

EXISTENCE RESULTS FOR THE HIGHER-ORDER Q-CURVATURE EQUATION

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ABSTRACT. We obtain existence results for the Q -curvature equation of order $2k$ on a closed Riemannian manifold of dimension $n \geq 2k + 1$, where $k \geq 1$ is an integer. We obtain these results under the assumptions that the Yamabe invariant of order $2k$ is positive and the Green's function of the corresponding operator is positive, which are satisfied in particular when the manifold is Einstein with positive scalar curvature. In the case where $2k + 1 \leq n \leq 2k + 3$ or the manifold is locally conformally flat, we assume moreover that the operator has positive mass. In the case where $n \geq 2k + 4$ and the manifold is not locally conformally flat, the results essentially reduce to the determination of the sign of a complicated constant depending only on n and k .

1. INTRODUCTION AND MAIN RESULTS

Given an integer $k \geq 1$, a smooth, closed Riemannian manifold (M, g) of dimension $n > 2k$ and a smooth positive function f in M , we consider the equation

$$P_{2k}u = f|u|^{2_k^*-2}u \quad \text{in } M, \tag{1.1}$$

where P_{2k} is the GJMS operator with leading part Δ^k , $\Delta := \delta d$ is the Laplace–Beltrami operator with nonnegative eigenvalues and $2_k^* := 2n/(n - 2k)$ is the critical Sobolev exponent. The so-called GJMS operators were discovered by Graham, Jenne, Mason and Sparling [18] by using a construction based on the Fefferman–Graham ambient metric [14, 15]. They provide a natural extension to higher orders of the Yamabe operator [42] ($k = 1$) and the Paneitz–Branson operator [4, 32] ($k = 2$). When u is positive, (1.1) arises in the problem of prescribing Branson's Q -curvature of order $2k$ in a given conformal class (see Branson [5]). More precisely, the positive solutions u to the equation (1.1) correspond to the conformal metrics $u^{4/(n-2k)}g$ with Q -curvature of order $2k$ equal to $\frac{2}{n-2k}f$.

Let Y_{2k} be the conformal invariant defined by

$$Y_{2k} := \inf_{\tilde{g} \in [g]} \left(\text{Vol}_{\tilde{g}}(M)^{-\frac{n-2k}{n}} \int_M Q_{2k, \tilde{g}} dv_{\tilde{g}} \right) = \inf_{\substack{u \in C^\infty(M) \\ u > 0 \text{ in } M}} \frac{\int_M u P_{2k}u dv_g}{\left(\int_M u^{2_k^*} dv_g \right)^{\frac{n-2k}{n}}},$$

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where $[g]$ is the conformal class of g , and $\text{Vol}_{\tilde{g}}(M)$, $dv_{\tilde{g}}$ and $Q_{2k,\tilde{g}}$ are the volume, volume element and Q -curvature of order $2k$, respectively, of (M, \tilde{g}) . Throughout this paper, we assume that $Y_{2k} > 0$. As is easily seen, this is equivalent to the coercivity of the operator P_{2k} , which is also equivalent to $\lambda_1(P_{2k}) > 0$, where $\lambda_1(P_{2k})$ is the first eigenvalue of P_{2k} .

In the case where $k = 1$, it is well-known that there exists at least one positive solution to the equation (1.1) with $f \equiv 1$ if and only if $Y_2 > 0$ (see the historic work of Aubin [2], Schoen [36], Trudinger [41] and Yamabe [42]). In the case where $k = 2$, the existence of at least one positive solution to this problem has been obtained under positivity assumptions on the scalar curvature and Q -curvature of order 4 (see Gursky and Malchiodi [21]) and later extended to the cases where $Y_2 > 0$ and $Y_4 > 0$ in dimension $n \geq 6$ (see Gursky, Hang and Lin [20]) and the case where $Y_2 > 0$ and $Q_4 > 0$ in dimension $n \geq 5$ (see Hang and Yang [22, 23]). This question has also been solved by Qing and Raske [33] in the locally conformally flat case for all orders $k \geq 2$, under a topological assumption on the Poincaré exponent of the holonomy representation of the fundamental group, using an approach introduced by Schoen [37] for $k = 1$. More general existence results have also been obtained in the case where $f \not\equiv 1$ (see among others Aubin [2], Escobar and Schoen [12], Hebey [24] and Hebey and Vaugon [25] for $k = 1$, Djadli, Hebey and Ledoux [10], Esposito and Robert [13] and Robert [34] for $k = 2$, Chen and Hou [9] for $k = 3$ and Robert [35] for higher orders).

We let W be the Weyl tensor of (M, g) and $|W|$ be the norm of W with respect to g . In the case where $2k + 1 \leq n \leq 2k + 3$ or (M, g) is locally conformally flat, assuming that $Y_{2k} > 0$, for every point $\xi \in M$, we let $m(\xi)$ be the mass of P_{2k} at ξ (see (3.2) for the definition of the mass). Our main result is the following:

Theorem 1.1. *Let $k \geq 1$ be an integer, (M, g) be a smooth, closed Riemannian manifold of dimension $n \geq 2k + 1$ and f be a smooth positive function in M . Assume that $Y_{2k} > 0$ and there exists a maximal point ξ of f such that*

$$\Delta f(\xi) = 0 \quad \text{if } n \geq 2k + 2 \quad (1.2)$$

and

$$\begin{cases} |W(\xi)|^2 f(\xi) + c(n, k) \Delta^2 f(\xi) > 0 & \text{if } n \geq 2k + 5 \\ W(\xi) \neq 0 & \text{if } n = 2k + 4 \\ m(\xi) > 0 & \text{if } 2k + 1 \leq n \leq 2k + 3, \end{cases} \quad (1.3)$$

where $c(n, k)$ is a positive constant depending only on n and k (see (2.65) for the value of $c(n, k)$). Then there exists a nontrivial solution $u \in C^{2k}(M)$ to the equation (1.1), which minimizes the energy functional (2.1). If moreover the Green's function of the operator P_{2k} is positive, then u is positive, which implies that the Q -curvature of order $2k$ of the metric $u^{4/(n-2k)}g$ is equal to $\frac{2}{n-2k}f$.

In particular, Theorem 1.1 extends to all orders previous results obtained by Aubin [2] for $k = 1$ (in this case, the positivity of the Green's function is not an issue), Esposito and Robert [13] for $k = 2$ and Chen and Hou [9] for $k = 3$.

In the case where f is constant, we obtain the following:

Theorem 1.2. *Let $k \geq 1$ be an integer and (M, g) be a smooth, closed Riemannian manifold of dimension $n \geq 2k + 1$. Assume that $Y_{2k} > 0$ and its Green's function is positive. Assume moreover that if $2k + 1 \leq n \leq 2k + 3$ or (M, g) is locally*

conformally flat, then $m(\xi) > 0$ for some point $\xi \in M$. Then there exists a conformal metric to g with constant Q -curvature of order $2k$.

Notice that Theorem 1.2 is a direct consequence of Theorem 1.1 in the case where (M, g) is not locally conformally flat of dimension $n \geq 2k+4$. A more general result about the locally conformally flat case will be stated in Section 3.

When (M, g) is Einstein, Fefferman and Graham [15, Proposition 7.9] (see also Gover [17] for a proof based on tractors) established the formula

$$P_{2k} = \prod_{j=1}^k \left(\Delta + \frac{(n+2j-2)(n-2j)}{4n(n-1)} S \right),$$

where S is the Scalar curvature of (M, g) . In this case, it is easy to see that if S is positive, then P_{2k} is coercive, and so $Y_{2k} > 0$. Furthermore, successive applications of the maximum principles yield that the Green's function of the operator P_{2k} is positive. Therefore, we obtain the following corollary of Theorem 1.1:

Corollary 1.1. *Let $k \geq 1$ be an integer and (M, g) be a smooth, closed Einstein manifold of positive scalar curvature and dimension $n \geq 2k+1$. Let f be a smooth positive function in M such that there exists a maximal point ξ of f satisfying (1.2) and (1.3). Then there exists a conformal metric to g with Q -curvature of order $2k$ equal to $\frac{2}{n-2k}f$.*

The positivity of the Green's function of the operator P_4 has been shown to be true by Gursky and Malchiodi [21] and Hang and Yang [22, 23] under positivity assumptions on the Q -curvature of order 4 and the scalar curvature or the Yamabe invariant of the manifold. Positivity results for the mass of P_4 have also been obtained by Gursky and Malchiodi [21], Hang and Yang [22], Humbert and Raulot [26] and Michel [31], thus extending the positive mass theorem obtained by Schoen and Yau [38–40] for $k = 1$. As far as the authors know, no such results have yet been obtained for higher orders. As regards the case where $n = 2k$, we point out that the problem of prescribing the Q -curvature involves a different equation than (1.1) which contains an exponential non-linearity. Some references in this case are Chang and Yang [8], Djadli and Malchiodi [11] and Li, Li and Liu [29] for $k = 2$ and Baird, Fardoun and Regbaoui [3] for higher orders.

The proofs of Theorems 1.1 and 1.2 are based on the approach introduced by Aubin [2] and Schoen [36] in the case where $k = 1$. This approach consists in deriving an asymptotic expansion for the energy functional associated with the equation (1.1), which we apply to a suitable family of test functions depending on a real parameter (see (2.1) for the energy functional; see (2.5) and (3.4) for the definitions of our families of test functions). To simplify the calculations of curvature terms, we use the conformal normal coordinates introduced by Lee and Parker [28] and later improved by Cao [7] and Günther [19]. Our proof also crucially relies on the derivation of an expression for the highest-order terms of the GJMS operators (see (2.7)), which we obtain by using Juhl's formulae [27]. In the case where $n \geq 2k+4$, the proof essentially reduces to determining the sign of a constant $C(n, k)$, which appears in the energy expansion (see (2.6)). In particular, we recover the values found in [9, 13] for $C(n, k)$ with $k \in \{2, 3\}$ (see Remark 2.1). We then conclude the proof by using a minimization result in the spirit of Aubin [2] (see Mazumdar [30, Theorem 3]). When the Green's function of the operator P_{2k} is positive,

by an application of the Green's representation formula, we obtain moreover that the minimizing solution is positive (see the argument in [30, end of Section 3]). We point out that at one place in the proof, namely in the very last computation to determine the sign of $C(n, k)$ (see (2.63)), we have used the computation software *Maple* to expand a complicated polynomial with integer coefficients.

The paper is organized as follows. In Section 2, we prove Theorem 1.1 in the case where $n \geq 2k + 4$. In Section 3, we complete the proof of Theorems 1.1 in the remaining case where $2k + 1 \leq n \leq 2k + 3$ and we state and prove a more general result in the case where g is conformally flat in some open subset of the manifold. Theorem 1.2 then directly follows from this new result together with Theorem 1.1.

2. PROOF OF THEOREM 1.1 IN THE CASE WHERE $n \geq 2k + 4$

Given an integer $k \geq 1$ and a smooth positive function f in M , we let $I_{k,f}$ be the energy functional defined as

$$I_{k,f,g}(u) := \frac{\int_M u P_{2k} u \, dv_g}{\left(\int_M f |u|^{2k} \, dv_g \right)^{\frac{n-2k}{n}}} \quad (2.1)$$

for all functions $u \in C^{2k}(M)$ such that $u \not\equiv 0$. We fix a point $\xi \in M$. By applying a conformal change of metric (see Cao [7], Günther [19] and Lee and Parker [28]), we may assume that

$$\det g(x) = 1 \quad \forall x \in \Omega \quad (2.2)$$

for some neighborhood Ω of the point ξ , where $\det g$ is the determinant of g in geodesic normal coordinates at ξ . In particular (see [28]), it follows from (2.2) that

$$\text{Ric}(\xi) = \text{Sym} \nabla \text{Ric}(\xi) = \text{Sym} \left(\text{Ric}_{ab;cd}(\xi) + \frac{2}{9} W_{eabf}(\xi) W_{cd}^e{}^f(\xi) \right) = 0, \quad (2.3)$$

where Sym stands for the symmetric part, Ric is the Ricci tensor, and $\text{Ric}_{ab;cd}$ and W_{eabf} are the coordinates of $\nabla^2 \text{Ric}$ and W , respectively, with the standard convention on raising and lowering indices. By taking traces in (2.3) and using Bianchi's identities, we obtain

$$S(\xi) = |\nabla S(\xi)| = 0, \quad \Delta S(\xi) = \frac{1}{6} |W(\xi)|^2 \quad \text{and} \quad \text{Ric}^{ab}{}_{;ab}(\xi) = -\frac{1}{12} |W(\xi)|^2 \quad (2.4)$$

Let $r_0 > 0$ be such that the injectivity radius of the metric g at the point ξ is greater than $3r_0$ and $B(\xi, 3r_0) \subset \Omega$, where $B(\xi, r_0)$ is the ball of center ξ and radius r_0 with respect to g . We then let χ be a smooth cutoff function in $[0, \infty)$ such that $\chi \equiv 1$ in $[0, r_0]$, $0 \leq \chi \leq 1$ in $(r_0, 2r_0)$ and $\chi \equiv 0$ in $[2r_0, \infty)$. For every $\mu > 0$, we then define our test functions as

$$U_\mu(x) := \chi(d_g(x, \xi)) \mu^{\frac{2k-n}{2}} U(\mu^{-1} \exp_\xi^{-1} x) \quad \forall x \in M, \quad (2.5)$$

where d_g is the geodesic distance with respect to g , \exp_ξ is the exponential map with respect to g at the point ξ and U is the function in \mathbb{R}^n (we identify $T_\xi M$ with \mathbb{R}^n) defined as

$$U(x) := (1 + |x|^2)^{-\frac{n-2k}{2}} \quad \forall x \in \mathbb{R}^n.$$

It is easy to verify that U is a solution of the equation

$$\Delta_0^k U = \left[\prod_{j=-k}^{k-1} (n+2j) \right] U^{2k^*-1} \quad \text{in } \mathbb{R}^n,$$

where Δ_0 is the Euclidean Laplacian.

Proposition 2.1. *Let $k \geq 1$ be an integer, (M, g) be a smooth, closed Riemannian manifold of dimension $n \geq 2k + 4$ and f be a smooth positive function in M . Assume that g satisfies (2.2) for some point $\xi \in M$. Let $I_{k,f,g}$ be as in (2.1) and U_μ be as in (2.5). Then there exists a positive constant $C(n, k)$ depending only on n and k (see (2.62) for the value of $C(n, k)$) such that as $\mu \rightarrow 0$,*

$$\begin{aligned} I_{k,f,g}(U_\mu) &= \omega_n^{\frac{2k}{n}} f(\xi)^{-\frac{n-2k}{n}} \left((2k-1)! B\left(\frac{n}{2} - k, 2k\right)^{-1} \right. \\ &\quad \times \left(1 + \frac{(n-2k)}{2n(n-2)} \frac{\Delta f(\xi)}{f(\xi)} \mu^2 \right. \\ &\quad \left. - \frac{(n-2k)}{4n(n-2)} \left(\frac{\Delta^2 f(\xi)}{2(n-4)f(\xi)} - \frac{(n-k)(\Delta f(\xi))^2}{n(n-2)f(\xi)^2} \right) \mu^4 \right. \\ &\quad \left. - C(n, k) \mu^4 \begin{cases} |W(\xi)|^2 \ln(1/\mu) + O(1) & \text{if } n = 2k + 4 \\ |W(\xi)|^2 + o(1) & \text{if } n > 2k + 4. \end{cases} \right), \quad (2.6) \end{aligned}$$

where ω_n is the volume of the standard n -dimensional sphere and B is the beta function defined as

$$B(a, b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} \quad \forall a, b > 0.$$

Proof of Proposition 2.1. We let P be the Schouten tensor defined as

$$P := \frac{1}{n-2} \left(\text{Ric} - \frac{S}{2(n-1)} g \right)$$

and B be the Bach tensor whose coordinates are given by

$$B_{ij} := P_{ab} W_i^a{}^b + P_{ij;a}{}^a - P_{ia;j}{}^a,$$

where W_{iajb} , P_{ab} and $P_{ij;ab}$ are the coordinates of W , P and $\nabla^2 P$, respectively. We let (\cdot, \cdot) be the multiple inner product induced by the metric g for the tensors of same rank, i.e. such that $(S, T) = S^{i_1 \dots i_l} T_{i_1 \dots i_l}$ for all tensors S and T of rank $l \in \mathbb{N}$. The first step in the proof of Proposition 2.1 is as follows:

Step 2.1. *For every $k \in \mathbb{N}$ such that $n \geq 2k + 1$, we have*

$$\begin{aligned} P_{2k} &= \Delta^k + k\Delta^{k-1}(J_1 \cdot) + k(k-1)\Delta^{k-2}(J_2 \cdot + (T_1, \nabla) + (T_2, \nabla^2)) \\ &\quad + k(k-1)(k-2)\Delta^{k-3}((T_3, \nabla^2) + (T_4, \nabla^3)) \\ &\quad + k(k-1)(k-2)(k-3)\Delta^{k-4}(T_5, \nabla^4) + Z, \quad (2.7) \end{aligned}$$

where Z is a smooth linear operator of order less than $2k - 4$ if $k \geq 3$, $Z := 0$ if $k \leq 2$, the functions J_1 and J_2 are defined as

$$J_1 := \frac{n-2}{4(n-1)} S$$

and

$$J_2 := \frac{1}{6} \left(\frac{3n^2 - 12n - 4k + 8}{16(n-1)^2} S^2 - (k+1)(n-4)|P|^2 - \frac{3n+2k-4}{4(n-1)} \Delta S \right),$$

and the tensors T_1, T_2, T_3, T_4 and T_5 are defined as

$$T_1 := \frac{n-2}{4(n-1)} \nabla S - \frac{2}{3} (k+1) \delta P,$$

$$T_2 := \frac{2}{3} (k+1) P,$$

$$T_3 := \frac{n-2}{6(n-1)} \nabla^2 S + \frac{(k+1)(n-2)}{6(n-1)} S P - \frac{k+1}{3} (\delta \nabla P + 2 \nabla \delta P + 2 R * P) - \frac{2}{15} (k+1)(k+2) \left(3 P^\# P + \frac{B}{n-4} \right),$$

$$T_4 := \frac{2}{3} (k+1) \nabla P$$

and

$$T_5 := \frac{2}{5} (k+1) \left(\frac{5k+7}{9} P \otimes P + \nabla^2 P \right),$$

where $\#$ stands for the musical isomorphism with respect to g (i.e. $P^\# := g^{-1} P$), and $\delta \nabla P$, $\nabla \delta P$ and $R * P$ stand for the covariant tensors whose coordinates are given by

$$(\delta \nabla P)_{ij} := -P_{ij;a}{}^a, \quad (\nabla \delta P)_{ij} := -P_i{}^a{}_{;aj} \quad \text{and} \quad (R * P)_{ij} := R_{ia}{}^a{}_b P_j{}^b + R_{ibja} P^{ab}, \quad (2.8)$$

where R_{ibja} , P_{ab} and $P_{ij;ab}$ are the coordinates of the Riemann tensor, P and $\nabla^2 P$, respectively.

Proof of Step 2.1. Throughout this proof, for every integer l , o^l stands for a linear operator of order less than l if $l > 0$ and $o^l := 0$ if $l \leq 0$. Juhl's formulae [27] (see also Fefferman and Graham [16]) give

$$\begin{aligned} P_{2k} &= M_2^k - \sum_{j=1}^{k-1} j(k-j) M_2^{j-1} M_4 M_2^{k-j-1} \\ &+ \frac{1}{4} \sum_{j=1}^{k-2} j(j+1)(k-j)(k-j-1) M_2^{j-1} M_6 M_2^{k-j-2} \\ &+ \sum_{j=2}^{k-2} (j+1)(k-j-1) \sum_{i=1}^{j-1} i(k-i) M_2^{i-1} M_4 M_2^{j-i-1} M_4 M_2^{k-j-2} + o^{2k-5}, \quad (2.9) \end{aligned}$$

where the operators M_2, M_4 and M_6 (which differ by a minus sign from the notations in [16, 27]) are defined as

$$M_2 := P_2 = \Delta + \mu_2,$$

$$M_4 := P_2^2 - P_4 = 4\delta P^\# d + \mu_4$$

and

$$M_6 := P_6 - 2P_4P_2 - 2P_2P_4 + 3P_2^3 = \delta A_6^\# d + \mu_6,$$

where μ_6 is a smooth function in M which we do not need explicitly, μ_2 and μ_4 are the functions defined as

$$\mu_2 := \frac{n-2}{4(n-1)} S \quad \text{and} \quad \mu_4 := \frac{\Delta S}{2(n-1)} + \frac{S^2}{4(n-1)^2} + (n-4)|P|^2$$

and A_6 is the tensor defined as

$$A_6 := 48 P^\# P + \frac{16}{n-4} B.$$

We point out that throughout this paper, we use the same sign convention for the Riemann tensor as in the paper of Lee and Parker [28], which is the opposite of the convention used by Juhl [27]. Straightforward expansions yield

$$\begin{aligned} M_2^k &= \Delta^k + \frac{n-2}{4(n-1)} \sum_{j=1}^k \Delta^{j-1} (S \Delta^{k-j}) \\ &\quad + \frac{(n-2)^2}{16(n-1)^2} \sum_{j=2}^k \sum_{i=1}^{j-1} \Delta^{i-1} (S \Delta^{j-i-1} (S \Delta^{k-j})) + o^{2k-5} \\ &= \Delta^k + \frac{n-2}{4(n-1)} \sum_{j=1}^k \Delta^{j-1} (S \Delta^{k-j}) + \frac{(n-2)^2}{16(n-1)^2} \sum_{j=2}^k (j-1) \Delta^{k-2} (S^2 \cdot) + o^{2k-4} \\ &= \Delta^k + \frac{n-2}{4(n-1)} \sum_{j=1}^k \Delta^{j-1} (S \Delta^{k-j}) + \frac{k(k-1)(n-2)^2}{32(n-1)^2} \Delta^{k-2} (S^2 \cdot) + o^{2k-4} \end{aligned} \tag{2.10}$$

and

$$\begin{aligned} M_2^{j-1} M_4 M_2^{k-j-1} &= 4 \Delta^{j-1} \delta P^\# d \Delta^{k-j-1} + \Delta^{j-1} (\mu_4 \Delta^{k-j-1}) \\ &\quad + \frac{n-2}{n-1} \sum_{i=1}^{j-1} \Delta^{i-1} (S \Delta^{j-i-1} \delta P^\# d \Delta^{k-j-1}) \\ &\quad + \frac{n-2}{n-1} \sum_{i=j+1}^{k-1} \Delta^{j-1} \delta P^\# d \Delta^{i-j-1} (S \Delta^{k-i-1}) + o^{2k-5} \\ &= 4 \Delta^{j-1} \delta P^\# d \Delta^{k-j-1} + \mu_4 \Delta^{k-2} - \frac{(k-2)(n-2)}{n-1} \Delta^{k-3} (S P, \nabla^2) + o^{2k-4} \end{aligned} \tag{2.11}$$

and

$$M_2^{j-1} M_6 M_2^{k-j-2} = \Delta^{j-1} \delta A_6^\# d \Delta^{k-j-2} + o^{2k-5} = -\Delta^{k-3} (A_6, \nabla^2) + o^{2k-4} \tag{2.12}$$

and

$$\begin{aligned} M_2^{i-1} M_4 M_2^{j-i-1} M_4 M_2^{k-j-2} &= 16 \Delta^{i-1} \delta P^\# d \Delta^{j-i-1} \delta P^\# d \Delta^{k-j-2} + o^{2k-5} \\ &= 16 \Delta^{k-4} (P \otimes P, \nabla^4) + o^{2k-4}. \end{aligned} \tag{2.13}$$

Furthermore, by induction, one can check that

$$S \Delta^j = \Delta^j (S \cdot) - j (\Delta S) \Delta^{j-1} + 2j \Delta^{j-1} (\nabla S, \nabla) + 2j(j-1) \Delta^{j-2} (\nabla^2 S, \nabla^2) + o^{2j-2} \tag{2.14}$$

and

$$\begin{aligned} \delta P^\# d\Delta^j &= \Delta^j ((\delta P, \nabla) - (P, \nabla^2)) + j\Delta^{j-1} ((\delta\nabla P + 2\nabla\delta P + 2R * P, \nabla^2) \\ &\quad - 2(\nabla P, \nabla^3)) - 2j(j-1)\Delta^{j-2} (\nabla^2 P, \nabla^4) + o^{2j}, \end{aligned} \quad (2.15)$$

where $\delta\nabla P$, $\nabla\delta P$ and $R * P$ are as in (2.8). The proof of (2.15) relies on the commutation formula

$$u_{;abcd} = u_{;cdab} + R^e{}_{cad} u_{;eb} + R^e{}_{abd} u_{;ce} + R^e{}_{cbd} u_{;ae} + R^e{}_{abc} u_{;de} + o^2 u,$$

which gives

$$\begin{aligned} \delta P^\# d\Delta u - \Delta\delta P^\# du &= (P^{bc} u_{;a}{}^a{}_{;b};c - (P^{bc} u_{;b};ca) {}^a \\ &= P^{bc} (u_{;a}{}^a{}_{;bc} - u_{;bca}{}^a) - P^{bc}{}_{;a}{}^a u_{;bc} - 2P^{bc}{}_{;ca} u_{;b}{}^a - 2P^{bc}{}_{;a} u_{;bc}{}^a + o^2 u \\ &= 2P^{bc} (R^d{}_{bc}{}^a u_{;ad} + R^d{}_{ac}{}^a u_{;bd}) - P^{bc}{}_{;a}{}^a u_{;bc} - 2P^{bc}{}_{;ca} u_{;b}{}^a - 2P^{bc}{}_{;a} u_{;bc}{}^a + o^2 u \\ &= (\delta\nabla P + 2\nabla\delta P + 2R * P, \nabla^2 u) - 2(\nabla P, \nabla^3 u) + o^2 u. \end{aligned}$$

By combining Faulhaber's formulae with (2.11)–(2.15), we obtain

$$\begin{aligned} \sum_{j=1}^k \Delta^{j-1} (S \Delta^{k-j}) &= k\Delta^{k-1} (S \cdot) - \frac{k(k-1)}{2} \Delta^{k-2} ((\Delta S) \cdot) \\ &\quad + k(k-1) \Delta^{k-2} (\nabla S, \nabla) + \frac{2k(k-1)(k-2)}{3} \Delta^{k-3} (\nabla^2 S, \nabla^2) + o^{2k-4} \end{aligned} \quad (2.16)$$

and

$$\begin{aligned} \sum_{j=1}^{k-1} j(k-j) M_2^{j-1} M_4 M_2^{k-j-1} &= k(k-1)(k+1) \left(\frac{2}{3} \Delta^{k-2} ((\delta P, \nabla) - (P, \nabla^2)) \right. \\ &\quad + \frac{k-2}{3} \Delta^{k-3} ((\delta\nabla P + 2\nabla\delta P + 2R * P, \nabla^2) - 2(\nabla P, \nabla^3)) + \frac{1}{6} \Delta^{k-2} (\mu_4 \cdot) \\ &\quad \left. - \frac{2(k-2)(k-3)}{5} \Delta^{k-4} (\nabla^2 P, \nabla^4) - \frac{(k-2)(n-2)}{6(n-1)} \Delta^{k-3} (SP, \nabla^2) \right) + o^{2k-4} \end{aligned} \quad (2.17)$$

and

$$\begin{aligned} \sum_{j=1}^{k-2} j(j+1)(k-j)(k-j-1) M_2^{j-1} M_6 M_2^{k-j-2} \\ = -\frac{k(k-1)(k-2)(k+1)(k+2)}{30} \Delta^{k-3} (A_6, \nabla^2) + o^{2k-4} \end{aligned} \quad (2.18)$$

and

$$\begin{aligned} \sum_{j=2}^{k-2} (j+1)(k-j-1) \sum_{i=1}^{j-1} i(k-i) M_2^{i-1} M_4 M_2^{j-i-1} M_4 M_2^{k-j-2} \\ = \frac{2k(k-1)(k-2)(k-3)(k+1)(5k+7)}{45} \Delta^{k-4} (P \otimes P, \nabla^4) + o^{2k-4}. \end{aligned} \quad (2.19)$$

Finally, (2.7) follows by putting together (2.9), (2.10) and (2.16)–(2.19). This ends the proof of Step 2.1. \square

The next step is as follows:

Step 2.2. Assume that $n \geq 2k + 4$ and $k \geq 3$. Then for every smooth linear operator Z of order less than $2k - 4$, as $\mu \rightarrow 0$,

$$\int_M U_\mu Z U_\mu dv_g = \begin{cases} \mathcal{O}(\mu^4) & \text{if } n = 2k + 4 \\ \mathfrak{o}(\mu^4) & \text{if } n > 2k + 4. \end{cases} \quad (2.20)$$

Proof of Step 2.2. By rewriting the integral in geodesic normal coordinates, we obtain

$$\int_M U_\mu Z U_\mu dv_g = \int_{B(0, 2r_0)} \tilde{U}_\mu \tilde{Z} \tilde{U}_\mu dx = \sum_{|\alpha| < 2k-4} \int_{B(0, 2r_0)} z_\alpha \tilde{U}_\mu \partial^\alpha \tilde{U}_\mu dx, \quad (2.21)$$

where

$$\tilde{U}_\mu(x) := \mu^{\frac{2k-n}{2}} U(x/\mu) \quad \text{and} \quad \tilde{Z}(x) := \sum_{|\alpha| < 2k-4} z_\alpha(x) \partial^\alpha \quad \forall x \in B(0, 2r_0) \quad (2.22)$$

for some smooth functions z_α in $B(0, 2r_0)$, where α is a multi-index. A straightforward change of variable then gives

$$\int_{B(0, 2r_0)} z_\alpha \tilde{U}_\mu \partial^\alpha \tilde{U}_\mu dx = \mu^{2k-|\alpha|} \int_{B(0, 2r_0/\mu)} z_\alpha(\mu x) U(x) \partial^\alpha U(x) dx. \quad (2.23)$$

An easy induction yields that for every multi-index α , there exists a constant C_α such that

$$|\partial^\alpha U(x)| \leq C_\alpha (1 + |x|^2)^{-\frac{n-2k+|\alpha|}{2}} \quad \forall x \in \mathbb{R}^n \quad (2.24)$$

It follows from (2.23) and (2.24) that

$$\begin{aligned} \int_{B(0, 2r_0)} z_\alpha \tilde{U}_\mu \partial^\alpha \tilde{U}_\mu dx &= \mathcal{O} \left(\mu^{2k-|\alpha|} \int_{B(0, 2r_0/\mu)} (1 + |x|^2)^{-n+2k-|\alpha|/2} dx \right) \\ &= \begin{cases} \mathcal{O}(\mu^{2k-|\alpha|}) & \text{if } |\alpha| > 4k - n \\ \mathcal{O}(\mu^{n-2k} \ln(1/\mu)) & \text{if } |\alpha| = 4k - n \\ \mathcal{O}(\mu^{n-2k}) & \text{if } |\alpha| < 4k - n. \end{cases} \end{aligned} \quad (2.25)$$

Finally, (2.20) follows from (2.21) and (2.25). \square

We then prove the following:

Step 2.3. Assume that $n \geq 2k + 4$ and g satisfies (2.2) for some point $\xi \in M$. Then, as $\mu \rightarrow 0$,

$$\int_M U_\mu \Delta^k U_\mu dv_g = 2^{2k-n} (2k-1)! \omega_n B\left(\frac{n}{2} - k, 2k\right)^{-1} + \mathcal{O}(\mu^{n-2k}). \quad (2.26)$$

If $k \geq 2$, then for every smooth function f in M ,

$$\begin{aligned} \int_M f U_\mu \Delta^{k-2} U_\mu dv_g &= \frac{2^{2k-n-1} (n-1)! (k-2)! \omega_n}{(n-2)(n-4)(n-2k-2)} f(\xi) \mu^4 \\ &\quad \times \sum_{l=k-2}^{2k-4} \frac{l!}{(l-k+2)!(2k-l-4)!(n+l-2k-1)!} B\left(\frac{n}{2} - k - 1, l+1\right)^{-1} \end{aligned}$$

$$\begin{aligned}
& \times \left\{ \begin{array}{ll} 2 \ln(1/\mu) & \text{if } n = 2k + 4 \text{ and } l = k - 2 \\ \text{B}\left(\frac{n}{2} + l - 2k, 2k - l - 2\right) & \text{otherwise} \end{array} \right\} \\
& + \left\{ \begin{array}{ll} \text{O}(\mu^4) & \text{if } n = 2k + 4 \\ \text{o}(\mu^4) & \text{if } n > 2k + 4, \end{array} \right. \quad (2.27)
\end{aligned}$$

for every smooth, covariant tensor T of rank 1,

$$\begin{aligned}
\int_M (T, \nabla U_\mu) \Delta^{k-2} U_\mu dv_g &= -\frac{2^{2k-n-2} (n-2k) (n-1)! (k-2)! \omega_n}{(n-2) (n-4) (n-2k-2)} T_{;a}^a(\xi) \mu^4 \\
& \times \sum_{l=k-2}^{2k-4} \frac{l!}{(l-k+2)! (2k-l-4)! (n+l-2k)!} \text{B}\left(\frac{n}{2} - k - 1, l + 1\right)^{-1} \\
& \times \left\{ \begin{array}{ll} 2 \ln(1/\mu) & \text{if } n = 2k + 4 \text{ and } l = k - 2 \\ \text{B}\left(\frac{n}{2} + l - 2k, 2k - l - 2\right) & \text{otherwise} \end{array} \right\} \\
& + \left\{ \begin{array}{ll} \text{O}(\mu^4) & \text{if } n = 2k + 4 \\ \text{o}(\mu^4) & \text{if } n > 2k + 4 \end{array} \right. \quad (2.28)
\end{aligned}$$

and for every smooth, covariant tensor T of rank 2,

$$\begin{aligned}
\int_M (T, \nabla^2 U_\mu) \Delta^{k-2} U_\mu dv_g &= \frac{2^{2k-n-4} (n-2k) (n-1)! (k-2)! \omega_n}{(n-2) (n-4) (n-2k-2)} \\
& \times \sum_{l=k-2}^{2k-4} \frac{l!}{(l-k+2)! (2k-l-4)! (n+l-2k+1)!} \text{B}\left(\frac{n}{2} - k - 1, l + 1\right)^{-1} \\
& \times \left(-2(n-4)(n+2l-2k) \text{B}\left(\frac{n}{2} - 2k + l + 1, 2k - l - 2\right) T_a^a(\xi) \mu^2 \right. \\
& + \left. ((n-2k+2)(T_{;ab}^{ab}(\xi) + T_{;ba}^{ab}(\xi)) - (n+2l-2k) T_a^a{}_{;b}^b(\xi)) \mu^4 \right. \\
& \times \left. \left\{ \begin{array}{ll} 2 \ln(1/\mu) & \text{if } n = 2k + 4 \text{ and } l = k - 2 \\ \text{B}\left(\frac{n}{2} + l - 2k, 2k - l - 2\right) & \text{otherwise} \end{array} \right\} \right) \\
& + \left\{ \begin{array}{ll} \text{O}(\mu^4) & \text{if } n = 2k + 4 \\ \text{o}(\mu^4) & \text{if } n > 2k + 4. \end{array} \right. \quad (2.29)
\end{aligned}$$

If $k \geq 3$, then for every smooth, covariant tensor T of rank 2,

$$\begin{aligned}
\int_M (T, \nabla^2 U_\mu) \Delta^{k-3} U_\mu dv_g &= -\frac{2^{2k-n-5} (n-2k) (n-1)! (k-3)! \omega_n}{(n-2) (n-4) (n-2k-2)} T_a^a(\xi) \mu^4 \\
& \times \sum_{l=k-3}^{2k-6} \frac{(n+2l-2k) l!}{(l-k+3)! (2k-l-6)! (n+l-2k+1)!} \text{B}\left(\frac{n}{2} - k - 1, l + 1\right)^{-1} \\
& \times \left\{ \begin{array}{ll} 2 \ln(1/\mu) & \text{if } n = 2k + 4 \text{ and } l = k - 3 \\ \text{B}\left(\frac{n}{2} + l - 2k + 1, 2k - l - 3\right) & \text{otherwise} \end{array} \right\} \\
& + \left\{ \begin{array}{ll} \text{O}(\mu^4) & \text{if } n = 2k + 4 \\ \text{o}(\mu^4) & \text{if } n > 2k + 4 \end{array} \right. \quad (2.30)
\end{aligned}$$

and for every smooth, covariant tensor T of rank 3,

$$\begin{aligned}
\int_M (T, \nabla^3 U_\mu) \Delta^{k-3} U_\mu dv_g &= \frac{2^{2k-n-6} (n-2k) (n-2k+2) (n-1)! (k-3)! \omega_n}{(n-2) (n-4) (n-2k-2)} \\
&\times (T_a^{ab}{}_{;b}(\xi) + T_a^{ba}{}_{;b}(\xi) + T_a^{ba}{}_{;a}(\xi)) \mu^4 \\
&\times \sum_{l=k-3}^{2k-6} \frac{(n+2l-2k) l!}{(l-k+3)! (2k-l-6)! (n+l-2k+2)!} \mathbf{B}\left(\frac{n}{2} - k - 1, l + 1\right)^{-1} \\
&\times \left\{ \begin{array}{ll} 2 \ln(1/\mu) & \text{if } n = 2k + 4 \text{ and } l = k - 3 \\ \mathbf{B}\left(\frac{n}{2} + l - 2k + 1, 2k - l - 3\right) & \text{otherwise} \end{array} \right\} \\
&+ \left\{ \begin{array}{ll} \mathcal{O}(\mu^4) & \text{if } n = 2k + 4 \\ \mathfrak{o}(\mu^4) & \text{if } n > 2k + 4. \end{array} \right. \tag{2.31}
\end{aligned}$$

If $k \geq 4$, then for every smooth, covariant tensor T of rank 4,

$$\begin{aligned}
\int_M (T, \nabla^4 U_\mu) \Delta^{k-4} U_\mu dv_g &= \frac{2^{2k-n-8} (n-2k) (n-2k+2) (n-1)! (k-4)! \omega_n}{3 (n-2) (n-4) (n-2k-2)} \\
&\times (T_a^{ab}{}_{;b}(\xi) + T_a^{ba}{}_{;b}(\xi) + T_a^{ba}{}_{;a}(\xi)) \mu^4 \\
&\times \sum_{l=k-4}^{2k-8} \frac{(n+2l-2k) (n+2l-2k+2) l!}{(l-k+4)! (2k-l-8)! (n+l-2k+3)!} \mathbf{B}\left(\frac{n}{2} - k - 1, l + 1\right)^{-1} \\
&\times \left\{ \begin{array}{ll} 2 \ln(1/\mu) & \text{if } n = 2k + 4 \text{ and } l = k - 4 \\ \mathbf{B}\left(\frac{n}{2} + l - 2k + 2, 2k - l - 4\right) & \text{otherwise} \end{array} \right\} \\
&+ \left\{ \begin{array}{ll} \mathcal{O}(\mu^4) & \text{if } n = 2k + 4 \\ \mathfrak{o}(\mu^4) & \text{if } n > 2k + 4. \end{array} \right. \tag{2.32}
\end{aligned}$$

Proof of Step 2.3. We let j and l be two integers such that

$$\max(2(k-l-2), 0) \leq j \leq k-l \quad \text{and} \quad \max(k-4, 0) \leq l \leq k$$

and T be a smooth, covariant tensor of rank j . By using geodesic normal coordinates, we obtain

$$\begin{aligned}
&\int_M (T, \nabla^j U_\mu) \Delta^l U_\mu dv_g - \int_{B(\xi, r_0)} (T, \nabla^j U_\mu) \Delta^l U_\mu dv_g \\
&= \int_{B(0, 2r_0) \setminus B(0, r_0)} \tilde{Z}_1 \tilde{U}_\mu \tilde{Z}_2 \tilde{U}_\mu dx \\
&= \sum_{|\alpha_1| \leq j} \sum_{|\alpha_2| \leq 2l} \int_{B(0, 2r_0) \setminus B(0, r_0)} z_{1, \alpha} z_{2, \alpha} (\partial^{\alpha_2} \tilde{U}_\mu) (\partial^{\alpha_1} \tilde{U}_\mu) dx, \tag{2.33}
\end{aligned}$$

where \tilde{U}_μ is as in (2.22) and

$$\tilde{Z}_1(x) := \sum_{|\alpha| \leq j} z_{1, \alpha}(x) \partial^\alpha \quad \text{and} \quad \tilde{Z}_2(x) := \sum_{|\alpha| \leq 2l} z_{2, \alpha}(x) \partial^\alpha \quad \forall x \in B(0, 2r_0)$$

for some smooth functions $z_{1,\alpha}$ and $z_{2,\alpha}$ in $B(0, 2r_0)$. By proceeding as in (2.23)–(2.25), we obtain

$$\int_{B(0,2r_0) \setminus B(0,r_0)} z_{1,\alpha} z_{2,\alpha} \partial^{\alpha_2} \tilde{U}_\mu \partial^{\alpha_1} \tilde{U}_\mu dx = O(\mu^{n-2k}). \quad (2.34)$$

It follows from (2.33) and (2.34) that

$$\int_M (T, \nabla^j U_\mu) \Delta^l U_\mu dv_g = \int_{B(\xi, r_0)} (T, \nabla^j U_\mu) \Delta^l U_\mu dv_g + O(\mu^{n-2k}). \quad (2.35)$$

By using (2.2) and rewriting the integral in the right-hand side of (2.35) in geodesic normal coordinates, we obtain

$$\int_{B(\xi, r_0)} (T, \nabla^j U_\mu) \Delta^l U_\mu dv_g = \sum_{j'=0}^j \int_{B(0, r_0)} \widehat{T}^{i_1 \dots i_{j'}} \circ \exp_\xi U_{\mu, i_1 \dots i_{j'}} \Delta_0^l U_\mu dx, \quad (2.36)$$

where $U_{\mu, i_1 \dots i_{j'}} := \partial^{(i_1 \dots i_{j'})} (U_\mu \circ \exp_\xi)$ and the tensor \widehat{T} is defined as

$$\widehat{T}^{i_1 \dots i_{j'}} := \begin{cases} T^{i_1 \dots i_j} & \text{if } j' = j \\ -\Gamma_{e_1 \dots e_j}^{i_1 \dots i_{j'}} T^{e_1 \dots e_j} & \text{if } j' < j \end{cases},$$

where $\Gamma_{e_1 \dots e_j}^{i_1 \dots i_{j'}}$ is the generalized Christoffel symbol such that $\Gamma_{e_1 \dots e_j}^{i_1 \dots i_{j'}}$ is symmetric in $i_1, \dots, i_{j'}$ and

$$u_{;e_1 \dots e_j} = u_{,e_1 \dots e_j} - \sum_{j'=0}^{j-1} \Gamma_{e_1 \dots e_j}^{i_1 \dots i_{j'}} u_{,i_1 \dots i_{j'}}$$

in geodesic normal coordinates. By using (2.36) together with a straightforward change of variable and a Taylor expansion, we then obtain

$$\begin{aligned} & \int_{B(\xi, r_0)} (T, \nabla^j U_\mu) \Delta^l U_\mu dv_g \\ &= \sum_{j'=0}^j \mu^{2k-2l-j'} \int_{B(0, r_0/\mu)} \widehat{T}^{i_1 \dots i_{j'}} (\exp_\xi(\mu x)) U_{,i_1 \dots i_{j'}}(x) \Delta_0^l U(x) dx \\ &= \sum_{j'=\max(2(k-l-2), 0)}^j \sum_{j''=0}^{j'+2l-2k+4} \frac{\mu^{2k-2l-j'+j''}}{j''!} \widehat{T}^{i_1 \dots i_{j'}, i_{j'+1} \dots i_{j'+j''}}(\xi) \\ & \quad \times \int_{B(0, r_0/\mu)} U_{,i_1 \dots i_{j'}, i_{j'+1} \dots i_{j'+j''}} \Delta_0^l U dx + O\left(\sum_{j'=0}^j \mu^{\max(5, 2k-2l-j')}\right) \\ & \quad \times \int_{B(0, r_0/\mu)} |x|^{\max(j'+2l-2k+5, 0)} |U_{,i_1 \dots i_{j'}} \Delta_0^l U| dx \Big). \end{aligned} \quad (2.37)$$

On the other hand, by using (2.24), we obtain

$$\begin{aligned}
& \mu^{\max(5, 2k-2l-j')} \int_{B(0, r_0/\mu)} |x|^{\max(j'+2l-2k+5, 0)} |U_{,i_1 \dots i_{j'}} \Delta_0^l U| dx \\
&= O \left(\mu^{\max(5, 2k-2l-j')} \int_{B(0, r_0/\mu)} |x|^{\max(j'+2l-2k+5, 0)} (1 + |x|^2)^{-\frac{2n+j'+2l-4k}{2}} dx \right) \\
&= \begin{cases} O(\mu^4) & \text{if } n = 2k + 4 \\ o(\mu^4) & \text{otherwise.} \end{cases} \tag{2.38}
\end{aligned}$$

It follows from (2.35), (2.37) and (2.38) that

$$\begin{aligned}
\int_M (T, \nabla^j U_\mu) \Delta^l U_\mu dv_g &= \sum_{j'=\max(2(k-l-2), 0)}^j \sum_{j''=0}^{j'+2l-2k+4} \frac{\mu^{2k-2l-j'+j''}}{j''!} \\
&\quad \times \widehat{T}^{i_1 \dots i_{j'}, i_{j'+1} \dots i_{j'+j''}}(\xi) \int_{B(0, r_0/\mu)} U_{,i_1 \dots i_{j'} x_{i_{j'+1}} \dots x_{i_{j'+j''}}} \Delta_0^l U dx \\
&\quad + \begin{cases} O(\mu^4) & \text{if } n = 2k + 4 \\ o(\mu^4) & \text{if } n > 2k + 4. \end{cases} \tag{2.39}
\end{aligned}$$

An easy induction gives

$$\begin{aligned}
U_{,i_1 \dots i_j}(x) &= \sum_{m=0}^{\lfloor j/2 \rfloor} \frac{2^{j-2m}}{m!(j-2m)!} \partial_r^{j-m} U(r) \\
&\quad \times \sum_{\sigma \in \mathfrak{S}(j)} \delta_{i_{\sigma(1)} i_{\sigma(2)}} \dots \delta_{i_{\sigma(2m-1)} i_{\sigma(2m)}} x_{i_{\sigma(2m+1)}} \dots x_{i_{\sigma(j)}} \quad \forall x \in \mathbb{R}^n, \tag{2.40}
\end{aligned}$$

where $r := |x|^2$, $U(r) := U(x) = (1+r)^{(2k-n)/2}$, $\mathfrak{S}(j)$ is the set of all permutations of $(1, \dots, j)$ and $\delta_{i_{\sigma(1)} i_{\sigma(2)}}, \dots, \delta_{i_{\sigma(2m-1)} i_{\sigma(2m)}}$ stand for the Kronecker symbols. Furthermore, it is easy to see that

$$\begin{aligned}
\partial_r^j U(r) &= (-1)^j 2^{-j} (n-2k)(n-2k+2) \dots (n-2k+2j-2) (1+r)^{-\frac{n-2k+2j}{2}} \\
&= \frac{2(-1)^j j!}{(n-2k-2)} B\left(\frac{n}{2} - k - 1, j+1\right)^{-1} (1+r)^{-\frac{n-2k+2j}{2}}. \tag{2.41}
\end{aligned}$$

Another induction yields

$$\Delta_0^l U(x) = \begin{cases} \frac{2^{2l+1} l!}{(n-2k-2)(k-l-1)!} \sum_{l'=l}^{2l} \frac{l'!(k+l-l'-1)!}{(l'-l)!(2l-l)!} \\ \quad \times B\left(\frac{n}{2} - k - 1, l'+1\right)^{-1} (1+r)^{-\frac{n+2l'-2k}{2}} & \text{if } l < k \\ 2^{2k} (2k-1)! B\left(\frac{n}{2} - k, 2k\right)^{-1} (1+r)^{-\frac{n+2k}{2}} & \text{if } l = k \end{cases} \tag{2.42}$$

for all $x \in \mathbb{R}^n$. In the case where $j = 0$, $l = k$ and $T \equiv 1$, it follows from (2.42) that

$$\int_{B(0, r_0/\mu)} U \Delta_0^k U dx = 2^{2k-1} (2k-1)! \omega_{n-1} B\left(\frac{n}{2} - k, 2k\right)^{-1} \int_0^{(r_0/\mu)^2} \frac{r^{\frac{n-2}{2}}}{(1+r)^n} dr, \tag{2.43}$$

where $\omega_{n-1} = \text{Vol}(\mathbb{S}^{n-1}, g_0)$ is the volume of the standard $(n-1)$ -dimensional sphere. On the other hand, in the case where $l < k$, by putting together (2.40)–(2.42), we obtain

$$\begin{aligned} \int_{B(0, r_0/\mu)} U_{,i_1 \dots i_{j'}, x_{i_{j'+1}} \dots x_{i_{j'+j''}}} \Delta_0^l U \, dx &= \frac{2^{2l+1} l!}{(n-2k-2)^2 (k-l-1)!} \sum_{l'=l}^{2l} \sum_{m=0}^{\lfloor j'/2 \rfloor} \\ &\frac{(-1)^{j'-m} 2^{j'-2m} l'! (k+l-l'-1)! (j'-m)!}{(l'-l)! (2l-l')! m! (j'-2m)!} \mathbf{B}\left(\frac{n}{2} - k - 1, l' + 1\right)^{-1} \\ &\times \mathbf{B}\left(\frac{n}{2} - k - 1, j' - m + 1\right)^{-1} \int_0^{(r_0/\mu)^2} \frac{r^{\frac{n+j'+j''-2m-2}{2}}}{(1+r)^{n+j'-m+l'-2k}} \, dr \sum_{\sigma \in \mathfrak{S}(j')} \delta_{i_{\sigma(1)} i_{\sigma(2)}} \\ &\dots \delta_{i_{\sigma(2m-1)} i_{\sigma(2m)}} \int_{\mathbb{S}^{n-1}} y_{i_{\sigma(2m+1)}} \dots y_{i_{\sigma(j')}} y_{i_{j'+1}} \dots y_{i_{j'+j''}} \, dv_{g_0}(y). \end{aligned} \quad (2.44)$$

A standard computation gives

$$\int_0^{(r_0/\mu)^2} \frac{r^{a-1} \, dr}{(1+r)^b} = \begin{cases} 2 \ln(1/\mu) + O(1) & \text{if } b = a \\ \mathbf{B}(a, b-a) + O(\mu^{2(b-a)}) & \text{if } b > a. \end{cases} \quad (2.45)$$

On the other hand, by using the fact (see for example Brendle [6, Proposition 28]) that for every homogeneous polynomial Φ of degree $j \geq 2$,

$$\int_{\mathbb{S}^{n-1}} \Phi(y) \, dv_{g_0}(y) = \frac{-1}{j(n+j-2)} \int_{\mathbb{S}^{n-1}} \Delta_0 \Phi(y) \, dv_{g_0}(y),$$

another induction yields that when j is even,

$$\begin{aligned} \int_{\mathbb{S}^{n-1}} y_{i_1} \dots y_{i_j} \, dv_{g_0}(y) &= \frac{(n-2)\omega_{n-1}}{2^{j+1}(j/2)!^2} \mathbf{B}\left(\frac{n-2}{2}, \frac{j+2}{2}\right) \\ &\times \sum_{\sigma \in \mathfrak{S}(j)} \delta_{i_{\sigma(1)} i_{\sigma(2)}} \dots \delta_{i_{\sigma(j-1)} i_{\sigma(j)}}. \end{aligned} \quad (2.46)$$

The integral in (2.46) vanishes when j is odd. By observing that

$$\omega_n = 2^{n-1} \mathbf{B}\left(\frac{n}{2}, \frac{n}{2}\right) \omega_{n-1}, \quad (2.47)$$

we obtain that for even j ,

$$\mathbf{B}\left(\frac{n-2}{2}, \frac{j+2}{2}\right) = \frac{2^{2-n} (n-1)! (j/2)! \omega_n}{(n-2)(n+j/2-1)! \omega_{n-1}} \mathbf{B}\left(\frac{n}{2}, \frac{n+j}{2}\right)^{-1}. \quad (2.48)$$

By using (2.45)–(2.48) together with the identity

$$\begin{aligned} &\mathbf{B}\left(\frac{n}{2}, \frac{n+j'+j''-2m}{2}\right)^{-1} \mathbf{B}\left(\frac{n+j'+j''-2m}{2}, \frac{n+j'-j''+2l'-4k}{2}\right) \\ &= \frac{\left(\frac{j'+j''}{2} + n - m - 1\right)!}{(n+j'-m+l'-2k-1)! \left(\frac{j''-j'}{2} + 2k - l' - 1\right)!} \\ &\times \mathbf{B}\left(\frac{n+j'-j''+2l'-4k}{2}, \frac{j''-j'+4k-2l'}{2}\right), \end{aligned}$$

we obtain that if $j' + j''$ is even, then

$$\begin{aligned}
& \int_0^{(r_0/\mu)^2} \frac{r^{n+j'+j''-2m-2}}{(1+r)^{n+j'-m+l'-2k}} dr \int_{\mathbb{S}^{n-1}} y_{i_{\sigma(2m+1)}} \cdots y_{i_{\sigma(j')}} y_{i_{j'+1}} \cdots y_{i_{j'+j''}} dv_{g_0}(y) \\
&= \frac{2^{1-n-j'-j''+2m} (n-1)! \omega_n}{(n+j'-m+l'-2k-1)! \left(\frac{j''-j'}{2} + 2k - l' - 1\right)! \left(\frac{j'+j''}{2} - m\right)!} \\
& \times \left\{ \begin{array}{ll} 2 \ln(1/\mu) + O(1) & \text{if } n + j' - j'' + 2l' - 4k = 0 \\ \text{B}\left(\frac{n+j'-j''+2l'-4k}{2}, \frac{j''-j'+4k-2l'}{2}\right) & \\ + O(\mu^{n+j'-j''+2l'-4k}) & \text{if } 0 < n + j' - j'' + 2l' - 4k < n \end{array} \right\} \\
& \times \sum_{\sigma' \in \mathfrak{S}(S_{j',j'',m,\sigma})} \delta_{i_{\sigma'(\sigma(2m+1))} i_{\sigma'(\sigma(2m+2))}} \cdots \delta_{i_{\sigma'(\sigma(j'-1))} i_{\sigma'(\sigma(j'))}} \\
& \times \delta_{i_{\sigma'(j'+1)} i_{\sigma'(j'+2)}} \cdots \delta_{i_{\sigma'(j'+j''-1)} i_{\sigma'(j'+j'')}} , \tag{2.49}
\end{aligned}$$

where

$$S_{j',j'',m,\sigma} := (\sigma(2m+1), \dots, \sigma(j'), j'+1, \dots, j'+j'')$$

and $\mathfrak{S}(S_{j',j'',m,\sigma})$ stands for the set of all permutations of $S_{j',j'',m,\sigma}$. In the case where $j = 0$, $l = k$ and $T \equiv 1$, (2.26) follows from (2.35), (2.36), (2.43), (2.45) and (2.47). On the other hand, in the case where $l < k$, by combining (2.39), (2.44) and (2.49) (and replacing j'' by $j' - 2m' + 2l - 2k + 4$ for $m' \in \{0, \dots, \lfloor j'/2 \rfloor + l - k + 2\}$ so that $j' + j''$ is even and $0 \leq j'' \leq j' + 2l - 2k + 4$), we obtain

$$\begin{aligned}
& \int_M (T, \nabla^j U_\mu) \Delta^l U_\mu dv_g = \frac{2^{2k-n-2} (n-1)! l! \omega_n}{(n-2k-2)^2 (k-l-1)!} \sum_{l'=l}^{2l} \sum_{j'=\max(2(k-l-2), 0)}^j \sum_{m=0}^{\lfloor j'/2 \rfloor} \\
& \sum_{m'=0}^{\lfloor j'/2 \rfloor + l - k + 2} \frac{2^{2m'-j'} l! (k+l-l'-1)! c(n, k, j', l, l', m, m') \mu^{4-2m'}}{(l'-l)! (2l-l')! (k+l-l'-m'+1)! (j'-2m'+2l-2k+4)!} \\
& \times \text{B}\left(\frac{n}{2} - k - 1, l' + 1\right)^{-1} \widehat{T}^{i_1 \dots i_{j'}, i_{j'+1} \dots i_{2(j'-m'+l-k+2)}}(\xi) \\
& \times \sum_{\sigma \in \mathfrak{S}(j')} \sum_{\sigma' \in \mathfrak{S}(S_{j',j'-2m'+2l-2k+4,m,\sigma})} \delta_{i_{\sigma(1)} i_{\sigma(2)}} \cdots \delta_{i_{\sigma(2m-1)} i_{\sigma(2m)}} \\
& \times \delta_{i_{\sigma'(\sigma(2m+1))} i_{\sigma'(\sigma(2m+2))}} \cdots \delta_{i_{\sigma'(\sigma(j'-1))} i_{\sigma'(\sigma(j'))}} \\
& \times \delta_{i_{\sigma'(j'+1)} i_{\sigma'(j'+2)}} \cdots \delta_{i_{\sigma'(2(j'-m'+l-k+2)-1)} i_{\sigma'(2(j'-m'+l-k+2))}} \\
& \times \left\{ \begin{array}{ll} 2 \ln(1/\mu) + O(1) & \text{if } n = 2k + 4, l' = l \text{ and } m' = 0 \\ \text{B}\left(\frac{n}{2} + m' + l' - l - k - 2, k + l - l' - m' + 2\right) + o(\mu^{2m'}) & \text{otherwise} \end{array} \right\} \\
& + \left\{ \begin{array}{ll} O(\mu^4) & \text{if } n = 2k + 4 \\ o(\mu^4) & \text{if } n > 2k + 4, \end{array} \right. \tag{2.50}
\end{aligned}$$

where

$$c(n, k, j', l, l', m, m') := (-1)^{j'-m} \mathbf{B} \left(\frac{n}{2} - k - 1, j' - m + 1 \right)^{-1} \\ \times \frac{(j' - m)!}{m! (j' - 2m)! (n + j' - m + l' - 2k - 1)! (j' - m - m' + l - k + 2)!}.$$

Straightforward computations yield

$$\sum_{\sigma \in \mathfrak{S}(j')} \sum_{\sigma' \in \mathfrak{S}(S_{j', j'', m, \sigma})} \delta_{i_{\sigma(1)} i_{\sigma(2)}} \cdots \delta_{i_{\sigma(2m-1)} i_{\sigma(2m)}} \delta_{i_{\sigma'(1)} i_{\sigma'(2)}} \cdots \delta_{i_{\sigma'(j'+j''-1)} i_{\sigma'(j'+j'')}} \\ \cdots \delta_{i_{\sigma'(j'+j''-1)} i_{\sigma'(j'+j'')}} \\ = \begin{cases} 1 & \text{if } j' = j'' = m = 0 \\ 2 \delta_{i_1 i_2} & \text{if } j' = j'' = 1 \text{ and } m = 0 \\ 2(2-m) \delta_{i_1 i_2} & \text{if } j' = 2, j'' = 0 \text{ and } m \leq 1 \\ 16(\delta_{i_1 i_2} \delta_{i_3 i_4} + \delta_{i_1 i_3} \delta_{i_2 i_4} + \delta_{i_1 i_4} \delta_{i_2 i_3}) & \text{if } j' = j'' = 2 \text{ and } m = 0 \\ 4 \delta_{i_1 i_2} \delta_{i_3 i_4} & \text{if } j' = j'' = 2 \text{ and } m = 1 \\ 2(4-2m)! (\delta_{i_1 i_2} \delta_{i_3 i_4} + \delta_{i_1 i_3} \delta_{i_2 i_4} + \delta_{i_1 i_4} \delta_{i_2 i_3}) & \text{if } j' = 3, j'' = 1 \text{ and } m \leq 1 \\ 8(4-2m)! (\delta_{i_1 i_2} \delta_{i_3 i_4} + \delta_{i_1 i_3} \delta_{i_2 i_4} + \delta_{i_1 i_4} \delta_{i_2 i_3}) & \text{if } j' = 4, j'' = 0 \text{ and } m \leq 2. \end{cases} \quad (2.51)$$

On the other hand, by using (2.3) and the fact that for all $a, b, c, d, e \in \{1, \dots, n\}$, in geodesic normal coordinates,

$$g_{ab}(\xi) = \delta_{ab}, \quad g_{ab,c}(\xi) = 0 \quad \text{and} \quad g_{ab,cd}(\xi) = \frac{1}{3} (\mathbf{R}_{acdb}(\xi) + \mathbf{R}_{adcb}(\xi)),$$

and

$$\Gamma_{bc}^a(\xi) = 0, \quad \Gamma_{bc,d}^a(\xi) = \frac{1}{3} (\mathbf{R}_{abdc}(\xi) + \mathbf{R}_{acdb}(\xi)) \quad \text{and} \quad \Gamma_{cde}^{ab}(\xi) = 0,$$

we obtain

$$\begin{cases} \widehat{T}(\xi) = T(\xi) & \text{if } j = 0 \\ \widehat{T}_{,a}^a(\xi) = T_{,a}^a(\xi) & \text{if } j = 1 \\ \left\{ \begin{array}{l} \widehat{T}_{,a}^a(\xi) = 0, \quad \widehat{T}_a^a(\xi) = T_a^a(\xi), \quad \widehat{T}_{a,b}^{a,b}(\xi) = T_{a,b}^{a,b}(\xi) \\ \text{and} \quad \widehat{T}_{,ab}^{ab}(\xi) = \widehat{T}_{,ba}^{ab}(\xi) = T_{,ab}^{ab}(\xi) \end{array} \right\} & \text{if } j = 2 \\ \left\{ \begin{array}{l} \widehat{T}_a^a(\xi) = 0 \quad \text{and} \quad \widehat{T}_a^{ab}(\xi) + \widehat{T}_a^{ba}(\xi) + \widehat{T}_b^{a,b}(\xi) \\ = T_a^{ab}(\xi) + T_a^{ba}(\xi) + T_b^{a,b}(\xi) \end{array} \right\} & \text{if } j = 3 \\ \widehat{T}_a^{a,b}(\xi) + \widehat{T}_{ab}^{ab}(\xi) + \widehat{T}_{ab}^{ba}(\xi) = T_a^{a,b}(\xi) + T_{ab}^{ab}(\xi) + T_{ab}^{ba}(\xi) & \text{if } j = 4. \end{cases} \quad (2.52)$$

We then obtain (2.27) by putting together (2.50), (2.51) and (2.52) and using the identities

$$c(n, k, 0, k-2, l', 0, 0) = \frac{n-2k-2}{2(n+l'-2k-1)!}$$

and

$$\begin{aligned} & B\left(\frac{n}{2} + l' - 2k, 2k - l' + 4\right) \\ &= \frac{4(2k - l' - 1)(2k - l' - 2)}{(n-2)(n-4)} B\left(\frac{n}{2} + l' - 2k, 2k - l' - 2\right). \end{aligned}$$

The estimates (2.28)–(2.32) follow in the same way from (2.50), (2.51) and (2.52) by using the identities

$$\begin{aligned} c(n, k, 1, k-2, l', 0, 0) &= -\frac{(n-2k-2)(n-2k)}{4(n+l'-2k)!}, \\ c(n, k, 2, k-2, l', 0, 0) &= \frac{(n-2k-2)(n-2k)(n-2k+2)}{32(n+l'-2k+1)!}, \\ 2c(n, k, 2, k-2, l', 0, 1) + c(n, k, 2, k-2, l', 1, 1) \\ &= 4c(n, k, 2, k-2, l', 0, 0) + c(n, k, 2, k-2, l', 1, 0) \\ &= 2c(n, k, 2, k-3, l', 0, 0) + c(n, k, 2, k-3, l', 1, 0) \\ &= -\frac{(n-2k-2)(n-2k)(n+2l'-2k)}{8(n+l'-2k+1)!}, \\ 24c(n, k, 3, k-3, l', 0, 0) + 2c(n, k, 3, k-3, l', 1, 0) \\ &= \frac{(n-2k-2)(n-2k)(n-2k+2)(n+2l'-2k)}{8(n+l'-2k+2)!}, \\ 24c(n, k, 4, k-4, l', 0, 0) + 2c(n, k, 4, k-4, l', 1, 0) + c(n, k, 4, k-4, l', 2, 0) \\ &= \frac{(n-2k-2)(n-2k)(n-2k+2)(n+2l'-2k)(n+2l'-2k+2)}{64(n+l'-2k+3)!} \end{aligned}$$

and

$$\begin{aligned} & B\left(\frac{n}{2} + l' - l - k - 2, k + l - l' + 2\right) \\ &= \frac{2(k+l-l'+1)}{n-2} B\left(\frac{n}{2} + l' - l - k - 2, k + l - l' + 1\right) \\ &= \frac{4(k+l-l'+1)(k+l-l')}{(n-2)(n-4)} B\left(\frac{n}{2} + l' - l - k - 2, k + l - l'\right). \end{aligned}$$

This ends the proof of Step 2.3. \square

As regards the integral in the denominator of $I_{k,f,g}(u)$, we obtain the following:

Step 2.4. *Assume that $n \geq 2k + 1$ and g satisfies (2.2) for some point $\xi \in M$. Then, for every smooth function f in M , as $\mu \rightarrow 0$,*

$$\int_M f U_\mu^{2k*} dv_g = \frac{\omega_n}{2^n} f(\xi) - \frac{\omega_n \Delta f(\xi) \mu^2}{2^{n+1}(n-2)} + \frac{\omega_n \Delta^2 f(\xi) \mu^4}{2^{n+3}(n-2)(n-4)} + o(\mu^4). \quad (2.53)$$

Proof of Step 2.4. By observing that U_μ^{2k*} does not depend on k in $B(0, r_0)$, we obtain that (2.53) is in fact identical to an estimate obtained by Esposito and Robert [13] in the case where $k = 2$ (note that in our case, $\text{Ric}(\xi) = 0$ and $\nabla S(\xi) = 0$ since we are working with conformal normal coordinates, see (2.3) and (2.4)). \square

We can now end the proof of Proposition 2.1 by putting together the results of Steps 2.1–2.4:

End of proof of Proposition 2.1. We assume that $k \geq 2$ and refer to Aubin [2] for the case where $k = 1$. By using (2.53), we obtain

$$\begin{aligned} \left(\int_M f U_\mu^{2k} dv_g \right)^{-\frac{n-2k}{n}} &= \left(\frac{\omega_n}{2^n} f(\xi) \right)^{-\frac{n-2k}{n}} \left[1 + \frac{(n-2k) \Delta f(\xi) \mu^2}{2n(n-2) f(\xi)} \right. \\ &\quad \left. - \frac{n-2k}{4n(n-2)} \left(\frac{\Delta^2 f(\xi)}{2(n-4) f(\xi)} - \frac{(n-k) (\Delta f(\xi))^2}{n(n-2) f(\xi)^2} \right) \mu^4 + o(\mu^4) \right]. \end{aligned} \quad (2.54)$$

We let $J_1, J_2, T_1, T_2, T_3, T_4, T_5$ and Z be as in Step 2.1. Since $k \geq 1$, by integrating by parts, we obtain

$$\begin{aligned} \int_M U_\mu \Delta^{k-1} (J_1 U_\mu) dv_g &= \int_M \Delta (J_1 U_\mu) \Delta^{k-2} U_\mu dv_g \\ &= \int_M (U_\mu \Delta J_1 - 2(\nabla J_1, \nabla U_\mu) + J_1 \Delta U_\mu) \Delta^{k-2} U_\mu dv_g. \end{aligned} \quad (2.55)$$

By integrating by parts again, it follows from (2.7) and (2.55) that

$$\begin{aligned} \int_M U_\mu P_{2k} U_\mu dv_g &= \int_M U_\mu \Delta^k U_\mu dv_g + k \int_M ((k-1) J_2 + \Delta J_1) U_\mu \\ &\quad + ((k-1) T_1 - 2\nabla J_1, \nabla U_\mu) + ((k-1) T_2 - J_1 g, \nabla^2 U_\mu) \Delta^{k-2} U_\mu dv_g \\ &\quad + k(k-1)(k-2) \int_M ((T_3, \nabla^2 U_\mu) + (T_4, \nabla^3 U_\mu)) \Delta^{k-3} U_\mu dv_g \\ &\quad + k(k-1)(k-2)(k-3) \int_M (T_5, \nabla^4 U_\mu) \Delta^{k-4} U_\mu dv_g + \int_M U_\mu Z U_\mu dv_g. \end{aligned} \quad (2.56)$$

By using (2.4), we obtain

$$P_a^a ; b^b (\xi) = P_{b;a}^a ; b^b (\xi) = P_a^b ; b^a (\xi) = -\frac{|W(\xi)|^2}{12(n-1)}. \quad (2.57)$$

By using (2.3), (2.4) and (2.57) together with straightforward computations, we obtain

$$(T_3)_a^a (\xi) = -\frac{n+3k+1}{36(n-1)} |W(\xi)|^2 = -\frac{n+3k+1}{36n(n-1)} |W(\xi)|^2 g_a^a$$

and

$$\begin{aligned} (T_4)_a^{ab} ; b (\xi) + (T_4)_a^{ba} ; b (\xi) + (T_4)_a^b ; a (\xi) &= -\frac{k+1}{6(n-1)} |W(\xi)|^2 \\ &= \frac{k+1}{(n-1)(n+2)} ((\nabla S \otimes g)_a^{ab} ; b (\xi) + (\nabla S \otimes g)_a^{ba} ; b (\xi) + (\nabla S \otimes g)_a^b ; a (\xi)) \end{aligned}$$

and

$$\begin{aligned} (T_5)_a^a b^b (\xi) + (T_5)_{ab}^{ab} (\xi) + (T_5)_{ab}^{ba} (\xi) &= -\frac{k+1}{10(n-1)} |W(\xi)|^2 \\ &= -\frac{(k+1) |W(\xi)|^2}{10n(n-1)(n+2)} ((g \otimes g)_a^a b^b (\xi) + (g \otimes g)_{ab}^{ab} (\xi) + (g \otimes g)_{ab}^{ba} (\xi)). \end{aligned}$$

By using these identities together with (2.30)–(2.32) and observing that

$$(\nabla S \otimes g, \nabla^3 U_\mu) = -\Delta(\nabla S, \nabla U_\mu) - 2(\nabla^2 S, \nabla^2 U_\mu) - (\nabla^3 S, \nabla U_\mu \otimes g),$$

and

$$(\nabla^2 S)_a^a(\xi) = -\frac{1}{6} |\mathbb{W}(\xi)|^2 = -\frac{1}{6n} |\mathbb{W}(\xi)|^2 g_a^a(\xi),$$

we obtain that for $k \geq 3$,

$$\begin{aligned} & \int_M (T_3, \nabla^2 U_\mu) \Delta^{k-3} U_\mu dv_g \\ &= \frac{n+3k+1}{36n(n-1)} |\mathbb{W}(\xi)|^2 \int_M (\Delta U_\mu) \Delta^{k-3} U_\mu dv_g + \begin{cases} O(\mu^4) & \text{if } n = 2k+4 \\ o(\mu^4) & \text{if } n > 2k+4 \end{cases} \\ &= \frac{n+3k+1}{36n(n-1)} |\mathbb{W}(\xi)|^2 \int_M U_\mu \Delta^{k-2} U_\mu dv_g + \begin{cases} O(\mu^4) & \text{if } n = 2k+4 \\ o(\mu^4) & \text{if } n > 2k+4 \end{cases} \end{aligned} \quad (2.58)$$

and

$$\begin{aligned} & \int_M (T_4, \nabla^3 U_\mu) \Delta^{k-3} U_\mu dv_g = -\frac{k+1}{(n-1)(n+2)} \int_M \left(\Delta(\nabla S, \nabla U_\mu) \right. \\ & \quad \left. + \frac{1}{3n} |\mathbb{W}(\xi)|^2 \Delta U_\mu + (\nabla^3 S, \nabla U_\mu \otimes g) \right) \Delta^{k-3} U_\mu dv_g + \begin{cases} O(\mu^4) & \text{if } n = 2k+4 \\ o(\mu^4) & \text{if } n > 2k+4 \end{cases} \\ &= -\frac{k+1}{3n(n-1)(n+2)} \int_M \left(|\mathbb{W}(\xi)|^2 U_\mu + 3n(\nabla S, \nabla U_\mu) \right) \Delta^{k-2} U_\mu dv_g \\ & \quad - \frac{k+1}{(n-1)(n+2)} \int_M U_\mu \Delta^{k-3} (\nabla^3 S, \nabla U_\mu \otimes g) dv_g + \begin{cases} O(\mu^4) & \text{if } n = 2k+4 \\ o(\mu^4) & \text{if } n > 2k+4, \end{cases} \end{aligned} \quad (2.59)$$

and for $k \geq 4$,

$$\begin{aligned} & \int_M (T_5, \nabla^4 U_\mu) \Delta^{k-4} U_\mu dv_g \\ &= -\frac{(k+1) |\mathbb{W}(\xi)|^2}{10n(n-1)(n+2)} \int_M (\Delta^2 U_\mu) \Delta^{k-4} U_\mu dv_g + \begin{cases} O(\mu^4) & \text{if } n = 2k+4 \\ o(\mu^4) & \text{if } n > 2k+4 \end{cases} \\ &= -\frac{(k+1) |\mathbb{W}(\xi)|^2}{10n(n-1)(n+2)} \int_M U_\mu \Delta^{k-2} U_\mu dv_g + \begin{cases} O(\mu^4) & \text{if } n = 2k+4 \\ o(\mu^4) & \text{if } n > 2k+4. \end{cases} \end{aligned} \quad (2.60)$$

It follows from (2.56) and (2.58)–(2.60) that

$$\begin{aligned} & \int_M U_\mu P_{2k} U_\mu dv_g = \int_M U_\mu \Delta^k U_\mu dv_g + k \int_M \left(\left((k-1) J_2 + \Delta J_1 \right. \right. \\ & \quad \left. \left. + \left(\frac{(k-1)(k-2)(n+3k+1)}{36n(n-1)} - \frac{(k-1)(k-2)(k+1)(3k+1)}{30n(n-1)(n+2)} \right) |\mathbb{W}(\xi)|^2 \right) U_\mu \right. \\ & \quad \left. + \left((k-1) T_1 - 2\nabla J_1 - \frac{(k+1)(k-1)(k-2)}{(n-1)(n+2)} \nabla S, \nabla U_\mu \right) \right. \\ & \quad \left. + ((k-1) T_2 - J_1 g, \nabla^2 U_\mu) \right) \Delta^{k-2} U_\mu dv_g \end{aligned}$$

$$\begin{aligned}
& + \int_M U_\mu \left(ZU_\mu - \frac{k(k+1)(k-1)(k-2)}{(n-1)(n+2)} \Delta^{k-3} (\nabla^3 S, \nabla U_\mu \otimes g) \right) dv_g \\
& + \begin{cases} \mathcal{O}(\mu^4) & \text{if } n = 2k + 4 \\ \mathfrak{o}(\mu^4) & \text{if } n > 2k + 4. \end{cases} \tag{2.61}
\end{aligned}$$

Straightforward computations together with (2.3), (2.4) and (2.57) give

$$\begin{aligned}
J_1(\xi) &= 0 \quad \text{and} \quad \Delta J_1(\xi) = \frac{n-2}{24(n-1)} |\mathbb{W}(\xi)|^2, \\
J_2(\xi) &= -\frac{3n+2k-4}{144(n-1)} |\mathbb{W}(\xi)|^2, \\
(T_1)_{;a}^a(\xi) &= -\frac{3n+4k-2}{72(n-1)} |\mathbb{W}(\xi)|^2, \\
(T_2)_a^a(\xi) &= 0
\end{aligned}$$

and

$$(T_2)_a^a{}_{;b}^b(\xi) = (T_2)^{ab}{}_{;ab}(\xi) = (T_2)^{ab}{}_{;ba}(\xi) = -\frac{k+1}{18(n-1)} |\mathbb{W}(\xi)|^2.$$

By using these identities together with (2.20), (2.27)–(2.29), (2.54) and (2.61), we obtain that (2.6) holds true with $C(n, k)$ defined as

$$\begin{aligned}
C(n, k) &:= \frac{(n-3)(n-5)!k!}{16(k-1)(n-2k-2)} \\
& \times \sum_{l=k-2}^{2k-4} \frac{l!}{(l-k+2)!(2k-l-4)!(n+l-2k+1)!} \left(8(n+l-2k)(n+l-2k+1) \right. \\
& \times \left(\frac{(k-1)(3n+2k-4)}{144} - \frac{n-2}{24} - \frac{(k-1)(k-2)(n+3k+1)}{36n} \right. \\
& + \left. \frac{(k-1)(k-2)(k+1)(3k+1)}{30n(n+2)} \right) + 4(n-2k)(n+l-2k+1) \left(\frac{n-2}{12} \right. \\
& - \left. \frac{(k-1)(3n+4k-2)}{72} + \frac{(k+1)(k-1)(k-2)}{6(n+2)} \right) \\
& + (n-2k) \left(\frac{(k+1)(k-1)(n-2k-2l+4)}{18} \right. \\
& - \left. \frac{(n-2)(2(n-2k+2) - n(n+2l-2k))}{24} \right) \Big) \\
& \times \mathbb{B} \left(\frac{n}{2} - k - 1, l + 1 \right)^{-1} \begin{cases} 2\chi_{\{l=k-2\}} & \text{if } n = 2k + 4 \\ \mathbb{B} \left(\frac{n}{2} + l - 2k, 2k - l - 2 \right) & \text{otherwise} \end{cases} \\
& = \frac{(n-3)(n-5)!k!}{5760n(n+2)(k-1)(n-2k-2)} \\
& \times \sum_{l=k-2}^{2k-4} \frac{l!c(n, k, l)}{(l-k+2)!(2k-l-4)!(n+l-2k+1)!} \\
& \times \mathbb{B} \left(\frac{n}{2} - k - 1, l + 1 \right)^{-1} \begin{cases} 2\chi_{\{l=k-2\}} & \text{if } n = 2k + 4 \\ \mathbb{B} \left(\frac{n}{2} + l - 2k, 2k - l - 2 \right) & \text{otherwise,} \end{cases} \tag{2.62}
\end{aligned}$$

where

$$\begin{aligned}
c(n, k, l) := & 4(n+l-2k)(n+l-2k+1)(5n(n+2)(k-1)(3n+2k-4) \\
& - 30n(n+2)(n-2) - 20(n+2)(k-1)(k-2)(n+3k+1) \\
& + 24(k-1)(k-2)(k+1)(3k+1)) + 20n(n-2k)(n+l-2k+1) \\
& \times (6(n+2)(n-2) - (n+2)(k-1)(3n+4k-2) + 12(k+1)(k-1)(k-2)) \\
& + 5n(n+2)(n-2k)(4(k+1)(k-1)(n-2k-2l+4) \\
& - 3(n-2)(2(n-2k+2) - n(n+2l-2k))).
\end{aligned}$$

By letting $k := 3 + a$, $n := 2k + 4 + b$ and $l := k - 2 + c$ and using the software *Maple* to expand the expression of $c(n, k, l)$, we then obtain

$$\begin{aligned}
c(n, k, l) = & 4(15ab^3 + 1200a^2b + 3880ab + 1920 + 10656a + 480b + 4528a^2 + 624a^3 \\
& + 40b^2 + 450ab^2 + 80a^3b + 80a^2b^2 + 32a^4)c^2 + 2(71552a^2b + 414912a + 500a^2b^3 \\
& + 247984ab + 31840a^3 + 53660ab^2 + 3200a^4 + 640a^3b^2 + 11020b^3 + 150ab^4 \\
& + 128a^5 + 9056a^3b + 660b^4 + 161440a^2 + 448a^4b + 15b^5 + 311040b + 10520a^2b^2 \\
& + 4830ab^3 + 426240 + 85840b^2)c + 128a^6 + 576a^5b + 1088a^4b^2 + 1020a^3b^3 \\
& + 560a^2b^4 + 150ab^5 + 15b^6 + 3904a^5 + 14720a^4b + 21896a^3b^2 + 15940a^2b^3 \\
& + 5640ab^4 + 720b^5 + 49408a^4 + 149280a^3b + 167032a^2b^2 + 81120ab^3 + 13780b^4 \\
& + 332096a^3 + 754720a^2b + 563824ab^2 + 134240b^3 + 1250304a^2 + 1900224ab \\
& + 704640b^2 + 2499840a + 1900800b + 2073600. \tag{2.63}
\end{aligned}$$

Since all the coefficients in this expression are positive, it follows that $C(n, k)$ is positive whenever $k \geq 3$, $n \geq 2k + 4$ and $l \geq k - 2$. Furthermore, in the case where $k = 2$ and $l = 0$, we find

$$c(n, 2, 0) = 5n(n+2)(n-4)^2(n^2 - 4n - 4) > 0 \quad \forall n \geq 8.$$

Therefore, in all cases, we find that $C(n, k)$ is positive. This ends the proof of Proposition 2.1. \square

We can now prove Theorem 1.1 by using Proposition 2.1.

Proof of Theorem 1.1 in the case where $n \geq 2k + 4$. Let $\xi \in M$ be a maximal point of f and $\tilde{g} = \varphi^{4/(n-2)}g$ be a conformal metric to g such that $\varphi(\xi) = 1$ and $\det \tilde{g}(x) = 1$ for all x in a neighborhood of the point ξ . Notice that since ξ is a maximal point of f , if $\Delta_g f(\xi) = 0$, then $\nabla^j f = 0$ for all $j \in \{1, 2, 3\}$. In particular, since $\varphi(\xi) = 1$, it follows that

$$\Delta_{\tilde{g}} f(\xi) = 0 \quad \text{and} \quad \Delta_{\tilde{g}}^2 f(\xi) = \Delta_g^2 f(\xi), \tag{2.64}$$

where Δ_g and $\Delta_{\tilde{g}}$ are the Laplace–Beltrami operators with respect to the metrics g and \tilde{g} , respectively, and the covariant derivatives, the Ricci tensor and the multiple inner product in the right-hand side of the second identity are with respect to the metric g . Let $c(n, k)$ be the constant defined as

$$c(n, k) := \begin{cases} 0 & \text{if } n = 2k + 4 \\ \frac{(n-2k)(2k-1)!}{8n(n-2)(n-4)C(n, k)} \text{B}\left(\frac{n}{2} - k, 2k\right)^{-1} & \text{if } n > 2k + 4, \end{cases} \tag{2.65}$$

where $C(n, k)$ is as in (2.6) (see also (2.62)). By applying Proposition 2.1 together with (2.64) and the fact that $|W|$ is conformally invariant, we then obtain that if (1.2) and (1.3) hold true, then

$$\inf_{u \in C^{2k}(M) \setminus \{0\}} I_{k, f, \tilde{g}}(u) < \omega_n^{\frac{2k}{n}} (2k-1)! B\left(\frac{n}{2} - k, 2k\right)^{-1} \left(\max_{x \in M} f(x)\right)^{-\frac{n-2k}{n}}. \quad (2.66)$$

On the other hand, by conformal invariance of the operator P_{2k} , we obtain

$$\inf_{u \in C^{2k}(M) \setminus \{0\}} I_{k, f, \tilde{g}}(u) = \inf_{u \in C^{2k}(M) \setminus \{0\}} I_{k, f, g}(u). \quad (2.67)$$

By putting together (2.66) and (2.67) and applying Theorem 3 of Mazumdar [30], we then obtain that the conclusions of Theorem 1.1 hold true. \square

Remark 2.1. From (2.62), we recover the values found by Esposito and Robert [13] for $C(n, 2)$ (see (21) and (22) in [13]) and Chen and Hou [9] for $C(n, 3)$ (see Proposition 3.1 in [9]), namely

$$C(n, 2) = \begin{cases} \frac{7}{12} & \text{if } n = 8 \\ \frac{(n-4)(n^2 - 4n - 4)}{96(n-6)(n-8)} & \text{if } n \geq 9 \end{cases}$$

and

$$C(n, 3) = \begin{cases} 21 & \text{if } n = 10 \\ \frac{(n-6)(3n^4 - 24n^3 - 4n^2 + 208n + 384)}{768(n-10)(n-8)} & \text{if } n \geq 11. \end{cases} \quad (2.68)$$

More precisely, to obtain (2.68) from (2.62), we use the identities

$$\begin{aligned} B(1, 2) &= \frac{1}{2}, \\ B\left(\frac{n}{2} - 4, 2\right)^{-1} B\left(\frac{n}{2} - 5, 3\right) &= \frac{4}{n-10} \quad \text{for } n \geq 11 \end{aligned}$$

and

$$B\left(\frac{n}{2} - 4, 3\right)^{-1} B\left(\frac{n}{2} - 4, 2\right) = \frac{n-4}{4} \quad \text{for } n \geq 11,$$

and to obtain (2.68) from the formulae in [9], we use the identities

$$B\left(\frac{n}{2} + 1, \frac{n}{2} - 5\right) = \frac{2^{5-n} n (n-3) (n-1) \omega_n}{(n-10)(n-8)(n-6) \omega_{n-1}}$$

and

$$3n^5 - 18n^4 - 52n^3 + 200n^2 + 800n + 768 = (n+2)(3n^4 - 24n^3 - 4n^2 + 208n + 384).$$

3. THE REMAINING CASES

This section is devoted to the proof of Theorem 1.1 in the remaining case where $2k+1 \leq n \leq 2k+3$ together with the following result in the case where g is conformally flat in some open subset of the manifold:

Theorem 3.1. *Let $k \geq 1$ be an integer, (M, g) be a smooth, closed Riemannian manifold of dimension $n \geq 2k + 1$ and f be a smooth positive function in M . Assume that $Y_{2k} > 0$ and there exists a maximal point ξ of f such that $m(\xi) > 0$ (see (3.2) for the definition of the mass), $\nabla^j f(\xi) = 0$ for all $j \in \{1, \dots, n - 2k\}$ and g is conformally flat in some neighborhood of the point ξ . Then there exists a nontrivial solution $u \in C^{2k}(M)$ to the equation (1.1), which minimizes the energy functional (2.1). If moreover the Green's function of the operator P_{2k} is positive, then u is positive, which implies that the Q -curvature of order $2k$ of the metric $u^{4/(n-2k)}g$ is equal to $\frac{2}{n-2k}f$.*

Notice that Theorem 1.2 is now a direct consequence of Theorems 1.1 and 3.1.

Throughout this section, we fix a point $\xi \in M$ and assume that $2k + 1 \leq n \leq 2k + 3$ or g is conformally flat in some neighborhood of ξ . In these cases, our proofs are based on the method of Schoen [36] for the resolution of the remaining cases of the Yamabe problem, which has been extended to the $k = 2$ case by Gursky and Malchiodi [21] and Hang and Yang [22, 23]. We consider a family of global test functions involving the Green's function and derive an expression for the energy functional $I_{k,f,g}$ (see (2.1)) associated with the equation (1.1). Then, analogously as in the case $n \geq 2k + 4$, by using the expansion obtained in Proposition 2.1, we obtain the existence of a nontrivial solution to the equation (1.1) under a positivity assumption on the mass of the operator P_{2k} .

We now discuss the definition of the mass. By applying a conformal change of metric, we may assume that

$$\begin{cases} g \text{ satisfies (2.2) in some neighborhood } \Omega \text{ of } \xi \text{ if } 2k + 1 \leq n \leq 2k + 3 \\ g \text{ is flat in some neighborhood } \Omega \text{ of } \xi \text{ if } n \geq 2k + 4. \end{cases} \quad (3.1)$$

Then, in the geodesic normal coordinates at ξ determined by g , the Green's function $G_{2k}(x) := G_{2k}(x, \xi)$ of the operator P_{2k} has the expansion

$$G_{2k}(x) = b_{n,k} d_g(x, \xi)^{2k-n} + m(\xi) + o(1) \quad (3.2)$$

as $x \rightarrow \xi$ (see Lee and Parker [28] for $k = 1$ and Michel [31] for $k \geq 2$), where $m(\xi) \in M$ is called the mass of the operator P_{2k} at the point ξ and the constant $b_{n,k}$ is defined as

$$b_{n,k}^{-1} := 2^{k-1} (k-1)! (n-2)(n-4) \cdots (n-2k) \omega_{n-1}.$$

It is important to point out that the sign of $m(\xi)$ does not depend on our choice of conformal metric (see Michel [31, Théorème 3.1]).

Now that the mass is defined, we consider the regular part of the Green's function, which plays a crucial role in the proofs of our theorems. It follows from (3.2) that there exists a continuous function h_{2k} in M such that $h_{2k}(\xi) = m(\xi)$ and

$$G_{2k}(x) = b_{n,k} d_g(x, \xi)^{2k-n} + h_{2k}(x) \quad \forall x \in M \setminus \{\xi\}. \quad (3.3)$$

Furthermore, we have that $h_{2k} \in C^\infty(\Omega)$ in the case where g is flat in Ω and $h_{2k} \in W^{2k,p}(\Omega)$ for all $p \in [1, n/(n-4)]$ if $n \geq 5$ and $p \in [1, \infty)$ if $n \in \{3, 4\}$ in the case where $2k + 1 \leq n \leq 2k + 3$ and g satisfies (2.2) in Ω . This follows from classical elliptic regularity theory (see Agmon, Douglis and Nirenberg [1]) together with the fact that $P_{2k}h_{2k} = \Delta^k h_{2k} = 0$ in Ω in the case where g is flat in Ω and $P_{2k}h_{2k} = O(d_g(\cdot, \xi)^{4-n})$ in Ω in the case where g satisfies (2.2) in Ω (see [31, Lemme 2.2]).

For every $\mu > 0$, letting χ and U_μ be as in Section 2, we consider the test functions V_μ defined as

$$V_\mu(x) := U_\mu(x) + b_{n,k}^{-1} \mu^{\frac{n-2k}{2}} (\chi(d_g(x, \xi)) h_{2k}(x) + (1 - \chi(d_g(x, \xi))) G_{2k}(x)) \quad (3.4)$$

for all $x \in M$. Note that $V_\mu \in W^{2k, 2n/(n+2k)}(M)$ so that in particular the integral $\int_M V_\mu P_{2k} V_\mu dv_g$ is well defined. We then obtain the following:

Proposition 3.1. *Let $k \geq 1$ be an integer, (M, g) be a smooth, closed Riemannian manifold of dimension $n \geq 2k + 1$, f be a smooth positive function in M and ξ be a point in M such that $\nabla^j f(\xi) = 0$ for all $j \in \{1, \dots, n - 2k\}$ and (3.1) holds true. Let $I_{k,f,g}$ be as in (2.1) and V_μ be as in (3.4). Then, as $\mu \rightarrow 0$,*

$$I_{k,f,g}(V_\mu) = \omega_n^{\frac{2k}{n}} (2k - 1)! \mathbf{B}\left(\frac{n}{2} - k, 2k\right)^{-1} f(\xi)^{-\frac{n-2k}{n}} \times \left(1 - b_{n,k}^{-1} \mathbf{B}\left(\frac{n}{2}, \frac{n}{2}\right)^{-1} \mathbf{B}\left(\frac{n}{2}, k\right) m(\xi) \mu^{n-2k} + o(\mu^{n-2k})\right). \quad (3.5)$$

Proof of Proposition 3.1. The first step in the proof is as follows:

Step 3.1. *Assume that g satisfies (3.1) for some point $\xi \in M$. Then, as $\mu \rightarrow 0$,*

$$\int_M V_\mu P_{2k} V_\mu dv_g = 2^{2k-n} \omega_n (2k - 1)! \mathbf{B}\left(\frac{n}{2} - k, 2k\right)^{-1} \times \left(1 + b_{n,k}^{-1} \mathbf{B}\left(\frac{n}{2}, \frac{n}{2}\right)^{-1} \mathbf{B}\left(\frac{n}{2}, k\right) m(\xi) \mu^{n-2k} + o(\mu^{n-2k})\right). \quad (3.6)$$

Proof of Step 3.1. We write

$$V_\mu(x) = b_{n,k}^{-1} \mu^{\frac{n-2k}{2}} G_{2k}(x) + \underbrace{\left(U_\mu(x) - \mu^{\frac{n-2k}{2}} \chi(d_g(x, \xi)) d_g(x, \xi)^{2k-n}\right)}_{W_\mu(x)}$$

for all $x \in M \setminus \{\xi\}$. Straightforward estimates give

$$\begin{aligned} \int_M V_\mu P_{2k} V_\mu dv_g &= \int_{B(\xi, 2r_0)} V_\mu P_{2k} W_\mu dv_g \\ &= \int_{B(\xi, r_0)} \left(U_\mu + b_{n,k}^{-1} \mu^{\frac{n-2k}{2}} h_{2k}\right) P_{2k} W_\mu dv_g \\ &\quad + \mathcal{O}\left(\mu^{\frac{n-2k}{2}} \int_{B(\xi, 2r_0) \setminus B(\xi, r_0)} |P_{2k} W_\mu| dv_g\right) \\ &= \int_{B(\xi, r_0)} U_\mu P_{2k} W_\mu dv_g + b_{n,k}^{-1} \mu^{\frac{n-2k}{2}} \int_{B(\xi, r_0)} h_{2k} P_{2k} W_\mu dv_g \\ &\quad + \mathcal{O}\left(\mu^{n-2k} \sum_{|\alpha| \leq 2k} \int_{B(0, 2r_0) \setminus B(0, r_0)} \left| \partial^\alpha \left[(\mu^2 + |x|^2)^{\frac{2k-n}{2}} - |x|^{2k-n} \right] \right| dx\right) \\ &= \int_{B(\xi, r_0)} U_\mu P_{2k} W_\mu dv_g + b_{n,k}^{-1} \mu^{\frac{n-2k}{2}} \int_{B(\xi, r_0)} h_{2k} P_{2k} W_\mu dv_g + o(\mu^{n-2k}). \quad (3.7) \end{aligned}$$

We claim that

$$|P_{2k} W_\mu - \Delta^k U_\mu| \leq C \mu^{\frac{n-2(k-2)}{2}} d_g(x, \xi)^{2-n} \quad \forall x \in B(\xi, r_0) \setminus \{\xi\} \quad (3.8)$$

for some constant C independent of x , μ and ξ . Assuming (3.8) and proceeding as in Step 2.3, it then follows from (3.7) that

$$\begin{aligned}
\int_M V_\mu P_{2k} V_\mu dv_g &= \int_{B(\xi, r_0)} U_\mu \Delta^k U_\mu dv_g + b_{n,k}^{-1} \mu^{\frac{n-2k}{2}} \int_{B(\xi, r_0)} h_{2k} \Delta^k U_\mu dv_g \\
&+ O\left(\mu^{n-2(k-1)} \int_{B(0, r_0)} |x|^{2-n} (\mu^2 + |x|^2)^{\frac{2k-n}{2}} dx\right) + o(\mu^{n-2k}) \\
&= \int_{B(0, r_0)} \tilde{U}_\mu \Delta_0^k \tilde{U}_\mu dx + b_{n,k}^{-1} \mu^{\frac{n-2k}{2}} \int_{B(0, r_0)} h_{2k} (\exp_\xi x) \Delta_0^k \tilde{U}_\mu dx + o(\mu^{n-2k}) \\
&= 2^{2k-1} (2k-1)! \omega_{n-1} B\left(\frac{n}{2} - k, 2k\right)^{-1} \left(\int_0^{(r_0/\mu)^2} \frac{r^{\frac{n-2}{2}}}{(1+r)^n} dr \right. \\
&\quad \left. + b_{n,k}^{-1} m(\xi) \mu^{n-2k} \int_0^{(r_0/\mu)^2} \frac{r^{\frac{n-2}{2}}}{(1+r)^{\frac{n+2k}{2}}} dr \right) + o(\mu^{n-2k}) \\
&= 2^{2k-n} (2k-1)! \omega_n B\left(\frac{n}{2} - k, 2k\right)^{-1} \\
&\quad \times \left(1 + b_{n,k}^{-1} B\left(\frac{n}{2}, \frac{n}{2}\right)^{-1} B\left(\frac{n}{2}, k\right) m(\xi) \mu^{n-2k} + o(\mu^{n-2k}) \right).
\end{aligned}$$

Therefore, it remains to prove (3.8) to complete the proof of Step 3.2. Notice that (3.8) is clearly satisfied with $C = 0$ in the case where $n \geq 2k + 4$ and g is flat in Ω . Therefore, we may assume in what follows that we are in the case where $2k + 1 \leq n \leq 2k + 3$ and g satisfies (2.2) in Ω . By using (2.7), we obtain

$$\begin{aligned}
P_{2k} &= \Delta^k + k \Delta^{k-1} (J_1 \cdot) + k(k-1) \Delta^{k-2} ((T_1, \nabla) + (T_2, \nabla^2)) \\
&\quad + k(k-1)(k-2) \Delta^{k-3} (T_4, \nabla^3) + Z, \quad (3.9)
\end{aligned}$$

where Z is a smooth linear operator of order less than $2k - 3$ if $k \geq 2$, $Z := 0$ if $k = 1$. By induction, one can check that

$$\Delta^{k-1} (J_1 \cdot) = J_1 \Delta^{k-1} - 2(k-1) (\nabla J_1, \nabla \Delta^{k-2}) + o^{2k-3}, \quad (3.10)$$

$$\Delta^{k-2} (T_1, \nabla) = (T_1, \nabla \Delta^{k-2}) + o^{2k-3}, \quad (3.11)$$

$$\Delta^{k-2} (T_2, \nabla^2) = (T_2, \nabla^2 \Delta^{k-2}) - 2(k-2) (\nabla T_2, \nabla^3 \Delta^{k-3}) + o^{2k-3} \quad (3.12)$$

and

$$\Delta^{k-3} (T_4, \nabla^3) = (T_4, \nabla^3 \Delta^{k-3}) + o^{2k-3}, \quad (3.13)$$

where o^{2k-3} is as in the proof of Step 2.1. It follows from (2.3), (2.4) and (3.9)–(3.13) that

$$\begin{aligned}
P_{2k} W_\mu &= \Delta^k W_\mu + \frac{2k(k-1)(k+1)}{3(n-2)} ((\text{Ric}, \nabla^2 \Delta^{k-2} W_\mu) - (\delta \text{Ric}, \nabla \Delta^{k-2} W_\mu) \\
&\quad - (k-2) (\nabla \text{Ric}, \nabla^3 \Delta^{k-2} W_\mu)) + O\left(d_g(\cdot, \xi)^2 |\nabla^{2k-2} W_\mu|\right) \\
&\quad + d_g(\cdot, \xi) |\nabla^{2k-3} W_\mu| + \sum_{j=0}^{2k-4} |\nabla^j W_\mu| \quad (3.14)
\end{aligned}$$

in $M \setminus \{\xi\}$, uniformly with respect to μ and ξ . By using geodesic normal coordinates together with (2.3) and a Taylor expansion, it follows from (3.14) that

$$\begin{aligned} (P_{2k}W_\mu - \Delta^k U_\mu)(\exp_\xi x) &= (P_{2k}W_\mu - \Delta^k W_\mu)(\exp_\xi x) \\ &= O\left(|x|^2 |\nabla^{2k-2}\widetilde{W}_\mu(x)| + |x| |\nabla^{2k-3}\widetilde{W}_\mu(x)| + \sum_{j=0}^{2k-4} |\nabla^j \widetilde{W}_\mu(x)|\right) \end{aligned} \quad (3.15)$$

uniformly with respect to $x \in B(0, r_0) \setminus \{0\}$, μ and ξ , where

$$\widetilde{W}_\mu(x) := \mu^{\frac{2k-n}{2}} U(x/\mu) - \mu^{\frac{n-2k}{2}} |x|^{2k-n}.$$

Similarly as in (2.40), we obtain that for every $j \in \mathbb{N}$, there exists a constant C_j independent of x and μ such that

$$\begin{aligned} |\nabla^j \widetilde{W}_\mu(x)| &= \mu^{\frac{2k-n-2j}{2}} |\nabla^j W(x/\mu)| \\ &\leq C_j \sum_{m=0}^{\lfloor j/2 \rfloor} \mu^{\frac{2k-n-4j+4m}{2}} r^{\frac{j-2m}{2}} |\partial_r^{j-m} W(r/\mu^2)| \end{aligned} \quad (3.16)$$

for all $x \in B(0, r_0) \setminus \{0\}$, where $r := |x|^2$ and

$$W(x) = W(r) := (1+r)^{(2k-n)/2} - r^{(2k-n)/2}.$$

Furthermore, it is easy to see that

$$|\partial_r^j W(r)| \leq C'_j r^{\frac{2k-n-2j-2}{2}} \quad (3.17)$$

for some constant C'_j independent of r . We then obtain (3.8) by putting together (3.15)–(3.17). This completes the proof of Step 3.1. \square

Step 3.2. Assume that g satisfies (3.1) for some point $\xi \in M$. Let f be a smooth function f in M such that $\nabla^j f(\xi) = 0$ for all $j \in \{1, \dots, n-2k\}$. Then, as $\mu \rightarrow 0$,

$$\begin{aligned} \int_M f |V_\mu|^{2k^*} dv_g &= \frac{\omega_n}{2^n} \left(f(\xi) + 2_k^* b_{n,k}^{-1} B\left(\frac{n}{2}, \frac{n}{2}\right)^{-1} B\left(\frac{n}{2}, k\right) f(\xi) m(\xi) \mu^{n-2k} \right. \\ &\quad \left. + o(\mu^{n-2k}) \right). \end{aligned} \quad (3.18)$$

Proof of Step 3.2. By using a Taylor expansion together with straightforward estimates, we obtain

$$\begin{aligned} \int_M f |V_\mu|^{2k^*} dv_g &= \int_{B(\xi, r_0)} f |U_\mu + b_{n,k}^{-1} \mu^{\frac{n-2k}{2}} h_{2k}|^{2k^*} dv_g + O(\mu^n) \\ &= f(\xi) \int_{B(\xi, r_0)} U_\mu^{2k^*} dv_g + 2_k^* b_{n,k}^{-1} \mu^{\frac{n-2k}{2}} \int_{B(\xi, r_0)} f h_{2k} U_\mu^{2k^*-1} dv_g \\ &\quad + O\left(\int_{B(0, r_0)} \left(|x|^{n-2k+1} \left(\frac{\mu}{\mu^2 + |x|^2}\right)^n + \mu^{n-2k} \left(\frac{\mu}{\mu^2 + |x|^2}\right)^{2k}\right) dx + \mu^n\right) \\ &= \frac{\omega_n}{2^n} f(\xi) + \frac{n\omega_{n-1}}{n-2k} b_{n,k}^{-1} f(\xi) m(\xi) \mu^{n-2k} \int_0^{(r_0/\mu)^2} \frac{r^{\frac{n-2}{2}} dr}{(1+r)^{\frac{n+2k}{2}}} + o(\mu^{n-2k}). \end{aligned} \quad (3.19)$$

Then (3.18) follows from (2.45), (2.47) and (3.19). \square

We can now end the proofs of Proposition 3.1 and Theorems 1.1 and 1.2.

End of proof of Proposition 3.1. We obtain (3.5) by putting together (3.6) and (3.18). \square

Proofs of Theorem 1.1 in the case where $2k + 1 \leq n \leq 2k + 3$ and of Theorem 3.1. Let $\xi \in M$ be a maximal point of f such that $\nabla^j f(\xi) = 0$ for all $j \in \{1, \dots, n - 2k\}$ (notice that for a maximal point, this is equivalent to (1.2) in the case where $2k + 1 \leq n \leq 2k + 3$). By applying Proposition 3.1 together with a conformal change of metric, we then obtain that if $m(\xi) > 0$, then there exists a function $V \in W^{2k, 2n/(n+2k)}(M) \setminus \{0\}$ such that

$$I_{k, f, g}(V) < \omega_n^{\frac{2k}{n}} (2k - 1)! \left(\frac{n}{2} - k, 2k\right)^{-1} \left(\max_{x \in M} f(x)\right)^{-\frac{n-2k}{n}}.$$

Notice that $W^{2k, 2n/(n+2k)}(M) \hookrightarrow L^{2^*_k}(M)$ so that a density argument gives

$$\inf_{u \in C^{2k}(M) \setminus \{0\}} I_{k, f, g}(u) < \omega_n^{\frac{2k}{n}} (2k - 1)! \left(\frac{n}{2} - k, 2k\right)^{-1} \left(\max_{x \in M} f(x)\right)^{-\frac{n-2k}{n}}.$$

We can then conclude the proofs of Theorems 1.1 and 3.1 by applying Theorem 3 of Mazumdar [30]. \square

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