CONTINUITY AND INJECTIVITY OF OPTIMAL MAPS

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Abstract. Figalli–Kim–McCann proved in [14] the continuity and injectivity of optimal maps under the assumption (B3) of nonnegative cross-curvature. In the recent [15, 16], they extend their results to the assumption (A3w) of Trudinger-Wang [34], and they prove, moreover, the Hölder continuity of these maps. We give here an alternative and independent proof of the extension to (A3w) of the continuity and injectivity of optimal maps based on the sole arguments of [14] and on new Alexandrov-type estimates for lower bounds.

1. Introduction

Given a cost function $c : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, $n \geq 2$, and two probability densities f and g in $\mathbb{R}^n$ with respect to Lebesgue's measure, Monge's problem of optimal transportation consists in finding a minimizer of the cost functional

$$\mathcal{C}(T) := \int_{\mathbb{R}^n} c(x, T(x))\, f(x)\, dx$$

among all measurable maps $T : \mathbb{R}^n \to \mathbb{R}^n$ pushing the measure with density f forward to the measure with density g i.e. $\int_\Gamma g = \int_{T^{-1}(\Gamma)} f$ for all Borel subsets Γ of $\mathbb{R}^n$. A solution of the optimal transportation problem is called an *optimal map*.

Figalli–Kim–McCann proved in [14] a continuity and injectivity result for optimal maps under the assumption (B3) of nonnegative cross-curvature of the cost function (extending the works by Loeper [29] and Trudinger–Wang [35] where the result was proved under the stronger condition (A3)). In the recent [15, 16], Figalli–Kim–McCann extend their results to the assumption (A3w) of Trudinger–Wang [34], and they prove, moreover, the Hölder continuity of optimal maps. In this paper, we give an alternative proof which was found independently of the extension to (A3w) of the continuity and injectivity of optimal maps based on the sole arguments of [14] and on new Alexandrov-type estimates for lower bounds (see Theorem 3.3 and Corollary 3.5). These estimates are, on some aspects, more general than the estimates obtained recently by Figalli–Kim–McCann [15]. Our proof relies, in particular, on invariance properties of the cost function under affine renormalization, properties which are observed but not used in [15]. The derivation of new estimates for lower bounds turns out to be the main difficulty in the extension of the regularity of optimal maps to (A3w). In the rest of the proof, we follow, and slightly adapt when necessary, the very nice ideas by Figalli–Kim–McCann [14]. The strategy of the proof had been developed originally by Caffarelli [4] for the Monge–Ampère equation. Several tricky obstacles arise when applying this strategy to more general cost functions. These obstacles have been overcome by Figalli–Kim–McCann [14–16] and, alternatively, in this paper as regards the extension from (B3) to (A3w) of the continuity and injectivity.

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In this paper, we assume (A0)–(A3w) below for a cost function $c : U \to \mathbb{R}$ in a bounded open subset U of $\mathbb{R}^n \times \mathbb{R}^n$.

(A0) $c \in C^4(U)$.

(A1) For any $(x,y) \in U$ and for any $(-p,-q) \in D_x c(x, U_x) \times D_y c(U_y, y)$, where $U_x := \{y' \in \mathbb{R}^n;\ (x,y') \in U\}$ and $U_y := \{x' \in \mathbb{R}^n;\ (x',y) \in U\}$, there exist unique $Y = Y(x,p)$ and $X = X(q,y)$ such that $D_x c(x,Y) = -p$ and $D_y c(X,y) = -q$.

(A2) For any $(x,y) \in U$, $\det D^2_{xy} c(x,y) \neq 0$.

(A3w) For any $(x,y) \in U$ and $(\xi,\eta) \in \mathbb{R}^n \times \mathbb{R}^n$ such that $\xi \perp \eta$, there holds

$$-D^2_{pp} A(x,p) \,.\, (\xi,\xi,\eta,\eta) \geq 0\,, \tag{1.1}$$

where $Y(x,p) = y$, i.e. $-p = D_x c(x,y)$, and $A(x,p) := D^2_{xx} c(x, Y(x,p))$.

The assumptions (A0)–(A3w) are the same assumptions under which Trudinger–Wang [34] obtained a smoothness result for optimal maps when the densities are smooth. See also the former reference by Ma–Trudinger–Wang [32] where the result was proved under assumption (A3) which consists in asking that the inequality in (1.1) is strict. In the same context of smooth densities, Loeper [29] proved that (A3w) is necessary for the continuity of optimal maps. For more general measures, under the assumption (A3), Loeper [29] proved the Hölder continuity of optimal maps, and Trudinger–Wang [35] gave a different proof for the continuity (these results follow from more general regularity results for potential functions proved in [29] and [35]). Liu [26] improved the result of Loeper [29] by obtaining a sharp Hölder exponent. Figalli–Kim–McCann, in their first paper on the question [14], proved the continuity and injectivity of optimal maps under the following assumption (B3) of nonnegative cross-curvature (now called (B4) in [15]).

(B3) For any curve $t \in [-1,1] \mapsto (D_y c(x(t), y(0)), D_x c(x(0), y(t)))$ which is an affinely parametrized line segment, there holds

$$\mathrm{cross}_{x(0),y(0)}\left[x'(0), y'(0)\right] := -\left.\frac{\partial^4}{\partial s^2 \partial t^2}\right|_{(s,t)=(0,0)} c(x(s), y(t)) \geq 0\,. \tag{1.2}$$

(A3w) is equivalent to asking that (1.2) holds provided $\left.\frac{\partial^2}{\partial s \partial t}\right|_{(s,t)=(0,0)} c(x(s), y(t)) = 0$. We refer to Kim–McCann [24, 25] for a detailed discussion on the notion of cross-curvature.

In addition to our assumptions on the cost function, we assume that the densities lie in two open subsets Ω^+ and Ω^* of $\mathbb{R}^n$ which satisfy (B) below.

(B) $\Omega^+ \times \Omega^* \Subset U$, Ω^+ and Ω^* are strongly c-convex with respect to each others.

Ω^+ and Ω^* are said to be *c-convex* (resp. *strongly c-convex*) with respect to each others if $D_x c(x, \Omega^*)$ and $D_y c(\Omega^+, y)$ are convex (resp. strongly convex) for all $(x,y) \in \Omega^+ \times \Omega^*$. A convex set K is said to be *strongly convex* if there exists $R > 0$ such that for any $x \in \partial K$, $K \subset B_{x-R\nu}(R)$ for some outer unit normal vector ν to a supporting hyperplane of K at x, where $B_{x-R\nu}(R)$ is the ball of center $x - R\nu$ and radius R. In case K is smooth, K is strongly convex if and only if all principal curvatures of its boundary are bounded below by $1/R$. Strong convexity is also called uniform convexity in Trudinger–Wang [34, 35]. Our main result states as follows. We refer to Section 2 and to the references therein for discussions on Kantorovich duality.

Theorem 1.1. *Assume that the cost function satisfies* (A0)–(A3w)*. Let Ω, Ω^+, and Ω^* be three bounded open subsets of $\mathbb{R}^n$ such that $\Omega \subset \Omega^+$ and such that* (B) *holds. Let $f \in L^1(\Omega^+)$ and $g \in L^1(\Omega^*)$ be two probability densities such that $f \in L^\infty(\Omega^+)$, $(1/f) \in L^\infty(\Omega)$, $g \in L^\infty(\Omega^*)$, and $(1/g) \in L^\infty(\Omega^*)$. Let $T : \overline{\Omega^+} \to \overline{\Omega^*}$ be the optimal map determined by Kantorovich duality. Then T is continuous and one-to-one in Ω.*

Regularity of optimal maps has been intensively studied. Historic references on this question for the cost $c(x,y) = -|x-y|^2$ are by Brenier [3], Caffarelli [4–9], Delanoë [10], and Urbas [36]. We also mention the related works by Wang [39,40] on the reflector antenna design problem. Another early result of regularity by Gangbo–McCann [22] addressed the case of the squared cost when restricted to the product of the boundaries of two strongly convex sets. For more general cost functions, without pretension of exhaustivity, recent advances on the regularity of optimal maps are by Delanoë–Ge [11,12], Figalli–Kim–McCann [13–16], Figalli–Loeper [17], Figalli–Rifford [18], Figalli–Rifford–Villani [19,20], Kim–McCann [24,25], Liu [26], Liu–Trudinger[27], Liu–Trudinger–Wang [28], Loeper [29,30], Loeper–Villani [31], Ma–Trudinger–Wang [32], and Trudinger–Wang [33–35]. More references can be found in Figalli–Kim–McCann [14,15]. We refer to [14,15] for an extensive discussion on the problem of regularity of optimal maps. We also refer to Gangbo–McCann [21], Urbas [37], and to the book by Villani [38] where one can find more general material on optimal transportation.

The paper is organized as follows. We begin with some preliminaries in Section 2. In particular, in Section 2, we state the regularity results for potential functions which imply Theorem 1.1. In Sections 3 and 4, we derive Alexandrov-type estimates for lower and upper bounds. We combine this material in Section 5 in order to prove the regularity of potential functions. As an appendix, we state some results on optimal transportation and convex sets.

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2. Preliminaries

2.1. **Potential functions.** We let U, Ω^+, Ω^*, c satisfy (A0)–(A3w), (B), and we let Ω be an open subset of Ω^+. By Kantorovich duality, it is known, see for instance Caffarelli [9], Gangbo–McCann [21], and Villani [38], that the solution of the optimal transportation problem is almost everywhere uniquely determined by a map $T : \overline{\Omega^+} \to \overline{\Omega^*}$ satisfying the relation

$$D_x c(x, T(x)) = -Du(x) \qquad \text{a.e. in } \overline{\Omega^+} \tag{2.1}$$

for some *c-convex* function u in $\overline{\Omega^+}$ focussing in $\overline{\Omega^*}$, i.e. such that there exists $v : \overline{\Omega^*} \to \mathbb{R}$ satisfying $v(y) = u^{c^*}(y)$ and $u(x) = v^c(x)$ for all $(x,y) \in \overline{\Omega^+} \times \overline{\Omega^*}$, where

$$u^{c^*}(y) := \sup_{x' \in \overline{\Omega^+}} \left(-c(x',y) - u(x')\right) \quad \text{and} \quad v^c(x) := \sup_{y' \in \overline{\Omega^*}} \left(-c(x,y') - v(y')\right). \tag{2.2}$$

The functions u and v are said to be *potential functions* for the optimal transportation problem. We refer, for instance, to Kim–McCann [24] and to the book by Villani [38] for more material on c-convex functions. In particular, c-convex functions are semiconvex, i.e. $x \mapsto u(x) + C|x|^2$ is locally convex in Ω^+ for some $C > 0$. We define the c-subdifferential of u at $x \in \overline{\Omega^+}$ by

$$\partial_c u(x) := \left\{y \in \overline{\Omega^*}; \quad u(x) + u^{c^*}(y) + c(x,y) = 0\right\}, \tag{2.3}$$

where u^{c^*} is as in (2.2). The function u is differentiable at some point $x \in \Omega$ if and only if $\partial_c u(x)$ is a singleton, and in this case, we get $\partial_c u(x) = \{T(x)\}$, where $T(x)$ is as in (2.1). In particular, T is continuous and one-to-one in Ω if and only if u is continuously differentiable and *strictly c-convex* in Ω, i.e. for any $y \in \partial_c u(\Omega)$, there exists a unique $x \in \Omega$ such that $y \in \partial_c u(x)$. Moreover, see Figalli–Kim–McCann [14, Lemma 3.1] (see also Ma–Trudinger–Wang [32]), we get $|\partial_c u| \geq \|f/g\|^{-1}_{L^\infty(\Omega\times\Omega^*)} \mathcal{L}^n$ in Ω and $|\partial_c u| \leq \|g/f\|_{L^\infty(\Omega^+\times\Omega^*)} \mathcal{L}^n$ in $\overline{\Omega^+}$, in the sense of measures, where $\mathcal{L}^n$ is Lebesgue's measure and $|\partial_c u(\Gamma)| = \mathcal{L}^n(\partial_c u(\Gamma))$ for all Borel subset Γ of $\overline{\Omega^+}$. In particular, in order to prove Theorem 1.1, we can assume that the potential function u satisfies (C) and (D) below.

- (C) u is c-convex in $\overline{\Omega^+}$ focussing in $\overline{\Omega^*}$.
- (D) There exist $\Lambda_1, \Lambda_2 > 0$ such that $|\partial_c u| \geq \Lambda_1 \mathcal{L}^n$ in Ω and $|\partial_c u| \leq \Lambda_2 \mathcal{L}^n$ in $\overline{\Omega^+}$ in the sense of measures, where $\mathcal{L}^n$ is the Lebesgue measure in $\mathbb{R}^n$.

In Theorems 2.1 and 2.2 below, we state our regularity results for potential functions. The proofs of Theorems 2.1 and 2.2 are left to Section 5.

Theorem 2.1. *Assume that U, Ω, Ω^+, Ω^*, c, u satisfy* (A0)–(A3w), (B), (C), *and* (D). *Then u is strictly c-convex in Ω.*

Theorem 2.2. *Assume that U, Ω, Ω^+, Ω^*, c, u satisfy* (A0)–(A3w), (B), (C), *and* (D). *Then u is continuously differentiable in Ω.*

Proof of Theorem 1.1. As discussed above, Theorem 1.1 follows from Theorems 2.1 and 2.2 by Kantorovich duality. □

2.2. **Convexity of sublevel sets.** We let U, Ω^+, Ω^*, c, u satisfy (A0)–(A3w), (B), (C), (D), and we let Ω be an open subset of Ω^+. We fix $y_0 \in \overline{\Omega^*}$. We define

$$U_0 := \{(-D_y c(x,y_0), y);\, (x,y) \in U \text{ and } (x,y_0) \in U\},\ \Omega_0 := -D_y c(\Omega, y_0),$$
$$\text{and } \Omega_0^+ := -D_y c\left(\Omega^+, y_0\right). \quad (2.4)$$

Clearly, Ω_0 is an open subset of Ω_0^+. We define a new cost function c_0 in U_0 by

$$c_0(q,y) := c(X(q,y_0), y) - c(X(q,y_0), y_0), \quad (2.5)$$

and we define a new function u_0 in Ω_0^+ by

$$u_0(q) := u(X(q,y_0)) + c(X(q,y_0), y_0), \quad (2.6)$$

where $X(q,y_0)$ is as in (A1). As is easily checked, after the above change of coordinates, U_0, Ω_0, Ω_0^+, c_0, u_0 satisfy the same conditions as U, Ω, Ω^+, c, u, except (A0) which is replaced by (A0w) below. In other words, U_0, Ω_0^+, c_0, u_0 satisfy (A0w)–(A3w), (B), (C), and (D).

(A0w) $c_0 \in C^3(U_0)$ and $D^4_{qqyy} c_0$ exists and is continuous in U_0.

Moreover, c_0 satisfies two additional conditions, namely that

$$D_q c_0(q,y_0) = 0 \quad \text{and} \quad D_y c_0(q,y_0) = -q \quad (2.7)$$

for all $q \in \overline{\Omega_0^+}$. In particular, it follows from (B) and (2.7) that Ω_0^+ is strongly convex. In Lemma 2.3 below, we state the property of convexity of sublevel sets. This property has been proved and used independently by Figalli–Kim–McCann [14] and Liu [26]. Lemma 2.3 is stated under the assumption (Bw) below, slightly weaker than (B).

(Bw) $\Omega^+ \times \Omega^* \Subset U$, Ω^+ and Ω^* are c-convex with respect to each others.

Lemma 2.3. *Let U, Ω^+, Ω^*, c, u satisfy (A0)–(A3w), (Bw), (C), and let $y_0 \in \overline{\Omega^*}$. Let U_0, Ω_0^+, c_0, u_0 be as in* (2.4)–(2.6). *Then for any $q, q' \in \overline{\Omega_0^+}$, there holds*

$$u_0([q,q']) \leq \max(u_0(q), u_0(q')).$$

In particular, u_0 has convex sublevel sets, i.e. for any $\lambda \in \mathbb{R}$, the set

$$\Omega_0^\lambda := \left\{q \in \overline{\Omega_0^+};\, u_0(q) \leq \lambda\right\} \quad (2.8)$$

is convex.

3. Alexandrov-type estimates for lower bounds

We let U, Ω^+, Ω^*, c, u satisfy (A0)–(A3w), (Bw), and (C). We fix $y_0 \in \overline{\Omega^*}$, and we let U_0, Ω_0^+, c_0, u_0 be as in (2.4)–(2.6). Moreover, for any compact convex subset K of $\overline{\Omega_0^+}$, we define

$$\sigma\left(c_0, y_0, K, \Omega^*\right) := \sup\left\{\frac{D^2_{pp}A_0\left(q,p\right).\left(\xi,\xi,\eta,\eta\right)}{\left|\langle\xi,\eta\rangle\right|};\right.$$
$$\left. q\in\left[q_-,q_+\right]\subset K,\ p\in\left[0,\eta\right],\ -\eta\in D_qc_0\left(q,\Omega^*\right),\ \text{and}\ \xi=q_+-q_-\right\}, \tag{3.1}$$

where $A_0\left(q,p\right) := D^2_{qq}c_0\left(q,Y_0\left(q,p\right)\right)$ and $D_qc_0\left(q,Y_0\left(q,p\right)\right)=-p$. We adopt the convention that $\sigma\left(c_0,y_0,K,\Omega^*\right)=0$ in case $\operatorname{diam}\left(K\right)=0$. Preliminary to this section, we prove the following lemma.

Lemma 3.1. *Let U, Ω^+, Ω^*, c satisfy* (A0)–(A3w), (Bw), *and let $y_0 \in \overline{\Omega^*}$. Let U_0, Ω_0^+, c_0 be as in* (2.4) *and* (2.5). *Then for any compact convex subset K of $\overline{\Omega_0^+}$, there holds*

$$0\le\sigma\left(c_0,y_0,K,\Omega^*\right)\le 2\left\|D^2_{pp}A_0\right\|_\infty\left\|D_qc_0\right\|_\infty \operatorname{diam}\left(K\right) \tag{3.2}$$

where $\sigma\left(c_0,y_0,\left[q,q'\right],\Omega^*\right)$ and A_0 are as in (3.1), *$\operatorname{diam}(K)$ and $\operatorname{diam}(\Omega^*)$ are the diameters of K and Ω^*.*

Proof. We get the first inequality in (3.2) by letting $\xi\to0$ in the definition of $\sigma\left(c_0,y_0,K,\Omega^*\right)$. Now, we prove the second inequality. We let $q_-,q_+\in K$, $q\in\left[q_-,q_+\right]$, $\xi=q_+-q_-$, $-\eta\in D_qc_0\left(q,\Omega^*\right)$, and $p\in\left[0,\eta\right]$. By (A3w), we get

$$D^2_{pp}A_0\left(q,p\right).\left(\xi,\xi,\eta,\eta\right)\le D^2_{pp}A_0\left(q,p\right).\left(\frac{\langle\xi,\eta\rangle}{\left|\eta\right|^2}\eta,\xi,\eta,\eta\right)$$
$$+D^2_{pp}A_0\left(q,p\right).\left(\xi-\frac{\langle\xi,\eta\rangle}{\left|\eta\right|^2}\eta,\frac{\langle\xi,\eta\rangle}{\left|\eta\right|^2}\eta,\eta,\eta\right)\le 2\left\|D^2_{pp}A_0\right\|_\infty\cdot\left|\langle\xi,\eta\rangle\right|\cdot\left|\xi\right|\cdot\left|\eta\right|. \tag{3.3}$$

Since K is convex and $q_-,q_+\in K$, we get $\left[q_-,q_+\right]\subset K$. Moreover, we get $-\eta\in D_qc_0\left(q,\Omega^*\right)$. It follows that

$$\left|\eta\right|\le\left\|D_qc_0\right\|_\infty \quad\text{and}\quad \left|\xi\right|\le\operatorname{diam}\left(K\right). \tag{3.4}$$

The second inequality in (3.2) then follows from (3.3) and (3.4). ☐

Our next lemma states as follows.

Lemma 3.2. *Let U, Ω^+, Ω^*, c satisfy* (A0)–(A3w), (Bw), *and let $y_0 \in \overline{\Omega^*}$. Let U_0, Ω_0^+, c_0 be as in* (2.4) *and* (2.5). *Then for any $q,q'\in\overline{\Omega_0^+}$ and $y\in\overline{\Omega^*}$ such that $D_qc_0\left(q',y\right).\left(q'-q\right)\ge0$, there holds*

$$D_qc_0\left(q',y\right).\left(q'-q\right)\le e^{\sigma\left(c_0,y_0,\left[q,q'\right],\Omega^*\right)/2}D_qc_0\left(q,y\right).\left(q'-q\right), \tag{3.5}$$

where $\sigma\left(c_0,y_0,\left[q,q'\right],\Omega^*\right)$ is as in (3.1).

Proof. For any $t\in[0,1]$, we define $\varphi\left(t\right):=c_0\left(q_t,y\right)$, where $q_t:=\left(1-t\right)q+tq'$. By (2.7), we get $D^2_{qq}c_0\left(q_t,y_0\right)=0$ and $D^3_{qqy}c_0\left(q_t,y_0\right)=0$. By (A0)–(A3w), (Bw), and the mean value theorem, it follows that there exists $s\in[0,1]$ such that

$$\varphi''\left(t\right)=D^2_{qq}c_0\left(q_t,y\right).\left(q'-q,q'-q\right)=\frac12D^2_{pp}A_0\left(q_t,sp\right).\left(q'-q,q'-q,p,p\right)$$
$$\le\frac12\sigma\left(c_0,y_0,\left[q,q'\right],\Omega^*\right)\left|\langle q'-q,p\rangle\right|=\frac12\sigma\left(c_0,y_0,\left[q,q'\right],\Omega^*\right)\left|\varphi'\left(t\right)\right|, \tag{3.6}$$

where $Y_0(q_t,p)=y$, i.e. $D_q c_0(q_t,y)=-p$, and $A_0(q_t,sp)$ is as in (3.1). Integrating (3.6), we get that either $\varphi'(t)<0$ for all $t\in[0,1]$ or $\varphi'(0)\geq 0$ and

$$\varphi'(t)\leq e^{\sigma(c_0,y_0,[q,q'],\Omega^*)t/2}\varphi'(0)$$

for all $t\in[0,1]$. In particular, in case $t=1$ and $\varphi'(1)\geq 0$, we get (3.5). □

In what follows, we often combine Lemma 3.2 with the mean value theorem. Doing so, we get that for any $q,q'\in\overline{\Omega_0^+}$ and $y\in\overline{\Omega^*}$, if $D_q c_0(q',y).(q'-q)\geq 0$, then

$$D_q c_0(q',y).(q'-q)\leq e^{\sigma(c_0,y_0,[q,q'],\Omega^*)/2}\left(c_0(q',y)-c_0(q,y)\right). \tag{3.7}$$

Similarly, we get that for any $q,q'\in\overline{\Omega_0^+}$ and $y\in\overline{\Omega^*}$, if $c_0(q',y)-c_0(q,y)\geq 0$, then

$$c_0(q',y)-c_0(q,y)\leq e^{\sigma(c_0,y_0,[q,q'],\Omega^*)/2}D_q c_0(q,y).(q'-q). \tag{3.8}$$

We state our general Alexandrov-type estimates for lower bounds in Theorem 3.3 below. In case the cost function satisfies the assumption (B3) of nonnegative cross-curvature, see Figalli–Kim–McCann [14], estimates for lower bounds can be deduced from the fact that in this case, the cost measure is dominated by the Monge–Ampère measure. Our estimates state as follows.

Theorem 3.3. *Let U, Ω^+, Ω^*, c, u satisfy* (A0)–(A3w), (Bw), (C)*, and let $y_0\in\overline{\Omega^*}$. Let U_0, Ω_0^+, c_0, u_0 be as in* (2.4)–(2.6)*. Let $\lambda\in\mathbb{R}$ be such that $\Omega_0^\lambda\neq\emptyset$ and $\Omega_0^\lambda\Subset\Omega_0^+$, where Ω_0^λ is as in* (2.8)*. Then for any $K\Subset\Omega_0^\lambda$ such that $K\neq\emptyset$, there holds*

$$\left(\lambda-\inf_K u_0\right)^n\geq C\left\|\det\left(D^2_{qy}c_0\right)^{-1}\right\|_\infty^{-1}\frac{e^{-n\sigma(c_0,y_0,\Omega_0^\lambda,\Omega^*)}d\left(K,\partial\Omega_0^\lambda\right)^{2n}}{\left(d\left(K,\partial\Omega_0^\lambda\right)+diam\left(K\right)\right)^n}\left|\partial_{c_0}u_0\left(K\right)\right| \tag{3.9}$$

for some $C=C(n)>0$, where $\sigma(c_0,y_0,\Omega_0^\lambda,\Omega^*)$ is as in (3.1)*, $diam(K)$ is the diameter of K, and $d\left(K,\partial\Omega_0^\lambda\right)$ is the distance between K and $\partial\Omega_0^\lambda$.*

Proof. For any $q\in K$, we define

$$E_0^\lambda(q):=\left\{y\in\overline{\Omega^*};\quad c_0(q,y)-c_0(q',y)\leq\lambda-u_0(q)\quad\forall q'\in\partial\Omega_0^\lambda\right\}.$$

Since $u_0=\lambda$ on $\partial\Omega_0^\lambda$, we have $\partial_{c_0}u_0(q)\subset E_0^\lambda(q)$. We claim that for any $q,q_0\in K$, and $y\in E_0^\lambda(q)$, there holds

$$\left|D_q c_0(q_0,y)\right|\leq e^{\sigma(c_0,y_0,\Omega_0^\lambda,\Omega^*)}\left(\lambda-u_0(q)\right)\frac{d\left(K,\partial\Omega_0^\lambda\right)+\operatorname{diam}(K)}{d\left(K,\partial\Omega_0^\lambda\right)^2}. \tag{3.10}$$

We prove this claim. Since Ω_0^λ is bounded and $q\in K\Subset\Omega_0^\lambda$, we get that there exists $q'\in\partial\Omega_0^\lambda$ such that

$$D_q c_0(q_0,y).(q_0-q')=\left|D_q c_0(q_0,y)\right|\cdot\left|q_0-q'\right|.$$

By (3.7), it follows that

$$\left|D_q c_0(q_0,y)\right|\cdot\left|q_0-q'\right|\leq e^{\sigma(c_0,y_0,[q_0,q'],\Omega^*)/2}\left(c_0(q_0,y)-c_0(q',y)\right). \tag{3.11}$$

Since $y\in E_0^\lambda(q)$ and $q'\in\partial\Omega_0^\lambda$, we get

$$c_0(q_0,y)-c_0(q',y)\leq c_0(q_0,y)-c_0(q,y)+\lambda-u_0(q). \tag{3.12}$$

If $q=q_0$, then it follows directly from (3.11) and (3.12) that

$$\left|D_q c_0(q_0,y)\right|\leq\frac{e^{\sigma(c_0,y_0,[q_0,q'],\Omega^*)/2}}{\left|q_0-q'\right|}\left(\lambda-u_0(q)\right)\leq\frac{e^{\sigma(c_0,y_0,\Omega_0^\lambda,\Omega^*)/2}}{d\left(K,\partial\Omega_0^\lambda\right)}\left(\lambda-u_0(q)\right), \tag{3.13}$$

and thus we get (3.10). Therefore, we can assume that $q \neq q_0$. Since Ω_0^λ is bounded and convex, we get that there exists a unique $\widetilde{q} \in \partial \Omega_0^\lambda$ such that

$$\frac{q-\widetilde{q}}{|q-\widetilde{q}|} = \frac{q_0 - q}{|q_0 - q|}\,.$$

By (3.7) and (3.8), it follows that if $c_0\left(q_0, y\right) - c_0\left(q, y\right) \geq 0$, then

$$c_0\left(q_0, y\right) - c_0\left(q, y\right) \leq e^{\sigma\left(c_0, y_0, [\widetilde{q}, q_0], \Omega^*\right)/2} \frac{|q_0 - q|}{|q - \widetilde{q}|} \left(c_0\left(q, y\right) - c_0\left(\widetilde{q}, y\right)\right). \tag{3.14}$$

Since $y \in E_0^\lambda\left(q\right)$ and $\widetilde{q} \in \partial \Omega_0^\lambda$, we get

$$c_0\left(q, y\right) - c_0\left(\widetilde{q}, y\right) \leq \lambda - u_0\left(q\right). \tag{3.15}$$

By (3.11)–(3.15), we get

$$\left|D_q c_0\left(q_0, y\right)\right| \cdot \left|q_0 - q'\right| \leq e^{\sigma\left(c_0, y_0, \Omega_0^\lambda, \Omega^*\right)} \left(\lambda - u_0\left(q\right)\right) \left(1 + \frac{|q_0 - q|}{|q - \widetilde{q}|}\right). \tag{3.16}$$

Our claim (3.10) follows from (3.16) in view of $\left|q_0 - q'\right| \geq d\left(K, \partial\Omega_0^\lambda\right)$, $\left|q - \widetilde{q}\right| \geq d\left(K, \partial\Omega_0^\lambda\right)$ and $\left|q_0 - q\right| \leq \operatorname{diam}\left(K\right)$. In particular, by (3.10), we get that

$$\left|D_q c_0\left(q_0, E_0^\lambda\left(K\right)\right)\right| \leq \left|B_0\left(R_0^\lambda\right)\right| = \left|B_0\left(1\right)\right| \left(R_0^\lambda\right)^n, \tag{3.17}$$

where

$$R_0^\lambda := e^{\sigma\left(c_0, y_0, \Omega_0^\lambda, \Omega^*\right)} \left(\lambda - \inf_K u_0\right) \frac{d\left(K, \partial\Omega_0^\lambda\right) + \operatorname{diam}\left(K\right)}{d\left(K, \partial\Omega_0^\lambda\right)^2}\,.$$

Moreover, since $\partial_{c_0} u_0\left(q\right) \subset E_0^\lambda\left(q\right)$ for all $q \in K$, we get

$$\left|\partial_{c_0} u_0\left(K\right)\right| \leq \left|E_0^\lambda\left(K\right)\right| \leq \left\| \det\left(D_{qy}^2 c_0\right)^{-1} \right\|_\infty \left|D_q c_0\left(q_0, E_0^\lambda\left(K\right)\right)\right|. \tag{3.18}$$

Finally, (3.9) follows from (3.17) and (3.18). $\square$

In what follows, we combine Theorem 3.3 with affine renormalizations. The invariance under affine renormalization of the optimal transportation problem with general costs have been observed but not used in Figalli–Kim–McCann [15]. We let $L : \mathbb{R}^n \to \mathbb{R}^n$ be an affine transformation such that $\det L \neq 0$. We define $L^* := \left|\det \overline{L}\right|^{2/n} \cdot {}^t\overline{L}^{-1}$, where ${}^t\overline{L}^{-1}$ and $\det \overline{L}$ are, respectively, the transpose of the inverse and the determinant of the linear part of L. In particular, we have $\det L^* = \det \overline{L}$. For simplicity, we denote $\det L := \det \overline{L}$. We define

$$\begin{aligned} U^L &:= \left\{\left(x, \left(L^*\right)^{-1} y\right);\, (x,y) \in U\right\}, \ \left(\Omega^*\right)^L := \left(L^*\right)^{-1} \Omega^*, \\ &\qquad c^L\left(x,y\right) := \left|\det L\right|^{-2/n} c\left(x, L^* y\right), \text{ and } u^L\left(x\right) := \left|\det L\right|^{-2/n} u\left(x\right). \end{aligned} \tag{3.19}$$

As is easily checked, U^L, $\left(\Omega^*\right)^L$, c^L, and u^L satisfy (A0)–(A3w), (Bw), (C). Moreover, given $y_0 \in \overline{\Omega^*}$, we define $y_0^L := \left(L^*\right)^{-1} y_0$, and similarly to (2.4)–(2.6), we define

$$\begin{aligned} U_0^L &:= \left\{\left(-D_y c^L\left(x, y_0^L\right), y\right);\, (x,y) \in U^L \text{ and } \left(x, y_0^L\right) \in U^L\right\}, \ \left(\Omega_0^+\right)^L := -D_y c^L\left(\Omega^+, y_0^L\right), \\ c_0^L\left(q,y\right) &:= c^L\left(X^L\left(q, y_0^L\right), y\right) - c^L\left(X^L\left(q, y_0^L\right), y_0^L\right), \\ u_0^L\left(q\right) &:= u^L\left(X^L\left(q, y_0^L\right)\right) + c^L\left(X^L\left(q, y_0^L\right), y_0^L\right), \ \left(\Omega_0^\lambda\right)^L := \left\{q \in \overline{\Omega_0^+};\, u_0^L\left(q\right) \leq \lambda^L\right\}, \end{aligned} \tag{3.20}$$

where $\lambda^L := |\det L|^{-2/n} \lambda$ and $X^L(q,y)$ is such that $D_y c^L\left(X^L(q,y), y\right) = q$. By direct calculations, we get $X^L(q,y) = X\left(\overline{L}q, L^* y\right)$, and thus

$$U_0^L = \left\{\left(\overline{L}^{-1} q, (L^*)^{-1} y\right);\, (q,y) \in U_0\right\},\ \left(\Omega_0^+\right)^L = \overline{L}^{-1} \Omega_0^+,\ c_0^L(q,y) = |\det L|^{-2/n} c_0\left(\overline{L}q, L^* y\right),$$
$$u_0^L(q) = |\det L|^{-2/n} u_0\left(\overline{L}q\right), \text{ and } \left(\Omega_0^\lambda\right)^L = \overline{L}^{-1} \Omega_0^\lambda. \quad (3.21)$$

Now, we prove the following key property of $\sigma(c_0, y_0, \Omega_0^\lambda, \Omega^*)$.

Lemma 3.4. *Let U, Ω^+, Ω^*, c satisfy* (A0)–(A3w), (Bw), *and let $y_0 \in \overline{\Omega^*}$. Let U_0, Ω_0^+, c_0 be as in* (2.4)–(2.5), *and U_0^L, $\left(\Omega_0^+\right)^L$, c_0^L be as in* (3.20)–(3.21). *For any affine transformation $L : \mathbb{R}^n \to \mathbb{R}^n$ and any compact convex subset K of $\overline{\Omega_0^+}$, there holds*

$$\sigma\left(c_0^L, y_0^L, K^L, (\Omega^*)^L\right) = \sigma\left(c_0, y_0, K, \Omega^*\right), \quad (3.22)$$

where $K^L := \overline{L}^{-1} K$.

Proof. We fix

$$q \in [q_-, q_+] \subset K,\ \xi := q_+ - q_-,\ -\eta \in D_q c_0\left(q, \Omega^*\right), \text{ and } p \in [0, \eta]. \quad (3.23)$$

We define

$$q^L := \overline{L}^{-1} q,\ q_\pm^L := \overline{L}^{-1} q_\pm,\ \xi^L := q_+^L - q_-^L,\ \eta^L := (L^*)^{-1} \eta, \text{ and } p^L := (L^*)^{-1} p. \quad (3.24)$$

By (3.21), (3.23), and (3.24), we get

$$q^L \in \left[q_-^L, q_+^L\right] \subset K^L,\ \xi^L = q_+^L - q_-^L,\ -\eta^L \in D_q c_0^L\left(q^L, (\Omega^*)^L\right), \text{ and } p^L \in \left[0, \eta^L\right]. \quad (3.25)$$

Now, we claim that

$$\frac{D_{pp}^2 A_0^L\left(q^L, p^L\right).\left(\xi^L, \xi^L, \eta^L, \eta^L\right)}{\left|\left\langle \xi^L, \eta^L \right\rangle\right|} = \frac{D_{pp}^2 A_0(q,p).(\xi, \xi, \eta, \eta)}{|\langle \xi, \eta \rangle|}, \quad (3.26)$$

where $A_0^L\left(q^L, p^L\right) := D_{qq}^2 c_0^L\left(q^L, Y_0^L\left(q^L, p^L\right)\right)$ and $D_q c_0^L\left(q^L, Y_0^L\left(q^L, p^L\right)\right) = -p^L$. From (3.21), we derive

$$Y_0^L\left(q^L, p^L\right) = (L^*)^{-1}\left(Y_0(q,p)\right),$$
$$A_0^L\left(q^L, p^L\right).\left(\xi^L, \xi^L\right) = |\det L|^{-2/n} A_0(q,p).(\xi, \xi),$$

and thus

$$D_{pp}^2 A_0^L\left(q^L, p^L\right).\left(\xi^L, \xi^L, \eta^L, \eta^L\right) = |\det L|^{-2/n} D_{pp}^2 A_0(q,p).(\xi, \xi, \eta, \eta). \quad (3.27)$$

Moreover, we have

$$\left\langle \xi^L, \eta^L \right\rangle = |\det L|^{-2/n} \langle \xi, \eta \rangle. \quad (3.28)$$

Our claim (3.26) then follows from (3.27) and (3.28). Finally, we deduce (3.22) from (3.25) and (3.26). $\square$

By combining Theorem 3.3 and Lemma 3.4, we get Corollary 3.5 below. We use this result in Section 5 in order to prove Theorems 2.1 and 2.2.

Corollary 3.5. *Let U, Ω^+, Ω^*, c, u satisfy* (A0)–(A3w), (Bw), (C), *and let $y_0 \in \overline{\Omega^*}$. Let U_0, Ω_0^+, c_0, u_0 be as in* (2.4)–(2.6). *Let $\lambda \in \mathbb{R}$ be such that $\Omega_0^\lambda \Subset \Omega_0^+$ and the interior of Ω_0^λ is nonempty, where Ω_0^λ is as in* (2.8). *Let E be a n-dimensional ellipsoid such that $E \subset \Omega_0^\lambda \subset nE$, where nE is the dilation of E with respect to its center of mass (the existence of such an ellipsoid is given by Theorem 6.5). Assume that there exist $\Lambda_1 > 0$, $\delta \in (0,1)$, and*

$q_0 \in \Omega_0^\lambda$ such that $|\partial_{c_0} u_0| \geq \Lambda_1 \mathcal{L}^n$ in $E_\delta := q_0 - c_E + \delta E$ in the sense of measures, where $\mathcal{L}^n$ is the Lebesgue measure in $\mathbb{R}^n$ and c_E is the center of E. Then there holds

$$\big(\lambda - \inf_{\Omega_0^\lambda} u_0\big)^n \geq C\Lambda_1 \big\| \det\big(D^2_{qy} c_0\big)^{-1} \big\|_\infty^{-1} e^{-n\sigma(c_0,y_0,\Omega_0^\lambda,\Omega^*)} \delta^{2n} \left|\Omega_0^\lambda\right|^2 \tag{3.29}$$

for some $C = C(n) > 0$, where $\sigma(c_0, y_0, \Omega_0^\lambda, \Omega^*)$ is as in (3.1).

Corollary 3.5 can be compared with the estimates obtained recently by Figalli–Kim–McCann [15]. Under similar conditions as in Corollary 3.5, Figalli–Kim–McCann [15, Theorem 6.2] prove that, provided that $\delta < \varepsilon_c / (4 \mathrm{diam}(E))$, where $\varepsilon_c > 0$ depends on the cost function, there holds

$$\big(\lambda - \inf_{\Omega_0^\lambda} u_0\big)^n \geq C\Lambda_1 \big\| \det\big(D^2_{qy} c_0\big)^{-1} \big\|_\infty^{-1} \delta^{2n} \left|\Omega_0^\lambda\right|^2 \tag{3.30}$$

for some $C = C(n) > 0$. Clearly, for δ small, the lower bound in (3.30) is better than ours in (3.29) because of the exponential term which depends on the cost function in (3.29). For δ fixed and $\mathrm{diam}(E)$ large, the lower bound in (3.29) is better since (3.30) is established provided that $\delta < \varepsilon_c / (4\mathrm{diam}(E))$. In general, the question whether our estimate or Figalli–Kim–McCann's one is better depends on δ, c, and $\mathrm{diam}(E)$.

Proof. By translation, we can assume that $c_E = 0$. We let $L : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation satisfying $E = L\big(\overline{B_0(1)}\big)$, where $B_0(1)$ is the ball of center 0 and radius 1. We define $y_0^L := (L^*)^{-1} y_0$, and we let U_0^L, $\big(\Omega_0^+\big)^L$, c_0^L, u_0^L, λ^L, and $\big(\Omega_0^\lambda\big)^L$ be as in (3.20)–(3.21). Since $E \subset \Omega_0^\lambda \subset nE$, we get $\overline{B_0(1)} \subset \big(\Omega_0^\lambda\big)^L \subset \overline{B_0(n)}$. We define

$$\begin{aligned} K_\delta :=& \big\{ q \in \Omega_0^\lambda \cap E_\delta\,;\quad d\big(L^{-1}q, \partial\big(\Omega_0^\lambda\big)^L\big) \geq \delta/(2n) \big\},\\ K_\delta^L := L^{-1} K_\delta =& \big\{ q \in \big(\Omega_0^\lambda\big)^L \cap \overline{B_{q_0^L}(\delta)}\,;\quad d\big(q, \partial\big(\Omega_0^\lambda\big)^L\big) \geq \delta/(2n) \big\}, \end{aligned}$$

where $q_0^L := L^{-1} q_0$. By Theorem 3.3, we get

$$\big(\lambda^L - \inf_{K_\delta^L} u_0^L\big)^n \geq C \big\| \det\big(D^2_{qy} c_0^L\big)^{-1} \big\|_\infty^{-1} \frac{e^{-n\sigma(c_0^L, y_0^L, (\Omega_0^\lambda)^L, (\Omega^*)^L)} d\big(K_\delta^L, \partial\big(\Omega_0^\lambda\big)^L\big)^{2n}}{\big(d\big(K_\delta^L, \partial\big(\Omega_0^\lambda\big)^L\big) + \mathrm{diam}\big(K_\delta^L\big)\big)^n} \big|\partial_{c_0^L} u_0^L\big(K_\delta^L\big)\big| \tag{3.31}$$

for some $C = C(n) > 0$. By definition of K_δ^L, we get $d\big(K_\delta^L, \partial\big(\Omega_0^\lambda\big)^L\big) \geq \delta/(2n)$ and $\mathrm{diam}\big(K_\delta^L\big) \leq 2\delta$, and thus

$$\frac{d\big(K_\delta^L, \partial\big(\Omega_0^\lambda\big)^L\big)^2}{d\big(K_\delta^L, \partial\big(\Omega_0^\lambda\big)^L\big) + \mathrm{diam}\big(K_\delta^L\big)} \geq \frac{\delta}{2n(1+4n)}\,. \tag{3.32}$$

By direct calculations, we get

$$\big(\lambda^L - \inf_{K_\delta^L} u_0^L\big)^n \leq \big(\lambda^L - \inf_{(\Omega_0^\lambda)^L} u_0^L\big)^n = |\det L|^{-2} \big(\lambda - \inf_{\Omega_0^\lambda} u_0\big)^n, \tag{3.33}$$

$$|\det L|^{-2} = \left|\Omega_0^\lambda\right|^{-2} \cdot \left|\big(\Omega_0^\lambda\big)^L\right|^2 \leq \left|\Omega_0^\lambda\right|^{-2} \cdot \left|B_0(n)\right|^2, \tag{3.34}$$

$$\big\| \det\big(D^2_{qy} c_0^L\big)^{-1} \big\|_\infty = \big\| \det\big(D^2_{qy} c_0\big)^{-1} \big\|_\infty\,, \tag{3.35}$$

$$\big|\partial_{c_0^L} u_0^L\big(K_\delta^L\big)\big| = \big|(L^*)^{-1}\big(\partial_{c_0} u_0\big(K_\delta\big)\big)\big| = |\det L|^{-1} \big|\partial_{c_0} u_0\big(K_\delta\big)\big|. \tag{3.36}$$

Since $K_\delta \subset E_\delta$, by assumption, we get

$$\big|\partial_{c_0} u_0\big(K_\delta\big)\big| \geq \Lambda_1 \big|K_\delta\big| = \Lambda_1 |\det L| \cdot \big|K_\delta^L\big|. \tag{3.37}$$

We claim that

$$\big|K_\delta^L\big| \geq C\delta^n \tag{3.38}$$

for some $C = C(n) > 0$. We prove this claim. By convexity of $\left(\Omega_0^\lambda\right)^L$ and since $B_0(1) \subset \left(\Omega_0^\lambda\right)^L$ and $q_0^L \in \left(\Omega_0^\lambda\right)^L$, we have

$$\operatorname{Hull}\left(B_0(1) \cup \left\{q_0^L\right\}\right) \subset \left(\Omega_0^\lambda\right)^L,$$

where $\operatorname{Hull} E$ is the convex hull of a set E. By definition of K_δ^L, it follows that

$$B_{q_0^L}(\delta) \cap \operatorname{Hull}\left(B_0\left(1 - \frac{\delta}{2n}\right) \cup \left\{q_0^L - \frac{\delta}{2n} \cdot \frac{q_0^L}{|q_0^L|}\right\}\right) \subset K_\delta^L. \tag{3.39}$$

Using that $\left|q_0^L\right| \le n$, we then deduce (3.38) from (3.39) by direct computations. Finally, (3.29) follows from (3.31)–(3.38) and Lemma 3.4. ☐

4. Alexandrov-type estimates for upper bounds

We let U, Ω^+, Ω^*, c, u satisfy (A0)–(A3w), (Bw), and (C). We fix $y_0 \in \overline{\Omega^*}$, and we let U_0, Ω_0^+, c_0, u_0 be as in (2.4)–(2.6). Contrary to the estimates for lower bounds in the previous section, the extension to (A3w) of the Alexandrov-type estimates for upper bounds relies on a simple adaptation of the arguments by Figalli–Kim–McCann [14]. Except for this adaptation (located in (4.7)–(4.8) in the proof of Theorem 4.2), and though we write the proofs in a slightly more direct way, we follow in this section the very nice ideas by Figalli–Kim–McCann [14]. We rewrite their arguments for the sake of completeness in our paper. Preliminary to the estimates for upper bounds, Figalli–Kim–McCann [14] establish the following Lipschitz estimate.

Lemma 4.1. *Let U, Ω^+, Ω^*, c satisfy* (A0)–(A2), (Bw), *and let $y_0 \in \overline{\Omega^*}$. Let U_0, Ω_0^+, c_0 be as in* (2.4) *and* (2.5). *Then for any $q, q' \in \overline{\Omega_0^+}$ and $y \in \overline{\Omega^*}$, there holds*

$$\left|D_q c_0(q', y) - D_q c_0(q, y)\right| \le \beta(c_0)^{-1} |q' - q| \left|D_q c_0(q, y)\right|, \tag{4.1}$$

where $\beta(c_0)^{-1} = \left\|D_{qqy}^3 c_0\right\|_\infty \left\|\left(D_{qy}^2 c_0\right)^{-1}\right\|_\infty$.

Proof. By the mean value theorem, we get that there exists $t \in [0,1]$ such that

$$\left|D_q c_0(q', y) - D_q c_0(q, y)\right|^2 = D_{qq}^2 c_0(q_t, y) . \left(D_q c_0(q', y) - D_q c_0(q, y), q' - q\right),$$

where $q_t = (1-t) q + t q'$. It follows from (2.7) that $D_q c_0(q, y_0) = 0$, $D_{qq}^2 c_0(q_t, y_0) = 0$, and $D_{qqy}^3 c_0(q_t, y_0) = 0$. By (Bw) and the mean value theorem, we then get that there exists $s \in [0,1]$ such that

$$\begin{aligned}
&\left|D_q c_0(q', y) - D_q c_0(q, y)\right|^2 \\
&\quad = D_{qqy}^3 c_0(q_t, Y_0(q, sp)) . \left(D_q c_0(q', y) - D_q c_0(q, y), q' - q, D_p Y_0(q, sp) . p\right) \\
&\quad \le \left\|D_{qqy}^3 c_0\right\|_\infty \left\|D_p Y_0\right\|_\infty \left|D_q c_0(q', y) - D_q c_0(q, y)\right| |q' - q| |p|,
\end{aligned} \tag{4.2}$$

where $Y_0(q, p) = y$, i.e. $D_q c_0(q, y) = -p$. In particular, we get $\left\|D_p Y_0\right\|_\infty = \left\|\left(D_{qy}^2 c_0\right)^{-1}\right\|_\infty$. (4.1) then follows from (4.2). ☐

Given a family of mutually orthogonal hyperplanes $(\Pi_1, \dots, \Pi_n)$ supporting a compact convex subset K of $\mathbb{R}^n$, we get the existence of a unique dual family of mutually orthogonal hyperplanes $(\Pi'_1, \dots, \Pi'_n)$ supporting K such that $\Pi'_i /\!/ \Pi_i$ and $\Pi'_i \ne \Pi_i$ for all $i = 1, \dots, n$. We let $\operatorname{Box}_{\Pi_1, \dots, \Pi_n}(K)$ be the compact subset of $\mathbb{R}^n$ delimited by the hyperplanes $(\Pi_1, \dots, \Pi_n)$ and $(\Pi'_1, \dots, \Pi'_n)$. We state a general result on Alexandrov-type estimates for upper bounds in Theorem 4.2 below. In particular, Theorem 4.2 extends the original work by Alexandrov [1] which was concerned with the special case of the cost function $c(x, y) = -|x - y|^2$. Here, the nonlinearity of the modified cost makes the general situation much trickier than in [1]. In the proof of Theorem 4.2, we mostly follow Figalli–Kim–McCann [14], except for the slight adaptation in (4.7)–(4.8) which extends the result to (A3w).

Theorem 4.2. *Let U, Ω^+, Ω^*, c, u satisfy* (A0)–(A3w), (Bw), (C), *and let $y_0 \in \overline{\Omega^*}$. Let U_0, Ω_0^+, c_0, u_0 be as in* (2.4)–(2.6). *Let $\lambda \in \mathbb{R}$, $q_0 \in \Omega_0^\lambda$, and $(\Pi_1, \dots, \Pi_n)$ be a family of mutually orthogonal hyperplanes supporting Ω_0^λ. Assume that $Box_{\Pi_1,\dots,\Pi_n}\left(\Omega_0^\lambda\right) \subset \Omega_0^+$. There exists δ_n depending only on n such that if $diam\left(\Omega_0^\lambda\right) \le \delta_n \beta(c_0)$, where $\beta(c_0)$ is as in Lemma 4.1 and $diam\left(\Omega_0^\lambda\right)$ is the diameter of Ω_0^λ, then*

$$(\lambda - u_0(q_0))^n \le C \left\|\det\left(D^2_{qy} c_0\right)\right\|_\infty \left(\prod_{i=1}^n d(q_0, \Pi_i)\right) \left|\partial_{c_0} u_0\left(\Omega_0^\lambda\right)\right| \tag{4.3}$$

for some $C = C(n) > 0$, where $d(q_0, \Pi_i)$ is the distance between q_0 and Π_i.

Proof. First, we claim that

$$E_0^\lambda := \left\{y \in \overline{\Omega^*};\quad c_0(q_0, y) - c_0(q, y) \le \lambda - u_0(q_0) \quad \forall q \in \partial\Omega_0^\lambda\right\} \subset \partial_{c_0} u_0\left(\Omega_0^\lambda\right) \tag{4.4}$$

(in the notations of Figalli–Kim–McCann [14], E_0^λ is the c_0-subdifferential at q_0 of the c_0-cone generated by q_0 and Ω_0^λ with height $\lambda - u_0(q_0)$). Indeed, since $u_0 = \lambda$ on $\partial\Omega_0^\lambda$, we get that if y belongs to E_0^λ, then the function $v_{0,y} : q \mapsto u_0(q) + c_0(q, y) - c_0(q_0, y) - u_0(q_0)$ is nonnegative on $\partial\Omega_0^\lambda$. Since $v_{0,y}(q_0) = 0$, $q_0 \in \Omega_0^\lambda$, and Ω_0^λ is bounded, it follows that $v_{0,y}$ admits a local minimum at some point $q_{0,y}$ in $\overline{\Omega_0^\lambda}$. By Corollary 6.3 in appendix, we then get that $q_{0,y}$ is a global minimum point of $v_{0,y}$ in $\overline{\Omega_0^+}$, i.e. $y \in \partial_{c_0} u_0(q_{0,y}) \subset \partial_{c_0} u_0\left(\Omega_0^\lambda\right)$. This ends the proof of (4.4). Now, for any $i = 1, \dots, n$, we choose $q_i \in \Pi_i \cap \partial\Omega_0^\lambda$ and we claim that there exists $y_i \in E_0^\lambda$ such that $D_q c_0(q_i, y_i)$ is normal to Π_i and

$$c_0(q_0, y_i) - c_0(q_i, y_i) = \lambda - u_0(q_0). \tag{4.5}$$

Indeed, by Corollary 6.2, we get $\partial u_0(q_i) = -D_q c_0(q_i, \partial_{c_0} u_0(q_i))$. In particular, by semiconvexity of u_0, since Ω_0^λ is a level set of u_0 and Π_i is a supporting hyperplane of Ω_0^λ, we get that there exists $y_i' \in \partial_{c_0} u_0(q_i)$ such that $D_q c_0(q_i, y_i')$ is normal to Π_i. Since $y_i' \in \partial_{c_0} u_0(q_i)$, $q_i \in \partial\Omega_0^\lambda$, and $u_0 = \lambda$ on $\partial\Omega_0^\lambda$, we get

$$c_0(q, y_i') - c_0(q_i, y_i') \ge \lambda - u_0(q) \tag{4.6}$$

for all $q \in \overline{\Omega_0^+}$. By (Bw), by continuity of the function $y \mapsto c_0(q_0, y) - c_0(q_i, y)$ on the c_0^*-segment connecting y_0 and y_i' with respect to q_i, and since, by (2.7), $c_0(q_0, y_0) - c_0(q_i, y_0) = 0$, we get that there exists $y_i \in \overline{\Omega^*}$ on this c_0^*-segment such that (4.5) holds. By (4.6), since $u_0 = \lambda$ on $\partial\Omega_0^\lambda$, and since, by (2.7), $c_0(q_i, y_0) - c_0(q, y_0) = 0$ for all $q \in \partial\Omega_0^\lambda$, by Theorem 6.1 in appendix, we get $c_0(q_i, y_i) - c_0(q, y_i) \le 0$ for all $q \in \partial\Omega_0^\lambda$. By (4.5), we then get $y_i \in E_0^\lambda$. Moreover, since $D_q c_0(q_i, y_i')$ is normal to Π_i and y_i belongs to the c_0-segment connecting y_0 and y_i' with respect to q_i, we get that $D_q c_0(q_i, y_i)$ is normal to Π_i. By Theorem 6.1, we get that the function $q \mapsto -c_0(q, y_i)$ has convex sublevel sets. By (4.5) and since $y_i \in E_0^\lambda$, $q_i \in \Pi_i$, and $D_q c_0(q_i, y_i)$ is normal to Π_i, it follows that

$$c_0(q_0, y_i) - c_0(q, y_i) \ge \lambda - u_0(q_0) \tag{4.7}$$

for all $q \in \Pi_i$. We let $q_i' \in \Pi_i$ be such that $|q_0 - q_i'| = d(q_0, \Pi_i)$. Since $\mathrm{Box}_{\Pi_1,\dots,\Pi_n}\left(\Omega_0^\lambda\right) \subset \Omega_0^+$, we get $q_i' \in \Omega_0^+$. By (4.7) and the mean value theorem, we get that there exists $t \in [0, 1]$ such that

$$D_q c_0(q_t, y_i) . (q_0 - q_i') \ge \lambda - u_0(q_0),$$

where $q_t = (1 - t) q_i' + t q_0$. Since $|q_0 - q_i'| = d(q_0, \Pi_i)$, it follows that

$$|D_q c_0(q_t, y_i)| \ge \frac{\lambda - u_0(q_0)}{d(q_0, \Pi_i)}. \tag{4.8}$$

Moreover, we get

$$|q_t - q_0| = (1-t)\,|q_0 - q_i'| = (1-t)\,d\left(q_0, \Pi_i\right) \le d\left(q_0, q_i\right) \le \operatorname{diam}\left(\Omega_0^\lambda\right). \tag{4.9}$$

It follows from (4.9) and Lemma 4.1 that

$$\left|D_q c_0\left(q_t, y_i\right)\right| \le \left(1 + \beta\left(c_0\right)^{-1} \operatorname{diam}\left(\Omega_0^\lambda\right)\right) \left|D_q c_0\left(q_0, y_i\right)\right|. \tag{4.10}$$

By Lemma 4.1, we also get

$$\begin{aligned}\left|\frac{D_q c_0\left(q_i, y_i\right)}{\left|D_q c_0\left(q_i, y_i\right)\right|} - \frac{D_q c_0\left(q_0, y_i\right)}{\left|D_q c_0\left(q_0, y_i\right)\right|}\right| &\le \frac{2\left|D_q c_0\left(q_i, y_i\right) - D_q c_0\left(q_0, y_i\right)\right|}{\left|D_q c_0\left(q_0, y_i\right)\right|} \le 2\beta\left(c_0\right)^{-1}\left|q_i - q_0\right| \\ &\le 2\beta\left(c_0\right)^{-1} \operatorname{diam}\left(\Omega_0^\lambda\right). \end{aligned}\tag{4.11}$$

By continuity of the determinant function, we get that there exists a constant δ_n depending only on n such that for any families $e = (e_1, \dots, e_n)$ and $f = (f_1, \dots, f_n)$ of vectors in $\mathbb{R}^n$, if e is orthogonal and $\left|\frac{e_i}{|e_i|} - \frac{f_i}{|f_i|}\right| \le 2\delta_n$ for all $i = 1, \dots, n$, then $|\det(f)| \ge \frac{1}{2} \cdot |f_1| \cdot |f_2| \cdots |f_n|$. By (4.11), it follows that if $\operatorname{diam}\left(\Omega_0^\lambda\right) \le \delta_n \beta\left(c_0\right)$, then

$$\prod_{i=1}^n \left|D_q c_0\left(q_0, y_i\right)\right| \le 2 \det\left(D_q c_0\left(q_0, y_i\right)\right)_{i=1,\dots,n} = 2n!\left|\operatorname{Hull}\left(D_q c_0\left(q_0, y_i\right)\right)_{i=1,\dots,n}\right|. \tag{4.12}$$

By Theorem 6.1 in appendix, we get that E_0^λ is c_0^*-convex with respect to q_0, and thus

$$\left|D_q c_0\left(q_0, E_0^\lambda\right)\right| \ge \left|\operatorname{Hull}\left(D_q c_0\left(q_0, y_i\right)\right)_{i=1,\dots,n}\right|. \tag{4.13}$$

Moreover, by (4.4), we get

$$\left|\partial_{c_0} u_0\left(\Omega_0^\lambda\right)\right| \ge \left|E_0^\lambda\right| \ge \left\|\det\left(D_{qy}^2 c_0\right)\right\|_\infty^{-1} \left|D_q c_0\left(q_0, E_0^\lambda\right)\right|. \tag{4.14}$$

Finally, (4.3) follows from (4.8)–(4.14). $\square$

Now, still following Figalli–Kim–McCann [14], we fix a supporting hyperplane Π of Ω_0^λ, and we claim that we can choose a family of mutually orthogonal hyperplanes $(\Pi_1, \dots, \Pi_n)$ supporting Ω_0^λ such that $\Pi_1 = \Pi$ and such that for any $q_0 \in \Omega_0^\lambda$, there holds

$$\prod_{i=2}^n d\left(q_0, \Pi_i\right) \le C\mathcal{H}^{n-1}\left(\pi_1\left(\Omega_0^\lambda\right)\right) \tag{4.15}$$

for some $C = C(n) > 0$, where $\pi_1\left(\Omega_0^\lambda\right)$ is the projection of Ω_0^λ onto Π. Indeed, in order to get (4.15), we can choose, for instance, the hyperplanes $(\Pi_2, \dots, \Pi_n)$ to be orthogonal to the axes of a $(n-1)$-dimensional ellipsoid E satisfying $E \subset \pi_1\left(\Omega_0^\lambda\right) \subset (n-1)\,E$ (which existence is given by Theorem 6.5). We then get the following corollary of Theorem 4.2 by applying Theorem 6.6 as in Figalli–Kim–McCann [14].

Corollary 4.3. *Let U, Ω^+, Ω^*, c, u satisfy* (A0)–(A3w), (Bw), (C), *and let $y_0 \in \overline{\Omega^*}$. Let U_0, Ω_0^+, c_0, u_0 be as in* (2.4)–(2.6). *Let $\lambda \in \mathbb{R}$, $q_0 \in \Omega_0^\lambda$, and $(\Pi_1, \dots, \Pi_n)$, $\Pi_1 = \Pi$, be a family of mutually orthogonal hyperplanes supporting Ω_0^λ and such that* (4.15) *holds. Assume that $Box_{\Pi_1,\dots,\Pi_n}\left(\Omega_0^\lambda\right) \subset \Omega_0^+$. If $\operatorname{diam}\left(\Omega_0^\lambda\right) \le \delta_n \beta\left(c_0\right)$, where δ_n and $\beta\left(c_0\right)$ are as in Lemma 4.1 and Theorem 4.2 and $\operatorname{diam}\left(\Omega_0^\lambda\right)$ is the diameter of Ω_0^λ, then*

$$\left(\lambda - u_0\left(q_0\right)\right)^n \le C\left\|\det\left(D_{qy}^2 c_0\right)\right\|_\infty \frac{d\left(q_0, \Pi\right)}{\ell_\Pi}\left|\Omega_0^\lambda\right|\left|\partial_{c_0} u_0\left(\Omega_0^\lambda\right)\right| \tag{4.16}$$

for some $C = C(n) > 0$, where $d\left(q_0, \Pi\right)$ is the distance between q_0 and Π, and ℓ_Π is the maximal length among all segments obtained by intersecting Ω_0^λ with an orthogonal line to Π.

Proof. By (4.15) and Theorem 6.6, we get

$$\ell_{\Pi} \prod_{i=2}^{n} d\left(q_0, \Pi_i\right) \leq C \left|\Omega_0^{\lambda}\right| \tag{4.17}$$

for some $C = C(n) > 0$. (4.16) then follows from (4.17) and Theorem 4.2. □

5. Proofs of Theorems 2.1 and 2.2

We let U, Ω^+, Ω^*, c, u satisfy (A0)–(A3w), (B), (C) (D), and we let Ω be an open subset of Ω^+. We prove Theorems 2.1 and 2.2 by combining the Alexandrov-type estimates for lower and upper bounds in Corollaries 3.5 and 4.3. The proofs of Theorems 2.1 and 2.2 follow a strategy developed by Caffarelli [4] for the Monge–Ampère equation and extended by Figalli–Kim–McCann [14] to the more general and more difficult setting of nonlinear cost functions.

Proof of Theorem 2.1. We assume by contradiction that there exists $y_0 \in \partial_c u(\Omega)$ such that $(\partial_c u)^{-1}(y_0) \cap \Omega$ is not a singleton. By Theorem 6.4 in appendix and since $(\partial_c u)^{-1}(y_0)$ is closed and Ω^+ is open and bounded, we get $y_0 \in \Omega^*$ and $(\partial_c u)^{-1}(y_0) \Subset \Omega^+$. We let U_0, Ω_0, Ω_0^+, c_0, u_0 be as in (2.4)–(2.6). By Lemma 2.3, we get that $(\partial_{c_0} u_0)^{-1}(y_0)$ is convex. Translating, if necessary, the set $(\partial_{c_0} u_0)^{-1}(y_0)$, we may assume that 0 is an exposed point of $(\partial_{c_0} u_0)^{-1}(y_0)$. Since $(\partial_{c_0} u_0)^{-1}(y_0) \cap \Omega_0$ is not a singleton, it follows that there exists $q_0 \in (\partial_{c_0} u_0)^{-1}(y_0) \cap \Omega_0 \backslash \{0\}$ and $v_0 \in (\partial_{c_0} u_0)^{-1}(y_0) \backslash \{0\}$ such that v_0 is normal to a supporting hyperplane of $(\partial_{c_0} u_0)^{-1}(y_0)$ at 0. We define

$$\begin{aligned} K_0 &:= \left\{q \in (\partial_{c_0} u_0)^{-1}(y_0)\,; \quad \langle q, v_0\rangle \leq \langle q_0, v_0\rangle\right\}, \\ K_0' &:= \left\{q \in (\partial_{c_0} u_0)^{-1}(y_0)\,; \quad \langle q, v_0\rangle \leq \gamma_0 \left|v_0\right|^2\right\}, \end{aligned}$$

for some $\gamma_0 \in (0,1)$ to be chosen small. In particular, we get $K_0' \subsetneq K_0$. Moreover, since $v_0 \in (\partial_{c_0} u_0)^{-1}(y_0) \backslash \{0\}$ is normal to a supporting hyperplane of $(\partial_{c_0} u_0)^{-1}(y_0)$ at 0, we get $\langle q, v_0\rangle \geq 0$ for all $q \in K_0$. For $\varepsilon > 0$ small, letting $y_\varepsilon = y_0 - \varepsilon v_0$, we define

$$\begin{aligned} &U_\varepsilon := \left\{(-D_y c(q, y_\varepsilon), y)\,; (q,y) \in U \text{ and } (q, y_\varepsilon) \in U\right\}, \; \Omega_\varepsilon := -D_y c(\Omega, y_\varepsilon), \\ &\Omega_\varepsilon^+ := -D_y c\left(\Omega^+, y_\varepsilon\right), \; q_\varepsilon := -D_y c_0(q_0, y_\varepsilon), \; q_\varepsilon' := -D_y c_0(\gamma_0 v_0, y_\varepsilon), \text{and}\, q_\varepsilon'' := -D_y c_0(0, y_\varepsilon). \end{aligned}$$

By (2.7) and since $c_0 \in C^1(U_0)$ and $y_\varepsilon \to y_0$, we get $\Omega_\varepsilon \to \Omega_0$, $\Omega_\varepsilon^+ \to \Omega_0^+$, $q_\varepsilon \to q_0$, $q_\varepsilon' \to \gamma_0 v_0$, and $q_\varepsilon'' \to 0$. We define a new cost function c_ε in U_ε by

$$c_\varepsilon(q, y) = c(X(q, y_\varepsilon), y) - c(X(q, y_\varepsilon), y_\varepsilon),$$

and we define a new function u_ε in Ω_ε^+ by

$$u_\varepsilon(q) = u(X(q, y_\varepsilon)) + c(X(q, y_\varepsilon), y_\varepsilon),$$

where $X(q, y_\varepsilon)$ is as in (A1). Moreover, we define

$$\begin{aligned} K_\varepsilon &:= \left\{q \in \overline{\Omega_\varepsilon^+}\,; \quad u_\varepsilon(q) \leq u_\varepsilon(q_\varepsilon)\right\}, \\ K_\varepsilon' &:= \left\{q \in \overline{\Omega_\varepsilon^+}\,; \quad u_\varepsilon(q) \leq u_\varepsilon(q_\varepsilon')\right\}. \end{aligned}$$

By Lemma 2.3, we get that K_ε and K_ε' are convex. We claim that $K_\varepsilon \to K_0$ and $K_\varepsilon' \to K_0'$ as $\varepsilon \to 0$. Indeed, we get

$$-D_y c_\varepsilon(K_\varepsilon, y_0) = \left\{q \in \overline{\Omega_0^+}\,; \quad u_0(q) \leq u_0(q_0) + c_0(q_0, y_\varepsilon) - c_0(q, y_\varepsilon)\right\}. \tag{5.1}$$

For all $q \in \overline{\Omega_0^+}$, by (2.7), we find

$$\begin{aligned} &c_0(q_0, y_\varepsilon) - c_0(q, y_\varepsilon) \\ &\quad = \left(D_y c_0(q_0, y_0) - D_y c_0(q, y_0)\right) . (y_\varepsilon - y_0) + \mathrm{O}\left(\left\|D^3_{qyy} c_0\right\|_\infty |q - q_0| \, |y_\varepsilon - y_0|^2\right) \\ &\quad = \varepsilon \langle q_0 - q, v_0\rangle + \mathrm{O}\left(\left\|D^3_{qyy} c_0\right\|_\infty \operatorname{diam}\left(\Omega_0^+\right) \varepsilon^2 |v_0|^2\right). \end{aligned} \tag{5.2}$$

It follows from (5.1) and (5.2) that $-D_y c_\varepsilon(K_\varepsilon, y_0) \to K_0$ as $\varepsilon \to 0$. In particular, since $K_0 \subset (\partial_{c_0} u_0)^{-1}(y_0) \Subset \Omega_0^+$, we get $-D_y c_\varepsilon(K_\varepsilon, y_0) \Subset \Omega_0^+$ for ε small. Since $c_0 \in C^1(U_0)$, $y_\varepsilon \to y_0$, we get $(-D_y c_\varepsilon(\cdot, y_0))^{-1} = -D_y c_0(\cdot, y_\varepsilon) \to -D_y c_0(\cdot, y_0) = \mathrm{id}_{\Omega_0^+}$ uniformly in Ω_0^+ as $\varepsilon \to 0$. It follows that $K_\varepsilon \to K_0$ as $\varepsilon \to 0$. In particular, since $K_0 \subset (\partial_{c_0} u_0)^{-1}(y_0) \Subset \Omega_0^+$ and $\Omega_\varepsilon^+ \to \Omega_0^+$, we get $K_\varepsilon \Subset \Omega_\varepsilon^+$ for ε small. Similarly, we get $K'_\varepsilon \to K'_0$ as $\varepsilon \to 0$. By definition of K_ε and K'_ε, we get either $K_\varepsilon \subset K'_\varepsilon$ or $K'_\varepsilon \subset K_\varepsilon$. Since their limits satisfy $K'_0 \subsetneq K_0$, we get $K'_\varepsilon \subsetneq K_\varepsilon$ for ε small. By Theorem 6.5, we get that there exists a n-dimensional ellipsoid E_ε such that $E_\varepsilon \subset K_\varepsilon \subset nE_\varepsilon$, where nE_ε is the dilation of E_ε with respect to its center of mass. Since $K_\varepsilon \to K_0$ as $\varepsilon \to 0$ and K_0 is bounded, we get that the diameter ot E_ε is uniformly bounded for ε small. Since $q_\varepsilon \to q_0$ and $\Omega_\varepsilon \to \Omega_0$ as $\varepsilon \to 0$, and since $q_0 \in \Omega_0$ and Ω_0 is open, it follows that there exists $\delta \in (0,1)$ such that $E_{\varepsilon,\delta} := (q_\varepsilon - c_{E_\varepsilon} + \delta E_\varepsilon) \subset \Omega_\varepsilon$ for ε small, where c_{E_ε} is the center of E_ε. In particular, by (D), we get that for any Borel subset Γ of $E_{\varepsilon,\delta}$, there holds

$$\begin{aligned} |\partial_{c_\varepsilon} u_\varepsilon(\Gamma)| &= |\partial_c u(X(\Gamma, y_\varepsilon))| \ge \Lambda_1 |X(\Gamma, y_\varepsilon)| \\ &\ge \Lambda_1 \left\|\det (D_q X)^{-1}\right\|_\infty^{-1} |\Gamma| = \Lambda_1 \left\|\det\left(D^2_{xy} c\right)\right\|_\infty^{-1} |\Gamma|. \end{aligned}$$

By Corollary 3.5, it follows that

$$\left(u_\varepsilon(q_\varepsilon) - \inf_{K_\varepsilon} u_\varepsilon\right)^n \ge C\Lambda_1 \left\|\det\left(D^2_{xy} c\right)\right\|_\infty^{-1} \left\| \det\left(D^2_{qy} c_\varepsilon\right)^{-1}\right\|_\infty^{-1} e^{-n\sigma(c_\varepsilon, y_\varepsilon, K_\varepsilon, \Omega^*)} \delta^{2n} |K_\varepsilon|^2 \tag{5.3}$$

for some $C = C(n) > 0$, where $\sigma(c_\varepsilon, y_\varepsilon, K_\varepsilon, \Omega^*)$ is as in (3.1). By (3.7), we get

$$\sigma\big(c_\varepsilon, y_\varepsilon, K_\varepsilon, \Omega^*\big) \le 2 \left\|D^2_{pp} A_\varepsilon\right\|_\infty \left\|D_q c_\varepsilon\right\|_\infty \operatorname{diam}(K_\varepsilon), \tag{5.4}$$

where $A_\varepsilon(q,p) := D^2_{qq} c_\varepsilon(q, Y_\varepsilon(q,p))$ and $D_q c_\varepsilon(q, Y_\varepsilon(q,p)) = -p$. Moreover, we get

$$\begin{aligned} \left\| \det\left(D^2_{qy} c_\varepsilon\right)^{-1} \right\|_\infty &\le \left\| \det\left(D^2_{xy} c\right)^{-1} \right\|_\infty \left\| \det (D_q X)^{-1} \right\|_\infty \\ &= \left\| \det\left(D^2_{xy} c\right)^{-1} \right\|_\infty \left\| \det\left(D^2_{xy} c\right) \right\|_\infty, \end{aligned} \tag{5.5}$$

$$\left\|D_q c_\varepsilon\right\|_\infty \le \left\|D_x c\right\|_\infty \left\|D_q X\right\|_\infty = \left\|D_x c\right\|_\infty \left\| \left(D^2_{xy} c\right)^{-1} \right\|_\infty, \tag{5.6}$$

$$\left\|D^2_{pp} A_\varepsilon\right\|_\infty \le \left\|D^2_{pp} A\right\|_\infty \left\|D_q X\right\|_\infty^2 \left\| (D_q X)^{-1} \right\|_\infty^2 = \left\|D^2_{pp} A\right\|_\infty \left\| \left(D^2_{xy} c\right)^{-1} \right\|_\infty^2 \left\|D^2_{xy} c\right\|_\infty^2. \tag{5.7}$$

By (5.3)–(5.7) and since $\operatorname{diam}(K_\varepsilon) \to \operatorname{diam}(K_0) < \infty$ as $\varepsilon \to 0$, we get

$$\left(u_\varepsilon(q_\varepsilon) - \inf_{K_\varepsilon} u_\varepsilon\right)^n \ge C |K_\varepsilon|^2. \tag{5.8}$$

for some $C > 0$ independent of ε. Since $0, \gamma_0 v_0, q_0 \in (\partial_{c_0} u_0)^{-1}(y_0)$, by (5.2), and since $\langle q, v_0\rangle \ge 0$ for all $q \in K_0$, we get

$$\begin{aligned} \frac{u_\varepsilon(q'_\varepsilon) - u_\varepsilon(q''_\varepsilon)}{u_\varepsilon(q_\varepsilon) - \inf_{K_\varepsilon} u_\varepsilon} &\ge \frac{c_0(\gamma_0 v_0, y_\varepsilon) - c_0(0, y_\varepsilon)}{\sup_{-q \in D_y c_\varepsilon(K_\varepsilon, y_0)} \left(c_0(q_0, y_\varepsilon) - c_0(q, y_\varepsilon)\right)} \\ &\ge \frac{\gamma_0 |v_0|^2}{\langle q_0, v_0\rangle} + \mathrm{O}\left(\left\|D^3_{qyy} c_0\right\|_\infty \operatorname{diam}\left(\Omega_0^+\right) \varepsilon |v_0|^2\right). \end{aligned} \tag{5.9}$$

Now, we end the proof by applying Theorem 4.2 and showing a contradiction with (5.8) and (5.9). For ε small, we define

$$\Pi_\varepsilon := \Big\{ q \in \mathbb{R}^n; \quad \langle q, v_0 \rangle = \inf_{q' \in K'_\varepsilon} \langle q', v_0 \rangle \Big\}.$$

We let $(\Pi_{1,\varepsilon}, \dots, \Pi_{n,\varepsilon})$ be a family of mutually orthogonal hyperplanes supporting K'_ε, such that $\Pi_{1,\varepsilon} = \Pi_\varepsilon$, and such that (4.15) holds with $\Omega_0^\lambda := K'_\varepsilon$. Since $0 \in \Omega_0^+$ is an exposed point of $(\partial_{c_0} u_0)^{-1}(y_0)$, since $v_0 \in (\partial_{c_0} u_0)^{-1}(y_0) \setminus \{0\}$ is normal to a supporting hyperplane of $(\partial_{c_0} u_0)^{-1}(y_0)$ at 0, and since $K'_\varepsilon \to K'_0$, $\Omega_\varepsilon^+ \to \Omega_0^+$, and $\beta(c_\varepsilon) \to \beta(c_0)$ as $\varepsilon \to 0$, we can choose γ_0 small enough so that $\mathrm{Box}_{\Pi_{1,\varepsilon},\dots,\Pi_{n,\varepsilon}}(K'_\varepsilon) \subset \Omega_\varepsilon^+$ and $\operatorname{diam}(K'_\varepsilon) < \delta_n \beta(c_\varepsilon)$, where $\beta(c_\varepsilon)$ and δ_n are as in Lemma 4.1 and Theorem 4.2. By Corollary 4.3, we then get

$$\big(u_\varepsilon(q'_\varepsilon) - u_\varepsilon(q''_\varepsilon)\big)^n \le C \left\| \det\left(D^2_{qy} c_\varepsilon\right) \right\|_\infty \frac{d(q''_\varepsilon, \Pi_\varepsilon)}{\ell_{\Pi_\varepsilon}} |K'_\varepsilon| \, |\partial_{c_\varepsilon} u_\varepsilon(K'_\varepsilon)| \tag{5.10}$$

for some $C = C(n) > 0$, where $d(q''_\varepsilon, \Pi_\varepsilon)$ is the distance between q''_ε and Π_ε, and ℓ_{Π_ε} is the maximal length among all segments obtained by intersecting K'_ε with a line spanned by the vector v_0. By (D), we get

$$\begin{aligned} |\partial_{c_\varepsilon} u_\varepsilon(K'_\varepsilon)| &= |\partial_c u(X(K'_\varepsilon, y_\varepsilon))| \le \Lambda_2 |X(K'_\varepsilon, y_\varepsilon)| \\ &\le \Lambda_2 \big\| \det(D_q X) \big\|_\infty |K'_\varepsilon| \le \Lambda_2 \big\| \det\left(D^2_{xy} c\right)^{-1} \big\|_\infty |K'_\varepsilon| \,. \end{aligned} \tag{5.11}$$

Moreover, we get

$$\begin{aligned} \left\| \det\left(D^2_{qy} c_\varepsilon\right) \right\|_\infty &\le \big\| \det\left(D^2_{xy} c\right) \big\|_\infty \big\| \det(D_q X) \big\|_\infty \\ &\le \big\| \det\left(D^2_{xy} c\right) \big\|_\infty \big\| \det\left(D^2_{xy} c\right)^{-1} \big\|_\infty . \end{aligned} \tag{5.12}$$

It follows from (5.10)–(5.12) that

$$\big(u_\varepsilon(q'_\varepsilon) - u_\varepsilon(q''_\varepsilon)\big)^n \le C \frac{d(q''_\varepsilon, \Pi_\varepsilon)}{\ell_{\Pi_\varepsilon}} |K'_\varepsilon|^2 \tag{5.13}$$

for some $C > 0$ independent of ε. By (5.8), (5.9), (5.13), and since $K'_\varepsilon \subset K_\varepsilon$, we get

$$d(q''_\varepsilon, \Pi_\varepsilon) \ge C \ell_{\Pi_\varepsilon} \tag{5.14}$$

for some $C > 0$ independent of ε. Since $K'_\varepsilon \to K'_0$ as $\varepsilon \to 0$, we get $\Pi_\varepsilon \to \Pi_0$ and $\ell_{\Pi_\varepsilon} \to \ell_{\Pi_0}$. Since $q''_\varepsilon \to 0 \in \Pi_0$ and $\ell_{\Pi_0} \ge \gamma_0 |v_0|^2$, we then get a contradiction by passing to the limit as $\varepsilon \to 0$ into (5.14). This ends the proof of Theorem 2.1. □

Now, using the strict c-convexity of u, we prove the continuous differentiability of u as follows.

Proof of Theorem 2.2. By Corollary 6.2 in appendix, we get that $\partial u(x) = -D_x c(x, \partial_c u(x))$ for all $x \in \Omega$. In particular, by semiconvexity of u, in order to prove that u is continuously differentiable in Ω, it suffices to prove that for any $x \in \Omega$, $\partial_c u(x)$ is a singleton. By contradiction, we assume that there exists $x_0 \in \Omega$ such that $\partial_c u(x_0)$ contains at least two distinct points. We let $y_0 \in \partial_c u(x_0)$ be such that $-D_x c(x_0, y_0)$ is an exposed point of $\partial u(x_0)$. We let U_0, Ω_0, Ω_0^+, c_0, u_0 be as in (2.4)–(2.6). We get

$$\partial u_0(q_0) = (\partial u(x_0) + D_x c(x_0, y_0)) . D_q X(q_0, y_0) , \tag{5.15}$$

where $X(q_0, y_0) = x_0$. In particular, we get that 0 is an exposed point of $\partial u_0(q_0)$. Since $\partial_c u(x_0)$ contains at least two distinct points, it follows that there exists $v_0 \in \partial u_0(q_0) \setminus \{0\}$ such that v_0 is normal to a supporting hyperplane of $\partial u_0(q_0)$ at 0. For $\varepsilon > 0$ small, we define

$$K_\varepsilon := \Big\{ q \in \overline{\Omega_0^+} ; \quad u_0(q) \le u_0(q_0) + \varepsilon \Big\} .$$

By continuity of u_0 and since, by Theorem 2.1, u_0 is strictly c-convex, we get $K_\varepsilon \to \{q_0\}$ as $\varepsilon \to 0$. For ε small, we define

$$\Pi_\varepsilon := \left\{ q \in \mathbb{R}^n; \quad \langle q, v_0 \rangle = \sup_{q' \in K_\varepsilon} \langle q', v_0 \rangle \right\}.$$

Applying Corollaries 3.5 and 4.3 in the same way as in the proof of Theorem 2.1, we then prove that

$$d\left(q_0, \Pi_\varepsilon\right) \ge C \ell_{\Pi_\varepsilon} \tag{5.16}$$

for some $C > 0$ independent of ε, where ℓ_{Π_ε} is the maximal length among all segments obtained by intersecting K_ε with a line spanned by the vector v_0. Now, we end the proof by estimating $d\left(q_0, \Pi_\varepsilon\right)$ and ℓ_{Π_ε} as $\varepsilon \to 0$ and showing a contradiction with (5.16). On the one hand, by (3.7), letting $y_0' \in \partial_{c_0} u_0\left(q_0\right)$ be such that $D_q c_0\left(q_0, y_0'\right) = -v_0$, we get that for any $q \in \Omega_0^+$, if $\langle q, v_0 \rangle \ge \langle q_0, v_0 \rangle$, then

$$\begin{aligned} u_0\left(q\right) - u_0\left(q_0\right) \ge c_0\left(q_0, y_0'\right) - c_0\left(q, y_0'\right) &\ge e^{-\sigma(c_0, y_0, \Omega_0^+, \Omega^*)/2} D_q c_0\left(q_0, y_0'\right) . \left(q_0 - q\right) \\ &= e^{-\sigma(c_0, y_0, \Omega_0^+, \Omega^*)/2} \left\langle q - q_0, v_0 \right\rangle . \end{aligned}$$

It follows that for ε small, there holds

$$K_\varepsilon \subset \left\{ q \in \mathbb{R}^n ; \quad \langle q - q_0, v_0 \rangle < \varepsilon e^{\sigma(c_0, y_0, \Omega_0^+, \Omega^*)/2} \right\}.$$

In particular, we get

$$d\left(q_0, \Pi_\varepsilon\right) \le \varepsilon e^{\sigma(c_0, y_0, \Omega_0^+, \Omega^*)/2} / \left|v_0\right| . \tag{5.17}$$

On the other hand, since $v_0 \in \partial u_0\left(q_0\right) \setminus \{0\}$ is normal to a supporting hyperplane of $\partial u_0\left(q_0\right)$ at 0 and since u_0 is semiconvex, by the max formula, see Borwein–Lewis [2, Corollary 6.1.2], we get

$$u_0\left(q_0 - \gamma v_0\right) - u_0\left(q_0\right) \le \gamma \max_{p \in \partial u_0(q_0)} \langle p, -v_0 \rangle + \mathrm{o}\left(\gamma\right) = \mathrm{o}\left(\gamma\right)$$

as $\gamma \to 0$. It follows that for ε small, there exists $\gamma_\varepsilon > 0$ such that $q_0 - \gamma_\varepsilon v_0 \in K_\varepsilon$ and $\gamma_\varepsilon / \varepsilon \to \infty$. In particular, we get $\ell_{\Pi_\varepsilon} / \varepsilon \to \infty$, which contradicts (5.16) and (5.17). This ends the proof of Theorem 2.2. ☐

6. Appendix

In this appendix, we gather some results on optimal transportation and convex sets that we use in this paper.

6.1. **The maximum principle for cost functions.** In Theorem 6.1 below, we state a maximum principle which was first established by Loeper [29], see also Kim–McCann [24], Trudinger–Wang [33, 35], and Villani [38] for other proofs and extensions. For any $x \in \overline{\Omega^+}$ and $y, y' \in \overline{\Omega^*}$, letting $D_x c\left(x, y\right) = -p$ and $D_x c\left(x, y'\right) = -p'$, i.e. $Y\left(x, p\right) = y$ and $Y\left(x, p'\right) = y'$, we say that $t \mapsto Y\left(x, \left(1 - t\right) p + t p'\right)$ is the *c^*-segment* connecting y and y' with respect to x. Loeper's maximum principle states as follows. Needless to say, a dual result can be obtained by exchanging the roles of x and y.

Theorem 6.1. *Let U, Ω^+, Ω^*, c satisfy* (A0)–(A3w) *and* (Bw)*. Let $x, x' \in \overline{\Omega^+}$ and $y, y' \in \overline{\Omega^*}$. Then for any $t \in [0, 1]$, there holds $f\left(t\right) := c\left(x, y_t\right) - c\left(x', y_t\right) \le \max\left(f\left(0\right), f\left(1\right)\right)$, where $t \mapsto y_t$ is the c^*-segment connecting y and y' with respect to x.*

For any $x \in \Omega^+$, we let $\partial_c u(x)$ be as in (2.3) and $\partial u(x)$ be the subdifferential of u at x defined by

$$\partial u(x) := \{y \in \mathbb{R}^n; \quad u(x') \geq u(x) + \langle y, x' - x \rangle + \mathrm{o}(|x' - x|) \text{ as } x' \to x\}. \tag{6.1}$$

We give two corollaries below of Theorem 6.1 which we use in this paper. These results were obtained by Loeper [29]. Here again, we mention the related references by Kim–McCann [24], Trudinger–Wang [33, 35], and Villani [38].

Corollary 6.2. *Let U, Ω^+, Ω^*, c, u satisfy* (A0)–(A3w), (Bw), *and* (C). *For any $x \in \Omega^+$, there holds $\partial u(x) = -D_x c(x, \partial_c u(x))$, where $\partial u(x)$ and $\partial_c u(x)$ are as in* (2.3) *and* (6.1). *In particular, $D_x c(x, \partial_c u(x))$ is convex.*

Corollary 6.3. *Let U, Ω^+, Ω^*, c, u satisfy* (A0)–(A3w), (Bw), *and* (C). *For any $y \in \overline{\Omega^*}$, any local minimum point of the function $x \mapsto u(x) + c(x, y)$ is a global minimum point in Ω^+.*

6.2. **On the image and inverse image of boundary points.** The assumption (B) of strong c-convexity of domains is used in this paper in order to apply Theorem 6.4 below, which is due to Figalli–Kim–McCann [14].

Theorem 6.4. *Let $\Omega \subset \Omega^+$ and Ω^* be bounded open subsets of $\mathbb{R}^n$. Let c and u satisfy* (A0)–(A2) *and* (C).

(i) If $|\partial_c u| \geq \Lambda_1 \mathcal{L}^n$ in Ω for some $\Lambda_1 > 0$ and Ω^* is strongly c^*-convex with respect to Ω, then interior points of Ω cannot be mapped by $\partial_c u$ to boundary points of Ω^*, i.e. $\partial_c u^{-1}(\partial \Omega^*) \cap \Omega = \emptyset$.

(ii) If $|\partial_c u| \leq \Lambda_2 \mathcal{L}^n$ in $\overline{\Omega^+}$ for some $\Lambda_2 > 0$ and Ω^+ is strongly c-convex with respect to Ω^*, then boundary points of Ω^+ cannot be mapped by $\partial_c u$ into interior points of Ω^*, i.e. $\partial_c u(\partial \Omega^+) \cap \Omega^* = \emptyset$.

6.3. **Two results on convex sets.** We also use in this paper the following two results on convex sets. The first one is John's Theorem [23]. We use this result in both the Alexandrov-type estimates for lower and upper bounds in Sections 3 and 4.

Theorem 6.5. *For any compact convex subset K of $\mathbb{R}^n$ with nonempty interior, there exists a n-dimensional ellipsoid such that $E \subset K \subset nE$, where nE is the dilation of E with respect to its center of mass.*

Another result on convex sets, Theorem 6.6 below, is used in the proof of the Alexandrov-type estimates for upper bounds Corollary 4.3. This result was established by Figalli–Kim–McCann [14].

Theorem 6.6. *Let K be a convex subset of $\mathbb{R}^n = \mathbb{R}^{n'} \times \mathbb{R}^{n''}$. Let π'' be the projection of $\mathbb{R}^n$ onto $\mathbb{R}^{n''}$. Let $x'' \in \pi''(K)$ and $K' = (\pi'')^{-1}(x'') \cap K$. Then there holds*

$$\mathcal{H}^{n'}(K') \mathcal{H}^{n''}(\pi''(K)) \leq C \mathcal{L}^n(K)$$

for some $C = C(n', n'') > 0$, where $\mathcal{H}^d$ is the d-dimensional Haussdorf measure and $\mathcal{L}^n$ is the n-dimensional Lebesgue measure.

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