EXISTENCE RESULTS FOR THE HIGHER-ORDER Q-CURVATURE EQUATION

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Abstract. We obtain existence results for the Q-curvature equation of arbitrary order $2k$ on a closed Riemannian manifold of dimension $n \geq 2k+1$, where $k \geq 1$ is an integer. We obtain these results under the assumptions that the operator is coercive and its Green's function is positive, which are satisfied for instance when the manifold is Einstein. In the case where $2k+1 \leq n \leq 2k+3$ or (M, g) is locally conformally flat, we assume moreover that the operator has positive mass. In the case where $n \geq 2k+4$ and (M, g) is not locally conformally flat, the results essentially reduce to the determination of the sign of a complicated constant depending only on n and k.

1. Introduction and main results

Given an integer $k \geq 1$, a smooth, closed Riemannian manifold (M, g) of dimension $n > 2k$ and a smooth positive function f in M, we consider the equation

$$P_{2k} u = f \left|u\right|^{2_k^* - 2} u \quad \text{in M}, \tag{1.1}$$

where P_{2k} is the GJMS operator with leading part Δ^k, $\Delta := \delta d$ is the Laplace–Beltrami operator with nonnegative eigenvalues and $2_k^* := 2n/\left(n-2k\right)$ is the critical Sobolev exponent. The so-called GJMS operators were discovered by Graham, Jenne, Mason and Sparling [18] by using a construction based on the Fefferman–Graham ambient metric [14, 15]. They provide a natural extension to higher orders of the Yamabe operator [42] ($k = 1$) and the Paneitz–Branson operator [4, 32] ($k = 2$). When u is positive, (1.1) arises in the problem of prescribing Branson's Q-curvature of order $2k$ in a given conformal class (see Branson [5]). More precisely, the positive solutions u to the equation (1.1) correspond to the conformal metrics $u^{4/(n-2k)} g$ with Q-curvature of order $2k$ equal to $\frac{2}{n-2k} f$.

Throughout this paper, we assume that the operator P_{2k} is *coercive* in the sense that there exists a constant $C > 0$ such that

$$\int_M u P_{2k} u \, dv_g \geq C \int_M u^2 dv_g$$

for all functions $u \in C^{2k}\left(M\right)$, where dv_g is the volume element with respect to g.

The existence of at least one positive solution to the equation (1.1) with $f \equiv 1$ has been completely solved in the case where $k = 1$ (see the historic work of

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Aubin [2], Schoen [36], Trudinger [41] and Yamabe [42]) and it has been solved to a large extent in the case where $k = 2$ (see Djadli, Hebey and Ledoux [10], Esposito and Robert [13], Gursky, Hang and Lin [20], Gursky and Malchiodi [21] Hang and Yang [22, 23] and Robert [34]). For $k = 3$, Chen and Hou [9] obtained the existence of at least one solution of (1.1) in the case of non-locally conformally flat manifolds of dimension $n \geq 10$. The solution obtained by Chen and Hou [9] is positive under the assumption that the Green's function of the operator P_6 is positive (see Mazumdar [30, Theorem 3]). This question has also been solved by Qing and Raske [33] in the locally conformally flat case for all orders $k \geq 2$, under a topological assumption on the Poincaré exponent of the holonomy representation of the fundamental group, using an approach introduced by Schoen [37] for $k = 1$. More general existence results have also been obtained in the case where $f \not\equiv 1$ (see among others Aubin [2], Escobar and Schoen [12], Hebey [24] and Hebey and Vaugon [25] for $k = 1$, Djadli, Hebey and Ledoux [10], Esposito and Robert [13] and Robert [34] for $k = 2$, Chen and Hou [9] for $k = 3$ and Robert [35] for higher orders).

We let Ric and W be the Ricci and Weyl curvature tensors of (M, g), $|\mathrm{W}|$ be the norm of W with respect to g and $(\cdot, \cdot)$ be the multiple inner product defined as $(S, T) = g^{i_1 j_1} \cdots g^{i_l j_l} S_{i_1 \dots i_l} T_{j_1 \dots j_l}$ for all covariant tensors S and T of rank $l \geq 1$. In the case where $2k+1 \leq n \leq 2k+3$ or (M, g) is locally conformally flat, assuming that the operator P_{2k} is coercive, for every point $\xi \in M$, we let $m(\xi)$ be the mass of P_{2k} at ξ (see (3.2) for the definition of the mass). Our main result is the following:

Theorem 1.1. *Let $k \geq 1$ be an integer, (M, g) be a smooth, closed Riemannian manifold of dimension $n \geq 2k+1$ and f be a smooth positive function in M. Assume that the operator P_{2k} is coercive and there exists a maximal point ξ of f such that*

$$\Delta f(\xi) = 0 \quad \text{if } n \geq 2k+2 \tag{1.2}$$

and

$$\begin{cases} |\mathrm{W}(\xi)|^2 f(\xi) + c(n,k)\, \Delta^2 f(\xi) > 0 & \text{if } n \geq 2k+5 \\ \mathrm{W}(\xi) \neq 0 & \text{if } n = 2k+4 \\ m(\xi) > 0 & \text{if } 2k+1 \leq n \leq 2k+3, \end{cases} \tag{1.3}$$

where $c(n,k)$ is a positive constant depending only on n and k (see (2.68) for the value of $c(n,k)$). Then there exists a nontrivial solution $u \in C^{2k}(M)$ to the equation (1.1), which minimizes the energy functional (2.1). If moreover the Green's function of the operator P_{2k} is positive, then u is positive, which implies that the Q-curvature of order $2k$ of the metric $u^{4/(n-2k)} g$ is equal to $\frac{2}{n-2k} f$.

In the case where f is constant, we obtain the following:

Theorem 1.2. *Let $k \geq 1$ be an integer and (M, g) be a smooth, closed Riemannian manifold of dimension $n \geq 2k+1$. Assume that the operator P_{2k} is coercive and its Green's function is positive. Assume moreover that if $2k+1 \leq n \leq 2k+3$ or (M, g) is locally conformally flat, then $m(\xi) > 0$ for some point $\xi \in M$. Then there exists a conformal metric to g with constant Q-curvature of order $2k$.*

Remark that Theorem 1.2 is a direct consequence of Theorem 1.1 in the case where (M, g) is not locally conformally flat of dimension $n \geq 2k+4$. A more general result about the locally conformally flat case will be stated in Section 3.

When (M, g) is Einstein, Fefferman and Graham [15, Proposition 7.9] (see also Gover [17] for a proof based on tractors) established the formula

$$P_{2k} = \prod_{j=1}^{k} \left(\Delta + \frac{(n+2j-2)(n-2j)}{4n(n-1)} \mathrm{S} \right),$$

where S is the Scalar curvature of (M, g). In this case, it is easy to see that if S is positive, then P_{2k} is coercive. Furthermore, successive applications of the maximum principles yield that the Green's function of the operator P_{2k} is positive. Therefore, we obtain the following corollary of Theorem 1.1:

Corollary 1.1. *Let $k \geq 1$ be an integer and (M, g) be a smooth, closed Einstein manifold of positive scalar curvature and dimension $n \geq 2k+1$. Let f be a smooth positive function in M such that there exists a maximal point ξ of f satisfying (1.2) and (1.3). Then there exists a conformal metric to g with Q-curvature of order $2k$ equal to $\frac{2}{n-2k} f$.*

The positivity of the Green's function of the operator P_4 has been shown to be true by Gursky and Malchiodi [21] and Hang and Yang [22, 23] under some positivity assumptions on the Q-curvature of order 4 and the scalar curvature or the Yamabe invariant of the manifold. Positivity results for the mass of P_4 have also been obtained by Gursky and Malchiodi [21], Hang and Yang [22], Humbert and Raulot [26] and Michel [31], thus extending the positive mass theorem obtained by Schoen and Yau [38–40] for $k = 1$. As far as the authors know, no such results have yet been obtained for higher orders. As regards the case where $n = 2k$, we point out that the problem of prescribing the Q-curvature involves a different equation than (1.1), which contains an exponential non-linearity. Possible references in this case are Chang and Yang [8], Djadli and Malchiodi [11] and Li, Li and Liu [29] for $k = 2$ and Baird, Fardoun and Regbaoui [3] for higher orders.

The proofs of Theorems 1.1 and 1.2 are based on the approach introduced by Aubin [2] and Schoen [36] in the case where $k = 1$. This approach consists in deriving an asymptotic expansion for the energy functional associated with the equation (1.1), which we apply to a suitable family of test functions depending on a real parameter (see (2.1) for the energy functional; see (2.5) and (3.4) for the definitions of our families of test functions. To simplify the calculations of curvature terms, we use the conformal normal coordinates introduced by Lee and Parker [28] and later improved by Cao [7] and Günther [19]. Our proof also crucially relies on the derivation of an expression for the highest-order terms of the GJMS operators (see (2.6)), which we obtain by using Juhl's formulae [27]. In the case where $n \geq 2k+4$, the proof essentially reduces to determine the sign of a constant $C(n, k)$ depending only on n and k, which appears in the energy expansion (see (2.6)). In particular, we recover the values found in [9, 13] for $C(n, k)$ with $k \in \{2, 3\}$. We then conclude the proof by using a general minimization result in the spirit of Aubin [2] (see Mazumdar [30, Theorem 3]). We point out that at one place in the proof, namely in the very last computation to determine the sign of $C(n, k)$ (see (2.66)), we have used the computation software *Maple* to expand a complicated polynomial with integer coefficients.

The paper is organized as follows. In Section 2, we prove Theorem 1.1 in the case where $n \geq 2k+4$. In Section 3, we complete the proof of Theorems 1.1 in the remaining case where $2k+1 \leq n \leq 2k+3$ and we state and prove a more general result in the case where g is conformally flat in some open subset of the manifold. Theorem 1.2 then directly follows from this new result together with Theorem 1.1.

2. Proof of Theorem 1.1 in the case where $n \geq 2k+4$

Given an integer $k \geq 1$ and a smooth positive function f in M, we let $I_{k,f}$ be the energy functional defined as

$$I_{k,f,g}(u) := \frac{\displaystyle\int_M u P_{2k} u \, dv_g}{\left(\displaystyle\int_M f |u|^{2^*_k} dv_g\right)^{\frac{n-2k}{n}}} \tag{2.1}$$

for all functions $u \in C^{2k}(M)$ such that $u \not\equiv 0$. We fix a point $\xi \in M$. By applying a conformal change of metric (see Cao [7], Günther [19] and Lee and Parker [28]), we may assume that

$$\det g(x) = 1 \quad \forall x \in \Omega \tag{2.2}$$

for some neighborhood Ω of the point ξ, where $\det g$ is the determinant of g in geodesic normal coordinates at ξ. In particular (see [28]), it follows from (2.2) that

$$\begin{aligned} &\mathrm{Ric}(\xi) = 0, \quad \mathrm{Sym}\,\nabla\,\mathrm{Ric}(\xi) = 0, \\ &\text{and} \quad \mathrm{Sym}\left(\mathrm{Ric}_{ab;cd}(\xi) + \frac{2}{9}\sum_{e,f=1}^{n} \mathrm{W}_{eabf}(\xi)\,\mathrm{W}_{ecdf}(\xi)\right) = 0 \end{aligned} \tag{2.3}$$

in normal coordinates at the point ξ, where Sym stands for the symmetric part and $\mathrm{Ric}_{ab;cd}$ and W_{eabf} are the coordinates of $\nabla^2\,\mathrm{Ric}$ and W, respectively. By taking traces in (2.3) and using Bianchi's identities, we obtain

$$\mathrm{S}(\xi) = 0, \quad \nabla\,\mathrm{S}(\xi) = 0 \quad \text{and} \quad \Delta\,\mathrm{S}(\xi) = -2\sum_{a,b=1}^{n} \mathrm{Ric}_{ab;ab}(\xi) = \frac{1}{6}\left|\mathrm{W}(\xi)\right|^2. \tag{2.4}$$

Let $r_0 > 0$ be such that the injectivity radius of the metric g at the point ξ is greater than $3r_0$ and $B(\xi, 3r_0) \subset \Omega$, where $B(\xi, r_0)$ is the ball of center ξ and radius $3r_0$ with respect to g. We then let χ be a smooth cutoff function in $[0, \infty)$ such that $\chi \equiv 1$ in $[0, r_0]$, $0 \leq \chi \leq 1$ in $(r_0, 2r_0)$ and $\chi \equiv 0$ in $[2r_0, \infty)$. For every $\mu > 0$, we then define our test functions as

$$U_\mu(x) := \chi(d_g(x, \xi))\,\mu^{\frac{2k-n}{2}} U\left(\mu^{-1} \exp_\xi^{-1} x\right) \quad \forall x \in M, \tag{2.5}$$

where d_g is the geodesic distance with respect to g, $\exp_\xi$ is the exponential map with respect to g at the point ξ and U is the function in $\mathbb{R}^n$ (we identify $T_\xi M$ with $\mathbb{R}^n$) defined as

$$U(x) = \left(1 + |x|^2\right)^{-\frac{n-2k}{2}} \quad \forall x \in \mathbb{R}^n.$$

It is easy to verify that U is a solution of the equation

$$\Delta_0^k U = \left[\prod_{j=-k}^{k-1} (n+2j)\right] U^{2^*_k - 1} \quad \text{in } \mathbb{R}^n,$$

where Δ_0 is the Euclidean Laplacian.

Proposition 2.1. *Let $k \geq 1$ be an integer, (M,g) be a smooth, closed Riemannian manifold of dimension $n \geq 2k+4$ and f be a smooth positive function in M. Assume that g satisfies (2.2) for some point $\xi \in M$. Let $I_{k,f,g}$ be as in (2.1) and U_μ be as in (2.5). Then there exists a positive constant $C(n,k)$ depending only on n and k (see (2.65) for the value of $C(n,k)$) such that as $\mu \to 0$,*

$$\begin{aligned} I_{k,f,g}(U_\mu) = \omega_n^{\frac{2k}{n}} f(\xi)^{-\frac{n-2k}{n}} \Bigg(& (2k-1)!\,\mathrm{B}\left(\frac{n}{2}-k,2k\right)^{-1} \\ & + \frac{(n-2k)(2k-1)!}{2n(n-2)}\,\mathrm{B}\left(\frac{n}{2}-k,2k\right)^{-1} \frac{\Delta f(\xi)}{f(\xi)}\mu^2 \\ - \frac{(n-2k)(2k-1)!}{4n(n-2)}\,\mathrm{B}\left(\frac{n}{2}-k,2k\right)^{-1} & \left(\frac{\Delta^2 f(\xi)}{2(n-4)f(\xi)} - \frac{(n-k)(\Delta f(\xi))^2}{n(n-2)f(\xi)^2}\right)\mu^4 \\ & - C(n,k)\,\mu^4 \left\{ \begin{array}{ll} |\mathrm{W}(\xi)|^2 \ln(1/\mu) + \mathrm{O}(1) & \text{if } n = 2k+4 \\ |\mathrm{W}(\xi)|^2 + \mathrm{o}(1) & \text{if } n > 2k+4. \end{array} \right\} \Bigg), \end{aligned} \tag{2.6}$$

where ω_n is the volume of the standard n-dimensional sphere and B is the beta function defined as

$$\mathrm{B}(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} \quad \forall a,b>0.$$

Proof of Proposition 2.1. We let P be the Schouten tensor defined as

$$\mathrm{P} := \frac{1}{n-2}\left(\mathrm{Ric} - \frac{\mathrm{S}}{2(n-1)}\,g\right)$$

and B be the Bach tensor whose coordinates are given by

$$\mathrm{B}_{ij} := g^{ab} g^{cd}\,\mathrm{P}_{ac}\,\mathrm{W}_{ibjd} + g^{ab}\left(\mathrm{P}_{ij;ab} - \mathrm{P}_{ia;jb}\right)$$

in Einstein's summation notation, where g^{ab}, P_{ac}, $\mathrm{P}_{ij;ab}$ and W_{ibjd} are the coordinates of g^{-1}, P, $\nabla^2\,\mathrm{P}$ and W, respectively. The first step in the proof of Proposition 2.1 is as follows:

Step 2.1. *For every $k \in \mathbb{N}$ such that $n \geq 2k+1$, we have*

$$\begin{aligned} P_{2k} = \Delta^k + k\Delta^{k-1}(f_1 \cdot) + k(k-1)\Delta^{k-2}\left(f_2 \cdot + (T_1,\nabla) + \left(T_2,\nabla^2\right)\right) & \\ + k(k-1)(k-2)\Delta^{k-3}\left(\left(T_3,\nabla^2\right) + \left(T_4,\nabla^3\right)\right) & \\ + k(k-1)(k-2)(k-3)\Delta^{k-4}\left(T_5,\nabla^4\right) + Z, & \end{aligned} \tag{2.7}$$

where Z is a smooth linear operator of order less than $2k-4$ if $k \geq 3$, $Z := 0$ if $k \leq 2$, the functions f_1 and f_2 are defined as

$$f_1 := \frac{n-2}{4(n-1)}\,\mathrm{S} \quad \text{and} \quad f_2 := \frac{1}{6}\left(\frac{3n^2-12n-4k+8}{16(n-1)^2}\,\mathrm{S}^2 - (k+1)(n-4)|\mathrm{P}|^2 - \frac{3n+2k-4}{4(n-1)}\Delta\,\mathrm{S}\right)$$

and the tensors T_1, T_2, T_3, T_4 and T_5 are defined as

$$T_1 := \frac{n-2}{4(n-1)}\nabla\,\mathrm{S} - \frac{2}{3}(k+1)\,\delta\,\mathrm{P}, \quad T_2 := \frac{2}{3}(k+1)\,\mathrm{P},$$

$$
\begin{aligned}
T_3 &:= \frac{n-2}{6(n-1)}\nabla^2 \mathrm{S} + \frac{(k+1)(n-2)}{6(n-1)}\mathrm{S}\,\mathrm{P} - \frac{k+1}{3}\left(\delta\nabla\mathrm{P} + 2\nabla\delta\mathrm{P} + 2\,\mathrm{R}*\mathrm{P}\right)\\
&\quad - \frac{2}{15}(k+1)(k+2)\left(3\,\mathrm{P}^{\#}\,\mathrm{P} + \frac{\mathrm{B}}{n-4}\right),\\
T_4 &:= \frac{2}{3}(k+1)\nabla\mathrm{P} \quad \textit{and} \quad T_5 := \frac{2}{5}(k+1)\left(\frac{5k+7}{9}\mathrm{P}\otimes\mathrm{P} + \nabla^2\mathrm{P}\right),
\end{aligned}
$$

where $\#$ stands for the musical isomorphism with respect to g (i.e. $\mathrm{P}^{\#} := g^{-1}\,\mathrm{P}$) and $\mathrm{R}*\mathrm{P}$ stands for the covariant tensor whose coordinates are given by

$$
(\mathrm{R}*\mathrm{P})_{ij} := g^{ab}g^{cd}\left(\mathrm{R}_{iabc}\,\mathrm{P}_{jd} + \mathrm{R}_{icja}\,\mathrm{P}_{bd}\right), \tag{2.8}
$$

where g^{ab}, R_{iacb} and P_{jd} are the coordinates of g^{-1} and the Riemann and Schouten curvature tensors, respectively.

Proof of Step 2.1. Throughout this proof, for every integer l, o^l stands for a linear operator of order less than l if $l>0$ and $\mathrm{o}^l := 0$ if $l \le 0$. Juhl's formulae [27] (see also Fefferman and Graham [16]) give

$$
\begin{aligned}
P_{2k} = M_2^k &- \sum_{j=1}^{k-1} j(k-j)\, M_2^{j-1} M_4 M_2^{k-j-1}\\
&\qquad + \frac{1}{4}\sum_{j=1}^{k-2} j(j+1)(k-j)(k-j-1)\, M_2^{j-1} M_6 M_2^{k-j-2}\\
+ \sum_{j=2}^{k-2}&(j+1)(k-j-1)\sum_{i=1}^{j-1} i(k-i)\, M_2^{i-1} M_4 M_2^{j-i-1} M_4 M_2^{k-j-2} + \mathrm{o}^{2k-5},
\end{aligned} \tag{2.9}
$$

where the operators M_2, M_4 and M_6 are defined as

$$
M_2 := \Delta + \mu_2, \quad M_4 := 4\delta\,\mathrm{P}^{\#} d + \mu_4 \quad \text{and} \quad M_6 := \delta A_6^{\#} d + \mu_6,
$$

where μ_6 is a smooth function in M which we do not need explicitly, μ_2 and μ_4 are the functions defined as

$$
\mu_2 := \frac{n-2}{4(n-1)}\mathrm{S} \quad \text{and} \quad \mu_4 := \frac{\Delta\,\mathrm{S}}{2(n-1)} + \frac{\mathrm{S}^2}{4(n-1)^2} + (n-4)\,|\mathrm{P}|^2
$$

and A_6 is the tensor defined as

$$
A_6 := 48\,\mathrm{P}^{\#}\,\mathrm{P} + \frac{16}{n-4}\mathrm{B}.
$$

We point out that throughout this paper, we use the same sign convention for the Riemann curvature tensor as in the paper of Lee and Parker [28], which is the opposite of the convention used by Fefferman and Graham [16] and Juhl [27]. Straightforward expansions yield

$$
\begin{aligned}
M_2^k &= \Delta^k + \frac{n-2}{4(n-1)}\sum_{j=1}^{k}\Delta^{j-1}\left(\mathrm{S}\,\Delta^{k-j}\right)\\
&\quad + \frac{(n-2)^2}{16(n-1)^2}\sum_{j=2}^{k}\sum_{i=1}^{j-1}\Delta^{i-1}\left(\mathrm{S}\,\Delta^{j-i-1}\left(\mathrm{S}\,\Delta^{k-j}\right)\right) + \mathrm{o}^{2k-5}
\end{aligned}
$$

$$= \Delta^k + \frac{n-2}{4(n-1)} \sum_{j=1}^{k} \Delta^{j-1} \left(\mathrm{S}\, \Delta^{k-j} \right) + \frac{(n-2)^2}{16(n-1)^2} \sum_{j=2}^{k} (j-1)\, \Delta^{k-2} \left(\mathrm{S}^2 \cdot \right) + \mathrm{o}^{2k-4}$$
$$= \Delta^k + \frac{n-2}{4(n-1)} \sum_{j=1}^{k} \Delta^{j-1} \left(\mathrm{S}\, \Delta^{k-j} \right) + \frac{k(k-1)(n-2)^2}{32(n-1)^2} \Delta^{k-2} \left(\mathrm{S}^2 \cdot \right) + \mathrm{o}^{2k-4}, \tag{2.10}$$

$$\begin{aligned} M_2^{j-1} M_4 M_2^{k-j-1} &= 4\Delta^{j-1} \delta\, \mathrm{P}^{\#}\, d\Delta^{k-j-1} + \Delta^{j-1} \left(\mu_4 \Delta^{k-j-1} \right) \\ &+ \frac{n-2}{n-1} \sum_{i=1}^{j-1} \Delta^{i-1} \left(\mathrm{S}\, \Delta^{j-i-1} \delta\, \mathrm{P}^{\#}\, d\Delta^{k-j-1} \right) \\ &+ \frac{n-2}{n-1} \sum_{i=j+1}^{k-1} \Delta^{j-1} \delta\, \mathrm{P}^{\#}\, d\Delta^{i-j-1} \left(\mathrm{S}\, \Delta^{k-i-1} \right) + \mathrm{o}^{2k-5} \\ &= 4\Delta^{j-1} \delta\, \mathrm{P}^{\#}\, d\Delta^{k-j-1} + \mu_4 \Delta^{k-2} - \frac{(k-2)(n-2)}{n-1} \Delta^{k-3} \left(\mathrm{S}\, \mathrm{P}, \nabla^2 \right) + \mathrm{o}^{2k-4}, \end{aligned} \tag{2.11}$$

$$M_2^{j-1} M_6 M_2^{k-j-2} = \Delta^{j-1} \delta A_6^{\#} d\Delta^{k-j-2} + \mathrm{o}^{2k-5} = -\Delta^{k-3} \left(A_6, \nabla^2 \right) + \mathrm{o}^{2k-4}. \tag{2.12}$$

and

$$\begin{aligned} M_2^{i-1} M_4 M_2^{j-i-1} M_4 M_2^{k-j-2} &= 16\Delta^{i-1} \delta\, \mathrm{P}^{\#}\, d\Delta^{j-i-1} \delta\, \mathrm{P}^{\#}\, d\Delta^{k-j-2} + \mathrm{o}^{2k-5} \\ &= 16\Delta^{k-4} \left(\mathrm{P} \otimes \mathrm{P}, \nabla^4 \right) + \mathrm{o}^{2k-4} \end{aligned} \tag{2.13}$$

Furthermore, by induction, one can check that

$$\mathrm{S}\, \Delta^j = \Delta^j (\mathrm{S}\, \cdot) - j \Delta\, \mathrm{S}\, \Delta^{j-1} + 2j \Delta^{j-1} (\nabla \mathrm{S}, \nabla) + 2j(j-1)\, \Delta^{j-2} \left(\nabla^2 \mathrm{S}, \nabla^2 \right) + \mathrm{o}^{2j-2} \tag{2.14}$$

and

$$\begin{aligned} \delta\, \mathrm{P}^{\#}\, d\Delta^j = \Delta^j \left((\delta\, \mathrm{P}, \nabla) - \left(\mathrm{P}, \nabla^2 \right) \right) + j \Delta^{j-1} \Big(& \left(\delta \nabla\, \mathrm{P} + 2 \nabla \delta\, \mathrm{P} + 2\, \mathrm{R} * \mathrm{P}, \nabla^2 \right) \\ & - 2 \left(\nabla\, \mathrm{P}, \nabla^3 \right) \Big) - 2j(j-1)\, \Delta^{j-2} \left(\nabla^2 \mathrm{P}, \nabla^4 \right) + \mathrm{o}^{2j}, \end{aligned} \tag{2.15}$$

where $\mathrm{R} * \mathrm{P}$ is as in (2.8). The proof of (2.15) relies on the commutation formula

$$\begin{aligned} u_{;abcd} &= \left(u_{;acb} + R^{e}_{abc} u_{;e} \right)_{;d} = u_{;acbd} + \mathrm{R}^{e}_{abc}\, u_{;ed} + \mathrm{o}^2\, u = u_{;cabd} + \mathrm{R}^{e}_{abc}\, u_{;de} + \mathrm{o}^2\, u \\ &= u_{;cadb} + \mathrm{R}^{e}_{abd}\, u_{;ce} + \mathrm{R}^{e}_{cbd}\, u_{;ae} + \mathrm{R}^{e}_{abc}\, u_{;de} + \mathrm{o}^2\, u \\ &= \left(u_{;cda} + R^{e}_{cab} u_{;e} \right)_{;b} + \mathrm{R}^{e}_{abd}\, u_{;ce} + \mathrm{R}^{e}_{cbd}\, u_{;ae} + \mathrm{R}^{e}_{abc}\, u_{;de} + \mathrm{o}^2\, u \\ &= u_{;cdab} + \mathrm{R}^{e}_{cad}\, u_{;eb} + \mathrm{R}^{e}_{abd}\, u_{;ce} + \mathrm{R}^{e}_{cbd}\, u_{;ae} + \mathrm{R}^{e}_{abc}\, u_{;de} + \mathrm{o}^2\, u \end{aligned}$$

where $\mathrm{R}^{e}_{abc} := g^{ef}\, \mathrm{R}_{fabc}$, which gives

$$\begin{aligned} & \delta P^{\#} d\Delta u - \Delta \delta P^{\#} du = g^{ab} g^{cc'} g^{dd'} \left(\left(P_{c'd'} u_{;abc} \right)_{;d} - \left(P_{c'd'} u_{;c} \right)_{;dab} \right) \\ &= g^{ab} g^{cc'} g^{dd'} \left(P_{c'd'} \left(u_{;abcd} - u_{;cdab} \right) - P_{c'd';ab} u_{;cd} - 2 P_{c'd';da} u_{;cb} - 2 P_{c'd';a} u_{;cdb} \right) \\ &+ o^2 u = g^{ab} g^{cc'} g^{dd'} \left(2 P_{c'd'} \left(\mathrm{R}^{e}_{cad}\, u_{;be} + \mathrm{R}^{e}_{abd}\, u_{;ce} \right) - P_{c'd';ab} u_{;cd} - 2 P_{c'd';da} u_{;cb} \right. \\ &\left. - 2 P_{c'd';a} u_{;cdb} \right) + \mathrm{o}^2\, u = \left(\delta \nabla\, \mathrm{P} + 2 \nabla \delta\, \mathrm{P} + 2\, \mathrm{R} * \mathrm{P}, \nabla^2 u \right) - 2 \left(\nabla\, \mathrm{P}, \nabla^3 u \right) + \mathrm{o}^2\, u. \end{aligned}$$

By using (2.11)–(2.15) together with Faulhaber's formula, which gives

$$\sum_{j=1}^{k} j=\frac{k\left(k+1\right)}{2},\quad \sum_{j=1}^{k} j^2=\frac{k\left(k+1\right)\left(2k+1\right)}{6},\quad \sum_{j=1}^{k} j^3=\frac{k^2\left(k+1\right)^2}{4},$$

$$\sum_{j=1}^{k} j^4=\frac{k\left(k+1\right)\left(2k+1\right)\left(3k^2+3k-1\right)}{30},\quad \sum_{j=1}^{k} j^5=\frac{k^2\left(k+1\right)^2\left(2k^2+2k-1\right)}{12},$$

we obtain

$$\begin{aligned}\sum_{j=1}^{k}\Delta^{j-1}\left(\mathrm{S}\,\Delta^{k-j}\right)&=k\Delta^{k-1}\left(\mathrm{S}\,\cdot\right)-\frac{k\left(k-1\right)}{2}\Delta^{k-2}\left(\Delta\,\mathrm{S}\,\cdot\right)\\&+k\left(k-1\right)\Delta^{k-2}\left(\nabla\,\mathrm{S},\nabla\right)+\frac{2k\left(k-1\right)\left(k-2\right)}{3}\Delta^{k-3}\left(\nabla^2\,\mathrm{S},\nabla^2\right)+\mathrm{o}^{2k-4},\end{aligned}\tag{2.16}$$

$$\begin{aligned}\sum_{j=1}^{k-1} j\left(k-j\right)M_2^{j-1}M_4M_2^{k-j-1}&=k\left(k-1\right)\left(k+1\right)\bigg(\frac{2}{3}\Delta^{k-2}\left(\left(\delta\,\mathrm{P},\nabla\right)-\left(\mathrm{P},\nabla^2\right)\right)\\&+\frac{k-2}{3}\Delta^{k-3}\left(\left(\delta\nabla\,\mathrm{P}+2\nabla\delta\,\mathrm{P}+2\,\mathrm{R}*\mathrm{P},\nabla^2\right)-2\left(\nabla\,\mathrm{P},\nabla^3\right)\right)+\frac{1}{6}\Delta^{k-2}\left(\mu_4\cdot\right)\\&-\frac{2\left(k-2\right)\left(k-3\right)}{5}\Delta^{k-4}\left(\nabla^2\,\mathrm{P},\nabla^4\right)-\frac{\left(k-2\right)\left(n-2\right)}{6\left(n-1\right)}\Delta^{k-3}\left(\mathrm{S}\,\mathrm{P},\nabla^2\right)\bigg)+\mathrm{o}^{2k-4},\end{aligned}\tag{2.17}$$

$$\begin{aligned}&\sum_{j=1}^{k-2} j\left(j+1\right)\left(k-j\right)\left(k-j-1\right)M_2^{j-1}M_6M_2^{k-j-2}\\&\qquad=-\frac{k\left(k-1\right)\left(k-2\right)\left(k+1\right)\left(k+2\right)}{30}\Delta^{k-3}\left(A_6,\nabla^2\right)+\mathrm{o}^{2k-4}\end{aligned}\tag{2.18}$$

and

$$\begin{aligned}&\sum_{j=2}^{k-2}\left(j+1\right)\left(k-j-1\right)\sum_{i=1}^{j-1} i\left(k-i\right)M_2^{i-1}M_4M_2^{j-i-1}M_4M_2^{k-j-2}\\&\quad=\frac{2k\left(k-1\right)\left(k-2\right)\left(k-3\right)\left(k+1\right)\left(5k+7\right)}{45}\Delta^{k-4}\left(\mathrm{P}\otimes\mathrm{P},\nabla^4\right)+\mathrm{o}^{2k-4}.\end{aligned}\tag{2.19}$$

Finally, (2.7) follows by putting together (2.9), (2.10) and (2.16)–(2.19). This ends the proof of Step 2.1. ☐

The next step is as follows:

Step 2.2. *Assume that $n\ge 2k+4$ and $k\ge 3$. Then for every smooth linear operator Z of order less than $2k-4$, as $\mu\to 0$,*

$$\int_M U_\mu Z U_\mu\,dv_g=\begin{cases}\mathrm{O}\left(\mu^4\right) & \text{if } n=2k+4\\ \mathrm{o}\left(\mu^4\right) & \text{if } n>2k+4.\end{cases}\tag{2.20}$$

Proof of Step 2.2. By rewriting the integral in geodesic normal coordinates, we obtain

$$\int_M U_\mu Z U_\mu\,dv_g=\int_{B(0,2r_0)}\widetilde{U}_\mu\widetilde{Z}\widetilde{U}_\mu\,dx=\sum_{|\alpha|<2k-4}\int_{B(0,2r_0)} z_\alpha\widetilde{U}_\mu\partial^\alpha\widetilde{U}_\mu\,dx,\tag{2.21}$$

where

$$\widetilde{U}_\mu\left(x\right):=\mu^{\frac{2k-n}{2}}U\left(x/\mu\right)\ \text{ and }\ \widetilde{Z}\left(x\right):=\sum_{|\alpha|<2k-4}z_\alpha\left(x\right)\partial^\alpha\quad\forall x\in B\left(0,2r_0\right)\tag{2.22}$$

for some smooth functions z_α in $B\left(0,2r_0\right)$. A straightforward change of variable then gives

$$\int_{B(0,2r_0)}z_\alpha\widetilde{U}_\mu\partial^{(\alpha)}\widetilde{U}_\mu\,dx=\mu^{2k-|\alpha|}\int_{B(0,2r_0/\mu)}z_\alpha\left(\mu x\right)U\left(x\right)\partial^{(\alpha)}U\left(x\right)dx.\tag{2.23}$$

An easy induction yields that for every multi-index α, there exists a constant C_α such that

$$\left|\partial^{(\alpha)}U\left(x\right)\right|\le C_\alpha\left(1+\left|x\right|^2\right)^{-\frac{n-2k+|\alpha|}{2}}\quad\forall x\in\mathbb{R}^n\tag{2.24}$$

It follows from (2.23) and (2.24) that

$$\begin{aligned}\int_{B(0,2r_0)}z_\alpha\widetilde{U}_\mu\partial^{(\alpha)}\widetilde{U}_\mu\,dx&=\mathrm{O}\left(\mu^{2k-|\alpha|}\int_{B(0,2r_0/\mu)}\left(1+\left|x\right|^2\right)^{-n+2k-|\alpha|/2}dx\right)\\&=\begin{cases}\mathrm{O}\left(\mu^{2k-|\alpha|}\right)&\text{if }\left|\alpha\right|>4k-n\\\mathrm{O}\left(\mu^{n-2k}\ln\left(1/\mu\right)\right)&\text{if }\left|\alpha\right|=4k-n\\\mathrm{O}\left(\mu^{n-2k}\right)&\text{if }\left|\alpha\right|<4k-n.\end{cases}\end{aligned}\tag{2.25}$$

Finally, (2.20) follows from (2.21) and (2.25). □

We then prove the following:

Step 2.3. *Assume that $n\ge 2k+4$ and g satisfies (2.2) for some point $\xi\in M$. Then, as $\mu\to 0$,*

$$\int_M U_\mu\Delta^kU_\mu\,dv_g=2^{2k-n}\left(2k-1\right)!\,\omega_n\,\mathrm{B}\left(\frac{n}{2}-k,2k\right)^{-1}+\mathrm{O}\left(\mu^{n-2k}\right).\tag{2.26}$$

If $k\ge 2$, then for every smooth function f in M,

$$\begin{aligned}\int_M fU_\mu\Delta^{k-2}U_\mu\,dv_g&=\frac{2^{2k-n-1}\left(n-1\right)!\left(k-2\right)!\,\omega_n}{\left(n-2\right)\left(n-4\right)\left(n-2k-2\right)}f\left(\xi\right)\mu^4\\&\times\sum_{l=k-2}^{2k-4}\frac{l!}{\left(l-k+2\right)!\left(2k-l-4\right)!\left(n+l-2k-1\right)!}\mathrm{B}\left(\frac{n}{2}-k-1,l+1\right)^{-1}\\&\times\left\{\begin{array}{ll}2\ln\left(1/\mu\right)&\text{if }n=2k+4\text{ and }l=k-2\\\mathrm{B}\left(\frac{n}{2}+l-2k,2k-l-2\right)&\text{otherwise}\end{array}\right\}\\&+\begin{cases}\mathrm{O}\left(\mu^4\right)&\text{if }n=2k+4\\\mathrm{o}\left(\mu^4\right)&\text{if }n>2k+4,\end{cases}\end{aligned}\tag{2.27}$$

for every smooth, covariant tensor T of rank 1,

$$\begin{aligned}\int_M\left(T,\nabla U_\mu\right)\Delta^{k-2}U_\mu\,dv_g&=-\frac{2^{2k-n-2}\left(n-2k\right)\left(n-1\right)!\left(k-2\right)!\,\omega_n}{\left(n-2\right)\left(n-4\right)\left(n-2k-2\right)}\sum_{a=1}^nT_{a;a}\left(\xi\right)\\&\times\mu^4\sum_{l=k-2}^{2k-4}\frac{l!}{\left(l-k+2\right)!\left(2k-l-4\right)!\left(n+l-2k\right)!}\mathrm{B}\left(\frac{n}{2}-k-1,l+1\right)^{-1}\end{aligned}$$

$$\times\begin{Bmatrix} 2\ln\left(1/\mu\right) & \text{if } n=2k+4 \text{ and } l=k-2\\ \mathrm{B}\left(\frac{n}{2}+l-2k,2k-l-2\right) & \text{otherwise}\end{Bmatrix} + \begin{cases}\mathrm{O}\left(\mu^4\right) & \text{if } n=2k+4\\ \mathrm{o}\left(\mu^4\right) & \text{if } n>2k+4\end{cases} \tag{2.28}$$

and for every smooth, covariant tensor T of rank 2,

$$\begin{aligned}\int_M \left(T,\nabla^2 U_\mu\right)\Delta^{k-2}U_\mu\,dv_g &= \frac{2^{2k-n-4}\left(n-2k\right)\left(n-1\right)!\left(k-2\right)!\,\omega_n}{\left(n-2\right)\left(n-4\right)\left(n-2k-2\right)}\\ &\times\sum_{l=k-2}^{2k-4}\frac{l!}{\left(l-k+2\right)!\left(2k-l-4\right)!\left(n+l-2k+1\right)!}\mathrm{B}\left(\frac{n}{2}-k-1,l+1\right)^{-1}\\ &\times\Bigg(-2\left(n-4\right)\left(n+2l-2k\right)\mathrm{B}\left(\frac{n}{2}-2k+l+1,2k-l-2\right)\sum_{a=1}^{n}T_{aa}\left(\xi\right)\mu^2\\ &+\left(\left(n-2k+2\right)\sum_{a,b=1}^{n}\left(T_{ab;ab}\left(\xi\right)+T_{ab;ba}\left(\xi\right)\right)-\left(n+2l-2k\right)\sum_{a,b=1}^{n}T_{aa;bb}\left(\xi\right)\right)\mu^4\\ &\times\begin{Bmatrix} 2\ln\left(1/\mu\right) & \text{if } n=2k+4 \text{ and } l=k-2\\ \mathrm{B}\left(\frac{n}{2}+l-2k,2k-l-2\right) & \text{otherwise}\end{Bmatrix}\Bigg)\\ &+\begin{cases}\mathrm{O}\left(\mu^4\right) & \text{if } n=2k+4\\ \mathrm{o}\left(\mu^4\right) & \text{if } n>2k+4.\end{cases}\end{aligned} \tag{2.29}$$

If $k\ge 3$, then for every smooth, covariant tensor T of rank 2,

$$\begin{aligned}\int_M \left(T,\nabla^2 U_\mu\right)\Delta^{k-3}U_\mu\,dv_g &= -\frac{2^{2k-n-5}\left(n-2k\right)\left(n-1\right)!\left(k-3\right)!\,\omega_n}{\left(n-2\right)\left(n-4\right)\left(n-2k-2\right)}\sum_{a=1}^{n}T_{aa}\left(\xi\right)\\ &\times\mu^4\sum_{l=k-3}^{2k-6}\frac{\left(n+2l-2k\right)l!}{\left(l-k+3\right)!\left(2k-l-6\right)!\left(n+l-2k+1\right)!}\mathrm{B}\left(\frac{n}{2}-k-1,l+1\right)^{-1}\\ &\times\begin{Bmatrix} 2\ln\left(1/\mu\right) & \text{if } n=2k+4 \text{ and } l=k-3\\ \mathrm{B}\left(\frac{n}{2}+l-2k+1,2k-l-3\right) & \text{otherwise}\end{Bmatrix}\\ &+\begin{cases}\mathrm{O}\left(\mu^4\right) & \text{if } n=2k+4\\ \mathrm{o}\left(\mu^4\right) & \text{if } n>2k+4\end{cases}\end{aligned} \tag{2.30}$$

and for every smooth, covariant tensor T of rank 3,

$$\begin{aligned}\int_M \left(T,\nabla^3 U_\mu\right)\Delta^{k-3}U_\mu\,dv_g &= \frac{2^{2k-n-6}\left(n-2k\right)\left(n-2k+2\right)\left(n-1\right)!\left(k-3\right)!\,\omega_n}{\left(n-2\right)\left(n-4\right)\left(n-2k-2\right)}\\ &\times\sum_{a,b=1}^{n}\left(T_{aab;b}\left(\xi\right)+T_{aba;b}\left(\xi\right)+T_{abb;a}\left(\xi\right)\right)\mu^4\\ &\times\sum_{l=k-3}^{2k-6}\frac{\left(n+2l-2k\right)l!}{\left(l-k+3\right)!\left(2k-l-6\right)!\left(n+l-2k+2\right)!}\mathrm{B}\left(\frac{n}{2}-k-1,l+1\right)^{-1}\end{aligned}$$

$$\times\left\{\begin{matrix}2\ln\left(1/\mu\right) & \text{if } n=2k+4 \text{ and } l=k-3\\ \mathrm{B}\left(\frac{n}{2}+l-2k+1,2k-l-3\right) & \text{otherwise}\end{matrix}\right\}$$
$$+\left\{\begin{matrix}\mathrm{O}\left(\mu^4\right) & \text{if } n=2k+4\\ \mathrm{o}\left(\mu^4\right) & \text{if } n>2k+4.\end{matrix}\right. \tag{2.31}$$

If $k\geq 4$, then for every smooth, covariant tensor T of rank 4,

$$\int_M\left(T,\nabla^4U_\mu\right)\Delta^{k-4}U_\mu\,dv_g=\frac{2^{2k-n-8}\left(n-2k\right)\left(n-2k+2\right)\left(n-1\right)!\left(k-4\right)!\,\omega_n}{3\left(n-2\right)\left(n-4\right)\left(n-2k-2\right)}$$
$$\times\sum_{a,b=1}^n\left(T_{aabb}\left(\xi\right)+T_{abab}\left(\xi\right)+T_{abba}\left(\xi\right)\right)\mu^4$$
$$\times\sum_{l=k-4}^{2k-8}\frac{\left(n+2l-2k\right)\left(n+2l-2k+2\right)l!}{\left(l-k+4\right)!\left(2k-l-8\right)!\left(n+l-2k+3\right)!}\mathrm{B}\left(\frac{n}{2}-k-1,l+1\right)^{-1}$$
$$\times\left\{\begin{matrix}2\ln\left(1/\mu\right) & \text{if } n=2k+4 \text{ and } l=k-4\\ \mathrm{B}\left(\frac{n}{2}+l-2k+2,2k-l-4\right) & \text{otherwise}\end{matrix}\right\}$$
$$+\left\{\begin{matrix}\mathrm{O}\left(\mu^4\right) & \text{if } n=2k+4\\ \mathrm{o}\left(\mu^4\right) & \text{if } n>2k+4.\end{matrix}\right. \tag{2.32}$$

Proof of Step 2.3. We let j and l be two integers such that

$$\max\left(2\left(k-l-2\right),0\right)\leq j\leq k-l\quad\text{and}\quad\max\left(k-4,0\right)\leq l\leq k$$

and T be a smooth, covariant tensor of rank j. By using geodesic normal coordinates, we obtain

$$\begin{aligned}\int_M&\left(T,\nabla^jU_\mu\right)\Delta^lU_\mu\,dv_g-\int_{B(\xi,r_0)}\left(T,\nabla^jU_\mu\right)\Delta^lU_\mu\,dv_g\\&=\int_{B(0,2r_0)\backslash B(0,r_0)}\widetilde{Z}_1\widetilde{U}_\mu\widetilde{Z}_2\widetilde{U}_\mu\,dx\\&=\sum_{|\alpha_1|\leq j}\sum_{|\alpha_2|\leq 2l}\int_{B(0,2r_0)\backslash B(0,r_0)}z_{1,\alpha}z_{2,\alpha}\partial^{\alpha_2}\widetilde{U}_\mu\partial^{\alpha_1}\widetilde{U}_\mu\,dx,\end{aligned}\tag{2.33}$$

where $\widetilde{U}_\mu$ is as in (2.22) and

$$\widetilde{Z}_1\left(x\right):=\sum_{|\alpha|\leq j}z_{1,\alpha}\left(x\right)\partial^\alpha\quad\text{and}\quad\widetilde{Z}_2\left(x\right):=\sum_{|\alpha|\leq 2l}z_{2,\alpha}\left(x\right)\partial^\alpha\quad\forall x\in B\left(0,2r_0\right)$$

for some smooth functions $z_{1,\alpha}$ and $z_{2,\alpha}$ in $B\left(0,2r_0\right)$. By proceeding as in (2.23)–(2.25), we obtain

$$\int_{B(0,2r_0)\backslash B(0,r_0)}z_{1,\alpha}z_{2,\alpha}\partial^{\alpha_2}\widetilde{U}_\mu\partial^{\alpha_1}\widetilde{U}_\mu\,dx=\mathrm{O}\left(\mu^{n-2k}\right).\tag{2.34}$$

It follows from (2.33) and (2.34) that

$$\int_M\left(T,\nabla^jU_\mu\right)\Delta^lU_\mu\,dv_g=\int_{B(\xi,r_0)}\left(T,\nabla^jU_\mu\right)\Delta^lU_\mu\,dv_g+\mathrm{O}\left(\mu^{n-2k}\right).\tag{2.35}$$

By using (2.2) and rewriting the integral in the right-hand side of (2.35) in geodesic normal coordinates, we obtain

$$\int_{B(\xi,r_0)} \left(T,\nabla^j U_\mu\right) \Delta^l U_\mu \, dv_g = \sum_{j'=0}^{j} \int_{B(0,r_0)} \widehat{T}^{i_1\dots i_{j'}} \circ \exp_\xi U_{\mu,i_1\dots i_{j'}} \Delta_0^l U_\mu \, dx, \tag{2.36}$$

where $U_{\mu,i_1\dots i_{j'}} := \partial^{(i_1\dots i_{j'})}\left(U_\mu \circ \exp_\xi\right)$ and the tensor $\widehat{T}$ is defined as

$$\widehat{T}^{i_1\dots i_{j'}} := \left.\begin{cases} \delta^{i_1}_{e_1}\cdots\delta^{i_j}_{e_j} & \text{if } j'=j \\ -\Gamma^{i_1\dots i_{j'}}_{e_1\dots e_j} & \text{if } j'<j \end{cases}\right\} g^{i_1'e_1}\cdots g^{i_j'e_j} T_{i_1'\dots i_j'},$$

where $\delta^{i_1}_{e_1},\dots,\delta^{i_j}_{e_j}$ stand for the Kronecker symbols and $\Gamma^{i_1\dots i_{j'}}_{e_1\dots e_j}$ is the generalized Christoffel symbol such that $\Gamma^{i_1\dots i_{j'}}_{e_1\dots e_j}$ is symmetric in $i_1,\dots,i_{j'}$ and

$$u_{;e_1\dots e_j} = u_{,e_1\dots e_j} - \sum_{j'=0}^{j-1} \Gamma^{i_1\dots i_{j'}}_{e_1\dots e_j} u_{,i_1\dots i_{j'}}.$$

By using (2.36) together with a straightforward change of variable and a Taylor expansion, we then obtain

$$\begin{aligned}
&\int_{B(\xi,r_0)} \left(T,\nabla^j U_\mu\right) \Delta^l U_\mu \, dv_g \\
&= \sum_{j'=0}^{j} \mu^{2k-2l-j'} \int_{B(0,r_0/\mu)} \widehat{T}^{i_1\dots i_{j'}} \left(\exp_\xi(\mu x)\right) U_{,i_1\dots i_{j'}}(x) \, \Delta_0^l U(x) \, dx \\
&= \sum_{j'=\max(2(k-l-2),0)}^{j} \; \sum_{j''=0}^{j'+2l-2k+4} \frac{\mu^{2k-2l-j'+j''}}{j''!} \sum_{i_1,\dots,i_{j'+j''}=1}^{n} \widehat{T}^{i_1\dots i_{j'}}{}_{,i_{j'+1}\dots i_{j'+j''}}(\xi) \\
&\quad \times \int_{B(0,r_0/\mu)} U_{,i_1\dots i_{j'}} x_{i_{j'+1}} \cdots x_{i_{j'+j''}} \Delta_0^l U \, dx + \mathrm{O}\Bigg(\sum_{j'=0}^{j} \mu^{\max(5,2k-2l-j')} \\
&\quad \times \int_{B(0,r_0/\mu)} |x|^{\max(j'+2l-2k+5,0)} \left|U_{,i_1\dots i_{j'}} \Delta_0^l U\right| dx\Bigg).
\end{aligned} \tag{2.37}$$

On the other hand, by using (2.24), we obtain

$$\begin{aligned}
&\mu^{\max(5,2k-2l-j')} \int_{B(0,r_0/\mu)} |x|^{\max(j'+2l-2k+5,0)} \left|U_{,i_1\dots i_{j'}} \Delta_0^l U\right| dx \\
&= \mathrm{O}\left(\mu^{\max(5,2k-2l-j')} \int_{B(0,r_0/\mu)} |x|^{\max(j'+2l-2k+5,0)} \left(1+|x|^2\right)^{-\frac{2n+j'+2l-4k}{2}} dx\right) \\
&= \begin{cases} \mathrm{O}\left(\mu^4\right) & \text{if } n=2k+4 \\ \mathrm{o}\left(\mu^4\right) & \text{otherwise.} \end{cases}
\end{aligned} \tag{2.38}$$

It follows from (2.35), (2.37) and (2.38) that

$$\int_M \left(T,\nabla^j U_\mu\right)\Delta^l U_\mu\,dv_g=\sum_{j'=\max(2(k-l-2),0)}^{j}\sum_{j''=0}^{j'+2l-2k+4}\frac{\mu^{2k-2l-j'+j''}}{j''!}$$
$$\times\sum_{i_1,\dots,i_{j'+j''}=1}^{n}\widehat{T}^{i_1\dots i_{j'}}{}_{,i_{j'+1}\dots i_{j'+j''}}(\xi)\int_{B(0,r_0/\mu)}U_{,i_1\dots i_{j'}}x_{i_{j'+1}}\cdots x_{i_{j'+j''}}\Delta_0^l\,U\,dx$$
$$+\begin{cases}\mathrm{O}\left(\mu^4\right) & \text{if } n=2k+4\\ \mathrm{o}\left(\mu^4\right) & \text{if } n>2k+4.\end{cases}\tag{2.39}$$

An easy induction gives

$$U_{,i_1\dots i_j}(x)=\sum_{m=0}^{\lfloor j/2\rfloor}\frac{2^{j-2m}}{m!\,(j-2m)!}\partial_r^{j-m}U(r)$$
$$\times\sum_{\sigma\in\mathfrak{S}(j)}\delta_{i_{\sigma(1)}i_{\sigma(2)}}\cdots\delta_{i_{\sigma(2m-1)}i_{\sigma(2m)}}x_{i_{\sigma(2m+1)}}\cdots x_{i_{\sigma(j)}}\quad\forall x\in\mathbb{R}^n,\tag{2.40}$$

where $r:=|x|^2$, $U(r):=U(x)=(1+r)^{(2k-n)/2}$ and $\mathfrak{S}(j)$ is the set of all permutations of $(1,\dots,j)$. Furthermore, it is easy to see that

$$\partial_r^j U(r)=(-1)^j\,2^{-j}\,(n-2k)\,(n-2k+2)\cdots(n-2k+2j-2)\,(1+r)^{-\frac{n-2k+2j}{2}}$$
$$=\frac{2\,(-1)^j\,j!}{(n-2k-2)}\,\mathrm{B}\left(\frac{n}{2}-k-1,j+1\right)^{-1}(1+r)^{-\frac{n-2k+2j}{2}}.\tag{2.41}$$

Another induction yields

$$\Delta_0^l\,U(x)=\begin{cases}\dfrac{2^{2l+1}\,l!}{(n-2k-2)\,(k-l-1)!}\displaystyle\sum_{l'=l}^{2l}\frac{l'!\,(k+l-l'-1)!}{(l'-l)!\,(2l-l')!}\\ \qquad\times\,\mathrm{B}\left(\dfrac{n}{2}-k-1,l'+1\right)^{-1}(1+r)^{-\frac{n+2l'-2k}{2}} & \text{if } l<k\\ 2^{2k}\,(2k-1)!\,\mathrm{B}\left(\dfrac{n}{2}-k,2k\right)^{-1}(1+r)^{-\frac{n+2k}{2}} & \text{if } l=k\end{cases}\tag{2.42}$$

for all $x\in\mathbb{R}^n$. In the case where $j=0$, $l=k$ and $T\equiv 1$, it follows from (2.42) that

$$\int_{B(0,r_0/\mu)}U\Delta_0^k\,U\,dx=2^{2k-1}\,(2k-1)!\,\omega_{n-1}\,\mathrm{B}\left(\frac{n}{2}-k,2k\right)^{-1}\int_0^{(r_0/\mu)^2}\frac{r^{\frac{n-2}{2}}}{(1+r)^n}\,dr,\tag{2.43}$$

where $\omega_{n-1}=\mathrm{Vol}\left(\mathbb{S}^{n-1},g_0\right)$ is the volume of the standard $(n-1)$-dimensional sphere. On the other hand, in the case where $l<k$, by putting together (2.40)–(2.42) , we obtain

$$\int_{B(0,r_0/\mu)}U_{,i_1\dots i_{j'}}x_{i_{j'+1}}\cdots x_{i_{j'+j''}}\Delta_0^l\,U\,dx=\frac{2^{2l+1}\,l!}{(n-2k-2)^2\,(k-l-1)!}\sum_{l'=l}^{2l}\sum_{m=0}^{\lfloor j'/2\rfloor}$$
$$\times\frac{(-1)^{j'-m}\,2^{j'-2m}\,l'!\,(k+l-l'-1)!\,(j'-m)!}{(l'-l)!\,(2l-l')!\,m!\,(j'-2m)!}\,\mathrm{B}\left(\frac{n}{2}-k-1,l'+1\right)^{-1}$$

$$\times \mathrm{B}\left(\frac{n}{2}-k-1, j'-m+1\right)^{-1} \int_0^{(r_0/\mu)^2} \frac{r^{\frac{n+j'+j''-2m-2}{2}}}{(1+r)^{n+j'-m+l'-2k}}\, dr \sum_{\sigma\in\mathfrak{S}(j')}$$

$$\delta_{i_{\sigma(1)}i_{\sigma(2)}}\cdots\delta_{i_{\sigma(2m-1)}i_{\sigma(2m)}} \int_{\mathbb{S}^{n-1}} y_{i_{\sigma(2m+1)}}\cdots y_{i_{\sigma(j')}} y_{i_{j'+1}}\cdots y_{i_{j'+j''}}\, dv_{g_0}(y)\,. \tag{2.44}$$

A standard computation gives

$$\int_0^{(r_0/\mu)^2} \frac{r^{a-1}dr}{(1+r)^b} = \begin{cases} 2\ln(1/\mu) + \mathrm{O}(1) & \text{if } b=a \\ \mathrm{B}(a,b-a) + \mathrm{O}\left(\mu^{2(b-a)}\right) & \text{if } b>a. \end{cases} \tag{2.45}$$

On the other hand, by using the fact (see for instance Brendle [6, Proposition 28]) that for every homogeneous polynomial Φ of degree $j\geq 2$,

$$\int_{\mathbb{S}^{n-1}} \Phi(y)\, dv_{g_0}(y) = \frac{-1}{j(n+j-2)}\int_{\mathbb{S}^{n-1}} \Delta_0\Phi(y)\, dv_{g_0}(y)\,,$$

another induction yields that when j is even,

$$\int_{\mathbb{S}^{n-1}} y_{i_1}\cdots y_{i_j}\, dv_{g_0}(y) = \frac{(n-2)\,\omega_{n-1}}{2^{j+1}(j/2)!^2}\,\mathrm{B}\left(\frac{n-2}{2},\frac{j+2}{2}\right) \times \sum_{\sigma\in\mathfrak{S}(j)} \delta_{i_{\sigma(1)}i_{\sigma(2)}}\cdots\delta_{i_{\sigma(j-1)}i_{\sigma(j)}}. \tag{2.46}$$

The integral in (2.46) vanishes when j is odd. By remarking that

$$\omega_n = 2^{n-1}\,\mathrm{B}\left(\frac{n}{2},\frac{n}{2}\right)\omega_{n-1}, \tag{2.47}$$

we obtain that for even j,

$$\mathrm{B}\left(\frac{n-2}{2},\frac{j+2}{2}\right) = \frac{2^{2-n}(n-1)!\,(j/2)!\,\omega_n}{(n-2)(n+j/2-1)!\,\omega_{n-1}}\,\mathrm{B}\left(\frac{n}{2},\frac{n+j}{2}\right)^{-1}. \tag{2.48}$$

By using (2.45)–(2.48) together with the identity

$$\mathrm{B}\left(\frac{n}{2},\frac{n+j'+j''-2m}{2}\right)^{-1}\mathrm{B}\left(\frac{n+j'+j''-2m}{2},\frac{n+j'-j''+2l'-4k}{2}\right) = \frac{\left(\frac{j'+j''}{2}+n-m-1\right)!}{(n+j'-m+l'-2k-1)!\left(\frac{j''-j'}{2}+2k-l'-1\right)!} \times \mathrm{B}\left(\frac{n+j'-j''+2l'-4k}{2},\frac{j''-j'+4k-2l'}{2}\right),$$

we obtain that if $j'+j''$ is even, then

$$\int_0^{(r_0/\mu)^2}\frac{r^{\frac{n+j'+j''-2m-2}{2}}}{(1+r)^{n+j'-m+l'-2k}}\,dr\int_{\mathbb{S}^{n-1}}y_{i_{\sigma(2m+1)}}\cdots y_{i_{\sigma(j')}}y_{i_{j'+1}}\cdots y_{i_{j'+j''}}\,dv_{g_0}(y)$$
$$=\frac{2^{1-n-j'-j''+2m}(n-1)!\,\omega_n}{(n+j'-m+l'-2k-1)!\left(\frac{j''-j'}{2}+2k-l'-1\right)!\left(\frac{j'+j''}{2}-m\right)!}$$
$$\times\left\{\begin{array}{ll}2\ln(1/\mu)+\mathrm{O}(1) & \text{if } n+j'-j''+2l'-4k=0\\ \mathrm{B}\left(\dfrac{n+j'-j''+2l'-4k}{2},\dfrac{j''-j'+4k-2l'}{2}\right) & \\ \qquad+\mathrm{O}\left(\mu^{n+j'-j''+2l'-4k}\right) & \text{if } 0<n+j'-j''+2l'-4k<n\end{array}\right\}$$
$$\times\sum_{\sigma'\in\mathfrak{S}(S_{j',j'',m,\sigma})}\delta_{i_{\sigma'(\sigma(2m+1))}i_{\sigma'(\sigma(2m+2))}}\cdots\delta_{i_{\sigma'(\sigma(j'-1))}i_{\sigma'(\sigma(j'))}}$$
$$\times\,\delta_{i_{\sigma'(j'+1)}i_{\sigma'(j'+2)}}\cdots\delta_{i_{\sigma'(j'+j''-1)}i_{\sigma'(j'+j'')}},\tag{2.49}$$

where

$$S_{j',j'',m,\sigma}:=\left(\sigma(2m+1),\dots,\sigma(j'),j'+1,\dots,j'+j''\right)$$

and $\mathfrak{S}(S_{j',j'',m,\sigma})$ stands for the set of all permutations of $S_{j',j'',m,\sigma}$. In the case where $j=0$, $l=k$ and $T\equiv 1$, (2.26) follows from (2.35), (2.36), (2.43), (2.45) and (2.47). On the other hand, in the case where $l<k$, by combining (2.39), (2.44) and (2.49) (and replacing j'' by $j'-2m'+2l-2k+4$ for $m'\in\{0,\dots,\lfloor j'/2\rfloor+l-k+2\}$ so that $j'+j''$ is even and $0\le j''\le j'+2l-2k+4$), we obtain

$$\int_M\left(T,\nabla^jU_\mu\right)\Delta^lU_\mu\,dv_g=\frac{2^{2k-n-2}(n-1)!\,l!\,\omega_n}{(n-2k-2)^2(k-l-1)!}\sum_{l'=l}^{2l}\sum_{j'=\max(2(k-l-2),0)}^{j}\sum_{m=0}^{\lfloor j'/2\rfloor}$$
$$\sum_{m'=0}^{\lfloor j'/2\rfloor+l-k+2}\frac{2^{2m'-j'}l'!\,(k+l-l'-1)!\,c(n,k,j',l,l',m,m')\,\mu^{4-2m'}}{(l'-l)!\,(2l-l')!\,(k+l-l'-m'+1)!\,(j'-2m'+2l-2k+4)!}$$
$$\times\mathrm{B}\left(\frac{n}{2}-k-1,l'+1\right)^{-1}\sum_{i_1,\dots,i_{2(j'-m'+l-k+2)}=1}^{n}\widehat{T}^{i_1\dots i_{j'}}{}_{,i_{j'+1}\dots i_{2(j'-m'+l-k+2)}}(\xi)$$
$$\times\sum_{\sigma\in\mathfrak{S}(j')}\sum_{\sigma'\in\mathfrak{S}(S_{j',j'-2m'+2l-2k+4,m,\sigma})}\delta_{i_{\sigma(1)}i_{\sigma(2)}}\cdots\delta_{i_{\sigma(2m-1)}i_{\sigma(2m)}}$$
$$\times\,\delta_{i_{\sigma'(\sigma(2m+1))}i_{\sigma'(\sigma(2m+2))}}\cdots\delta_{i_{\sigma'(\sigma(j'-1))}i_{\sigma'(\sigma(j'))}}$$
$$\times\,\delta_{i_{\sigma'(j'+1)}i_{\sigma'(j'+2)}}\cdots\delta_{i_{\sigma'(2(j'-m'+l-k+2)-1)}i_{\sigma'(2(j'-m'+l-k+2))}}$$
$$\times\left\{\begin{array}{ll}2\ln(1/\mu)+\mathrm{O}(1) & \text{if } n=2k+4,\ l'=l\text{ and } m'=0\\ \mathrm{B}\left(\dfrac{n}{2}+m'+l'-l-k-2,k+l-l'-m'+2\right)+\mathrm{o}\left(\mu^{2m'}\right) & \text{otherwise}\end{array}\right\}$$
$$+\left\{\begin{array}{ll}\mathrm{O}\left(\mu^4\right) & \text{if } n=2k+4\\ \mathrm{o}\left(\mu^4\right) & \text{if } n>2k+4,\end{array}\right.\tag{2.50}$$

where

$$c\left(n,k,j',l,l',m,m'\right):=\left(-1\right)^{j'-m}\mathrm{B}\left(\frac{n}{2}-k-1,j'-m+1\right)^{-1}\\ \times\frac{\left(j'-m\right)!}{m!\left(j'-2m\right)!\left(n+j'-m+l'-2k-1\right)!\left(j'-m-m'+l-k+2\right)!}.$$

Straightforward computations yield

$$\begin{aligned}&\sum_{\sigma\in\mathfrak{S}(j')}\sum_{\sigma'\in\mathfrak{S}(S_{j',j'',m,\sigma})}\delta_{i_{\sigma(1)}i_{\sigma(2)}}\cdots\delta_{i_{\sigma(2m-1)}i_{\sigma(2m)}}\delta_{i_{\sigma'(\sigma(2m+1))}i_{\sigma'(\sigma(2m+2))}}\cdots\\&\qquad\cdots\delta_{i_{\sigma'(\sigma(j'-1))}i_{\sigma'(\sigma(j'))}}\delta_{i_{\sigma'(j'+1)}i_{\sigma'(j'+2)}}\cdots\delta_{i_{\sigma'(j'+j''-1)}i_{\sigma'(j'+j'')}}\\&=\begin{cases}1 & \text{if } j'=j''=m=0\\ 2\,\delta_{i_1i_2} & \text{if } j'=j''=1\text{ and }m=0\\ 2\left(2-m\right)\delta_{i_1i_2} & \text{if } j'=2,\ j''=0\text{ and }m\le1\\ 16\left(\delta_{i_1i_2}\delta_{i_3i_4}+\delta_{i_1i_3}\delta_{i_2i_4}+\delta_{i_1i_4}\delta_{i_2i_3}\right) & \text{if } j'=j''=2\text{ and }m=0\\ 4\,\delta_{i_1i_2}\delta_{i_3i_4} & \text{if } j'=j''=2\text{ and }m=1\\ 2\left(4-2m\right)!\left(\delta_{i_1i_2}\delta_{i_3i_4}+\delta_{i_1i_3}\delta_{i_2i_4}+\delta_{i_1i_4}\delta_{i_2i_3}\right) & \text{if } j'=3,\ j''=1\text{ and }m\le1\\ 8\left(4-2m\right)!\left(\delta_{i_1i_2}\delta_{i_3i_4}+\delta_{i_1i_3}\delta_{i_2i_4}+\delta_{i_1i_4}\delta_{i_2i_3}\right) & \text{if } j'=4,\ j''=0\text{ and }m\le2.\end{cases}\end{aligned} \tag{2.51}$$

On the other hand, by using (2.3) and the fact that for all $a,b,c,d,e\in\{1,\dots,n\}$,

$$g^{ab}\left(\xi\right)=\delta^{ab},\quad g^{ab}{}_{,c}\left(\xi\right)=0,\quad g^{ab}{}_{,cd}\left(\xi\right)=-\frac{1}{3}\left(\mathrm{R}_{acdb}\left(\xi\right)+\mathrm{R}_{adcb}\left(\xi\right)\right),$$
$$\Gamma^a_{bc}\left(\xi\right)=0,\quad \Gamma^a_{bc,d}\left(\xi\right)=\frac{1}{3}\left(\mathrm{R}_{abdc}\left(\xi\right)+\mathrm{R}_{acdb}\left(\xi\right)\right)\quad\text{and}\quad\Gamma^{ab}_{cde}\left(\xi\right)=0,$$

we obtain

$$\widehat{T}\left(\xi\right)=T\left(\xi\right)\quad\text{if } j=0,\qquad\sum_{a=1}^n\widehat{T}^a{}_{,a}\left(\xi\right)=\sum_{a=1}^nT_{a;a}\left(\xi\right)\quad\text{if } j=1, \tag{2.52}$$

$$\left.\begin{cases}\displaystyle\sum_{a=1}^n\widehat{T}^a{}_{,a}\left(\xi\right)=0,\quad\sum_{a=1}^n\widehat{T}^{aa}\left(\xi\right)=\sum_{a=1}^nT_{aa}\left(\xi\right),\\ \displaystyle\sum_{a,b=1}^n\widehat{T}^{aa}{}_{,bb}\left(\xi\right)=\sum_{a,b=1}^nT_{aa;bb}\left(\xi\right)\quad\text{and}\\ \displaystyle\sum_{a,b=1}^n\widehat{T}^{ab}{}_{,ab}\left(\xi\right)=\sum_{a,b=1}^n\widehat{T}^{ab}{}_{,ba}\left(\xi\right)=\sum_{a,b=1}^nT_{ab;ab}\left(\xi\right)\end{cases}\right\}\quad\text{if } j=2, \tag{2.53}$$

$$\left.\begin{cases}\displaystyle\sum_{a=1}^n\widehat{T}^{aa}\left(\xi\right)=0\quad\text{and}\quad\sum_{a,b=1}^n\Big(\widehat{T}^{aab}{}_{,b}\left(\xi\right)+\widehat{T}^{aba}{}_{,b}\left(\xi\right)\\ \displaystyle\quad+\widehat{T}^{abb}{}_{,a}\left(\xi\right)\Big)=\sum_{a,b=1}^n\left(T_{aab;b}\left(\xi\right)+T_{aba;b}\left(\xi\right)+T_{abb;a}\left(\xi\right)\right)\end{cases}\right\}\quad\text{if } j=3, \tag{2.54}$$

and

$$\sum_{a,b=1}^{n}\left(\widehat{T}^{aabb}(\xi)+\widehat{T}^{abab}(\xi)+\widehat{T}^{abba}(\xi)\right)$$
$$=\sum_{a,b=1}^{n}\left(T_{aabb}(\xi)+T_{abab}(\xi)+T_{abba}(\xi)\right)\quad\text{if } j=4. \tag{2.55}$$

We then obtain (2.27) by putting together (2.50), (2.51) and (2.52) and using the identities

$$c\left(n,k,0,k-2,l',0,0\right)=\frac{n-2k-2}{2\left(n+l'-2k-1\right)!}$$

and

$$\mathrm{B}\left(\frac{n}{2}+l'-2k,2k-l'+4\right)$$
$$=\frac{4\left(2k-l'-1\right)\left(2k-l'-2\right)}{\left(n-2\right)\left(n-4\right)}\mathrm{B}\left(\frac{n}{2}+l'-2k,2k-l'-2\right).$$

The estimates (2.28)–(2.32) follow in the same way from (2.50), (2.51) and (2.52)–(2.55) by using the identities

$$\begin{aligned}
&c\left(n,k,1,k-2,l',0,0\right)=-\frac{\left(n-2k-2\right)\left(n-2k\right)}{4\left(n+l'-2k\right)!},\\
&c\left(n,k,2,k-2,l',0,0\right)=\frac{\left(n-2k-2\right)\left(n-2k\right)\left(n-2k+2\right)}{32\left(n+l'-2k+1\right)!},\\
&2c\left(n,k,2,k-2,l',0,1\right)+c\left(n,k,2,k-2,l',1,1\right)\\
&\qquad=4c\left(n,k,2,k-2,l',0,0\right)+c\left(n,k,2,k-2,l',1,0\right)\\
&\qquad=2c\left(n,k,2,k-3,l',0,0\right)+c\left(n,k,2,k-3,l',1,0\right)\\
&\qquad=-\frac{\left(n-2k-2\right)\left(n-2k\right)\left(n+2l'-2k\right)}{8\left(n+l'-2k+1\right)!},\\
&24c\left(n,k,3,k-3,l',0,0\right)+2c\left(n,k,3,k-3,l',1,0\right)\\
&\qquad=\frac{\left(n-2k-2\right)\left(n-2k\right)\left(n-2k+2\right)\left(n+2l'-2k\right)}{8\left(n+l'-2k+2\right)!}\\
&24c\left(n,k,4,k-4,l',0,0\right)+2c\left(n,k,4,k-4,l',1,0\right)+c\left(n,k,4,k-4,l',2,0\right)\\
&\qquad=\frac{\left(n-2k-2\right)\left(n-2k\right)\left(n-2k+2\right)\left(n+2l'-2k\right)\left(n+2l'-2k+2\right)}{64\left(n+l'-2k+3\right)!}.
\end{aligned}$$

and

$$\begin{aligned}
&\mathrm{B}\left(\frac{n}{2}+l'-l-k-2,k+l-l'+2\right)\\
&\qquad=\frac{2\left(k+l-l'+1\right)}{n-2}\mathrm{B}\left(\frac{n}{2}+l'-l-k-2,k+l-l'+1\right)\\
&\qquad=\frac{4\left(k+l-l'+1\right)\left(k+l-l'\right)}{\left(n-2\right)\left(n-4\right)}\mathrm{B}\left(\frac{n}{2}+l'-l-k-2,k+l-l'\right).
\end{aligned}$$

This ends the proof of Step 2.3. $\square$

As regards the integral in the denominator of $I_{k,f,g}(u)$, we obtain the following:

Step 2.4. *Assume that $n \geq 2k+1$ and g satisfies (2.2) for some point $\xi \in M$. Then, for every smooth function f in M, as $\mu \to 0$,*

$$\int_M f U_\mu^{2_k^*} dv_g = \frac{\omega_n}{2^n} f(\xi) - \frac{\omega_n \Delta f(\xi) \mu^2}{2^{n+1}(n-2)} + \frac{\omega_n \Delta^2 f(\xi) \mu^4}{2^{n+3}(n-2)(n-4)} + \mathrm{o}\left(\mu^4\right). \quad (2.56)$$

Proof of Step 2.4. By remarking that $U_\mu^{2_k^*}$ does not depend on k in $B(0, r_0)$, we obtain that (2.56) is in fact identical to an estimate obtained by Esposito and Robert [13] in the case where $k=2$ (note that in our case, $\mathrm{Ric}(\xi) = 0$ and $\nabla \mathrm{S}(\xi) = 0$ since we are working with conformal normal coordinates, see (2.3) and (2.4)). □

We can now end the proof of Proposition 2.1 by putting together the results of Steps 2.1– 2.4:

End of proof of Proposition 2.1. We assume that $k \geq 2$ and refer to Aubin [2] for the case where $k=1$. By using (2.56), we obtain

$$\left(\int_M f U_\mu^{2_k^*} dv_g\right)^{-\frac{n-2k}{n}} = \left(\frac{\omega_n}{2^n} f(\xi)\right)^{-\frac{n-2k}{n}} \left[1 + \frac{(n-2k)\Delta f(\xi) \mu^2}{2n(n-2) f(\xi)} - \frac{n-2k}{4n(n-2)} \left(\frac{\Delta^2 f(\xi)}{2(n-4) f(\xi)} - \frac{(n-k)(\Delta f(\xi))^2}{n(n-2) f(\xi)^2}\right) \mu^4 + \mathrm{o}\left(\mu^4\right)\right]. \quad (2.57)$$

We let f_1, f_2, T_1, T_2, T_3, T_4, T_5 and Z be as in Step 2.1. Since $k \geq 1$, by integrating by parts, we obtain

$$\begin{aligned}\int_M U_\mu \Delta^{k-1}(f_1 U_\mu) dv_g &= \int_M \Delta(f_1 U_\mu) \Delta^{k-2} U_\mu \, dv_g \\ &= \int_M (U_\mu \Delta f_1 - 2(\nabla f_1, \nabla U_\mu) + f_1 \Delta U_\mu) \Delta^{k-2} U_\mu \, dv_g. \quad (2.58)\end{aligned}$$

By integrating by parts again, it follows from (2.7) and (2.58) that

$$\begin{aligned}\int_M U_\mu P_{2k} U_\mu \, dv_g &= \int_M U_\mu \Delta^k U_\mu \, dv_g + k \int_M \big(((k-1) f_2 + \Delta f_1) U_\mu \\ &\quad + ((k-1) T_1 - 2\nabla f_1, \nabla U_\mu) + \big((k-1) T_2 - f_1 g, \nabla^2 U_\mu\big)\big) \Delta^{k-2} U_\mu \, dv_g \\ &\quad + k(k-1)(k-2) \int_M \big(\left(T_3, \nabla^2 U_\mu\right) + \left(T_4, \nabla^3 U_\mu\right)\big) \Delta^{k-3} U_\mu \, dv_g \\ &\quad + k(k-1)(k-2)(k-3) \int_M \left(T_5, \nabla^4 U_\mu\right) \Delta^{k-4} U_\mu dv_g + \int_M U_\mu Z U_\mu \, dv_g. \quad (2.59)\end{aligned}$$

By using (2.4), we obtain

$$\sum_{a,b=1}^n \mathrm{P}_{aa;bb}(\xi) = \sum_{a,b=1}^n \mathrm{P}_{ab;ab}(\xi) = \sum_{a,b=1}^n \mathrm{P}_{ab;ba}(\xi) = -\frac{|\mathrm{W}(\xi)|^2}{12(n-1)}. \quad (2.60)$$

By using (2.3), (2.4) and (2.60) together with straightforward computations, we obtain

$$\sum_{a=1}^n (T_3)_{aa}(\xi) = -\frac{n+3k+1}{36(n-1)} |\mathrm{W}(\xi)|^2 = -\frac{n+3k+1}{36n(n-1)} |\mathrm{W}(\xi)|^2 \sum_{a=1}^n g_{aa}(\xi),$$

$$\sum_{a,b=1}^{n}\left((T_4)_{aab;b}(\xi)+(T_4)_{aba;b}(\xi)+(T_4)_{abb;a}(\xi)\right)=-\frac{k+1}{6(n-1)}\left|\mathrm{W}(\xi)\right|^2$$
$$=\frac{k+1}{(n-1)(n+2)}\sum_{a,b=1}^{n}\left((\nabla S\otimes g)_{aab;b}(\xi)+(\nabla S\otimes g)_{aba;b}(\xi)+(\nabla S\otimes g)_{abb;a}(\xi)\right)$$

and

$$\sum_{a,b=1}^{n}\left((T_5)_{aabb}(\xi)+(T_5)_{abab}(\xi)+(T_5)_{abba}(\xi)\right)=-\frac{k+1}{10(n-1)}\left|\mathrm{W}(\xi)\right|^2$$
$$=-\frac{(k+1)\left|\mathrm{W}(\xi)\right|^2}{10n(n-1)(n+2)}\sum_{a,b=1}^{n}\left((g\otimes g)_{aabb}(\xi)+(g\otimes g)_{abab}(\xi)+(g\otimes g)_{abba}(\xi)\right).$$

By using these identities together with (2.30)–(2.32) and remarking that

$$\left(\nabla S\otimes g,\nabla^3U_\mu\right)=-\Delta\left(\nabla S,\nabla U_\mu\right)-2\left(\nabla^2S,\nabla^2U_\mu\right)-\left(\nabla^3S,\nabla U_\mu\otimes g\right),$$

and

$$\sum_{a=1}^{n}\left(\nabla^2S\right)_{aa}(\xi)=-\frac{1}{6}\left|\mathrm{W}(\xi)\right|^2=-\frac{1}{6n}\left|\mathrm{W}(\xi)\right|^2\sum_{a=1}^{n}g_{aa}(\xi),$$

we obtain that for $k\geq 3$,

$$\begin{aligned}&\int_M\left(T_3,\nabla^2U_\mu\right)\Delta^{k-3}U_\mu\,dv_g\\
&\quad=\frac{n+3k+1}{36n(n-1)}\left|\mathrm{W}(\xi)\right|^2\int_M\Delta U_\mu\Delta^{k-3}U_\mu\,dv_g+\begin{cases}\mathrm{O}\left(\mu^4\right)&\text{if } n=2k+4\\ \mathrm{o}\left(\mu^4\right)&\text{if } n>2k+4\end{cases}\\
&\quad=\frac{n+3k+1}{36n(n-1)}\left|\mathrm{W}(\xi)\right|^2\int_M U_\mu\Delta^{k-2}U_\mu\,dv_g+\begin{cases}\mathrm{O}\left(\mu^4\right)&\text{if } n=2k+4\\ \mathrm{o}\left(\mu^4\right)&\text{if } n>2k+4,\end{cases}\end{aligned}\tag{2.61}$$

$$\begin{aligned}&\int_M\left(T_4,\nabla^3U_\mu\right)\Delta^{k-3}U_\mu\,dv_g=-\frac{k+1}{(n-1)(n+2)}\int_M\Bigg(\Delta\left(\nabla S,\nabla U_\mu\right)\\
&\quad+\frac{1}{3n}\left|\mathrm{W}(\xi)\right|^2\Delta U_\mu+\left(\nabla^3S,\nabla U_\mu\otimes g\right)\Bigg)\Delta^{k-3}U_\mu\,dv_g+\begin{cases}\mathrm{O}\left(\mu^4\right)&\text{if } n=2k+4\\ \mathrm{o}\left(\mu^4\right)&\text{if } n>2k+4\end{cases}\\
&\quad=-\frac{k+1}{3n(n-1)(n+2)}\int_M\Big(\left|\mathrm{W}(\xi)\right|^2U_\mu+3n\left(\nabla S,\nabla U_\mu\right)\Big)\Delta^{k-2}U_\mu\,dv_g\\
&\quad-\frac{k+1}{(n-1)(n+2)}\int_M U_\mu\Delta^{k-3}\left(\nabla^3S,\nabla U_\mu\otimes g\right)dv_g+\begin{cases}\mathrm{O}\left(\mu^4\right)&\text{if } n=2k+4\\ \mathrm{o}\left(\mu^4\right)&\text{if } n>2k+4\end{cases}\end{aligned}\tag{2.62}$$

and for $k\geq 4$,

$$\begin{aligned}&\int_M\left(T_5,\nabla^4U_\mu\right)\Delta^{k-4}U_\mu\,dv_g\\
&\quad=-\frac{(k+1)\left|\mathrm{W}(\xi)\right|^2}{10n(n-1)(n+2)}\int_M\left(\Delta^2U_\mu\right)\Delta^{k-4}U_\mu\,dv_g+\begin{cases}\mathrm{O}\left(\mu^4\right)&\text{if } n=2k+4\\ \mathrm{o}\left(\mu^4\right)&\text{if } n>2k+4\end{cases}\\
&\quad=-\frac{(k+1)\left|\mathrm{W}(\xi)\right|^2}{10n(n-1)(n+2)}\int_M U_\mu\Delta^{k-2}U_\mu\,dv_g+\begin{cases}\mathrm{O}\left(\mu^4\right)&\text{if } n=2k+4\\ \mathrm{o}\left(\mu^4\right)&\text{if } n>2k+4.\end{cases}\end{aligned}\tag{2.63}$$

It follows from (2.59) and (2.61)–(2.63) that

$$\begin{aligned}\int_M U_\mu P_{2k} U_\mu \, dv_g = \int_M U_\mu \Delta^k U_\mu \, dv_g + k \int_M \bigg(\bigg((k-1) f_2 + \Delta f_1 \\ + \left(\frac{(k-1)(k-2)(n+3k+1)}{36n(n-1)} - \frac{(k-1)(k-2)(k+1)(3k+1)}{30n(n-1)(n+2)} \right) |\mathrm{W}(\xi)|^2 \bigg) U_\mu \\ + \left((k-1) T_1 - 2\nabla f_1 - \frac{(k+1)(k-1)(k-2)}{(n-1)(n+2)} \nabla S, \nabla U_\mu \right) \\ + \left((k-1) T_2 - f_1 g, \nabla^2 U_\mu \right) \bigg) \Delta^{k-2} U_\mu \, dv_g \\ + \int_M U_\mu \left(Z U_\mu - \frac{k(k+1)(k-1)(k-2)}{(n-1)(n+2)} \Delta^{k-3} \left(\nabla^3 S, \nabla U_\mu \otimes g \right) \right) dv_g \\ + \begin{cases} \mathrm{O}\left(\mu^4\right) & \text{if } n = 2k+4 \\ \mathrm{o}\left(\mu^4\right) & \text{if } n > 2k+4. \end{cases} \qquad (2.64)\end{aligned}$$

By using (2.3), (2.4) and (2.60) together with straightforward computations, we obtain

$$f_1(\xi) = 0, \quad \Delta f_1(\xi) = \frac{n-2}{24(n-1)} |\mathrm{W}(\xi)|^2, \quad f_2(\xi) = -\frac{3n+2k-4}{144(n-1)} |\mathrm{W}(\xi)|^2,$$
$$\sum_{a=1}^n (T_1)_{a;a}(\xi) = -\frac{3n+4k-2}{72(n-1)} |\mathrm{W}(\xi)|^2, \quad \sum_{a=1}^n (T_2)_{aa}(\xi) = 0$$

and

$$\sum_{a,b=1}^n (T_2)_{aa;bb}(\xi) = \sum_{a,b=1}^n (T_2)_{ab;ab}(\xi) = \sum_{a,b=1}^n (T_2)_{ab;ba}(\xi) = -\frac{k+1}{18(n-1)} |\mathrm{W}(\xi)|^2.$$

By using these identities together with (2.20), (2.27)–(2.29), (2.57) and (2.64), we obtain that (2.6) holds true with $C(n,k)$ defined as

$$\begin{aligned}C(n,k) := \frac{(n-3)(n-5)!\,k!}{16(k-1)(n-2k-2)} \\ \times \sum_{l=k-2}^{2k-4} \frac{l!}{(l-k+2)!(2k-l-4)!(n+l-2k+1)!} \bigg(8(n+l-2k)(n+l-2k+1) \\ \times \bigg(\frac{(k-1)(3n+2k-4)}{144} - \frac{n-2}{24} - \frac{(k-1)(k-2)(n+3k+1)}{36n} \\ + \frac{(k-1)(k-2)(k+1)(3k+1)}{30n(n+2)} \bigg) + 4(n-2k)(n+l-2k+1) \bigg(\frac{n-2}{12} \\ - \frac{(k-1)(3n+4k-2)}{72} + \frac{(k+1)(k-1)(k-2)}{6(n+2)} \bigg) \\ + (n-2k) \bigg(\frac{(k+1)(k-1)(n-2k-2l+4)}{18} \\ - \frac{(n-2)(2(n-2k+2) - n(n+2l-2k))}{24} \bigg) \bigg)\end{aligned}$$

$$\times \mathrm{B}\left(\frac{n}{2}-k-1, l+1\right)^{-1} \begin{cases} 2\chi_{\{l=k-2\}} & \text{if } n = 2k+4 \\ \mathrm{B}\left(\frac{n}{2}+l-2k, 2k-l-2\right) & \text{otherwise} \end{cases}$$

$$= \frac{(n-3)\,(n-5)!\,k!}{5760n\,(n+2)\,(k-1)\,(n-2k-2)}$$

$$\times \sum_{l=k-2}^{2k-4} \frac{l!\,c\,(n,k,l)}{(l-k+2)!\,(2k-l-4)!\,(n+l-2k+1)!}$$

$$\times \mathrm{B}\left(\frac{n}{2}-k-1, l+1\right)^{-1} \begin{cases} 2\chi_{\{l=k-2\}} & \text{if } n = 2k+4 \\ \mathrm{B}\left(\frac{n}{2}+l-2k, 2k-l-2\right) & \text{otherwise,} \end{cases} \tag{2.65}$$

where

$$\begin{aligned} c\,(n,k,l) := {} & 4\,(n+l-2k)\,(n+l-2k+1)\,(5n\,(n+2)\,(k-1)\,(3n+2k-4) \\ & - 30n\,(n+2)\,(n-2) - 20\,(n+2)\,(k-1)\,(k-2)\,(n+3k+1) \\ & + 24\,(k-1)\,(k-2)\,(k+1)\,(3k+1)) + 20n\,(n-2k)\,(n+l-2k+1) \\ & \times (6\,(n+2)\,(n-2) - (n+2)\,(k-1)\,(3n+4k-2) + 12\,(k+1)\,(k-1)\,(k-2)) \\ & + 5n\,(n+2)\,(n-2k)\,(4\,(k+1)\,(k-1)\,(n-2k-2l+4) \\ & - 3\,(n-2)\,(2\,(n-2k+2) - n\,(n+2l-2k))). \end{aligned}$$

By letting $k := 3+a$, $n := 2k+4+b$ and $l := k-2+c$ and using the software *Maple* to expand the expression of $c\,(n,k,l)$, we then obtain

$$\begin{aligned} c\,(n,k,l) = {} & 4(15ab^3+1200a^2b+3880ab+1920+10656a+480b+4528a^2+624a^3 \\ & +40b^2+450ab^2+80a^3b+80a^2b^2+32a^4)c^2+2(71552a^2b+414912a+500a^2b^3+247984ab \\ & +31840a^3+53660ab^2+3200a^4+640a^3b^2+11020b^3+150ab^4+128a^5+9056a^3b+660b^4 \\ & +161440a^2+448a^4b+15b^5+311040b+10520a^2b^2+4830ab^3+426240+85840b^2)c \\ & +128a^6+576a^5b+1088a^4b^2+1020a^3b^3+560a^2b^4+150ab^5+15b^6+3904a^5+14720a^4b \\ & +21896a^3b^2+15940a^2b^3+5640ab^4+720b^5+49408a^4+149280a^3b+167032a^2b^2 \\ & +81120ab^3+13780b^4+332096a^3+754720a^2b+563824ab^2+134240b^3+1250304a^2 \\ & +1900224ab+704640b^2+2499840a+1900800b+2073600. \end{aligned} \tag{2.66}$$

Since all the coefficients in this expression are positive, it follows that $C\,(n,k)$ is positive whenever $k \geq 3$, $n \geq 2k+4$ and $l \geq k-2$. Furthermore, in the case where $k=2$ and $l=0$, we find

$$c\,(n,2,0) = 5n(n+2)(n-4)^2(n^2-4n-4) > 0 \quad \forall n \geq 8.$$

Therefore, in all cases, we find that $C\,(n,k)$ is positive. This ends the proof of Proposition 2.1. □

We can now prove Theorem 1.1 by using Proposition 2.1.

Proof of Theorem 1.1 in the case where $n \geq 2k+4$. Let $\xi \in M$ be a maximal point of f and $\widetilde{g} = \varphi^{4/(n-2)}g$ be a conformal metric to g such that $\varphi\,(\xi) = 1$ and $\det \widetilde{g}\,(x) = 1$ for all x in a neighborhood of the point ξ. Remark that since ξ

is a maximal point of f, if $\Delta_g f(\xi) = 0$, then $\nabla^j f = 0$ for all $j \in \{1,2,3\}$. In particular, since $\varphi(\xi) = 1$, it follows that

$$\Delta_{\widetilde{g}} f(\xi) = 0 \quad \text{and} \quad \Delta_{\widetilde{g}}^2 f(\xi) = \Delta_g^2 f(\xi), \tag{2.67}$$

where Δ_g and $\Delta_{\widetilde{g}}$ are the Laplace–Beltrami operators with respect to the metrics g and $\widetilde{g}$, respectively, and the covariant derivatives, the Ricci tensor and the multiple inner product in the right-hand side of the second identity are with respect to the metric g. Let $c(n,k)$ be the constant defined as

$$c(n,k) := \begin{cases} 0 & \text{if } n = 2k+4 \\ \dfrac{(n-2k)(2k-1)!}{8n(n-2)(n-4)C(n,k)} \mathrm{B}\left(\dfrac{n}{2}-k, 2k\right)^{-1} & \text{if } n > 2k+4, \end{cases} \tag{2.68}$$

where $C(n,k)$ is as in (2.6) (see also (2.65)). By applying Proposition 2.1 together with (2.67) and the fact that $|W|$ is conformally invariant, we then obtain that if (1.2) and (1.3) hold true, then

$$\inf_{u \in C^{2k}(M)\backslash\{0\}} I_{k,f,\widetilde{g}}(u) < \omega_n^{\frac{2k}{n}} (2k-1)! \,\mathrm{B}\left(\frac{n}{2}-k, 2k\right)^{-1} \left(\max_{x\in M} f(x)\right)^{-\frac{n-2k}{n}}. \tag{2.69}$$

On the other hand, by conformal invariance of the operator P_{2k}, we obtain

$$\inf_{u \in C^{2k}(M)\backslash\{0\}} I_{k,f,\widetilde{g}}(u) = \inf_{u \in C^{2k}(M)\backslash\{0\}} I_{k,f,g}(u). \tag{2.70}$$

By putting together (2.69) and (2.70) and applying Theorem 3 of Mazumdar [30], we then obtain that the conclusions of Theorem 1.1 hold true. □

3. The remaining cases

This section is devoted to the proof of Theorem 1.1 in the remaining case where $2k+1 \le n \le 2k+3$ together with the following result in the case where g is conformally flat in some open subset of the manifold:

Theorem 3.1. *Let $k \ge 1$ be an integer, (M,g) be a smooth, closed Riemannian manifold of dimension $n \ge 2k+1$ and f be a smooth positive function in M. Assume that the operator P_{2k} is coercive and there exists a maximal point ξ of f such that $m(\xi) > 0$ (see (3.2) for the definition of the mass), $\nabla^j f(\xi) = 0$ for all $j \in \{1,\dots,n-2k\}$ and g is conformally flat in some neighborhood of the point ξ. Then there exists a nontrivial solution $u \in C^{2k}(M)$ to the equation (1.1), which minimizes the energy functional (2.1). If moreover the Green's function of the operator P_{2k} is positive, then u is positive, which implies that the Q-curvature of order $2k$ of the metric $u^{4/(n-2k)}g$ is equal to $\frac{2}{n-2k}f$.*

Remark that Theorem 1.2 is now a direct consequence of Theorems 1.1 and 3.1.

Throughout this section, we fix a point $\xi \in M$ and assume that $2k+1 \le n \le 2k+3$ or g is conformally flat in some neighborhood of ξ. In these cases, our proofs are based on the method of Schoen [36] for the resolution of the remaining cases of the Yamabe problem, which has been extended to the $k=2$ case by Gursky and Malchiodi [21] and Hang and Yang [22,23]. We consider a family of global test functions involving the Green's function and derive an expression for the energy functional $I_{k,f,g}$ (see (2.1)) associated with the equation (1.1). Then, analogously as in the case $n \ge 2k+4$, by using the expansion obtained in Proposition 2.1, we obtain the existence of a nontrivial solution to the equation (1.1) under a positivity assumption on the mass of the operator P_{2k}.

We now discuss the definition of the mass. By applying a conformal change of metric, we may assume that

$$\begin{cases} g \text{ satisfies (2.2) in some neighborhood } \Omega \text{ of } \xi \text{ if } 2k+1 \le n \le 2k+3 \\ g \text{ is flat in some neighborhood } \Omega \text{ of } \xi \text{ if } n \ge 2k+4. \end{cases} \tag{3.1}$$

Then, in the geodesic normal coordinates at ξ determined by g, the Green's function $G_{2k}(x) := G_{2k}(x,\xi)$ of the operator P_{2k} has the expansion

$$G_{2k}(x) = b_{n,k}\, d_g(x,\xi)^{2k-n} + m(\xi) + \mathrm{o}(1) \tag{3.2}$$

as $x \to \xi$ (see Lee and Parker [28] for $k=1$ and Michel [31] for $k \ge 2$), where $m(\xi) \in M$ is called the mass of the operator P_{2k} at the point ξ and the constant $b_{n,k}$ is defined as

$$b_{n,k}^{-1} := 2^{k-1}(k-1)!\,(n-2)(n-4)\cdots(n-2k)\,\omega_{n-1}.$$

It is important to point out that the sign of $m(\xi)$ does not depend on our choice of conformal metric (see Michel [31, Théorème 3.1]).

Now that the mass is defined, we consider the regular part of the Green's function, which plays a crucial role in the proofs of our theorems. It follows from (3.2) that there exists a continuous function h_{2k} in M such that $h_{2k}(\xi) = m(\xi)$ and

$$G_{2k}(x) = b_{n,k}\, d_g(x,\xi)^{2k-n} + h_{2k}(x) \qquad \forall x \in M \backslash \{\xi\}. \tag{3.3}$$

Furthermore, we have that $h_{2k} \in C^\infty(\Omega)$ in the case where g is flat in Ω and $h_{2k} \in W^{2k,p}(\Omega)$ for all $p \in [1, n/(n-4))$ if $n \ge 5$ and $p \in [1,\infty)$ if $n \in \{3,4\}$ in the case where $2k+1 \le n \le 2k+3$ and g satisfies (2.2) in Ω. This follows from classical elliptic regularity theory (see Agmon, Douglis and Nirenberg [1]) together with the fact that $P_{2k}h_{2k} = \Delta^k h_{2k} = 0$ in Ω in the case where g is flat in Ω and $P_{2k}h_{2k} = \mathrm{O}(d_g(\cdot,\xi)^{4-n})$ in Ω in the case where g satisfies (2.2) in Ω (see [31, Lemme 2.2]).

For every $\mu > 0$, letting χ and U_μ be as in Section 2, we consider the test functions V_μ defined as

$$V_\mu(x) := U_\mu(x) + b_{n,k}^{-1}\, \mu^{\frac{n-2k}{2}} \left(\chi(d_g(x,\xi)) h_{2k}(x) + (1-\chi(d_g(x,\xi)))\, G_{2k}(x)\right) \tag{3.4}$$

for all $x \in M$. Note that $V_\mu \in W^{2k,2n/(n+2k)}(M)$ so that in particular the integral $\int_M V_\mu P_{2k} V_\mu \, dv_g$ is well defined. We then obtain the following:

Proposition 3.1. *Let $k \ge 1$ be an integer, (M,g) be a smooth, closed Riemannian manifold of dimension $n \ge 2k+1$, f be a smooth positive function in M and ξ be a point in M such that $\nabla^j f(\xi) = 0$ for all $j \in \{1,\dots,n-2k\}$ and (3.1) holds true. Let $I_{k,f,g}$ be as in (2.1) and V_μ be as in (3.4). Then, as $\mu \to 0$,*

$$\begin{aligned} I_{k,f,g}(V_\mu) = \omega_n^{\frac{2k}{n}}\,(2k-1)!\,\mathrm{B}\left(\frac{n}{2}-k, 2k\right)^{-1} f(\xi)^{-\frac{n-2k}{n}} \\ \times \left(1 - b_{n,k}^{-1}\,\mathrm{B}\left(\frac{n}{2},\frac{n}{2}\right)^{-1} \mathrm{B}\left(\frac{n}{2},k\right) m(\xi)\,\mu^{n-2k} + \mathrm{o}\left(\mu^{n-2k}\right)\right). \end{aligned} \tag{3.5}$$

Proof of Proposition 3.1. The first step in the proof is as follows:

Step 3.1. *Assume that g satisfies (3.1) for some point $\xi \in M$. Then, as $\mu \to 0$,*

$$\int_M V_\mu P_{2k} V_\mu \, dv_g = 2^{2k-n} \omega_n (2k-1)! \, \mathrm{B}\left(\frac{n}{2} - k, 2k\right)^{-1} \times \left(1 + b_{n,k}^{-1} \mathrm{B}\left(\frac{n}{2}, \frac{n}{2}\right)^{-1} \mathrm{B}\left(\frac{n}{2}, k\right) m(\xi) \, \mu^{n-2k} + \mathrm{o}\left(\mu^{n-2k}\right)\right). \tag{3.6}$$

Proof of Step 3.1. We write

$$V_\mu(x) = b_{n,k}^{-1} \mu^{\frac{n-2k}{2}} G_{2k}(x) + \Big(\underbrace{U_\mu(x) - \mu^{\frac{n-2k}{2}} \chi(d_g(x,\xi)) \, d_g(x,\xi)^{2k-n}}_{W_\mu(x)}\Big)$$

for all $x \in M \backslash \{\xi\}$. Straightforward estimates give

$$\begin{aligned}
\int_M V_\mu P_{2k} V_\mu \, dv_g &= \int_{B(\xi, 2r_0)} V_\mu P_{2k} W_\mu \, dv_g \\
&= \int_{B(\xi, r_0)} \left(U_\mu + b_{n,k}^{-1} \mu^{\frac{n-2k}{2}} h_{2k}\right) P_{2k} W_\mu \, dv_g \\
&\quad + \mathrm{O}\left(\mu^{\frac{n-2k}{2}} \int_{B(\xi, 2r_0) \backslash B(\xi, r_0)} |P_{2k} W_\mu| \, dv_g\right) \\
&= \int_{B(\xi, r_0)} U_\mu P_{2k} W_\mu \, dv_g + b_{n,k}^{-1} \mu^{\frac{n-2k}{2}} \int_{B(\xi, r_0)} h_{2k} P_{2k} W_\mu \, dv_g \\
&\quad + \mathrm{O}\left(\mu^{n-2k} \sum_{|\alpha| \le 2k} \int_{B(0, 2r_0) \backslash B(0, r_0)} \left|\partial^\alpha \left[\left(\mu^2 + |x|^2\right)^{\frac{2k-n}{2}} - |x|^{2k-n}\right]\right| dx\right) \\
&= \int_{B(\xi, r_0)} U_\mu P_{2k} W_\mu \, dv_g + b_{n,k}^{-1} \mu^{\frac{n-2k}{2}} \int_{B(\xi, r_0)} h_{2k} P_{2k} W_\mu \, dv_g + \mathrm{o}(\mu^{n-2k}).
\end{aligned} \tag{3.7}$$

We claim that

$$\left|P_{2k} W_\mu - \Delta^k U_\mu\right| \le C \mu^{\frac{n-2(k-2)}{2}} d_g(x,\xi)^{2-n} \quad \forall x \in B(\xi, r_0) \setminus \{\xi\} \tag{3.8}$$

for some constant C independent of x, μ and ξ. Assuming (3.8) and proceeding as in Step 2.3, it then follows from (3.7) that

$$\begin{aligned}
\int_M V_\mu P_{2k} V_\mu \, dv_g &= \int_{B(\xi, r_0)} U_\mu \Delta^k U_\mu \, dv_g + b_{n,k}^{-1} \mu^{\frac{n-2k}{2}} \int_{B(\xi, r_0)} h_{2k} \Delta^k U_\mu \, dv_g \\
&\quad + \mathrm{O}\left(\mu^{n-2(k-1)} \int_{B(0, r_0)} |x|^{2-n} \left(\mu^2 + |x|^2\right)^{\frac{2k-n}{2}} dx\right) + \mathrm{o}\left(\mu^{n-2k}\right) \\
&= \int_{B(0, r_0)} \widetilde{U}_\mu \Delta_0^k \widetilde{U}_\mu \, dx + b_{n,k}^{-1} \mu^{\frac{n-2k}{2}} \int_{B(0, r_0)} h_{2k}\left(\exp_\xi x\right) \Delta_0^k \widetilde{U}_\mu \, dx + \mathrm{o}\left(\mu^{n-2k}\right) \\
&= 2^{2k-1} (2k-1)! \, \omega_{n-1} \mathrm{B}\left(\frac{n}{2} - k, 2k\right)^{-1} \left(\int_0^{(r_0/\mu)^2} \frac{r^{\frac{n-2}{2}}}{(1+r)^n} dr \right. \\
&\quad \left. + b_{n,k}^{-1} m(\xi) \, \mu^{n-2k} \int_0^{(r_0/\mu)^2} \frac{r^{\frac{n-2}{2}}}{(1+r)^{\frac{n+2k}{2}}} dr\right) + \mathrm{o}\left(\mu^{n-2k}\right)
\end{aligned}$$

$$= 2^{2k-n}\left(2k-1\right)!\,\omega_n B\left(\frac{n}{2}-k,2k\right)^{-1}$$
$$\times\left(1+b_{n,k}^{-1}\,\mathrm{B}\left(\frac{n}{2},\frac{n}{2}\right)^{-1}\mathrm{B}\left(\frac{n}{2},k\right)m\left(\xi\right)\mu^{n-2k}+\mathrm{o}\left(\mu^{n-2k}\right)\right).$$

Therefore, it remains to prove (3.8) to complete the proof of Step 3.2. Remark that (3.8) is clearly satisfied with $C=0$ in the case where $n\geq 2k+4$ and g is flat in Ω. Therefore, we may assume in what follows that we are in the case where $2k+1\leq n\leq 2k+3$ and g satisfies (2.2) in Ω. By using (2.7), we obtain

$$P_{2k}=\Delta^k+k\Delta^{k-1}\left(f_1\cdot\right)+k\left(k-1\right)\Delta^{k-2}\left(\left(T_1,\nabla\right)+\left(T_2,\nabla^2\right)\right)$$
$$+k\left(k-1\right)\left(k-2\right)\Delta^{k-3}\left(T_4,\nabla^3\right)+Z,\quad(3.9)$$

where Z is a smooth linear operator of order less than $2k-3$ if $k\geq 2$, $Z:=0$ if $k=1$. By induction, one can check that

$$\Delta^{k-1}\left(f_1\cdot\right)=f_1\Delta^{k-1}-2\left(k-1\right)\left(\nabla f_1,\nabla\Delta^{k-2}\right)+\mathrm{o}^{2k-3},\quad(3.10)$$

$$\Delta^{k-2}\left(T_1,\nabla\right)=\left(T_1,\nabla\Delta^{k-2}\right)+\mathrm{o}^{2k-3},\quad(3.11)$$

$$\Delta^{k-2}\left(T_2,\nabla^2\right)=\left(T_2,\nabla^2\Delta^{k-2}\right)-2\left(k-2\right)\left(\nabla T_2,\nabla^3\Delta^{k-3}\right)+\mathrm{o}^{2k-3}\quad(3.12)$$

and

$$\Delta^{k-3}\left(T_4,\nabla^3\right)=\left(T_4,\nabla^3\Delta^{k-3}\right)+\mathrm{o}^{2k-3},\quad(3.13)$$

where o^{2k-3} is as in the proof of Step 2.1. It follows from (2.3), (2.4) and (3.9)–(3.13) that

$$P_{2k}W_\mu=\Delta^kW_\mu+\frac{2k\left(k-1\right)\left(k+1\right)}{3\left(n-2\right)}\Big(\left(\mathrm{Ric},\nabla^2\Delta^{k-2}W_\mu\right)-\left(\delta\,\mathrm{Ric},\nabla\Delta^{k-2}W_\mu\right)$$
$$-\left(k-2\right)\left(\nabla\,\mathrm{Ric},\nabla^3\Delta^{k-2}W_\mu\right)\Big)+\mathrm{O}\Big(d_g\left(\cdot,\xi\right)^2\left|\nabla^{2k-2}W_\mu\right|$$
$$+d_g\left(\cdot,\xi\right)\left|\nabla^{2k-3}W_\mu\right|+\sum_{j=0}^{2k-4}\left|\nabla^jW_\mu\right|\Big)\quad(3.14)$$

in $M\backslash\left\{\xi\right\}$, uniformly with respect to μ and ξ. By using geodesic normal coordinates together with (2.3) and a Taylor expansion, it follows from (3.14) that

$$\left(P_{2k}W_\mu-\Delta^kU_\mu\right)\left(\exp_\xi x\right)=\left(P_{2k}W_\mu-\Delta^kW_\mu\right)\left(\exp_\xi x\right)$$
$$=\mathrm{O}\Big(\left|x\right|^2\left|\nabla^{2k-2}\widetilde{W}_\mu\left(x\right)\right|+\left|x\right|\left|\nabla^{2k-3}\widetilde{W}_\mu\left(x\right)\right|+\sum_{j=0}^{2k-4}\left|\nabla^j\widetilde{W}_\mu\left(x\right)\right|\Big)\quad(3.15)$$

uniformly with respect to $x\in B\left(0,r_0\right)\backslash\left\{0\right\}$, μ and ξ, where

$$\widetilde{W}_\mu\left(x\right):=\mu^{\frac{2k-n}{2}}U\left(x/\mu\right)-\mu^{\frac{n-2k}{2}}\left|x\right|^{2k-n}.$$

Similarly as in (2.40), we obtain that for every $j\in\mathbb{N}$, there exists a constant C_j independing of x and μ such that

$$\left|\nabla^j\widetilde{W}_\mu\left(x\right)\right|=\mu^{\frac{2k-n-2j}{2}}\left|\nabla^jW\left(x/\mu\right)\right|$$
$$\leq C_j\sum_{m=0}^{\lfloor j/2\rfloor}\mu^{\frac{2k-n-4j+4m}{2}}r^{\frac{j-2m}{2}}\left|\partial_r^{j-m}W\left(r/\mu^2\right)\right|\quad(3.16)$$

for all $x \in B(0, r_0) \setminus \{0\}$, where $r := |x|^2$ and

$$W(x) = W(r) := (1+r)^{(2k-n)/2} - r^{(2k-n)/2}.$$

Furthermore, it is easy to see that

$$\left|\partial_r^j W(r)\right| \le C'_j r^{\frac{2k-n-2j-2}{2}} \tag{3.17}$$

for some constant C'_j independend of r. We then obtain (3.8) by putting together (3.15)–(3.17). This completes the proof of Step 3.1. $\square$

Step 3.2. *Assume that g satisfies (3.1) for some point $\xi \in M$. Let f be a smooth function f in M such that $\nabla^j f(\xi) = 0$ for all $j \in \{1, \dots, n-2k\}$. Then, as $\mu \to 0$,*

$$\int_M f|V_\mu|^{2_k^*} dv_g = \frac{\omega_n}{2^n}\bigg(f(\xi) + 2_k^* b_{n,k}^{-1} \mathrm{B}\left(\frac{n}{2}, \frac{n}{2}\right)^{-1} \mathrm{B}\left(\frac{n}{2}, k\right) f(\xi)\, m(\xi)\, \mu^{n-2k} + \mathrm{o}\left(\mu^{n-2k}\right)\bigg). \tag{3.18}$$

Proof of Step 3.2. By using a Taylor expansion together with straightforward estimates, we obtain

$$\begin{aligned}
\int_M f|V_\mu|^{2_k^*} dv_g &= \int_{B(\xi,r_0)} f\left|U_\mu + b_{n,k}^{-1} \mu^{\frac{n-2k}{2}} h_{2k}\right|^{2_k^*} dv_g + \mathrm{O}(\mu^n) \\
&= f(\xi) \int_{B(\xi,r_0)} U_\mu^{2_k^*} dv_g + 2_k^* b_{n,k}^{-1} \mu^{\frac{n-2k}{2}} \int_{B(\xi,r_0)} f h_{2k} U_\mu^{2_k^*-1} dv_g \\
&\quad + \mathrm{O}\left(\int_{B(0,r_0)} \left(|x|^{n-2k+1}\left(\frac{\mu}{\mu^2+|x|^2}\right)^n + \mu^{n-2k}\left(\frac{\mu}{\mu^2+|x|^2}\right)^{2k}\right) dx + \mu^n\right) \\
&= \frac{\omega_n}{2^n} f(\xi) + \frac{n\,\omega_{n-1}}{n-2k} b_{n,k}^{-1} f(\xi)\, m(\xi)\, \mu^{n-2k} \int_0^{(r_0/\mu)^2} \frac{r^{\frac{n-2}{2}} dr}{(1+r)^{\frac{n+2k}{2}}} + \mathrm{o}\left(\mu^{n-2k}\right).
\end{aligned} \tag{3.19}$$

Then (3.18) follows from (2.45), (2.47) and (3.19). $\square$

We can now end the proofs of Proposition 3.1 and Theorems 1.1 and 1.2.

End of proof of Proposition 3.1. We obtain (3.5) by putting together (3.6) and (3.18). $\square$

Proofs of Theorem 1.1 in the case where $2k+1 \le n \le 2k+3$ and of Theorem 3.1. Let $\xi \in M$ be a maximal point of f such that $\nabla^j f(\xi) = 0$ for all $j \in \{1, \dots, n-2k\}$ (remark that for a maximal point, this is equivalent to (1.2) in the case where $2k+1 \le n \le 2k+3$). By applying Proposition 3.1 together with a conformal change of metric, we then obtain that if $m(\xi) > 0$, then there exists a function $V \in W^{2k,2n/(n+2k)}(M) \setminus \{0\}$ such that

$$I_{k,f,g}(V) < \omega_n^{\frac{2k}{n}} (2k-1)!\, \mathrm{B}\left(\frac{n}{2} - k, 2k\right)^{-1} \left(\max_{x \in M} f(x)\right)^{-\frac{n-2k}{n}}.$$

Remark that $W^{2k,2n/(n+2k)}(M) \hookrightarrow L^{2_k^*}(M)$ so that a density argument gives

$$\inf_{u \in C^{2k}(M) \setminus \{0\}} I_{k,f,g}(u) < \omega_n^{\frac{2k}{n}} (2k-1)!\, \mathrm{B}\left(\frac{n}{2} - k, 2k\right)^{-1} \left(\max_{x \in M} f(x)\right)^{-\frac{n-2k}{n}}.$$

We can then conclude the proofs of Theorems 1.1 and 3.1 by applying Theorem 3 of Mazumdar [30]. $\square$

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