_\varepsilon\left(W_{\delta_{j,\varepsilon}(t_j),\sigma_j,\xi}\right) \\ &- \sum_{j=1}^k\sum_{l\neq j}\left(-1\right)^{j+l}f_\varepsilon\left(W_{\delta_{j,\varepsilon}(t_j),\sigma_j,\xi}\right)W_{\delta_{l,\varepsilon}(t_l),\sigma_l,\xi}\Bigg)dv_g\,, \end{aligned} \tag{3.5}$$

where $F_\varepsilon\left(u\right)$ is as in (2.16). We are led to estimate each term in (3.5). We use the techniques developed by Aubin [2] in order to estimate the first terms in (3.5). By Cartan's expansion of the metric in geodesic normal coordinates, we get that for any $\alpha,\beta = 1,\dots,n$, for y close to

0, there hold

$$g^{\alpha\beta}\left(\exp_\xi y\right)=\delta^{\alpha\beta}+\frac{1}{3}\delta^{\gamma\beta}R^{\alpha}_{\mu\gamma\nu}\left(\xi\right)y^\mu y^\nu+\mathrm{O}\left(\left|y\right|^3\right) \tag{3.6}$$

$$\sqrt{\left|\exp_\xi^* g\left(y\right)\right|}=1-\frac{1}{6}R_{\mu\nu}\left(\xi\right)y^\mu y^\nu+\mathrm{O}\left(\left|y\right|^3\right), \tag{3.7}$$

where the real numbers $\delta^{\alpha\beta}$ are the Kronecker symbols, the function $\left|\exp_\xi^* g\right|$ is the determinant of the metric, the functions $g^{\alpha\beta}$ are the components of g^{-1}, the functions $R^{\alpha}_{\mu\beta\nu}$ are the components of the Riemann curvature tensor, and the functions $R_{\mu\nu}$ are the components of the Ricci curvature tensor in geodesic normal coordinates. For any $j=1,\dots,k$, using (3.6) and (3.7) together with symmetry properties of the components of the Riemann curvature tensor, we find

$$\begin{aligned}
&\int_M\left|\nabla W_{\delta_{j,\varepsilon}(t_j),\sigma_j,\xi}\right|_g^2dv_g\\
&=n^{\frac{n-2}{2}}\left(n-2\right)^{\frac{n+2}{2}}\omega_{n-1}\int_0^{\frac{r_0}{2\delta_{j,\varepsilon}(t_j)}}\frac{r^{n+1}}{\left(1+r^2\right)^n}\bigg(1-\frac{n-2}{6n}\operatorname{Ric}_g\left(\xi\right).\left(\Psi_\xi\left(\sigma_j\right),\Psi_\xi\left(\sigma_j\right)\right)\delta_{j,\varepsilon}\left(t_j\right)^2\\
&\quad-\frac{1}{6n}\operatorname{Scal}_g\left(\xi\right)\delta_{j,\varepsilon}\left(t_j\right)^2r^2+\mathrm{O}\left(\delta_{j,\varepsilon}\left(t_j\right)^3r^3\right)\bigg)dr+\mathrm{O}\left(\delta_{j,\varepsilon}\left(t_j\right)^{n-2}\right)\\
&=K_n^{-n}\bigg(1-\frac{n-2}{6n}\chi_n\operatorname{Ric}_g\left(\xi\right).\left(\Psi_\xi\left(\sigma_j\right),\Psi_\xi\left(\sigma_j\right)\right)t_j^2\varepsilon^{\frac{n+4j-6}{n-2}}\\
&\quad-\frac{\theta_n\pi_{n,j}}{6n}\operatorname{Scal}_g\left(\xi\right)t_j^2\varepsilon^{\frac{n+4j-6}{n-2}}\bigg)+\mathrm{o}\left(\varepsilon^{\frac{n+4j-6}{n-2}}\right),
\end{aligned}\tag{3.8}$$

$$\begin{aligned}
&\int_M hW^2_{\delta_{j,\varepsilon}(t_j),\sigma_j,\xi}dv_g=n^{\frac{n-2}{2}}\left(n-2\right)^{\frac{n-2}{2}}\omega_{n-1}h\left(\xi\right)\delta_{j,\varepsilon}(t_j)^2\int_0^{\frac{r_0}{2\delta_{j,\varepsilon}(t_j)}}\frac{r^{n-1}}{\left(1+r^2\right)^{n-2}}\\
&\times\left(1+\mathrm{O}\left(\delta_{j,\varepsilon}\left(t_j\right)^2r^2\right)\right)dr+\mathrm{O}\left(\delta_{j,\varepsilon}\left(t_j\right)^{n-2}\right)=\frac{2\gamma_n\pi_{n,j}}{n}K_n^{-n}h\left(\xi\right)t_j^2\varepsilon^{\frac{n+4j-6}{n-2}}+\mathrm{o}\left(\varepsilon^{\frac{n+4j-6}{n-2}}\right)
\end{aligned}\tag{3.9}$$

$$\begin{aligned}
&\frac{1}{2^*-\varepsilon}\int_M W^{2^*-\varepsilon}_{\delta_{j,\varepsilon}(t_j),\sigma_j,\xi}dv_g=\frac{\left(n\left(n-2\right)\right)^{\frac{n-2}{4}\left(2^*-\varepsilon\right)}}{2^*-\varepsilon}\omega_{n-1}\delta_{j,\varepsilon}\left(t_j\right)^{\frac{n-2}{2}\varepsilon}\\
&\times\int_0^{\frac{r_0}{2\delta_{j,\varepsilon}(t_j)}}\frac{r^{n-1}}{\left(1+r^2\right)^{\frac{n-2}{2}\left(2^*-\varepsilon\right)}}\bigg(1-\frac{1}{6}\operatorname{Ric}_g\left(\xi\right).\left(\Psi_\xi\left(\sigma_j\right),\Psi_\xi\left(\sigma_j\right)\right)\delta_{j,\varepsilon}\left(t_j\right)^2\\
&\quad-\frac{1}{6n}\operatorname{Scal}_g\left(\xi\right)\delta_{j,\varepsilon}\left(t_j\right)^2r^2+\mathrm{O}\left(\delta_{j,\varepsilon}\left(t_j\right)^3r^3\right)\bigg)dr+\mathrm{O}\left(\delta_{j,\varepsilon}\left(t_j\right)^n\right)\\
&=\frac{n-2}{2n}K_n^{-n}\bigg(1+\frac{n-2}{2}\varepsilon\ln\mu_\varepsilon+\frac{n+4j-6}{4}\varepsilon\ln\varepsilon+\frac{n-2}{2}\varepsilon\ln t_j+\frac{2\beta_n}{n-2}\varepsilon\\
&\quad-\frac{\chi_n}{6\left(n-2\right)}\operatorname{Scal}_g\left(\xi\right)t_j^2\varepsilon^{\frac{n+4j-6}{n-2}}-\frac{\chi_n}{6}\operatorname{Ric}_g\left(\xi\right).\left(\Psi_\xi\left(\sigma_j\right),\Psi_\xi\left(\sigma_j\right)\right)t_j^2\varepsilon^{\frac{n+4j-6}{n-2}}\bigg)+\mathrm{o}\left(\varepsilon\right),
\end{aligned}\tag{3.10}$$

where Scal_g is the scalar curvature, Ric_g is the Ricci curvature, Ψ_ξ is as in Section 2, ω_n (resp. ω_{n-1}) is the volume of the unit n–sphere (resp. $(n-1)$–sphere), K_n is as in (3.2), β_n is as in (3.4), and

$$\theta_n:=\begin{cases}6 & \text{if } n=4\,,\\ \dfrac{n+2}{n-4} & \text{if } n\ge5\,,\end{cases}\qquad \gamma_n:=\begin{cases}3 & \text{if } n=4\,,\\ \dfrac{2\left(n-1\right)}{\left(n-2\right)\left(n-4\right)} & \text{if } n\ge5\,,\end{cases}\tag{3.11}$$

$$\chi_n := \begin{cases} 0 & \text{if } n = 4\,, \\ 1 & \text{if } n \geq 5\,, \end{cases} \qquad \pi_{n,j} := \begin{cases} 2j-1 & \text{if } n = 4\,, \\ 1 & \text{if } n \geq 5\,. \end{cases} \tag{3.12}$$

For any $l > j$, using (3.1), we find

$$\int_M h W_{\delta_{j,\varepsilon}(t_j),\sigma_j,\xi} W_{\delta_{l,\varepsilon}(t_l),\sigma_l,\xi} dv_g = \mathrm{O}\left(\frac{\delta_{l,\varepsilon}(t_l)^{\frac{n-2}{2}}}{\delta_{j,\varepsilon}(t_j)^{\frac{n-6}{2}}} \int_0^{\frac{r_0}{\delta_{j,\varepsilon}(t_j)}} \frac{r dr}{(1+r^2)^{\frac{n-2}{2}}}\right) = \mathrm{O}\left(\varepsilon^{l-\frac{n-6}{n-2}(j-1)}\right). \tag{3.13}$$

For any $l > j$, changing variables, we find

$$\begin{aligned} &\int_M \left\langle \nabla W_{\delta_{j,\varepsilon}(t_j),\sigma_j,\xi}, \nabla W_{\delta_{l,\varepsilon}(t_l),\sigma_l,\xi}\right\rangle_g dv_g \\ &\qquad = -n^{\frac{n-2}{4}}(n-2)^{\frac{n+2}{4}}\left(\frac{\delta_{l,\varepsilon}(t_l)}{\delta_{j,\varepsilon}(t_j)}\right)^{\frac{n-2}{2}} \int_{\mathbb{R}^n} \frac{\langle \nabla U(y), y+\sigma_j\rangle}{|y+\sigma_j|^n} dy\,(1+\mathrm{o}(1)), \end{aligned} \tag{3.14}$$

where the function U is as in (2.4). Similarly, for any $l \neq j$, we find

$$\begin{aligned} &\int_M W^{2^*-1-\varepsilon}_{\delta_{j,\varepsilon}(t_j),\sigma_j,\xi} W_{\delta_{l,\varepsilon}(t_l),\sigma_l,\xi} dv_g \\ &= \begin{cases} (n(n-2))^{\frac{n-2}{4}} \left(\frac{\delta_{l,\varepsilon}(t_l)}{\delta_{j,\varepsilon}(t_j)}\right)^{\frac{n-2}{2}} \int_{\mathbb{R}^n} \frac{U(y)^{2^*-1}}{|y+\sigma_j|^{n-2}} dy\,(1+\mathrm{o}(1)) & \text{if } j < l, \\ \frac{(n(n-2))^{\frac{n-2}{4}}}{\left(1+|\sigma_l|^2\right)^{\frac{n-2}{2}}} \left(\frac{\delta_{j,\varepsilon}(t_j)}{\delta_{l,\varepsilon}(t_l)}\right)^{\frac{n-2}{2}} \int_{\mathbb{R}^n} U(y)^{2^*-1} dy\,(1+\mathrm{o}(1)) & \text{if } j > l, \end{cases} \end{aligned} \tag{3.15}$$

as $\varepsilon \to 0$. Regarding the last integral in (3.15), we find

$$\int_{\mathbb{R}^n} U(y)^{2^*-1} dy = (n(n-2))^{\frac{n+2}{4}} \omega_{n-1} \int_0^{+\infty} \frac{r^{n-1} dr}{(1+r^2)^{\frac{n+2}{2}}} = n^{\frac{n-2}{4}} (n-2)^{\frac{n+2}{4}} \omega_{n-1}\,. \tag{3.16}$$

Moreover, since the function $y \mapsto ((n-2)\,\omega_{n-1})^{-1} |y+\sigma_j|^{2-n}$ is the Green's function for the Euclidean Laplace operator at the point $-\sigma_j$, and since the function U is a solution to the equation $\Delta_{\mathrm{Eucl}} U = |U|^{2^*-2} U$ in $\mathbb{R}^n$, we get

$$-\int_{\mathbb{R}^n} \frac{\langle \nabla U(y), y+\sigma_j\rangle}{|y+\sigma_j|^n} dy = \frac{1}{n-2} \int_{\mathbb{R}^n} \frac{U(y)^{2^*-1}}{|y+\sigma_j|^{n-2}} dy = \omega_{n-1} U(-\sigma_j)\,. \tag{3.17}$$

By (3.2) and (3.14)–(3.17), we get

$$\begin{aligned} \int_M \left\langle \nabla W_{\delta_{j,\varepsilon}(t_j),\sigma_j,\xi}, \nabla W_{\delta_{l,\varepsilon}(t_l),\sigma_l,\xi}\right\rangle_g dv_g &= \int_M W^{2^*-1-\varepsilon}_{\delta_{j,\varepsilon}(t_j),\sigma_j,\xi} W_{\delta_{l,\varepsilon}(t_l),\sigma_l,\xi} dv_g\,(1+\mathrm{o}(1)) \\ &= \begin{cases} \frac{2^n \omega_{n-1} K_n^{-n}}{n\omega_n \left(1+|\sigma_j|^2\right)^{\frac{n-2}{2}}} \left(\frac{t_l}{t_j}\right)^{\frac{n-2}{2}} \varepsilon^{l-j}(1+\mathrm{o}(1)) & \text{if } j < l, \\ \frac{2^n \omega_{n-1} K_n^{-n}}{n\omega_n \left(1+|\sigma_l|^2\right)^{\frac{n-2}{2}}} \left(\frac{t_j}{t_l}\right)^{\frac{n-2}{2}} \varepsilon^{j-l}(1+\mathrm{o}(1)) & \text{if } j > l, \end{cases} \end{aligned} \tag{3.18}$$

as $\varepsilon \to 0$. Finally, using the same procedure as in Musso–Pistoia [30] (see also Ge–Musso–Pistoia [20]) which consists in estimating the integral on different annuli, we get

$$\int_M \Bigg(F_\varepsilon \left(\mathrm{Tower}_{\delta_\varepsilon(t),\sigma,\xi}\right) - \sum_{j=1}^{k} F_\varepsilon \left(W_{\delta_{j,\varepsilon}(t_j),\sigma_j,\xi}\right) - \sum_{j=1}^{k} \sum_{l\neq j} (-1)^{j+l} f_\varepsilon \left(W_{\delta_{j,\varepsilon}(t_j),\sigma_j,\xi}\right) W_{\delta_{l,\varepsilon}(t_l),\sigma_l,\xi} \Bigg) dv_g = \mathrm{o}\left(\varepsilon\right). \tag{3.19}$$

(3.3) follows from (3.8)–(3.13), (3.18), and (3.19). ☐

It follows from Proposition 2.1 that for ε small, for any (t,σ,ξ) in $(\mathbb{R}_+^*)^k \times (\mathbb{R}^n)^{k-1} \times N$, there holds

$$DJ_\varepsilon \left(\mathrm{Tower}_{\delta_\varepsilon(t),\sigma,\xi} + \phi_{\delta_\varepsilon(t),\sigma,\xi}\right) = \sum_{i=0}^{n} \sum_{j=1}^{k} \lambda_{i,j,\delta_\varepsilon(t),\sigma,\xi} \left\langle Z_{i,\delta_{j,\varepsilon}(t_j),\sigma_j,\xi}, \cdot \right\rangle_h \tag{3.20}$$

for some real numbers $\lambda_{i,j,\delta_\varepsilon(t),\sigma,\xi}$, where $\delta_\varepsilon(t)$ is as in (2.11). We estimate the real numbers $\lambda_{i,j,\delta_\varepsilon(t),\sigma,\xi}$ in Lemma 3.2 below.

Lemma 3.2. *If $n \geq 4$, then for any compact subset A of $(\mathbb{R}_+^*)^k \times (\mathbb{R}^n)^{k-1} \times N$, there exists a positive constant C_A such that for ε small, for any (t,σ,ξ) in A, and any $i = 0,\dots,n$ and $j = 1,\dots,k$, there holds*

$$\left|\lambda_{i,j,\delta_\varepsilon(t),\sigma,\xi}\right| \leq C_A \varepsilon\,, \tag{3.21}$$

where $\lambda_{i,j,\delta_\varepsilon(t),\sigma,\xi}$ is as in (3.20) and $\delta_\varepsilon(t)$ is as in (2.11).

Proof. All our estimates in this proof are uniform with respect to (t,σ,ξ) in compact subsets of $(\mathbb{R}_+^*)^k \times (\mathbb{R}^n)^{k-1} \times N$ and with respect to ε in $(0,\varepsilon_0)$ for some fixed positive real number ε_0. For any $i,l = 0,\dots,n$ and $j,m = 1,\dots,k$, we find

$$\left\langle Z_{i,\delta_{j,\varepsilon}(t_j),\sigma_j,\xi}, Z_{l,\delta_{m,\varepsilon}(t_m),\sigma_m,\xi} \right\rangle_h \longrightarrow \left\|\nabla V_i\right\|_2^2 \delta_{il}\delta_{jm} \tag{3.22}$$

as $\varepsilon \to 0$, where the functions V_i are as in (2.5)–(2.6) and the real numbers δ_{il} and δ_{jm} are the Kronecker symbols. By (3.20) and (3.22), for any $i = 0,\dots,n$ and $j = 1,\dots,k$, we get

$$DJ_\varepsilon \left(\mathrm{Tower}_{\delta_\varepsilon(t),\sigma,\xi} + \phi_{\delta_\varepsilon(t),\sigma,\xi}\right).Z_{i,\delta_{j,\varepsilon}(t_j),\sigma_j,\xi} = \lambda_{i,j,\delta_\varepsilon(t),\sigma,\xi} \left\|\nabla V_i\right\|_2^2 + \mathrm{o}\left(\sum_{l=0}^{n} \sum_{m=1}^{k} \left|\lambda_{l,m,\delta_\varepsilon(t),\sigma,\xi}\right|\right) \tag{3.23}$$

as $\varepsilon \to 0$. On the other hand, since the function $\phi_{\delta_\varepsilon(t),\sigma,\xi}$ belongs to $K^\perp_{\delta_\varepsilon(t),\sigma,\xi}$, we get

$$\begin{aligned} DJ_\varepsilon &\left(\mathrm{Tower}_{\delta_\varepsilon(t),\sigma,\xi} + \phi_{\delta_\varepsilon(t),\sigma,\xi}\right).Z_{i,\delta_{j,\varepsilon}(t_j),\sigma_j,\xi} \\ &= \left\langle \mathrm{Tower}_{\delta_\varepsilon(t),\sigma,\xi} - i^*\left(f_\varepsilon\left(\mathrm{Tower}_{\delta_\varepsilon(t),\sigma,\xi}\right)\right), Z_{i,\delta_{j,\varepsilon}(t_j),\sigma_j,\xi} \right\rangle_h \\ &\quad + (-1)^{j-1} \left\langle Z_{i,\delta_{j,\varepsilon}(t_j),\sigma_j,\xi} - (-1)^{j-1} i^*\left(f'_\varepsilon\left(\mathrm{Tower}_{\delta_\varepsilon(t),\sigma,\xi}\right) Z_{i,\delta_{j,\varepsilon}(t_j),\sigma_j,\xi}\right), \phi_{\delta_\varepsilon(t),\sigma,\xi} \right\rangle_h \\ &\quad - \int_M \Big(f_\varepsilon\left(\mathrm{Tower}_{\delta_\varepsilon(t),\sigma,\xi} + \phi_{\delta_\varepsilon(t),\sigma,\xi}\right) - f_\varepsilon\left(\mathrm{Tower}_{\delta_\varepsilon(t),\sigma,\xi}\right) \\ &\qquad - f'_\varepsilon\left(\mathrm{Tower}_{\delta_\varepsilon(t),\sigma,\xi}\right) \phi_{\delta_\varepsilon(t),\sigma,\xi} \Big) Z_{i,\delta_{j,\varepsilon}(t_j),\sigma_j,\xi}\, dv_g\,. \end{aligned} \tag{3.24}$$

We are led to estimate each terms in (3.24). By Lemma 4.1, we get

$$\begin{aligned}&\left\langle \mathrm{Tower}_{\delta_\varepsilon(t),\sigma,\xi} - i^*\left(f_\varepsilon\left(\mathrm{Tower}_{\delta_\varepsilon(t),\sigma,\xi}\right)\right), Z_{0,\delta_{j,\varepsilon}(t_j),\sigma_j,\xi}\right\rangle_h\\ &\quad = (-1)^{j-1} t_j \frac{d}{dt_j} J_\varepsilon\left(\mathrm{Tower}_{\delta_\varepsilon(t),\sigma,\xi}\right) - \sum_{i=1}^n \sigma_{ji}\frac{d}{d\sigma_{ji}} J_\varepsilon\left(\mathrm{Tower}_{\delta_\varepsilon(t),\sigma,\xi}\right)\end{aligned} \tag{3.25}$$

and

$$\left\langle \mathrm{Tower}_{\delta_\varepsilon(t),\sigma,\xi} - i^*\left(f_\varepsilon\left(\mathrm{Tower}_{\delta_\varepsilon(t),\sigma,\xi}\right)\right), Z_{i,\delta_{j,\varepsilon}(t_j),\sigma_j,\xi}\right\rangle_h = (-1)^{j-1}\frac{d}{d\sigma_{ji}} J_\varepsilon\left(\mathrm{Tower}_{\delta_\varepsilon(t),\sigma,\xi}\right) \tag{3.26}$$

for all $i = 1, \dots, n$. By Cauchy–Schwarz inequality and Lemmas 3.1, 4.1, 5.1, it follows from (3.25) and (3.26) that

$$\left\langle \mathrm{Tower}_{\delta_\varepsilon(t),\sigma,\xi} - i^*\left(f_\varepsilon\left(\mathrm{Tower}_{\delta_\varepsilon(t),\sigma,\xi}\right)\right), Z_{i,\delta_{j,\varepsilon}(t_j),\sigma_j,\xi}\right\rangle_h = \mathrm{O}(\varepsilon) \tag{3.27}$$

for all $i = 0, \dots, n$. By Cauchy–Schwarz inequality, Proposition 2.1, and Lemma 5.1, we get

$$\left\langle Z_{i,\delta_{j,\varepsilon}(t_j),\sigma_j,\xi} - (-1)^{j-1} i^*\left(f'_\varepsilon\left(\mathrm{Tower}_{\delta_\varepsilon(t),\sigma,\xi}\right) Z_{i,\delta_{j,\varepsilon}(t_j),\sigma_j,\xi}\right), \phi_{\delta_\varepsilon(t),\sigma,\xi}\right\rangle_h = \mathrm{o}(\varepsilon) \tag{3.28}$$

as $\varepsilon \to 0$, for all $i = 0, \dots, n$. As is easily checked, there exists a positive real number C such that for ε small, there holds

$$\left|f'_\varepsilon(u+v) - f'_\varepsilon(u)\right| \le C\begin{cases} |v|\left(|u|^{2^*-3-\varepsilon} + |v|^{2^*-3-\varepsilon}\right) & \text{if } n = 4,5,\\ \min\left(|u|^{2^*-3-\varepsilon}|v|, |v|^{2^*-2-\varepsilon}\right) & \text{if } n \ge 6,\end{cases} \tag{3.29}$$

for all real numbers u and v. In case $n = 4, 5$, by the Mean Value Theorem, (3.29), Hölder's inequality, Proposition 2.1, and Lemma 5.1, we get

$$\begin{aligned}&\int_M \left(f_\varepsilon\left(\mathrm{Tower}_{\delta_\varepsilon(t),\sigma,\xi} + \phi_{\delta_\varepsilon(t),\sigma,\xi}\right) - f_\varepsilon\left(\mathrm{Tower}_{\delta_\varepsilon(t),\sigma,\xi}\right) - f'_\varepsilon\left(\mathrm{Tower}_{\delta_\varepsilon(t),\sigma,\xi}\right)\phi_{\delta_\varepsilon(t),\sigma,\xi}\right) Z_{i,\delta_{j,\varepsilon}(t_j),\sigma_j,\xi}\, dv_g\\ &= \mathrm{O}\left(\left\|\phi_{\delta_\varepsilon(t),\sigma,\xi}\right\|^2_{2^*-\varepsilon}\left\|Z_{i,\delta_{j,\varepsilon}(t_j),\sigma_j,\xi}\right\|_{2^*-\varepsilon}\left(\left\|\mathrm{Tower}_{\delta_\varepsilon(t),\sigma,\xi}\right\|^{2^*-3-\varepsilon}_{2^*-\varepsilon} + \left\|\phi_{\delta_\varepsilon(t),\sigma,\xi}\right\|^{2^*-3-\varepsilon}_{2^*-\varepsilon}\right)\right) = \mathrm{o}(\varepsilon)\end{aligned} \tag{3.30}$$

as $\varepsilon \to 0$. Now, we assume that $n \ge 6$. For any $j = 1, \dots, k$, we define the annulus

$$A_{j,\delta_\varepsilon(t),\xi} := B_\xi\left(\sqrt{\delta_{j-1,\varepsilon}(t_{j-1})\,\delta_{j,\varepsilon}(t_j)}\right)\setminus B_\xi\left(\sqrt{\delta_{j,\varepsilon}(t_j)\,\delta_{j+1,\varepsilon}(t_{j+1})}\right), \tag{3.31}$$

where $\delta_{0,\varepsilon}(t_0) := r_0^2/\delta_{1,\varepsilon}(t_1)$, $\delta_{j,\varepsilon}(t_j)$ is as in (2.11) for all $j = 1, \dots, k$, $\delta_{k+1,\varepsilon}(t_{k+1}) := 0$, and $B_\xi(r)$ is the geodesic ball of center ξ and radius r with respect to the metric g. By the Mean Value Theorem, (3.29), and Hölder's inequality, we get that for any $l \ne j$, there holds

$$\begin{aligned}&\int_{A_{l,\delta_\varepsilon(t),\xi}} \left(f_\varepsilon\left(\mathrm{Tower}_{\delta_\varepsilon(t),\sigma,\xi} + \phi_{\delta_\varepsilon(t),\sigma,\xi}\right) - f_\varepsilon\left(\mathrm{Tower}_{\delta_\varepsilon(t),\sigma,\xi}\right)\right.\\ &\qquad \left. - f'_\varepsilon\left(\mathrm{Tower}_{\delta_\varepsilon(t),\sigma,\xi}\right)\phi_{\delta_\varepsilon(t),\sigma,\xi}\right) Z_{i,\delta_{j,\varepsilon}(t_j),\sigma_j,\xi}\, dv_g\\ &\qquad\quad = \mathrm{O}\left(\left\|\phi_{\delta_\varepsilon(t),\sigma,\xi}\right\|^{2^*-1-\varepsilon}_{2^*-\varepsilon}\left\|Z_{i,\delta_{j,\varepsilon}(t_j),\sigma_j,\xi}\mathbf{1}_{A_{l,\delta_\varepsilon(t),\xi}}\right\|_{2^*-\varepsilon}\right).\end{aligned} \tag{3.32}$$

For any $l \ne j$, a rough estimate gives

$$\left\|Z_{i,\delta_{j,\varepsilon}(t_j),\sigma_j,\xi}\mathbf{1}_{A_{l,\delta_\varepsilon(t),\xi}}\right\|_{2^*-\varepsilon} = \mathrm{O}\left(\sqrt{\varepsilon}\right). \tag{3.33}$$

By (3.32), (3.33), Proposition 2.1, and Lemma 5.1, we get

$$\int_{M\setminus A_{j,\delta_\varepsilon(t),\xi}} \left(f_\varepsilon\left(\mathrm{Tower}_{\delta_\varepsilon(t),\sigma,\xi}+\phi_{\delta_\varepsilon(t),\sigma,\xi}\right)-f_\varepsilon\left(\mathrm{Tower}_{\delta_\varepsilon(t),\sigma,\xi}\right)\right.$$
$$\left.-f'_\varepsilon\left(\mathrm{Tower}_{\delta_\varepsilon(t),\sigma,\xi}\right)\phi_{\delta_\varepsilon(t),\sigma,\xi}\right)Z_{i,\delta_{j,\varepsilon}(t_j),\sigma_j,\xi}dv_g=\mathrm{o}\left(\varepsilon\right) \tag{3.34}$$

as $\varepsilon\to 0$. Moreover, we get

$$\begin{aligned}&\int_{A_{j,\delta_\varepsilon(t),\xi}} \left(f_\varepsilon\left(\mathrm{Tower}_{\delta_\varepsilon(t),\sigma,\xi}+\phi_{\delta_\varepsilon(t),\sigma,\xi}\right)-f_\varepsilon\left(\mathrm{Tower}_{\delta_\varepsilon(t),\sigma,\xi}\right)\right.\\
&\qquad\left.-f'_\varepsilon\left(\mathrm{Tower}_{\delta_\varepsilon(t),\sigma,\xi}\right)\phi_{\delta_\varepsilon(t),\sigma,\xi}\right)Z_{i,\delta_{j,\varepsilon}(t_j),\sigma_j,\xi}dv_g\\
&\quad\le\int_{A_{j,\delta_\varepsilon(t),\xi}}\Big|\left(f_\varepsilon\left(\mathrm{Tower}_{\delta_\varepsilon(t),\sigma,\xi}+\phi_{\delta_\varepsilon(t),\sigma,\xi}\right)-f_\varepsilon\left(\mathrm{Tower}_{\delta_\varepsilon(t),\sigma,\xi}\right)\right.\\
&\qquad\left.-f'_\varepsilon\left(W_{\delta_{j,\varepsilon}(t_j),\sigma_j,\xi}\right)\phi_{\delta_\varepsilon(t),\sigma,\xi}\right)Z_{i,\delta_{j,\varepsilon}(t_j),\sigma_j,\xi}\Big|dv_g\\
&\qquad+\int_{A_{j,\delta_\varepsilon(t),\xi}}\Big|\left(f'_\varepsilon\left(\mathrm{Tower}_{\delta_\varepsilon(t),\sigma,\xi}\right)-f'_\varepsilon\left(W_{\delta_{j,\varepsilon}(t_j),\sigma_j,\xi}\right)\right)\phi_{\delta_\varepsilon(t),\sigma,\xi}Z_{i,\delta_{j,\varepsilon}(t_j),\sigma_j,\xi}\Big|dv_g\,.\end{aligned} \tag{3.35}$$

By (3.29) and Hölder's inequality, we get

$$\begin{aligned}&\int_{A_{j,\delta_\varepsilon(t),\xi}} \left(f_\varepsilon\left(\mathrm{Tower}_{\delta_\varepsilon(t),\sigma,\xi}+\phi_{\delta_\varepsilon(t),\sigma,\xi}\right)-f_\varepsilon\left(\mathrm{Tower}_{\delta_\varepsilon(t),\sigma,\xi}\right)\right.\\
&\qquad\left.-f'_\varepsilon\left(\mathrm{Tower}_{\delta_\varepsilon(t),\sigma,\xi}\right)\phi_{\delta_\varepsilon(t),\sigma,\xi}\right)Z_{i,\delta_{j,\varepsilon}(t_j),\sigma_j,\xi}dv_g\\
&\quad=\mathrm{O}\Bigg(\left\|\phi_{\delta_\varepsilon(t),\sigma,\xi}\right\|_{2^*-\varepsilon}\Bigg(\sum_{l\neq j}\left\|W^{2^*-3-\varepsilon}_{\delta_{j,\varepsilon}(t_j),\sigma_j,\xi}W_{\delta_{l,\varepsilon}(t_l),\sigma_l,\xi}Z_{i,\delta_{j,\varepsilon}(t_j),\sigma_j,\xi}\mathbf{1}_{A_{j,\delta_\varepsilon(t),\xi}}\right\|_{\frac{2^*-\varepsilon}{2^*-1-\varepsilon}}\\
&\qquad+\left\|W^{2^*-3-\varepsilon}_{\delta_{j,\varepsilon}(t_j),\sigma_j,\xi}Z_{i,\delta_{j,\varepsilon}(t_j),\sigma_j,\xi}\right\|_{\frac{2^*-\varepsilon}{2^*-2-\varepsilon}}\left\|\phi_{\delta_\varepsilon(t),\sigma,\xi}\right\|_{2^*-\varepsilon}\Bigg)\Bigg)\,.\end{aligned} \tag{3.36}$$

For any $l\neq j$, we find

$$\left\|W^{2^*-3-\varepsilon}_{\delta_{j,\varepsilon}(t_j),\sigma_j,\xi}W_{\delta_{l,\varepsilon}(t_l),\sigma_l,\xi}Z_{i,\delta_{j,\varepsilon}(t_j),\sigma_j,\xi}\mathbf{1}_{A_{j,\delta_\varepsilon(t),\xi}}\right\|_{\frac{2^*-\varepsilon}{2^*-1-\varepsilon}}=\begin{cases}\mathrm{O}\left(\varepsilon\left|\ln\varepsilon\right|^{\frac{2}{3}}\right) & \text{if } n=6,\\ \mathrm{O}\left(\varepsilon^{\frac{n+2}{2(n-2)}}\right) & \text{if } n\ge 7,\end{cases} \tag{3.37}$$

$$\left\|W^{2^*-3-\varepsilon}_{\delta_{j,\varepsilon}(t_j),\sigma_j,\xi}Z_{i,\delta_{j,\varepsilon}(t_j),\sigma_j,\xi}\right\|_{\frac{2^*-\varepsilon}{2^*-1-\varepsilon}}=\mathrm{O}\left(1\right). \tag{3.38}$$

By (3.36), (3.37), (3.38), Proposition 2.1, and Lemma 5.1, we get

$$\int_{A_{j,\delta_\varepsilon(t),\xi}} \left(f_\varepsilon\left(\mathrm{Tower}_{\delta_\varepsilon(t),\sigma,\xi}+\phi_{\delta_\varepsilon(t),\sigma,\xi}\right)-f_\varepsilon\left(\mathrm{Tower}_{\delta_\varepsilon(t),\sigma,\xi}\right)\right.$$
$$\left.-f'_\varepsilon\left(\mathrm{Tower}_{\delta_\varepsilon(t),\sigma,\xi}\right)\phi_{\delta_\varepsilon(t),\sigma,\xi}\right)Z_{i,\delta_{j,\varepsilon}(t_j),\sigma_j,\xi}dv_g=\mathrm{o}\left(\varepsilon\right) \tag{3.39}$$

as $\varepsilon\to 0$. By (3.23)–(3.28), (3.30), (3.34), (3.39), for any $i=0,\dots,n$ and $j=1,\dots,k$, we get

$$\lambda_{i,j,\delta_\varepsilon(t),\sigma,\xi}=\mathrm{O}\left(\varepsilon\right)+\mathrm{o}\left(\sum_{l=0}^{n}\sum_{m=1}^{k}\left|\lambda_{l,m,\delta_\varepsilon(t),\sigma,\xi}\right|\right) \tag{3.40}$$

as $\varepsilon\to 0$. We then get (3.21). This ends the proof of Lemma 3.2. □

In Lemma 3.3 below, we show that the first order terms in the asymptotic expansion of $\mathcal{J}_\varepsilon\left(t,\sigma,\xi\right)$ defined in (2.17) are the same as for $J_\varepsilon\left(\mathrm{Tower}_{\delta_\varepsilon(t),\sigma,\xi}\right)$. This result, together with Lemma 3.1, concludes the proof of the asymptotic expansion (2.18).

Lemma 3.3. *If $n \geq 4$, then there holds*

$$\mathcal{J}_\varepsilon (t, \sigma, \xi) = J_\varepsilon \left(\mathrm{Tower}_{\delta_\varepsilon(t),\sigma,\xi}\right) + \mathrm{o}(\varepsilon) \tag{3.41}$$

as $\varepsilon \to 0$, C^1–uniformly with respect to (t, σ, ξ) in compact subsets of $(\mathbb{R}_+^*)^k \times (\mathbb{R}^n)^{k-1} \times N$.

All our estimates in the proofs below are uniform with respect to (t, σ, ξ) in compact subsets of $(\mathbb{R}_+^*)^k \times (\mathbb{R}^n)^{k-1} \times N$ and with respect to ε in $(0, \varepsilon_0)$ for some fixed positive real number ε_0.

Proof of the C^0–part of (3.41). We get

$$\begin{aligned}\mathcal{J}_\varepsilon (t, \sigma, \xi) - J_\varepsilon \left(\mathrm{Tower}_{\delta_\varepsilon(t),\sigma,\xi}\right) &= \left\langle \mathrm{Tower}_{\delta_\varepsilon(t),\sigma,\xi} - i^* \left(f_\varepsilon \left(\mathrm{Tower}_{\delta_\varepsilon(t),\sigma,\xi}\right)\right), \phi_{\delta_\varepsilon(t),\sigma,\xi}\right\rangle_h \\ &\quad + \frac{1}{2} \left\|\phi_{\delta_\varepsilon(t),\sigma,\xi}\right\|_h^2 - \int_M \Big(F_\varepsilon \left(\mathrm{Tower}_{\delta_\varepsilon(t),\sigma,\xi} + \phi_{\delta_\varepsilon(t),\sigma,\xi}\right) - F_\varepsilon \left(\mathrm{Tower}_{\delta_\varepsilon(t),\sigma,\xi}\right) \\ &\quad - f_\varepsilon \left(\mathrm{Tower}_{\delta_\varepsilon(t),\sigma,\xi}\right) \phi_{\delta_\varepsilon(t),\sigma,\xi}\Big) dv_g\,. \end{aligned} \tag{3.42}$$

By Cauchy–Schwarz inequality, Proposition 2.1, and Lemma 5.1,

$$\left\langle \mathrm{Tower}_{\delta_\varepsilon(t),\sigma,\xi} - i^* \left(f_\varepsilon \left(\mathrm{Tower}_{\delta_\varepsilon(t),\sigma,\xi}\right)\right), \phi_{\delta_\varepsilon(t),\sigma,\xi}\right\rangle_h + \frac{1}{2} \left\|\phi_{\delta_\varepsilon(t),\sigma,\xi}\right\|_h^2 = \mathrm{o}(\varepsilon) \tag{3.43}$$

as $\varepsilon \to 0$. Now, we estimate the last term in (3.42). By the Mean Value Theorem, Hölder's inequality, Proposition 2.1, and Lemma 5.1, we get

$$\begin{aligned}&\int_M \left(F_\varepsilon \left(\mathrm{Tower}_{\delta_\varepsilon(t),\sigma,\xi} + \phi_{\delta_\varepsilon(t),\sigma,\xi}\right) - F_\varepsilon \left(\mathrm{Tower}_{\delta_\varepsilon(t),\sigma,\xi}\right) - f_\varepsilon \left(\mathrm{Tower}_{\delta_\varepsilon(t),\sigma,\xi}\right) \phi_{\delta_\varepsilon(t),\sigma,\xi}\right) dv_g \\ &\quad = \mathrm{O}\left(\left\|\phi_{\delta_\varepsilon(t),\sigma,\xi}\right\|_{2^*-\varepsilon}^2 \left(\left\|\mathrm{Tower}_{\delta_\varepsilon(t),\sigma,\xi}\right\|_{2^*-\varepsilon}^{2^*-2-\varepsilon} + \left\|\phi_{\delta_\varepsilon(t),\sigma,\xi}\right\|_{2^*-\varepsilon}^{2^*-2-\varepsilon}\right)\right) = \mathrm{o}(\varepsilon)\end{aligned} \tag{3.44}$$

as $\varepsilon \to 0$. The C^0–part of (3.41) follows from (3.42), (3.43), and (3.44). ☐

Proof of the C^1–part of (3.41) *with respect to t and σ.* We let ϱ stand either for t_j or σ_{ji} for some $i = 1, \dots, n$ and $j = 1, \dots, k$. By Lemma 4.1, we get

$$\frac{d}{d\varrho} \mathrm{Tower}_{\delta_\varepsilon(t),\sigma,\xi}(x) = \sum_{i=0}^n \sum_{j=1}^k \nu_{i,t_j,\sigma_j} Z_{i,\delta_{j,\varepsilon}(t_j),\sigma_j,\xi}(x)\,, \tag{3.45}$$

where the functions $Z_{i,\delta_{j,\varepsilon}(t_j),\sigma_j,\xi}$ are as in (2.7) and the real numbers $\nu_{i,t_j\sigma_j}$ are uniformly bounded with respect to (t, σ) in compact subsets of $(\mathbb{R}_+^*)^k \times (\mathbb{R}^n)^{k-1}$. By (3.45) and since the function $\phi_{\delta_\varepsilon(t),\sigma,\xi}$ belongs to $K_{\delta_\varepsilon(t),\sigma,\xi}^\perp$, we get

$$\begin{aligned}&\frac{d}{d\varrho}\left(\mathcal{J}_\varepsilon (t, \sigma, \xi) - J_\varepsilon \left(\mathrm{Tower}_{\delta_\varepsilon(t),\sigma,\xi}\right)\right) = DJ_\varepsilon \left(\mathrm{Tower}_{\delta_\varepsilon(t),\sigma,\xi} + \phi_{\delta_\varepsilon(t),\sigma,\xi}\right) . \left(\frac{d}{d\varrho}\phi_{\delta_\varepsilon(t),\sigma,\xi}\right) \\ &+ \sum_{i=0}^n \sum_{j=1}^k \nu_{i,t_j\sigma_j} \Bigg((-1)^{j-1} \left\langle Z_{i,\delta_{j,\varepsilon}(t_j),\sigma_j,\xi} - (-1)^{j-1} i^* \left(f'_\varepsilon \left(\mathrm{Tower}_{\delta_\varepsilon(t),\sigma,\xi}\right) Z_{i,\delta_{j,\varepsilon}(t_j),\sigma_j,\xi}\right), \phi_{\delta_\varepsilon(t),\sigma,\xi}\right\rangle_h \\ &\qquad - \int_M \Big(f_\varepsilon \left(\mathrm{Tower}_{\delta_\varepsilon(t),\sigma,\xi} + \phi_{\delta_\varepsilon(t),\sigma,\xi}\right) - f_\varepsilon \left(\mathrm{Tower}_{\delta_\varepsilon(t),\sigma,\xi}\right) \\ &\qquad - f'_\varepsilon \left(\mathrm{Tower}_{\delta_\varepsilon(t),\sigma,\xi}\right) \phi_{\delta_\varepsilon(t),\sigma,\xi}\Big) Z_{i,\delta_{j,\varepsilon}(t_j),\sigma_j,\xi} dv_g\Bigg).\end{aligned} \tag{3.46}$$

By (3.20), we get

$$DJ_\varepsilon\left(\mathrm{Tower}_{\delta_\varepsilon(t),\sigma,\xi}+\phi_{\delta_\varepsilon(t),\sigma,\xi}\right)\cdot\left(\frac{d}{d\varrho}\phi_{\delta_\varepsilon(t),\sigma,\xi}\right)=\sum_{i=0}^{n}\sum_{j=1}^{k}\lambda_{i,j,\delta_\varepsilon(t),\sigma,\xi}\left\langle Z_{i,\delta_{j,\varepsilon}(t_j),\sigma_j,\xi},\frac{d}{d\varrho}\phi_{\delta_\varepsilon(t),\sigma,\xi}\right\rangle_h, \tag{3.47}$$

where the functions $Z_{i,\delta_{j,\varepsilon}(t_j),\sigma_j,\xi}$ are as in (2.7) and the real numbers $\lambda_{i,j,\delta_\varepsilon(t),\sigma,\xi}$ are as in (3.20). Since the function $\phi_{\delta_\varepsilon(t),\sigma,\xi}$ belongs to $K^\perp_{\delta_\varepsilon(t),\sigma,\xi}$, differentiating $\langle Z_{i,\delta_{j,\varepsilon}(t_j),\sigma_j,\xi},\phi_{\delta_\varepsilon(t),\sigma,\xi}\rangle_h$ with respect to ϱ, we get

$$\left\langle Z_{i,\delta_{j,\varepsilon}(t_j),\sigma_j,\xi},\frac{d}{d\varrho}\phi_{\delta_\varepsilon(t),\sigma,\xi}\right\rangle_h=-\left\langle \frac{d}{d\varrho}Z_{i,\delta_{j,\varepsilon}(t_j),\sigma_j,\xi},\phi_{\delta_\varepsilon(t),\sigma,\xi}\right\rangle_h. \tag{3.48}$$

By (3.47), (3.48), and Cauchy–Schwarz inequality, we get

$$\begin{aligned}&\left|DJ_\varepsilon\left(\mathrm{Tower}_{\delta_\varepsilon(t),\sigma,\xi}+\phi_{\delta_\varepsilon(t),\sigma,\xi}\right)\cdot\left(\frac{d}{d\varrho}\phi_{\delta_\varepsilon(t),\sigma,\xi}\right)\right|\\&\qquad\le\sum_{i=0}^{n}\sum_{j=1}^{k}\left|\lambda_{i,j,\delta_\varepsilon(t),\sigma,\xi}\right|\left\|\frac{d}{d\varrho}Z_{i,\delta_{j,\varepsilon}(t_j),\sigma_j,\xi}\right\|_h\left\|\phi_{\delta_\varepsilon(t),\sigma,\xi}\right\|_h.\end{aligned} \tag{3.49}$$

For any $i=0,\dots,n$ and $j=1,\dots,k$, we find

$$\left\|\frac{d}{d\varrho}Z_{i,\delta_{j,\varepsilon}(t_j),\sigma_j,\xi}\right\|_h=\mathrm{O}(1). \tag{3.50}$$

By (3.49), (3.50), Proposition 2.1, and Lemmas 3.2 and 5.1, we get

$$DJ_\varepsilon\left(\mathrm{Tower}_{\delta_\varepsilon(t),\sigma,\xi}+\phi_{\delta_\varepsilon(t),\sigma,\xi}\right)\cdot\left(\frac{d}{d\varrho}\phi_{\delta_\varepsilon(t),\sigma,\xi}\right)=\mathrm{o}(\varepsilon) \tag{3.51}$$

as $\varepsilon\to0$. The C^1–part of (3.41) with respect to t and σ follows from (3.28), (3.30), (3.34), (3.39), (3.46), and (3.51). □

Proof of the C^1–part of (3.41) *with respect to ξ.* For any $i=1,\dots,n$, by (3.20), we get

$$\begin{aligned}&\frac{d}{dy_i}\mathcal{J}_\varepsilon\left(t,\sigma,\exp_\xi y\right)\Big|_{y=0}\\&\quad=\sum_{l=0}^{n}\sum_{j=1}^{k}\lambda_{l,j,\delta_\varepsilon(t),\sigma,\xi}\left\langle Z_{l,\delta_{j,\varepsilon}(t_j),\sigma_j,\xi},\frac{d}{dy_i}\left(\mathrm{Tower}_{\delta_\varepsilon(t),\sigma,\exp_\xi y}+\phi_{\delta_\varepsilon(t),\sigma,\exp_\xi y}\right)\Big|_{y=0}\right\rangle_h,\end{aligned} \tag{3.52}$$

where the functions $Z_{l,\delta_{j,\varepsilon}(t_j),\sigma_j,\xi}$ are as in (2.7) and the real numbers $\lambda_{i,j,\delta_\varepsilon(t),\sigma,\xi}$ are as in (3.20). For any $i=1,\dots,n$, $l=0,\dots,n$, and $j=1,\dots,k$, by Lemma 4.1 and Cauchy–Schwarz inequality, we get

$$\begin{aligned}&\left|\left\langle Z_{l,\delta_{j,\varepsilon}(t_j),\sigma_j,\xi},\frac{d}{dy_i}\mathrm{Tower}_{\delta_\varepsilon(t),\sigma,\exp_\xi y}\Big|_{y=0}+\frac{d}{dy_i}\mathrm{Tower}_{\delta_\varepsilon(t),\sigma,\xi}\left(\exp_\xi y\right)\Big|_{y=\exp_\xi^{-1}x}\right\rangle\right|\\&\qquad\le\sum_{m=1}^{k}\left\|Z_{l,\delta_{j,\varepsilon}(t_j),\sigma_j,\xi}\right\|_h\left(\left\|\gamma_{i,\delta_{m,\varepsilon}(t_m),\sigma_m,\xi}\right\|_h+\left\|\widetilde{\gamma}_{i,\delta_{m,\varepsilon}(t_m),\sigma_m,\xi}\right\|_h\right)=\mathrm{O}\left(\sqrt{\varepsilon}\right),\end{aligned} \tag{3.53}$$

where the functions $\gamma_{i,\delta_{m,\varepsilon}(t_m),\sigma_m,\xi}$ are as in Lemma 4.1. Since the function $\phi_{\delta_\varepsilon(t),\sigma,\exp_\xi y}$ belongs to $K^\perp_{\delta_\varepsilon(t),\sigma,\exp_\xi y}$, differentiating $\left\langle Z_{l,\delta_{j,\varepsilon}(t_j),\sigma_j,\exp_\xi y}, \phi_{\delta_\varepsilon(t),\sigma,\exp_\xi y}\right\rangle_h = 0$ with respect to y_i, we get

$$\left\langle Z_{l,\delta_{j,\varepsilon}(t_j),\sigma_j,\xi}, \frac{d}{dy_i}\phi_{\delta_\varepsilon(t),\sigma,\exp_\xi y}\bigg|_{y=0}\right\rangle_h = -\left\langle \frac{d}{dy_i} Z_{l,\delta_{j,\varepsilon}(t_j),\sigma_j,\exp_\xi y}\bigg|_{y=0}, \phi_{\delta_\varepsilon(t),\sigma,\xi}\right\rangle_h . \tag{3.54}$$

Proceeding as in the proof of Lemma 4.1, we find

$$\left\| \frac{d}{dy_i} Z_{l,\delta_{j,\varepsilon}(t_j),\sigma_j,\exp_\xi y}\bigg|_{y=0} + \frac{d}{dy_i} Z_{l,\delta_{j,\varepsilon}(t_j),\sigma_j,\xi}\left(\exp_\xi y\right)\bigg|_{y=\exp_\xi^{-1} x}\right\|_h = \mathrm{O}\left(1\right). \tag{3.55}$$

By (3.54), (3.55), Proposition 2.1, Lemma 5.1, and Cauchy–Schwarz inequality, we get

$$\begin{aligned}&\left|\left\langle Z_{l,\delta_{j,\varepsilon}(t_j),\sigma_j,\xi}, \frac{d}{dy_i}\phi_{\delta_\varepsilon(t),\sigma,\exp_\xi y}\bigg|_{y=0}\right\rangle_h - \left\langle \frac{d}{dy_i} Z_{l,\delta_{j,\varepsilon}(t_j),\sigma_j,\xi}\left(\exp_\xi y\right)\bigg|_{y=\exp_\xi^{-1} x}, \phi_{\delta_\varepsilon(t),\sigma,\xi}\right\rangle_h\right| \\ &\le \left\| \frac{d}{dy_i} Z_{l,\delta_{j,\varepsilon}(t_j),\sigma_j,\exp_\xi y}\bigg|_{y=0} + \frac{d}{dy_i} Z_{l,\delta_{j,\varepsilon}(t_j),\sigma_j,\xi}\left(\exp_\xi y\right)\bigg|_{y=\exp_\xi^{-1} x}\right\|_h \left\|\phi_{\delta_\varepsilon(t),\sigma,\xi}\right\|_h = \mathrm{o}\left(1\right).\end{aligned} \tag{3.56}$$

Since there holds $Z_{l,\delta_{j,\varepsilon}(t_j),\sigma_j,\xi} \equiv 0$ in $M \backslash B_\xi\left(r_0\right)$, where the real number r_0 is as in Section 2, integrating by parts, we get

$$\begin{aligned}&\left\langle \frac{d}{dy_i} Z_{l,\delta_{j,\varepsilon}(t_j),\sigma_j,\xi}\left(\exp_\xi y\right)\bigg|_{y=\exp_\xi^{-1} x}, \phi_{\delta_\varepsilon(t),\sigma,\xi}\right\rangle_h \\ &\quad + \left\langle Z_{l,\delta_{j,\varepsilon}(t_j),\sigma_j,\xi}, \frac{d}{dy_i}\phi_{\delta_\varepsilon(t),\sigma,\xi}\left(\exp_\xi y\right)\bigg|_{y=\exp_\xi^{-1} x}\right\rangle_h + \int_{B_0(r_0)} \frac{d}{dy_\alpha}\left(Z_{l,\delta_{j,\varepsilon}(t_j),\sigma_j,\xi}\left(\exp_\xi y\right)\right) \\ &\qquad\qquad \times \frac{d}{dy_\beta}\left(\phi_{\delta_\varepsilon(t),\sigma,\xi}\left(\exp_\xi y\right)\right) \frac{d}{dy_i}\left(g^{\alpha\beta}\left(\exp_\xi y\right)\sqrt{\left|\exp_\xi^* g\left(y\right)\right|}\right) dy\Bigg) \\ &+ \int_{B_0(r_0)} Z_{l,\delta_{j,\varepsilon}(t_j),\sigma_j,\xi}\left(\exp_\xi y\right)\phi_{\delta_\varepsilon(t),\sigma,\xi}\left(\exp_\xi y\right)\frac{d}{dy_i}\left(h\left(\exp_\xi y\right)\sqrt{\left|\exp_\xi^* g\left(y\right)\right|}\right) dy = 0,\end{aligned} \tag{3.57}$$

where the function $\left|\exp_\xi^* g\right|$ is the determinant of the metric and the functions $g^{\alpha\beta}$ are the components of g^{-1} in geodesic normal coordinates. By (3.57), Proposition 2.1, Lemma 5.1, and Cauchy–Schwarz inequality, we get

$$\begin{aligned}&\left\langle \frac{d}{dy_i} Z_{l,\delta_{j,\varepsilon}(t_j),\sigma_j,\xi}\left(\exp_\xi y\right)\bigg|_{y=\exp_\xi^{-1} x}, \phi_{\delta_\varepsilon(t),\sigma,\xi}\right\rangle_h + \left\langle Z_{l,\delta_{j,\varepsilon}(t_j),\sigma_j,\xi}, \frac{d}{dy_i}\phi_{\delta_\varepsilon(t),\sigma,\xi}\left(\exp_\xi y\right)\bigg|_{y=\exp_\xi^{-1} x}\right\rangle_h \\ &\quad = \mathrm{O}\left(\left\|Z_{l,\delta_{j,\varepsilon}(t_j),\sigma_j,\xi}\right\|_{1,2}\left\|\phi_{\delta_\varepsilon(t),\sigma,\xi}\right\|_{1,2}\right) = \mathrm{o}\left(1\right).\end{aligned} \tag{3.58}$$

By (3.52), (3.53), (3.56), (3.58), and Lemma 3.2, we get

$$\begin{aligned}&\frac{d}{dy_i}\mathcal{J}_\varepsilon\left(t,\sigma,\exp_\xi y\right)\bigg|_{y=0} = -\sum_{l=1}^n\sum_{j=1}^k \lambda_{l,j,\delta_\varepsilon(t),\sigma,\xi} \\ &\left\langle Z_{l,\delta_{j,\varepsilon}(t_j),\sigma_j,\xi}, \frac{d}{dy_i}\left(\mathrm{Tower}_{\delta_\varepsilon(t),\sigma,\xi}\left(\exp_\xi y\right) + \phi_{\delta_\varepsilon(t),\sigma,\xi}\left(\exp_\xi y\right)\right)\bigg|_{y=\exp_\xi^{-1} x}\right\rangle_h + \mathrm{o}\left(\varepsilon\right)\end{aligned} \tag{3.59}$$

as $\varepsilon \to 0$. From now on, we fix a real number r_1 such that $r_0 < r_1 < i_g$, where r_0 is as in Section 2 and i_g is the injectivity radius of the manifold. We let η be a smooth cutoff function such that $0 \le \eta \le 1$ in $\mathbb{R}^n$, $\eta = 1$ in $B_0(r_0)$, and $\eta = 0$ in $M \backslash B_0(r_1)$. By (3.20), (3.59), and since there holds $Z_{l,\delta_{j,\varepsilon}(t_j),\sigma_j,\xi} \equiv 0$ in $M \backslash B_\xi(r_0)$ for all $l = 0, \dots, n$ and $j = 1, \dots, n$, we get

$$\frac{d}{dy_i}\mathcal{J}_\varepsilon\left(t,\sigma,\exp_\xi y\right)\bigg|_{y=0} + DJ_\varepsilon\left(\mathrm{Tower}_{\delta_\varepsilon(t),\sigma,\xi} + \phi_{\delta_\varepsilon(t),\sigma,\xi}\right).\Bigg(\frac{d}{dy_i}\,\mathrm{Tower}_{\delta_\varepsilon(t),\sigma,\xi}\left(\exp_\xi y\right)\bigg|_{y=\exp_\xi^{-1}x} + \eta\left(\exp_\xi^{-1}x\right)\frac{d}{dy_i}\phi_{\delta_\varepsilon(t),\sigma,\xi}\left(\exp_\xi y\right)\bigg|_{y=\exp_\xi^{-1}x}\Bigg) = \mathrm{o}(\varepsilon) \quad (3.60)$$

as $\varepsilon \to 0$. Integrating by parts, we get

$$\begin{aligned}
&DJ_\varepsilon\left(\mathrm{Tower}_{\delta_\varepsilon(t),\sigma,\xi} + \phi_{\delta_\varepsilon(t),\sigma,\xi}\right)\\
&\quad.\left(\eta\left(\exp_\xi^{-1}x\right)\frac{d}{dy_i}\left(\mathrm{Tower}_{\delta_\varepsilon(t),\sigma,\xi}\left(\exp_\xi y\right) + \phi_{\delta_\varepsilon(t),\sigma,\xi}\left(\exp_\xi y\right)\right)\bigg|_{y=\exp_\xi^{-1}x}\right)\\
&= \int_M \left\langle \nabla\phi_{\delta_\varepsilon(t),\sigma,\xi}, \nabla\eta\left(\exp_\xi^{-1}(x)\right)\right\rangle_g \frac{d}{dy_i}\phi_{\delta_\varepsilon(t),\sigma,\xi}\left(\exp_\xi y\right)\bigg|_{y=\exp_\xi^{-1}x} dv_g\\
&- \frac{1}{2}\int_{\mathbb{R}^n}\frac{d}{dy_\alpha}\left(\mathrm{Tower}_{\delta_\varepsilon(t),\sigma,\xi}\left(\exp_\xi y\right) + \phi_{\delta_\varepsilon(t),\sigma,\xi}\left(\exp_\xi y\right)\right)\\
&\quad\times\frac{d}{dy_\beta}\left(\mathrm{Tower}_{\delta_\varepsilon(t),\sigma,\xi}\left(\exp_\xi y\right) + \phi_{\delta_\varepsilon(t),\sigma,\xi}\left(\exp_\xi y\right)\right)\frac{d}{dy_i}\left(g^{\alpha\beta}\left(\exp_\xi y\right)\eta(y)\sqrt{\left|\exp_\xi^* g(y)\right|}\right)dy\\
&- \frac{1}{2}\int_{\mathbb{R}^n}\left(\mathrm{Tower}_{\delta_\varepsilon(t),\sigma,\xi}\left(\exp_\xi y\right) + \phi_{\delta_\varepsilon(t),\sigma,\xi}\left(\exp_\xi y\right)\right)^2\frac{d}{dy_i}\left(h\left(\exp_\xi y\right)\eta(y)\sqrt{\left|\exp_\xi^* g(y)\right|}\right)dy\\
&+ \int_{\mathbb{R}^n}F_\varepsilon\left(\mathrm{Tower}_{\delta_\varepsilon(t),\sigma,\xi}\left(\exp_\xi y\right) + \phi_{\delta_\varepsilon(t),\sigma,\xi}\left(\exp_\xi y\right)\right)\frac{d}{dy_i}\left(\eta(y)\sqrt{\left|\exp_\xi^* g(y)\right|}dy\right), \qquad (3.61)
\end{aligned}$$

where $F_\varepsilon(u)$ is as in (2.16), the function $\left|\exp_\xi^* g\right|$ is the determinant of the metric, and the functions $g^{\alpha\beta}$ are the components of g^{-1} in geodesic normal coordinates. We are led to estimates each term in (3.61). First, by Proposition 2.1 and Lemma 5.1, we get

$$\int_M \left\langle \nabla\phi_{\delta_\varepsilon(t),\sigma,\xi}, \nabla\eta\left(\exp_\xi^{-1}(x)\right)\right\rangle_g \frac{d}{dy_i}\phi_{\delta_\varepsilon(t),\sigma,\xi}\left(\exp_\xi y\right)\bigg|_{y=\exp_\xi^{-1}x} dv_g = \mathrm{O}\left(\left\|\nabla\phi_{\delta_\varepsilon(t),\sigma,\xi}\right\|_2^2\right) = \mathrm{o}(\varepsilon) \quad (3.62)$$

as $\varepsilon \to 0$. By Cartan's expansion of the metric in geodesic normal coordinates, we get that for any $\alpha, \beta = 1, \dots, n$, for y close to 0, there hold

$$g^{\alpha\beta}\left(\exp_\xi y\right) = \delta^{\alpha\beta} + \frac{1}{3}\delta^{\gamma\beta}R^\alpha_{\mu\gamma\nu}(\xi)\,y^\mu y^\nu + \frac{1}{6}\delta^{\gamma\beta}R^\alpha_{\mu\gamma\nu,\sigma}(\xi)\,y^\mu y^\nu y^\sigma + \mathrm{O}\left(|y|^4\right), \qquad (3.63)$$

$$\sqrt{\left|\exp_\xi^* g(y)\right|} = 1 - \frac{1}{6}R_{\mu\nu}(\xi)\,y^\mu y^\nu - \frac{1}{12}R_{\mu\nu,\sigma}(\xi)\,y^\mu y^\nu y^\sigma + \mathrm{O}\left(|y|^4\right), \qquad (3.64)$$

where the real numbers $\delta^{\alpha\beta}$ are the Kronecker symbols, the functions $R^\alpha_{\mu\beta\nu}$ are the components of the Riemann curvature tensor, the functions $R_{\mu\nu}$ are the components of the Ricci curvature tensor in geodesic normal coordinates. By (3.63) and (3.64), we get

$$\begin{aligned}
&\int_{\mathbb{R}^n}\frac{d}{dy_\alpha}\left(\mathrm{Tower}_{\delta_\varepsilon(t),\sigma,\xi}\left(\exp_\xi y\right) + \phi_{\delta_\varepsilon(t),\sigma,\xi}\left(\exp_\xi y\right)\right)\\
&\quad\times\frac{d}{dy_\beta}\left(\mathrm{Tower}_{\delta_\varepsilon(t),\sigma,\xi}\left(\exp_\xi y\right) + \phi_{\delta_\varepsilon(t),\sigma,\xi}\left(\exp_\xi y\right)\right)\frac{d}{dy_i}\left(g^{\alpha\beta}\left(\exp_\xi y\right)\eta(y)\sqrt{\left|\exp_\xi^* g(y)\right|}\right)dy
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{j=1}^{k} \int_{\mathbb{R}^n} \frac{d}{dy_\alpha} \left(W_{\delta_{j,\varepsilon(t_j)},\sigma_j,\xi}\left(\exp_\xi y\right)\right) \frac{d}{dy_\beta} \left(W_{\delta_{j,\varepsilon(t_j)},\sigma_j,\xi}\left(\exp_\xi y\right)\right) \\
&\quad \times \frac{d}{dy_i}\left(g^{\alpha\beta}\left(\exp_\xi y\right)\sqrt{\left|\exp_\xi^* g\left(y\right)\right|}\right) dy \\
&\quad + \mathrm{O}\left(\sum_{j=1}^{k}\sum_{l\neq j}\int_M \left|\nabla W_{\delta_{j,\varepsilon(t_j)},\sigma_j,\xi}\right|_g \left|\nabla W_{\delta_{l,\varepsilon(t_l)},\sigma_l,\xi}\right|_g d_g\left(x,\xi\right) dv_g\right) \\
&+ \mathrm{O}\left(\sum_{j=1}^{k}\int_M \left|\nabla W_{\delta_{j,\varepsilon(t_j)},\sigma_j,\xi}\right|_g \left|\nabla \phi_{\delta_\varepsilon(t),\sigma,\xi}\right|_g d_g\left(x,\xi\right) dv_g\right) + \mathrm{O}\left(\int_M \left|\nabla \phi_{\delta_\varepsilon(t),\sigma,\xi}\right|_g^2 dv_g\right) \quad (3.65)
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{d}{dy_i}\left(g^{\alpha\beta}\left(\exp_\xi y\right)\sqrt{\left|\exp_\xi^* g\left(y\right)\right|}\right) &= \frac{1}{3}\left(\delta^{\gamma\beta}R^\alpha_{i\gamma\mu}\left(\xi\right) + \delta^{\gamma\beta}R^\alpha_{\mu\gamma i}\left(\xi\right) - \delta^{\alpha\beta}R_{i\mu}\left(\xi\right)\right) y^\mu \\
+ \frac{1}{12}\big(2\delta^{\gamma\beta}R^\alpha_{i\gamma\mu,\nu}\left(\xi\right) + 2\delta^{\gamma\beta}R^\alpha_{\mu\gamma i,\nu}\left(\xi\right) &+ 2\delta^{\gamma\beta}R^\alpha_{\mu\gamma\nu,i}\left(\xi\right) - 2\delta^{\alpha\beta}R_{i\mu,\nu}\left(\xi\right) - \delta^{\alpha\beta}R_{\mu\nu,i}\left(\xi\right)\big) y^\mu y^\nu \\
&+ \mathrm{O}\left(\left|y\right|^3\right). \quad (3.66)
\end{aligned}
$$

Moreover, it follows from the second Bianchi estimate that

$$
2\delta^{\mu\nu}R_{i\mu,\nu}\left(\xi\right) = \frac{d}{dy_i}\mathrm{Scal}_g\left(\exp_\xi y\right)\Big|_{y=0}. \quad (3.67)
$$

Using (3.66) and (3.67) together with symmetry properties of the components of the Riemann curvature tensor, we find

$$
\begin{aligned}
&\int_{\mathbb{R}^n} \frac{d}{dy_\alpha}\left(W_{\delta_{j,\varepsilon(t_j)},\sigma_j,\xi}\left(\exp_\xi y\right)\right)\frac{d}{dy_\beta}\left(W_{\delta_{j,\varepsilon(t_j)},\sigma_j,\xi}\left(\exp_\xi y\right)\right)\frac{d}{dy_i}\left(g^{\alpha\beta}\left(\exp_\xi y\right)\sqrt{\left|\exp_\xi^* g\left(y\right)\right|}\right) dy \\
&= -\frac{1}{12} n^{\frac{n-4}{2}}\left(n-2\right)^{\frac{n+2}{2}}\omega_{n-1}\int_0^{\frac{r_0}{2\delta_{j,\varepsilon}(t_j)}} \frac{r^{n+1}}{\left(1+r^2\right)^n}\Bigg(4\left(n-2\right)\mathrm{Ric}_g\left(\xi\right).\left(\Psi_\xi\left(\sigma_j\right),\Psi_\xi\left(e_i\right)\right)\delta_{j,\varepsilon}\left(t_j\right) \\
&\quad + \left(n-2\right)\left(2\left\langle \nabla \mathrm{Ric}_g\left(\exp_\xi y\right).\left(\Psi_\xi\left(\sigma_j\right),\Psi_\xi\left(e_i\right)\right)\big|_{y=0},\sigma_j\right\rangle\right. \\
&\quad \left.+ \frac{d}{dy_i}\mathrm{Ric}_g\left(\exp_\xi y\right).\left(\Psi_\xi\left(\sigma_j\right),\Psi_\xi\left(\sigma_j\right)\right)\Big|_{y=0}\right)\delta_{j,\varepsilon}\left(t_j\right)^2 \\
&\quad + 2\frac{d}{dy_i}\mathrm{Scal}_g\left(\exp_\xi y\right)\Big|_{y=0}\delta_{j,\varepsilon}\left(t_j\right)^2 r^2 + \mathrm{O}\left(\delta_{j,\varepsilon}\left(t_j\right)^3 r^3\right)\Bigg) dr + \mathrm{O}\left(\delta_{j,\varepsilon}\left(t_j\right)^{n-2}\right) \\
&= -\frac{n-2}{12n}K_n^{-n}\Bigg(4\,\mathrm{Ric}_g\left(\xi\right).\left(\Psi_\xi\left(\sigma_j\right),\Psi_\xi\left(e_i\right)\right)t_j\mu_\varepsilon\varepsilon^{\frac{n+4j-6}{2(n-2)}} \\
&\quad + \chi_n\left(2\left\langle \nabla \mathrm{Ric}_g\left(\exp_\xi y\right).\left(\Psi_\xi\left(\sigma_j\right),\Psi_\xi\left(e_i\right)\right)\big|_{y=0},\sigma_j\right\rangle\right. \\
&\quad \left.+ \frac{d}{dy_i}\mathrm{Ric}_g\left(\exp_\xi y\right).\left(\Psi_\xi\left(\sigma_j\right),\Psi_\xi\left(\sigma_j\right)\right)\Big|_{y=0}\right)t_j^2\varepsilon^{\frac{n+4j-6}{n-2}} \\
&\quad + \frac{2\theta_n\pi_{n,j}}{n-2}\frac{d}{dy_i}\mathrm{Scal}_g\left(\exp_\xi y\right)\Big|_{y=0} t_j^2\varepsilon^{\frac{n+4j-6}{n-2}}\Bigg) + \mathrm{o}\left(\varepsilon^{\frac{n+4j-6}{n-2}}\right), \quad (3.68)
\end{aligned}
$$

where Scal_g is the scalar curvature, Ric_g is the Ricci curvature, Ψ_ξ is as in Section 2, ω_{n-1} is the volume of the unit $(n-1)$–sphere, K_n is as in (3.2), θ_n, χ_n, and $\pi_{n,j}$ are as in (3.11)–(3.12). For any $l > j$, using (3.1), we find

$$\int_M \left|\nabla W_{\delta_{j,\varepsilon}(t_j),\sigma_j,\xi}\right|_g \left|\nabla W_{\delta_{l,\varepsilon}(t_l),\sigma_l,\xi}\right|_g d_g(x,\xi)\, dv_g = \mathrm{O}\left(\frac{\delta_{l,\varepsilon}(t_l)^{\frac{n-2}{2}}}{\delta_{j,\varepsilon}(t_j)^{\frac{n-4}{2}}}\int_0^{\frac{r_0}{\delta_{j,\varepsilon}(t_j)}} \frac{r dr}{(1+r^2)^{\frac{n-1}{2}}}\right)$$
$$= \mathrm{O}\left(\mu_\varepsilon \varepsilon^{l-\frac{n-4}{n-2}j+\frac{n-6}{2(n-2)}}\right). \tag{3.69}$$

For any $j = 1, \dots, k$, by Cauchy–Schwarz inequality, we get

$$\int_M \left|\nabla W_{\delta_{j,\varepsilon}(t_j),\sigma_j,\xi}\right|_g \left|\nabla \phi_{\delta_\varepsilon(t),\sigma,\xi}\right|_g d_g(x,\xi)\, dv_g \le \left\|\left(\nabla W_{\delta_{j,\varepsilon}(t_j),\sigma_j,\xi}\right) d_g(x,\xi)\right\|_2 \left\|\nabla \phi_{\delta_\varepsilon(t),\sigma,\xi}\right\|_2 . \tag{3.70}$$

Using (3.1), we find

$$\int_M \left|\nabla W_{\delta_{j,\varepsilon}(t_j),\sigma_j,\xi}\right|_g^2 d_g(x,\xi)^2\, dv_g = \mathrm{O}\left(\delta_{j,\varepsilon}(t_j)^2 \int_0^{\frac{r_0}{\delta_{j,\varepsilon}(t_j)}} \frac{r^{n+1} dr}{(1+r^2)^{n-1}}\right) = \mathrm{O}\left(\varepsilon^{\frac{n+4j-6}{n-2}}\right). \tag{3.71}$$

By (3.70), (3.71), Proposition 2.1, and Lemma 5.1, we get

$$\int_M \left|\nabla W_{\delta_{j,\varepsilon}(t_j),\sigma_j,\xi}\right|_g \left|\nabla \phi_{\delta_\varepsilon(t),\sigma,\xi}\right|_g d_g(x,\xi)\, dv_g = \mathrm{o}(\varepsilon) \quad \text{and} \quad \int_M \left|\nabla \phi_{\delta_\varepsilon(t),\sigma,\xi}\right|_g^2 dv_g = \mathrm{o}(\varepsilon) \tag{3.72}$$

as $\varepsilon \to 0$. By (3.65), (3.68), (3.69), and (3.72), we get

$$\begin{aligned}
&\int_{\mathbb{R}^n} \frac{d}{dy_\alpha}\left(\mathrm{Tower}_{\delta_\varepsilon(t),\sigma,\xi}\left(\exp_\xi y\right) + \phi_{\delta_\varepsilon(t),\sigma,\xi}\left(\exp_\xi y\right)\right)\\
&\quad \times \frac{d}{dy_\beta}\left(\mathrm{Tower}_{\delta_\varepsilon(t),\sigma,\xi}\left(\exp_\xi y\right) + \phi_{\delta_\varepsilon(t),\sigma,\xi}\left(\exp_\xi y\right)\right) \frac{d}{dy_i}\left(g^{\alpha\beta}\left(\exp_\xi y\right)\eta(y)\sqrt{\left|\exp_\xi^* g(y)\right|}\right) dy\\
&= -\frac{n-2}{12n} K_n^{-n}\Bigg(4\sum_{j=1}^k \mathrm{Ric}_g(\xi).\left(\Psi_\xi(\sigma_j), \Psi_\xi(e_i)\right) t_j \varepsilon^{\frac{n+4j-6}{2(n-2)}}\\
&\quad + \chi_n\Bigg(2\left\langle \nabla \mathrm{Ric}_g\left(\exp_\xi y\right).\left(\Psi_\xi(\sigma_1), \Psi_\xi(e_i)\right)\Big|_{y=0}, \sigma_1\right\rangle\\
&\quad + \frac{d}{dy_i}\mathrm{Ric}_g\left(\exp_\xi y\right).\left(\Psi_\xi(\sigma_1), \Psi_\xi(\sigma_1)\right)\Big|_{y=0}\Bigg) t_1^2\varepsilon + \frac{2\theta_n}{n-2}\frac{d}{dy_i}\mathrm{Scal}_g\left(\exp_\xi y\right)\Big|_{y=0} t_1^2\varepsilon\Bigg) + \mathrm{o}(\varepsilon)
\end{aligned} \tag{3.73}$$

as $\varepsilon \to 0$, where θ_n and χ_n are as in (3.11)–(3.12). By (3.64), we get

$$\begin{aligned}
&\int_{\mathbb{R}^n} \left(\mathrm{Tower}_{\delta_\varepsilon(t),\sigma,\xi}\left(\exp_\xi y\right) + \phi_{\delta_\varepsilon(t),\sigma,\xi}\left(\exp_\xi y\right)\right)^2 \frac{d}{dy_i}\left(h\left(\exp_\xi y\right)\eta(y)\sqrt{\left|\exp_\xi^* g(y)\right|}\right) dy\\
&= \frac{d}{dy_i} h\left(\exp_\xi y\right)\Big|_{y=0} \sum_{j=1}^k \int_M W_{\delta_{j,\varepsilon}(t_j),\sigma_j,\xi}^2 dv_g + \mathrm{O}\left(\sum_{j=1}^k \int_M W_{\delta_{j,\varepsilon}(t_j),\sigma_j,\xi}^2 d_g(x,\xi)\, dv_g\right)\\
&\quad + \mathrm{O}\left(\sum_{j=1}^k \sum_{l\ne j} \int_M W_{\delta_{j,\varepsilon}(t_j),\sigma_j,\xi} W_{\delta_{l,\varepsilon}(t_l),\sigma_l,\xi} dv_g\right) + \mathrm{O}\left(\sum_{j=1}^k \int_M W_{\delta_{j,\varepsilon}(t_j),\sigma_j,\xi}\phi_{\delta_\varepsilon(t),\sigma,\xi} dv_g\right)\\
&\quad + \mathrm{O}\left(\int_M \phi_{\delta_\varepsilon(t),\sigma,\xi}^2 dv_g\right)
\end{aligned} \tag{3.74}$$

and

$$\int_M W^2_{\delta_{j,\varepsilon}(t_j),\sigma_j,\xi} dv_g = n^{\frac{n-2}{2}} (n-2)^{\frac{n-2}{2}} \omega_{n-1} \delta_{j,\varepsilon}(t_j)^2 \int_0^{\frac{r_0}{2\delta_{j,\varepsilon}(t_j)}} \frac{r^{n-1}}{(1+r^2)^{n-2}} \left(1 + \mathrm{O}\left(\delta_{j,\varepsilon}(t_j)^2 r^2\right)\right) dr$$

$$+ \mathrm{O}\left(\delta_{j,\varepsilon}(t_j)^{n-2}\right) = \frac{2\pi_{n,j}\gamma_n}{n} K_n^{-n} t_j^2 \varepsilon^{\frac{n+4j-6}{n-2}} + \mathrm{o}\left(\varepsilon^{\frac{n+4j-6}{n-2}}\right) \quad (3.75)$$

for all $j = 1, \dots, k$, where ω_{n-1} is the volume of the unit $(n-1)$–sphere, K_n is as in (3.2), γ_n and $\pi_{n,j}$ are as in (3.11)–(3.12). For any $j = 1, \dots, k$, using (3.1), we find

$$\int_M W^2_{\delta_{j,\varepsilon}(t_j),\sigma_j,\xi} d_g(x,\xi)\, dv_g = \mathrm{O}\left(\delta_{j,\varepsilon}(t_j)^3 \int_0^{\frac{r_0}{\delta_{j,\varepsilon}(t_j)}} \frac{r^n dr}{(1+r^2)^{n-2}}\right)$$

$$= \begin{cases} \mathrm{O}\left(\mu_\varepsilon^2 \varepsilon^{2j-1}\right) & \text{if } n = 4, \\ \mathrm{O}\left(\varepsilon^{\frac{4j-1}{2}} |\ln \varepsilon|\right) & \text{if } n = 5, \\ \mathrm{O}\left(\varepsilon^{\frac{3(n+4j-6)}{2(n-2)}}\right) & \text{if } n \geq 6. \end{cases} \quad (3.76)$$

For any $l > j$, we find

$$\int_M W_{\delta_{j,\varepsilon}(t_j),\sigma_j,\xi} W_{\delta_{l,\varepsilon}(t_l),\sigma_l,\xi} dv_g = \mathrm{O}\left(\frac{\delta_{l,\varepsilon}(t_l)^{\frac{n-2}{2}}}{\delta_{j,\varepsilon}(t_j)^{\frac{n-6}{2}}} \int_0^{\frac{r_0}{\delta_{j,\varepsilon}(t_j)}} \frac{r dr}{(1+r^2)^{\frac{n-2}{2}}}\right) = \mathrm{O}\left(\varepsilon^{l - \frac{n-6}{n-2}(j-1)}\right). \quad (3.77)$$

For any $j = 1, \dots, k$, by Cauchy–Schwarz inequality, (3.1), Proposition 2.1, and Lemma 5.1, we get

$$\int_M W_{\delta_{j,\varepsilon}(t_j),\sigma_j,\xi} \phi_{\delta_\varepsilon(t),\sigma,\xi} dv_g = \mathrm{O}\left(\left\|W_{\delta_{j,\varepsilon}(t_j),\sigma_j,\xi}\right\|_2 \left\|\phi_{\delta_\varepsilon(t),\sigma,\xi}\right\|_2\right)$$

$$= \mathrm{O}\left(\delta_{j,\varepsilon}(t_j) \left\|U \mathbf{1}_{B_0(r_0/\delta_{j,\varepsilon}(t_j))}\right\|_2 \left\|\phi_{\delta_\varepsilon(t),\sigma,\xi}\right\|_2\right) \quad (3.78)$$

Moreover, by Proposition 2.1 and Lemma 5.1, we get

$$\int_M \left|\phi_{\delta_\varepsilon(t),\sigma,\xi}\right|_g^2 dv_g = \mathrm{o}(\varepsilon) \quad (3.79)$$

as $\varepsilon \to 0$. By (3.74)–(3.79), we get

$$\int_{\mathbb{R}^n} \left(\mathrm{Tower}_{\delta_\varepsilon(t),\sigma,\xi}\left(\exp_\xi y\right) + \phi_{\delta_\varepsilon(t),\sigma,\xi}\left(\exp_\xi y\right)\right)^2 \frac{d}{dy_i}\left(h\left(\exp_\xi y\right) \eta(y) \sqrt{\left|\exp_\xi^* g(y)\right|}\right) dy$$

$$= \frac{2\gamma_n}{n} K_n^{-n} \frac{d}{dy_i}\left(h\left(\exp_\xi y\right)\right)\Big|_{y=0} t_1^2 \varepsilon + \mathrm{o}(\varepsilon) \quad (3.80)$$

as $\varepsilon \to 0$, where γ_n is as in (3.11). Using (3.64) and letting $A_{j,\delta_\varepsilon(t),\xi}$ be as in (3.31) for all $j = 1, \dots, k$, we get

$$\Bigg| \int_{\mathbb{R}^n} \Bigg(F_\varepsilon\left(\mathrm{Tower}_{\delta_\varepsilon(t),\sigma,\xi}\left(\exp_\xi y\right) + \phi_{\delta_\varepsilon(t),\sigma,\xi}\left(\exp_\xi y\right)\right)$$

$$- \frac{1}{2^* - \varepsilon} \sum_{j=1}^k W_{\delta_{j,\varepsilon}(t_j),\sigma_j,\xi}\left(\exp_\xi y\right)^{2^*-\varepsilon} \Bigg) \frac{d}{dy_i}\left(\eta(y) \sqrt{\left|\exp_\xi^* g(y)\right|}\right) dy \Bigg|$$

$$\leq \sum_{j=1}^{k}\Bigg(\int_{A_{j,\delta_\varepsilon(t),\xi}}\left|F_\varepsilon\left(\mathrm{Tower}_{\delta_\varepsilon(t),\sigma,\xi}+\eta\left(\exp_\xi^{-1}x\right)\phi_{\delta_\varepsilon(t),\sigma,\xi}\right)-\frac{1}{2^*-\varepsilon}W^{2^*-\varepsilon}_{\delta_{j,\varepsilon}(t_j),\sigma_j,\xi}\right|d_g\left(x,\xi\right)dv_g$$
$$+\frac{1}{2^*-\varepsilon}\sum_{l\neq j}\int_{A_{j,\delta_\varepsilon(t),\xi}}W^{2^*-\varepsilon}_{\delta_{l,\varepsilon}(t_l),\sigma_l,\xi}d_g\left(x,\xi\right)dv_g\Bigg)+\frac{1}{2^*-\varepsilon}\int_{M\backslash B_\xi(r_0)}\phi^{2^*-\varepsilon}_{\delta_\varepsilon(t),\sigma,\xi}dv_g. \quad (3.81)$$

By the Mean Value Theorem, for any $j=1,\dots,k$, we get

$$\int_{A_{j,\delta_\varepsilon(t),\xi}}\left|F_\varepsilon\left(\mathrm{Tower}_{\delta_\varepsilon(t),\sigma,\xi}+\eta\left(\exp_\xi^{-1}x\right)\phi_{\delta_\varepsilon(t),\sigma,\xi}\right)-\frac{1}{2^*-\varepsilon}W^{2^*-\varepsilon}_{\delta_{j,\varepsilon}(t_j),\sigma_j,\xi}\right|d_g\left(x,\xi\right)dv_g$$
$$=\mathrm{O}\Bigg(\sum_{l\neq j}\int_{A_{j,\delta_\varepsilon(t),\xi}}W^{2^*-1-\varepsilon}_{\delta_{j,\varepsilon}(t_j),\sigma_j,\xi}W_{\delta_{l,\varepsilon}(t_l),\sigma_l,\xi}d_g\left(x,\xi\right)dv_g+\sum_{l\neq j}\int_{A_{j,\delta_\varepsilon(t),\xi}}W^{2^*-\varepsilon}_{\delta_{l,\varepsilon}(t_l),\sigma_l,\xi}d_g\left(x,\xi\right)dv_g$$
$$+\int_{A_{j,\delta_\varepsilon(t),\xi}}W^{2^*-1-\varepsilon}_{\delta_{j,\varepsilon}(t_j),\sigma_j,\xi}\phi_{\delta_\varepsilon(t),\sigma,\xi}d_g\left(x,\xi\right)dv_g+\int_{A_{j,\delta_\varepsilon(t),\xi}}\phi^{2^*-\varepsilon}_{\delta_\varepsilon(t),\sigma,\xi}dv_g\Bigg). \quad (3.82)$$

For any $j=1,\dots,k$, using (3.64) and (3.67) together with symmetry properties of the components of the Riemann curvature tensor, we find

$$\frac{1}{2^*-\varepsilon}\int_{\mathbb{R}^n}W^{2^*-\varepsilon}_{\delta_{j,\varepsilon}(t_j),\sigma_j,\xi}\frac{d}{dy_i}\sqrt{\left|\exp_\xi^* g\left(y\right)\right|}dy$$
$$=-\frac{\left(n\left(n-2\right)\right)^{\frac{n-2}{4}\left(2^*-\varepsilon\right)}\omega_{n-1}}{12\left(2^*-\varepsilon\right)}\delta_{j,\varepsilon}\left(t_j\right)^{\frac{n-2}{2}\varepsilon}\int_0^{\frac{r_0}{2\delta_{j,\varepsilon}(t_j)}}\frac{r^{n-1}}{\left(1+r^2\right)^{\frac{n-2}{2}\left(2^*-\varepsilon\right)}}$$
$$\times\Bigg(4\,\mathrm{Ric}_g\left(\xi\right).\left(\Psi_\xi\left(\sigma_j\right),\Psi_\xi\left(e_i\right)\right)\delta_{j,\varepsilon}\left(t_j\right)+\Bigg(2\left\langle\nabla\,\mathrm{Ric}_g\left(\exp_\xi y\right).\left(\Psi_\xi\left(\sigma_j\right),\Psi_\xi\left(e_i\right)\right)\Big|_{y=0},\sigma_j\right\rangle$$
$$+\frac{d}{dy_i}\mathrm{Ric}_g\left(\exp_\xi y\right).\left(\Psi_\xi\left(\sigma_j\right),\Psi_\xi\left(\sigma_j\right)\right)\Big|_{y=0}\Bigg)\delta_{j,\varepsilon}\left(t_j\right)^2$$
$$+\frac{2}{n}\frac{d}{dy_i}\mathrm{Scal}_g\left(\exp_\xi y\right)\Big|_{y=0}\delta_{j,\varepsilon}\left(t_j\right)^2r^2+\mathrm{O}\left(\delta_{j,\varepsilon}\left(t_j\right)^3r^3\right)\Bigg)dr+\mathrm{O}\left(\delta_{j,\varepsilon}\left(t_j\right)^n\right)$$
$$=-\frac{n-2}{24n}K_n^{-n}\Bigg(4\,\mathrm{Ric}_g\left(\xi\right).\left(\Psi_\xi\left(\sigma_j\right),\Psi_\xi\left(e_i\right)\right)t_j\mu_\varepsilon\varepsilon^{\frac{n+4j-6}{2(n-2)}}$$
$$+\chi_n\Bigg(2\left\langle\nabla\,\mathrm{Ric}_g\left(\exp_\xi y\right).\left(\Psi_\xi\left(\sigma_j\right),\Psi_\xi\left(e_i\right)\right)\Big|_{y=0},\sigma_j\right\rangle$$
$$+\frac{d}{dy_i}\mathrm{Ric}_g\left(\exp_\xi y\right).\left(\Psi_\xi\left(\sigma_j\right),\Psi_\xi\left(\sigma_j\right)\right)\Big|_{y=0}+\frac{2}{n-2}\frac{d}{dy_i}\mathrm{Scal}_g\left(\exp_\xi y\right)\Big|_{y=0}\Bigg)t_j^2\varepsilon^{\frac{n+4j-6}{n-2}}\Bigg)$$
$$+\mathrm{o}\left(\varepsilon\right). \quad (3.83)$$

For any $l\neq j$, using (3.1), we find

$$\int_M W^{2^*-1-\varepsilon}_{\delta_{j,\varepsilon}(t_j),\sigma_j,\xi}W_{\delta_{l,\varepsilon}(t_l),\sigma_l,\xi}d_g\left(x,\xi\right)dv_g$$
$$=\begin{cases}\mathrm{O}\left(\dfrac{\delta_{l,\varepsilon}\left(t_l\right)^{\frac{n-2}{2}}}{\delta_{j,\varepsilon}\left(t_j\right)^{\frac{n-4}{2}}}\displaystyle\int_0^{\frac{r_0}{\delta_{j,\varepsilon}(t_j)}}\frac{r^2dr}{\left(1+r^2\right)^{\frac{n-2}{2}\left(2^*-1-\varepsilon\right)}}\right) & \text{if } j<l,\\ \mathrm{O}\left(\dfrac{\delta_{j,\varepsilon}\left(t_j\right)^{\frac{n-2}{2}}}{\delta_{l,\varepsilon}\left(t_l\right)^{\frac{n-4}{2}}}\displaystyle\int_0^{\frac{r_0}{\delta_{j,\varepsilon}(t_j)}}\frac{r^{n-1}dr}{\left(1+r^2\right)^{\frac{n-2}{2}\left(2^*-1-\varepsilon\right)}}\right) & \text{if } j>l,\end{cases}$$

$$= \begin{cases} \mathrm{O}\left(\mu_\varepsilon \varepsilon^{l-\frac{n-4}{n-2}j+\frac{n-6}{2(n-2)}}\right) & \text{if } j<l, \\ \mathrm{O}\left(\mu_\varepsilon \varepsilon^{j-\frac{n-4}{n-2}l+\frac{n-6}{2(n-2)}}\right) & \text{if } j>l, \end{cases} \tag{3.84}$$

$$\int_{A_{j,\delta_\varepsilon(t),\xi}} W^{2^*-\varepsilon}_{\delta_{l,\varepsilon}(t_l),\sigma_l,\xi} d_g(x,\xi)\, dv_g = \mathrm{O}\left(\delta_{l,\varepsilon}(t_l) \int_{\frac{\sqrt{\delta_{j,\varepsilon}(t_j)\delta_{j+1,\varepsilon}(t_{j+1})}}{\delta_{l,\varepsilon}(t_l)}}^{\frac{\sqrt{\delta_{j-1,\varepsilon}(t_{j-1})\delta_{j,\varepsilon}(t_j)}}{\delta_{l,\varepsilon}(t_l)}} \frac{r^n dr}{(1+r^2)^{\frac{n-2}{2}(2^*-\varepsilon)}}\right)$$

$$= \begin{cases} \mathrm{O}\left(\mu_\varepsilon \varepsilon^{\frac{2n}{n-2}l-\frac{2(n-1)}{n-2}j-\frac{n+4}{2(n-2)}}\right) & \text{if } j<l, \\ \mathrm{O}\left(\mu_\varepsilon \varepsilon^{\frac{2(n+1)}{n-2}j-\frac{2n}{n-2}l-\frac{n+8}{2(n-2)}}\right) & \text{if } j>l. \end{cases} \tag{3.85}$$

For any $j=1,\dots,k$, by Hölder's inequality and Sobolev's inequality, we get

$$\int_M W^{2^*-1-\varepsilon}_{\delta_{j,\varepsilon}(t_j),\sigma_j,\xi} \phi_{\delta_\varepsilon(t),\sigma,\xi} d_g(x,\xi)\, dv_g = \mathrm{O}\left(\left\| W^{2^*-1-\varepsilon}_{\delta_{j,\varepsilon}(t_j),\sigma_j,\xi} d_g(x,\xi)\right\|_{\frac{2^*-\varepsilon}{2^*-1-\varepsilon}} \left\|\phi_{\delta_\varepsilon(t),\sigma,\xi}\right\|_h\right). \tag{3.86}$$

Using (3.1), we find

$$\int_M W^{2^*-\varepsilon}_{\delta_{j,\varepsilon}(t_j),\sigma_j,\xi} d_g(x,\xi)^{\frac{2^*-\varepsilon}{2^*-1-\varepsilon}}\, dv_g = \mathrm{O}\left(\delta_{j,\varepsilon}(t_j)^{\frac{2^*-\varepsilon}{2^*-1-\varepsilon}} \int_0^{\frac{r_0}{\delta_{j,\varepsilon}(t_j)}} \frac{r^{n-1+\frac{2^*-\varepsilon}{2^*-1-\varepsilon}}dr}{(1+r^2)^{\frac{n-2}{2}(2^*-\varepsilon)}}\right)$$
$$= \mathrm{O}\left(\mu_\varepsilon^{\frac{2n}{n+2}} \varepsilon^{\frac{n(n+4j-6)}{(n+2)(n-2)}}\right). \tag{3.87}$$

By (3.86), (3.87), Proposition 2.1, and Lemma 5.1, we get

$$\int_M W^{2^*-1-\varepsilon}_{\delta_{j,\varepsilon}(t_j),\sigma_j,\xi} \phi_{\delta_\varepsilon(t),\sigma,\xi} d_g(x,\xi)\, dv_g = \mathrm{o}(\varepsilon) \quad \text{and} \quad \int_M \phi^{2^*-\varepsilon}_{\delta_\varepsilon(t),\sigma,\xi} dv_g = \mathrm{o}(\varepsilon) \tag{3.88}$$

as $\varepsilon\to 0$. By (3.81)–(3.88), we get

$$\int_{\mathbb{R}^n} F_\varepsilon\left(\mathrm{Tower}_{\delta_\varepsilon(t),\sigma,\xi}(\exp_\xi y) + \phi_{\delta_\varepsilon(t),\sigma,\xi}(\exp_\xi y)\right) \frac{d}{dy_i}\left(\eta(y)\sqrt{\left|\exp_\xi^* g(y)\right|}\right) dy$$
$$= -\frac{n-2}{24n} K_n^{-n}\Bigg(4\sum_{j=1}^k \mathrm{Ric}_g(\xi).(\Psi_\xi(\sigma_j),\Psi_\xi(e_i))\, t_j \varepsilon^{\frac{n+4j-6}{2(n-2)}}$$
$$+ \chi_n\Bigg(2\left\langle \nabla\mathrm{Ric}_g(\exp_\xi y).(\Psi_\xi(\sigma_1),\Psi_\xi(e_i))\big|_{y=0}, \sigma_1\right\rangle$$
$$+ \frac{d}{dy_i}\mathrm{Ric}_g(\exp_\xi y).(\Psi_\xi(\sigma_1),\Psi_\xi(\sigma_1))\Big|_{y=0} + \frac{2}{n-2}\frac{d}{dy_i}\mathrm{Scal}_g(\exp_\xi y)\Big|_{y=0}\Bigg) t_1^2\varepsilon\Bigg) + \mathrm{o}(\varepsilon) \tag{3.89}$$

as $\varepsilon\to 0$, where χ_n is as in (3.12). By (3.61), (3.73), (3.80), and (3.89), we get

$$DJ_\varepsilon\left(\mathrm{Tower}_{\delta_\varepsilon(t),\sigma,\xi} + \phi_{\delta_\varepsilon(t),\sigma,\xi}\right)$$
$$.\left(\eta(\exp_\xi^{-1} x)\frac{d}{dy_i}\left(\mathrm{Tower}_{\delta_\varepsilon(t),\sigma,\xi}(\exp_\xi y) + \phi_{\delta_\varepsilon(t),\sigma,\xi}(\exp_\xi y)\right)\Big|_{y=\exp_\xi^{-1}x}\right)$$
$$= \frac{\gamma_n}{n} K_n^{-n}\varepsilon t_1^2 \frac{d}{dy_i}\left(\frac{n-2}{4(n-1)}\mathrm{Scal}_g(\exp_\xi y) - h(\exp_\xi y)\right)\Big|_{y=0} + \mathrm{o}(\varepsilon) \tag{3.90}$$

as $\varepsilon \to 0$, where K_n is as in (3.2) and γ_n is as in (3.11). By (3.60), (3.62), and (3.90), we get that (3.41) is C^1–uniform with respect to ξ. □

4. First derivatives estimates

In Lemma 4.1 below, we give pointwise estimates for the first derivatives of the functions $W_{\delta_{j,\varepsilon}(t_j),\sigma_j,\xi}$.

Lemma 4.1. *Let A be a compact subset of $(\mathbb{R}_+^*)^k \times (\mathbb{R}^n)^{k-1} \times N$. For ε small, for any (t,σ,ξ) in A, for any $i = 1,\dots,n$ and $j = 1,\dots,k$, and for any point x in M, there hold*

$$\frac{d}{dt_j} W_{\delta_{j,\varepsilon}(t_j),\sigma_j,\xi} = \frac{1}{t_j}\bigg(Z_{0,\delta_{j,\varepsilon}(t_j),\sigma_j,\xi} + \sum_{i=1}^n \sigma_{ji} Z_{i,\delta_{j,\varepsilon}(t_j),\sigma_j,\xi} \bigg), \tag{4.1}$$

$$\frac{d}{d\sigma_{ji}} W_{\delta_{j,\varepsilon}(t_j),\sigma_j,\xi} = Z_{i,\delta_{j,\varepsilon}(t_j),\sigma_j,\xi}\,, \tag{4.2}$$

$$\frac{d}{dy_i} W_{\delta_{j,\varepsilon}(t_j),\sigma_j,\exp_\xi y}\bigg|_{y=0} = \frac{1}{\delta_{j,\varepsilon}(t_j)} Z_{i,\delta_{j,\varepsilon}(t_j),\sigma_j,\xi} + \gamma_{i,\delta_{j,\varepsilon}(t_j),\sigma_j,\xi}\,, \tag{4.3}$$

$$\frac{d}{dy_i} W_{\delta_{j,\varepsilon}(t_j),\sigma_j,\xi}\left(\exp_\xi y\right)\bigg|_{y=\exp_\xi^{-1} x} = -\frac{1}{\delta_{j,\varepsilon}(t_j)} Z_{i,\delta_{j,\varepsilon}(t_j),\sigma_j,\xi}(x) + \widetilde{\gamma}_{i,\delta_{j,\varepsilon}(t_j),\sigma_j,\xi}\,, \tag{4.4}$$

where $\delta_\varepsilon(t)$ is as in (2.11), the functions $Z_{i,\delta_{j,\varepsilon}(t_j),\sigma_j,\xi}$ are as in (2.7), and the functions $\gamma_{i,\delta_{j,\varepsilon}(t_j),\sigma_j,\xi}$ and $\widetilde{\gamma}_{i,\delta_{j,\varepsilon}(t_j),\sigma_j,\xi}$ are such that

$$\left\|\gamma_{i,\delta_{j,\varepsilon}(t_j),\sigma_j,\xi}\right\|_h \le C_A \varepsilon^{\frac{n+4j-6}{2(n-2)}}, \tag{4.5}$$

$$\left\|\widetilde{\gamma}_{i,\delta_{j,\varepsilon}(t_j),\sigma_j,\xi}\right\|_h \le C_A \mu_\varepsilon^{\frac{n-2}{2}} \varepsilon^{\frac{n+4j-6}{4}}, \tag{4.6}$$

for some positive constant C_A independent of ε, t, σ, and ξ.

Proof. We get (4.1) and (4.2) by straightforward computations. We prove (4.3) and (4.5). All our estimates in this proof are uniform with respect to (t,σ,ξ) in compact subsets of $(\mathbb{R}_+^*)^k \times (\mathbb{R}^n)^{k-1} \times N$ and with respect to ε in $(0,\varepsilon_0)$ for some fixed positive real number ε_0. For y close to 0, we identify $T_{\exp_\xi y}M$ with $\mathbb{R}^n$ thanks to a local orthonormal frame, parallel at ξ. For any $j = 1,\dots,k$, we get

$$\begin{aligned}\frac{d}{dy_i} W_{\delta_{j,\varepsilon}(t_j),\sigma_j,\exp_\xi y}(x)\bigg|_{y=0} &= W_{\delta_{j,\varepsilon}(t_j),\sigma_j,\xi}(x)\, \frac{d}{dy_i}\Big(\ln\chi\Big(d_{g_{\exp_\xi y}}\left(x,\exp_\xi y\right)\Big)\Big)\bigg|_{y=0} \\ &\quad - \frac{n^{\frac{n-2}{4}}(n-2)^{\frac{n+2}{4}}}{\delta_{j,\varepsilon}(t_j)} \sum_{j=1}^n Z_{j,\delta_{j,\varepsilon}(t_j),\sigma_j,\xi}(x)\, \frac{d}{dy_i}\big(\widetilde{\exp}_{\xi,\exp_\xi y}^{-1}(x)\big)_j\bigg|_{y=0}.\end{aligned} \tag{4.7}$$

Regarding the first term in the right-hand side of (4.7), we find

$$\begin{aligned}\left\|W_{\delta_{j,\varepsilon}(t_j),\sigma_j,\xi}\, \frac{d}{dy_i}\Big(\ln\chi\Big(d_{g_{\exp_\xi y}}\left(x,\exp_\xi y\right)\Big)\Big)\bigg|_{y=0}\right\|_{1,2} &= \mathrm{O}\Big(\left\|\nabla W_{\delta_{j,\varepsilon}(t_j),\sigma_j,\xi}\mathbf{1}_{B_\xi(r_0)\setminus B_\xi(r_0/2)}\right\|_2 \\ &\quad + \left\|W_{\delta_{j,\varepsilon}(t_j),\sigma_j,\xi}\mathbf{1}_{B_\xi(r_0)\setminus B_\xi(r_0/2)}\right\|_2\Big) = \mathrm{O}\Big(\delta_{j,\varepsilon}(t_j)^{\frac{n-2}{2}}\Big).\end{aligned} \tag{4.8}$$

Regarding the second term in the right-hand side of (4.7), we claim that for any $i, j = 1, \dots, n$, for y close to 0, there holds

$$\frac{d}{dy_i}\left(\widetilde{\exp}^{-1}_{\xi,\exp_\xi y}\left(\exp_\xi \eta\right)\right)_j\Big|_{y=0} = -\delta_{ij} + \mathrm{O}\left(|\eta|^2\right), \tag{4.9}$$

where the real numbers δ_{ij} are the Kronecker symbols. We prove this claim. For any $j = 1, \dots, n$, for y, η close to 0, we define

$$\mathcal{E}_{j,\xi}(\eta, y) := \left(\exp^{-1}_{\exp_\xi y} \exp_\xi \eta\right)_j.$$

Clearly, $\mathcal{E}_{j,\xi}(\eta, y)$ is smooth with respect to ξ, η, and y. In order to prove the Taylor expansion (4.9), we compute the first and second order derivatives of $\mathcal{E}_{j,\xi}(\eta, y)$ with respect to η and y. Since the frame is parallel at ξ, we get $\mathcal{E}_{j,\xi}(0, y) = -y_j$. Differentiating this equation gives

$$\frac{\partial \mathcal{E}_{j,\xi}}{\partial y_i}(0,0) = -\delta_{ij} \quad \text{and} \quad \frac{\partial^2 \mathcal{E}_{j,\xi}}{\partial y_k \partial y_i}(0,0) = 0 \tag{4.10}$$

for all $i, j, k = 1, \dots, n$. We also remark that $\mathcal{E}_{j,\xi}(\eta, 0) = \eta_j$, and thus we get

$$\frac{\partial \mathcal{E}_{j,\xi}}{\partial \eta_i}(0,0) = \delta_{ij} \quad \text{and} \quad \frac{\partial^2 \mathcal{E}_{j,\xi}}{\partial \eta_k \partial \eta_i}(0,0) = 0 \tag{4.11}$$

for all $i, j, k = 1, \dots, n$. As a third equation, we find $\mathcal{E}_{j,\xi}(\eta, \eta) = 0$. Differentiating this equation and using (4.10) and (4.11), we find

$$\frac{\partial^2 \mathcal{E}_{j,\xi}}{\partial \eta_k \partial y_i}(0,0) = -\frac{1}{2}\left(\frac{\partial^2 \mathcal{E}_{j,\xi}}{\partial y_k \partial y_i}(0,0) + \frac{\partial^2 \mathcal{E}_{j,\xi}}{\partial \eta_k \partial \eta_i}(0,0)\right) = 0 \tag{4.12}$$

for all $i, j, k = 1, \dots, n$. (4.9) follows from (4.10) and (4.12). By (4.7), (4.8), and (4.9), we get

$$\begin{aligned}\left\|\frac{d}{dy_i} W_{\delta_{j,\varepsilon}(t_j),\sigma_j,\exp_\xi y}\Big|_{y=0} - \frac{n^{\frac{n-2}{4}}(n-2)^{\frac{n+2}{4}}}{\delta_{j,\varepsilon}(t_j)} Z_{i,\delta_{j,\varepsilon}(t_j),\sigma_j,\xi}\right\|_{1,2} &= \mathrm{O}\Big(\delta_{j,\varepsilon}(t_j)^{\frac{n-2}{2}} \\ + \delta_{j,\varepsilon}(t_j)^{-1}\left\|d_{g_\xi}(x,\xi)^2 \nabla Z_{j,\delta_{j,\varepsilon}(t_j),\sigma_j,\xi}\right\|_2 &+ \delta_{j,\varepsilon}(t_j)^{-1}\left\|d_{g_\xi}(x,\xi) Z_{j,\delta_{j,\varepsilon}(t_j),\sigma_j,\xi}\right\|_2\Big).\end{aligned} \tag{4.13}$$

We find

$$\int_M d_{g_\xi}(x,\xi)^4 \left|\nabla Z_{j,\delta_{j,\varepsilon}(t_j),\sigma_j,\xi}\right|^2_{g_\xi} dv_{g_\xi} = \mathrm{O}\left(\delta_{j,\varepsilon}(t_j)^4\right), \tag{4.14}$$

$$\int_M d_{g_\xi}(x,\xi)^2 Z^2_{j,\delta_{j,\varepsilon}(t_j),\sigma_j,\xi} dv_{g_\xi} = \mathrm{O}\left(\delta_{j,\varepsilon}(t_j)^4\right). \tag{4.15}$$

(4.3) and (4.5) follow from (4.13), (4.14), and (4.15). Now, we prove (4.4) and (4.6). For any $j = 1, \dots, k$, we get

$$\frac{d}{dy_i} W_{\delta_{j,\varepsilon}(t_j),\sigma_j,\xi}\left(\exp_\xi y\right)\Big|_{y=\exp_\xi^{-1} x} = W_{\delta_{j,\varepsilon}(t_j),\sigma_j,\xi}(x)\, \frac{d}{dy_i}\left(\ln \chi(|y|)\right)\Big|_{y=0} - \frac{1}{\delta_{j,\varepsilon}(t_j)} Z_{i,\delta_{j,\varepsilon}(t_j),\sigma_j,\xi}(x). \tag{4.16}$$

We find

$$\begin{aligned}\left\|W_{\delta_{j,\varepsilon}(t_j),\sigma_j,\xi}(x)\, \frac{d}{dy_i}\left(\ln \chi(|y|)\right)\Big|_{y=0}\right\|_{1,2} &= \mathrm{O}\Big(\left\|\nabla W_{\delta_{j,\varepsilon}(t_j),\sigma_j,\xi} \mathbf{1}_{B_\xi(r_0)\setminus B_\xi(r_0/2)}\right\|_2 \\ &+ \left\|W_{\delta_{j,\varepsilon}(t_j),\sigma_j,\xi} \mathbf{1}_{B_\xi(r_0)\setminus B_\xi(r_0/2)}\right\|_2\Big) = \mathrm{O}\left(\delta_{j,\varepsilon}(t_j)^{\frac{n-2}{2}}\right).\end{aligned} \tag{4.17}$$

Finally, (4.4) and (4.6) follow from (4.16) and (4.17). □

5. Error estimates

This section is devoted to the error estimates. We state our estimates as follows.

Lemma 5.1. *For any compact subset A of $(\mathbb{R}_+^*)^k \times (\mathbb{R}^n)^{k-1} \times N$, there exists a positive constant C_A such that for ε small, for any (t, σ, ξ) in A, and for any $j = 1, \dots, k$, there hold*

$$\begin{aligned}&\left\| i^* \left(f_\varepsilon \left(\mathrm{Tower}_{\delta_\varepsilon(t),\sigma,\xi} \right) \right) - \mathrm{Tower}_{\delta_\varepsilon(t),\sigma,\xi} \right\|_h \\ &\qquad \le C_A \begin{cases} \mu_\varepsilon \sqrt{\varepsilon} & \text{if } n = 4, \\ \varepsilon^{\frac{3}{4}} & \text{if } n = 5, \\ \varepsilon \left| \ln \varepsilon \right| & \text{if } n = 6, \\ \varepsilon^{\frac{n+2}{2(n-2)}} & \text{if } n \ge 7, \end{cases}\end{aligned} \tag{5.1}$$

$$\begin{aligned}&\left\| (-1)^{j-1} \, i^* \left(f_\varepsilon' \left(\mathrm{Tower}_{\delta_\varepsilon(t),\sigma,\xi} \right) Z_{0,\delta_{j,\varepsilon}(t_j),\sigma_j,\xi} \right) - Z_{0,\delta_{j,\varepsilon}(t_j),\sigma_j,\xi} \right\|_h \\ &\qquad \le C_A \begin{cases} \mu_\varepsilon \sqrt{\varepsilon} & \text{if } n = 4 \text{ and } j = 1, \\ \varepsilon^{\frac{3}{4}} & \text{if } n = 5 \text{ and } j = 1, \\ \varepsilon \left| \ln \varepsilon \right| & \text{if } n = 6 \text{ or } (n = 4, 5 \text{ and } j \ge 2), \\ \varepsilon^{\frac{n+2}{2(n-2)}} & \text{if } n \ge 7, \end{cases}\end{aligned} \tag{5.2}$$

$$\begin{aligned}&\left\| (-1)^{j-1} \, i^* \left(f_\varepsilon' \left(\mathrm{Tower}_{\delta_\varepsilon(t),\sigma,\xi} \right) Z_{i,\delta_{j,\varepsilon}(t_j),\sigma_j,\xi} \right) - Z_{i,\delta_{j,\varepsilon}(t_j),\sigma_j,\xi} \right\|_h \\ &\qquad \le C_A \begin{cases} \varepsilon \left| \ln \varepsilon \right| & \text{if } n = 4, 5, 6, \\ \varepsilon^{\frac{n+2}{2(n-2)}} & \text{if } n \ge 7, \end{cases}\end{aligned} \tag{5.3}$$

for all $i = 1, \dots, n$, where μ_ε and $\delta_\varepsilon(t)$ are as in (2.11).

Proof. All our estimates in this proof are uniform with respect to (t, σ, ξ) in compact subsets of $(\mathbb{R}_+^*)^k \times (\mathbb{R}^n)^{k-1} \times N$ and with respect to ε in $(0, \varepsilon_0)$ for some fixed positive real number ε_0. First, we prove (5.1). By continuity of i^*, we get

$$\begin{aligned}&\left\| i^* \left(f_\varepsilon \left(\mathrm{Tower}_{\delta_\varepsilon(t),\sigma,\xi} \right) \right) - \mathrm{Tower}_{\delta_\varepsilon(t),\sigma,\xi} \right\|_h \\ &\qquad = \mathrm{O} \left(\left\| f_\varepsilon \left(\mathrm{Tower}_{\delta_\varepsilon(t),\sigma,\xi} \right) - \Delta_g \mathrm{Tower}_{\delta_\varepsilon(t),\sigma,\xi} - h \mathrm{Tower}_{\delta_\varepsilon(t),\sigma,\xi} \right\|_{\frac{2n}{n+2}} \right).\end{aligned}$$

It follows that

$$\begin{aligned}&\left\| i^* \left(f_\varepsilon \left(\mathrm{Tower}_{\delta_\varepsilon(t),\sigma,\xi} \right) \right) - \mathrm{Tower}_{\delta_\varepsilon(t),\sigma,\xi} \right\|_h \\ &\qquad = \mathrm{O} \Bigg(\left\| f_\varepsilon \left(\mathrm{Tower}_{\delta_\varepsilon(t),\sigma,\xi} \right) - \sum_{j=1}^k (-1)^{j-1} f_\varepsilon \left(W_{\delta_{j,\varepsilon}(t_j),\sigma_j,\xi} \right) \right\|_{\frac{2n}{n+2}} \\ &\qquad\qquad + \sum_{j=1}^k \left\| f_\varepsilon \left(W_{\delta_{j,\varepsilon}(t_j),\sigma_j,\xi} \right) - \Delta_g W_{\delta_{j,\varepsilon}(t_j),\sigma_j,\xi} - h W_{\delta_{j,\varepsilon}(t_j),\sigma_j,\xi} \right\|_{\frac{2n}{n+2}} \Bigg).\end{aligned} \tag{5.4}$$

By similar computations as in Musso–Pistoia [30] and Ge–Musso–Pistoia [20], we get

$$\left\| f_\varepsilon \left(\mathrm{Tower}_{\delta_\varepsilon(t),\sigma,\xi} \right) - \sum_{j=1}^k (-1)^{j-1} f_\varepsilon \left(W_{\delta_{j,\varepsilon}(t_j),\sigma_j,\xi} \right) \right\|_{\frac{2n}{n+2}} = \begin{cases} \mathrm{O}(\varepsilon) & \text{if } n = 4, 5, \\ \mathrm{O}\left(\varepsilon \left| \ln \varepsilon \right|^{\frac{2}{3}} \right) & \text{if } n = 6, \\ \mathrm{O}\left(\varepsilon^{\frac{n+2}{2(n-2)}} \right) & \text{if } n \ge 7. \end{cases} \tag{5.5}$$

In order to estimate the second term in (5.4), we denote $\chi_\xi(x) := \chi(d_g(x,\xi))$ and $u_{\delta,\sigma,\xi}(x) := \delta^{\frac{2-n}{2}} u\left(\delta^{-1}\exp_\xi^{-1}(x) - \sigma\right)$ for all points x in M and all functions $u : \mathbb{R}^n \to \mathbb{R}$. Since U is a solution to the equation $\Delta_{\mathrm{Eucl}} U = |U|^{2^*-2} U$ in $\mathbb{R}^n$, we get

$$\begin{aligned}
&\left\| f_\varepsilon\left(W_{\delta_{j,\varepsilon}(t_j),\sigma_j,\xi}\right) - \Delta_g W_{\delta_{j,\varepsilon}(t_j),\sigma_j,\xi} - h W_{\delta_{j,\varepsilon}(t_j),\sigma_j,\xi}\right\|_{\frac{2n}{n+2}} \\
&\quad \le \left\| \chi_\xi^{2^*-1-\varepsilon}\left(U^{2^*-1-\varepsilon}_{\delta_{j,\varepsilon}(t_j),\sigma_j,\xi} - U^{2^*-1}_{\delta_{j,\varepsilon}(t_j),\sigma_j,\xi}\right)\right\|_{\frac{2n}{n+2}} + \left\| \left(\chi_\xi^{2^*-1-\varepsilon} - \chi_\xi\right) U^{2^*-1}_{\delta_{j,\varepsilon}(t_j),\sigma_j,\xi}\right\|_{\frac{2n}{n+2}} \\
&\quad + \left\| \chi_\xi \left(\delta_{j,\varepsilon}(t_j)^{-2} \left(\Delta_{\mathrm{Eucl}} U\right)_{\delta_{j,\varepsilon}(t_j),\sigma_j,\xi} - \Delta_g U_{\delta_{j,\varepsilon}(t_j),\sigma_j,\xi}\right)\right\|_{\frac{2n}{n+2}} + \left\| U_{\delta_{j,\varepsilon}(t_j),\sigma_j,\xi} \Delta_g \chi_\xi \right\|_{\frac{2n}{n+2}} \\
&\quad + 2\left\| \left\langle \nabla\chi_\xi, \nabla U_{\delta_{j,\varepsilon}(t_j),\sigma_j,\xi}\right\rangle_g \right\|_{\frac{2n}{n+2}} + \left\| h\chi_\xi U_{\delta_{j,\varepsilon}(t_j),\sigma_j,\xi}\right\|_{\frac{2n}{n+2}} . \qquad (5.6)
\end{aligned}$$

We are led to estimate each terms in (5.6). We find

$$\begin{aligned}
&\int_M \left| \chi_\xi^{2^*-1-\varepsilon}\left(U^{2^*-1-\varepsilon}_{\delta_{j,\varepsilon}(t_j),\sigma_j,\xi} - U^{2^*-1}_{\delta_{j,\varepsilon}(t_j),\sigma_j,\xi}\right)\right|^{\frac{2n}{n+2}} dv_g \\
&\quad = \mathrm{O}\left(\varepsilon^{\frac{2n}{n+2}} \int_0^{\frac{r_0}{\delta_{j,\varepsilon}(t_j)}} \frac{r^{n-1}\left(|\ln\varepsilon| + \ln\left(1+r^2\right)\right)^{\frac{2n}{n+2}} dr}{\left(1+r^2\right)^n}\right) = \mathrm{O}\left(\varepsilon^{\frac{2n}{n+2}} |\ln\varepsilon|^{\frac{2n}{n+2}}\right), \qquad (5.7)
\end{aligned}$$

$$\int_M \left|\left(\chi_\xi^{2^*-1-\varepsilon} - \chi_\xi\right) U^{2^*-1}_{\delta_{j,\varepsilon}(t_j),\sigma_j,\xi}\right|^{\frac{2n}{n+2}} dv_g = \mathrm{O}\left(\int_{\frac{r_0}{2\delta_{j,\varepsilon}(t_j)}}^{\frac{r_0}{\delta_{j,\varepsilon}(t_j)}} \frac{r^{n-1} dr}{\left(1+r^2\right)^n}\right) = \mathrm{O}\left(\mu_\varepsilon^n \varepsilon^{\frac{n(n+4j-6)}{2(n-2)}}\right), \quad (5.8)$$

$$\begin{aligned}
\int_M \left| U_{\delta_{j,\varepsilon}(t_j),\sigma_j,\xi} \Delta_g \chi_\xi\right|^{\frac{2n}{n+2}} dv_g &= \mathrm{O}\left(\delta_{j,\varepsilon}(t_j)^{\frac{4n}{n+2}} \int_{\frac{r_0}{2\delta_{j,\varepsilon}(t_j)}}^{\frac{r_0}{\delta_{j,\varepsilon}(t_j)}} \frac{r^{n-1} dr}{\left(1+r^2\right)^{\frac{n(n-2)}{n+2}}}\right) \\
&= \mathrm{O}\left(\mu_\varepsilon^{\frac{n(n-2)}{n+2}} \varepsilon^{\frac{n(n+4j-6)}{2(n+2)}}\right), \qquad (5.9)
\end{aligned}$$

$$\begin{aligned}
\int_M \left|\left\langle \nabla\chi_\xi, \nabla U_{\delta_{j,\varepsilon}(t_j),\sigma_j,\xi}\right\rangle_g\right|^{\frac{2n}{n+2}} dv_g &= \mathrm{O}\left(\delta_{j,\varepsilon}(t_j)^{\frac{2n}{n+2}} \int_{\frac{r_0}{2\delta_{j,\varepsilon}(t_j)}}^{\frac{r_0}{\delta_{j,\varepsilon}(t_j)}} \frac{r^{n-1+\frac{2n}{n+2}} dr}{\left(1+r^2\right)^{\frac{n(n-2)}{n+2}}}\right) \\
&= \mathrm{O}\left(\mu_\varepsilon^{\frac{n(n-2)}{n+2}} \varepsilon^{\frac{n(n+4j-6)}{2(n+2)}}\right). \qquad (5.10)
\end{aligned}$$

Using the fact that, in geodesic normal coordinates, there holds $\Delta_g = -g^{\alpha\beta}\left(\frac{\partial^2}{\partial x_\alpha \partial x_\beta} - \Gamma^\gamma_{\alpha\beta}\frac{\partial}{\partial x_\gamma}\right)$, where the functions $g^{\alpha\beta}$ are the components of g^{-1} and the functions $\Gamma^\gamma_{\alpha\beta}$ are the Christoffel symbols of the metric g, using (3.1) together with Cartan's expansion of the metric, we find

$$\begin{aligned}
&\int_M \left| \chi_\xi\left(\delta_{j,\varepsilon}(t_j)^{-2}\left(\Delta_{\mathrm{Eucl}} U\right)_{\delta_{j,\varepsilon}(t_j),\sigma_j,\xi} - \Delta_g U_{\delta_{j,\varepsilon}(t_j),\sigma_j,\xi}\right)\right|^{\frac{2n}{n+2}} dv_g \\
&\quad = \mathrm{O}\left(\delta_{j,\varepsilon}(t_j)^{\frac{4n}{n+2}} \int_0^{\frac{r_0}{\delta_{j,\varepsilon}(t_j)}} \frac{r^{n-1+\frac{4n}{n+2}} dr}{\left(1+r^2\right)^{\frac{n^2}{n+2}}}\right) = \begin{cases} \mathrm{O}\left(\mu_\varepsilon^{\frac{4}{3}} \varepsilon^{\frac{2(2j-1)}{3}}\right) & \text{if } n=4, \\ \mathrm{O}\left(\varepsilon^{\frac{5(4j-1)}{14}}\right) & \text{if } n=5, \\ \mathrm{O}\left(\varepsilon^{\frac{3j}{2}} |\ln\varepsilon|\right) & \text{if } n=6, \\ \mathrm{O}\left(\varepsilon^{\frac{2n(n+4j-6)}{(n+2)(n-2)}}\right) & \text{if } n \ge 7. \end{cases} \qquad (5.11)
\end{aligned}$$

It remains to estimate the last term in (5.6). We find

$$\begin{aligned}\int_M \left|h\chi_\xi U_{\delta_{j,\varepsilon}(t_j),\sigma_j,\xi}\right|^{\frac{2n}{n+2}} dv_g &= \mathrm{O}\left(\delta_{j,\varepsilon}(t_j)^{\frac{4n}{n+2}} \int_0^{\frac{r_0}{\delta_{j,\varepsilon}(t_j)}} \frac{r^{n-1}dr}{(1+r^2)^{\frac{n(n-2)}{n+2}}}\right)\\ &= \begin{cases} \mathrm{O}\left(\mu_\varepsilon^{\frac{4}{3}}\varepsilon^{\frac{2(2j-1)}{3}}\right) & \text{if } n=4,\\ \mathrm{O}\left(\varepsilon^{\frac{5(4j-1)}{14}}\right) & \text{if } n=5,\\ \mathrm{O}\left(\varepsilon^{\frac{3j}{2}}\left|\ln\varepsilon\right|\right) & \text{if } n=6,\\ \mathrm{O}\left(\varepsilon^{\frac{2n(n+4j-6)}{(n+2)(n-2)}}\right) & \text{if } n\geq 7.\end{cases}\end{aligned} \tag{5.12}$$

For any $j=1,\dots,k$, by (5.6)–(5.12), we get

$$\begin{aligned}&\left\|f_\varepsilon\left(W_{\delta_{j,\varepsilon}(t_j),\sigma_j,\xi}\right) - \Delta_g W_{\delta_{j,\varepsilon}(t_j),\sigma_j,\xi} - hW_{\delta_{j,\varepsilon}(t_j),\sigma_j,\xi}\right\|_{\frac{2n}{n+2}}\\ &= \begin{cases} \mathrm{O}\left(\mu_\varepsilon\sqrt{\varepsilon}\right) & \text{if } n=4 \text{ and } j=1,\\ \mathrm{O}\left(\varepsilon^{\frac{3}{4}}\right) & \text{if } n=5 \text{ and } j=1,\\ \mathrm{O}\left(\varepsilon\left|\ln\varepsilon\right|\right) & \text{if } n\geq 6 \text{ or } (n=4,5 \text{ and } j\geq 2).\end{cases}\end{aligned} \tag{5.13}$$

Finally, (5.1) follows from (5.4), (5.5), and (5.13). Now, we prove (5.2) and (5.3). By continuity of i^*, we get

$$\begin{aligned}&\left\|(-1)^{j-1} i^*\left(f_\varepsilon'\left(\mathrm{Tower}_{\delta_\varepsilon(t),\sigma,\xi}\right)Z_{i,\delta_{j,\varepsilon}(t_j),\sigma_j,\xi}\right) - Z_{i,\delta_{j,\varepsilon}(t_j),\sigma_j,\xi}\right\|_h\\ &= \mathrm{O}\left(\left\|(-1)^{j-1} f_\varepsilon'\left(\mathrm{Tower}_{\delta_\varepsilon(t),\sigma,\xi}\right)Z_{i,\delta_{j,\varepsilon}(t_j),\sigma_j,\xi} - \Delta_g Z_{i,\delta_{j,\varepsilon}(t_j),\sigma_j,\xi} - hZ_{i,\delta_{j,\varepsilon}(t_j),\sigma_j,\xi}\right\|_{\frac{2n}{n+2}}\right)\end{aligned} \tag{5.14}$$

It follows that

$$\begin{aligned}&\left\|(-1)^{j-1} i^*\left(f_\varepsilon'\left(\mathrm{Tower}_{\delta_\varepsilon(t),\sigma,\xi}\right)Z_{i,\delta_{j,\varepsilon}(t_j),\sigma_j,\xi}\right) - Z_{i,\delta_{j,\varepsilon}(t_j),\sigma_j,\xi}\right\|_h\\ &\quad= \mathrm{O}\Bigg(\left\|\left((-1)^{j-1} f_\varepsilon'\left(\mathrm{Tower}_{\delta_\varepsilon(t),\sigma,\xi}\right) - f_\varepsilon'\left(W_{\delta_{j,\varepsilon}(t_j),\sigma_j,\xi}\right)\right)Z_{i,\delta_{j,\varepsilon}(t_j),\sigma_j,\xi}\right\|_{\frac{2n}{n+2}}\\ &\qquad+ \left\|f_\varepsilon'\left(W_{\delta_{j,\varepsilon}(t_j),\sigma_j,\xi}\right)Z_{i,\delta_{j,\varepsilon}(t_j),\sigma_j,\xi} - \Delta_g Z_{i,\delta_{j,\varepsilon}(t_j),\sigma_j,\xi} - hZ_{i,\delta_{j,\varepsilon}(t_j),\sigma_j,\xi}\right\|_{\frac{2n}{n+2}}\Bigg).\end{aligned} \tag{5.15}$$

Similar computations as in Musso–Pistoia [30] and Ge–Musso–Pistoia [20] give

$$\begin{aligned}&\left\|\left((-1)^{j-1} f_\varepsilon'\left(\mathrm{Tower}_{\delta_\varepsilon(t),\sigma,\xi}\right) - f_\varepsilon'\left(W_{\delta_{j,\varepsilon}(t_j),\sigma_j,\xi}\right)\right)Z_{i,\delta_{j,\varepsilon}(t_j),\sigma_j,\xi}\right\|_{\frac{2n}{n+2}}\\ &= \begin{cases} \mathrm{O}\left(\varepsilon\right) & \text{if } n=4,5,\\ \mathrm{O}\left(\varepsilon\left|\ln\varepsilon\right|^{\frac{2}{3}}\right) & \text{if } n=6,\\ \mathrm{O}\left(\varepsilon^{\frac{n+2}{2(n-2)}}\right) & \text{if } n\geq 7.\end{cases}\end{aligned} \tag{5.16}$$

Since V_i is a solution to the equation $\Delta_{\text{Eucl}} V_i = (2^* - 1) U^{2^*-2} V_i$ in $\mathbb{R}^n$, we get

$$\begin{aligned}
&\left\| f'_\varepsilon \left(W_{\delta_{j,\varepsilon}(t_j),\sigma_j,\xi} \right) Z_{i,\delta_{j,\varepsilon}(t_j),\sigma_j,\xi} - \Delta_g Z_{i,\delta_{j,\varepsilon}(t_j),\sigma_j,\xi} - h Z_{i,\delta_{j,\varepsilon}(t_j),\sigma_j,\xi} \right\|_{\frac{2n}{n+2}} \\
&\quad \leq (2^* - 1 - \varepsilon) \left\| \chi_\xi^{2^*-1-\varepsilon} \left(U^{2^*-2-\varepsilon}_{\delta_{j,\varepsilon}(t_j),\sigma_j,\xi} - U^{2^*-2}_{\delta_{j,\varepsilon}(t_j),\sigma_j,\xi} \right) V_{i,\delta_{j,\varepsilon}(t_j),\sigma_j,\xi} \right\|_{\frac{2n}{n+2}} \\
&\qquad + (2^* - 1 - \varepsilon) \left\| \left(\chi_\xi^{2^*-1-\varepsilon} - \chi_\xi \right) U^{2^*-2}_{\delta_{j,\varepsilon}(t_j),\sigma_j,\xi} V_{i,\delta_{j,\varepsilon}(t_j),\sigma_j,\xi} \right\|_{\frac{2n}{n+2}} \\
&\quad + \left\| \chi_\xi \left(\delta_{j,\varepsilon}(t_j)^{-2} \left(\Delta_{\text{Eucl}} V_i \right)_{\delta_{j,\varepsilon}(t_j),\sigma_j,\xi} - \Delta_g V_{i,\delta_{j,\varepsilon}(t_j),\sigma_j,\xi} \right) \right\|_{\frac{2n}{n+2}} + \left\| V_{i,\delta_{j,\varepsilon}(t_j),\sigma_j,\xi} \Delta_g \chi_\xi \right\|_{\frac{2n}{n+2}} \\
&\qquad + 2 \left\| \left\langle \nabla \chi_\xi, \nabla V_{i,\delta_{j,\varepsilon}(t_j),\sigma_j,\xi} \right\rangle_g \right\|_{\frac{2n}{n+2}} + \left\| h \chi_\xi V_{i,\delta_{j,\varepsilon}(t_j),\sigma_j,\xi} \right\|_{\frac{2n}{n+2}} \qquad (5.17)
\end{aligned}$$

with the same notations as in (5.6) and $V_{i,\delta_{j,\varepsilon}(t_j),\sigma_j,\xi} = (V_i)_{\delta_{j,\varepsilon}(t_j),\sigma_j,\xi}$. Since $|V_0(y)| \leq \frac{n-2}{2} U(y)$, $|\nabla V_0(y)| \leq \frac{n-2}{2} |\nabla U(y)|$, and $|\nabla^2 V_0(y)| \leq C \left(1 + |y|^2\right)^{-n/2}$ for all points y in $\mathbb{R}^n$, by (5.7)–(5.12) and (5.17), we get

$$\begin{aligned}
&\left\| f'_\varepsilon \left(W_{\delta_{j,\varepsilon}(t_j),\sigma_j,\xi} \right) Z_{0,\delta_{j,\varepsilon}(t_j),\sigma_j,\xi} - \Delta_g Z_{0,\delta_{j,\varepsilon}(t_j),\sigma_j,\xi} - h Z_{0,\delta_{j,\varepsilon}(t_j),\sigma_j,\xi} \right\|_{\frac{2n}{n+2}} \\
&\quad = \begin{cases} \mathrm{O}\left(\mu_\varepsilon \sqrt{\varepsilon} \right) & \text{if } n = 4 \text{ and } j = 1, \\ \mathrm{O}\left(\varepsilon^{\frac{3}{4}} \right) & \text{if } n = 5 \text{ and } j = 1, \\ \mathrm{O}\left(\varepsilon \left| \ln \varepsilon \right| \right) & \text{if } n \geq 6 \text{ or } (n = 4, 5 \text{ and } j \geq 2). \end{cases} \qquad (5.18)
\end{aligned}$$

For any $i = 1, \dots, n$, we find

$$\begin{aligned}
&\int_M \left| \chi_\xi^{2^*-1-\varepsilon} \left(U^{2^*-2-\varepsilon}_{\delta_{j,\varepsilon}(t_j),\sigma_j,\xi} - U^{2^*-2}_{\delta_{j,\varepsilon}(t_j),\sigma_j,\xi} \right) V_{i,\delta_{j,\varepsilon}(t_j),\sigma_j,\xi} \right|^{\frac{2n}{n+2}} dv_g \\
&\quad = \mathrm{O}\left(\varepsilon^{\frac{2n}{n+2}} \int_0^{\frac{r_0}{\delta_{j,\varepsilon}(t_j)}} \frac{r^{n-1+\frac{2n}{n+2}} \left(|\ln \varepsilon| + \ln(1 + r^2) \right)^{\frac{2n}{n+2}} dr}{(1 + r^2)^{\frac{n(n+4)}{n+2}}} \right) = \mathrm{O}\left(\varepsilon^{\frac{2n}{n+2}} |\ln \varepsilon|^{\frac{2n}{n+2}} \right), \qquad (5.19)
\end{aligned}$$

$$\begin{aligned}
\int_M \left| \left(\chi_\xi^{2^*-1-\varepsilon} - \chi_\xi \right) U^{2^*-2}_{\delta_{j,\varepsilon}(t_j),\sigma_j,\xi} V_{i,\delta_{j,\varepsilon}(t_j),\sigma_j,\xi} \right|^{\frac{2n}{n+2}} dv_g &= \mathrm{O}\left(\int_{\frac{r_0}{2\delta_{j,\varepsilon}(t_j)}}^{\frac{r_0}{\delta_{j,\varepsilon}(t_j)}} \frac{r^{n-1+\frac{2n}{n+2}} dr}{(1 + r^2)^{\frac{n(n+4)}{n+2}}} \right) \\
&= \mathrm{O}\left(\mu_\varepsilon^{\frac{n(n+4)}{n+2}} \varepsilon^{\frac{n(n+4)(n+4j-6)}{2(n+2)(n-2)}} \right), \qquad (5.20)
\end{aligned}$$

$$\begin{aligned}
\int_M \left| V_{i,\delta_{j,\varepsilon}(t_j),\sigma_j,\xi} \Delta_g \chi_\xi \right|^{\frac{2n}{n+2}} dv_g &= \mathrm{O}\left(\delta_{j,\varepsilon}(t_j)^{\frac{4n}{n+2}} \int_{\frac{r_0}{2\delta_{j,\varepsilon}(t_j)}}^{\frac{r_0}{\delta_{j,\varepsilon}(t_j)}} \frac{r^{n-1+\frac{2n}{n+2}} dr}{(1 + r^2)^{\frac{n^2}{n+2}}} \right) \\
&= \mathrm{O}\left(\mu_\varepsilon^{\frac{n^2}{n+2}} \varepsilon^{\frac{n^2(n+4j-6)}{2(n+2)(n-2)}} \right), \qquad (5.21)
\end{aligned}$$

$$\begin{aligned}
\int_M \left| \left\langle \nabla \chi_\xi, \nabla V_{i,\delta_{j,\varepsilon}(t_j),\sigma_j,\xi} \right\rangle_g \right|^{\frac{2n}{n+2}} dv_g &= \mathrm{O}\left(\delta_{j,\varepsilon}(t_j)^{\frac{2n}{n+2}} \int_{\frac{r_0}{2\delta_{j,\varepsilon}(t_j)}}^{\frac{r_0}{\delta_{j,\varepsilon}(t_j)}} \frac{r^{n-1} dr}{(1 + r^2)^{\frac{n^2}{n+2}}} \right) \\
&= \mathrm{O}\left(\mu_\varepsilon^{\frac{n^2}{n+2}} \varepsilon^{\frac{n^2(n+4j-6)}{2(n+2)(n-2)}} \right). \qquad (5.22)
\end{aligned}$$

Similarly to (5.11), we find

$$\int_M \left|\chi_\xi \left(\delta_{j,\varepsilon}(t_j)^{-2}\left(\Delta_{\mathrm{Eucl}} V_i\right)_{\delta_{j,\varepsilon}(t_j),\sigma_j,\xi} - \Delta_g V_{i,\delta_{j,\varepsilon}(t_j),\sigma_j,\xi}\right)\right|^{\frac{2n}{n+2}} dv_g$$
$$= \mathrm{O}\left(\delta_{j,\varepsilon}(t_j)^{\frac{4n}{n+2}} \int_0^{\frac{r_0}{\delta_{j,\varepsilon}(t_j)}} \frac{r^{n-1+\frac{4n}{n+2}} dr}{(1+r^2)^{\frac{n(n+1)}{n+2}}}\right) = \mathrm{O}\left(\mu_\varepsilon^{\frac{2(n-2)}{n+2}} \varepsilon^{\frac{2n(n+4j-6)}{(n+2)(n-2)}}\right). \quad (5.23)$$

Moreover, we find

$$\int_M \left|h\chi_\xi V_{i,\delta_{j,\varepsilon}(t_j),\sigma_j,\xi}\right|^{\frac{2n}{n+2}} dv_g = \mathrm{O}\left(\delta_{j,\varepsilon}(t_j)^{\frac{4n}{n+2}} \int_0^{\frac{r_0}{\delta_{j,\varepsilon}(t_j)}} \frac{r^{n-1+\frac{2n}{n+2}} dr}{(1+r^2)^{\frac{n^2}{n+2}}}\right)$$
$$= \mathrm{O}\left(\mu_\varepsilon^{-2} \delta_{j,\varepsilon}(t_j)^{\frac{4n}{n+2}}\right) = \mathrm{O}\left(\mu_\varepsilon^{\frac{2(n-2)}{n+2}} \varepsilon^{\frac{2n(n+4j-6)}{(n+2)(n-2)}}\right). \quad (5.24)$$

For any $i = 1, \dots, n$ and $j = 1, \dots, k$, by (5.17) and (5.19)–(5.24), we get

$$\left\|f'_\varepsilon\left(W_{\delta_{j,\varepsilon}(t_j),\sigma_j,\xi}\right) Z_{i,\delta_{j,\varepsilon}(t_j),\sigma_j,\xi} - \Delta_g Z_{i,\delta_{j,\varepsilon}(t_j),\sigma_j,\xi} - h Z_{i,\delta_{j,\varepsilon}(t_j),\sigma_j,\xi}\right\|_{\frac{2n}{n+2}} = \mathrm{O}\left(\varepsilon\left|\ln \varepsilon\right|\right). \quad (5.25)$$

Finally, (5.2) and (5.3) follow from (5.15), (5.16), (5.18), and (5.25). This ends the proof of Lemma 5.1. □

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