NON-SYNCHRONIZED SOLUTIONS TO NONLINEAR ELLIPTIC SCHRÖDINGER SYSTEMS ON A CLOSED RIEMANNIAN MANIFOLD

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Abstract. On a smooth, closed Riemannian manifold, we study the question of proportionality of components, also called synchronization, of vector-valued solutions to nonlinear elliptic Schrödinger systems with constant coefficients. In particular, we obtain bifurcation results showing the existence of branches of non-synchronized solutions emanating from the constant solutions.

1. Introduction

On a smooth, closed Riemannian manifold (M,g) of dimension n, we consider vector-valued solutions $(u_1,u_2)\in C^2(M)^2$ to elliptic systems of the form

$$\begin{cases}\Delta_g u_1 = F_1(u_1,u_2) & \text{in } M\\ \Delta_g u_2 = F_2(u_1,u_2)\,, & \text{in } M\end{cases}\tag{1.1}$$

where F_1 and F_2 are C^1 functions and $\Delta_g := -\operatorname{div}\nabla$ is the Laplace–Beltrami operator. In particular, we are interested in the stationary nonlinear Schrödinger system

$$\begin{cases}\Delta_g u_1+\lambda_1 u_1 = a_{11}u_1^{q-1}+a_{12}u_2^{q-2}u_1 & \text{in } M\\ \Delta_g u_2+\lambda_2 u_2 = a_{21}u_1^{q-2}u_2+a_{22}u_2^{q-1} & \text{in } M\\ u_1,u_2>0 & \text{in } M,\end{cases}\tag{1.2}$$

where $\lambda_1,\lambda_2,a_{11},a_{12},a_{21},a_{22}\in\mathbb{R}$ and $q\in(2,\infty)$. In the cubic case $q=4$, the system (1.2) arises in particular in nonlinear optics (see

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for instance Akhmediev and Ankiewicz [1] and Kanna and Lakshmanan [20]) and the Hartree–Fock theory for Bose–Einstein condensates (see Esry, Greene, Burke and Bohn [12] and Timmermans [33]). Such systems have received considerable attention from mathematicians in recent years. Among many others, let us refer for instance to the work of Bartsch, Dancer and Wang [2], Clapp and Pistoia [5, 6], Gladiali, Grossi and Troestler [14, 15], Guo, Li and Wei [16], Guo and Liu [17], Li and Ma [22], Li and Villavert [23, 24], Lin and Wei [25], Liu and Wang [26], Peng, Peng and Wang [29], Sirakov [31], Soave and Zilio [32], Terracini and Verzini [34] and Wei and Wu [36] in the case where $M = \mathbb{R}^n$ (note that when (M, g) is the standard round sphere, $n \geq 3$, $\lambda_1 = \lambda_2 = n\,(n-2)/4$ and $q = 2n/\,(n-2)$, we can use stereographic projection to write (1.2) as a system in $\mathbb{R}^n$) and Chen and Zou [4], Clapp, Pistoia and Tavares [7], Druet and Hebey [10] and Druet, Hebey and Vétois [11] in the case of a more general manifold.

In this paper, we are interested in the question of proportionality of components, also called synchronization, of solutions to the system (1.2). A solution (u_1, u_2) of (1.2) is said to be *synchronized* if there exists a constant $\Lambda > 0$ such that $u_2 \equiv \Lambda u_1$ in M. This question has been studied for instance by Montaru, Sirakov and Souplet [28] and Quittner and Souplet [30] in the case of systems in domains of $\mathbb{R}^n$. It also naturally arises in the case of a closed manifold.

It is easy to see that every synchronized solution $(u_1, u_2) = (u_1, \Lambda u_1)$ of the system (1.2) is such that u_1 is constant in the case where $\lambda_1 \neq \lambda_2$ and u_1 is a solution of the equation

$$\begin{cases} \Delta_g u_1 + \lambda_1 u_1 = \mu_1 u_1^{q-2} & \text{in } M \\ u_1 > 0 & \text{in } M, \end{cases} \tag{1.3}$$

where $\mu_1 := a_{11} + a_{12}\Lambda^{q-2} = a_{21} + a_{22}\Lambda^{q-2}$, in the case where $\lambda_1 = \lambda_2$. We know from a result of Bidaut-Véron and Véron [3] that the equation (1.3) does not have any non-constant solutions when

$$2 < q \leq 2^* \quad \text{and} \quad \begin{cases} \dfrac{n-1}{n}\,(q-2)\,\lambda_1 g \leq \mathrm{Ric}_g & \text{if } q < 2^* \\ \dfrac{n-1}{n}\,(q-2)\,\lambda_1 g < \mathrm{Ric}_g & \text{if } q = 2^*, \end{cases} \tag{1.4}$$

where $2^* := \infty$ if $n \leq 2$, $2^* := 2n/\,(n-2)$ if $n \geq 3$, Ric_g is the Ricci curvature of the manifold and the latter inequalities are in the sense of bilinear forms. On the other hand, existence results of non-constant solutions to equation (1.3) abound in the case where (1.4) is not satisfied (see for instance Chen, Wei and Yan [9], Hebey and

Vaugon [18], Hebey and Wei [19], Micheletti, Pistoia and Vétois [27] and Vétois and Wang [35].)

For simplicity, in this introduction, we state our results in the case of the sphere $(\mathbb{S}^n, g_0)$, where g_0 is the standard round metric. Furthermore, we assume that $\lambda_1 = \lambda_2$ and $a_{12} = a_{21}$, namely we consider the system

$$\begin{cases} \Delta_{g_0} u_1 + \lambda u_1 = a u_1^{q-1} + b u_2^{q-2} u_1 & \text{in } \mathbb{S}^n \\ \Delta_{g_0} u_2 + \lambda u_2 = b u_1^{q-2} u_2 + c u_2^{q-1} & \text{in } \mathbb{S}^n \\ u_1, u_2 > 0 & \text{in } \mathbb{S}^n, \end{cases} \tag{1.5}$$

where $\lambda, a, b, c \in \mathbb{R}$ and $q \in (2, \infty)$. In the Euclidean space, this case has been studied for instance by Clapp and Pistoia [5] (via stereographic projection, the system studied in [5] matches with (1.5) when $n = 4$, $\lambda = 2$ and the parameters α and β in [5] are equal to 2). We refer to Sections 2 and 3 for results applying to more general systems and more general manifolds. For the system (1.5), we obtain the following:

Theorem 1.1. *Let $\lambda, a, b, c \in \mathbb{R}$ and $q \in (2, \infty)$.*

(i) *If either $c \le b \le a$ or $a \le b \le c$ and at least one of the two inequalities is strict, then the system (1.5) has no solutions.*

(ii) *If either [$a < b$ and $c < b$] or $a = b = c$, then every solution of (1.5) is synchronized.*

(iii) *There exist real numbers λ, a, b and c such that $\lambda > 0$, $a = c > b > 0$ and (1.5) has non-synchronized solutions. More precisely, we have the following result: for every $\lambda, a, b \in C^1([-\delta, \delta])$, $\delta > 0$, if the following conditions hold:*

(A1) $\lambda(0)(a(0) + b(0)) > 0$,

(A2) $\lambda(0) \notin \left\{\frac{2j(2j+n-1)}{q-2} :\ j \in \mathbb{N}\right\}$, *where* $\mathbb{N} := \{1, 2, \dots\}$,

(A3) $\beta(0) := \lambda(0)\frac{a(0)-b(0)}{a(0)+b(0)} \in \left\{\frac{j(j+n-1)}{q-2} :\ j \in \mathbb{N}\right\}$ *and* $\beta'(0) \ne 0$,

then there exists a C^1 branch (see Definition 2.1) of non-synchronized solutions to (1.5) with $\lambda = \lambda(\alpha)$, $c = a = a(\alpha)$ and $b = b(\alpha)$ emanating from the constant solution at $\alpha = 0$.

Theorem 1.1 (iii) extends a previous result obtained by Gladiali, Grossi and Troestler [14] for systems with Sobolev critical growth in $\mathbb{R}^n$, which, via stereographic projection, corresponds to the case where $n \ge 3$, $\lambda = n(n-2)/4$ and $q = 2n/(n-2)$. Like in [14], our approach is based on the bifurcation theory at eigenvalues of odd multiplicity. Unlike in [14], by taking advantage of our closed manifold setting, we perform our constructions in $C^{1,\theta}(M)$, $\theta \in (0, 1)$, instead of Sobolev spaces, which allows us to treat the case of systems with supercritical growth.

Theorem 1.1 (iii) is proven in Section 2 (as a particular case of Theorem 2.3) and Theorem 1.1 (i) and (ii) are proven in Section 3 (as a particular cases of Theorem 3.1 (i) and (ii)).

2. Bifurcation results

This section is devoted to bifurcation results showing the existence of branches of non-synchronized solutions for systems like (1.2).

Definition 2.1. *Let (M,g) be a smooth, closed Riemannian manifold, Ω be an open set in $\mathbb{R}^2$, $I:=[-\delta,\delta]$, $\delta>0$, and $F_1,F_2\in C^1(I\times\Omega)$. Consider the system*

$$\begin{cases}\Delta_g u_1=F_1(\alpha,u_1,u_2) & \text{in } M\\ \Delta_g u_2=F_2(\alpha,u_1,u_2) & \text{in } M,\end{cases}\tag{2.1}$$

where $\alpha\in I$. Assume that for every $\alpha\in I$, there exists a solution $(\overline{u}_1(\alpha),\overline{u}_2(\alpha))\in C^2(M)^2$ of (2.1) such that $(\overline{u}_1(\alpha),\overline{u}_2(\alpha))\to(\overline{u}_1(0),\overline{u}_2(0))$ in $C^2(M)^2$ as $\alpha\to 0$. Let $\mathcal{S}$ be the set of all solutions $(\alpha,u_1,u_2)\in I\times C^2(M)^2$ to (2.1) such that $(u_1,u_2)\neq(\overline{u}_1(\alpha),\overline{u}_2(\alpha))$. We say that the solution $(0,\overline{u}_1(0),\overline{u}_2(0))$ is a bifurcation point *of (2.1) if $(0,\overline{u}_1(0),\overline{u}_2(0))\in\overline{\mathcal{S}}$, where $\overline{\mathcal{S}}$ stands for the closure of $\mathcal{S}$ in $I\times C^2(M)^2$. Furthermore, we say that a subset $\mathcal{B}\subseteq\mathcal{S}$ is a C^1* branch *of solutions to (2.1) emanating from $(0,\overline{u}_1(0),\overline{u}_2(0))$ if $\mathcal{B}\neq\emptyset$ and $\mathcal{B}\cup\{(0,\overline{u}_1(0),\overline{u}_2(0))\}$ is a C^1 curve in $I\times C^2(M)^2$.*

In the case of the sphere, we obtain Theorem 1.1 (iii). In the case of a more general manifold, we obtain the following:

Theorem 2.2. *Let (M,g) be a smooth, closed Riemannian manifold, $I:=[-\delta,\delta]$, $\delta>0$, $\lambda_1,\lambda_2,a_{11},a_{12},a_{21},a_{22}\in C^1(I)$ and $q\in(2,\infty)$. Consider the system*

$$\begin{cases}\Delta_g u_1+\lambda_1(\alpha)u_1=a_{11}(\alpha)u_1^{q-1}+a_{12}(\alpha)u_2^{q-2}u_1 & \text{in } M\\ \Delta_g u_2+\lambda_2(\alpha)u_2=a_{21}(\alpha)u_1^{q-2}u_2+a_{22}(\alpha)u_2^{q-1} & \text{in } M\\ u_1,u_2>0 & \text{in } M,\end{cases}\tag{2.2}$$

where $\alpha\in I$. Assume that the following conditions hold:

- (B1) *For every $\alpha\in I$, the system (1.2) has a unique constant solution $(\overline{u}_1(\alpha),\overline{u}_2(\alpha))$.*
- (B2) *For every $\alpha\in I$, the matrix*

$$\mathcal{A}(\alpha):=\begin{pmatrix}a_{11}(\alpha)\overline{u}_1(\alpha)^{q-2} & a_{12}(\alpha)\overline{u}_2(\alpha)^{q-3}\overline{u}_1(\alpha)\\ a_{21}(\alpha)\overline{u}_1(\alpha)^{q-3}\overline{u}_2(\alpha) & a_{22}(\alpha)\overline{u}_2(\alpha)^{q-2}\end{pmatrix}$$

has two distinct, non-zero, real eigenvalues $\beta_1(\alpha)$ and $\beta_2(\alpha)$.

(B3) $\mathcal{H}_1^* \times \mathcal{H}_2^*$ *has odd dimension, where*

$$\mathcal{H}_i^* := \left\{\varphi \in C^2(M) : \Delta_g \varphi = (q-2)\,\beta_i(0)\,\varphi \text{ in } M\right\} \quad \forall i \in \{1,2\}.$$

(B4) *For every* $i \in \{1,2\}$, *if* $\mathcal{H}_i^* \neq \{0\}$, *then* $\beta_i'(0) \neq 0$ *and either* $\lambda_1(0) \neq \lambda_2(0)$ *or* $\beta_i(0) \neq \lambda_1(0) = \lambda_2(0)$.

Then the solution $(0, \overline{u}_1(0), \overline{u}_2(0))$ *is a bifurcating point of the system* (2.2). *Furthermore, there exists a neighborhood* $\mathcal{N}$ *of* $(0, \overline{u}_1(0), \overline{u}_2(0))$ *in* $I \times C^2(M)^2$ *such that for every solution* $(\alpha, u_1, u_2) \in \mathcal{N}$ *of* (2.2), *if* $(u_1, u_2) \neq (\overline{u}_1(\alpha), \overline{u}_2(\alpha))$, *then* (u_1, u_2) *is non-synchronized. If moreover* $\mathcal{H}_1^* \times \mathcal{H}_2^*$ *has dimension one, then there exists a* C^1 *branch of non-synchronized solutions to* (2.2) *emanating from* $(0, \overline{u}_1(0), \overline{u}_2(0))$.

Both Theorem 1.1 (iii) and Theorem 2.2 follow from the following general bifurcation result for systems of the form (2.1):

Theorem 2.3. *Let* (M,g) *be a smooth, closed Riemannian manifold,* Ω *be an open set in* $\mathbb{R}^2$ *such that* $(0,0) \in \Omega$, $I := [-\delta, \delta]$, $\delta > 0$, *and* $F_1, F_2 \in C^1(I \times \Omega)$ *such that* $\partial_\alpha \partial_{u_i} F_j$ *exists and is continuous in* $I \times \Omega$ *for all* $i, j \in \{1,2\}$, *where we denote by* (α, u_1, u_2) *a point in* $I \times \Omega$. *Assume that the following conditions hold:*

(C1) $F_1(\alpha, 0, 0) = F_2(\alpha, 0, 0) = 0$ *for all* $\alpha \in I$.
(C2) $\partial_{u_2} F_1(\alpha, 0, 0) = \partial_{u_1} F_2(\alpha, 0, 0) = 0$ *for all* $\alpha \in I$.
(C3) *There exist two closed subspaces* $\mathcal{H}_1$ *and* $\mathcal{H}_2$ *of* $C^{1,\theta}(M)$, $\theta \in (0,1)$, *and two open subsets* $\mathcal{U}_1 \subseteq \mathcal{H}_1$ *and* $\mathcal{U}_2 \subseteq \mathcal{H}_2$ *which contain 0 and satisfy the following conditions:*
  - $(u_1(x), u_2(x)) \in \Omega$ *for all* $x \in M$ *and* $(u_1, u_2) \in \mathcal{U}_1 \times \mathcal{U}_2$.
  - $(\Delta_g + 1)^{-1} u_i \in \mathcal{H}_i$ *for all* $u_i \in \mathcal{U}_i$ *and* $i \in \{1,2\}$.
  - $(\Delta_g + 1)^{-1} F_i(\alpha, u_1, u_2) \in \mathcal{H}_i$ *for all* $(\alpha, u_1, u_2) \in I \times \mathcal{U}_1 \times \mathcal{U}_2$ *and* $i \in \{1,2\}$.

(C4) $\mathcal{H}_1^* \times \mathcal{H}_2^*$ *has odd dimension, where*

$$\mathcal{H}_i^* := \{\varphi \in \mathcal{H}_i : \ \Delta_g \varphi = \partial_{u_i} F_i(0,0,0)\,\varphi \text{ in } M\} \quad \forall i \in \{1,2\}.$$

(C5) *For every* $i \in \{1,2\}$, *if* $\mathcal{H}_i^* \neq \{0\}$, *then* $\partial_\alpha \partial_{u_i} F_i(0,0,0) \neq 0$.

Then there exists a sequence of solutions $((\alpha_m, u_{1,m}, u_{2,m}))_{m \in \mathbb{N}}$ *to the system* (2.1) *such that* $(\alpha_m, u_{1,m}, u_{2,m}) \in I \times ((\mathcal{U}_1 \times \mathcal{U}_2) \setminus \{(0,0)\})$ *and* $(\alpha_m, u_{1,m}, u_{2,m}) \to (0,0,0)$ *in* $I \times C^2(M)^2$ *as* $m \to \infty$. *Furthermore, every such sequence* $((\alpha_m, u_{1,m}, u_{2,m}))_{m \in \mathbb{N}}$ *is such that up to a subsequence,*

$$u_{i,m} = \varepsilon_m \varphi_i + \mathrm{o}(\varepsilon_m) \quad \text{in } \mathcal{H}_i \quad \forall i \in \{1,2\} \tag{2.3}$$

as $m \to \infty$ *for some* $(\varphi_1, \varphi_2) \in (\mathcal{H}_1^* \times \mathcal{H}_2^*) \setminus \{(0,0)\}$ *and* $\varepsilon_m > 0$ *such that* $\varepsilon_m \to 0$. *If moreover* $\mathcal{H}_1^* \times \mathcal{H}_2^*$ *has dimension one and* $\partial_{u_i} \partial_{u_j} F_k$ *exists and is continuous in* $I \times \Omega$ *for all* $i, j, k \in \{1,2\}$, *then there exists* *a neighborhood $\mathcal{N}$ of $(0,0,0)$ in $I\times\mathcal{U}_1\times\mathcal{U}_2$ such that the set of solutions $(\alpha,u_1,u_2)\in\mathcal{N}\setminus(I\times\{(0,0)\})$ to (2.1) is a C^1 branch emanating from $(0,0,0)$ whose tangent line at $(0,0,0)$ is directed by some vector in $\mathbb{R}\times((\mathcal{H}_1^*\times\mathcal{H}_2^*)\setminus\{(0,0)\})$.*

Proof of Theorem 2.3. Let

$$\mathcal{H}:=\mathcal{H}_1\times\mathcal{H}_2,\quad \mathcal{H}^*:=\mathcal{H}_1^*\times\mathcal{H}_2^*\quad\text{and}\quad \mathcal{U}:=\mathcal{U}_1\times\mathcal{U}_2.$$

By replacing $\mathcal{U}_1$ and $\mathcal{U}_2$ by smaller sets if necessary, we may assume that there exists a compact set $K\subset\Omega$ such that $(u_1(x),u_2(x))\in K$ for all $x\in M$ and $(u_1,u_2)\in\mathcal{U}$. The solutions $(u_1,u_2)\in\mathcal{U}$ to (2.1) are given by the zeros of the function $\mathcal{T}:I\times\mathcal{U}\to\mathcal{H}$ defined by

$$\mathcal{T}(\alpha,u_1,u_2):=\mathcal{I}(u_1,u_2)-\mathcal{K}_\alpha(u_1,u_2),\quad\text{where}\quad \mathcal{I}(u_1,u_2):=\begin{pmatrix}u_1\\u_2\end{pmatrix}$$

and

$$\mathcal{K}_\alpha(u_1,u_2):=\begin{pmatrix}(\Delta_g+1)^{-1}[F_1(\alpha,u_1,u_2)+u_1]\\(\Delta_g+1)^{-1}[F_2(\alpha,u_1,u_2)+u_2]\end{pmatrix}$$

for all $(\alpha,u_1,u_2)\in I\times\mathcal{U}$. By the assumption (C3), we have that the functions $\mathcal{K}_\alpha$ and $\mathcal{T}$ are well-defined. In what follows, we write a point in $I\times\mathcal{U}$ as (α,U).

Step 1. We begin with proving that $\mathcal{K}_\alpha$ is compact, $\mathcal{T}\in C^1(I\times\mathcal{U})$ and $D_U\,\partial_\alpha\mathcal{T}$ exists and is continuous in $I\times\mathcal{U}$.

Suppose that $((u_{1,m},u_{2,m}))_{m\in\mathbb{N}}$ is a bounded sequence in $\mathcal{U}$. Then the sequences $(u_{1,m})_m$ and $(u_{2,m})_m$ are bounded in $C^{1,\theta}(M,K)$ and up to a subsequence, $(u_{1,m},u_{2,m})\to(u_{1,0},u_{2,0})$ in $C^1(M,K)^2$ as $m\to\infty$. Let $\alpha\in I$ and

$$\begin{pmatrix}\tilde u_{1,m}\\\tilde u_{2,m}\end{pmatrix}:=\mathcal{K}_\alpha(u_{1,m},u_{2,m})$$

so that

$$\begin{pmatrix}\Delta_g\tilde u_{1,m}+\tilde u_{1,m}\\\Delta_g\tilde u_{2,m}+\tilde u_{2,m}\end{pmatrix}=\begin{pmatrix}F_1(\alpha,u_{1,m},u_{2,m})+u_{1,m}\\F_2(\alpha,u_{1,m},u_{2,m})+u_{2,m}\end{pmatrix}.$$

Since $F_1,F_2\in C^1(I\times\Omega)$, $u_{1,0},u_{2,0}\in C^1(M,K)$ and $K\subset\Omega$, by standard elliptic estimates, we obtain that there exists $(\tilde u_{1,0},\tilde u_{2,0})\in C^2(M)^2$ satisfying

$$\begin{pmatrix}\Delta_g\tilde u_{1,0}+\tilde u_{1,0}\\\Delta_g\tilde u_{2,0}+\tilde u_{2,0}\end{pmatrix}=\begin{pmatrix}F_1(\alpha,u_{1,0},u_{2,0})+u_{1,0}\\F_2(\alpha,u_{1,0},u_{2,0})+u_{2,0}\end{pmatrix}.$$

For $i=1,2$, we then obtain

$$(\Delta_g+1)(\tilde u_{i,m}-\tilde u_{i,0})=F_i(\alpha,u_{1,m},u_{2,m})-F_i(\alpha,u_{1,0},u_{2,0})+u_{i,m}-u_{i,0}=\mathrm{o}(1)\quad(2.4)$$

uniformly in M. Therefore, $(\tilde{u}_{1,m}, \tilde{u}_{2,m}) \to (\tilde{u}_{1,0}, \tilde{u}_{2,0})$ in $\mathcal{H}$ and so $\mathcal{K}_\alpha$ is compact.

Now, if $(\alpha_m, u_{1,m}, u_{2,m}) \to (\alpha_0, u_{1,0}, u_{2,0})$ in $I \times \mathcal{U}$, then, arguing as above, we obtain that $\mathcal{K}_{\alpha_m}(u_{1,m}, u_{2,m}) \to \mathcal{K}_{\alpha_0}(u_{1,0}, u_{2,0})$ in $\mathcal{H}$. This shows that $\mathcal{T}$ is continuous in $I \times \mathcal{U}$.

For every $(\alpha, u_1, u_2) \in I \times \mathcal{U}$, we have

$$\partial_\alpha \mathcal{T}(\alpha, u_1, u_2) = -\begin{pmatrix} (\Delta_g + 1)^{-1} [\partial_\alpha F_1(\alpha, u_1, u_2)] \\ (\Delta_g + 1)^{-1} [\partial_\alpha F_2(\alpha, u_1, u_2)] \end{pmatrix},$$

$$D_U \mathcal{T}(\alpha, u_1, u_2)[(v_1, v_2)]$$
$$= \begin{pmatrix} v_1 - (\Delta_g + 1)^{-1} [\partial_{u_1} F_1(\alpha, u_1, u_2) v_1 + \partial_{u_2} F_1(\alpha, u_1, u_2) v_2 + v_1] \\ v_2 - (\Delta_g + 1)^{-1} [\partial_{u_1} F_2(\alpha, u_1, u_2) v_1 + \partial_{u_2} F_2(\alpha, u_1, u_2) v_2 + v_2] \end{pmatrix}$$

and

$$D_U \partial_\alpha \mathcal{T}(\alpha, u_1, u_2)[(v_1, v_2)]$$
$$= -\begin{pmatrix} (\Delta_g + 1)^{-1} [\partial_{u_1}\partial_\alpha F_1(\alpha, u_1, u_2) v_1 + \partial_{u_2}\partial_\alpha F_1(\alpha, u_1, u_2) v_2] \\ (\Delta_g + 1)^{-1} [\partial_{u_1}\partial_\alpha F_2(\alpha, u_1, u_2) v_1 + \partial_{u_2}\partial_\alpha F_2(\alpha, u_1, u_2) v_2] \end{pmatrix}.$$

If $(\alpha_m, u_{1,m}, u_{2,m}) \to (\alpha_0, u_{1,0}, u_{2,0})$ in $I \times \mathcal{U}$, then for $i, j = 1, 2$, by using the regularity assumptions on F, we obtain that

$$\partial_\alpha F_j(\alpha_m, u_{1,m}, u_{2,m}) \longrightarrow \partial_\alpha F_j(\alpha_0, u_{1,0}, u_{2,0}),$$
$$\partial_{u_i} F_j(\alpha_m, u_{1,m}, u_{2,m}) \longrightarrow \partial_{u_i} F_j(\alpha_0, u_{1,0}, u_{2,0})$$

and

$$\partial_{u_i}\partial_\alpha F_j(\alpha_m, u_{1,m}, u_{2,m}) \longrightarrow \partial_{u_i}\partial_\alpha F_j(\alpha_0, u_{1,0}, u_{2,0})$$

uniformly in M. Then, arguing as in (2.4), we obtain that $\mathcal{T} \in C^1(I \times \mathcal{U})$ and $D_U \partial_\alpha \mathcal{T}$ is continuous in $I \times \mathcal{U}$.

Step 2. We now establish the main bifurcation results.

We have $\mathcal{T}(\alpha, 0, 0) = (0, 0)$ for all $\alpha \in I$ by the assumption (C1). Furthermore, by the assumption (C2), we have

$$D_U \mathcal{T}(0, 0, 0)[(v_1, v_2)] = \begin{pmatrix} v_1 - (\Delta_g + 1)^{-1} [\partial_{u_1} F_1(0, 0, 0) v_1 + v_1] \\ v_2 - (\Delta_g + 1)^{-1} [\partial_{u_2} F_2(0, 0, 0) v_2 + v_2] \end{pmatrix}.$$

Then $(v_1, v_2) \in \ker D_U \mathcal{T}(0, 0, 0)$ if and only if $(v_1, v_2) \in \mathcal{H}^*$. By the assumption (C4), it follows that $D_U \mathcal{T}(0, 0, 0)$ has a nontrivial kernel consisting of eigenfunctions of Δ_g.

Now, $\ker D_U \mathcal{T}(0, 0, 0)$ has odd dimension by the assumption (C4). This along with the results of Step 1 allows to apply Theorem A of Westreich [37] (see also Theorems II.3.3 and II.4.4 and the statements (II.4.29) and (II.4.31) in Kielhöfer's book [21]), which gives that the solution $(0,0,0)$ is a bifurcation point of the system (2.1) in $I\times(\mathcal{H}\setminus\{(0,0)\})$ provided the following condition holds:

$$\Big[D_U\partial_\alpha\mathcal{T}(0,0,0)[(v_1,v_2)]\in\operatorname{range}(D_U\mathcal{T}(0,0,0))\\ \text{and}\quad (v_1,v_2)\in\ker(D_U\mathcal{T}(0,0,0))\Big]\iff(v_1,v_2)=(0,0). \quad (2.5)$$

Remark that by standard elliptic estimates, the C^2 topology in Definition 2.1 can be replaced without loss of generality by the $C^{1,\theta}$ topology. If moreover $\mathcal{H}^*$ has dimension one and $\partial_{u_i}\partial_{u_j}F_k$ exists and is continuous in $I\times\Omega$ for all $i,j,k\in\{1,2\}$, then the last part of Theorem 2.3 follows from Theorem 1.7 of Crandall–Rabinowitz [8] (see also Kielhöfer [21, Theorem I.5.1]).

We now show that the condition (2.5) holds for the function $\mathcal{T}$ under our assumptions on F. Let $(v_1,v_2)\in\ker D_U\mathcal{T}(0,0,0)$ and $(w_1,w_2):=D_U\partial_\alpha\mathcal{T}(0,0,0)[(v_1,v_2)]$. Then

$$\begin{pmatrix}\Delta_g v_1\\ \Delta_g v_2\end{pmatrix}=\begin{pmatrix}\partial_{u_1}F_1(0,0,0)\,v_1\\ \partial_{u_2}F_2(0,0,0)\,v_2\end{pmatrix}.$$

Furthermore, by the assumption (C2), we obtain

$$\begin{pmatrix}\Delta_g w_1+w_1\\ \Delta_g w_2+w_2\end{pmatrix}=-\begin{pmatrix}\partial_{u_1}\partial_\alpha F_1(0,0,0)\,v_1\\ \partial_{u_2}\partial_\alpha F_2(0,0,0)\,v_2\end{pmatrix}.$$

So then

$$\begin{pmatrix}(\partial_{u_1}F_1(0,0,0)+1)(\Delta_g w_1+w_1)\\ (\partial_{u_2}F_2(0,0,0)+1)(\Delta_g w_2+w_2)\end{pmatrix}\\ =-\begin{pmatrix}\partial_{u_1}\partial_\alpha F_1(0,0,0)(\Delta_g v_1+v_1)\\ \partial_{u_2}\partial_\alpha F_2(0,0,0)(\Delta_g v_2+v_2)\end{pmatrix},$$

which gives

$$\begin{pmatrix}(\partial_{u_1}F_1(0,0,0)+1)\,w_1\\ (\partial_{u_2}F_2(0,0,0)+1)\,w_2\end{pmatrix}=-\begin{pmatrix}\partial_{u_1}\partial_\alpha F_1(0,0,0)\,v_1\\ \partial_{u_2}\partial_\alpha F_2(0,0,0)\,v_2\end{pmatrix}.$$

Now, if we suppose that $(w_1,w_2)\in\operatorname{range}(D_U\mathcal{T}(0,0,0))$, then by the assumption (C5), we obtain that $(v_1,v_2)\in\operatorname{range}(D_U\mathcal{T}(0,0,0))$ and so there exists $(\varphi_1,\varphi_2)\in\mathcal{H}$ such that

$$\begin{pmatrix}\Delta_g\varphi_1-\partial_{u_1}F_1(0,0,0)\,\varphi_1\\ \Delta_g\varphi_2-\partial_{u_2}F_2(0,0,0)\,\varphi_2\end{pmatrix}=\begin{pmatrix}\Delta_g v_1+v_1\\ \Delta_g v_2+v_2\end{pmatrix}.$$

For $i = 1, 2$, straightforward integrations by parts then yield

$$\int_M \left(|\nabla v_i|^2 + v_i^2\right) dv_g = \int_M v_i \left(\Delta_g \varphi_i - \partial_{u_i} F_i (0,0,0)\, \varphi_i\right) dv_g$$
$$= \int_M \left(\Delta_g v_i - \partial_{u_i} F_i (0,0,0)\, v_i\right) \varphi_i dv_g = 0,$$

where dv_g is the volume element with respect to the metric g. It follows that $(v_1, v_2) = (0, 0)$. Hence condition (2.5) is satisfied.

Step 3. Finally we prove the expansion (2.3).

Let $((\alpha_m, u_{1,m}, u_{2,m}))_{m\in\mathbb{N}}$ be a sequence of solutions to (2.1), such that $(\alpha_m, u_{1,m}, u_{2,m}) \in I \times (\mathcal{U}\setminus\{(0,0)\})$ and $(\alpha_m, u_{1,m}, u_{2,m}) \to (0,0,0)$ in $I \times C^2(M)^2$ as $m \to \infty$. For $i = 1, 2$, consider the sequence

$$w_{i,m} := \varepsilon_m^{-1} u_{i,m}, \quad \text{where} \quad \varepsilon_m := \max\left(\|u_{1,m}\|_{C^{1,\theta}}, \|u_{2,m}\|_{C^{1,\theta}}\right)$$

so that

$$\max\left(\|w_{1,m}\|_{C^{1,\theta}}, \|w_{2,m}\|_{C^{1,\theta}}\right) = 1.$$

Since $(\alpha_m, u_{1,m}, u_{2,m})$ satisfies (2.1), it follows by our assumptions on F that

$$\Delta_g w_{i,m} = \varepsilon_m^{-1} F_i (\alpha_m, u_{1,m}, u_{2,m}) = \partial_{u_i} F_i(\alpha_m, 0, 0) w_{i,m} + \mathrm{o}(1)$$

uniformly in M. Then, by standard elliptic theory, it follows that up to a subsequence $w_{i,m} \to \varphi_i$ in $\mathcal{H}_i$ for some function $\varphi_i \in \mathcal{H}_i$ satisfying

$$\Delta_g \varphi_i = \partial_{u_i} F_i (0,0,0)\, \varphi_i \quad \text{in } M.$$

Hence φ_i belongs to $\mathcal{H}_i^*$ and further $\max\left(\|\varphi_1\|_{C^{1,\theta}}, \|\varphi_2\|_{C^{1,\theta}}\right) = 1$. It follows that $(\varphi_1, \varphi_2) \in \mathcal{H}^*\setminus\{(0,0)\}$ and

$$u_{i,m} = \varepsilon_m w_{i,m} = \varepsilon_m (\varphi_i + \mathrm{o}(1)) \quad \text{in } \mathcal{H}_i.$$

This completes the proof of Theorem 2.3. □

We can now prove Theorem 1.1 (iii) and Theorem 2.2 by using Theorem 2.3. We start with proving Theorem 2.2.

Proof of Theorem 2.2. First note that the system in (2.2) can be rewritten as

$$\begin{pmatrix} \Delta_g u_1 \\ \Delta_g u_2 \end{pmatrix} = \begin{pmatrix} \tilde{F}_1 (\alpha, u_1, u_2) \\ \tilde{F}_2 (\alpha, u_1, u_2) \end{pmatrix},$$

where

$$\tilde{F}(\alpha, u_1, u_2) = \begin{pmatrix} \tilde{F}_1 (\alpha, u_1, u_2) \\ \tilde{F}_2 (\alpha, u_1, u_2) \end{pmatrix}$$
$$:= \begin{pmatrix} a_{11}(\alpha) |u_1|^{q-2} u_1 + a_{12}(\alpha) |u_2|^{q-2} u_1 - \lambda_1(\alpha) u_1 \\ a_{21}(\alpha) |u_1|^{q-2} u_2 + a_{22}(\alpha) |u_2|^{q-2} u_2 - \lambda_2(\alpha) u_2 \end{pmatrix}.$$

Now let's transform this system so as to apply Theorem 2.3. For every $\alpha \in I$, the unique constant solution to (2.2) (which existence follows from the assumption (B1)) is given by

$$\begin{cases} \overline{u}_1(\alpha) = \left(\dfrac{\lambda_1(\alpha) a_{22}(\alpha) - \lambda_2(\alpha) a_{12}(\alpha)}{a_{11}(\alpha) a_{22}(\alpha) - a_{21}(\alpha) a_{12}(\alpha)} \right)^{1/(q-2)} \\ \overline{u}_2(\alpha) = \left(\dfrac{\lambda_2(\alpha) a_{11}(\alpha) - \lambda_1(\alpha) a_{21}(\alpha)}{a_{11}(\alpha) a_{22}(\alpha) - a_{21}(\alpha) a_{12}(\alpha)} \right)^{1/(q-2)} \end{cases}$$

and it satisfies

$$\begin{cases} a_{11}(\alpha) \overline{u}_1(\alpha)^{q-2} + a_{12}(\alpha) \overline{u}_2(\alpha)^{q-2} = \lambda_1(\alpha) & \text{in } M \\ a_{21}(\alpha) \overline{u}_1(\alpha)^{q-2} + a_{22}(\alpha) \overline{u}_2(\alpha)^{q-2} = \lambda_2(\alpha) & \text{in } M. \end{cases}$$

We look for solutions of (1.2) bifurcating from $(\overline{u}_1(\alpha), \overline{u}_2(\alpha))$. By the assumption (B2), for every $\alpha \in I$, the matrix $\mathcal{A}(\alpha)$ has two distinct, non-zero, real eigenvalues $\beta_1(\alpha)$ and $\beta_2(\alpha)$ given by

$$\{\beta_1(\alpha), \beta_2(\alpha)\} := \left\{ \frac{a_{11}(\alpha) \overline{u}_1(\alpha)^{q-2} + a_{22}(\alpha) \overline{u}_2(\alpha)^{q-2}}{2} \pm \frac{\sqrt{D(\alpha)}}{2} \right\},$$

where

$$D(\alpha) := \left(a_{11}(\alpha) \overline{u}_1(\alpha)^{q-2} - a_{22}(\alpha) \overline{u}_2(\alpha)^{q-2} \right)^2 + 4 a_{12}(\alpha) a_{21}(\alpha) \overline{u}_1(\alpha)^{q-2} \overline{u}_2(\alpha)^{q-2}.$$

Let $\mathcal{P}(\alpha)$ be the 2×2 matrix such that

$$\mathcal{A}(\alpha) = \mathcal{P}(\alpha)^{-1} \begin{pmatrix} \beta_1(\alpha) & 0 \\ 0 & \beta_2(\alpha) \end{pmatrix} \mathcal{P}(\alpha).$$

Consider $(u_1, u_2) \in C^2(M)^2$ and let

$$\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} := P(\alpha) \begin{pmatrix} u_1 - \overline{u}_1(\alpha) \\ u_2 - \overline{u}_2(\alpha) \end{pmatrix}.$$

We then define

$$F(\alpha, v_1, v_2) = \begin{pmatrix} F_1(\alpha, v_1, v_2) \\ F_2(\alpha, v_1, v_2) \end{pmatrix} := P(\alpha) \tilde{F}(\alpha, u_1, u_2), \tag{2.6}$$

where

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} \overline{u}_1(\alpha) \\ \overline{u}_2(\alpha) \end{pmatrix} + P(\alpha)^{-1} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}. \tag{2.7}$$

We then obtain that the system (1.2) is equivalent to

$$\begin{pmatrix} \Delta_g v_1 \\ \Delta_g v_2 \end{pmatrix} = \begin{pmatrix} F_1(\alpha, v_1, v_2) \\ F_2(\alpha, v_1, v_2) \end{pmatrix}. \tag{2.8}$$

Next we apply Theorem 2.3 to (2.8). Note that the condition (C1) of Theorem 2.3 is satisfied by (2.8). Furthermore, since $(\overline{u}_1(\alpha), \overline{u}_2(\alpha)) \in (0,\infty)^2$ for all $\alpha \in I$, by continuity, we obtain that there exists $\delta_0 > 0$ such that $(u_1, u_2) \in (0,\infty)^2$ for all $(\alpha, v_1, v_2) \in I \times (-\delta_0, \delta_0)^2$. In particular, letting $\Omega := (-\delta_0, \delta_0)^2$, we then obtain that $F \in C^1(I \times \Omega)$, $\partial_{u_i}\partial_\alpha F_j$ and $\partial_{u_i}\partial_{u_j} F_k$ exist and are continuous in $I \times \Omega$ for all $i, j, k \in \{1, 2\}$ and the condition (C3) is satisfied with $\mathcal{H}_1 = \mathcal{H}_2 := C^{1,\theta}(M)$ and $\mathcal{U}_1 = \mathcal{U}_2 := C^{1,\theta}(M, (-\delta_0, \delta_0))$. The condition (C2) is also satisfied as we obtain differentiating

$$\begin{aligned} D_{(v_1,v_2)} F(\alpha, 0, 0) &= \mathcal{P}(\alpha)\left[(q-2)\mathcal{A}(\alpha)\right]\mathcal{P}^{-1}(\alpha) \\ &= (q-2)\begin{pmatrix} \beta_1(\alpha) & 0 \\ 0 & \beta_2(\alpha) \end{pmatrix}. \end{aligned}$$

The assumptions (B3) and (B4) then imply that the conditions (C4) and (C5) of Theorem 2.3 are also satisfied.

By applying Theorem 2.3 and reversing the above change of function, we then obtain that the solution $(0, \overline{u}_1(0), \overline{u}_2(0))$ is a bifurcation point of the system (2.2). Furthermore, we obtain that for every sequence $((\alpha_m, u_{1,m}, u_{2,m}))_{m\in\mathbb{N}}$ of solutions to (2.2), if $(u_{1,m}, u_{2,m}) \neq (\overline{u}_1(0), \overline{u}_2(0))$ and $(\alpha_m, u_{1,m}, u_{2,m}) \to (0, \overline{u}_1(0), \overline{u}_2(0))$ in $I \times C^2(M)^2$ as $m \to \infty$, then up to a subsequence,

$$u_{i,m} = \overline{u}_i(\alpha_m) + \varepsilon_m\left(q_{i1}\varphi_1 + q_{i2}\varphi_2 + \mathrm{o}(1)\right) \quad \text{in } \mathcal{H}_i \quad \forall i \in \{1, 2\},$$

where $(q_{ij})_{1\le i,j\le 2} := P(0)^{-1}$, for some $(\varphi_1, \varphi_2) \in \mathcal{H}^* \backslash \{(0,0)\}$ and $\varepsilon_m > 0$ such that $\varepsilon_m \to 0$. By the assumptions (B2) and (B4), we have that for $i = 1, 2$, either $\varphi_i \equiv 0$ or φ_i is not constant in M. Also by assumption (B4) we have that if $\varphi_i \neq 0$ for $i = 1, 2$, then either $\lambda_1(0) \neq \lambda_2(0)$ or $\beta_i(0) \neq \lambda_1(0) = \lambda_2(0)$, which implies $\overline{u}_2(0)\, q_{1i} \neq \overline{u}_1(0)\, q_{2i}$. In particular, we obtain that $(u_{1,m}, u_{2,m})$ is non-synchronized. Therefore, we obtain that there exists a neighborhood $\mathcal{N}$ of $(0, \overline{u}_1(0), \overline{u}_2(0))$ in $I \times C^2(M)^2$ such that for every solution $(\alpha, u_1, u_2) \in \mathcal{N}$ of (2.2), if $(u_1, u_2) \neq (\overline{u}_1(\alpha), \overline{u}_2(\alpha))$, then (u_1, u_2) is non-synchronized. If moreover $\mathcal{H}^*$ has dimension one, then it follows from the last part of Theorem 2.3 that there exists a C^1 branch of non-synchronized solutions to (2.2) emanating from $(0, \overline{u}_1(0), \overline{u}_2(0))$. This completes the proof of Theorem 2.2. □

Proof of Theorem 1.1 (iii). We proceed as in the proof of Theorem 2.2. By the assumptions (A1) and (A3) along with the continuity of λ, a and b, letting δ be smaller if necessary, we may assume that $a(\alpha) > b(\alpha)$ and $\lambda(\alpha)(a(\alpha) + b(\alpha)) > 0$ for all $\alpha \in I$. Then the unique constant solution for the system (1.5) is given by

$$\begin{pmatrix} \overline{u}_1(\alpha) \\ \overline{u}_2(\alpha) \end{pmatrix} := \overline{u}(\alpha) \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \text{where } \overline{u}(\alpha) := \left(\frac{\lambda(\alpha)}{a(\alpha)+b(\alpha)} \right)^{1/(q-2)}$$

and it satisfies

$$\begin{cases} a(\alpha)\,\overline{u}_1(\alpha)^{q-2} + b(\alpha)\,\overline{u}_2(\alpha)^{q-2} = \lambda(\alpha) \\ b(\alpha)\,\overline{u}_1(\alpha)^{q-2} + a(\alpha)\,\overline{u}_2(\alpha)^{q-2} = \lambda(\alpha). \end{cases}$$

We look for solutions of (1.5) bifurcating from $(\overline{u}_1(\alpha), \overline{u}_2(\alpha))$. For (1.5), the eigenvalues $\beta_1(\alpha)$ and $\beta_2(\alpha)$ of $\mathcal{A}(\alpha)$ are given by

$$\beta_1(\alpha) = \lambda(\alpha) \quad \text{and} \quad \beta_2(\alpha) = \lambda(\alpha)\,\frac{a(\alpha)-b(\alpha)}{a(\alpha)+b(\alpha)}.$$

We let F and $\Omega = (-\delta_0, \delta_0)^2$ be defined similarly as in the proof of Theorem 2.2, so that in particular, the conditions (C1) and (C2) of Theorem 2.3 are satisfied. In this case, we find

$$\mathcal{P}(\alpha) = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

Next, we choose the appropriate subspaces $\mathcal{H}_1$ and $\mathcal{H}_2$ and open subsets $\mathcal{U}_1 \subseteq \mathcal{H}_1$ and $\mathcal{U}_2 \subseteq \mathcal{H}_2$. For this, we use an idea from Gladiali, Grossi and Troestler [14]. Consider the reflexion $\hat{v}$ across the equator $\{x_n = 0\}$ of the sphere $\mathbb{S}^n$ defined by

$$\hat{v}(x) := v(x_1, \dots, x_n, -x_{n+1}) \quad \forall x = (x_1, \dots, x_n, x_{n+1}) \in \mathbb{S}^n$$

for all functions $v : \mathbb{S}^n \to \mathbb{R}$. By stereographic projection along with a conformal change of metric, this corresponds to the Kelvin transform in $\mathbb{R}^n$. We let $N_0 := (0, \dots, 0, 1)$ and $j_0 \in \mathbb{N}$ be such that

$$(q-2)\,\beta_2(0) = \lambda_{j_0} := j_0(j_0 + n - 1)$$

i.e. the j_0-th eigenvalue of Δ_{g_0} on $\mathbb{S}^n$ (the existence of j_0 is given by the assumption (A3)). We then define $\mathcal{H}_1$ and $\mathcal{H}_2$ as

$$\mathcal{H}_1 := \left\{ v \in C^{1,\theta}(\mathbb{S}^n) : v \text{ is radial with respect to } N_0 \text{ and } \hat{v} = v \right\}$$

and

$$\mathcal{H}_2 := \begin{cases} \left\{ v \in C^{1,\theta}(\mathbb{S}^n) : v \text{ is radial w.r.t. } N_0 \text{ and } \hat{v} = v \right\} & \text{if } j_0 \text{ is even} \\ \left\{ v \in C^{1,\theta}(\mathbb{S}^n) : v \text{ is radial w.r.t. } N_0 \text{ and } \hat{v} = -v \right\} & \text{if } j_0 \text{ is odd.} \end{cases}$$

For $i=1,2$, we take $\mathcal{U}_i := C^{1,\theta}\left(\mathbb{S}^n,(-\delta_0,\delta_0)\right)\cap\mathcal{H}_i$. For every $(v_1,v_2)\in\mathcal{U}$, letting (u_1,u_2) be as in (2.7), we then obtain

$$\begin{aligned}\begin{pmatrix}\hat{u}_1\\ \hat{u}_2\end{pmatrix} &= \begin{pmatrix}\overline{u}_1(\alpha)\\ \overline{u}_2(\alpha)\end{pmatrix}+\frac{1}{2}\begin{pmatrix}\hat{v}_1+\hat{v}_2\\ \hat{v}_1-\hat{v}_2\end{pmatrix}\\ &= \begin{cases}\begin{pmatrix}\overline{u}_1(\alpha)\\ \overline{u}_2(\alpha)\end{pmatrix}+\frac{1}{2}\begin{pmatrix}v_1+v_2\\ v_1-v_2\end{pmatrix}=\begin{pmatrix}u_1\\ u_2\end{pmatrix} & \text{if } j_0 \text{ is even}\\ \begin{pmatrix}\overline{u}_1(\alpha)\\ \overline{u}_2(\alpha)\end{pmatrix}+\frac{1}{2}\begin{pmatrix}v_1-v_2\\ v_1+v_2\end{pmatrix}=\begin{pmatrix}u_2\\ u_1\end{pmatrix} & \text{if } j_0 \text{ is odd.}\end{cases}\end{aligned}$$

It follows that

$$\begin{aligned}F(\alpha,\hat{v}_1,\hat{v}_2) &= \begin{pmatrix}F_1(\alpha,\hat{v}_1,\hat{v}_2)\\ F_2(\alpha,\hat{v}_1,\hat{v}_2)\end{pmatrix}\\ &= \begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}\begin{pmatrix}a(\alpha)\hat{u}_1^{q-1}+b(\alpha)\hat{u}_2^{q-2}\hat{u}_1-\lambda(\alpha)\hat{u}_1\\ b(\alpha)\hat{u}_1^{q-2}\hat{u}_2+a(\alpha)\hat{u}_2^{q-1}-\lambda(\alpha)\hat{u}_2\end{pmatrix}\\ &= \begin{cases}F(\alpha,v_1,v_2) & \text{if } j_0 \text{ is even}\\ \begin{pmatrix}F_1(\alpha,v_1,v_2)\\ -F_2(\alpha,v_1,v_2)\end{pmatrix} & \text{if } j_0 \text{ is odd.}\end{cases}\end{aligned}$$

This along with standard elliptic regularity and symmetry arguments gives that the condition (C3) is satisfied.

Recall that the spherical harmonics φ satisfying $\Delta_{g_0}\varphi=\lambda_{j_0}\varphi$ are given by the restriction to $\mathbb{S}^n$ of the harmonic polynomials of degree j_0 in $\mathbb{R}^{n+1}$. In particular, up to a constant factor, the unique such function φ_{N_0,j_0} that is radial with respect to N_0 is given by the Jacobi polynomial

$$\begin{aligned}\varphi_{N_0,j_0}(x_1,\dots,x_{n+1}) := \sum_{j=0}^{j_0}&\begin{pmatrix}j_0+(n-2)/2\\ j\end{pmatrix}\begin{pmatrix}j_0+(n-2)/2\\ j_0-j\end{pmatrix}\\ &\times\left(\frac{x_{n+1}-1}{2}\right)^{j_0-j}\left(\frac{x_{n+1}+1}{2}\right)^{j}\quad\forall x\in\mathbb{S}^n\end{aligned}$$

(see Gladiali [13]). So then the assumptions (A2) and (A3) give that $\mathcal{H}_1^*=\{0\}$ and $\mathcal{H}_2^*$ has dimension one. In particular, we obtain that the condition (C4) is satisfied. Furthermore, the condition (C5) follows from (A3).

By applying Theorem 2.3, we then obtain that there exists a C^1 branch of solutions to (2.2) emanating from $(0,\overline{u}(0),\overline{u}(0))$ whose tangent line at $(0,\overline{u}(0),\overline{u}(0))$ is directed by some vector of the form $(\mu,\varphi,-\varphi)$, where $\mu\in\mathbb{R}$ and $\varphi\in\mathcal{H}_2^*\backslash\{0\}$. In particular, we obtain that near $(0, \overline{u}(0), \overline{u}(0))$, the solutions on this branch are non-synchronized, which completes the proof of Theorem 1.1 (iii). □

3. Synchronization and non-existence results

In this section, we prove the following results, which extend Theorem 1.1 to more general systems and manifolds:

Theorem 3.1. *(Case $\lambda_1 = \lambda_2$) Let $\lambda, a_{11}, a_{12}, a_{21}, a_{22} \in \mathbb{R}$, $q \in (2, \infty)$ and (M, g) be a smooth, closed Riemannian manifold. Consider the system (1.2) with $\lambda_1 = \lambda_2 = \lambda$.*

(i) *If either $[a_{21} \leq a_{11}$ and $a_{22} \leq a_{12}]$ or $[a_{11} \leq a_{21}$ and $a_{12} \leq a_{22}]$ and at least one of the two inequalities is strict, then (1.2) has no solutions.*
(ii) *If either $[a_{11} < a_{21}$ and $a_{22} < a_{12}]$ or $[a_{11} = a_{21}$ and $a_{22} = a_{12}]$, then every solution of (1.2) is synchronized.*
(iii) *Assuming that there exists at least one non-zero eigenvalue of Δ_g with odd multiplicity, Theorem 2.2 provides examples of real numbers $\lambda, a_{11}, a_{12}, a_{21}, a_{22} > 0$ such that $a_{21} < a_{11}$, $a_{12} < a_{22}$ and (1.2) has non-synchronized solutions.*

Theorem 3.2. *(Case $\lambda_1 < \lambda_2$) Let $\lambda_1, \lambda_2, a_{11}, a_{12}, a_{21}, a_{22} \in \mathbb{R}$, $q \in (2, \infty)$ and (M, g) be a smooth, closed Riemannian manifold. Consider the system (1.2). Assume that $\lambda_1 < \lambda_2$.*

(i) *If $[a_{21} \leq a_{11}$ and $a_{22} \leq a_{12}]$, then (1.2) has no solutions.*
(ii) *Assuming that there exists at least one non-zero eigenvalue of Δ_g with odd multiplicity, Theorem 2.2 provides examples of real numbers $\lambda_1, \lambda_2, a_{11}, a_{12}, a_{21}, a_{22} > 0$ such that $\lambda_2 > \lambda_1 > 0$ and (1.2) has non-synchronized solutions in each of the following cases: $[a_{21} < a_{11}$ and $a_{12} < a_{22}]$, $[a_{11} < a_{21}$ and $a_{22} < a_{12}]$, $[a_{11} < a_{21}$ and $a_{12} < a_{22}]$, $[a_{11} = a_{21}$ and $a_{12} < a_{22}]$ and $[a_{11} < a_{21}$ and $a_{12} = a_{22}]$.*

Proof of Theorem 3.1 (i) and Theorem 3.2 (i). Suppose that there exists a solution (u_1, u_2) of (1.2). We define

$$v(x) := \frac{u_1(x)}{u_2(x)} \qquad \forall x \in M.$$

We then obtain

$$\Delta_g v = \frac{\Delta_g u_1}{u_2} + 2\frac{\langle \nabla_g u_1, \nabla_g u_2 \rangle_g}{u_2^2} - \frac{u_1 \Delta_g u_2}{u_2^2} - 2\frac{u_1 |\nabla_g u_2|_g^2}{u_2^3}$$

$$= (a_{11} - a_{21}) u_1^{q-2} v + (a_{12} - a_{22}) u_2^{q-2} v + (\lambda_2 - \lambda_1) v + 2\frac{\langle \nabla_g v, \nabla_g u_2 \rangle_g}{u_2}$$

in M. In the case where $a_{21} \leq a_{11}$, $a_{22} \leq a_{12}$, $\lambda_1 \leq \lambda_2$ and one of these inequalities is strict, we then obtain

$$\Delta_g v > 2 \left\langle \nabla_g v, \nabla_g \left[\ln u_2\right] \right\rangle_g \quad \text{in } M,$$

which is in contradiction with the minimum principle. Similarly, in the case where $a_{11} \leq a_{21}$, $a_{12} \leq a_{22}$, $\lambda_2 \leq \lambda_1$ and one of these inequalities is strict, we obtain

$$\Delta_g v < 2 \left\langle \nabla_g v, \nabla_g \left[\ln u_2\right] \right\rangle_g \quad \text{in } M,$$

which is in contradiction with the maximum principle. This proves Theorem 3.1 (i) and Theorem 3.2 (i). □

Proof of Theorem 3.1 (ii). Suppose first that $a_{11} < a_{21}$, $a_{22} < a_{12}$ and $\lambda_1 = \lambda_2$. Let v be as in the previous proof and $x_1, x_2 \in M$ be such that

$$v\left(x_1\right) = \min\left\{v\left(x\right) : \ x \in M\right\} \quad \text{and} \quad v\left(x_2\right) = \max\left\{v\left(x\right) : \ x \in M\right\}.$$

We then obtain

$$0 \geq \Delta_g v\left(x_1\right) = \left[\left(a_{12} - a_{22}\right) u_2\left(x_1\right)^{q-2} - \left(a_{21} - a_{11}\right) u_1\left(x_1\right)^{q-2}\right] v\left(x_1\right)$$

and

$$0 \leq \Delta_g v\left(x_2\right) = \left[\left(a_{12} - a_{22}\right) u_2\left(x_2\right)^{q-2} - \left(a_{21} - a_{11}\right) u_1\left(x_2\right)^{q-2}\right] v\left(x_2\right),$$

which imply

$$v\left(x_2\right) \leq \left(\frac{a_{12} - a_{22}}{a_{21} - a_{11}}\right)^{1/(q-2)} \leq v\left(x_1\right).$$

It follows that v is constant in M.

Now suppose that $a_{11} = a_{21}$, $a_{22} = a_{12}$ and $\lambda_1 = \lambda_2$. In this case, we have

$$\Delta_g v = 2 \left\langle \nabla_g v, \nabla_g \left[\ln u_2\right] \right\rangle_g \quad \text{in } M.$$

It then follows from the maximum principle that v is constant in M. This completes the proof of Theorem 3.1 (ii). □

Proof of Theorem 3.1 (iii) and Theorem 3.2 (ii). We choose our examples of the form $a_{11}\left(\alpha\right) = \lambda_1\left(\alpha\right) a$, $a_{12}\left(\alpha\right) = \lambda_1\left(\alpha\right) b$, $a_{21}\left(\alpha\right) = \lambda_2\left(\alpha\right) b$ and $a_{22}\left(\alpha\right) = \lambda_2\left(\alpha\right) a$ for some $a, b > 0$ and $\lambda_1, \lambda_2 \in C^1\left(I, \left(0, \infty\right)\right)$, where $I := \left[-\delta, \delta\right]$ for some $\delta > 0$ to be chosen small. As is easily seen, for every $\alpha \in I$, if

$$a \neq b \quad \text{and} \quad D\left(\alpha\right) := \left(\lambda_1\left(\alpha\right) - \lambda_2\left(\alpha\right)\right)^2 a^2 + 4\lambda_1\left(\alpha\right) \lambda_2\left(\alpha\right) b^2 > 0,$$

then the system (1.2) has a unique constant solution given by

$$\begin{pmatrix} \overline{u}_1\left(\alpha\right) \\ \overline{u}_2\left(\alpha\right) \end{pmatrix} = \left(a + b\right)^{-1/(q-2)} \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

and the matrix $A(\alpha)$ has two distinct, non-zero real eigenvalues given by

$$\{\beta_1(\alpha),\beta_2(\alpha)\}=\left\{\frac{(\lambda_1(\alpha)+\lambda_2(\alpha))\,a\pm\sqrt{D(\alpha)}}{2(a+b)}\right\}.$$

Now, we treat each case separately and choose a,b,λ_1 and λ_2 in such a way that $a\neq b$, $D>0$, $(q-2)\,\beta_1(0)\notin\operatorname{spec}(\Delta_g)$, $(q-2)\,\beta_2(0)=\lambda_0$, $\beta_2'(0)\neq 0$ and either $\lambda_1(0)\neq\lambda_2(0)$ or $\beta_2(0)\neq\lambda_1(0)=\lambda_2(0)$, where $\operatorname{spec}(\Delta_g)$ stands for the spectrum of Δ_g and λ_0 is a non-zero eigenvalue of Δ_g with odd multiplicity (which existence is given by assumption). Choosing δ small enough, we can then apply Theorem 2.2.

Case $\lambda_1=\lambda_2$, $a_{21}<a_{11}$ *and* $a_{12}<a_{22}$. Choose for instance $a:=\frac{\lambda_0+\lambda}{q-2}$, $b:=\frac{\lambda-\lambda_0}{q-2}$ and $\lambda_1(\alpha)=\lambda_2(\alpha):=\frac{\lambda(\alpha+1)}{q-2}$ for some $\lambda\in(\lambda_0,\infty)\setminus\operatorname{spec}(\Delta_g)$. Then $\lambda_1=\lambda_2>0$, $0<a_{21}=a_{12}<\frac{\lambda\lambda_1}{q-2}<a_{11}=a_{22}$, $a\neq b$, $D=4\lambda_1^2b^2>0$, $\beta_1(0)=\frac{\lambda}{q-2}$ (so that $(q-2)\,\beta_1(0)\notin\operatorname{spec}(\Delta_g)$) and $\beta_2(0)=\beta_2'(0)=\frac{\lambda_0}{q-2}>\frac{\lambda}{q-2}=\lambda_1(0)=\lambda_2(0)$.

Case $\lambda_1<\lambda_2$, $a_{21}<a_{11}$ *and* $a_{12}<a_{22}$. Choose for instance $a:=1$, $b:=\varepsilon$, $\lambda_1(\alpha):=\frac{2(1+\varepsilon)\lambda_0(\alpha+1)}{(2+\varepsilon+\varepsilon\sqrt{5+4\varepsilon})(q-2)}$ and $\lambda_2(\alpha):=(1+\varepsilon)\,\lambda_1(\alpha)$ for some small $\varepsilon>0$. Then $0<\lambda_1<\lambda_2$, $0<a_{21}=\varepsilon(1+\varepsilon)\,\lambda_1<\lambda_1=a_{11}$, $0<a_{12}=\varepsilon\lambda_1<(1+\varepsilon)\,\lambda_1=a_{22}$, $a\neq b$, $D=\varepsilon^2(5+4\varepsilon)\,\lambda_1^2>0$, $\beta_1(0)=\frac{(2+\varepsilon-\varepsilon\sqrt{5+4\varepsilon})\lambda_0}{(2+\varepsilon+\varepsilon\sqrt{5+4\varepsilon})(q-2)}<\frac{\lambda_0}{q-2}$, $\beta_1(0)\to\frac{\lambda_0}{q-2}$ as $\varepsilon\to 0$ (so that $(q-2)\,\beta_1(0)\notin\operatorname{spec}(\Delta_g)$) and $\beta_2(0)=\beta_2'(0)=\frac{\lambda_0}{q-2}$.

Case $\lambda_1<\lambda_2$, $a_{11}<a_{21}$ *and* $a_{22}<a_{12}$. Choose for instance $a:=1$, $b:=\sqrt{6}$, $\lambda_1(\alpha):=\frac{(1+\sqrt{6})\lambda_0(\alpha+1)}{5(q-2)}$ and $\lambda_2(\alpha):=2\lambda_1(\alpha)$. Then $0<\lambda_1<\lambda_2$, $0<a_{11}=\lambda_1<2\sqrt{6}\lambda_1=a_{21}$, $0<a_{22}=2\lambda_1<\sqrt{6}\lambda_1=a_{12}$, $a\neq b$, $D=49\lambda_1^2>0$, $\beta_1(0)=\frac{-2\lambda_0}{5(q-2)}<0$ (so that $(q-2)\,\beta_1(0)\notin\operatorname{spec}(\Delta_g)$) and $\beta_2(0)=\beta_2'(0)=\frac{\lambda_0}{q-2}$.

Case $\lambda_1<\lambda_2$, $a_{11}<a_{21}$ *and* $a_{12}<a_{22}$. Choose for instance $a:=1$, $b:=2$, $\lambda_1(\alpha):=\frac{3\lambda_0(\alpha+1)}{(3+2\sqrt{6})(q-2)}$ and $\lambda_2(\alpha):=5\lambda_1(\alpha)$. Then $0<\lambda_1<\lambda_2$, $0<a_{11}=\lambda_1<10\lambda_1=a_{21}$, $0<a_{12}=2\lambda_1<5\lambda_1=a_{22}$, $a\neq b$, $D=96\lambda_1^2>0$, $\beta_1(0)=\frac{(3-2\sqrt{6})\lambda_0}{(3+2\sqrt{6})(q-2)}<0$ (so that $(q-2)\,\beta_1(0)\notin\operatorname{spec}(\Delta_g)$) and $\beta_2(0)=\beta_2'(0)=\frac{\lambda_0}{q-2}$.

Case $\lambda_1<\lambda_2$, $a_{11}=a_{21}$ *and* $a_{12}<a_{22}$. Choose for instance $a:=1+\varepsilon$, $b:=1$, $\lambda_1(\alpha):=\frac{2(2+\varepsilon)\lambda_0(\alpha+1)}{((2+\varepsilon)(1+\varepsilon)+\sqrt{(1+\varepsilon)(4+\varepsilon^2+\varepsilon^3)})(q-2)}$ and $\lambda_2(\alpha):=(1+\varepsilon)\,\lambda_1(\alpha)$ for some small $\varepsilon>0$. Then $0<\lambda_1<\lambda_2$, $a_{11}=a_{21}=(1+\varepsilon)\,\lambda_1>0$, $0<a_{12}=\lambda_1<(1+\varepsilon)^2\lambda_1=a_{22}$, $a\neq b$, $D=$

$(1+\varepsilon)(4+\varepsilon^2+\varepsilon^3)\lambda_1^2 > 0$, $\beta_1(0) = \frac{((2+\varepsilon)(1+\varepsilon)-\sqrt{(1+\varepsilon)(4+\varepsilon^2+\varepsilon^3)})\lambda_0}{((2+\varepsilon)(1+\varepsilon)+\sqrt{(1+\varepsilon)(4+\varepsilon^2+\varepsilon^3)})(q-2)} > 0$, $\beta_1(0) \to 0$ as $\varepsilon \to 0$ (so that $(q-2)\beta_1(0) \notin \operatorname{spec}(\Delta_g)$) and $\beta_2(0) = \beta_2'(0) = \frac{\lambda_0}{q-2}$.

Case $\lambda_1 < \lambda_2$, $a_{11} < a_{21}$ *and* $a_{12} = a_{22}$. Choose for instance $a := 1$, $b := 3$, $\lambda_1(\alpha) := \frac{2\lambda_0(\alpha+1)}{(1+\sqrt{7})(q-2)}$ and $\lambda_2(\alpha) := 3\lambda_1(\alpha)$. Then $0 < \lambda_1 < \lambda_2$, $0 < a_{11} = \lambda_1 < 9\lambda_1 = a_{21}$, $a_{12} = a_{22} = 3\lambda_1 > 0$, $a \neq b$, $D = 112\lambda_1^2 > 0$, $\beta_1(0) = \frac{(1-\sqrt{7})\lambda_0}{(1+\sqrt{7})(q-2)} < 0$ (so that $(q-2)\beta_1(0) \notin \operatorname{spec}(\Delta_g)$) and $\beta_2(0) = \beta_2'(0) = \frac{\lambda_0}{q-2}$.

The results then follow by applying Theorem 2.2. □

References

[1] N. Akhmediev and A. Ankiewicz, *Partially Coherent Solitons on a Finite Background*, Phys. Rev. Lett. **82** (1999), no. 13, 2661–2664.

[2] T. Bartsch, N. Dancer, and Z.-Q. Wang, *A Liouville theorem, a-priori bounds, and bifurcating branches of positive solutions for a nonlinear elliptic system*, Calc. Var. Partial Differential Equations **37** (2010), no. 3-4, 345–361.

[3] M.-F. Bidaut-Véron and L. Véron, *Nonlinear elliptic equations on compact Riemannian manifolds and asymptotics of Emden equations*, Invent. Math. **106** (1991), no. 3, 489–539.

[4] Y.-H. Chen and W. Zou, *Vector solutions for two coupled Schrödinger equations on Riemannian manifolds*, J. Math. Phys. **60** (2019), no. 5, Art. 051502, 21 p.

[5] M. Clapp and A. Pistoia, *Existence and phase separation of entire solutions to a pure critical competitive elliptic system*, Calc. Var. Partial Differential Equations **57** (2018), no. 1, Art. 23, 20 p.

[6] ———, *Fully nontrivial solutions to elliptic systems with mixed couplings*, Nonlinear Anal. **216** (2022), Art. 112694, 19 p.

[7] M. Clapp, A. Pistoia, and H. Tavares, *Yamabe systems, optimal partitions, and nodal solutions to the Yamabe equation*, arXiv:2106.00579 (2021).

[8] M. G. Crandall and P. H. Rabinowitz, *Bifurcation from simple eigenvalues*, J. Functional Analysis **8** (1971), 321–340.

[9] W. Chen, J. Wei, and S. Yan, *Infinitely many solutions for the Schrödinger equations in $\mathbb{R}^n$ with critical growth*, J. Differential Equations **252** (2012), no. 3, 2425–2447.

[10] O. Druet and E. Hebey, *Stability for strongly coupled critical elliptic systems in a fully inhomogeneous medium*, Anal. PDE **2** (2009), no. 3, 305–359.

[11] O. Druet, E. Hebey, and J. Vétois, *Bounded stability for strongly coupled critical elliptic systems below the geometric threshold of the conformal Laplacian*, J. Funct. Anal. **258** (2010), no. 3, 999–1059.

[12] B. D. Esry, C. H. Greene, J. P. Burke, and J. L. Bohn, *Hartree-Fock theory for double condensates*, Phys. Rev. Lett. **78** (1997), no. 19, 3594–3597.

[13] F. Gladiali, *Separation of branches of $O(N-1)$-invariant solutions for a semilinear elliptic equation*, J. Math. Anal. Appl. **453** (2017), no. 1, 159–173.

[14] F. Gladiali, M. Grossi, and C. Troestler, *Entire radial and nonradial solutions for systems with critical growth*, Calc. Var. Partial Differential Equations **57** (2018), no. 2, Art. 53, 26 p.
[15] ———, *A non-variational system involving the critical Sobolev exponent. The radial case*, J. Anal. Math. **138** (2019), no. 2, 643–671.
[16] Y. Guo, B. Li, and J. Wei, *Entire nonradial solutions for non-cooperative coupled elliptic system with critical exponents in* $\mathbb{R}^3$, J. Differential Equations **256** (2014), no. 10, 3463–3495.
[17] Y. Guo and J. Liu, *Liouville type theorems for positive solutions of elliptic system in* $\mathbb{R}^N$, Comm. Partial Differential Equations **33** (2008), no. 1-3, 263–284.
[18] E. Hebey and M. Vaugon, *Meilleures constantes dans le théorème d'inclusion de Sobolev et multiplicité pour les problèmes de Niremberg et Yamabe*, Indiana Univ. Math. J. **41** (1992), no. 2, 377–407 (French).
[19] E. Hebey and J. Wei, *Resonant states for the static Klein-Gordon-Maxwell-Proca system*, Math. Res. Lett. **19** (2012), no. 4, 953-967.
[20] T. Kanna and M. Lakshmanan, *Exact soliton solutions, shape changing collisions, and partially coherent solitons in coupled nonlinear Schrdinger equations*, Phys. Rev. Lett. **86** (2001), no. 22, 5043–5046.
[21] H. Kielhöfer, *Bifurcation theory*, 2nd ed., Applied Mathematical Sciences, vol. 156, Springer, New York, 2012.
[22] C. Li and L. Ma, *Uniqueness of positive bound states to Schrödinger systems with critical exponents*, SIAM J. Math. Anal. **40** (2008), no. 3, 1049–1057.
[23] C. Li and J. Villavert, *A degree theory framework for semilinear elliptic systems*, Proc. Amer. Math. Soc. **144** (2016), no. 9, 3731–3740.
[24] ———, *Existence of positive solutions to semilinear elliptic systems with supercritical growth*, Comm. Partial Differential Equations **41** (2016), no. 7, 1029–1039.
[25] T.-C. Lin and J. Wei, *Ground state of* N *coupled nonlinear Schrödinger equations in* $\mathbb{R}^n$, $n \leq 3$, Comm. Math. Phys. **255** (2005), no. 3, 629–653.
[26] Z. Liu and Z.-Q. Wang, *Multiple bound states of nonlinear Schrödinger systems*, Comm. Math. Phys. **282** (2008), no. 3, 721–731.
[27] A. M. Micheletti, A. Pistoia, and J. Vétois, *Blow-up solutions for asymptotically critical elliptic equations*, Indiana Univ. Math. J. **58** (2009), no. 4, 1719–1746.
[28] A. Montaru, B. Sirakov, and P. Souplet, *Proportionality of components, Liouville theorems and a priori estimates for noncooperative elliptic systems*, Arch. Ration. Mech. Anal. **213** (2014), no. 1, 129–169.
[29] S. Peng, Y. Peng, and Z. Wang, *On elliptic systems with Sobolev critical growth*, Calc. Var. Partial Differential Equations **55** (2016), no. 6, Art. 142, 30 p.
[30] P. Quittner and P. Souplet, *Symmetry of components for semilinear elliptic systems*, SIAM J. Math. Anal. **44** (2012), no. 4, 2545–2559.
[31] B. Sirakov, *Least energy solitary waves for a system of nonlinear Schrödinger equations in* $\mathbb{R}^n$, Comm. Math. Phys. **271** (2007), no. 1, 199–221.
[32] N. Soave and A. Zilio, *Uniform bounds for strongly competing systems: the optimal Lipschitz case*, Arch. Ration. Mech. Anal. **218** (2015), no. 2, 647–697.
[33] E. Timmermans, *Phase separation of Bose-Einstein condensates*, Phys. Rev. Lett. **81** (1998), no. 26, 5718–5721.

[34] S. Terracini and G. Verzini, *Multipulse phases in k-mixtures of Bose-Einstein condensates*, Arch. Ration. Mech. Anal. **194** (2009), no. 3, 717–741.

[35] J. Vétois and S. Wang, *Infinitely many solutions for cubic nonlinear Schrödinger equations in dimension four*, Adv. Nonlinear Anal. **8** (2019), no. 1, 715–724.

[36] J. Wei and Y. Wu, *Ground states of nonlinear Schrödinger systems with mixed couplings*, J. Math. Pures Appl. (9) **141** (2020), 50–88.

[37] D. Westreich, *Bifurcation at eigenvalues of odd multiplicity*, Proc. Amer. Math. Soc. **41** (1973), 609–614.

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