

# BLOWING-UP SOLUTIONS FOR SECOND-ORDER CRITICAL ELLIPTIC EQUATIONS: THE IMPACT OF THE SCALAR CURVATURE

FRÉDÉRIC ROBERT AND JÉRÔME VÉTOIS

ABSTRACT. Given a closed manifold  $(M^n, g)$ ,  $n \geq 3$ , Olivier Druet [5, 7] proved that a necessary condition for the existence of energy-bounded blowing-up solutions to perturbations of the equation

$$\Delta_g u + h_0 u = u^{\frac{n+2}{n-2}}, \quad u > 0 \text{ in } M$$

is that  $h_0 \in C^1(M)$  touches the Yamabe potential somewhere when  $n \geq 4$  (the condition is different for  $n = 6$ ). In this paper, we prove that Druet's condition is also sufficient provided we add its natural differentiable version. For  $n \geq 6$ , our arguments are local. For the low dimensions  $n \in \{4, 5\}$ , our proof requires to introduce a suitable mass that is defined only where Druet's condition holds. This mass carries global information both on  $h_0$  and  $(M, g)$ .

## 1. INTRODUCTION AND MAIN RESULTS

Let  $(M, g)$  be a compact Riemannian manifold of dimension  $n \geq 3$  without boundary and  $h_0 \in C^p(M)$ ,  $1 \leq p \leq \infty$ . We consider the equation

$$(1) \quad \Delta_g u + h_0 u = u^{2^*-1}, \quad u > 0 \text{ in } M,$$

where  $\Delta_g := -\operatorname{div}_g(\nabla)$  is the Laplace–Beltrami operator and  $2^* := \frac{2n}{n-2}$ . We investigate the existence of families  $(h_\epsilon)_{\epsilon>0} \in C^p(M)$  and  $(u_\epsilon)_{\epsilon>0} \in C^2(M)$  satisfying

$$(2) \quad \Delta_g u_\epsilon + h_\epsilon u_\epsilon = u_\epsilon^{2^*-1}, \quad u_\epsilon > 0 \text{ in } M \text{ for all } \epsilon > 0,$$

and such that  $h_\epsilon \rightarrow h_0$  in  $C^p(M)$  and  $\max_M u_\epsilon \rightarrow \infty$  as  $\epsilon \rightarrow 0$ . We say that  $(u_\epsilon)_{\epsilon>0}$  *blows up* at some point  $\xi_0 \in M$  as  $\epsilon \rightarrow 0$  if for all  $r > 0$ ,  $\lim_{\epsilon \rightarrow 0} \max_{B_r(\xi_0)} u_\epsilon = +\infty$ . Druet [5, 7] obtained the following necessary condition for blow-up:

**Theorem 1.1** (Druet [5, 7]). *Let  $(M, g)$  be a compact Riemannian manifold of dimension  $n \geq 4$ . Let  $h_0 \in C^1(M)$  be such that  $\Delta_g + h_0$  is coercive. Assume that there exist families  $(h_\epsilon)_{\epsilon>0} \in C^1(M)$  and  $(u_\epsilon)_{\epsilon>0} \in C^2(M)$  satisfying (2) and such that  $h_\epsilon \rightarrow h_0$  strongly in  $C^1(M)$  and  $u_\epsilon \rightharpoonup u_0$  weakly in  $L^{2^*}(M)$ . Assume that  $(u_\epsilon)_\epsilon$  blows-up. Then there exists  $\xi_0 \in M$  such that  $(u_\epsilon)_\epsilon$  blows up at  $\xi_0$  and*

$$(3) \quad (h_0 - c_n \operatorname{Scal}_g)(\xi_0) = 0 \text{ if } n \neq 6 \text{ and } (h_0 - c_n \operatorname{Scal}_g - 2u_0)(\xi_0) = 0 \text{ if } n = 6.$$

Furthermore, if  $n \in \{4, 5\}$ , then  $u_0 \equiv 0$ .

Here  $c_n := \frac{n-2}{4(n-1)}$  and  $\operatorname{Scal}_g$  is the Scalar curvature of  $(M, g)$ . This result does not hold in dimension  $n = 3$ . Indeed, Hebey–Wei [12] constructed examples of

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blowing-up solutions to (2) on the standard sphere  $(\mathbb{S}^3, g_0)$ , which are bounded in  $L^{2^*}(\mathbb{S}^3)$  but do not satisfy (3).

This paper is concerned with the converse of Theorem 1.1 in dimensions  $n \geq 4$ . For the sake of clarity, we state separately our results in the cases  $u_0 \equiv 0$  in dimension  $n \geq 4$  (Theorem 1.2) and  $u_0 > 0$  in dimension  $n \geq 6$  (Theorem 1.3):

**Theorem 1.2** ( $u_0 \equiv 0$ ). *Let  $(M, g)$  be a compact Riemannian manifold of dimension  $n \geq 4$ . Let  $h_0 \in C^p(M)$ ,  $1 \leq p \leq \infty$ , be such that  $\Delta_g + h_0$  is coercive. Assume that there exists a point  $\xi_0 \in M$  such that*

$$(4) \quad (h_0 - c_n \operatorname{Scal}_g)(\xi_0) = |\nabla(h_0 - c_n \operatorname{Scal}_g)(\xi_0)| = 0.$$

*Then there exist families  $(h_\epsilon)_{\epsilon>0} \in C^p(M)$  and  $(u_\epsilon)_{\epsilon>0} \in C^2(M)$  satisfying (2) and such that  $h_\epsilon \rightarrow h_0$  strongly in  $C^p(M)$ ,  $u_\epsilon \rightarrow 0$  weakly in  $L^{2^*}(M)$  and  $(u_\epsilon)_{\epsilon>0}$  blows up at  $\xi_0$ .*

**Theorem 1.3** ( $u_0 > 0$ ). *Let  $(M, g)$  be a compact Riemannian manifold of dimension  $n \geq 6$ . Let  $h_0 \in C^p(M)$ ,  $1 \leq p \leq \infty$ , be such that  $\Delta_g + h_0$  is coercive. Assume that there exist a solution  $u_0 \in C^2(M)$  of (1) and a point  $\xi_0 \in M$  such that*

$$(5) \quad \begin{cases} (h_0 - c_n \operatorname{Scal}_g)(\xi_0) = |\nabla(h_0 - c_n \operatorname{Scal}_g)(\xi_0)| = 0 & \text{if } n \neq 6; \\ (h_0 - 2u_0 - c_6 \operatorname{Scal}_g)(\xi_0) = |\nabla(h_0 - 2u_0 - c_6 \operatorname{Scal}_g)(\xi_0)| = 0 & \text{if } n = 6; \end{cases}$$

*Then there exist families  $(h_\epsilon)_{\epsilon>0} \in C^p(M)$  and  $(u_\epsilon)_{\epsilon>0} \in C^2(M)$  satisfying (2) and such that  $h_\epsilon \rightarrow h_0$  strongly in  $C^p(M)$ ,  $u_\epsilon \rightarrow u_0$  weakly in  $L^{2^*}(M)$  and  $(u_\epsilon)_{\epsilon>0}$  blows up at  $\xi_0$ .*

Compared with Theorem 1.1, we have assumed here that condition (3) is also satisfied at order 1. However, this stronger condition is actually expected to be necessary for the existence of blowing-up solutions (see Theorem 12.1 in the last section of this paper and the discussion in Druet [7, Section 2.5]). Note that we do not make any nondegeneracy assumptions, neither on the solution  $u_0$ , nor on the critical point  $\xi_0$ .

We refer to Section 11 for examples of functions  $h_0$  and  $u_0$  satisfying the assumptions of Theorem 1.3. Recently, Premoselli–Thizy [20] obtained a beautiful example of blowing-up solutions showing that in dimension  $n \in \{4, 5\}$ , condition (4) may not be satisfied at all blow-up points.

When  $h_0 \equiv c_n \operatorname{Scal}_g$ , that is when (1) is the Yamabe equation, several examples of blowing-up solutions have been obtained. In the perturbative case, that is when  $h_\epsilon \neq c_n \operatorname{Scal}_g$ , examples of blowing-up solutions have been obtained by Druet–Hebey [8], Esposito–Pistoia–Vétois [10], Morabito–Pistoia–Vaira [19], Pistoia–Vaira [21] and Robert–Vétois [24]. In the nonperturbative case  $h_\epsilon \equiv c_n \operatorname{Scal}_g$ , we refer to Brendle [1] and Brendle–Marques [2] regarding the non-compactness of Yamabe metrics. When solutions blow-up not only pointwise but also in energy, the function  $h_0 - c_n \operatorname{Scal}_g$  may not vanish (see Chen–Wei–Yan [3] for  $n \geq 5$  and Vétois–Wang [29] for  $n = 4$ ).

When there does not exist any blowing-up solutions to the equations (2), then equation (1) is *stable*. We refer to the survey of Druet [7] and the book of Hebey [11] for exhaustive studies of the various concepts of stability. Stability also arises in the Lin–Ni–Takagi problem (see for instance del Pino–Musso–Roman–Wei [4] for a recent reference on this topic). In Geometry, stability is linked to the problem of

compactness of the Yamabe equation (see Schoen [27, 28], Li–Zhu [17], Druet [6], Marques [18], Li–Zhang [15, 16], Khuri–Marques–Schoen [13]).

Let us give some general considerations about the proofs. Theorem 1.1 yields *local* information on blow-up points. It is essentially the consequence of the concentration of the  $L^2$ -norm of the solutions at one of the blow-up points when  $n \geq 4$ . However, in our construction, the problem may be *both local and global*. Indeed, we reduce the problem to finding critical points of a functional defined on a finite-dimensional space. The first term in the asymptotic expansion of the reduced functional is local. This is due to the  $L^2$ -concentration of the standard bubble in the definition of our ansatz. The second term in the expansion plays a decisive role for obtaining critical points. For the high dimensions  $n \geq 6$ , this term is also local (see e.g. (30)). However, for  $n \in \{4, 5\}$ , the second term is global and we are then compelled to introduce a suitable notion of mass, which carries global information on  $h_0$  and  $(M, g)$ , and to add a corrective term to the standard bubble (see (49)) in order to obtain a reasonable expansion (see e.g. (60)). Unlike the case where  $n = 3$  or  $h_0 \equiv c_n \text{Scal}_g$ , the mass is not defined at all points in the manifold, but only at the points where the condition (5) is satisfied.

More precisely, Theorems 1.2 and 1.3 are consequences of Theorems 1.4 and 1.5 below. The latter are the core results of our paper. In these theorems, we fix a linear perturbation  $h_\epsilon = h_0 + \epsilon f$  for some function  $f \in C^p(M)$ . Furthermore, we specify the behavior of the blowing-up solutions that we obtain. Let  $H_1^2(M)$  be the completion of  $C^\infty(M)$  for the norm  $\|\cdot\|_{H_1^2} := \|\cdot\|_2 + \|\nabla \cdot\|_2$ . We say that  $(u_\epsilon)_\epsilon$  blows up with one bubble at some point  $\xi_0 \in M$  if  $u_\epsilon = u_0 + U_{\delta_\epsilon, \xi_\epsilon} + o(1)$  as  $\epsilon \rightarrow 0$  in  $H_1^2(M)$ , where  $u_0 \in H_1^2(M)$  is such that  $u_\epsilon \rightharpoonup u_0$  weakly in  $H_1^2(M)$ ,  $U_{\delta_\epsilon, \xi_\epsilon}$  is as in (17),  $(\delta_\epsilon, \xi_\epsilon) \rightarrow (0, \xi_0)$  and  $o(1) \rightarrow 0$  strongly in  $H_1^2(M)$  as  $\epsilon \rightarrow 0$ . For convenience, for every  $h_0, u_0 \in C^0(M)$ , we define

$$(6) \quad \varphi_{h_0} := h_0 - c_n \text{Scal}_g \quad \text{and} \quad \varphi_{h_0, u_0} := \begin{cases} h_0 - c_n \text{Scal}_g & \text{if } n \neq 6 \\ h_0 - 2u_0 - c_n \text{Scal}_g & \text{if } n = 6. \end{cases}$$

Our first result deals with the case where  $u_0 \equiv 0$  in dimension  $n \geq 4$ :

**Theorem 1.4.** *Let  $(M, g)$  be a compact Riemannian manifold of dimension  $n \geq 4$ . Let  $h_0 \in C^p(M)$ ,  $p \geq 2$ , be such that  $\Delta_g + h_0$  is coercive. Assume that there exists a point  $\xi_0 \in M$  satisfying (4). Assume in addition that  $\xi_0$  is a nondegenerate critical point of  $h_0 - c_n \text{Scal}_g$  and*

$$(7) \quad K_{h_0}(\xi_0) := \begin{cases} m_{h_0}(\xi_0) & \text{if } n = 4, 5 \\ \Delta_g(h_0 - c_n \text{Scal}_g)(\xi_0) + \frac{c_n}{6} |\text{Weyl}_g(\xi_0)|_g^2 & \text{if } n \geq 6 \end{cases} \neq 0,$$

where  $m_{h_0}(\xi_0)$  is the mass of  $\Delta_g + h_0$  at the point  $\xi_0$  (see Proposition-Definition 6.1), and  $\text{Weyl}_g$  is the Weyl curvature tensor of the manifold. We fix a function  $f \in C^p(M)$  such that  $f(\xi_0) \times K_{h_0}(\xi_0) > 0$ . Then for small  $\epsilon > 0$ , there exists  $u_\epsilon \in C^2(M)$  satisfying

$$(8) \quad \Delta_g u_\epsilon + (h_0 + \epsilon f)u_\epsilon = u_\epsilon^{2^*-1} \quad \text{in } M, \quad u_\epsilon > 0,$$

and such that  $u_\epsilon \rightharpoonup 0$  weakly in  $L^{2^*}(M)$  and  $(u_\epsilon)_\epsilon$  blows up with one bubble at  $\xi_0$ .

The definition of  $K_{h_0}(\xi_0)$  outlines the major difference between high- and low-dimensions that was mentioned above: for  $n \geq 6$ , it is a local quantity, but for  $n \in \{4, 5\}$ , it carries global information (see Section 6 for more discussions).

Next we deal with the case where  $u_0 > 0$  in dimension  $n \geq 6$ . We assume that  $u_0$  is *nondegenerate*, that is, for every  $\phi \in H_1^2(M)$ ,

$$(9) \quad \Delta_g \phi + h_0 \phi = (2^* - 1)u_0^{2^*-2} \phi \iff \phi \equiv 0.$$

**Theorem 1.5.** *Let  $(M, g)$  be a compact Riemannian manifold of dimension  $n \geq 6$ . Let  $h_0 \in C^p(M)$ ,  $p \geq 2$ , be such that  $\Delta_g + h_0$  is coercive. Assume that there exist a nondegenerate solution  $u_0 \in C^2(M)$  to equation (1) and  $\xi_0 \in M$  satisfying (5). Assume in addition that  $\xi_0$  is a nondegenerate critical point of  $\varphi_{h_0, u_0}$  and*

$$(10) \quad K_{h_0, u_0}(\xi_0) := \left. \begin{array}{ll} \Delta_g \varphi_{h_0, u_0}(\xi_0) + \frac{c_6}{6} |\text{Weyl}_g(\xi_0)|_g^2 & \text{if } n = 6 \\ u_0(\xi_0) & \text{if } 7 \leq n \leq 9 \\ 672u_0(\xi_0) + \Delta_g \varphi_{h_0, u_0}(\xi_0) + \frac{c_{10}}{6} |\text{Weyl}_g(\xi_0)|_g^2 & \text{if } n = 10 \\ \Delta_g \varphi_{h_0, u_0}(\xi_0) + \frac{c_n}{6} |\text{Weyl}_g(\xi_0)|_g^2 & \text{if } n \geq 11 \end{array} \right\} \neq 0.$$

We fix a function  $f \in C^p(M)$  such that

$$(11) \quad K_{h_0, u_0}(\xi_0) \times \left. \begin{array}{ll} [f + 2(\Delta_g + h_0 - 2u_0)^{-1}(fu_0)](\xi_0) & \text{if } n = 6 \\ f(\xi_0) & \text{if } n > 6 \end{array} \right\} > 0.$$

Then for small  $\epsilon > 0$ , there exists  $u_\epsilon \in C^2(M)$  satisfying (8) and such that  $u_\epsilon \rightharpoonup u_0$  weakly in  $L^{2^*}(M)$  and  $(u_\epsilon)_\epsilon$  blows up with one bubble at  $\xi_0$ .

The paper is organized as follows. In Section 2, we introduce our notations and discuss the general setting of the problem, including general  $C^1$ -estimates. In Sections 3, 4 and 5, we then compute a  $C^1$ -asymptotic expansion of the energy functional in the case where  $n \geq 6$ . In Section 6, we discuss the specific setting of dimensions  $n \in \{4, 5\}$  and we define the mass of  $\Delta_g + h_0$  in this case. In Section 7, we then deal with the  $C^1$ -asymptotic expansion of the energy functional when  $n \in \{4, 5\}$ . In Sections 8 and 9, we complete the proofs of Theorems 1.4 and 1.5, respectively. In Section 10, we then prove Theorems 1.2 and 1.3. In Section 11, we discuss the question of existence of functions  $h_0$  and  $u_0$  satisfying the assumptions of Theorem 1.3. Finally, in Section 12, we deal with the necessity of condition (4) on the gradient. The interested reader can find some computational details in [26].

## 2. NOTATIONS AND GENERAL SETTING

We follow the notations and definitions of Robert–Vétois [23].

**2.1. Euclidean setting.** We define

$$U_{1,0}(x) := \left( \frac{\sqrt{n(n-2)}}{1+|x|^2} \right)^{\frac{n-2}{2}} \quad \text{for all } x \in \mathbb{R}^n,$$

so that  $U_{1,0}$  is a positive solution to the equation  $\Delta_{\text{Eucl}} U = U^{2^*-1}$  in  $\mathbb{R}^n$ , where  $\text{Eucl}$  stands for the Euclidean metric. For every  $\delta > 0$  and  $\xi \in \mathbb{R}^n$ , we define

$$(12) \quad U_{\delta, \xi}(x) := \delta^{-\frac{n-2}{2}} U(\delta^{-1}(x - \xi)) = \left( \frac{\sqrt{n(n-2)}\delta}{\delta^2 + |x - \xi|^2} \right)^{\frac{n-2}{2}} \quad \text{for all } x \in \mathbb{R}^n.$$

As one checks,

$$(13) \quad \partial_\delta U_{\delta,\xi}(x) = \sqrt{n(n-2)}^{\frac{n-2}{2}} \frac{n-2}{2} \delta^{\frac{n-2}{2}-1} \frac{|x-\xi|^2 - \delta^2}{(\delta^2 + |x-\xi|^2)^{n/2}}$$

$$(14) \quad \text{and } \partial_{\xi_i} U_{\delta,\xi}(x) = \sqrt{n(n-2)}^{\frac{n-2}{2}} (n-2) \delta^{\frac{n-2}{2}} \frac{(x-\xi)_i}{(\delta^2 + |x-\xi|^2)^{n/2}}$$

for all  $i = 1, \dots, n$ . We denote  $p = (p_0, p_1, \dots, p_n) := (\delta, \xi) \in (0, \infty) \times \mathbb{R}^n$ .

**2.2. Riemannian setting.** We fix  $N > n - 2$  to be chosen large later. It follows from Lee–Parker [14] that there exists a function  $\Lambda \in C^\infty(M \times M)$  such that, defining  $\Lambda_\xi := \Lambda(\xi, \cdot)$ , we have

$$(15) \quad \Lambda_\xi > 0, \quad \Lambda_\xi(\xi) = 1 \text{ and } \nabla \Lambda_\xi(\xi) = 0 \text{ for all } \xi \in M$$

and, defining the metric  $g_\xi := \Lambda_\xi^{2^*-2} g$  conformal to  $g$ , we have

$$(16) \quad \text{Scal}_{g_\xi}(\xi) = 0, \quad \nabla \text{Scal}_{g_\xi}(\xi) = 0, \quad \Delta_g \text{Scal}_{g_\xi}(\xi) = \frac{1}{6} |\text{Weyl}_g(\xi)|_g^2$$

and  $dv_{g_\xi}(x) = (1 + O(|x|^N)) dx$  via the chart  $\exp_\xi^{g_\xi}$  around 0,

where  $dx$  is the Euclidean volume element,  $dv_{g_\xi}$  is the Riemannian volume element of  $(M, g_\xi)$  and  $\exp_\xi^{g_\xi}$  is the exponential chart at  $\xi$  for the metric  $g_\xi$ . There exists  $r_0 > 0$  such that the injectivity radius of the metric  $g_\xi$  satisfies  $i_{g_\xi}(M) \geq 3r_0$  for all  $\xi \in M$ . We let  $\chi \in C^\infty(\mathbb{R})$  be such that  $\chi(t) = 1$  for  $t \leq r_0$ ,  $\chi(t) = 0$  for  $t \geq 2r_0$  and  $0 \leq \chi \leq 1$ . For every  $\delta > 0$  and  $\xi \in M$ , we define the bubble

$$(17) \quad U_{\delta,\xi}(x) := \chi(d_{g_\xi}(x, \xi)) \Lambda_\xi(x) \left( \frac{\delta \sqrt{n(n-2)}}{\delta^2 + d_{g_\xi}(x, \xi)^2} \right)^{\frac{n-2}{2}}.$$

**2.3. General reduction theorem.** For every  $1 \leq q \leq \infty$ , we let  $\|\cdot\|_q$  be the usual norm of  $L^q(M)$ . For every  $h \in C^0(M)$ , we define

$$J_h(u) := \frac{1}{2} \int_M (|\nabla u|_g^2 + hu^2) dv_g - \frac{1}{2^*} \int_M u_+^{2^*} dv_g \text{ for all } u \in H_1^2(M),$$

where  $u_+ := \max(u, 0)$ . We let  $(\delta, \xi) \mapsto B_{h,\delta,\xi} = B_h(\delta, \xi)$  be a function in  $C^1((0, \infty) \times M, H_1^2(M))$  such that for every  $\delta > 0$ , there exists  $\epsilon(\delta) > 0$  independent of  $h$  and  $\xi$  such that

$$(18) \quad \|B_{h,\delta,\xi}\|_{H_1^2} + \delta \|\partial_p B_{h,\delta,\xi}\|_{H_1^2} < \epsilon(\delta) \text{ for all } p = (\delta, \xi) \in (0, \infty) \times M$$

and  $\epsilon(\delta) \rightarrow 0$  as  $\delta \rightarrow 0$ . The function  $B_{h,\delta,\xi}$  will be fixed later. We also let  $\tilde{u}_0 \in C^2(M)$ . We define

$$W_{h,\tilde{u}_0,\delta,\xi} := \tilde{u}_0 + U_{\delta,\xi} + B_{h,\delta,\xi}.$$

We fix a point  $\xi_0 \in M$  and a function  $h_0 \in C^0(M)$  such that  $\Delta_g + h_0$  is coercive. We let  $u_0 \in C^2(M)$  be a solution of the equation

$$\Delta_g u_0 + h_0 u_0 = u_0^{2^*-1}, \quad u_0 \geq 0 \text{ in } M.$$

It follows from the strong maximum principle that either  $u_0 \equiv 0$  or  $u_0 > 0$ . We assume that  $u_0$  is nondegenerate (see (9)). It then follows from [23] that there exist  $\epsilon_0 > 0$ ,  $U_0 \subset M$  a small open neighborhood of  $\xi_0$  and  $\Phi_{h,\tilde{u}_0} \in C^1((0, \epsilon_0) \times U_0, H_1^2(M))$  such that, when  $\|h - h_0\|_\infty < \epsilon_0$  and  $\|\tilde{u}_0 - u_0\|_{C^2} < \epsilon_0$ , we have

$$(19) \quad \|\Phi_{h,\tilde{u}_0,\delta,\xi}\|_{H_1^2} \leq C \|R_{\delta,\xi}\|_{\frac{2n}{n+2}}$$

for all  $(\delta, \xi) \in (0, \epsilon_0) \times U_0$ , where  $C > 0$  does not depend on  $(h, \tilde{u}_0, \delta, \xi)$ ,  $\Phi_{h, \tilde{u}_0, \delta, \xi} := \Phi_{h, \tilde{u}_0}(\delta, \xi)$  and

$$(20) \quad R_{\delta, \xi} := (\Delta_g + h)W_{h, \tilde{u}_0, \delta, \xi} - (W_{h, \tilde{u}_0, \delta, \xi})_+^{2^* - 1}.$$

Furthermore, for every  $(\delta_0, \xi_0) \in (0, \epsilon_0) \times U_0$ , we have

$$(21) \quad J'_h(W_{h, \tilde{u}_0, \delta_0, \xi_0} + \Phi_{h, \tilde{u}_0, \delta_0, \xi_0}) = 0$$

$$\iff (\delta_0, \xi_0) \text{ is a critical point of } (\delta, \xi) \mapsto J_h(W_{h, \tilde{u}_0, \delta, \xi} + \Phi_{h, \tilde{u}_0, \delta, \xi})$$

$$(22) \quad \text{and } J_h(W_{h, \tilde{u}_0, \delta, \xi} + \Phi_{h, \tilde{u}_0, \delta, \xi}) = J_h(W_{h, \tilde{u}_0, \delta, \xi}) + O(\|\Phi_{h, \tilde{u}_0, \delta, \xi}\|_{H^2}^2)$$

uniformly in  $(\delta, \xi) \in (0, \epsilon_0) \times U_0$  and  $(h, \tilde{u}_0)$  such that  $\|h - h_0\|_\infty < \epsilon_0$  and  $\|\tilde{u}_0 - u_0\|_{C^2} < \epsilon_0$ . Moreover, assuming that

$$(23) \quad |B_{h, \delta, \xi}(x)| + \delta |\partial_p B_{h, \delta, \xi}(x)| \leq C(U_{\delta, \xi}(x) + \delta \tilde{U}_{\delta, \xi}(x)) \text{ for all } x \in M,$$

$$\text{where } \tilde{U}_{\delta, \xi}(x) := \left( \frac{\delta \sqrt{n(n-2)}}{\delta^2 + d_{g_\xi}(x, \xi)^2} \right)^{\frac{n-2}{2}},$$

we have (see Esposito–Pistoia–Vétois [10])

$$(24) \quad \partial_p J_h(W + \Phi) = \partial_p J_h(W) + O(\delta^{-1} \|\Phi\|_{H^2} (\|R\|_{H^2}^{\frac{2n}{n+2}} + \delta \|\partial_p R\|_{H^2}^{\frac{2n}{n+2}} + \|\Phi\|_{H^2})) \\ + O(\mathbf{1}_{n \geq 7} \delta^{-1} \|\Phi\|_{H^2}^{2^* - 1}),$$

where, to avoid unnecessarily heavy notations, we drop the indices  $(h, \tilde{u}_0, \delta, \xi)$ , so that  $W := W_{h, \tilde{u}_0, \delta, \xi}$ ,  $\Phi := \Phi_{h, \tilde{u}_0, \delta, \xi}$ ,  $R = R_{\delta, \xi}$ , etc.

#### 2.4. Conventions:

- The differentiation in  $(\delta, \xi)$  is denoted by  $\partial_p$ , and the differentiation in  $x \in M$  (or  $\mathbb{R}^n$ ) by  $\partial_x$ .
- For every  $\xi \in U_0$ , we identify the tangent space  $T_\xi M$  with  $\mathbb{R}^n$ .
- $C$  denotes a positive constant that depends on  $n$ ,  $(M, g)$ ,  $\xi_0 \in M$ , the functions  $h_0, u_0 \in C^2(M)$  and  $A > 0$  such that  $\|h\|_{C^2} < A$  and  $\lambda_1(\Delta_g + h) > 1/A$ . When  $u_0 > 0$ , we assume that  $\|u_0\|_{C^2} < A$  and  $u_0 > 1/A$ .
- For every  $f, g \in \mathbb{R}$ , the notations  $f = O(g)$  and  $f = o(g)$  will stand for  $|f| \leq C|g|$  and  $|f| \leq C\epsilon(h, \delta, \xi)|g|$ , respectively, where  $\epsilon(h, \delta, \xi) \rightarrow 0$  as  $h \rightarrow h_0$  in  $C^2(M)$ ,  $\delta \rightarrow 0$  and  $\xi \rightarrow \xi_0$ .

### 3. ENERGY AND REMAINDER ESTIMATES: THE CASE $n \geq 6$ AND $u_0 \equiv \tilde{u}_0 \equiv 0$

In this section, we consider the case  $n \geq 6$  and  $u_0 \equiv \tilde{u}_0 \equiv 0$ . We set  $B_{h, \delta, \xi} \equiv 0$ . Then  $W_{h, \tilde{u}_0, \delta, \xi} = W_{\delta, \xi} \equiv U_{\delta, \xi}$ . We prove the following estimates for  $R = R_{\delta, \xi}$ :

**Proposition 3.1.** *Assume that  $n \geq 6$  and  $u_0 \equiv \tilde{u}_0 \equiv 0$ . Then*

$$(25) \quad \|R\|_{\frac{2n}{n+2}} + \delta \|\partial_p R\|_{\frac{2n}{n+2}} \leq C \begin{cases} \delta^2 + D_{h, \xi} \delta^2 (\ln(1/\delta))^{2/3} & \text{if } n = 6 \\ \delta^{\frac{n-2}{2}} + D_{h, \xi} \delta^2 & \text{if } 7 \leq n \leq 9 \\ \delta^4 (\ln(1/\delta))^{3/5} + D_{h, \xi} \delta^2 & \text{if } n = 10 \\ \delta^4 + D_{h, \xi} \delta^2 & \text{if } n \geq 11, \end{cases}$$

where  $D_{h, \xi} := \|h - h_0\|_\infty + d_g(\xi, \xi_0)^2$ .

*Proof of Proposition 3.1.* Let  $L_g := \Delta_g + c_n \text{Scal}_g$  be the conformal Laplacian. For a metric  $g' = w^{4/(n-2)}g$  conformal to  $g$  ( $w \in C^\infty(M)$  is positive), the conformal invariance law gives that  $L_{g'}\phi = w^{-(2^*-1)}L_g(w\phi)$  for all  $\phi \in C^\infty(M)$ . Therefore, we have

$$R = (\Delta_g + h)U - U^{2^*-1} = \Lambda_\xi^{2^*-1}(\Delta_{g_\xi}(\Lambda_\xi^{-1}U) - (\Lambda_\xi^{-1}U)^{2^*-1}) + \hat{h}_\xi U,$$

where  $\varphi_h$  is as in (6) and

$$(26) \quad \hat{h}_\xi := \varphi_h + c_n \Lambda_\xi^{2^*-2} \text{Scal}_{g_\xi}.$$

Via the exponential chart, using the radial symmetry of  $U_{\delta,0} : \mathbb{R}^n \rightarrow \mathbb{R}$ , we obtain

$$\Delta_{g_\xi}(\Lambda_\xi^{-1}U) - (\Lambda_\xi^{-1}U)^{2^*-1} = \Delta_{\text{Eucl}}U_{\delta,0} + \frac{\partial_r \sqrt{|g_\xi|}}{\sqrt{|g_\xi|}} \partial_r U_{\delta,0} - U_{\delta,0}^{2^*-1} = \frac{\partial_r \sqrt{|g_\xi|}}{\sqrt{|g_\xi|}} \partial_r U_{\delta,0}$$

around 0. It then follows from (16) that

$$(27) \quad R(x) = \hat{h}_\xi(x)U(x) + \delta^{\frac{n-2}{2}} \Theta_{\delta,\xi}(x), \text{ where } |\Theta_{\delta,\xi}(x)| + |\partial_p \Theta_{\delta,\xi}(x)| \leq C$$

for all  $(\delta, \xi) \in (0, \infty) \times U_0$  and  $x \in M$ . Computations then yield (25). This ends the proof of Proposition 3.1.  $\square$

Plugging together (22), (19), (24) and (25), we obtain

$$(28) \quad J_h(W + \Phi) = J_h(W) + \text{O} \begin{pmatrix} \delta^4 + D_{h,\xi}^2 \delta^4 (\ln(1/\delta))^{4/3} & \text{if } n = 6 \\ \delta^{n-2} + D_{h,\xi}^2 \delta^4 & \text{if } 7 \leq n \leq 9 \\ \delta^8 (\ln(1/\delta))^{6/5} + D_{h,\xi}^2 \delta^4 & \text{if } n = 10 \\ \delta^8 + D_{h,\xi}^2 \delta^4 & \text{if } n \geq 11 \end{pmatrix}$$

$$(29) \quad \text{and } \partial_{p_i} J_h(W + \Phi) = \partial_{p_i} J_h(W)$$

$$+ \text{O}(\delta^{-1}) \begin{cases} \delta^4 + D_{h,\xi}^2 \delta^4 (\ln(1/\delta))^{4/3} & \text{if } n = 6 \\ (\delta^{\frac{n-2}{2}} + D_{h,\xi} \delta^2)^{2^*-1} & \text{if } 7 \leq n \leq 9 \\ (\delta^4 (\ln(1/\delta))^{3/5} + D_{h,\xi} \delta^2)^{2^*-1} & \text{if } n = 10 \\ (\delta^4 + D_{h,\xi} \delta^2)^{2^*-1} & \text{if } n \geq 11 \end{cases}$$

for all  $i = 0, \dots, n$ . We now estimate  $J_h(W + \Phi)$ :

**Proposition 3.2.** *Assume that  $n \geq 6$  and  $u_0 \equiv \tilde{u}_0 \equiv 0$ . Then*

$$(30) \quad J_h(W + \Phi) = \frac{1}{n} \int_{\mathbb{R}^n} U_{1,0}^{2^*} dx + \frac{1}{2} \varphi_h(\xi) \delta^2 \int_{\mathbb{R}^n} U_{1,0}^2 dx \\ - \frac{1}{4n} \begin{cases} 24^2 \omega_5 K_{h_0}(\xi_0) \delta^4 \ln(1/\delta) + \text{O}(\delta^4 (\ln(1/\delta) + D_{h,\xi}^2 (\ln(1/\delta))^{4/3})) & \text{if } n = 6 \\ K_{h_0}(\xi_0) \delta^4 \int_{\mathbb{R}^n} |x|^2 U_{1,0}^2 dx + \text{o}(\delta^4) & \text{if } n \geq 7 \end{cases}$$

as  $\delta \rightarrow 0$ ,  $\xi \rightarrow \xi_0$  and  $h \rightarrow h_0$  in  $C^2(M)$ , where  $K_{h_0}(\xi_0)$  is as in (7).

*Proof of Proposition 3.2.* Integrating by parts, we obtain

$$J_h(U) = \frac{1}{2} \int_M [(\Delta_g + h)U - U^{2^*-1}]U dv_g + \frac{1}{n} \int_M U^{2^*} dv_g.$$

It follows from (27) and the volume estimate (16) that

$$J_h(U) = \frac{1}{2} \int_M \hat{h}_\xi U^2 dv_g + \frac{1}{n} \int_{\mathbb{R}^n} U_{1,0}^{2^*} dx + O(\delta^{n-2}).$$

With the change of metric, (16) and (17) yield

$$\int_M \hat{h}_\xi U^2 dv_g = \int_{B_{r_0}(\xi)} \hat{h}_\xi U^2 dv_g + O(\delta^{n-2}) = \int_{B_{r_0}(0)} A_{h,\xi} U_{\delta,0}^2 dx + O(\delta^{n-2}),$$

where  $A_{h,\xi}(x) := (\hat{h}_\xi \Lambda_\xi^{2-2^*})(\exp^{g_\xi}(x))$ . Using the radial symmetry of  $U_{\delta,0}$  together with Taylor expansions and the explicit form of  $U_{\delta,0}$ , we obtain

$$(31) \quad J_h(U) = \frac{1}{n} \int_{\mathbb{R}^n} U_{1,0}^{2^*} dx + \frac{1}{2} \varphi_h(\xi) \delta^2 \int_{\mathbb{R}^n} U_{1,0}^2 dx \\ - \frac{1}{4n} \begin{cases} 24^2 \omega_5 K_{h_0}(\xi_0) \delta^4 \ln(1/\delta) + o(\delta^4 \ln(1/\delta)) & \text{if } n = 6 \\ K_{h_0}(\xi_0) \delta^4 \int_{\mathbb{R}^n} |x|^2 U_{1,0}^2 dx + o(\delta^4) & \text{if } n \geq 7. \end{cases}$$

Plugging together (28) and (31), we obtain (30). This proves Proposition 3.2.  $\square$

With some extra care, arguing similarly as Esposito–Pistoia–Vétois [10], see also additional details are in [26], we also obtain the differentiable version:

**Proposition 3.3.** *Assume that  $n \geq 6$  and  $u_0 \equiv \tilde{u}_0 \equiv 0$ . Then*

$$(32) \quad \partial_\delta J_h(W + \Phi) = \varphi_h(\xi) \delta \int_{\mathbb{R}^n} U_{1,0}^2 dx \\ - \frac{1}{n} \begin{cases} 24^2 \omega_5 K_{h_0}(\xi_0) \delta^3 \ln(1/\delta) + o(\delta^3 \ln(1/\delta)) + O(D_{h,\xi}^2 \delta^3 (\ln(1/\delta))^{4/3}) & \text{if } n = 6 \\ K_{h_0}(\xi_0) \delta^3 \int_{\mathbb{R}^n} |x|^2 U_{1,0}^2 dx + o(\delta^3) + O(D_{h,\xi}^{2^*-1} \delta^{\frac{n+6}{n-2}}) & \text{if } n \geq 7 \end{cases}$$

$$(33) \quad \text{and } \partial_{\xi_i} J_h(W + \Phi) = \frac{1}{2} \partial_{\xi_i} \varphi_h(\xi) \delta^2 \int_{\mathbb{R}^n} U_{1,0}^2 dx \\ + O \begin{pmatrix} o(\delta^3 \ln(1/\delta)) + O(D_{h,\xi}^2 \delta^3 (\ln(1/\delta))^{4/3}) & \text{if } n = 6 \\ o(\delta^3) + O(D_{h,\xi}^{2^*-1} \delta^{\frac{n+6}{n-2}}) & \text{if } n \geq 7 \end{pmatrix}$$

for all  $i = 1, \dots, n$ , as  $\delta \rightarrow 0$ ,  $\xi \rightarrow \xi_0$  and  $h \rightarrow h_0$  in  $C^2(M)$ .

Theorem 1.4 for  $n \geq 6$  will be proved in Section 8.

#### 4. ENERGY AND REMAINDER ESTIMATES: THE CASE $n \geq 7$ AND $u_0, \tilde{u}_0 > 0$

In this section, we assume that  $u_0, \tilde{u}_0 > 0$  and  $n \geq 7$ , that is  $2^* - 1 < 2$ . As in the previous case, we set  $B_{h,\delta,\xi} \equiv 0$ , so that  $W_{h,\tilde{u}_0,\delta,\xi} = W_{\tilde{u}_0,\delta,\xi} \equiv \tilde{u}_0 + U_{\delta,\xi}$  and we are in the framework of Section 2. We prove the following estimates for  $R = R_{\delta,\xi}$ :

**Proposition 4.1.** *Assume that  $n \geq 7$  and  $u_0, \tilde{u}_0 > 0$ . Then*

$$(34) \quad \|R\|_{\frac{2n}{n+2}} \leq C \|\Delta_g \tilde{u}_0 + h \tilde{u}_0 - \tilde{u}_0^{2^*-1}\|_\infty + C(D_{h,\xi} + \delta^2 + \delta^{\frac{n-6}{n-2}}) \delta^2 \quad \text{and} \quad \|\partial_p R\|_{\frac{2n}{n+2}} \leq C\delta,$$

where  $D_{h,\xi}$  is as in (25).



*Proof of Proposition 4.1.* We have

$$R = (\Delta_g \tilde{u}_0 + h\tilde{u}_0 - \tilde{u}_0^{2^*-1}) + R^0 - ((\tilde{u}_0 + U)^{2^*-1} - \tilde{u}_0^{2^*-1} - U^{2^*-1})$$

$$\text{and } \partial_{p_i} R = \partial_{p_i} R^0 - (2^* - 1)((\tilde{u}_0 + U)^{2^*-2} - U^{2^*-2}) \partial_{p_i} U$$

for all  $i = 1, \dots, n$ , where  $R^0 := \Delta_g U + hU - U^{2^*-1}$ . Arguing as in [22], we then obtain (34), which proves Proposition 4.1.  $\square$

Plugging (34) together with (22), (19) and (24), we obtain

$$(35) \quad J_h(W + \Phi) = J_h(W) + O(\|\Delta_g \tilde{u}_0 + h\tilde{u}_0 - \tilde{u}_0^{2^*-1}\|_\infty^2 + D_{h,\xi}^2 \delta^4 + \delta^8 + \delta^{n-2})$$

$$(36) \quad \text{and } \partial_{p_i} J_h(W + \Phi) = \partial_{p_i} J_h(W) + O(\|\Delta_g \tilde{u}_0 + h\tilde{u}_0 - \tilde{u}_0^{2^*-1}\|_{\frac{2n}{n+2}} \delta \\ + \|\Delta_g \tilde{u}_0 + h\tilde{u}_0 - \tilde{u}_0^{2^*-1}\|_\infty^{2^*-1} \delta^{-1} + (D_{h,\xi} + \delta^2 + \delta^{\frac{n-6}{2}})^{2^*-1} \delta^{\frac{n+6}{2}} + D_{h,\xi} \delta^3 + \delta^5 + \delta^{n/2})$$

for all  $i = 0, \dots, n$ . We now estimate  $J_h(W + \Phi)$ :

**Proposition 4.2.** *Assume that  $n \geq 7$  and  $u_0, \tilde{u}_0 > 0$ . Then*

$$(37) \quad J_h(W + \Phi) = J_h(\tilde{u}_0) + \frac{1}{n} \int_{\mathbb{R}^n} U_{1,0}^{2^*} dx + \frac{1}{2} \varphi_h(\xi) \delta^2 \int_{\mathbb{R}^n} U_{1,0}^2 dx \\ - \frac{1}{4n} K_{h_0}(\xi_0) \delta^4 \int_{\mathbb{R}^n} |x|^2 U_{1,0}^2 dx + o(\delta^4) - u_0(\xi_0) \delta^{\frac{n-2}{2}} \int_{\mathbb{R}^n} U_{1,0}^{2^*-1} dx \\ + O(\|\Delta_g \tilde{u}_0 + h\tilde{u}_0 - \tilde{u}_0^{2^*-1}\|_\infty^2 + \delta^{\frac{n-2}{2}} (\|\Delta_g \tilde{u}_0 + h\tilde{u}_0 - \tilde{u}_0^{2^*-1}\|_\infty + \|\tilde{u}_0 - u_0\|_\infty + o(1)))$$

as  $\delta \rightarrow 0$ ,  $\xi \rightarrow \xi_0$  and  $h \rightarrow h_0$  in  $C^2(M)$ .

*Proof of Proposition 4.2.* We first write

$$J_h(\tilde{u}_0 + U) = J_h(\tilde{u}_0) + J_h(U) - \int_M \tilde{u}_0 U^{2^*-1} dv_g + \int_M (\Delta_g \tilde{u}_0 + h\tilde{u}_0 - \tilde{u}_0^{2^*-1}) U dv_g \\ - \frac{1}{2^*} \int_M ((\tilde{u}_0 + U)^{2^*} - \tilde{u}_0^{2^*} - U^{2^*} - 2^* \tilde{u}_0^{2^*-1} U - 2^* \tilde{u}_0 U^{2^*-1}) dv_g.$$

We then argue as in [22] to estimate the last term in the right-hand side of this identity. The third term is estimated as in [25]. The second term is estimated by using the expansion (31). Then (35) yields (37), thus proving Proposition 4.2.  $\square$

In the same spirit, arguing again as in Esposito-Pistoia-Vétois [10], see again details in [26], we obtain the following estimates for the derivatives of  $J_h(W + \Phi)$ :

**Proposition 4.3.** *Assume that  $n \geq 7$  and  $u_0, \tilde{u}_0 > 0$ . Then*

$$(38) \quad \partial_\delta J_h(W + \Phi) = \varphi_h(\xi) \delta \int_{\mathbb{R}^n} U_{1,0}^2 dx - \frac{1}{n} K_{h_0}(\xi_0) \delta^3 \int_{\mathbb{R}^n} |x|^2 U_{1,0}^2 dx + o(\delta^3) \\ - \frac{n-2}{2} u_0(\xi_0) \delta^{\frac{n-4}{2}} \int_{\mathbb{R}^n} U_{1,0}^{2^*-1} dx + O(\delta^{\frac{n-4}{2}} (\|\tilde{u}_0 - u_0\|_\infty + o(1)) \\ + \|\Delta_g \tilde{u}_0 + h\tilde{u}_0 - \tilde{u}_0^{2^*-1}\|_\infty \delta + \|\Delta_g \tilde{u}_0 + h\tilde{u}_0 - \tilde{u}_0^{2^*-1}\|_\infty^{2^*-1} \delta^{-1} + D_{h,\xi}^{2^*-1} \delta^{\frac{n+6}{2}})$$

$$(39) \quad \text{and } \partial_{\xi_i} J_h(W + \Phi) = \frac{1}{2} \partial_{\xi_i} \varphi_h(\xi) \delta^2 \int_{\mathbb{R}^n} U_{1,0}^2 dx + o(\delta^3) \\ + O(\delta^{\frac{n-4}{2}} (\|\tilde{u}_0 - u_0\|_\infty + o(1)) + \|\Delta_g \tilde{u}_0 + h\tilde{u}_0 - \tilde{u}_0^{2^*-1}\|_\infty \delta \\ + \|\Delta_g \tilde{u}_0 + h\tilde{u}_0 - \tilde{u}_0^{2^*-1}\|_\infty^{2^*-1} \delta^{-1} + D_{h,\xi}^{2^*-1} \delta^{\frac{n+6}{2}})$$

for all  $i = 1, \dots, n$ , as  $\delta \rightarrow 0$ ,  $\xi \rightarrow \xi_0$  and  $h \rightarrow h_0$  in  $C^2(M)$ .

Theorem 1.5 for  $n \geq 7$  will be proved in Section 9.

#### 5. ENERGY AND REMAINDER ESTIMATES: THE CASE $n = 6$ AND $u_0, \tilde{u}_0 > 0$

In this section, we assume that  $u_0, \tilde{u}_0 > 0$  and  $n = 6$ , that is  $2^* - 1 = 2$ . Here again, we set  $B_{h,\delta,\xi} \equiv 0$ , so that  $W_{h,\tilde{u}_0,\delta,\xi} = W_{\tilde{u}_0,\delta,\xi} \equiv \tilde{u}_0 + U_{\delta,\xi}$  and we are in the framework of Section 2. The remark underlying this section is that since  $2^* - 1 = 2$ ,

$$\Delta_g(u_0 + U) + h(u_0 + U) - (u_0 + U)^2 = \Delta_g U + (h - 2u_0)U - U^2.$$

Therefore, to obtain a good approximation of the blowing-up solution, it suffices to subtract a perturbation of  $2u_0$  to the potential. With this remark, setting

$$\bar{h} := h - 2\tilde{u}_0 \text{ and } \bar{h}_0 := h_0 - 2u_0$$

and arguing as in the case where  $n \geq 7$ , we obtain the following:

**Proposition 5.1.** *Assume that  $n = 6$  and  $u_0, \tilde{u}_0 > 0$ . Then*

$$(40) \quad J_h(W + \Phi) = J_h(\tilde{u}_0) + \frac{1}{n} \int_{\mathbb{R}^n} U_{1,0}^{2^*} dx + \frac{1}{2} \varphi_{h,\tilde{u}_0}(\xi) \delta^2 \int_{\mathbb{R}^n} U_{1,0}^2 dx \\ - 24\omega_5 K_{h_0,u_0}(\xi_0) \delta^4 \ln(1/\delta) + O(\|\Delta_g \tilde{u}_0 + h\tilde{u}_0 - \tilde{u}_0^2\|_\infty^2 + \|\Delta_g \tilde{u}_0 + h\tilde{u}_0 - \tilde{u}_0^2\|_\infty \delta^2) \\ + O(\delta^4 \ln(1/\delta)(o(1) + \bar{D}_{h,\xi}^2 (\ln(1/\delta))^{1/3})),$$

$$(41) \quad \partial_\delta J_h(W + \Phi) = \varphi_{h,\tilde{u}_0}(\xi) \delta \int_{\mathbb{R}^n} U_{1,0}^2 dx - 96\omega_5 K_{h_0,u_0}(\xi_0) \delta^3 \ln(1/\delta) \\ + O(\|\Delta_g \tilde{u}_0 + h\tilde{u}_0 - \tilde{u}_0^2\|_\infty \delta + \|\Delta_g \tilde{u}_0 + h\tilde{u}_0 - \tilde{u}_0^2\|_\infty^2 \delta^{-1}) \\ + O(\delta^3 \ln(1/\delta)(o(1) + \bar{D}_{h,\xi}^2 (\ln(1/\delta))^{1/3})),$$

$$(42) \quad \text{and } \partial_{\xi_i} J_h(W + \Phi) = \frac{1}{2} \partial_{\xi_i} \varphi_{h,\tilde{u}_0}(\xi) \delta^2 \int_{\mathbb{R}^n} U_{1,0}^2 dx \\ + O(\|\Delta_g \tilde{u}_0 + h\tilde{u}_0 - \tilde{u}_0^2\|_\infty \delta + \|\Delta_g \tilde{u}_0 + h\tilde{u}_0 - \tilde{u}_0^2\|_\infty^2 \delta^{-1})$$

$$(43) \quad + O(\delta^3 \ln(1/\delta)(o(1) + \bar{D}_{h,\xi}^2 (\ln(1/\delta))^{1/3}))$$

for all  $i = 1, \dots, n$ , as  $\delta \rightarrow 0$ ,  $\xi \rightarrow \xi_0$  and  $h \rightarrow h_0$  in  $C^2(M)$ , where  $\varphi_{h,\tilde{u}_0}$  and  $K_{h_0,u_0}(\xi_0)$  are as in (6), (10) and  $\bar{D}_{h,\xi} := \|\bar{h} - \bar{h}_0\|_\infty + d_g(\xi, \xi_0)^2$ .

Here again, additional details are in [26].

#### 6. SETTING AND DEFINITION OF THE MASS IN DIMENSIONS $n = 4, 5$

Following the computations in (27), we obtain the following:

**Lemma 6.1.** *There exist two smooth functions  $(\xi, x) \mapsto f_i(\xi, x)$ ,  $i = 1, 2$ , on  $M \times M$  such that for every  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  that is radially symmetric, we have*

$$(\Delta_g + h)(\chi(r)\Lambda_\xi(x)f(r)) = \Lambda_\xi(x)^{2^*-1} \chi \Delta_{\text{Eucl}}(f(r)) + f_1(\xi, x)f'(r) + f_2(\xi, x)f(r) \\ + \hat{h}_\xi \chi(x)\Lambda_\xi(x)f(r)$$

for all  $x \in M \setminus \{\xi\}$ , where  $r := d_{g_\xi}(x, \xi)$  and  $\hat{h}_\xi$  is as in (26). Furthermore,  $f_i(\xi, x) = 0$  when  $d_g(x, \xi) \geq r_0$  and there exists  $C_N > 0$  such that

$$|f_1(\xi, x)(x)| \leq C_N d_g(x, \xi)^{N-1} \text{ and } |f_2(\xi, x)| \leq C_N d_g(x, \xi)^{N-2} \text{ for all } x, \xi \in M.$$

We define

$$\Gamma_\xi(x) := \frac{\chi(d_{g_\xi}(x, \xi))\Lambda_\xi(x)}{(n-2)\omega_{n-1}d_{g_\xi}(x, \xi)^{n-2}}$$

for all  $x \in M \setminus \{\xi\}$ . It follows from Lemma 6.1 and the definition (12) that

(44)

$$\Delta_g U_{\delta, \xi} + h U_{\delta, \xi} = U_{\delta, \xi}^{2^*-1} + F_\delta(\xi, x) \delta^{\frac{n-2}{2}} + \hat{h}_\xi U_{\delta, \xi}$$

and  $(\Delta_g + h)\Gamma_\xi = \delta_\xi + \frac{F_0(\xi, x)}{k_n} + \hat{h}_\xi \Gamma_\xi$ , where  $k_n := (n-2)\omega_{n-1}\sqrt{n(n-2)}^{\frac{n-2}{2}}$ ,

$\delta_\xi$  is the Dirac mass at  $\xi$  and  $(t, \xi, x) \rightarrow F_t(\xi, x)$  is of class  $C^p$  on  $[0, \infty) \times M \times M$ , with  $p$  being as large as we want provided we choose  $N$  large enough. This includes  $t = 0$  and, therefore,

$$(45) \quad \lim_{t \rightarrow 0} F_t = F_0 \text{ in } C^p(M \times M).$$

For every  $t \geq 0$ , we define  $\beta_{h,t,\xi} \in H_1^2(M)$  as the unique solution to

$$(46) \quad (\Delta_g + h)\beta_{h,t,\xi} = -\frac{F_t(\xi, \cdot)}{k_n} - \hat{h}_\xi \begin{cases} \frac{U_{t,\xi}}{k_n t^{\frac{n-2}{2}}} & \text{if } t > 0 \\ \Gamma_\xi & \text{if } t = 0. \end{cases}$$

Since  $N > n - 2$  and  $n \leq 5$ , the right-hand-side is uniformly bounded in  $L^q(M)$  for some  $q > \frac{2n}{n+2}$ , independently of  $t \geq 0$ ,  $\xi \in U_0$  and  $h \in C^2(M)$  satisfying  $\|h\|_\infty < A$  and  $\lambda_1(\Delta_g + h) > 1/A$ . Therefore,  $\beta_{h,t,\xi}$  is well defined and we have

$$(47) \quad \|\beta_{h,t,\xi} - \beta_{h,0,\xi}\|_{H_1^2} = o(1) \text{ as } t \rightarrow 0$$

uniformly in  $\xi$  and  $h$ . Furthermore, we have  $\beta_{h,t,\xi} \in C^2(M)$  when  $t > 0$ . As one checks, with these definitions, we obtain that

$$G_{h,\xi} := \Gamma_\xi + \beta_{h,0,\xi}$$

is the Green's function of the operator  $\Delta_g + h$  at the point  $\xi$ .

**Proposition-Definition 6.1.** *Assume that  $n \in \{4, 5\}$  and  $N > n - 2$ . Let  $h \in C^2(M)$  be such that  $\Delta_g + h$  is coercive. Assume that there exists  $\xi \in M$  such that  $\varphi_h(\xi) = |\nabla \varphi_h(\xi)| = 0$ , where  $\varphi_h$  is as in (6). Then  $\beta_{h,0,\xi} \in C^0(M)$ . Furthermore,  $\beta_{h,0,\xi}(\xi)$  does not depend on the choice of  $N > n - 2$  and  $g_\xi$  satisfying (15) and (16). We then define the mass of  $\Delta_g + h$  at the point  $\xi$  as  $m_h(\xi) := \beta_{h,0,\xi}(\xi)$ .*

*Proof of Proposition-Definition 6.1.* We have  $\hat{h}_\xi(x)\Gamma_\xi(x) = O(d_g(x, \xi)^{4-n})$  since  $\varphi_h(\xi) = |\nabla \varphi_h(\xi)| = 0$ . We also have  $F_0(\xi, x) = O(d_g(x, \xi)^{N-n})$ . When  $N > n$ , this implies that  $\beta_{h,0,\xi} \in C^0(M)$ . One has that  $\beta_{h,0,\xi}(\xi)$  is independent of  $N$  and  $g_\xi$  (see Lee–Parker [14]). This ends the proof of Proposition-Definition 6.1.  $\square$

Note that when  $h \equiv c_n \text{Scal}_g$ , the mass  $m_{c_n \text{Scal}_g}(\xi)$  is defined for all  $\xi \in M$ , and one recovers the concept of mass of Schoen–Yau when  $n = 3, 4, 5$ . When  $h \equiv c_n \text{Scal}_g$  and the manifold is locally conformally flat, here again, one recovers the concept of Schoen–Yau. We now prove a differentiation result that will allow us to obtain Theorem 1.2:

**Proposition 6.1.** *Assume that  $n \in \{4, 5\}$ . Let  $h \in C^2(M)$  be such that  $\Delta_g + h$  is coercive. Assume that there exists  $\xi \in M$  such that  $\varphi_h(\xi) = |\nabla \varphi_h(\xi)| = 0$ . Let  $H \in C^2(M)$  be such that  $H(\xi) = |\nabla H(\xi)| = 0$ . Then  $m_{h+\epsilon H}(\xi)$  is well defined for small  $\epsilon \in \mathbb{R}$  and differentiable in  $\epsilon$ . Furthermore,*

$$\partial_\epsilon(m_{h+\epsilon H}(\xi))|_0 = - \int_M H G_{h,\xi}^2 dv_g.$$

*Proof of Proposition 6.1.* We set  $G_{h,\xi} = G_{c_n \text{Scal}_g,\xi} + \hat{\beta}_{h,\xi}$ , where  $\hat{\beta}_{h,\xi}$  is such that

$$(48) \quad (\Delta_g + h)\hat{\beta}_{h,\xi} = -\varphi_h G_{c_n \text{Scal}_g,\xi}, \quad \hat{\beta}_{h,\xi} \in H_1^2(M).$$

Under the assumptions of the proposition, we have  $\hat{\beta}_{h,\xi} \in C^0(M)$  and

$$\hat{\beta}_{h,\xi}(\xi) = - \int_M \varphi_h G_{c_n \text{Scal}_g,\xi} G_{h,\xi} dv_g.$$

As one checks, we have  $m_h(\xi) = m_{c_n \text{Scal}_g}(\xi) - \hat{\beta}_{h,\xi}(\xi)$ . Elliptic theory gives that  $h \mapsto \hat{\beta}_{h,\xi}$  is differentiable. Differentiating (48) with respect to  $h$  in the direction  $H$  yields

$$\partial_\epsilon(m_{h+\epsilon H}(\xi))|_{\epsilon=0} = - \int_M H G_{h,\xi}^2 dv_g.$$

This ends the proof of Proposition 6.1.  $\square$

## 7. ENERGY AND REMAINDER ESTIMATES IN DIMENSIONS $n = 4, 5$

Here we assume that  $n \leq 5$ ,  $u_0 \equiv \tilde{u}_0 \equiv 0$  and (4) is satisfied. We define

$$(49) \quad W_{h,\tilde{u}_0,\delta,\xi} = W_{h,\delta,\xi} := U_{\delta,\xi} + B_{h,\delta,\xi}, \quad \text{where } B_{h,\delta,\xi} := k_n \delta^{\frac{n-2}{2}} \beta_{h,\delta,\xi}.$$

Our first step is to obtain estimates for  $\beta_{h,\delta,\xi}$  and its derivatives in  $H_1^2(M)$ :

**Proposition 7.1.** *For  $n \in \{4, 5\}$ , let  $B_{h,\delta,\xi}$  be as in (49). Then (18) holds.*

*Proof of Proposition 7.1.* It follows from (47) that  $\|\beta_{h,\delta,\xi}\|_{H_1^2} \leq C$ . Differentiating (46) in  $\xi_i$ ,  $i = 1, \dots, n$ , we obtain

$$(\Delta_g + h)(\partial_{\xi_i} \beta_{h,\delta,\xi}) = -\frac{1}{k_n} \left( \partial_{\xi_i} F_\delta(\xi, \cdot) + \partial_{\xi_i} \hat{h}_\xi \frac{U_{\delta,\xi}}{\delta^{\frac{n-2}{2}}} + \hat{h}_\xi \frac{\partial_{\xi_i} U_{\delta,\xi}}{\delta^{\frac{n-2}{2}}} \right).$$

Using the explicit pointwise control that we have for the right-hand side of this identity, elliptic theory yields (18) for the derivative in  $\xi_i$ . We then apply the same method to estimate the derivative in  $\delta$ . This proves Proposition 7.1.  $\square$

We now need a pointwise control for  $\beta_{h,\delta,\xi}$  and its derivatives.

**Proposition 7.2.** *We have*

$$(50) \quad |\beta_{h,\delta,\xi}(x)| \leq C \begin{cases} 1 + |\ln(\delta^2 + d_g(x, \xi)^2)| & \text{if } n = 4 \\ (\delta^2 + d_g(x, \xi)^2)^{-1/2} & \text{if } n = 5, \end{cases}$$

$$(51) \quad |\partial_\delta \beta_{h,\delta,\xi}(x)| \leq C + C D_{h,\xi} \delta \ln(1/\delta) (\delta^2 + d_g(x, \xi)^2)^{-\frac{n-2}{2}} \quad \text{and}$$

(52)

$$|\partial_{\xi_i} \beta_{h,\delta,\xi}(x)| \leq C + C \begin{cases} D_{h,\xi} (\delta^2 + d_g(x, \xi)^2)^{-1/2} & \text{if } n = 4 \\ |\ln(\delta^2 + d_g(x, \xi)^2)| + D_{h,\xi} (\delta^2 + d_g(x, \xi)^2)^{-1} & \text{if } n = 5 \end{cases}$$

for all  $i = 1, \dots, n$ , where  $D_{h,\xi}$  is as in (25).

*Proof of Proposition 7.2.* Green's representation formula together with (46) yields (53)

$$\beta_{h,\delta,\xi}(x) = - \int_M G_{h,x}(y) \left( \frac{F_\delta(\xi, y)}{k_n} + \hat{h}_\xi \frac{\chi(d_{g_\xi}(y, \xi)) \Lambda_\xi(y)}{(n-2)\omega_{n-1}(\delta^2 + d_{g_\xi}(y, \xi)^2)^{\frac{n-2}{2}}} \right) dv_g(y)$$

for all  $x \in M$ . Standard estimates of the Green's function give  $0 < G_{h,x}(y) \leq Cd_g(x, y)^{2-n}$  for all  $x, y \in M$ ,  $x \neq y$ . Proposition 7.2 then follows from these estimates together with (45) and Giraud's lemma (see [9]).  $\square$

When the mass is defined at  $\xi$ , that is  $\varphi_h(\xi) = |\nabla\varphi_h(\xi)| = 0$ , then  $\beta_{h,\delta,\xi}$  is bounded. Otherwise, it is not. For instance, when  $n = 5$ , it behaves like  $(h - c_n \text{Scal}_g)(\xi) d_g(x, \xi)^{-1}$  as  $x \rightarrow \xi$  and  $\delta \rightarrow 0$ .

It follows from Proposition 7.2 that (18) is satisfied and therefore, we are in the framework of Section 2. Since  $n \leq 5$ , we then obtain

$$(54) \quad J_h(W + \Phi) = J_h(W) + O(\|R\|_{\frac{2n}{n+2}}^2)$$

$$(55) \quad \text{and } \partial_p J_h(W + \Phi) = \partial_p J_h(W) + O(\delta^{-1} \|R\|_{\frac{2n}{n+2}} (\|R\|_{\frac{2n}{n+2}} + \delta \|\partial_p R\|_{\frac{2n}{n+2}})),$$

where  $R = R_{\delta,\xi}$  is as in (20). We prove the following estimates for  $R$ :

**Proposition 7.3.** *We have*

$$(56) \quad \|R\|_{\frac{2n}{n+2}} + \delta \|\partial_p R\|_{\frac{2n}{n+2}} \leq C \begin{cases} \delta^2 \ln(1/\delta) & \text{if } n = 4 \\ D_{h,\xi} \delta^2 \ln(1/\delta) + \delta^2 & \text{if } n = 5. \end{cases}$$

*Proof of Proposition 7.3.* Note that since  $n < 6$ , we have  $2^* > 3$ . The definitions (46), (49) combined with (44) yield

$$(57) \quad R = U^{2^*-1} - (U + B)_+^{2^*-1} = -(2^* - 1)U^{2^*-2}B + O(U^{2^*-3}B^2 + |B|^{2^*-1}),$$

where we have used that  $U \geq 0$ . Therefore,

$$\|R\|_{\frac{2n}{n+2}} \leq C \|U^{2^*-2}B\|_{\frac{2n}{n+2}} + \||B|^{2^*-1}\|_{\frac{2n}{n+2}}.$$

As regards the gradient term, letting  $i \in \{0, \dots, n\}$ , we have

$$\partial_{p_i} R = -(2^* - 1)((U + B)_+^{2^*-2} - U^{2^*-2})\partial_{p_i} U + (U + B)_+^{2^*-2}\partial_{p_i} B.$$

We then obtain

$$\delta \|\partial_{p_i} R\| \leq C \tilde{U}^{2^*-2} |B| + C \tilde{U} |B|^{2^*-2} + C \delta \|\partial_{p_i} B\| \tilde{U}^{2^*-2},$$

where  $\tilde{U} = \tilde{U}_{\delta,\xi}$  is as in (23). Since  $B = k_n \delta^{\frac{n-2}{2}} \beta$ , Proposition 7.2 together with (50) and long but easy computations yields (56), thus proving Proposition 7.3.  $\square$

With (56), the estimates (54) and (55) become

$$(58) \quad J_h(W + \Phi) = J_h(W) + O \begin{pmatrix} \delta^4 (\ln(1/\delta))^2 & \text{if } n = 4 \\ \delta^4 + D_{h,\xi}^2 \delta^4 (\ln(1/\delta))^2 & \text{if } n = 5 \end{pmatrix}$$

$$(59) \quad \text{and } \partial_{p_i} J_h(W + \Phi) = \partial_{p_i} J_h(W) + O \begin{pmatrix} \delta^3 (\ln(1/\delta))^2 & \text{if } n = 4 \\ \delta^3 + D_{h,\xi}^2 \delta^3 (\ln(1/\delta))^2 & \text{if } n = 5 \end{pmatrix}.$$

**Proposition 7.4.** *We have*

$$(60) \quad J_h(W + \Phi) = \frac{1}{n} \int_{\mathbb{R}^n} U_{1,0}^{2^*} dx + \frac{1}{2} \varphi_h(\xi) \begin{cases} 8\omega_{n-1} \delta^2 \ln(1/\delta) & \text{if } n = 4 \\ \delta^2 \int_{\mathbb{R}^n} U_{1,0}^2 dx & \text{if } n = 5 \end{cases} \\ - \frac{k_n^2}{2} m_{h_0}(\xi_0) \delta^{n-2} + o(\delta^{n-2})$$

as  $\delta \rightarrow 0$ ,  $\xi \rightarrow \xi_0$  and  $h \rightarrow h_0$  in  $C^2(M)$ .

*Proof of Proposition 7.4.* We have

$$(61) \quad J_h(W) = \frac{1}{2} \int_M RW dv_g + \left( \frac{1}{2} - \frac{1}{2^*} \right) \int_M W_+^{2^*} dv_g.$$

Using that  $U \geq 0$ , we obtain

$$(62) \quad W_+^{2^*} = (U + B)_+^{2^*} = U^{2^*} + 2^* B U^{2^*-1} + O(B^2 U^{2^*-2} + |B|^{2^*}).$$

Plugging (57) and (62) into (61), and using (23), we obtain

$$J_h(W) = \frac{1}{n} \int_M U^{2^*} dv_g - \frac{1}{2} \int_M B U^{2^*-1} dv_g \\ + O \left( \int_M (\tilde{U}^{2^*-2} B^2 + \tilde{U} |B|^{2^*-1} + |B|^{2^*}) dv_g \right).$$

Since  $B = k_n \delta^{\frac{n-2}{2}} \beta$ , the pointwise estimate (50) and the definition (12) yield

$$J_h(W) = \frac{1}{n} \int_{\mathbb{R}^n} U_{1,0}^{2^*} dx - \frac{1}{2} \int_M B U^{2^*-1} dv_g + O \begin{pmatrix} \delta^4 (\ln(1/\delta))^3 & \text{if } n = 4 \\ \delta^4 & \text{if } n = 5 \end{pmatrix}.$$

The definitions (46) and (49) of  $\beta$  and  $B$  yield

$$(63) \quad \Delta_g B + hB = U^{2^*-1} - (\Delta_g U + hU) \text{ in } M.$$

Therefore, we obtain

$$\int_M B U^{2^*-1} dv_g = \int_M (|\nabla B|^2 + hB^2) dv_g - \delta^{\frac{n-2}{2}} \int_M F_\delta(\xi, \cdot) U dv_g - \int_M \hat{h}_\xi U^2 dv_g.$$

Since  $B = k_n \delta^{\frac{n-2}{2}} \beta$ , using (47) and (45) together with integration theory yields

$$(64) \quad \int_M B U^{2^*-1} dv_g = \delta^{n-2} k_n^2 \left( \int_M (|\nabla \beta_{h,0,\xi}|^2 + h\beta_{h,0,\xi}^2) dv_g \right. \\ \left. - \frac{1}{k_n} \int_M F_0(\xi, \cdot) \Gamma_\xi dv_g \right) - \int_M \hat{h}_\xi U^2 dv_g + o(\delta^{n-2}).$$

Taking the exponential chart at  $\xi$ , we write

$$\int_M \hat{h}_\xi U^2 dv_g = \hat{h}_\xi(\xi) \int_M U^2 dv_g + \partial_{\xi_i} \hat{h}_\xi(\xi) \int_M x^i U^2 dv_g \\ + \int_M (\hat{h}_\xi - \hat{h}_\xi(\xi) - \partial_{\xi_i} \hat{h}_\xi(\xi) x^i) U^2 dv_g.$$

As one checks, there exists  $C > 0$  such that

$$|\hat{h}_\xi - \hat{h}_\xi(\xi) - \partial_{\xi_i} \hat{h}_\xi(\xi) x^i| U^2 \leq C \delta^{n-2} d_g(\xi, x)^{6-2n}$$

for all  $x, \xi \in M$ ,  $x \neq \xi$ . Since  $n < 6$  and  $\xi$  remains in a neighborhood of  $\xi$  (so that the exponential chart remains nicely bounded), integration theory then yields

$$\int_M (\hat{h}_\xi - \hat{h}_\xi(\xi) - \partial_{\xi_i} \hat{h}_\xi(\xi) x^i) U^2 dv_g = \delta^{n-2} k_n^2 \int_M (\hat{h}_\xi - \hat{h}_\xi(\xi) - \partial_{\xi_i} \hat{h}_\xi(\xi) x^i) \Gamma_\xi^2 dv_g + o(\delta^{n-2}).$$

Furthermore, letting  $\xi \rightarrow \xi_0$ ,  $h \rightarrow h_0$  and using (4), we obtain

$$(65) \quad \int_M (\hat{h}_\xi - \hat{h}_\xi(\xi) - \partial_{\xi_i} \hat{h}_\xi(\xi) x^i) U^2 dv_g = \delta^{n-2} k_n^2 \int_M (\hat{h}_0)_{\xi_0} \Gamma_{\xi_0}^2 dv_g + o(\delta^{n-2}).$$

Via the exponential chart, using the radial symmetry of  $U$ , we obtain

$$\int_M x^i U^2 dv_g = O\left(\int_{B_{r_0}(0)} |x|^2 \left(\frac{\delta}{\delta^2 + |x|^2}\right)^{n-2} dx\right) = O(\delta^{n-2})$$

since  $n < 6$ . Using estimates on  $U_{\delta, \xi}$  together with the above estimates, we obtain

$$\int_M \hat{h}_\xi U^2 dv_g = \hat{h}_\xi(\xi) \begin{cases} 8\omega_{n-1} \delta^2 \ln(1/\delta) & \text{if } n = 4 \\ \delta^2 \int_{\mathbb{R}^n} U_{1,0}^2 dx & \text{if } n = 5 \end{cases} + \delta^{n-2} k_n^2 \int_M (\hat{h}_0)_{\xi_0} \Gamma_{\xi_0}^2 dv_g + o(\delta^{n-2}).$$

Combining this estimate with (64), we obtain

$$\int_M BU^{2^*-1} dv_g = -\hat{h}_\xi(\xi) \begin{cases} 8\omega_{n-1} \delta^2 \ln(1/\delta) & \text{if } n = 4 \\ \delta^2 \int_{\mathbb{R}^n} U_{1,0}^2 dx & \text{if } n = 5 \end{cases} + \delta^{n-2} k_n^2 I_{h_0, \xi_0} + o(\delta^{n-2}),$$

$$(66) \quad \text{where } I_{h_0, \xi_0} := \int_M (|\nabla \beta_{h_0, 0, \xi_0}|^2 + h_0 \beta_{h_0, 0, \xi_0}^2) dv_g - \frac{1}{k_n} \int_M F_0(\xi, \cdot) \Gamma_{\xi_0} dv_g - \int_M (\hat{h}_0)_{\xi_0} \Gamma_{\xi_0}^2 dv_g.$$

Integrating by parts and using the definition (46), we obtain

$$I_{h_0, \xi_0} = \int_M G_{h_0, \xi_0} (\Delta_g \beta_{h_0, 0, \xi_0} + h_0 \beta_{h_0, 0, \xi_0}) dv_g = \beta_{h_0, 0, \xi_0}(\xi_0) = m_{h_0}(\xi_0).$$

Putting these results together yields (7.4), which proves Proposition 7.4.  $\square$

With similar arguments as in the proof of Proposition 7.4, we obtain the estimates of the derivatives:

**Proposition 7.5.** *We have*

$$(67) \quad \partial_\delta J_h(W + \Phi) = \varphi_h(\xi) \begin{cases} 8\omega_{n-1} \delta \ln(1/\delta) & \text{if } n = 4 \\ \delta \int_{\mathbb{R}^n} U_{1,0}^2 dx & \text{if } n = 5 \end{cases} - \frac{n-2}{2} k_n^2 m_{h_0}(\xi_0) \delta^{n-3} + o(\delta^{n-3})$$

$$(68) \quad \text{and } \partial_{\xi_i} J_h(W + \Phi) = \frac{1}{2} \partial_{\xi_i} \varphi_h(\xi) \left\{ \begin{array}{ll} 8\omega_{n-1} \delta^2 \ln(1/\delta) & \text{if } n = 4 \\ \delta^2 \int_{\mathbb{R}^n} U_{1,0}^2 dx & \text{if } n = 5 \end{array} \right\} \\ + O \left( \begin{array}{ll} \delta^2 + D_{h,\xi} \delta^2 \ln(1/\delta) & \text{if } n = 4 \\ \delta^3 + D_{h,\xi} \delta^2 & \text{if } n = 5 \end{array} \right)$$

for all  $i = 1, \dots, n$ , as  $\delta \rightarrow 0$ ,  $\xi \rightarrow \xi_0$  and  $h \rightarrow h_0$  in  $C^2(M)$ .

Here again, we refer the reader to [26] for additional details.

## 8. PROOF OF THEOREM 1.4

We let  $h_0, f \in C^p(M)$ ,  $p \geq 2$ , and  $\xi_0 \in M$  satisfy the assumptions of Theorem 1.4. For small  $\epsilon > 0$  and  $\tau \in \mathbb{R}^n$ , we define

$$(69) \quad h_\epsilon := h_0 + \epsilon f \text{ and } \xi_\epsilon(\tau) := \exp_{\xi_0}^{g_{\xi_0}}(\sqrt{\epsilon}\tau).$$

We fix  $R > 0$  and  $0 < a < b$  to be chosen later.

**8.1. Proof of Theorem 1.4 for  $n \geq 6$ .** In this case, we let  $(\delta_\epsilon)_{\epsilon > 0} > 0$  be such that  $\delta_\epsilon \rightarrow 0$  as  $\epsilon \rightarrow 0$ . We define

$$(70) \quad \delta_\epsilon(t) := \delta_\epsilon t \text{ and } F_\epsilon(t, \tau) := J_{h_\epsilon}(U_{\delta_\epsilon(t), \xi_\epsilon(\tau)} + \Phi_{h_\epsilon, 0, \delta_\epsilon(t), \xi_\epsilon(\tau)})$$

for all  $(t, \tau) \in (a, b) \times \mathbb{R}^n$  such that  $|\tau| < R$ . With (4), we obtain

$$\varphi_{h_\epsilon}(\xi_\epsilon(\tau)) = \frac{1}{2} \nabla^2 \varphi_{h_0}(\xi_0)[\tau, \tau] \epsilon + f(\xi_0) \epsilon + o(\epsilon) \\ \text{and } \nabla \varphi_{h_\epsilon}(\xi_\epsilon(\tau)) = \nabla^2 \varphi_{h_0}(\xi_0)[\tau, \cdot] \sqrt{\epsilon} + o(\sqrt{\epsilon})$$

as  $\epsilon \rightarrow 0$  uniformly in  $|\tau| < R$ . We first assume that  $n \geq 7$ . In this case, we set  $\delta_\epsilon := \sqrt{\epsilon}$ . It follows from (30) that

$$(71) \quad \lim_{\epsilon \rightarrow 0} \frac{F_\epsilon(t, \tau) - \frac{1}{n} \int_{\mathbb{R}^n} U_{1,0}^{2^*} dx}{\epsilon^2} = E_0(t, \xi) \text{ in } C_{\text{loc}}^0((0, \infty) \times \mathbb{R}^n),$$

$$\text{where } E_0(t, \tau) := C_n \left( \frac{1}{2} \nabla^2 \varphi_{h_0}(\xi_0)[\tau, \tau] + f(\xi_0) \right) t^2 - D_n K_{h_0}(\xi_0) t^4,$$

for some constants  $C_n, D_n > 0$ . Furthermore, we have

$$\partial_t F_\epsilon(t, \tau) = \sqrt{\epsilon} (\partial_\delta J_{h_\epsilon}(U_{\delta_\epsilon(t), \xi_\epsilon(\tau)} + \Phi_{\delta_\epsilon(t), \xi_\epsilon(\tau)})) \\ \text{and } \partial_{\tau_i} F_\epsilon(t, \tau) = \sqrt{\epsilon} (\partial_{\xi_i} J_{h_\epsilon}(U_{\delta_\epsilon(t), \xi_\epsilon(\tau)} + \Phi_{\delta_\epsilon(t), \xi_\epsilon(\tau)})).$$

Therefore, it follows from (32) and (33) that the limit in (71) actually holds in  $C_{\text{loc}}^1((0, \infty) \times \mathbb{R}^n)$ . Assuming that  $f(\xi_0) \times K_{h_0}(\xi_0) > 0$ , we then obtain  $t_0 > 0$  such that  $(t_0, 0)$  is a nondegenerate critical point of  $E_0$ . Then, there exists a critical point  $(t_\epsilon, \tau_\epsilon)$  of  $F_\epsilon$  such that  $(t_\epsilon, \tau_\epsilon) \rightarrow (t_0, 0)$  as  $\epsilon \rightarrow 0$ . Then (21) yields

$$u_\epsilon := U_{\delta_\epsilon(t_\epsilon), \xi_\epsilon(\tau_\epsilon)} + \Phi_{h_\epsilon, 0, \delta_\epsilon(t_\epsilon), \xi_\epsilon(\tau_\epsilon)}$$

is a solution to (8) satisfying the conclusion of Theorem 1.4 when  $n \geq 7$ . In the case where  $n = 6$ , the proof is similar by choosing  $\delta_\epsilon$  such that  $\delta_\epsilon^2 \ln(1/\delta_\epsilon) = \epsilon$ .  $\square$



8.2. **Proof of Theorem 1.4 for  $n \in \{4, 5\}$ .** When  $n \in \{4, 5\}$ , we define

$$F_\epsilon(t, \tau) := J_{h_\epsilon}(U_{\delta_\epsilon(t), \xi_\epsilon(\tau)} + B_{h_\epsilon, \delta_\epsilon(t), \xi_\epsilon(\tau)} + \Phi_{h_\epsilon, 0, \delta_\epsilon(t), \xi_\epsilon(\tau)}),$$

where  $\delta_\epsilon(t)$  will be chosen differently depending on the dimension.

**Case  $n = 5$ .** In this case, we set  $\delta_\epsilon(t) := t\epsilon$ . It follows from (60) that

$$\lim_{\epsilon \rightarrow 0} \frac{F_\epsilon(t, \tau) - \frac{1}{n} \int_{\mathbb{R}^n} U_{1,0}^{2^*} dx}{\epsilon^3} = E_0(t, \xi) \text{ in } C_{\text{loc}}^0((0, \infty) \times \mathbb{R}^n),$$

$$\text{where } E_0(t, \tau) := C_5 \left( \frac{1}{2} \nabla^2 \varphi_{h_0}(\xi_0)(\tau, \tau) + f(\xi_0) \right) t^2 - \frac{k_n^2}{2} m_{h_0}(\xi_0) t^3.$$

It follows from the  $C^1$ -estimates of Proposition 7.5 that the convergence holds in  $C_{\text{loc}}^1((0, \infty) \times \mathbb{R}^n)$ . Assuming that  $f(\xi_0) \times m_{h_0}(\xi_0) > 0$ , we conclude as when  $n \geq 7$ .

**Case  $n = 4$ .** We set  $\delta_\epsilon(t) := e^{-t/\epsilon}$ . The  $C^1$ -estimates of Proposition 7.5 yield

$$\lim_{\epsilon \rightarrow 0} (-\epsilon \delta_\epsilon(t)^{-2} \partial_t F_\epsilon(t, \tau), \delta_\epsilon(t)^{-2} \partial_\tau F_\epsilon(t, \tau)) = (\psi_0(t, \tau), \psi_1(t, \tau))$$

in  $C_{\text{loc}}^0((0, \infty) \times \mathbb{R}^n)$ , where

$$\psi_0(t, \tau) := C_4 \left( \frac{1}{2} \nabla^2 \varphi_{h_0}(\xi_0)(\tau, \tau) + f(\xi_0) \right) t - \frac{n-2}{2} k_n^2 m_{h_0}(\xi_0)$$

$$\text{and } \psi_1(t, \tau) := \frac{1}{2} C_4 \nabla^2 \varphi_{h_0}(\xi_0)[\tau, \cdot] t.$$

Arguing as in Esposito–Pistoia–Vétois [10], we then obtain the existence of a critical point of  $J_{h_\epsilon}$  which satisfies the conclusion of Theorem 1.4.  $\square$

## 9. PROOF OF THEOREM 1.5

We let  $h_0, f \in C^p(M)$ ,  $p \geq 2$ ,  $u_0 \in C^2(M)$  and  $\xi_0 \in M$  satisfy the assumptions of Theorem 1.5. We let  $h_\epsilon$  be as in (8). We let  $\xi_\epsilon(\tau)$  and  $\delta_\epsilon(t)$  be as in (69) and (70). Since  $u_0$  is nondegenerate, there exists  $(u_{0,\epsilon})_{\epsilon>0} \in C^2(M)$  such that

$$(72) \quad \Delta_g u_{0,\epsilon} + h_\epsilon u_{0,\epsilon} = u_{0,\epsilon}^{2^*-1}, \quad u_{0,\epsilon} > 0 \text{ in } M \text{ for small } \epsilon > 0$$

and then  $\|u_{0,\epsilon} - u_0\|_{C^2} \leq C\epsilon$ . We let  $0 < a < b$ ,  $R > 0$  to be fixed later. We define

$$F_\epsilon(t, \tau) := J_{h_\epsilon}(u_{0,\epsilon} + U_{\delta_\epsilon(t), \xi_\epsilon(\tau)} + \Phi_{h_\epsilon, u_{0,\epsilon}, \delta_\epsilon(t), \xi_\epsilon(\tau)})$$

for all  $(t, \tau) \in (a, b) \times \mathbb{R}^n$  such that  $|\tau| < R$ . With (37), we obtain for  $n \geq 7$ ,

$$F_\epsilon(t, \tau) = J_{h_\epsilon}(u_{0,\epsilon}) + \frac{1}{n} \int_{\mathbb{R}^n} U_{1,0}^{2^*} dx + C_n \left( \frac{1}{2} \nabla^2 \varphi_{h_0}(\xi_0)(\tau, \tau) + f(\xi_0) \right) t^2 \epsilon \delta_\epsilon^2$$

$$+ o(\epsilon \delta_\epsilon^2) - D_n K_{h_0}(\xi_0) t^4 \delta_\epsilon^4 + o(\delta_\epsilon^4) - B_n u_0(\xi_0) t^{\frac{n-2}{2}} \delta_\epsilon^{\frac{n-2}{2}} + o(\delta_\epsilon^{\frac{n-2}{2}})$$

as  $\epsilon \rightarrow 0$  uniformly in  $a < t < b$  and  $|\tau| < R$ , where  $B_n, C_n, D_n > 0$ . We set  $\delta_\epsilon := \epsilon^{\frac{2}{n-6}}$  if  $7 \leq n \leq 10$  and  $\delta_\epsilon := \sqrt{\epsilon}$  if  $n \geq 11$  and then argue as in the proof of Theorem 1.4. In the case where  $n = 6$ , remarking that  $2^* - 1 = 2$ , differentiating (72) in  $\epsilon$  and using the nondegeneracy of  $u_0$ , we obtain

$$(\partial_\epsilon u_{0,\epsilon})|_0 = -(\Delta_g + h_0 - 2u_0)^{-1}(f u_0).$$

It follows that  $\varphi_{h_\epsilon, u_\epsilon} = h_\epsilon - 2u_{0,\epsilon} - c_n \text{Scal}_g = \varphi_{h_0, u_0} + \tilde{f}\epsilon + o(\epsilon)$  as  $\epsilon \rightarrow 0$ , where  $\tilde{f} := f + 2(\Delta_g + h_0 - 2u_0)^{-1}(f u_0)$ . We then conclude as in the previous cases.  $\square$

## 10. PROOFS OF THEOREMS 1.2 AND 1.3

**10.1. Proof of Theorem 1.2.** We let  $h_0 \in C^p(M)$ ,  $1 \leq p \leq \infty$ , and  $\xi_0 \in M$  be such that  $\Delta_g + h_0$  is coercive and (4) is satisfied. We first easily construct a suitable approximation  $(\tilde{h}_\epsilon)_{\epsilon>0} \in C^{\max\{2,p\}}(M)$  such that  $\tilde{h}_\epsilon \rightarrow h_0$  in  $C^p(M)$  as  $\epsilon \rightarrow 0$ ,  $\varphi_{\tilde{h}_\epsilon}(\xi_0) = \varphi_{h_0}(\xi_0) = 0$ ,  $|\nabla \varphi_{\tilde{h}_\epsilon}(\xi_0)| = |\nabla \varphi_{h_0}(\xi_0)| = 0$  and for small  $\epsilon > 0$ ,  $\xi_0$  is a nondegenerate critical point of  $\varphi_{\tilde{h}_\epsilon}$ . Using Proposition 6.1, we can assume moreover that for small  $\epsilon > 0$ ,  $K_{\tilde{h}_\epsilon}(\xi_0) \neq 0$  and the sign of  $K_{\tilde{h}_\epsilon}(\xi_0)$  is independent of  $\epsilon$ . We fix  $f_0 \in C^\infty(M)$  such that  $f_0(\xi_0) \times K_{\tilde{h}_\epsilon}(\xi_0) > 0$  for small  $\epsilon > 0$ . It then follows from Theorem 1.4 that there exist  $\alpha_\epsilon > 0$  and  $(\tilde{u}_{\epsilon,\alpha})_{0<\alpha<\alpha_\epsilon}$  such that

$$\Delta_g \tilde{u}_{\epsilon,\alpha} + (\tilde{h}_\epsilon + \alpha f_0) \tilde{u}_{\epsilon,\alpha} = \tilde{u}_{\epsilon,\alpha}^{2^*-1}, \quad \tilde{u}_{\epsilon,\alpha} > 0 \text{ in } M,$$

$\tilde{u}_{\epsilon,\alpha} \rightharpoonup 0$  weakly in  $L^{2^*}(M)$  and  $(\tilde{u}_{\epsilon,\alpha})_\alpha$  blows up with one bubble at  $\xi_0$  as  $\alpha \rightarrow 0$ . A diagonal argument then yields  $(h_\epsilon)_\epsilon$  and  $(u_\epsilon)_\epsilon$  such that Theorem 1.2 holds.  $\square$

**10.2. Proof of Theorem 1.3.** We let  $h_0 \in C^p(M)$ ,  $1 \leq p \leq \infty$ , such that  $\Delta_g + h_0$  is coercive,  $u_0 \in C^2(M)$ ,  $u_0 > 0$  and  $\xi_0 \in M$  such that (1) and (5) are satisfied.

**Lemma 10.1.** *There exists a neighborhood  $\Omega_0$  of  $\xi_0$  and families  $(\tilde{h}_\epsilon)_{\epsilon>0} \in C^p(M)$  and  $(\tilde{u}_\epsilon)_{\epsilon>0} \in C^2(M)$  such that  $\tilde{h}_\epsilon \rightarrow h_0$  in  $C^p(M)$ ,  $\tilde{u}_\epsilon \rightarrow u_0$  in  $C^2(M)$ ,  $\tilde{h}_\epsilon \equiv h_0$  and  $\tilde{u}_\epsilon \equiv u_0$  in  $\Omega_0$  and  $\tilde{u}_\epsilon$  is a nondegenerate solution of*

$$\Delta_g \tilde{u}_\epsilon + \tilde{h}_\epsilon \tilde{u}_\epsilon = \tilde{u}_\epsilon^{2^*-1}, \quad \tilde{u}_\epsilon > 0 \text{ in } M \text{ for all } k \in \mathbb{N}.$$

*Proof of Lemma 10.1.* For all  $v \in C^{p+2}(M)$  such that  $v > -u_0$ , we define  $u(v) := u_0 + v$  and  $h(v)$  such that

$$\Delta_g u(v) + h(v)u(v) = u(v)^{2^*-1} \text{ in } M.$$

By elliptic regularity, we have  $h(v) \rightarrow h_0$  in  $C^p(M)$  and  $u(v) \rightarrow u_0$  in  $C^2(M)$  as  $v \rightarrow 0$  in  $C^{p+2}(M)$ . We assume by contradiction that for every neighborhood  $\Omega$  of  $\xi_0$ , there exists a neighborhood  $V_\Omega$  of 0 in  $C^{p+2}(M)$  such that for every  $v \in V_\Omega$ , if  $v \equiv 0$  in  $\Omega$ , then  $u(v)$  is degenerate i.e. there exists  $\phi(v) \in K_v \setminus \{0\}$ , where

$$K_v := \{\phi \in H_1^2(M) : \Delta_g \phi + h(v)\phi = (2^* - 1)u(v)^{2^*-2}\phi \text{ in } M\}.$$

We can assume that  $\phi(v) \in \mathbb{S}_{K_v} := \{\phi \in K_v : \|\phi\|_{H_1^2} = 1\}$ . Then there exists  $\phi_v \in K_0$  and  $(t_k)_{k \in \mathbb{N}} > 0$  such that  $t_k \rightarrow 0$  and  $\phi(t_k v) \rightarrow \phi_v$  strongly in  $C^1(M)$ , so  $\phi_v \in \mathbb{S}_{K_0}$ . We then define  $\psi_k(v) := t_k^{-1}(\phi(t_k v) - \phi_v)$ . As one checks,

$$(73) \quad \Delta_g \psi_k(v) + h_0 \psi_k(v) = (2^* - 1)u_0^{2^*-2} \psi_k(v) + f_k(v)\phi(t_k v) \text{ in } M$$

for a suitable sequence  $(f_k)_k$  satisfying  $f_k(v) = u_0^{-1}L_0(v) + o(1)$  as  $k \rightarrow +\infty$ , where

$$L_0(v) := \Delta_g v + h_0 v - (1 - (2^* - 2)^2)u_0^{2^*-2}v.$$

We then obtain that there exists  $\psi_v \in K_0^\perp$  such that  $\Pi_{K_0^\perp}(\psi_k(v)) \rightharpoonup \psi_v$  weakly in  $H_1^2(M)$ , where  $\Pi_{K_0^\perp}$  is the projection onto  $K_0^\perp$ . Passing to the limit in (73) yields

$$\Delta_g \psi_v + h_0 \psi_v = (2^* - 1)u_0^{2^*-2} \psi_v + u_0^{-1}L_0(v)\phi_v \text{ in } M.$$

Since  $\phi_v \in K_0$ , multiplying this equation by  $\phi_v$  and integrating by parts yields

$$(74) \quad \int_M u_0^{-1}L_0(v)\phi_v^2 dv_g = 0.$$

We now construct  $v$  contradicting (74). For every  $\epsilon > 0$ , we choose  $\Omega := B_\epsilon(\xi_0)$  and we consider the neighborhood  $V_{B_\epsilon(\xi_0)}$  of 0 in  $C^{p+2}(M)$  and  $\chi \in C^\infty(\mathbb{R})$  be such that  $\chi(t) = 0$  for  $t \leq 1$  and  $\chi(t) = 1$  for  $t \geq 2$ . We define

$$v_\epsilon(x) := C_\epsilon \chi(d_g(x, \xi_0)/\epsilon) u_0(x) \text{ for all } x \in M \text{ and } \epsilon > 0.$$

For  $C_\epsilon \rightarrow 0$  suitably chosen,  $v_\epsilon \equiv 0$  in  $B_\epsilon(\xi_0)$  and  $v_\epsilon \in V_{B_\epsilon(\xi_0)}$ , so the above analysis applies. Up to a subsequence, we then obtain the existence of  $\phi_0 \in K_0$  such that

$$\lim_{\epsilon \rightarrow 0} \phi_{v_\epsilon} = \phi_0 \neq 0 \text{ in } C^2(M).$$

Applying (74) to  $\phi_{v_\epsilon}$ , integrating by parts and passing to the limit then yields a contradiction since  $C_\epsilon^{-1} v_\epsilon \rightarrow u_0$  in  $L^2(M)$ . This ends the proof of Lemma 10.1.  $\square$

We can now end the proof of Theorem 1.3. Letting  $\Omega_0, (\tilde{h}_\epsilon)_{\epsilon>0}$  and  $(\tilde{u}_\epsilon)_{\epsilon>0}$  be given by Lemma 10.1, we have  $\varphi_{\tilde{h}_\epsilon, \tilde{u}_\epsilon} \equiv \varphi_{h_0, u_0}$  in  $\Omega_0$  and so  $\varphi_{\tilde{h}_\epsilon, \tilde{u}_\epsilon}(\xi_0) = |\nabla \varphi_{\tilde{h}_\epsilon, \tilde{u}_\epsilon}(\xi_0)| = 0$ . Theorem 1.3 then follows by mimicking the proof of Theorem 1.2.

## 11. EXAMPLES OF $h_0$ AND $u_0$ SATISFYING THE ASSUMPTIONS OF THEOREM 1.3.

**Proposition 11.1.** *Let  $(M, g)$  be a compact Riemannian manifold of dimension  $n \geq 3$ . Then there exists  $\epsilon_0 \geq 0$  depending only on  $n$  and  $(M, g)$  such that  $\epsilon_0 > 0$  if  $(M, g)$  is not conformally diffeomorphic to the standard sphere,  $\epsilon_0 = 0$  otherwise, and for every  $\varphi_0 \in C^p(M)$ ,  $1 \leq p \leq \infty$ , satisfying  $\varphi_0(\xi_0) = |\nabla \varphi_0(\xi_0)| = 0$  and*

$$\begin{cases} \varphi_0 \leq \epsilon_0 \text{ and } \lambda_1(\Delta_g + \varphi_0 + c_n \text{Scal}_g) > 0 & \text{if } n \neq 6 \\ \lambda_1(\Delta_g + \varphi_0 + c_n \text{Scal}_g) < 0 & \text{if } n = 6, \end{cases}$$

there exists a solution  $u_0 \in C^2(M)$  of the equation (1) which satisfies (5) with  $h_0 := \varphi_0 + c_n \text{Scal}_g$  if  $n \neq 6$  and  $h_0 := \varphi_0 + c_n \text{Scal}_g + 2u_0$  if  $n = 6$ .

*Proof of Proposition 11.1:* Since  $(M, g)$  is aspherical, its Yamabe quotient is below the quotient of the round sphere. This property persists when adding a small perturbation  $\varphi_0$ . It is standard that this property yields the existence of a solution to the problem, which proves the proposition when  $n \neq 6$ . When  $n = 6$ , since  $2^* - 1 = 2$ , we can rewrite the equation (1) as  $\Delta_g u + (h_0 - 2u)u = -u^2$ . A classical variational method then yields the existence of a solution to (1). This ends the proof of Proposition 11.1.  $\square$

## 12. NECESSITY OF THE CONDITION ON THE GRADIENT

**Theorem 12.1.** *Let  $(M, g)$  be a compact Riemannian manifold of dimension  $n \geq 4$ . Let  $h_0 \in C^1(M)$  be such that  $\Delta_g + h_0$  is coercive. Assume that there exist  $(h_\epsilon)_{\epsilon>0} \in C^p(M)$ ,  $(u_\epsilon)_{\epsilon>0} \in C^2(M)$  satisfying (2) and such that  $h_\epsilon \rightarrow h_0$  strongly in  $C^1(M)$ . Assume that  $(M, g)$  is locally conformally flat. If  $(u_\epsilon)_\epsilon$  blows up with one bubble at some point  $\xi_0 \in M$  and  $u_\epsilon \rightarrow 0$  weakly as  $\epsilon \rightarrow 0$ , then (4) holds true.*

*Proof of Theorem 12.1.* Theorem 1.1 yields  $\varphi_{h_0}(\xi_0) = 0$ . With the conformal flatness, we can assume that  $\xi_0 \in \mathbb{R}^n$  and there exists  $(\hat{u}_\epsilon)_\epsilon \in C^2(B_2(\xi_0))$  such that

$$\Delta_{\text{Eucl}} \hat{u}_\epsilon + \hat{h}_\epsilon \hat{u}_\epsilon = \hat{u}_\epsilon^{2^*-1}, \quad \hat{u}_\epsilon > 0 \text{ in } B_2(\xi_0) \subset \mathbb{R}^n, \quad \hat{h}_\epsilon := (h_\epsilon - c_n \text{Scal}_g) \Lambda^{2-2^*}$$

for some function  $\Lambda > 0$ . It follows from [9] that for some constant  $C > 0$  we have

$$(75) \quad \frac{1}{C} \left( \frac{\delta_\epsilon}{\delta_\epsilon^2 + |x - \xi_\epsilon|^2} \right)^{\frac{n-2}{2}} \leq \hat{u}_\epsilon(x) \leq C \left( \frac{\delta_\epsilon}{\delta_\epsilon^2 + |x - \xi_\epsilon|^2} \right)^{\frac{n-2}{2}} \text{ for all } x \in M$$

for small  $\epsilon > 0$  and  $\xi_\epsilon \rightarrow \xi_0$ . Differentiating the Pohozaev identity on  $B_1(\xi_\epsilon)$  yields

$$\begin{aligned} & \frac{1}{2} \int_{B_1(\xi_\epsilon)} \partial_{x_i} \hat{h}_\epsilon \hat{u}_\epsilon^2 dx \\ &= \int_{\partial B_1(\xi_\epsilon)} \left( \frac{(x - \xi_\epsilon)_i}{|x - \xi_\epsilon|} \left( \frac{|\nabla \hat{u}_\epsilon|^2 + \hat{h}_\epsilon \hat{u}_\epsilon^2}{2} - \frac{\hat{u}_\epsilon^{2^*}}{2^*} \right) - \left\langle \frac{x - \xi_\epsilon}{|x - \xi_\epsilon|}, \nabla \hat{u}_\epsilon \right\rangle \partial_{x_i} \hat{u}_\epsilon \right) d\sigma(x). \end{aligned}$$

By standard elliptic theory and (75), we then obtain

$$(76) \quad \int_{B_1(\xi_\epsilon)} \partial_{x_i} \hat{h}_\epsilon \hat{u}_\epsilon^2 dx = O(\delta_\epsilon^{n-2}) \text{ as } \epsilon \rightarrow 0.$$

Estimating the left-hand side of (76) with (75) then gives  $\nabla \varphi_{h_0}(\xi_0) = 0$ . This ends the proof of Theorem 12.1.  $\square$

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FRÉDÉRIC ROBERT, UNIVERSITÉ DE LORRAINE, CNRS, IECL, F-54000 NANCY, FRANCE  
*Email address:* frederic.robert@univ-lorraine.fr

JÉRÔME VÉTOIS, MCGILL UNIVERSITY, DEPARTMENT OF MATHEMATICS AND STATISTICS, 805  
SHERBROOKE STREET WEST, MONTREAL, QUEBEC H3A 0B9, CANADA  
*Email address:* jerome.vetois@mcgill.ca