Fundamental solutions for anisotropic elliptic equations: existence and a priori estimates

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Abstract

We study anisotropic equations such as $-\sum_{i=1}^n \partial_{x_i}(|\partial_{x_i}u|^{p_i-2}\partial_{x_i}u) = \delta_0$ (with Dirac mass δ_0 at 0) in a domain $\Omega \subset \mathbb{R}^n$ $(n \geq 2)$ with $0 \in \Omega$ and $u|_{\partial\Omega} = 0$. Suppose that $p_i \in (1,\infty)$ for all i with their harmonic mean p satisfying either Case 1: $p < n$ and $\max_{1\leq i\leq n}\{p_i\} < \frac{p(n-1)}{n-p}$ or Case 2: $p = n$ and Ω is bounded. We establish the existence of a suitable notion of fundamental solution (or Green's function), together with sharp pointwise upper bound estimates near zero via an anisotropic Moser-type iteration scheme. As critical tools, we derive generalized anisotropic Sobolev inequalities and estimates in weak Lebesgue spaces.

1 Introduction and main result

Anisotropic elliptic equations have received much attention in recent years (see, for example, [2, 5, 10, 11, 15, 17, 21, 22, 24, 25, 33, 37, 42, 48] and their references). Time-dependent versions of these equations have been used as mathematical models to describe the spread of an epidemic disease, see [4, 6]. Such evolution models also arise in fluid dynamics when the media has different conductivities in different directions, see [1, 2].

We study anisotropic elliptic equations with right-hand side the Dirac mass δ_0 at zero in a domain Ω of $\mathbb{R}^n$ with $n \geq 2$ and $0 \in \Omega$, namely

$$\begin{cases} -\Delta_{\overrightarrow{p}} u = \delta_0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.1}$$

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We use $\Delta_{\overrightarrow{p}} u$ to denote $\operatorname{div}(\mathbf{A}(\nabla u))$, where $\mathbf{A}(\nabla u)$ is the vector field whose ith component is given by $|\partial_i u|^{p_i-2}\partial_i u$ with $p_i > 1$ and $1 \le i \le n$. As usual, $\nabla u = (\partial_1 u, \dots, \partial_n u)$ stands for the gradient of u. Let $\operatorname{meas}(E)$ denote the measure of a measurable set $E \subset \mathbb{R}^n$.

Let $\overrightarrow{p} := (p_1, \dots, p_n)$. Without loss of generality, we assume throughout that

$$1 < p_1 \le p_2 \le \ldots \le p_n < \infty. \tag{1.2}$$

In what follows, p denotes the *harmonic mean* of $p_1, \dots, p_n$, that is

$$\frac{1}{p} := \frac{1}{n}\sum_{i=1}^{n}\frac{1}{p_i}. \tag{1.3}$$

By a *fundamental solution* (or *Green's function*) of (1.1), we mean any non-negative solution of (1.1). Our main result gives the existence of a fundamental solution of (1.1) in appropriate spaces and sharp pointwise upper bound estimates near zero in two cases:

CASE 1: *Let Ω be an arbitrary domain of $\mathbb{R}^n$. Assume that*

$$p < n \text{ and } p_n < p_*, \text{ where } p_* := p\,(n-1)\,/(n-p)\,. \tag{1.4}$$

CASE 2: *Let $p = n$ and Ω be a bounded domain.*

The difficulty of this problem arises from the anisotropic character of (1.1) since there is no explicit formula known for a fundamental solution of (1.1) when p_i are *not all equal*. Our fundamental solution lies in a suitable anisotropic space, denoted by $\mathcal{T}_0^{1,\overrightarrow{p}}(\Omega)$, which is a natural generalization of the spaces introduced by Bénilan *et al.* [7] for this type of problem when p_i are all equal. For $n \ge 2$ and $R > 0$, we define $\mathcal{E}_{\overrightarrow{p}}(R)$ as follows

$$\mathcal{E}_{\overrightarrow{p}}(R) := \left\{x \in \mathbb{R}^n :\ 0 < \sum_{i=1}^{n}|x_i|^{s_i} < R\right\} \text{ and } s_i := \begin{cases} \dfrac{p_i p_*}{p_* - p_i} & \text{in Case 1,} \\ p_i & \text{in Case 2.} \end{cases} \tag{1.5}$$

In Section 2, we define the space $\mathcal{T}_0^{1,\overrightarrow{p}}(\Omega)$ and introduce a notion of weak solution of (1.1), whose existence is proved by Theorem 1.1 below.

Theorem 1.1. *Let Case 1 or Case 2 hold and $R > 0$ be such that $\mathcal{E}_{\overrightarrow{p}}(2R) \subset \Omega$. Then* (1.1) *admits a non-negative weak solution $\Phi \in \mathcal{T}_0^{1,\overrightarrow{p}}(\Omega) \cap W^{1,\infty}_{\mathrm{loc}}(\Omega \setminus \{0\})$ (and, hence, Φ is continuous in $\Omega \setminus \{0\}$) with the following properties:*

1. There exists a positive constant $C_0 = C_0(n, \overrightarrow{p})$ such that

$$\Phi(x) \le C_0 \left(\sum_{i=1}^{n}|x_i|^{\frac{p_i}{p_*-p_i}}\right)^{-1} \quad \textit{for all } x \in \mathcal{E}_{\overrightarrow{p}}(R) \textit{ in Case 1.} \tag{1.6}$$

2. *There exist positive constants* $a = a(n, \overrightarrow{p}, \text{meas}\,(\Omega))$ *and* $b = b(n, \overrightarrow{p})$ *so that*

$$\Phi\,(x) \leq a + b \left| \ln \left(\sum_{i=1}^{n} |x_i|^{p_i} \right) \right| \quad \textit{for all } x \in \mathcal{E}_{\overrightarrow{p}}\,(R) \quad \textit{in Case 2.} \tag{1.7}$$

Remark 1.2. *In Case 1 of Theorem 1.1, we also obtain gradient estimates for the weak solution* Φ*. More precisely, there exists a positive constant* $C_1 = C_1\,(n, \overrightarrow{p})$ *such that*

$$\sum_{i=1}^{n} |\partial_i \Phi(x)|^{p_i} \leq C_1 \left(\sum_{i=1}^{n} |x_i|^{\frac{p_i p_*}{p_* - p_i}} \right)^{-1} \quad \textit{for a.e. } x \in \mathcal{E}_{\overrightarrow{p}}\,(R)\,. \tag{1.8}$$

The proof of the gradient estimates in (1.8) *follows essentially from* (1.6) *using rescaling arguments and Lieberman's results [33], see Corollary 5.3.*

The main contribution of this paper is to establish the upper bound estimates in Theorem 1.1. The asymptotic behaviour near zero of a fundamental solution of (1.1) is an open question when p_i are not all equal. In Case 1, Namlyeyeva–Shishkov–Skrypnik conjectured in [37] that the asymptotic behaviour of a fundamental (or source-type) solution of (1.1) near zero is determined by the function U_0 defined by

$$U_0(x) := \left(\sum_{i=1}^{n} |x_i|^{\frac{p_i}{p_* - p_i}} \right)^{-1} \quad \text{for } x = (x_1, \ldots, x_n) \neq 0. \tag{1.9}$$

By [37, Theorem 2.2], we only know that for our fundamental solution Φ of (1.1), the limit inferior of $\Phi(x)/U_0(x)$ is finite, whereas its limit superior is greater than zero as x tends to zero. Note that U_0 is given by (1.9) in Case 1, whereas in Case 2 we define U_0 by

$$U_0(x) := -\ln \left(\sum_{i=1}^{n} |x_i|^{p_i} \right) \quad \text{for } x \in \mathbb{R}^n \setminus \{0\}. \tag{1.10}$$

In Theorem 1.1 we prove that the limit superior of $\Phi(x)/U_0(x)$ is *finite* as x approaches zero. Section 5 contains the major technical work which goes into proving (1.6) and (1.7). We outline the strategy in Section 1.1.

We prove the existence of a fundamental solution of (1.1) in Proposition 4.1 (see Section 4) using an approach inspired by Bénilan *et al.* [7] (see also Boccardo–Gallöuet [8, 9]). The idea is to consider approximate problems for which distributional solutions exist (see Lemma 3.1) and show that we can pass to the limit in the weak formulation to obtain a non-negative weak solution of (1.1). This plan is validated by Lemma 4.2, whose proof relies essentially on local estimates in *weak* Lebesgue spaces in Lemma 3.3. These estimates in Case 2 of Theorem 1.1 are new and more delicate since they require a refined anisotropic Sobolev inequality, also established in this paper: See Lemma B.1 in Appendix B, whose critical ingredient is an inequality of Moser–Trudinger type.

In Case 1, it is usually assumed that Ω is bounded and $p_1 > \frac{p(n-1)}{n(p-1)}$, so that a fundamental solution Φ of (1.1) can be obtained in $W_0^{1,1}(\Omega)$ (see Bendahmane–Karlsen [5] and Boccardo–Gallouët–Marcellini [10]). In this case, using an anisotropic Sobolev embedding (see [5, Theorem 2.1]), jointly with $\partial_i \Phi \in L^{q_i}(\Omega)$ for all $q_i \in [1, \frac{p_i(p_*-1)}{p_*})$ and $i = 1, \dots, n$ (see [10]), we find that $\Phi \in L^q(\Omega)$ for all $q \in [1, p_* - 1)$. This result was improved in [5], where the fundamental solution Φ was shown to belong to the Marcinkiewicz space $M^{p_*-1}(\Omega)$ (or, equivalently, the weak Lebesgue space $L^{p_*-1,\infty}(\Omega)$, see Section 3.2) and $\partial_i \Phi \in M^{\frac{p_i(p_*-1)}{p_*}}(\Omega)$ for $i = 1, \dots, n$. However, if $p_n \le \frac{p(n-1)}{n(p-1)}$, which implies that $p \in (1, 2 - \frac{1}{n}]$, the fundamental solution is not expected to belong to $W^{1,1}(\Omega)$ and thus the notions of weak derivatives and distributional solutions do not apply anymore (see [5]). In Theorem 1.1, we remove the restriction $p_1 > \frac{p(n-1)}{n(p-1)}$ in Case 1 and extend the existence of a fundamental solution of (1.1) on any domain Ω. Hence, our fundamental solution must be understood in an appropriate weak sense (see Definition 2.3) rather than in a distribution sense considered in [5, 10] (see also Corollary 4.4).

We do not know whether the fundamental solution of (1.1) is unique in our general framework. However, in a more restrictive setting, it is known that the fundamental solution of the p-Laplacian operator, $\operatorname{div}(|\nabla u|^{p-2} \nabla u)$ with $1 < p \le n$, is unique due to work of Kichenassamy–Véron [30] based on rescaling techniques and a strong maximum principle. For a different approach, we refer to Trudinger–Wang [45, 46].

The properties of the fundamental solutions of elliptic equations and their constructions have been investigated in many contexts. In the case of a linear operator, see the pioneer work of Krasovskiĭ [31]. For a construction in the context of a Riemannian manifold, see Druet–Hebey–Robert [20, Appendix A]. Recent progress in the study of Green's functions for the biharmonic or polyharmonic operators has been made, for instance, by Dall'Acqua–Sweers [18] and Grunau–Robert [28].

The study of the local behaviour of singular solutions to nonlinear elliptic equations relies heavily on the properties of fundamental solutions. In the case of quasilinear equations, see the seminal papers by Serrin [40, 41]. For background and other important contributions on the topic of singularities of solutions, we refer to Véron [47]. More recent progress on the classification of the isolated singularities of solutions has been made for nonlinear equations involving, for example, divergence-form operators (Brandolini *et al.* [12]) or Hardy–Sobolev operators (Cîrstea [16]). The fundamental solutions also play a critical role for singular solutions of fully nonlinear uniformly elliptic equations (see e.g., Labutin [32], Felmer–Quaas [23], Armstrong–Sirakov–Smart [3] and their references).

It is worth mentioning that for anisotropic elliptic equations there is little known about the local behaviour of singular solutions. By analogy with the p-Laplacian, it is expected that the properties of fundamental solutions will have many consequences for the singular solutions of anisotropic elliptic equations.

1.1 Sketch of the proof of (1.6) and (1.7)

We assume that p_i are *not* all equal, otherwise the estimates are trivial since an explicit fundamental solution can be found (see Section 2.1). Let $\mathcal{E}_{\overrightarrow{p}}(R)$ be given by (1.5). For $\lambda \in (0,1)$ and $r \in (0,R)$, we define $\mathcal{A}_r(\lambda)$ as the set $\mathcal{E}_{\overrightarrow{p}}((1+\lambda)r)\backslash\overline{\mathcal{E}_{\overrightarrow{p}}((1-\lambda)r)}$. Since $\mathcal{E}_{\overrightarrow{p}}(R)$ is included in $\bigcup_{0<r<R}\mathcal{A}_r(1/2)$, the estimates in (1.6) and (1.7) would follow from

$$\|\Phi\|_{L^\infty(\mathcal{A}_r(1/2))} \leq \begin{cases} Cr^{-1/p_*} & \text{in Case 1,} \\ a+b|\ln r| & \text{in Case 2.} \end{cases} \tag{1.11}$$

Using the weak solution Φ_ε of the approximate problem (3.2) in Lemma 3.1, it is enough to show that (1.11) holds for Φ_ε instead of Φ. This requires a delicate analysis, which is carried out separately for Case 1 (see Proposition 5.5) and Case 2 (see Proposition 5.6).

Our approach is based on an anisotropic version of the Moser-type iteration scheme. One difficulty arises in the running step of the iteration process, which unlike the isotropic case, does not give a "pure" reverse Hölder inequality between two precisely determined Lebesgue norms. Our anisotropic analogue renders one Lebesgue norm being dominated by one out of n possible different Lebesgue norms raised to different powers as follows.

Proposition 1.3 (An anisotropic reverse Hölder inequality)**.** *Let $m=n/(n-p)$ in Case 1 and $m>1/(n-1)$ in Case 2. For every $\Gamma>m(p-1)$ and $0<\lambda<\lambda'\leq 3/4$, we have*

$$\|\Phi_\varepsilon\|^{\frac{\Gamma}{m}}_{L^\Gamma(\mathcal{A}_r(\lambda))} \leq C r^{\frac{1}{m}-\frac{n-p}{n}} \max_{i=1,\dots,n}\left(\frac{r^{\frac{p_i(n-p)}{p(n-1)}-1}}{(\lambda'-\lambda)^{p_i}}\|\Phi_\varepsilon\|^{\frac{\Gamma}{m}+p_i-p}_{L^{\frac{\Gamma}{m}+p_i-p}(\mathcal{A}_r(\lambda'))}\right), \tag{1.12}$$

where C is a positive constant of the form $c(n,\overrightarrow{p})\max\{1,(\Gamma/m-p+1)^{-p_n}\}\Gamma^p$.

The proof of Proposition 1.3 (see Section 5.1) relies essentially on: (1) a weighted anisotropic Sobolev inequality (see Lemma A.1 in Appendix A), which is applied to $\eta\Phi_\varepsilon$ for some suitable function η in $C^1_c(\Omega_\varepsilon\setminus B_\varepsilon(0))$ and (2) key estimates derived by using $\eta^{\frac{\Gamma}{m}-p+p_i}\Phi_\varepsilon^{\frac{\Gamma}{m}-p+1}$ as a test function in the weak formulation of the approximate problem (3.2). Since the definition of m in Proposition 1.3 is different in Case 1 compared with Case 2, the iteration scheme needs to be devised carefully for each case as follows:

(I) In Case 1, the condition on Γ reads as $\Gamma>p_*-1$. Since C in (1.12) blows-up as Γ decreases to p_*-1, in the running step we shall require $\Gamma\geq p_*-1+\delta$ for some fixed positive $\delta=\delta(n,\overrightarrow{p})$. The choice of such a δ is possible because $p_n<p_*$.

(II) In Case 2, we apply Proposition 1.3 for $m=2$ and $\Gamma\geq q-1$, where q is arbitrarily larger than some value $C_0>0$, say $C_0=\max\{2n+1,2(p_n-n+1)\}$. Unlike Case 1, the exact value of the threshold is not essential as long as it is big enough.

With each subsequent iteration, we would need to ensure that Proposition 1.3 can be applied to *all* norms in the right-hand side of (1.12), that is for all $i = 1, \dots, n$ we require

$$\Gamma/m + p_i - p \geq \begin{cases} p_* - 1 + \delta & \text{in Case 1,} \\ q - 1 & \text{in Case 2.} \end{cases} \tag{1.13}$$

Let $(\Gamma_k)_{k\geq 1}$ satisfy $\lim_{k\to\infty} \Gamma_k = \infty$ and $\lambda_k = 1/2 + (1/2)^{k+1}$ for $k \geq 1$. For $k \geq k_0$ large and N a fixed large positive integer independent of k, we apply Proposition 1.3 with $\Gamma = \Gamma_k$, $\lambda = \lambda_{k+N+1}$ and $\lambda' = \lambda_{k+N}$. Let ℓ denote the maximum number of iterations permitted under the restrictions in (1.13). Due to the anisotropy in (1.12), we cannot determine the precise number of iterations ℓ. However, with a careful choice of Γ_k and N, we can guarantee that $k < \ell(k) \leq k + N$. After running our iteration scheme, jointly with various estimates from above, we find that $\|\Phi_\varepsilon\|_{L^{\Gamma_k}(\mathcal{A}_r(\lambda_{k+N+1}))}$ is dominated by

$$\begin{cases} \tilde{C} r^{\frac{\sum_{j=1}^{\ell} m^j \left(p_{\tau_j} - p_*\right)}{p_* \Gamma_k}} \|\Phi_\varepsilon\|_{L^{\Gamma_{k,\tau_1\dots\tau_\ell}}(\mathcal{A}_r(\lambda_{k+N-\ell+1}))}^{\frac{m^\ell \Gamma_{k,\tau_1\dots\tau_\ell}}{\Gamma_k}} & \text{in Case 1, see (5.32),} \\ \tilde{C}^{\frac{2^\ell}{\Gamma_k}} q^{\frac{n\sum_{j=1}^{\ell} 2^j}{\Gamma_k}} r^{-\frac{\sum_{j=1}^{\ell} 2^{j-1}}{\Gamma_k}} \|\Phi_\varepsilon\|_{L^{\Gamma_{k,\tau_1\dots\tau_\ell}}(\mathcal{A}_r(\lambda_{k+N-\ell+1}))}^{\frac{2^\ell \Gamma_{k,\tau_1\dots\tau_\ell}}{\Gamma_k}} & \text{in Case 2, see (5.51).} \end{cases}$$

Here, $\tilde{C}$ denotes a positive constant depending only on n and $\overrightarrow{p}$, while $\tau_j \in \{1, 2, \dots, n\}$ is in some sense a maximizer for each $j = 1, \dots, \ell$. For example, τ_1 is the index for which the right-hand side of (1.12) reaches its maximum and Γ_{k,τ_1} is given by $\Gamma_k/m + p_{\tau_1} - p$.

From the definition of ℓ, we have $\Gamma_{k,\tau_1\dots\tau_\ell} < p_* - 1 + \delta$ in Case 1 and $\Gamma_{k,\tau_1\dots\tau_\ell} < q - 1$ in Case 2. To further bound $\|\Phi_\varepsilon\|_{L^{\Gamma_{k,\tau_1\dots\tau_\ell}}(\mathcal{A}_r(\lambda_{k+N-\ell+1}))}$ from above, we invoke interpolation inequalities, together with the local estimates in weak Lebesgue spaces found in Lemma 3.3. To conclude (1.11) in Case 1, we let $k \to \infty$. In case 2, after some further manipulations we can also pass to the limit, leading to (5.59) in which we choose $q = C_0 \max\{|\ln r|, 1\}$ to reach (1.11). This concludes the sketch of the proof of the upper bound estimates in Theorem 1.1. We mention that some calculations could be simplified to certain degree by using a renormalisation argument. However, our anisotropic Moser scheme could be applied in cases when the renormalisation argument does not work.

The Moser iteration scheme represents a milestone in the development of the regularity theory of elliptic equations (see Gilbarg–Trudinger [26] or Han–Lin [29] for more details). We mention that for anisotropic equations, other Moser-type iteration schemes, which are different from ours, have been used to derive local boundedness or gradient estimates of solutions. See, for instance, works by Fusco–Sbordone [25], Lieberman [33], or the more recent paper by Cupini–Marcellini–Mascolo [17] and the references therein.

1.2 Plan of the paper

In Section 2, we define the appropriate anisotropic Sobolev spaces for our problem. In Section 3, we construct the above-mentioned family of solutions Φ_ε to approximate problems (see Lemma 3.1) for which we also establish key estimates in weak Lebesgue spaces (see Lemma 3.3). Section 4 is dedicated to the proof of the existence of a fundamental solution of (1.1) (see Proposition 4.1). In Section 5, we prove our upper bound estimates by performing an iteration scheme as explained above. In Appendix A, we prove a weighted anisotropic Sobolev inequality, see Lemma A.1, which is invoked in the proof of the anisotropic reverse Hölder inequality of Proposition 1.3. Finally, in Appendix B, we establish another anisotropic inequality based on an inequality of Moser–Trudinger type, see Lemma B.1, which is essential in the derivation of our estimates in weak Lebesgue spaces pertaining to Case 2 of Theorem 1.1.

2 Functional spaces

In this section, we give a suitable notion of weak solution for studying (1.1). To this end, we need to introduce appropriate anisotropic spaces since the solutions of (1.1) cannot be defined in general in the usual distribution sense.

2.1 Motivation

If $p_i = 2 \leq n$ for all $i = 1, \ldots, n$, then $\Delta_{\overrightarrow{p}} u$ gives the usual Laplacian, whose fundamental solution is well-known. When all p_i are equal to $p \in (1, n]$, a non-negative solution of (1.1) can still be found explicitly. For simplicity, let $\Omega = \left\{ \sum_{i=1}^n |x_i|^{\frac{p}{p-1}} < R^{\frac{1}{n-1}} \right\}$. By direct inspection, a fundamental solution of (1.1) takes the form

$$\Phi(x) = \begin{cases} C\left(\left(\displaystyle\sum_{i=1}^n |x_i|^{\frac{p}{p-1}}\right)^{\frac{p-n}{p}} - R^{\frac{p-n}{p(n-1)}}\right) & \text{if } 1 < p < n, \\ C\left(\ln\left(R^{\frac{1}{n-1}}\right) - \ln\left(\displaystyle\sum_{i=1}^n |x_i|^{\frac{n}{n-1}}\right)\right) & \text{if } p = n \end{cases} \tag{2.1}$$

for all $x = (x_1, \ldots, x_n) \in \Omega \setminus \{0\}$ and some normalizing constant $C = C(n,p) > 0$.

If all p_i are equal to $p \in (1, 2 - 1/n]$, then the fundamental solution Φ in (2.1) does *not* belong to $W^{1,1}_{\text{loc}}(\Omega)$. Indeed, a simple calculation gives that

$$|\nabla \Phi| = C\left(\frac{n-p}{p-1}\right)\left(\sum_{i=1}^n |x_i|^{\frac{p}{p-1}}\right)^{-\frac{n}{p}}\left(\sum_{i=1}^n |x_i|^{\frac{2}{p-1}}\right)^{\frac{1}{2}},$$

which shows that $|\nabla \Phi| \in L^1_{\text{loc}}(\Omega)$ if and only if $p \in (2 - 1/n, n)$. It follows that for $p \in (1, 2 - 1/n]$, we cannot take the gradient of Φ in the anisotropic $\overrightarrow{p}$-Laplacian in the usual distribution sense. We shall tackle this difficulty as in Bénilan *et al.* [7] by introducing the space $\mathcal{T}^{1,1}_{\text{loc}}(\Omega)$, which allows us to give a sense to the gradient of u even if it is not locally integrable in general. This will be done in Section 2.2, see Definition 2.1.

2.2 Anisotropic Sobolev spaces

We assume that (1.2) holds and Ω is an open subset of $\mathbb{R}^n$ with $n \geq 2$. For any $h > 0$ and $u : \Omega \to \mathbb{R}$, let $T_h u$ denote the *truncation of* u *at height* h, namely

$$T_h u : \Omega \to \mathbb{R}, \quad T_h u\,(x) := \begin{cases} u(x) & \text{if } |u\,(x)| \leq h, \\ h\,\mathrm{sgn}(u(x)) & \text{if } |u\,(x)| > h. \end{cases} \tag{2.2}$$

Let $\mathbf{1}_E$ denote the characteristic function of a measurable set E in $\mathbb{R}^n$.

Definition 2.1. *(i) The functional space* $\mathcal{T}^{1,1}_{\mathrm{loc}}(\Omega)$ *is defined as the set of all measurable functions* $u : \Omega \to \mathbb{R}$ *whose truncated function* $T_h u$ *belongs to* $W^{1,1}_{\mathrm{loc}}(\Omega)$ *for all* $h > 0$.

(ii) For any $u \in \mathcal{T}^{1,1}_{\mathrm{loc}}(\Omega)$*, the gradient* ∇u *is defined as the* unique *measurable function* $v : \Omega \to \mathbb{R}^n$ *such that for all* $h > 0$*, we have*

$$\nabla(T_h u) = v\mathbf{1}_{\{|u|<h\}} \quad \textit{a.e. in } \Omega. \tag{2.3}$$

For the existence and uniqueness (up to a set of measure zero) of v, see [7, Lemma 2.1].

Remark 2.2. *(i) The set* $\mathcal{T}^{1,1}_{\mathrm{loc}}(\Omega)$ *is not even a vector space, although if* $u_1 \in \mathcal{T}^{1,1}_{\mathrm{loc}}(\Omega)$ *and* $u_2 \in W^{1,1}_{\mathrm{loc}}(\Omega) \cap L^\infty_{\mathrm{loc}}(\Omega)$*, then* $u_1 + u_2 \in \mathcal{T}^{1,1}_{\mathrm{loc}}(\Omega)$ *(see [7, p. 245]).*

(ii) The above definition of derivative for $u \in \mathcal{T}^{1,1}_{\mathrm{loc}}(\Omega)$ *is* not *a definition in the sense of distributions since, in general, if* $u \in \mathcal{T}^{1,1}_{\mathrm{loc}}(\Omega) \cap L^1_{\mathrm{loc}}(\Omega)$*, then* ∇u *need not belong to* $L^1_{\mathrm{loc}}(\Omega)$*. However, for* $u \in \mathcal{T}^{1,1}_{\mathrm{loc}}(\Omega)$ *we have* $\nabla u \in L^1_{\mathrm{loc}}(\Omega)$ *if and only if* $u \in W^{1,1}_{\mathrm{loc}}(\Omega)$*, and in this case,* v *in (2.3) coincides with* ∇u *in the usual weak sense, that is*

$$\partial_i(T_h u) = \mathbf{1}_{\{|u|<h\}}\partial_i u \;\; \textit{a.e. in } \Omega \;\; \textit{for every } h > 0 \;\; \textit{and all } i = 1, \ldots, n.$$

Given the definition of $\mathcal{T}^{1,1}_{\mathrm{loc}}(\Omega)$, we now introduce a concept of weak solution of (1.1).

Definition 2.3. *We say a function* $u : \Omega \to \mathbb{R}$ *is a* weak solution *of* (1.1) *if* $u \in \mathcal{T}^{1,1}_{\mathrm{loc}}(\Omega)$ *and* $|\partial_i u|^{p_i-1} \in L^1_{\mathrm{loc}}(\Omega)$ *for all* $i = 1, \ldots, n$ *such that*

$$\sum_{i=1}^{n} \int_\Omega \partial_i u\,|\partial_i u|^{p_i-2}\,\partial_i\varphi\,dx = \varphi\,(0) \quad \textit{for all } \varphi \in C^\infty_c(\Omega)\,. \tag{2.4}$$

We next introduce other functional spaces needed in our paper:

(i) Let $\mathcal{T}^{1,\overrightarrow{p}}_{\mathrm{loc}}(\Omega)$ be the set of all $u \in \mathcal{T}^{1,1}_{\mathrm{loc}}(\Omega)$ such that $\partial_i(T_h u) \in L^{p_i}_{\mathrm{loc}}(\Omega)$ for every $h > 0$ and all $i = 1, \ldots, n$. Notice that $W^{1,\overrightarrow{p}}_{\mathrm{loc}}(\Omega) \subset \mathcal{T}^{1,\overrightarrow{p}}_{\mathrm{loc}}(\Omega)$ and

$$\mathcal{T}^{1,\overrightarrow{p}}_{\mathrm{loc}}(\Omega) \cap L^\infty_{\mathrm{loc}}(\Omega) = W^{1,\overrightarrow{p}}_{\mathrm{loc}}(\Omega) \cap L^\infty_{\mathrm{loc}}(\Omega).$$

(ii) The set $\mathcal{T}^{1,\overrightarrow{p}}(\Omega)$ consists of all functions $u \in \mathcal{T}^{1,1}_{\mathrm{loc}}(\Omega)$ with the property that $\partial_i(T_h u) \in L^{p_i}(\Omega)$ for every $h > 0$ and all $i = 1, \ldots, n$.

(iii) We define $\mathcal{T}_0^{1,\overrightarrow{p}}(\Omega)$ as the set of functions $u \in \mathcal{T}^{1,\overrightarrow{p}}(\Omega)$ such that for any $h > 0$, there exists a sequence $(\varphi_k)_{k\in\mathbb{N}}$ in $C_c^\infty(\Omega)$ such that as $k \to \infty$, it holds

$$\begin{cases} \varphi_k \to T_h u & \text{in } L^1_{\text{loc}}(\Omega), \\ \partial_i \varphi_k \to \partial_i(T_h u) & \text{in } L^{p_i}(\Omega) \text{ for all } i = 1, \ldots, n. \end{cases} \tag{2.5}$$

Note that, similar to Bénilan *et al.* [7], in the definition of $\mathcal{T}^{1,\overrightarrow{p}}(\Omega)$ and $\mathcal{T}_0^{1,\overrightarrow{p}}(\Omega)$ we do not require $T_h u$ to belong to any $L^q(\Omega)$ with $q \geq 1$. This condition is obviously satisfied when Ω is bounded, but it makes a real difference when Ω is unbounded.

As in Appendix II of [7], it can be proved that the above definition of $u \in \mathcal{T}_0^{1,\overrightarrow{p}}(\Omega)$ is equivalent to the following: The function $u \in \mathcal{T}^{1,\overrightarrow{p}}(\Omega)$ and there exists a sequence $(\zeta_k)_k$ in $C_c^\infty(\Omega)$ such that as $k \to \infty$, we have

$$\begin{cases} \zeta_k \to u \ \text{ a.e. in } \Omega; \\ \partial_i(T_h(\zeta_k)) \to \partial_i(T_h u) \ \text{ in } L^{p_i}(\Omega) \text{ for every } h > 0 \text{ and all } i = 1, \ldots, n. \end{cases} \tag{2.6}$$

Note that the main difference between (2.5) and (2.6) is that in the former case the sequence $(\varphi_k)_k$ depends on h.

(iv) Let $W^{1,\overrightarrow{p}}(\Omega)$ denote the set of all functions $u \in L^p(\Omega)$ whose weak partial derivative $\partial_i u$ exists and belongs to $L^{p_i}(\Omega)$ for each $i = 1, \ldots, n$. The anisotropic Sobolev space $W^{1,\overrightarrow{p}}(\Omega)$ is a reflexive and separable Banach space equipped with the norm

$$\|u\|_{W^{1,\overrightarrow{p}}(\Omega)} := \|u\|_{L^p(\Omega)} + \sum_{i=1}^n \|\partial_i u\|_{L^{p_i}(\Omega)}.$$

We write $u_n \to u$ in $W^{1,\overrightarrow{p}}_{\text{loc}}(\Omega)$ to mean $u_n \to u$ in $W^{1,\overrightarrow{p}}(\omega)$ for each $\omega \Subset \Omega$. By $\omega \Subset \Omega$, we mean that ω is an open set such that $\overline{\omega}$ is compact and $\overline{\omega} \subset \Omega$.

(v) The set $W_0^{1,\overrightarrow{p}}(\Omega)$ is defined as the closure of $C_c^\infty(\Omega)$ with respect to $\|\cdot\|_{W^{1,\overrightarrow{p}}(\Omega)}$. Hence, we have the inclusion $W_0^{1,\overrightarrow{p}}(\Omega) \subset \mathcal{T}_0^{1,\overrightarrow{p}}(\Omega)$. If $T_h u \in W_0^{1,\overrightarrow{p}}(\Omega)$ for every $h > 0$, then $u \in \mathcal{T}_0^{1,\overrightarrow{p}}(\Omega)$; The converse is also true provided that Ω is bounded. It can be proved that the definition of $u \in W_0^{1,\overrightarrow{p}}(\Omega)$ is equivalent to $u \in \mathcal{T}_0^{1,\overrightarrow{p}}(\Omega)$ and $\partial_i u \in L^{p_i}(\Omega)$ for all $i = 1, \ldots, n$ when Ω is *bounded*.

We recall the following Sobolev inequality due to Troisi (see Theorem 1.2 in [43]):

Lemma 2.4. *Let p be given by* (1.3). *Then there exists a constant $c > 0$ such that*

$$\|u\|_{L^q(\Omega)} \leq c \prod_{i=1}^n \|\partial_i u\|^{1/n}_{L^{p_i}(\Omega)} \quad \textit{for all } u \in C_c^\infty(\Omega), \tag{2.7}$$

where $c = c(n, \overrightarrow{p})$ and $q = np/(n-p)$ if $p < n$, while $c = c(n, \overrightarrow{p}, q, \operatorname{meas}(\operatorname{supp} u))$ and q is any positive number if $p \geq n$.

Remark 2.5. *If $p < n$, then* (2.7) *holds for any $u \in W_0^{1,\overrightarrow{p}}(\Omega)$ by a density argument.*

Remark 2.6. *If $p < n$, then $\mathcal{T}_0^{1,\overrightarrow{p}}(\Omega) \subset L_0(\Omega)$, where $L_0(\Omega)$ denotes the set of measurable functions $u : \Omega \to \mathbb{R}$ such that the set $\{|u| > \varepsilon\}$ has finite measure for every $\varepsilon > 0$. Indeed, $\partial_i T_h u \in L^{p_i}(\Omega)$ for every $h > 0$ so that $T_h u \in L^{np/(n-p)}(\Omega)$ (by a density argument using (2.7)). Thus, $T_h u \in L_0(\Omega)$ for every $h > 0$.*

From our definition of $W^{1,\overrightarrow{p}}(\Omega)$ and $W_0^{1,\overrightarrow{p}}(\Omega)$, we recover the usual Sobolev space $W^{1,p}(\Omega)$ and $W_0^{1,p}(\Omega)$, respectively when $p_i = p$ for all $i = 1, \ldots, n$. On the other hand, if Ω is *bounded* and $1 < p < \infty$, we infer that (2.7) is true with $q = p$ and any $u \in W_0^{1,\overrightarrow{p}}(\Omega)$ so that an equivalent norm on $W_0^{1,\overrightarrow{p}}(\Omega)$ can be taken as $|||u||| = \sum_{i=1}^n \|\partial_i u\|_{L^{p_i}(\Omega)}$.

We mention in passing that there are other versions of anisotropic spaces, which may not coincide with the ones introduced here (see, for example, Nikol'skiĭ [38], Rákosník [39] and Troisi [43]).

3 Auxiliary tools

In Lemma 3.1 we establish the existence of a family of approximate solutions for which later in Lemma 3.3 we give crucial uniform estimates in weak Lebesgue spaces. These estimates in Case 2 ($p = n$ and Ω is bounded) are new and different from those in Case 1 ($p < n$). In the former case, they are obtained via anisotropic Sobolev inequalities of Moser–Trudinger type (see Lemma B.1 in Appendix B). For the reader's convenience, in Section 3.2 we introduce the *weak* Lebesgue spaces and their properties.

3.1 Approximate solutions

Let Ω be a domain of $\mathbb{R}^n$ such that $0 \in \Omega$. In this section, we are in either Case 1 (see (1.4)) or Case 2 ($p = n$ and Ω is bounded). To prove the existence of weak solutions of (1.1), we consider suitable approximate problems for which existence results can be obtained easily.

We use $B_{1/\varepsilon}(0)$ to denote the ball of center 0 and radius $1/\varepsilon$. Since Ω may be unbounded (when $p < n$), we approximate Ω using a sequence of *bounded domains* Ω_ε with $\Omega_\varepsilon \to \Omega$ as $\varepsilon \searrow 0$. We fix $\varepsilon_0 \in (0,1)$. For $\varepsilon \in (0, \varepsilon_0]$, we define

$$\Omega_\varepsilon := \begin{cases} \Omega \cap B_{1/\varepsilon}(0) & \text{if } p < n, \\ \Omega & \text{if } p = n. \end{cases}$$

Moreover, for every $\varepsilon \in (0, \varepsilon_0]$, we construct a function f_ε with the following properties:

$$\begin{cases} f_\varepsilon \in C_c^\infty(B_\varepsilon(0)),\ f_\varepsilon \geq 0, \text{ and } \|f_\varepsilon\|_{L^1(\Omega_\varepsilon)} \leq 1, \\ f_\varepsilon \overset{\star}{\rightharpoonup} \delta_0 \text{ in the sense of measures as } \varepsilon \to 0. \end{cases} \tag{3.1}$$

Let $f_1 \in C_c^\infty(B_1(0))$ be such that $f_1 \geq 0$ and $\int_{B_1(0)} f_1(x)\,dx = 1$. For any $\varepsilon \in (0, \varepsilon_0]$, we set $f_\varepsilon(x) := \varepsilon^{-n} f_1(\varepsilon^{-1}x)$ for all $x \in B_\varepsilon(0)$ and $f_\varepsilon = 0$ on $\mathbb{R}^n \setminus B_\varepsilon(0)$. Hence, f_ε satisfies (3.1).

Lemma 3.1. *Let $\varepsilon \in (0, \varepsilon_0]$ be arbitrary. Then the problem*

$$\begin{cases} -\Delta_{\overrightarrow{p}} \Phi_\varepsilon = f_\varepsilon & \text{in } \Omega_\varepsilon, \\ \Phi_\varepsilon = 0 & \text{on } \partial\Omega_\varepsilon \end{cases} \tag{3.2}$$

admits a non-negative weak solution $\Phi_\varepsilon \in W_0^{1,\overrightarrow{p}}(\Omega_\varepsilon) \cap L^\infty_{\text{loc}}(\Omega_\varepsilon)$ in the sense that

$$\sum_{i=1}^n \int_{\Omega_\varepsilon} \partial_i \Phi_\varepsilon |\partial_i \Phi_\varepsilon|^{p_i-2} \partial_i \varphi\,dx = \int_{\Omega_\varepsilon} f_\varepsilon\,\varphi\,dx \quad \text{for all } \varphi \in W_0^{1,\overrightarrow{p}}(\Omega_\varepsilon)\,. \tag{3.3}$$

Furthermore, for every $h > 0$, the solution Φ_ε satisfies

$$\sum_{i=1}^n \int_{\{\Phi_\varepsilon \leq h\}} |\partial_i \Phi_\varepsilon|^{p_i}\,dx \leq h. \tag{3.4}$$

Proof. For $u \in W_0^{1,\overrightarrow{p}}(\Omega_\varepsilon)$ fixed, we define $A_\varepsilon u : W_0^{1,\overrightarrow{p}}(\Omega_\varepsilon) \to \mathbb{R}$ by

$$\langle A_\varepsilon u, v\rangle := \sum_{i=1}^n \int_{\Omega_\varepsilon} \partial_i u |\partial_i u|^{p_i-2} \partial_i v\,dx \quad \text{for every } v \in W_0^{1,\overrightarrow{p}}(\Omega_\varepsilon)\,.$$

Let p_i' denote the Hölder conjugate of p_i, that is $p_i' = p_i/(p_i - 1)$ for $i = 1, \ldots, n$. Clearly, $A_\varepsilon u$ belongs to the dual of $W_0^{1,\overrightarrow{p}}(\Omega_\varepsilon)$, denoted by $W^{-1,\overrightarrow{p'}}(\Omega_\varepsilon)$ with $\overrightarrow{p'} := (p_1', \ldots, p_n')$. One can easily check that the operator $A_\varepsilon : W_0^{1,\overrightarrow{p}}(\Omega_\varepsilon) \to W^{-1,\overrightarrow{p'}}(\Omega_\varepsilon)$ is *bounded*, *monotone*, *coercive* and *hemicontinuous* (see Bendahmane–Karlsen [5] for more details). Then, A_ε is a surjective operator (see Lions [34, Chapter 2, Theorem 2.1]). Let $B_\varepsilon \varphi$ denote the right-hand side of (3.3). Since $B_\varepsilon \in W^{-1,\overrightarrow{p'}}(\Omega_\varepsilon)$, the surjectivity of A_ε proves the existence of $\Phi_\varepsilon \in W_0^{1,\overrightarrow{p}}(\Omega_\varepsilon)$ such that $A_\varepsilon \Phi_\varepsilon = B_\varepsilon$, that is (3.3) holds. Moreover, by Fusco–Sbordone [25, Remark 3.5], we obtain that $\Phi_\varepsilon \in L^\infty_{\text{loc}}(\Omega_\varepsilon)$.

To prove that $\Phi_\varepsilon \geq 0$ a.e. in Ω_ε, we denote $N_\varepsilon := \{x \in \Omega_\varepsilon : \Phi_\varepsilon(x) < 0\}$. If $\operatorname{meas}(N_\varepsilon) \neq 0$, then by using $\Phi_\varepsilon \mathbf{1}_{N_\varepsilon}$ as a test function in (3.3), we find that

$$\sum_{i=1}^n \int_{N_\varepsilon} |\partial_i \Phi_\varepsilon|^{p_i}\,dx = \int_{N_\varepsilon} f_\varepsilon \Phi_\varepsilon\,dx \leq 0\,, \tag{3.5}$$

which implies that $\Phi_\varepsilon = 0$ a.e. in N_ε. This contradiction shows that $\Phi_\varepsilon \geq 0$ a.e. in Ω_ε.

For any $h > 0$, we have $T_h \Phi_\varepsilon = \min\{\Phi_\varepsilon, h\} \in W_0^{1,\overrightarrow{p}}(\Omega_\varepsilon)$. By using the truncated function $T_h \Phi_\varepsilon$ as a test function in (3.3), we obtain that

$$\begin{aligned} \sum_{i=1}^n \int_{\{\Phi_\varepsilon \leq h\}} |\partial_i \Phi_\varepsilon|^{p_i}\,dx &= \sum_{i=1}^n \int_{\Omega_\varepsilon} \partial_i \Phi_\varepsilon\, |\partial_i \Phi_\varepsilon|^{p_i-2}\, \partial_i(T_h \Phi_\varepsilon)\,dx \\ &= \int_{\Omega_\varepsilon} f_\varepsilon\, T_h \Phi_\varepsilon\,dx \leq h \int_{\Omega_\varepsilon} f_\varepsilon\,dx \leq h\,, \end{aligned} \tag{3.6}$$

which proves (3.4). This ends the proof of Lemma 3.1. □

3.2 Weak Lebesgue spaces

The weak $L^\infty(\Omega)$ space is by definition the usual $L^\infty(\Omega)$ space. Therefore, unless otherwise stated, we assume throughout this subsection that $0 < q < \infty$. Recall that $L^q(\Omega)$ denotes the set of all real-valued measurable functions u on Ω such that $|u|^q$ is integrable. The quasi-norm of such a function $u \in L^q(\Omega)$ is defined by

$$\|u\|_{L^q(\Omega)} := \left(\int_\Omega |u(x)|^q\,dx\right)^{\frac{1}{q}}.$$

Whenever $1 \le q < \infty$, the Minkowski inequality holds:

$$\|f+g\|_{L^q(\Omega)} \le \|f\|_{L^q(\Omega)} + \|g\|_{L^q(\Omega)} \quad \text{for all } f, g \in L^q(\Omega), \tag{3.7}$$

whereas for $0 < q < 1$, the inequality (3.7) is reversed when $f, g \ge 0$. However, for the case $0 < q < 1$, the inequality (3.7) is replaced by the following

$$\|f+g\|_{L^q(\Omega)} \le 2^{\frac{1-q}{q}}\left(\|f\|_{L^q(\Omega)} + \|g\|_{L^q(\Omega)}\right) \quad \text{for all } f, g \in L^q(\Omega).$$

Note that $L^q(\Omega)$ are Banach spaces for $q \ge 1$ and quasi-Banach spaces for $q \in (0,1)$.

For a measurable function $u : \Omega \to \mathbb{R}$, the *distribution function* of u is the function d_u defined on $[0,\infty)$ as follows

$$d_u(h) := \text{meas}(\{x \in \Omega : \quad |u(x)| > h\}). \tag{3.8}$$

The distribution function $h \longmapsto d_u(h)$ is a decreasing function.

Definition 3.2. *For $0 < q < \infty$, we define the weak $L^q(\Omega)$ space, denoted by $L^{q,\infty}(\Omega)$, as the set of all measurable functions $u : \Omega \to \mathbb{R}$ such that*

$$\|u\|_{L^{q,\infty}(\Omega)} = \sup_{h>0}\left\{h\, d_u(h)^{\frac{1}{q}}\right\} < \infty. \tag{3.9}$$

As an analogue of (3.7), for any $0 < q < \infty$ we have

$$\|f+g\|_{L^{q,\infty}(\Omega)} \le \max\{2, 2^{1/q}\}\left(\|f\|_{L^{q,\infty}(\Omega)} + \|g\|_{L^{q,\infty}(\Omega)}\right) \quad \text{for all } f, g \in L^{q,\infty}(\Omega).$$

Notice that $\|\cdot\|_{L^{q,\infty}(\Omega)}$ does not define a norm for $q \in (0,\infty)$, but a quasi-norm in $L^{q,\infty}(\Omega)$. It can be shown that $L^{q,\infty}(\Omega)$ is a complete quasi-normed space for $q \in (0,\infty)$.

The weak $L^q(\Omega)$ spaces are larger than the usual $L^q(\Omega)$ spaces, that is

$$L^q(\Omega) \subseteq L^{q,\infty}(\Omega) \quad \text{for } 0 < q < \infty. \tag{3.10}$$

Indeed, we have $\|u\|_{L^{q,\infty}(\Omega)} \le \|u\|_{L^q(\Omega)}$ for any $u \in L^q(\Omega)$ by Chebyshev's inequality:

$$h^q\, d_u(h) \le \int_{\{x\in\Omega:\ |u(x)|>h\}} |u(x)|^q\,dx.$$

Furthermore, the inclusion in (3.10) is strict. Indeed, for any $q \in (0, \infty)$, the function $u(x) = |x|^{-n/q}$ is not in $L^q(\mathbb{R}^n)$, whereas u belongs to $L^{q,\infty}(\mathbb{R}^n)$ with $\|u\|_{L^{q,\infty}(\mathbb{R}^n)}$ being the measure of the unit ball in $\mathbb{R}^n$ to the $(1/q)$th power.

If Ω is bounded, then for $0 < q_0 < q < \infty$, we have $L^{q,\infty}(\Omega) \subseteq L^{q_0}(\Omega)$ and

$$\|u\|_{L^{q_0}(\Omega)} \leq \left(\frac{q}{q - q_0}\right)^{\frac{1}{q_0}} (\text{meas}\,(\Omega))^{\frac{1}{q_0} - \frac{1}{q}} \|u\|_{L^{q,\infty}(\Omega)} \quad \text{for all } u \in L^{q,\infty}(\Omega). \tag{3.11}$$

The inequality in (3.11) will be applied several times in the paper. For more details on weak Lebesgue spaces, we refer to Grafakos [27] .

3.3 Key estimates in weak Lebesgue spaces

When $p < n$, Bendahmane–Karlsen [5] proved (3.12) under more general structure conditions on the anisotropic operator, but with a constant depending on the domain. In our framework, we prove that the constant in (3.12) depends only on n and $\overrightarrow{p}$. Furthermore, the case $p = n$ included in Lemma 3.3 is new compared with [5].

Lemma 3.3. *For any $\varepsilon \in (0, \varepsilon_0]$, let Φ_ε be the weak solution of* (3.2).

(1) If (1.4) *holds, then there exists a positive constant $C = C\left(n, \overrightarrow{p}\right)$ such that*

$$\sum_{i=1}^{n} \|\partial_i \Phi_\varepsilon\|_{L^{\frac{p_i(p_*-1)}{p_*},\infty}(\Omega_\varepsilon)} + \|\Phi_\varepsilon\|_{L^{p_*-1,\infty}(\Omega_\varepsilon)} \leq C. \tag{3.12}$$

(2) If $p = n$ and Ω is bounded, then there exists a positive constant $C = C\left(n, \overrightarrow{p}\right)$ such that for all $q \geq 1$, we have

$$\begin{cases} \|\Phi_\varepsilon\|_{L^{q,\infty}(\Omega)} \leq \max\left\{Cq, (\text{meas}\,(\Omega))^{1+\frac{1}{q}}\right\}, \\ \displaystyle\sum_{i=1}^{n} \|\partial_i \Phi_\varepsilon\|^{\frac{p_i q}{q+1}}_{L^{\frac{p_i q}{q+1},\infty}(\Omega)} \leq (Cq)^q + n\,(\text{meas}\,(\Omega))^{q+1}. \end{cases} \tag{3.13}$$

Proof. (1) We proceed using essentially the same ideas as in [5]. Let $p^* := np/(n-p)$.

Since $T_h\Phi_\varepsilon \in W_0^{1,\overrightarrow{p}}(\Omega_\varepsilon)$ and $p < n$, by Remark 2.5 and (3.6), there exists a constant $C = C\left(n, \overrightarrow{p}\right) > 0$ such that

$$\|T_h\Phi_\varepsilon\|_{L^{p^*}(\Omega_\varepsilon)} \leq C \prod_{i=1}^{n} \|\partial_i(T_h\Phi_\varepsilon)\|^{1/n}_{L^{p_i}(\Omega_\varepsilon)} \leq C h^{1/p} \quad \text{for every } h > 0. \tag{3.14}$$

We conclude (3.12) by showing that

$$\|\Phi_\varepsilon\|_{L^{p_*-1,\infty}(\Omega_\varepsilon)} \leq C^{\frac{p^*}{p_*-1}} \ \text{ and } \ \|\partial_i\Phi_\varepsilon\|_{L^{\frac{p_i(p_*-1)}{p_*},\infty}(\Omega_\varepsilon)} \leq (C^{p^*}+1)^{\frac{p_*}{p_i(p_*-1)}} \tag{3.15}$$

for each $i = 1, \dots, n$. From (2.2), (3.8), and (3.14), we obtain that

$$h\, d_{\Phi_\varepsilon}(h)^{\frac{1}{p^*}} = \left(\int_{\{\Phi_\varepsilon > h\}} (T_h \Phi_\varepsilon)^{p^*}\, dx\right)^{\frac{1}{p^*}} \le \|T_h\Phi_\varepsilon\|_{L^{p^*}(\Omega_\varepsilon)} \le C h^{\frac{1}{p}} \quad \text{for all } h > 0. \tag{3.16}$$

The first inequality in (3.15) follows from (3.16) since

$$d_{\Phi_\varepsilon}(h) \le C^{p^*} h^{1-p_*} \quad \text{for every } h > 0. \tag{3.17}$$

Moreover, using (3.6) and (3.17), for each $i = 1, \dots, n$ and $h > 0$, we find that

$$\begin{aligned} d_{\partial_i \Phi_\varepsilon}(h) &\le d_{\Phi_\varepsilon}\left(h^{p_i/p_*}\right) + \text{meas}\left(\left\{x \in \Omega_\varepsilon :\ \Phi_\varepsilon(x) \le h^{p_i/p_*} \text{ and } |\partial_i \Phi_\varepsilon(x)| > h\right\}\right) \\ &\le \left(C^{p^*} + 1\right) h^{-p_i(p_*-1)/p_*}. \end{aligned}$$

This implies the second inequality in (3.15). Hence, the assertion of (a) holds.

(2) By Lemma B.1 in Appendix B, there exists a constant $c = c(n, \overrightarrow{p}) > 0$ such that for all $q \ge 1$ and $h > 0$, we have

$$\|T_h\Phi_\varepsilon\|^n_{L^{\frac{(q+1)n}{n-1}}(\Omega)} \le c(q+1)^{n-1}\left(\text{meas}(\Omega) + \sum_{i=1}^n \int_{\{\Phi_\varepsilon \le h\}} |\partial_i \Phi_\varepsilon|^{p_i}\, dx\right)^{\frac{q+n}{q+1}}. \tag{3.18}$$

Since $T_h\Phi_\varepsilon = \min\{\Phi_\varepsilon, h\}$, by using (3.8) we see that

$$\|T_h\Phi_\varepsilon\|^n_{L^{\frac{(q+1)n}{n-1}}(\Omega)} \ge \left(\int_{\{\Phi_\varepsilon > h\}} h^{\frac{n(q+1)}{n-1}}\, dx\right)^{\frac{n-1}{q+1}} = h^n d_{\Phi_\varepsilon}(h)^{\frac{n-1}{q+1}} \quad \text{for all } h > 0. \tag{3.19}$$

From (3.4), (3.18), and (3.19), we infer that

$$h^n d_{\Phi_\varepsilon}(h)^{\frac{n-1}{q+1}} \le c(q+1)^{n-1}(\text{meas}(\Omega) + h)^{\frac{q+n}{q+1}} \quad \text{for every } h > 0.$$

Hence, for all $h > 0$, we have

$$\begin{aligned} h\, d_{\Phi_\varepsilon}(h)^{\frac{1}{q}} &\le c^{\frac{q+1}{q(n-1)}}(q+1)^{\frac{q+1}{q}}\left(h^{-1}\text{meas}(\Omega) + 1\right)^{\frac{q+n}{q(n-1)}} \\ &\le (c+1)^{\frac{2}{n-1}} 6q \left(h^{-1}\text{meas}(\Omega) + 1\right)^{\frac{n+1}{n-1}}. \end{aligned} \tag{3.20}$$

If $0 < h < \text{meas}(\Omega)$, then $h\, d_{\Phi_\varepsilon}(h)^{\frac{1}{q}} \le (\text{meas}(\Omega))^{\frac{1}{q}+1}$, whereas for $h \ge \text{meas}(\Omega)$ we obtain from (3.20) that $h\, d_{\Phi_\varepsilon}(h)^{\frac{1}{q}} \le Cq$, where $C = C(n, \overrightarrow{p}) > 1$. It follows that

$$\|\Phi_\varepsilon\|_{L^{q,\infty}(\Omega)} = \sup_{h>0}\left\{h\, d_{\Phi_\varepsilon}(h)^{\frac{1}{q}}\right\} \le \max\left\{Cq, (\text{meas}(\Omega))^{1+\frac{1}{q}}\right\}.$$

This proves the first inequality in (3.13). Using (3.4) and (3.8), for any $h > 0$, we find that

$$\begin{aligned} d_{\partial_i\Phi_\varepsilon}(h) &= \text{meas}(\{x \in \Omega :\ |\partial_i\Phi_\varepsilon(x)| > h\}) \\ &\le \text{meas}\left(\left\{x \in \Omega :\ \Phi_\varepsilon(x) \le h^{\frac{p_i}{q+1}} \text{ and } |\partial_i\Phi_\varepsilon(x)| > h\right\}\right) + d_{\Phi_\varepsilon}\left(h^{\frac{p_i}{q+1}}\right) \\ &\le h^{-p_i}\int_{\left\{\Phi_\varepsilon \le h^{\frac{p_i}{q+1}}\right\}} |\partial_i\Phi_\varepsilon|^{p_i}\, dx + d_{\Phi_\varepsilon}\left(h^{\frac{p_i}{q+1}}\right) \le h^{-\frac{p_i q}{q+1}}\left(1 + \|\Phi_\varepsilon\|^q_{L^{q,\infty}(\Omega)}\right). \end{aligned}$$

Hence, for every $i = 1, \ldots, n$, we obtain that

$$\|\partial_i \Phi_\varepsilon\|^{\frac{p_i q}{q+1}}_{L^{\frac{p_i q}{q+1},\infty}(\Omega)} = \sup_{h>0} \left\{ h^{\frac{p_i q}{q+1}} d_{\partial_i \Phi_\varepsilon}(h) \right\} \leq 1 + \max \left\{ (Cq)^q, (\text{meas}\,(\Omega))^{q+1} \right\}. \quad (3.21)$$

The second inequality in (3.13) follows from (3.21). This completes the proof. □

Remark 3.4. *In both cases of Lemma 3.3, we find that*

$$\lim_{h\to\infty} d_{\Phi_\varepsilon}(h) = 0 \quad \textit{and} \quad \lim_{h\to\infty} d_{\partial_i \Phi_\varepsilon}(h) = 0 \quad \textit{for every } i = 1, \ldots, n, \quad (3.22)$$

where all limits hold uniformly with respect to $\varepsilon \in (0, \varepsilon_0]$.

4 Existence of a fundamental solution

In this section, we adapt ideas of Bénilan *et al.* [7] to show that, under the assumptions of Theorem 1.1, problem (1.1) admits a fundamental solution $\Phi \in \mathcal{T}_0^{1,\overrightarrow{p}}(\Omega)$. This is obtained using an approximation procedure as in Boccardo–Gallöuet [8, 9] and Dall'Aglio [19]. For more general existence and regularity results for nonlinear measure data problems, see Mingione [35].

Proposition 4.1. *Suppose that either Case 1 or Case 2 in Theorem 1.1 holds. Then* (1.1) *has a non-negative weak solution* $\Phi \in \mathcal{T}_0^{1,\overrightarrow{p}}(\Omega)$.

We later prove that Φ in Proposition 4.1 belongs to $W^{1,\infty}_{\text{loc}}(\Omega \setminus \{0\})$ (see Remark 5.2).

Proof. Let $(\varepsilon_k)_{k\in\mathbb{N}}$ be a sequence in $(0, \varepsilon_0]$ such that $\varepsilon_k \searrow 0$ as $k \to \infty$. For any $k \in \mathbb{N}$, let Φ_{ε_k} be as in Lemma 3.1. We define $\Phi_{\varepsilon_k} = 0$ on $\mathbb{R}^n \setminus \Omega_{\varepsilon_k}$. For every $h > 0$ and $i = 1, \ldots, n$, we have the following facts about $(T_h \Phi_{\varepsilon_k})_k$:

(a) From (2.2) and $\Phi_{\varepsilon_k} \in W_0^{1,\overrightarrow{p}}(\Omega_{\varepsilon_k})$, we have $T_h \Phi_{\varepsilon_k} \in W_0^{1,\overrightarrow{p}}(\Omega_{\varepsilon_k})$ for all $k \in \mathbb{N}$ and

$$\partial_i T_h \Phi_{\varepsilon_k} = \mathbf{1}_{\{\Phi_{\varepsilon_k} < h\}} \partial_i \Phi_{\varepsilon_k} \text{ a.e. in } \Omega. \quad (4.1)$$

(b) For any $1 \leq q \leq \infty$, the family $(T_h \Phi_{\varepsilon_k})_k$ is uniformly bounded in $L^q_{\text{loc}}(\mathbb{R}^n)$.

(c) The family $(\partial_i T_h \Phi_{\varepsilon_k})_k$ is uniformly bounded in $L^{p_i}(\mathbb{R}^n)$ (see (3.4)).

(d) Up to a subsequence, $T_h \Phi_{\varepsilon_k}$ converges in $L^q_{\text{loc}}(\mathbb{R}^n)$ for all $q \in [1, \infty)$.

We need only prove (d). Fix $R > 0$. From (c), we see that $(T_h \Phi_{\varepsilon_k})_k$ is uniformly bounded in $W^{1,p_1}(B_R(0))$. So, up to a subsequence, $(T_h \Phi_{\varepsilon_k})_k$ converges in $L^q(B_R(0))$ for every $q \in [1, p_1]$ due to the compactness of the embedding $W^{1,p_1}(B_R(0)) \Subset L^{p_1}(B_R(0))$. Hence, using (b) and an interpolation between $L^1(B_R(0))$ and $L^\infty(B_R(0))$, we get that, up to a subsequence, $(T_h \Phi_{\varepsilon_k})_k$ converges in $L^q(B_R(0))$ for all $q \in [1, \infty)$. Since $R > 0$ is arbitrary, by a diagonal argument, we conclude (d).

In the framework of Proposition 4.1, we establish the following auxiliary result.

Lemma 4.2. *Fix K an arbitrary compact subset of Ω. Let $\varepsilon_0 > 0$ be small such that $K \subset \Omega_{\varepsilon_k}$ for every $k \in \mathbb{N}$. Then, for every $i = 1, \dots, n$, we have:*

(i) The family $\left\{|\partial_i \Phi_{\varepsilon_k}|^{p_i-2}\, \partial_i \Phi_{\varepsilon_k}\right\}_k$ is uniformly bounded in $L^1(K)$ and

$$\lim_{h\to\infty} \sup_{k\in\mathbb{N}} \int_{\left\{x\in\Omega_{\varepsilon_k}:\ |\partial_i\Phi_{\varepsilon_k}|>h\right\}} |\partial_i \Phi_{\varepsilon_k}|^{p_i-1}\, dx = 0. \tag{4.2}$$

(ii) There exists a measurable function $\Phi : \Omega \to \mathbb{R}$ such that, up to a subsequence,

$$\Phi_{\varepsilon_k} \to \Phi \text{ locally in measure and a.e. in } \Omega \text{ as } k \to \infty. \tag{4.3}$$

(iii) Moreover, $\Phi \in \mathcal{T}^{1,1}_{\mathrm{loc}}(\Omega)$ and, up to a subsequence, $\partial_i \Phi_{\varepsilon_k}$ satisfies

$$\partial_i \Phi_{\varepsilon_k} \to \partial_i \Phi \text{ locally in measure and a.e. in } \Omega \text{ as } k \to \infty. \tag{4.4}$$

Remark 4.3. *Let $h > 0$ be arbitrary. From (d) and (4.3), we get that, up to a subsequence,*

$$T_h \Phi_{\varepsilon_k} \to T_h \Phi \quad \text{in } L^q_{\mathrm{loc}}(\mathbb{R}^n) \text{ and a.e. in } \Omega \text{ for every } q \in [1,\infty). \tag{4.5}$$

We postpone the proof of Lemma 4.2 to complete the proof of Proposition 4.1.

Proof of Proposition 4.1 concluded. We first show that Φ is a weak solution of (1.1) (see Definition 2.3). Let $\varphi \in C^\infty_c(\Omega)$ be arbitrary. Then $\operatorname{supp}\varphi \subset \Omega_{\varepsilon_{k_0}}$ if $k_0 \in \mathbb{N}$ is large enough. Since Φ_{ε_k} is a weak solution of (3.2) with $\varepsilon = \varepsilon_k$ and $\Omega = \Omega_{\varepsilon_k}$, we have

$$\sum_{i=1}^n \int_{\Omega_{\varepsilon_k}} \partial_i \Phi_{\varepsilon_k} |\partial_i \Phi_{\varepsilon_k}|^{p_i-2} \partial_i \varphi\, dx = \int_{\Omega_\varepsilon} f_{\varepsilon_k}\, \varphi\, dx \quad \text{for every } k \geq k_0. \tag{4.6}$$

From Lemma 4.2(i), the family $\left\{|\partial_i \Phi_{\varepsilon_k}|^{p_i-2}\, \partial_i \Phi_{\varepsilon_k}\right\}_k$ is uniformly integrable in $L^1(K)$. In view of Lemma 4.2(iii), by Vitali's convergence theorem (see Brezis [13, p. 122]), we conclude that $|\partial_i \Phi|^{p_i-1} \in L^1_{\mathrm{loc}}(\Omega)$ and, up to a subsequence of ε_k, relabelled ε_k, we have

$$|\partial_i \Phi_{\varepsilon_k}|^{p_i-2}\, \partial_i \Phi_{\varepsilon_k} \to |\partial_i \Phi|^{p_i-2}\, \partial_i \Phi \ \text{ in } \ L^1_{\mathrm{loc}}(\Omega) \ \text{ as } \ k \to \infty. \tag{4.7}$$

Using (4.7) and (3.1), we can pass to the limit in (4.6) to conclude that

$$\sum_{i=1}^n \int_\Omega \partial_i \Phi |\partial_i \Phi|^{p_i-2} \partial_i \varphi\, dx = \varphi(0) \quad \text{for every } \varphi \in C^\infty_c(\Omega).$$

Hence, Φ is a weak solution of (1.1). We next show that $\Phi \in \mathcal{T}^{1,\vec{p}}_0(\Omega)$. Since $T_h \Phi_{\varepsilon_k} \in W^{1,\vec{p}}_0(\Omega_{\varepsilon_k})$ for any $h > 0$ and $k \in \mathbb{N}$, we find that there exists $\varphi_{h,k} \in C^\infty_c(\Omega_{\varepsilon_k})$ such that

$$\|\varphi_{h,k} - T_h \Phi_{\varepsilon_k}\|_{W^{1,\vec{p}}_0(\Omega_{\varepsilon_k})} \leq 1/k. \tag{4.8}$$

For $h > 0$ fixed, since $(\partial_i T_h \Phi_{\varepsilon_k})_k$ is uniformly bounded in $L^{p_i}(\Omega)$, we deduce that $(\partial_i \varphi_{h,k})_k$ is uniformly bounded in $L^{p_i}(\Omega)$. By Lemma 4.2(iii), up to a subsequence, it holds $\partial_i T_h \Phi_{\varepsilon_k} \to \partial_i T_h \Phi$ a.e. in Ω as $k \to \infty$. Thus, using (4.5) and (4.8), we get that, up to a subsequence, $(\varphi_{h,k})_k$ satisfies

$$\varphi_{h,k} \to T_h\Phi \text{ in } L^1_{\text{loc}}(\Omega) \text{ and } \partial_i\varphi_{h,k} \rightharpoonup \partial_i T_h\Phi \text{ in } L^{p_i}(\Omega) \text{ as } k \to \infty \text{ for } i = 1, \ldots, n.$$

By Mazur's lemma (see Brezis [13, p. 61]), there exists $\left(\widetilde{\varphi}_{h,k}\right)_k$ in $C_c^\infty(\Omega)$ such that

$$\widetilde{\varphi}_{h,k} \to T_h\Phi \text{ in } L^1_{\text{loc}}(\Omega) \text{ and } \partial_i\widetilde{\varphi}_{h,k} \to \partial_i T_h\Phi \text{ in } L^{p_i}(\Omega) \text{ as } k \to \infty$$

for all $i = 1, \ldots, n$. This proves that $\Phi \in \mathcal{T}_0^{1,\overrightarrow{p}}(\Omega)$. This completes the proof. □

Proof of Lemma 4.2. (i) Let $i = 1, \ldots, n$ be fixed arbitrarily. Let $q_i = p_i(p_* - 1)/p_*$ in Case 1 and $q_i > \max\{p_i - 1, 1\}$ in Case 2. If $m_i := q_i/(q_i - p_i + 1)$, then $m_i > 0$ in both cases. Hence, using (3.8) and (3.11), for every $k \in \mathbb{N}$ and $h > 0$, we have

$$\begin{cases} \displaystyle\int_K |\partial_i\Phi_{\varepsilon_k}|^{p_i-1}\,dx \le m_i\,(\text{meas}\,(K))^{1/m_i}\,\|\partial_i\Phi_{\varepsilon_k}\|^{p_i-1}_{L^{q_i,\infty}(K)}, \\ \displaystyle\int_{\{x\in\Omega_{\varepsilon_k}:\ |\partial_i\Phi_{\varepsilon_k}|>h\}} |\partial_i\Phi_{\varepsilon_k}|^{p_i-1}\,dx \le m_i\left(d_{\partial_i\Phi_{\varepsilon_k}}(h)\right)^{1/m_i}\|\partial_i\Phi_{\varepsilon_k}\|^{p_i-1}_{L^{q_i,\infty}(\Omega_{\varepsilon_k})}. \end{cases}$$

By Lemma 3.3 and Remark 3.4, we conclude the assertion of (i).

(ii) We prove that, up to a subsequence, $(\Phi_{\varepsilon_k})_k$ is a Cauchy sequence with respect to convergence in measure in K: For every $\nu, \tau \in (0, \infty)$, there exists $N_{\nu,\tau} \in \mathbb{N}$ such that

$$\text{meas}\left(\left\{x \in K:\ \left|\Phi_{\varepsilon_{k'}}(x) - \Phi_{\varepsilon_k}(x)\right| > \nu\right\}\right) < \tau \quad \text{for all } k, k' \ge N_{\nu,\tau}. \tag{4.9}$$

Let ν, τ and h be fixed arbitrarily in $(0, \infty)$. For $k, k' \in \mathbb{N}$, we use the notation $d_{\Phi_{\varepsilon_k}}(h)$ and $d_{\Phi_{\varepsilon_{k'}}}(h)$ as in (3.8). We define $I_{h,\nu,k,k'}$ as follows

$$I_{h,\nu,k,k'} := \text{meas}\left(\left\{x \in K:\ \Phi_{\varepsilon_k}(x) \le h,\ \Phi_{\varepsilon_{k'}}(x) \le h \text{ and } \left|(\Phi_{\varepsilon_{k'}} - \Phi_{\varepsilon_k})(x)\right| > \nu\right\}\right).$$

Then for every $h > 0$, we have the following estimates:

$$\begin{cases} \text{meas}\left(\left\{x \in K:\ \left|(\Phi_{\varepsilon_{k'}} - \Phi_{\varepsilon_k})(x)\right| > \nu\right\}\right) \le d_{\Phi_{\varepsilon_k}}(h) + d_{\Phi_{\varepsilon_{k'}}}(h) + I_{h,\nu,k,k'}, \\ I_{h,\nu,k,k'} \le \text{meas}\left(\{x \in K:\ |(T_h\Phi_{\varepsilon_{k'}} - T_h\Phi_{\varepsilon_k})(x)| > \nu\}\right). \end{cases} \tag{4.10}$$

From (d) above, we infer that, up to a subsequence, $(T_h\Phi_{\varepsilon_k})_k$ is a Cauchy sequence with respect to convergence in measure in K. Using Remark 3.4 and (4.10), we conclude up to a subsequence, $(\Phi_{\varepsilon_k})_k$ is a Cauchy sequence with respect to convergence in measure in K. Hence, up to a subsequence, $(\Phi_{\varepsilon_k})_k$ converges in measure in K to a measurable function $\Phi: K \to \mathbb{R}$. By Riesz Theorem, we can further pass to a subsequence, relabelled ε_k, such that $\Phi_{\varepsilon_k} \to \Phi$ a.e. in K. By a diagonal argument, we conclude the proof of (ii).

(iii) Let $i = 1, \dots, n$ be arbitrary. We show that, up to a subsequence, $\big(\partial_i \Phi_{\varepsilon_k}\big)_k$ is a Cauchy sequence with respect to convergence in measure in K.

For any $\nu, h \in (0, \infty)$ and $k, k' \in \mathbb{N}$, we introduce the following notation:

$$\begin{cases} U_{\nu,k,k'} := \big\{x \in K : \quad \big|\left(\partial_i \Phi_{\varepsilon_{k'}} - \partial_i \Phi_{\varepsilon_k}\right)(x)\,\big| > \nu\big\}, \\ V_{h,k,k'} := \big\{x \in K : \quad |\partial_i \Phi_{\varepsilon_k}(x)| \le h,\ |\partial_i \Phi_{\varepsilon_{k'}}(x)| \le h\big\}, \\ W_{h,k,k'} := \big\{x \in K : \quad |\left(\Phi_{\varepsilon_{k'}} - \Phi_{\varepsilon_k}\right)(x)\,| \le 1/h\big\}, \\ J_{h,\nu,k,k'} := \text{meas}\left(U_{\nu,k,k'} \cap V_{h,k,k'} \cap W_{h,k,k'}\right). \end{cases} \tag{4.11}$$

We want to get an upper bound estimate for meas $(U_{\nu,k,k'})$. Let $d_{\partial_i \Phi_{\varepsilon_k}}(h)$ and $d_{\partial_i \Phi_{\varepsilon_{k'}}}(h)$ be given as in (3.8). Then for all $h, \nu > 0$ and $k, k' \in \mathbb{N}$, we find that

$$\text{meas}\left(U_{\nu,k,k'}\right) \le d_{\partial_i \Phi_{\varepsilon_k}}(h) + d_{\partial_i \Phi_{\varepsilon_{k'}}}(h) + \text{meas}(K \setminus W_{h,k,k'}) + J_{h,\nu,k,k'}. \tag{4.12}$$

We next show that $J_{h,\nu,k,k'} \to 0$ as $h \to \infty$ uniformly with respect to $k, k' \in \mathbb{N}$. By testing problem (3.2) with the truncated function $T_{1/h}\big(\Phi_{\varepsilon_{k'}} - \Phi_{\varepsilon_k}\big)$, we obtain that

$$\sum_{i=1}^{n} \int_{W_{h,k,k'}} \big(|\partial_i \Phi_{\varepsilon_{k'}}|^{p_i-2} \partial_i \Phi_{\varepsilon_{k'}} - |\partial_i \Phi_{\varepsilon_k}|^{p_i-2} \partial_i \Phi_{\varepsilon_k}\big) \big(\partial_i \Phi_{\varepsilon_{k'}} - \partial_i \Phi_{\varepsilon_k}\big)\, dx \le \frac{2}{h}. \tag{4.13}$$

Indeed, the left-hand side (LHS) of (4.13) satisfies

$$\text{(LHS) of (4.13)} = \int_{\mathbb{R}^n} \big(f_{\varepsilon_{k'}} - f_{\varepsilon_k}\big) T_{1/h}\big(\Phi_{\varepsilon_{k'}} - \Phi_{\varepsilon_k}\big)\, dx \le \frac{1}{h} \int_{\mathbb{R}^n} \big|f_{\varepsilon_{k'}} - f_{\varepsilon_k}\big|\, dx \le \frac{2}{h}\,.$$

Moreover, there exists a positive constant C, independent of $h > 0$, such that

$$\big||s|^{p_i-2}\, s - |t|^{p_i-2}\, t\big| \ge \begin{cases} C h^{p_i-2}\, |s - t| & \text{if } p_i < 2 \\ C\, |s - t|^{p_i-1} & \text{if } p_i \ge 2 \end{cases} \tag{4.14}$$

for any $s, t \in (-h, h)$. By (4.11), (4.13) and (4.14), we infer that

$$\begin{aligned} 0 \le J_{h,\nu,k,k'} &\le \nu^{-\max\{2,p_i\}} \int_{V_{h,k,k'} \cap W_{h,k,k'}} \big(\partial_i \Phi_{\varepsilon_{k'}} - \partial_i \Phi_{\varepsilon_k}\big)^{\max\{2,p_i\}}\, dx \\ &\le \frac{2}{C \nu^{\max\{2,p_i\}} h^{p_i - \max\{2,p_i\}+1}} \to 0 \ \text{ as } h \to \infty. \end{aligned}$$

Hence, using Remark 3.4, (4.9) and (4.12), we find that, up to a subsequence, $(\partial_i \Phi_{\varepsilon_k})_k$ is a Cauchy sequence with respect to convergence in measure in K. By Riesz Theorem, we can pass to a subsequence such that $\partial_i \Phi_{\varepsilon_k} \to \Psi_i$ a.e. in K for some measurable function $\Psi_i : K \to \mathbb{R}$ and all $i = 1, \dots, n$. A standard diagonal argument gives that $\Psi_i : \Omega \to \mathbb{R}$ is measurable and, up to a subsequence, $\partial_i \Phi_{\varepsilon_k}$ satisfies for $i = 1, \dots, n$

$$\partial_i \Phi_{\varepsilon_k} \to \Psi_i \ \text{ locally in measure and a.e. in } \Omega \ \text{ as } k \to \infty. \tag{4.15}$$

We next prove that $\Phi \in \mathcal{T}^{1,1}_{\text{loc}}(\Omega)$ and $\Psi_i = \partial_i \Phi$ for all $i = 1, \ldots, n$. Let $h > 0$ and $R > 0$ be arbitrarily fixed. From (c), we have $(\partial_i T_h \Phi_{\varepsilon_k})_k$ is uniformly bounded in $L^{p_i}(\mathbb{R}^n)$ so that, up to a subsequence, $\partial_i T_h \Phi_{\varepsilon_k}$ converges weakly to some function $\Psi_{i,h}$ in $L^{p_i}(\mathbb{R}^n)$ (and, hence, in $L^{p_1}_{\text{loc}}(\mathbb{R}^n)$). Using (4.5) for $q = p_1$ and the compactness of the embedding $W^{1,p_1}(B_R(0)) \Subset L^{p_1}(B_R(0))$ for all $R > 0$, we get that $T_h \Phi \in W^{1,p_1}(B_R(0))$ and, up to a subsequence, $T_h \Phi_{\varepsilon_k} \rightharpoonup T_h \Phi$ in $W^{1,p_1}(B_R(0))$. We thus deduce that $\Psi_{i,h} = \partial_i T_h \Phi$ and, hence, up to a subsequence,

$$\partial_i T_h \Phi_{\varepsilon_k} \rightharpoonup \partial_i T_h \Phi \ \text{ in } L^{p_i}(\mathbb{R}^n). \tag{4.16}$$

In particular, we have $\Phi \in \mathcal{T}^{1,1}_{\text{loc}}(\Omega)$. From (4.1) and (4.15), we find that, up to a subsequence, $\partial_i T_h \Phi_{\varepsilon_k} \to \mathbf{1}_{\{\Phi < h\}} \Psi_i$ a.e. in Ω. This, jointly with (4.16), implies that $\partial_i T_h \Phi = \mathbf{1}_{\{\Phi < h\}} \Psi_i$ a.e. in $\Omega \cap B_R(0)$ for every $R > 0$. Thus $\partial_i \Phi = \Psi_i$ a.e. in Ω for $i = 1, \ldots, n$, which together with (4.15), proves (iii). This finishes the proof of Lemma 4.2. □

We next discuss the situations when our fundamental solution Φ in Proposition 4.1 becomes a distributional solution.

Corollary 4.4. *If in Case 1 of Theorem 1.1 we let Ω be bounded and $p_1 > \frac{p(n-1)}{n(p-1)}$, then Φ in Proposition 4.1 is a $W^{1,\overrightarrow{q}}_0(\Omega)$-distributional solution of* (1.1)*, where $\overrightarrow{q} = (q_1, \ldots, q_n)$ and $1 \le q_i < p_i \frac{n(p-1)}{p(n-1)}$ for $i = 1, \ldots, n$. The same conclusion holds in Case 2.*

Proof. The same argument applies for both Case 1 and Case 2. Let $i \in \{1, \ldots, n\}$ be fixed arbitrarily. Since $\mathcal{T}^{1,\overrightarrow{p}}_0(\Omega) \subset \mathcal{T}^{1,\overrightarrow{q}}_0(\Omega)$, by (v) in Section 2.2, it is enough to show that $\partial_i \Phi \in L^{q_i}(\Omega)$ with $1 \le q_i < p_i \frac{n(p-1)}{p(n-1)}$. Let $(\Phi_{\varepsilon_k})_k$ be such that (4.4) holds. By (3.11) and the weak Lebesgue estimates in Lemma 3.3, we infer that $(\partial_i \Phi_{\varepsilon_k})_k$ is uniformly bounded in $L^{q_i}(\Omega)$ and, hence, up to a subsequence, $\partial_i \Phi_{\varepsilon_k}$ converges weakly in $L^{q_i}(\Omega)$. Since $\partial_i \Phi_{\varepsilon_k} \to \partial_i \Phi$ a.e. in Ω, we conclude that $\partial_i \Phi \in L^{q_i}(\Omega)$ for $i = 1, \ldots, n$. □

5 Sharp upper bound estimates

Let Φ be the non-negative fundamental solution of (1.1), which was constructed in Section 4. In Theorem 5.1, we prove that Φ satisfies (1.11) from which we get (1.6) in Case 1 and (1.7) in Case 2. We follow the sketch of the proof outlined in Section 1.1. For $\lambda \in (0, 1)$ and $r > 0$, we define $\mathcal{A}_r(\lambda) := \mathcal{E}_{\overrightarrow{p}}((1+\lambda)r) \setminus \overline{\mathcal{E}_{\overrightarrow{p}}((1-\lambda)r)}$, where $\mathcal{E}_{\overrightarrow{p}}(R)$ is given by (1.5).

Theorem 5.1. *Under the assumptions of Theorem 1.1, the fundamental solution Φ given by Proposition 4.1 satisfies* (1.11) *for every $r \in (0, R)$.*

Remark 5.2. *For any compact subset K in $\Omega \setminus \{0\}$, we deduce from* (1.11) *that $\Phi = T_h \Phi$ on K for $h > 0$ large so that $\Phi \in W^{1,\overrightarrow{p}}(K) \cap L^\infty(K)$. It follows from Lieberman (see [33]) that $\Phi \in W^{1,\infty}_{\mathrm{loc}}(\Omega \setminus \{0\})$ and, hence, $\Phi \in C(\Omega \setminus \{0\})$.*

Our proof uses an iteration scheme of Moser-type (see Section 5.2 for Case 1 and Section 5.3 for Case 2), whose running step is given by Proposition 1.3 proved in Section 5.1.

Using (1.11), we next prove our gradient estimates in (1.8).

Corollary 5.3 (Gradient estimates)**.** *In Case 1, for the fundamental solution Φ constructed in Proposition 4.1, there exists a positive constant $C_1 = C_1\left(n, \overrightarrow{p}\right)$ such that* (1.8) *holds.*

Proof. We define $\Phi_r(x) := r^{\frac{n-p}{p(n-1)}} \Phi\big(r^{\frac{1}{s_1}} x_1, \dots, r^{\frac{1}{s_n}} x_n\big)$ for all $r \in (0,R)$ and $x \in \mathcal{E}_{\overrightarrow{p}}(2)$, where s_i is given by (1.5). Rescaling the equation satisfied by Φ, we get that

$$\operatorname{div}\left(\mathbf{A}(\nabla \Phi_r)\right) = 0 \quad \text{in } \mathcal{E}_{\overrightarrow{p}}(2) \setminus \mathcal{E}_{\overrightarrow{p}}(1/8)\,.$$

Using (1.11), we find that there exists a constant $c = c(n, \overrightarrow{p}) > 0$ such that

$$\Phi_r(x) \le c \quad \text{for all } x \in \mathcal{E}_{\overrightarrow{p}}(3/2) \backslash \mathcal{E}_{\overrightarrow{p}}(1/4)\,.$$

From Lieberman's gradient estimates in [33], there exists a constant $C = C\left(n, \overrightarrow{p}\right) > 0$ such that $|\nabla \Phi_r(x)| \le C$ for a.e. $x \in \mathcal{E}_{\overrightarrow{p}}(1) \backslash \mathcal{E}_{\overrightarrow{p}}(1/2)$. Hence, we have

$$\sum_{i=1}^n |\partial_i \Phi(x)|^{p_i} = \sum_{i=1}^n \left| r^{-\frac{1}{p_i}} \partial_i \Phi_r\big(r^{-\frac{1}{s_1}} x_1, \dots, r^{-\frac{1}{s_n}} x_n\big)\right|^{p_i} \le r^{-1} \sum_{i=1}^n C^{p_i}$$

for a.e. $x \in \mathcal{E}_{\overrightarrow{p}}(r) \backslash \mathcal{E}_{\overrightarrow{p}}(r/2)$. In view of $\mathcal{E}_{\overrightarrow{p}}(R) \subset \bigcup_{0<r<R} \mathcal{E}_{\overrightarrow{p}}(r) \backslash \mathcal{E}_{\overrightarrow{p}}(r/2)$, we get the desired estimate (1.8). □

Remark 5.4. *Since in Case 1, the function U_0 in* (1.9) *belongs to $L^{p_*-1,\infty}(\Omega)$, we infer from* (1.6) *and* (1.8) *that $\Phi \in L^{p_*-1,\infty}(\Omega)$ and $\partial_i \Phi \in L^{\frac{p_i(p_*-1)}{p_*},\infty}(\Omega)$ for $i = 1, \dots, n$.*

5.1 Proof of Proposition 1.3

In this subsection, we are in either Case 1 or Case 2 of Theorem 1.1. Let $\varepsilon \in (0, \varepsilon_0]$ and $r > 0$ be such that $\overline{\mathcal{A}_r(3/4)} \subset \Omega_\varepsilon \setminus B_\varepsilon(0)$. We use m to denote $n/(n-p)$ in Case 1 and any number greater than $1/(n-1)$ in Case 2. Let $\Gamma > m(p-1)$ be arbitrary.

We define β and Θ_j with $j = 1, \dots, n$ as follows

$$\beta = \frac{\Gamma}{m} - p \quad \text{and} \quad \Theta_j := \frac{r^{\frac{p_j(n-p)}{p(n-1)} - 1}}{(\lambda' - \lambda)^{p_j}} \|\Phi_\varepsilon\|^{\beta + p_j}_{L^{\beta+p_j}(\mathcal{A}_r(\lambda'))}. \tag{5.1}$$

Our assumption on Γ gives that $\beta > -1$.

We show that (1.12) holds, meaning that there exists a positive constant $c = c(n, \overrightarrow{p})$ such that for every $0 < \lambda < \lambda' \leq 3/4$, we have

$$\|\Phi_\varepsilon\|_{L^\Gamma(\mathcal{A}_r(\lambda))}^{\frac{\Gamma}{m}} \leq c \max\{1, (\beta+1)^{-p_n}\}\Gamma^p r^{\frac{1}{m}-\frac{n-p}{n}} \max_{j=1,\ldots,n} \Theta_j. \tag{5.2}$$

We split the proof into four steps. We choose a suitable function $\eta \in C_c^1(\Omega_\varepsilon \setminus B_\varepsilon(0))$ as in Step 1 below. We can suppose that $\Phi_\varepsilon \geq \nu > 0$ a.e. on $\operatorname{supp}\eta$ for some $\nu > 0$. If this is not true, we can consider $\Phi_\varepsilon + \nu$ in our argument and then let $\nu \to 0$.

Step 1. *We construct a function $\eta \in C^1(\Omega_\varepsilon \setminus B_\varepsilon(0))$ with the following properties:*

(a) $0 \leq \eta \leq 1$ *in* Ω_ε, $\eta = 0$ *in* $\Omega_\varepsilon \setminus \mathcal{A}_r(\lambda')$ *and* $\eta = 1$ *in* $\mathcal{A}_r(\lambda)$.

(b) $\eta^{\frac{\beta+p_1}{p_n}} \in C^1(\Omega_\varepsilon)$ *and there exists a positive constant* $c = c(n, \overrightarrow{p})$ *such that*

$$\eta^{\frac{\beta+p_1}{p_n}-1}|\partial_j \eta| \leq \frac{c}{p_1+\beta} \frac{r^{\frac{n-p}{p(n-1)}-\frac{1}{p_j}}}{\lambda'-\lambda} \quad \text{for every } j = 1, \ldots, n.$$

To this aim, we choose a function $\eta_1 \in C^1([0,\infty))$ such that $0 \leq \eta_1 \leq 1$ in $[0,\infty)$, as well as $\eta_1 = 0$ on $[0, 1-\lambda'] \cup [1+\lambda', \infty)$ and $\eta_1 = 1$ on $[1-\lambda, 1+\lambda]$. We further assume that $\eta_2 := \eta_1^{(\beta+p_1)/p_n} \in C^1[0,\infty)$ and $|\eta_2'| \leq 2/(\lambda'-\lambda)$. We now define

$$\eta(x) = \eta_1\left(r^{-1}\sum_{i=1}^n |x_i|^{s_i}\right) \quad \text{for all } x \in \Omega_\varepsilon, \tag{5.3}$$

where s_i is given by (1.5). It is easy to check that η in (5.3) satisfies (a) and (b).

Notation. For every $i, j = 1, \ldots, n$, we define V_{ij} and T_{ij} by

$$V_{ij} := \int_{\Omega_\varepsilon} \eta^{\beta+p_i-p_j} |\partial_j \eta|^{p_j} \Phi_\varepsilon^{\beta+p_j}\, dx \ \text{ and } \ T_{ij} := \int_{\Omega_\varepsilon} |\partial_j \Phi_\varepsilon|^{p_j} \eta^{\beta+p_i} \Phi_\varepsilon^\beta\, dx. \tag{5.4}$$

Step 2. *There exists a constant $C_0 = C_0(n, \overrightarrow{p}) > 0$ such that for every $i = 1, \ldots, n$,*

$$T_{ii} \leq C_0 \sum_{j=1}^n \left(\frac{\beta+p_i}{\beta+1}\right)^{p_j} V_{ij} < \infty. \tag{5.5}$$

From Step 1 and the definition of Θ_j in (5.1), we obtain that

$$\max_{i=1,\ldots,n} V_{ij} \leq \int_{\Omega_\varepsilon} \eta^{\left(\frac{\beta+p_1}{p_n}-1\right)p_j} |\partial_j \eta|^{p_j} \Phi_\varepsilon^{\beta+p_j}\, dx \leq \left(\frac{c}{\beta+p_1}\right)^{p_j} \Theta_j < \infty. \tag{5.6}$$

Since $\beta + p_i > (\beta+p_1)/p_n$, using Step 1 and our assumption $\overline{\mathcal{A}_r(\lambda')} \subset \Omega_\varepsilon \setminus B_\varepsilon(0)$, we have $\eta^{\beta+p_i} \in C_c^1(\Omega_\varepsilon \setminus B_\varepsilon(0))$. Taking $\eta^{\beta+p_i}\Phi_\varepsilon^{\beta+1}$ as a test function in (3.3), we get

$$\sum_{j=1}^n T_{ij} + \frac{\beta+p_i}{\beta+1} \sum_{j=1}^n \int_{\Omega_\varepsilon} |\partial_j \Phi_\varepsilon|^{p_j-2} (\partial_j \Phi_\varepsilon)(\partial_j \eta)\, \eta^{\beta+p_i-1} \Phi_\varepsilon^{\beta+1}\, dx = 0,$$

which implies that

$$\sum_{j=1}^{n} T_{ij} \leq \frac{\beta+p_i}{\beta+1} \sum_{j=1}^{n} \int_{\Omega_\varepsilon} |\partial_j \Phi_\varepsilon|^{p_j-1} |\partial_j \eta| \, \eta^{\beta+p_i-1} \Phi_\varepsilon^{\beta+1} dx \,. \tag{5.7}$$

Applying Young's inequality $ab \leq a^{p_j}/p_j + b^{p_j'}/p_j'$ in the following situation

$$p_j' = \frac{p_j}{p_j-1}, \; a = \tau^{-1+1/p_j} |\partial_j \eta| \eta^{-1} \Phi_\varepsilon, \; b = \tau^{1-1/p_j} |\partial_j \Phi_\varepsilon|^{p_j-1}, \text{ and } \tau = \frac{\beta+1}{\beta+p_i},$$

we arrive at

$$|\partial_j \Phi_\varepsilon|^{p_j-1} |\partial_j \eta| \, \eta^{-1} \Phi_\varepsilon \leq \frac{(\beta+1)(p_j-1)}{(\beta+p_i)p_j} |\partial_j \Phi_\varepsilon|^{p_j} + \frac{1}{p_j} \left(\frac{\beta+1}{\beta+p_i} \right)^{1-p_j} |\partial_j \eta|^{p_j} \, \eta^{-p_j} \Phi_\varepsilon^{p_j}.$$

Then, the right-hand side (RHS) of (5.7) satisfies

$$\text{(RHS) of (5.7)} \leq \sum_{j=1}^{n} \frac{p_j-1}{p_j} T_{ij} + \sum_{j=1}^{n} \frac{1}{p_j} \left(\frac{\beta+p_i}{\beta+1} \right)^{p_j} V_{ij}, \tag{5.8}$$

where V_{ij} and T_{ij} are given by (5.4). From (5.7) and (5.8), we infer that

$$\sum_{j=1}^{n} \frac{1}{p_j} T_{ij} \leq \sum_{j=1}^{n} \frac{1}{p_j} \left(\frac{\beta+p_i}{\beta+1} \right)^{p_j} V_{ij}. \tag{5.9}$$

The claim of Step 2 follows immediately from (5.9).

Step 3. *There exists a constant* $C_1 = C_1 \left(n, \overrightarrow{p}\right) > 0$ *such that for every* $i = 1, \dots, n$

$$H_i := \left\| (\eta \Phi_\varepsilon)^{\beta/p_i} \partial_i (\eta \Phi_\varepsilon) \right\|_{L^{p_i}(\Omega_\varepsilon)}^{p_i} \leq C_1 \max\{1, (\beta+1)^{-p_n}\} \max_{j=1,\dots,n} \Theta_j, \tag{5.10}$$

where Θ_j *is defined by* (5.1).

Indeed, for any $i = 1, \dots, n$, we find that

$$|\partial_i (\eta \Phi_\varepsilon)|^{p_i} \leq 2^{p_i-1} \left(\eta^{p_i} |\partial_i \Phi_\varepsilon|^{p_i} + \Phi_\varepsilon^{p_i} |\partial_i \eta|^{p_i} \right).$$

Hence, by (5.6) and Step 2, there exist positive constants C_0, $\tilde{C}_0$, and C_1, depending only on n and $\overrightarrow{p}$, such that

$$\begin{aligned} H_i &\leq 2^{p_i-1} \left(T_{ii} + V_{ii} \right) \leq 2^{p_i-1} (C_0+1) \sum_{j=1}^{n} \left(\frac{\beta+p_i}{\beta+1} \right)^{p_j} V_{ij} \\ &\leq \tilde{C}_0 \sum_{j=1}^{n} (\beta+1)^{-p_j} \left(\frac{p_n+\beta}{p_1+\beta} \right)^{p_j} \Theta_j \leq C_1 \max\{1, (\beta+1)^{-p_n}\} \max_{j=1,\dots,n} \Theta_j. \end{aligned} \tag{5.11}$$

This proves the claim of Step 3.

Step 4. *Proof of* (5.2) *concluded.*

We apply Lemma A.1 in Appendix A with $\Omega = \Omega_\varepsilon$ and $\varphi = \eta\Phi_\varepsilon$. Clearly, we have $\varphi \in L^\infty(\Omega_\varepsilon)$ and $\operatorname{supp}\varphi$ is a compact subset of Ω_ε. We denote $\xi = (\beta + p)/p$. The assumption $\Gamma > m(p-1)$ gives that $1 + p(\xi - 1)/p_1 > 0$.

If $\xi \geq 1$ (i.e., $\Gamma \geq mp$), then $\eta\Phi_\varepsilon \in W^{1,\overrightarrow{p}}(\Omega_\varepsilon)$ since $\Phi_\varepsilon \in W^{1,\overrightarrow{p}}(\Omega_\varepsilon)$ and $\eta \in C^1_c(\Omega_\varepsilon)$. If $\xi < 1$, then we find that $(\eta\Phi_\varepsilon)^{1+\frac{p(\xi-1)}{p_1}} \in W^{1,\overrightarrow{p}}(\Omega_\varepsilon)$ since $\Phi_\varepsilon^{1+\frac{p(\xi-1)}{p_1}} \in W^{1,\overrightarrow{p}}(\Omega_\varepsilon)$ (using that $\Phi_\varepsilon \geq \nu > 0$ a.e. on $\operatorname{supp}\eta$) and $\eta^{1+\frac{p(\xi-1)}{p_1}} \in C^1_c(\Omega_\varepsilon)$ (see Step 1).

Then, by Lemma A.1 in Appendix A, there exists a constant $c = c(n, \overrightarrow{p}) > 0$ so that

$$\|\eta\Phi_\varepsilon\|^{\frac{\Gamma}{m}}_{L^\Gamma(\Omega_\varepsilon)} \leq \begin{cases} c\Gamma^p \prod_{i=1}^n H_i^{\frac{p}{p_i n}} & \text{in Case } 1, \\ c\Gamma^n(\operatorname{meas}(\mathcal{A}_r(\lambda')))^{\frac{1}{m}} \prod_{i=1}^n H_i^{\frac{1}{p_i}} & \text{in Case } 2, \end{cases} \tag{5.12}$$

where H_i is defined by (5.10). Since $\eta = 1$ in $\mathcal{A}_r(\lambda)$ and $\sum_{i=1}^n 1/p_i = n/p$, from (5.12) and Step 3, we reach (5.2). This concludes the proof of Proposition 1.3. □

5.2 Proof of Theorem 5.1 in Case 1

Throughout this subsection, we assume that (1.4) holds. Recall that $m := n/(n-p)$. Let $\varepsilon \in (0, \varepsilon_0]$ and $r > 0$ be such that $\overline{\mathcal{A}_r(3/4)} \subset \Omega_\varepsilon \setminus B_\varepsilon(0)$. In what follows, we denote

$$\|\cdot\|_{L^q(\mathcal{A}_r(\lambda))} := \|\cdot\|_{q;\lambda}. \tag{5.13}$$

To keep the notation short, we shall not include r since it is fixed everywhere in the proof. The desired estimate (1.11) follows from (5.14) below applied to Φ_{ε_k} in Lemma 4.2 and passing to the limit, up to a subsequence, as $k \to \infty$.

Proposition 5.5. *There exists a positive constant $C = C(n, \overrightarrow{p})$ such that*

$$\|\Phi_\varepsilon\|_{L^\infty(\mathcal{A}_r(1/2))} \leq Cr^{-1/p_*}. \tag{5.14}$$

Proof. For any $\Gamma_* > p_* - 1$ and $i = 1, \dots, n$, we define $\Gamma_{*,i}$ as follows

$$\Gamma_{*,i} := \sigma_i(\Gamma_*) = \Gamma_*/m + p_i - p. \tag{5.15}$$

In this proof, we fix $\delta = \delta(n, \overrightarrow{p})$ so that $0 < \delta < (p_* - p_n)n/(2n-p)$. Hence, we have

$$\sigma_i(p_* - 1 + \delta) < p_* - 1 - \delta \quad \text{for all } i = 1, \dots, n. \tag{5.16}$$

By Proposition 1.3, we know that for any $\Gamma_* \geq p_* - 1 + \delta$, there exists a positive constant $C = C(n, \overrightarrow{p})$ such that

$$\|\Phi_\varepsilon\|_{\Gamma_*;\,\lambda} \leq \left[C\Gamma_*^p \max_{i=1,\dots,n} \left(\frac{r^{\frac{p_i}{p_*}-1}}{(\lambda' - \lambda)^{p_i}} \|\Phi_\varepsilon\|^{\Gamma_{*,i}}_{\Gamma_{*,i};\,\lambda'} \right) \right]^{m/\Gamma_*}. \tag{5.17}$$

The proof of Proposition 5.5 consists of iterating (5.17). We provide the details below.

The iterative scheme. Let $\lambda_k := 1/2 + (1/2)^{k+1}$ for every $k \geq 1$. Hence, we have

$$\lambda_1 = 3/4 \text{ and } \lambda_{k+1} = \lambda_k - 2^{-k-2} \text{ for all } k \geq 1. \tag{5.18}$$

We see that $(\lambda_k)_{k\geq 1}$ is decreasing and converges to $1/2$ as $k \to \infty$. Since $p_n < p_*$ and $m > 1$, we can find a large positive integer N so that

$$n\,(p_* - p_1)\,/p + \delta < m^N n\,(p_* - p_n)\,/p. \tag{5.19}$$

We construct a sequence $(\Gamma_k)_{k\geq 1}$ such that

$$n\,(p_* - p_1)\,/p + \delta < \Gamma_k/m^k < m^N n\,(p_* - p_n)\,/p. \tag{5.20}$$

Since $\Gamma_k \to \infty$ as $k \to \infty$, there exists a large integer $k_0 \geq 1$ such that $\Gamma_k > p_* - 1 + \delta$ for every $k > k_0$. In what follows, we fix $k > k_0$. For each $i_1, i_2 \in \{1, \ldots, n\}$, we define Γ_{k,i_1} as in (5.15) and $\Gamma_{k,i_1 i_2} = \sigma_{i_2}(\Gamma_{k,i_1})$, that is

$$\Gamma_{k,i_1} := \Gamma_k/m + p_{i_1} - p, \quad \Gamma_{k,i_1 i_2} := \Gamma_{k,i_1}/m + p_{i_2} - p. \tag{5.21}$$

We also need to introduce $\mathcal{B}_{i_1}$ and $\mathcal{B}_{i_1 i_2}$ as follows

$$\begin{cases} \mathcal{B}_{i_1} := 2^{\frac{m(k+N+2)p_{i_1}}{\Gamma_k}}\, r^{\frac{m(p_{i_1}-p_*)}{p_*\Gamma_k}} \|\Phi_\varepsilon\|_{\Gamma_{k,i_1};\,\lambda_{k+N}}^{\frac{m\Gamma_{k,i_1}}{\Gamma_k}}, \\ \mathcal{B}_{i_1 i_2} := \Gamma_{k,i_1}^{\frac{m^2 p}{\Gamma_k}}\, 2^{\frac{\sum_{j=1}^2 m^j (k+N-j+3)p_{i_j}}{\Gamma_k}}\, r^{\frac{\sum_{j=1}^2 m^j (p_{i_j}-p_*)}{p_*\Gamma_k}} \|\Phi_\varepsilon\|_{\Gamma_{k,i_1 i_2};\,\lambda_{k+N-1}}^{\frac{m^2\Gamma_{k,i_1 i_2}}{\Gamma_k}}. \end{cases} \tag{5.22}$$

Let $\tau_1, \tau_2 \in \{1, \ldots, n\}$ be such that $\mathcal{B}_{\tau_1} = \max_{i_1=1,\ldots,n} \mathcal{B}_{i_1}$ and $\mathcal{B}_{\tau_1\tau_2} = \max_{i_2=1,\ldots,n} \mathcal{B}_{\tau_1 i_2}$. We first apply (5.17) with $\Gamma_* = \Gamma_k$, $\lambda = \lambda_{k+N+1}$ and $\lambda' = \lambda_{k+N}$. We obtain that

$$\|\Phi_\varepsilon\|_{\Gamma_k;\,\lambda_{k+N+1}} \leq C^{\frac{m}{\Gamma_k}}\, \Gamma_k^{\frac{mp}{\Gamma_k}}\, \mathcal{B}_{\tau_1}. \tag{5.23}$$

Our choice of Γ_k ensures that $\Gamma_{k,i_1} > p_* - 1 + \delta$ for all $i_1 = 1, \ldots, n$. Thus in (5.23), we can continue using (5.17) to estimate each $\|\Phi_\varepsilon\|_{\Gamma_{k,i_1};\lambda_{k+N}}$ with $i_1 = 1, \ldots, n$ as follows

$$\|\Phi_\varepsilon\|_{\Gamma_{k,i_1};\lambda_{k+N}}^{\frac{m\Gamma_{k,i_1}}{\Gamma_k}} \leq C^{\frac{m^2}{\Gamma_k}}\, \Gamma_{k,i_1}^{\frac{m^2 p}{\Gamma_k}} \max_{i_2=1,\ldots,n} \left(2^{\frac{m^2(k+N+1)p_{i_2}}{\Gamma_k}}\, r^{\frac{m^2(p_{i_2}-p_*)}{p_*\Gamma_k}} \|\Phi_\varepsilon\|_{\Gamma_{k,i_1 i_2};\,\lambda_{k+N-1}}^{\frac{m^2\Gamma_{k,i_1 i_2}}{\Gamma_k}} \right).$$

This, jointly with (5.23), leads to

$$\|\Phi_\varepsilon\|_{\Gamma_k;\,\lambda_{k+N+1}} \leq C^{\frac{m+m^2}{\Gamma_k}}\, \Gamma_k^{\frac{mp}{\Gamma_k}}\, \mathcal{B}_{\tau_1\tau_2}. \tag{5.24}$$

If this iteration process works ℓ times, it gives rise to $\Gamma_{k,i_1\ldots i_\ell}$, $\mathcal{A}_{i_1\ldots i_\ell}$ and $\mathcal{B}_{i_1\ldots i_\ell}$, namely

$$\begin{cases} \Gamma_{k,i_1\ldots i_\ell} := \Gamma_{k,i_1\ldots i_{\ell-1}}/m + p_{i_\ell} - p, \\ \mathcal{A}_{i_1\ldots i_\ell} := \left(\prod_{j=2}^{\ell} \Gamma_{k,i_1\ldots i_{j-1}}^{\frac{m^j p}{\Gamma_k}} \right) 2^{\frac{\sum_{j=1}^\ell m^j (k+N-j+3)p_{i_j}}{\Gamma_k}}, \\ \mathcal{B}_{i_1\ldots i_\ell} := \mathcal{A}_{i_1\ldots i_\ell}\, r^{\frac{\sum_{j=1}^\ell m^j \left(p_{i_j}-p_*\right)}{p_*\Gamma_k}} \|\Phi_\varepsilon\|_{\Gamma_{k,i_1\ldots i_\ell};\,\lambda_{k+N-\ell+1}}^{\frac{m^\ell\Gamma_{k,i_1\ldots i_\ell}}{\Gamma_k}}. \end{cases} \tag{5.25}$$

For the definition of Γ_{k,i_1} and Γ_{k,i_1,i_2}, see (5.21), whereas for $\mathcal{B}_{i_1}$ and $\mathcal{B}_{i_1 i_2}$ see (5.22). For each $j \in \{2, \ldots, \ell\}$, let $\tau_j \in \{1, \ldots, n\}$ be such that

$$\mathcal{B}_{\tau_1 \ldots \tau_j} = \max_{i_j \in \{1, \ldots, n\}} \mathcal{B}_{\tau_1 \ldots \tau_{j-1} i_j}.$$

After ℓ uses of (5.17), we arrive at

$$\|\Phi_\varepsilon\|_{\Gamma_k;\, \lambda_{k+N+1}} \leq C^{\frac{\sum_{j=1}^{\ell} m^j}{\Gamma_k}} \Gamma_k^{\frac{mp}{\Gamma_k}} \mathcal{B}_{\tau_1 \ldots \tau_\ell}. \tag{5.26}$$

Let ℓ denote the maximum number for which (5.17) can be used iteratively, that is

$$\Gamma_{k,\tau_1 \ldots \tau_j} \geq p_* - 1 + \delta \quad \text{for all } 1 \leq j \leq \ell - 1 \text{ and } \Gamma_{k,\tau_1 \ldots \tau_\ell} < p_* - 1 + \delta. \tag{5.27}$$

We next prove that $k < \ell \leq k + N$. Using (5.25), we find that

$$\Gamma_{k,i_1 \ldots i_j} = \Gamma_k / m^j + \sum_{\nu=1}^{j} m^{\nu - j} \left(p_{i_\nu} - p \right) \quad \text{for all } 1 \leq j \leq \ell. \tag{5.28}$$

Since $p_1 \leq p_i \leq p_n$ for all $i = 1, \ldots, n$, it follows from (5.28) that

$$\Gamma_k / m^j - n \left(p - p_1 \right) / p \leq \Gamma_{k,i_1 \ldots i_j} \leq \Gamma_k / m^j + n \left(p_n - p \right) / p. \tag{5.29}$$

This, jointly with (5.20), implies that for any $i_1, \ldots, i_{k+N} \in \{1, \ldots, n\}$, we have

$$\Gamma_{k,i_1 \ldots i_j} > p_* - 1 + \delta \quad \text{for all } j = 1, \ldots, k \quad \text{and} \quad \Gamma_{k,i_1 \ldots i_{k+N}} < p_* - 1. \tag{5.30}$$

Using (5.30), we conclude that $k < \ell \leq k + N$. Thus from (5.20) and (5.29), we see that

$$\frac{m^\ell}{\Gamma_k} < \frac{m^N p}{n(p_* - p_1)}, \;\; \Gamma_k < m^{N+k}(p_* - 1) \;\; \text{and} \;\; \Gamma_{k,\tau_1 \ldots \tau_{j-1}} < m^{N+k-j+1}(p_* - 1) \tag{5.31}$$

for all $j = 2, \ldots, \ell$. Using (5.31) and a simple calculation, we obtain that

$$\begin{aligned} \frac{1}{\Gamma_k} \sum_{j=1}^{\ell} (N + k - j + 1) m^j &= \frac{m}{(m-1)\Gamma_k} \left[(N + k - \ell) m^\ell - (N + k) + \sum_{j=1}^{\ell} m^j \right] \\ &< \frac{m}{(m-1)\Gamma_k} \left(N m^\ell + \sum_{j=1}^{\ell} m^j \right) < C_{n,\overrightarrow{p}}, \end{aligned}$$

where $C_{n,\overrightarrow{p}} > 0$ is a positive constant depending only on n and $\overrightarrow{p}$.

Let $\mathcal{A}_{i_1 \ldots i_\ell}$ be defined by (5.25). It follows that

$$\Gamma_k^{\frac{mp}{\Gamma_k}} \mathcal{A}_{\tau_1 \ldots \tau_\ell} \leq p_*^{\frac{p \sum_{j=1}^{\ell} m^j}{\Gamma_k}} m^{\frac{p \sum_{j=1}^{\ell} (N+k-j+1) m^j}{\Gamma_k}} 2^{\frac{p_n \sum_{j=1}^{\ell} m^j (k+N-j+3)}{\Gamma_k}},$$

which is dominated from above by a positive constant depending on n and $\overrightarrow{p}$. Therefore, using (5.25) and (5.26), we infer that there exists a constant $\tilde{C}=\tilde{C}\left(n,\overrightarrow{p}\right)>0$ such that

$$\|\Phi_\varepsilon\|_{\Gamma_k;\,\lambda_{k+N+1}}\le\tilde{C}r^{\frac{1}{p_*\Gamma_k}\sum_{j=1}^{\ell}m^j\left(p_{\tau_j}-p_*\right)}\|\Phi_\varepsilon\|_{\Gamma_{k,\tau_1\dots\tau_\ell};\,\lambda_{k+N-\ell+1}}^{\frac{m^\ell\Gamma_{k,\tau_1\dots\tau_\ell}}{\Gamma_k}}. \tag{5.32}$$

We now estimate the (quasi-)norms in the right hand side of (5.32).

CLAIM. *There exists a constant* $c=c(n,\overrightarrow{p})>0$ *such that*

$$\|\Phi_\varepsilon\|_{\Gamma_{k,\tau_1\dots\tau_\ell};\,\lambda_{k+N-\ell+1}}\le c\,r^{\frac{p_*-1-\Gamma_{k,\tau_1\dots\tau_\ell}}{p_*\Gamma_{k,\tau_1\dots\tau_\ell}}}. \tag{5.33}$$

Suppose for the moment that (5.33) has been proved. We show how to conclude the proof of Proposition 5.5. Plugging (5.33) into (5.32), then using (5.28), we find positive constants $\tilde{C}$, c, and $\tilde{c}$, depending only on n and $\overrightarrow{p}$, such that

$$\|\Phi_\varepsilon\|_{\Gamma_k;\,\lambda_{k+N+1}}\le\tilde{C}c^{\frac{\Gamma_{k,\tau_1\dots\tau_\ell}}{\Gamma_k}m^\ell}r^{\frac{p_*-1-\Gamma_k}{p_*\Gamma_k}}\le\tilde{c}\,r^{\frac{p_*-1-\Gamma_k}{p_*\Gamma_k}}. \tag{5.34}$$

Letting $k\to\infty$ in (5.34), we obtain (5.14).

Proof of (5.33). From (5.27), we have

$$p_*-1+\delta>\Gamma_{k,\tau_1\dots\tau_\ell}=\Gamma_{k,\tau_1\dots\tau_{\ell-1}}/m+p_{\tau_\ell}-p>(p_*-1)/m+p_1-p=p_1-1. \tag{5.35}$$

We further distinguish two cases:

CASE A: Let $\Gamma_{k,\tau_1\dots\tau_\ell}\le p_*-1-\delta$.

In view of (5.35), we obtain that

$$\left(\frac{p_*-1}{p_*-1-\Gamma_{k,\tau_1\dots\tau_\ell}}\right)^{\frac{1}{\Gamma_{k,\tau_1\dots\tau_\ell}}}\le c_0,\quad\text{where } c_0:=\left(\frac{p_*-1}{\delta}\right)^{\frac{1}{p_1-1}}. \tag{5.36}$$

From (5.18) and $k+N-\ell\ge0$, we infer that $\operatorname{meas}\left(\mathcal{A}_r\left(\lambda_{k+N-\ell+1}\right)\right)\le\operatorname{meas}\left(\mathcal{A}_r\left(3/4\right)\right)$. Thus, using (3.11) with $q_0=\Gamma_{k,i_1\dots i_\ell}$, $q=p_*-1$, and $\Omega=\mathcal{A}_r\left(\lambda_{k+N-\ell+1}\right)$, we find that

$$\|\Phi_\varepsilon\|_{\Gamma_{k,\tau_1\dots\tau_\ell};\,\lambda_{k+N-\ell+1}}\le c_0(\operatorname{meas}\left(\mathcal{A}_r\left(3/4\right)\right))^{\frac{p_*-1-\Gamma_{k,\tau_1\dots\tau_\ell}}{(p_*-1)\Gamma_{k,\tau_1\dots\tau_\ell}}}\|\Phi_\varepsilon\|_{L^{p_*-1,\infty}(\mathcal{A}_r(\lambda_{k+N-\ell+1}))}.$$

This, together with (3.12), proves the claim of (5.33) in Case A.

CASE B: Let $p_*-1-\delta<\Gamma_{k,\tau_1\dots\tau_\ell}<p_*-1+\delta$.

We choose $\theta\in(0,1)$ with the following property:

$$\frac{1}{\Gamma_{k,\tau_1\dots\tau_\ell}}=\frac{1-\theta}{p_*-1-\delta}+\frac{\theta}{p_*-1+\delta}.$$

By the interpolation inequality between Lebesgue spaces, we deduce that

$$\|\Phi_\varepsilon\|_{\Gamma_{k,\tau_1\ldots\tau_\ell};\,\lambda_{k+N-\ell+1}} \leq \|\Phi_\varepsilon\|^{1-\theta}_{p_*-1-\delta;\,\lambda_{k+N-\ell+1}} \|\Phi_\varepsilon\|^{\theta}_{p_*-1+\delta;\,\lambda_{k+N-\ell+1}}. \tag{5.37}$$

From (5.17), jointly with (5.18) and $k < \ell \leq k+N$, we infer the existence of a positive constant $c_1 = c_1(n, \overrightarrow{p})$ such that

$$\|\Phi_\varepsilon\|_{p_*-1+\delta;\,\lambda_{k+N-\ell+1}} \leq c_1 \max_{i=1,\ldots,n} \left(r^{\frac{p_i}{p_*}-1} \|\Phi_\varepsilon\|^{\sigma_i(p_*-1+\delta)}_{\sigma_i(p_*-1+\delta);\,\lambda_{k+N-\ell}} \right)^{\frac{m}{p_*-1+\delta}}. \tag{5.38}$$

We next use the notation c_0 as in (5.36) and $c_2 := [(p_*-1)/\delta]^{1/(p_*-1-\delta)}$. Since (5.16) holds, as in Case A, we employ (3.11) to find that

$$\begin{cases} \|\Phi_\varepsilon\|_{\sigma_i(p_*-1+\delta);\,\lambda_{k+N-\ell}} \leq c_0(\mathrm{meas}\,(\mathcal{A}_r(3/4)))^{\frac{p_*-1-\sigma_i(p_*-1+\delta)}{(p_*-1)\sigma_i(p_*-1+\delta)}} \|\Phi_\varepsilon\|_{L^{p_*-1,\infty}(\mathcal{A}_r(\lambda_{k+N-\ell}))}, \\ \|\Phi_\varepsilon\|_{p_*-1-\delta;\,\lambda_{k+N-\ell+1}} \leq c_2(\mathrm{meas}\,(\mathcal{A}_r\,(3/4)))^{\frac{\delta}{(p_*-1)(p_*-1-\delta)}} \|\Phi_\varepsilon\|_{L^{p_*-1,\infty}(\mathcal{A}_r(\lambda_{k+N-\ell+1}))}. \end{cases}$$

Hence, from (5.37) and (5.38) together with (3.12), we conclude (5.33) in Case B. □

5.3 Proof of Theorem 5.1 in Case 2

In this subsection, we let $p = n$ and Ω be bounded. Let $\varepsilon \in (0, \varepsilon_0]$ and $r > 0$ be such that $\overline{\mathcal{A}_r\,(3/4)} \subset \Omega \setminus B_\varepsilon\,(0)$. We deduce (1.11) by applying (5.39) below to Φ_{ε_k} in Lemma 4.2 and passing to the limit, up to a subsequence, as $k \to \infty$.

Proposition 5.6. *There exist positive constants $a = a(\mathrm{meas}\,(\Omega), n, \overrightarrow{p})$ and $b = b(n, \overrightarrow{p})$ such that*

$$\|\Phi_\varepsilon\|_{L^\infty(\mathcal{A}_r(1/2))} \leq a + b|\ln r|. \tag{5.39}$$

Proof. Let $N \geq 1$ be a large integer such that $2^N > 2\,(p_n - p_1) + 1$.

Let $q \geq \max\{2n+1, 2(p_n - n + 1)\}$ be arbitrary. It follows that

$$\frac{q-1+2(n-p_1)}{q-1-2(p_n-n)} \leq 2(p_n - p_1) + 1 < 2^N. \tag{5.40}$$

We use the notation $\|\cdot\|_{q;\,\lambda}$ in the same way as in (5.13). By Proposition 1.3 with $m = 2$, there exists a constant $C = C(n, \overrightarrow{p}) > 0$ such that for every $\Gamma \geq q-1$, we have

$$\|\Phi_\varepsilon\|^{\Gamma}_{\Gamma;\,\lambda} \leq C^2 \Gamma^{2n} r^{-1} \max_{i_1=1,\ldots,n} \left((\lambda' - \lambda)^{-2p_{i_1}} \|\Phi_\varepsilon\|^{2\left(\frac{\Gamma}{2}+p_{i_1}-n\right)}_{\frac{\Gamma}{2}+p_{i_1}-n;\,\lambda'} \right). \tag{5.41}$$

We define $(\lambda_k)_{k\geq 1}$ as in the proof of Proposition 5.5 so that (5.18) holds. In view of (5.40), we can choose Γ_k with the property that

$$2^k[q-1+2(n-p_1)] < \Gamma_k < 2^{k+N}[q-1-2(p_n-n)] \quad \text{for all } k \geq 1. \tag{5.42}$$

For any $i_1, i_2, \dots, i_{k+N} \in \{1, \dots, n\}$, we define

$$\begin{cases} \Gamma_{k,i_1} := \Gamma_k/2 + p_{i_1} - n, \\ \Gamma_{k,i_1\dots i_j} := \Gamma_{k,i_1\dots i_{j-1}}/2 + p_{i_j} - n \quad \text{for every } j = 2, \dots, k+N. \end{cases} \tag{5.43}$$

Then, we deduce that

$$\Gamma_{k,i_1\dots i_j} = \Gamma_k/2^j + \sum_{s=1}^{j} 2^{s-j}(p_{i_s} - n) \quad \text{for every } j = 1, \dots, k+N.$$

This gives us the following estimate

$$\Gamma_k/2^j - 2(n - p_1) \le \Gamma_{k,i_1\dots i_j} \le \Gamma_k/2^j + 2(p_n - n). \tag{5.44}$$

Hence, our choice of Γ_k in (5.42) implies that

$$\Gamma_{k,i_1\dots i_j} > q - 1 \ \text{ for every } j = 1, \dots, k, \ \text{ whereas } \Gamma_{k,i_1\dots i_{k+N}} < q - 1. \tag{5.45}$$

Let $\tau_1 \in \{1, \dots, n\}$ be such that $\mathcal{D}_{\tau_1} = \max_{i_1=1,\dots,n} \mathcal{D}_{i_1}$, where $\mathcal{D}_{i_1}$ is defined by

$$\mathcal{D}_{i_1} := 2^{2(k+N+2)p_{i_1}} \|\Phi_\varepsilon\|_{\Gamma_{k,i_1};\ \lambda_{k+N}}^{2\Gamma_{k,i_1}}.$$

Using (5.41) with $\Gamma = \Gamma_k$, $\lambda = \lambda_{k+N+1}$, and $\lambda' = \lambda_{k+N}$, we find that

$$\|\Phi_\varepsilon\|_{\Gamma_k;\ \lambda_{k+N+1}}^{\Gamma_k} \le C^2 \Gamma_k^{2n} r^{-1} \mathcal{D}_{\tau_1}. \tag{5.46}$$

Since $\Gamma_{k,\tau_1} \ge q - 1$, we can use again (5.41) to estimate $\|\Phi_\varepsilon\|_{\Gamma_{k,\tau_1};\ \lambda_{k+N}}$ as follows

$$\|\Phi_\varepsilon\|_{\Gamma_{k,\tau_1};\ \lambda_{k+N}}^{2\Gamma_{k,\tau_1}} \le C^{2^2} \Gamma_{k,\tau_1}^{2^2 n} r^{-2} \max_{i_2=1,\dots,n} \left(2^{2^2(k+N+1)p_{i_2}} \|\Phi_\varepsilon\|_{\Gamma_{k,\tau_1 i_2};\ \lambda_{k+N-1}}^{2^2\Gamma_{k,\tau_1 i_2}} \right). \tag{5.47}$$

For each $i_1, i_2 \in \{1, \dots, n\}$, we define $\mathcal{D}_{i_1 i_2}$ by

$$\mathcal{D}_{i_1 i_2} := \Gamma_{k,i_1}^{2^2 n} 2^{\sum_{j=1}^{2} 2^j (k+N-j+3) p_{i_j}} \|\Phi_\varepsilon\|_{\Gamma_{k,i_1 i_2};\lambda_{k+N-1}}^{2^2\Gamma_{k,i_1 i_2}}.$$

Let $\tau_2 \in \{1, \dots, n\}$ be such that $\mathcal{D}_{\tau_1\tau_2} = \max_{i_2=1,\dots,n} \mathcal{D}_{\tau_1 i_2}$.

From (5.46) and (5.47), we deduce that

$$\|\Phi_\varepsilon\|_{\Gamma_k;\ \lambda_{k+N+1}}^{\Gamma_k} \le C^{\sum_{j=1}^{2} 2^j} \Gamma_k^{2n} r^{-\sum_{j=1}^{2} 2^{j-1}} \mathcal{D}_{\tau_1\tau_2}.$$

If this iteration process continues ℓ times, we obtain $\mathcal{D}_{i_1\dots i_\ell}$ given by

$$\mathcal{D}_{i_1\dots i_\ell} = \left(\prod_{j=2}^{\ell} \Gamma_{k,i_1\dots i_{j-1}}^{2^j n} \right) 2^{\sum_{j=1}^{\ell} 2^j (k+N-j+3) p_{i_j}} \|\Phi_\varepsilon\|_{\Gamma_{k,i_1\dots i_\ell};\ \lambda_{k+N-\ell+1}}^{2^\ell \Gamma_{k,i_1\dots i_\ell}}. \tag{5.48}$$

For every $j = 2, \dots, \ell$, we define $\tau_j \in \{1, \dots, n\}$ such that

$$\mathcal{D}_{\tau_1\dots\tau_j} = \max_{i_j=1,\dots,n} \mathcal{D}_{\tau_1\dots\tau_{j-1} i_j}.$$

We continue the above iterative process, denoting by ℓ the maximum number of iterations of (5.41), in the sense that

$$\Gamma_{k,\tau_1\ldots\tau_j} \geq q-1 \text{ for all } j=1,\ldots,\ell-1 \text{ and } \Gamma_{k,\tau_1\ldots\tau_\ell} < q-1. \tag{5.49}$$

From (5.45), we infer that $k<\ell\leq k+N$. Hence, after ℓ uses of (5.41), we obtain that

$$\|\Phi_\varepsilon\|^{\Gamma_k}_{\Gamma_k;\,\lambda_{k+N+1}} \leq C^{\sum_{j=1}^{\ell} 2^j}\Gamma_k^{2n}\, r^{-\sum_{j=1}^{\ell} 2^{j-1}}\mathcal{D}_{\tau_1\ldots\tau_\ell}. \tag{5.50}$$

CLAIM: *There exists a positive constant, still denoted by $C=C(n,\overrightarrow{p})$, such that*

$$\|\Phi_\varepsilon\|^{\Gamma_k}_{\Gamma_k;\,\lambda_{k+N+1}} \leq C^{2^\ell} q^{n\sum_{j=1}^{\ell} 2^j}\, r^{-\sum_{j=1}^{\ell} 2^{j-1}}\|\Phi_\varepsilon\|^{2^\ell\Gamma_{k,\tau_1\ldots\tau_\ell}}_{\Gamma_{k,\tau_1\ldots\tau_\ell};\,\lambda_{k+N-\ell+1}}. \tag{5.51}$$

We conclude (5.51) by using (5.50), combined with the following inequality

$$\Gamma_k^{2n}\left(\prod_{j=2}^{\ell}\Gamma^{2^j n}_{k,\tau_1\ldots\tau_{j-1}}\right)2^{\sum_{j=1}^{\ell}2^j(k+N-j+3)p_{\tau_j}} \leq \bar{C}^{2^\ell} q^{n\sum_{j=1}^{\ell}2^j} \tag{5.52}$$

for some constant $\bar{C}=\bar{C}(n,\overrightarrow{p})>0$. We next prove (5.52). Since $\ell>k$, we get that

$$\sum_{j=1}^{\ell} 2^j(k+N-j+3) = 2^{\ell+1}(k+N-\ell+4)-2(k+N+4) < 2^{\ell+1}(N+4). \tag{5.53}$$

From the inequalities in (5.42) and (5.44), we derive that

$$\Gamma_k < 2^{k+N}q \text{ and } \Gamma_{k,\tau_1\ldots\tau_{j-1}} < 2^{k+N-j+1}q \text{ for every } j=2,\ldots,\ell.$$

It follows that

$$\Gamma_k^{2n}\prod_{j=2}^{\ell}\Gamma^{2^j n}_{k,\tau_1\ldots\tau_{j-1}} \leq 2^{n\sum_{j=1}^{\ell}2^j(k+N-j+1)} q^{n\sum_{j=1}^{\ell}2^j}.$$

This, jointly with (5.53), proves (5.52). Hence, the assertion of (5.51) holds.

Proof of (5.39) *concluded.* Since $\Gamma_{k,\tau_1\ldots\tau_\ell}<q-1$ (see (5.49)), by using (3.11), we find that $\|\Phi_\varepsilon\|_{\Gamma_{k,\tau_1\ldots\tau_\ell};\,\lambda_{k+N-\ell+1}}$ is dominated by

$$q^{\frac{1}{\Gamma_{k,\tau_1\ldots\tau_\ell}}}\left(\text{meas}\,(\mathcal{A}_r(\lambda_{k+N-\ell+1}))\right)^{\frac{q-\Gamma_{k,\tau_1\ldots\tau_\ell}}{q\Gamma_{k,\tau_1\ldots\tau_\ell}}}\|\Phi_\varepsilon\|_{L^{q,\infty}(\mathcal{A}_r(\lambda_{k+N-\ell+1}))}.$$

Hence, using Lemma 3.3(2), for some constant $C=C(n,\overrightarrow{p})\geq 1$, we have

$$\|\Phi_\varepsilon\|_{\Gamma_{k,\tau_1\ldots\tau_\ell};\,\lambda_{k+N-\ell+1}} \leq q^{\frac{1}{\Gamma_{k,\tau_1\ldots\tau_\ell}}}(Cr)^{\frac{q-\Gamma_{k,\tau_1\ldots\tau_\ell}}{q\Gamma_{k,\tau_1\ldots\tau_\ell}}}Z_q, \tag{5.54}$$

where we denote $Z_q := Cq+(\text{meas}\,(\Omega))^{1+\frac{1}{q}}$. Using (5.54) into (5.51), we conclude that

$$\|\Phi_\varepsilon\|_{\Gamma_k;\lambda_{k+N+1}} \leq (Z_q)^{\frac{2^\ell}{\Gamma_k}\Gamma_{k,\tau_1\ldots\tau_\ell}} C^{\frac{2^\ell}{\Gamma_k}\left(2-\frac{\Gamma_{k,\tau_1\ldots\tau_\ell}}{q}\right)} r^{\frac{q-2^\ell\Gamma_{k,\tau_1\ldots\tau_\ell}}{q\Gamma_k}} q^{\frac{2^\ell(2n+1)-2n}{\Gamma_k}}. \tag{5.55}$$

From $k < \ell \le k + N$ and the choice of Γ_k in (5.42), we obtain that

$$2^{-N}/q < 2^\ell/\Gamma_k < 2^{N+1}/q. \tag{5.56}$$

Furthermore, in view of the estimates in (5.44), we also find that

$$\frac{2^\ell \Gamma_{k,\tau_1\ldots\tau_\ell}}{\Gamma_k} = 1 + \frac{G_k}{q}, \text{ where } G_k \text{ satisfies } -2^{N+2}n \le G_k \le 2^{N+2}p_n. \tag{5.57}$$

We now pass to a subsequence of G_k, still denoted by G_k, for which $\lim_{k\to\infty} G_k = \alpha$ for some $\alpha \in [-2^{N+2}n, 2^{N+2}p_n]$. Then using (5.56) and (5.57) in (5.55), we arrive at

$$\|\Phi_\varepsilon\|_{\Gamma_k;\lambda_{k+N+1}} \le (Z_q)^{1+\frac{2^{N+2}p_n}{q}} C^{\frac{2^{N+2}-1}{q}+\frac{2^{N+2}n}{q^2}} q^{\frac{2^{N+1}(2n+1)}{q}-\frac{2n}{\Gamma_k}} r^{\frac{1}{\Gamma_k}-\frac{1}{q}-\frac{G_k}{q^2}}.$$

Since $\lim_{k\to\infty} \Gamma_k = \infty$ and $\lim_{k\to\infty} \lambda_k = 1/2$, by letting $k \to \infty$, we obtain that

$$\|\Phi_\varepsilon\|_{L^\infty(\mathcal{A}_r(1/2))} \le (Z_q)^{1+\frac{2^{N+2}p_n}{q}} C^{\frac{2^{N+2}}{q}(1+\frac{n}{q})} q^{\frac{2^{N+1}(2n+1)}{q}} r^{-\frac{1}{q}-\frac{\alpha}{q^2}}. \tag{5.58}$$

Set $C_0 := \max\{2n+1, 2(p_n - n + 1)\}$. Recall that $q \ge C_0$ is arbitrarily fixed. Hence, using (5.58), there exists a positive constant $C' = C'(n, \overrightarrow{p})$ such that

$$\|\Phi_\varepsilon\|_{L^\infty(\mathcal{A}_r(1/2))} \le C' \left[q + (\operatorname{meas}(\Omega))^{1+\frac{1+2^{N+2}p_n}{q}+\frac{2^{N+2}p_n}{q^2}} \right] r^{-\frac{1}{q}-\frac{\alpha}{q^2}} \tag{5.59}$$

for any $q \ge C_0$. Finally, by choosing $q = C_0 \max\{|\ln r|, 1\}$ in (5.59), we conclude that

$$\|\Phi_\varepsilon\|_{L^\infty(\mathcal{A}_r(1/2))} \le a + b|\ln r|,$$

where $a = a(n, \overrightarrow{p}, \operatorname{meas}(\Omega))$ and $b = b(n, \overrightarrow{p})$ are positive constants. Hence, the assertion of (5.39) holds, which completes the proof of Proposition 5.6. □

A Weighted anisotropic Sobolev inequalities

Here, as in the whole paper, we continue to assume (1.2). In Lemma A.1, we establish a weighted version of Troisi's [43] anisotropic Sobolev inequality. For the use in the proof of Proposition 1.3, we make explicit an estimate of the constant with respect to the parameter ξ. As before, p is given by (1.3). We prove the following result.

Lemma A.1. *Let Ω be an open subset of $\mathbb{R}^n$ with $n \ge 2$. Let $p \le n$ and $\xi > 1 - p_1/p$.*

If $\varphi \in L^\infty(\Omega)$ satisfies $\varphi|\varphi|^{\min\left\{0, \frac{(\xi-1)p}{p_1}\right\}} \in W^{1,\overrightarrow{p}}(\Omega)$ and $\operatorname{supp}\varphi$ is a compact subset of Ω, then there exists a positive constant C, depending only on n and $\overrightarrow{p}$, such that:

(i) If $p < n$ and $p^* := np/(n-p)$, then we have

$$\|\varphi\|^\xi_{L^{p^*\xi}(\Omega)} \le C\xi \prod_{i=1}^n \left\| |\varphi|^{\frac{p(\xi-1)}{p_i}} \partial_i \varphi \right\|^{1/n}_{L^{p_i}(\operatorname{supp}\varphi)} < \infty. \tag{A.1}$$

(ii) If $p = n$, then for all $q > n/(n-1)$, it holds

$$\|\varphi\|^{\xi}_{L^{q\xi}(\Omega)} \le Cq\xi \left(\operatorname{meas}(\operatorname{supp}\varphi)\right)^{\frac{1}{q}} \prod_{i=1}^{n} \left\| |\varphi|^{\frac{n(\xi-1)}{p_i}} \partial_i\varphi \right\|^{1/n}_{L^{p_i}(\operatorname{supp}\varphi)} < \infty. \tag{A.2}$$

Proof. We first show that each side in the inequalities of (A.1) and (A.2) is well-defined. Since $\varphi \in L^\infty(\Omega)$ and $\operatorname{supp}\varphi$ is a compact subset of Ω, we have $\varphi \in L^\tau(\Omega)$ for all $\tau > 0$. We next introduce α_i and σ_i such that

$$\alpha_i := \sigma_i + 1 + p(\xi-1)/p_i \;\text{ with } \sigma_i \ge 0 \;\text{ for } i = 1, \ldots, n. \tag{A.3}$$

From $\xi > 1 - p_1/p$, we see that $1 + p(\xi-1)/p_i > 0$ and thus $\alpha_i > 0$ for $i = 1, \ldots, n$. The right-hand side of (A.1) and (A.2), respectively is well-defined by proving the following.

CLAIM 1: *We have $\varphi \in W^{1,\vec{p}}(\Omega)$ and $\partial_i(\varphi|\varphi|^{\alpha_i-1}) \in L^{p_i}(\Omega)$ for $i = 1, \ldots, n$ with*

$$\partial_i(\varphi|\varphi|^{\alpha_i-1}) = \alpha_i|\varphi|^{\alpha_i-1}\partial_i\varphi \quad \text{on } \operatorname{supp}\varphi. \tag{A.4}$$

We prove this claim. For $\theta > 1$ and $M > 0$, let $G_{\theta,M} = G \in C^1(\mathbb{R})$ be such that $G(t) = t|t|^{\theta-1}$ for $|t| \le M$. We also assume that there exists $C > 0$ such that $|G'(s)| \le C$ for every $s \in \mathbb{R}$. By adapting the proof of Proposition 9.5 in Brezis [13, p. 270], we deduce that if $u \in W^{1,\vec{p}}(\Omega) \cap L^\infty(\Omega)$, then for $\theta > 1$ and $M = \|u\|_{L^\infty(\Omega)}$, we have

$$G \circ u \in W^{1,\vec{p}}(\Omega) \;\text{ and }\; \partial_i(G \circ u) = (G' \circ u)\, \partial_i u \quad \text{for } i = 1, \ldots, n. \tag{A.5}$$

To establish Claim 1, we distinguish two cases: If $\xi \ge 1$, then $\varphi \in W^{1,\vec{p}}(\Omega)$ by our assumption and we conclude the claim by using (A.5) with $u = \varphi$ and $\theta = \alpha_i$ for each $i = 1, \ldots, n$. If $\xi < 1$, then Claim 1 follows from (A.5) with $u = \varphi|\varphi|^{(\xi-1)p/p_1}$ and $\theta = 1/(1 + (\xi-1)p/p_1)$, respectively $\theta = \alpha_i/(1 + (\xi-1)p/p_1)$ for $i = 1, \ldots, n$.

CLAIM 2: *If α_i is given by* (A.3) *and $p_i' = p_i/(p_i - 1)$ for every $i = 1, \ldots, n$, then*

$$\left(\int_\Omega |\varphi(x)|^{\frac{\sum_{i=1}^n \alpha_i}{n-1}}\, dx\right)^{n-1} \le \prod_{i=1}^{n} \left(\alpha_i \left\| |\varphi|^{\frac{p(\xi-1)}{p_i}} \partial_i\varphi \right\|_{L^{p_i}(\operatorname{supp}\varphi)} \left\| |\varphi|^{\sigma_i} \right\|_{L^{p_i'}(\Omega)}\right). \tag{A.6}$$

Indeed, by Hölder's inequality, we obtain that

$$\int_{\operatorname{supp}\varphi} |\varphi|^{\alpha_i-1}|\partial_i\varphi|\, dx \le \left\| |\varphi|^{\sigma_i} \right\|_{L^{p_i'}(\Omega)} \left\| |\varphi|^{\frac{p(\xi-1)}{p_i}} \partial_i\varphi \right\|_{L^{p_i}(\operatorname{supp}\varphi)}. \tag{A.7}$$

To conclude (A.6), it suffices to show that

$$\left(\int_\Omega |\varphi(x)|^{\frac{\sum_{i=1}^n \alpha_i}{n-1}}\, dx\right)^{n-1} \le \prod_{i=1}^{n} \left(\alpha_i \int_{\operatorname{supp}\varphi} |\varphi|^{\alpha_i-1}|\partial_i\varphi|\, dx\right). \tag{A.8}$$

In what follows, we extend φ by 0 outside Ω. As in Corollary 8.11 in Brezis [13, p. 215], we deduce that $t \longmapsto |\varphi(x_1, \ldots, x_{i-1}, t, x_{i+1}, \ldots, x_n)|^{\alpha_i}$ belongs to the usual Sobolev space $W^{1,p_i}(\mathbb{R})$ for $i=1,\dots,n$. Since $\operatorname{supp}\varphi$ is a compact subset of Ω, for a.e. $x\in\mathbb{R}^n$, we have the representation (see Theorem 8.2 in [13, p. 204])

$$|\varphi(x)|^{\alpha_i}=\int_{-\infty}^{x_i}\partial_i\left(|\varphi(x_1,\dots,x_{i-1},t,x_{i+1},\dots,x_n)|^{\alpha_i}\right)\,dt\le\int_{\mathbb{R}}|\partial_i(|\varphi|^{\alpha_i})|\,dx_i$$

for all $i=1,\dots,n$. For simplicity, let $\hat{x}_i=(x_1,\dots,x_{i-1},x_{i+1},\dots,x_n)$ and $V_i(\hat{x}_i)$ denote

$$V_i(\hat{x}_i):=\int_{\mathbb{R}}|\partial_i(|\varphi|^{\alpha_i})|\,dx_i=\alpha_i\int_{\mathbb{R}}|\varphi|^{\alpha_i-1}|\partial_i\varphi|\,dx_i\quad\text{for } i=1,\dots,n.$$

Therefore, we find that

$$|\varphi(x)|^{\frac{\sum_{i=1}^n\alpha_i}{n-1}}\le\prod_{i=1}^n\left(V_i(\hat{x}_i)\right)^{\frac{1}{n-1}}\quad\text{for all } x\in\mathbb{R}^n. \tag{A.9}$$

If $n=2$, then by integrating (A.9) with respect to x_1, then x_2, we obtain (A.8).

If $n\ge3$, then we integrate the inequality (A.9) successively with respect to the variables x_1, $x_2,\dots,x_n$, using each time the generalized Hölder inequality

$$\int_{\mathbb{R}}\prod_{i=1}^{n-1}f_i(t)\,dt\le\prod_{i=1}^{n-1}\|f_i\|_{L^{n-1}(\mathbb{R})},\quad\text{where } f_i\in L^{n-1}(\mathbb{R})\text{ for every } i=1,\dots,n-1.$$

We thus obtain that

$$\begin{aligned}\int_{\mathbb{R}}|\varphi(x)|^{\frac{\sum_{i=1}^n\alpha_i}{n-1}}\,dx_1&\le\left(V_1(\hat{x}_1)\right)^{\frac{1}{n-1}}\int_{\mathbb{R}}\prod_{i=2}^n\left(V_i(\hat{x}_i)\right)^{\frac{1}{n-1}}\,dx_1\\&\le\left(V_1(\hat{x}_1)\right)^{\frac{1}{n-1}}\prod_{i=2}^n\left[\int_{\mathbb{R}}V_i(\hat{x}_i)\,dx_1\right]^{\frac{1}{n-1}}.\end{aligned}$$

A similar integration over the variables $x_2,\dots,x_n$ leads to (A.8). This proves Claim 2.

Proof of Lemma A.1 concluded. Let $q>n/(n-1)$ if $p=n$. Choose $\sigma_i>0$ such that

$$p_i'\sigma_i=\alpha:=\begin{cases}\xi p^* & \text{if } p<n,\\ \xi[q-n/(n-1)] & \text{if } p=n.\end{cases} \tag{A.10}$$

With this choice of σ_i, let α_i be given by (A.3) for $i=1,\dots,n$. Notice that

$$\prod_{i=1}^n\left\||\varphi|^{\sigma_i}\right\|_{L^{p_i'}(\Omega)}=\left(\int_\Omega|\varphi|^\alpha\,dx\right)^{\frac{n(p-1)}{p}}. \tag{A.11}$$

By (A.10), there exists a constant $c=c\,(n,\overrightarrow{p})>0$ such that for all $i=1,\dots,n$

$$1+p(\xi-1)/p_i<\alpha_i\le\begin{cases}c\,\xi & \text{if } p<n,\\ c\,q\xi & \text{if } p=n.\end{cases} \tag{A.12}$$

Proof of (i). Let $p < n$. Since $\sum_{i=1}^n \alpha_i/(n-1) = p^*\xi$, from (A.6) and (A.11), we have

$$\left(\int_\Omega |\varphi(x)|^{p^*\xi}\,dx\right)^{\frac{1}{p^*}} \leq \prod_{i=1}^n \left(\alpha_i \left\| |\varphi|^{p(\xi-1)/p_i}\,\partial_i\varphi\right\|_{L^{p_i}(\operatorname{supp}\varphi)}\right)^{\frac{1}{n}}. \tag{A.13}$$

From (A.12) and (A.13), we conclude the assertion of (A.1).

Proof of (ii). Let $p = n$. Since $\sum_{i=1}^n \alpha_i/(n-1) = q\xi$, Hölder's inequality gives that

$$\int_\Omega |\varphi(x)|^\alpha\,dx \leq \left(\int_\Omega |\varphi(x)|^{\frac{\sum_{i=1}^n \alpha_i}{n-1}}\,dx\right)^{\frac{\alpha}{q\xi}} (\operatorname{meas}(\operatorname{supp}\varphi))^{1-\frac{\alpha}{q\xi}}. \tag{A.14}$$

From (A.10), it holds $q - \alpha/\xi = n/(n-1)$. Then, (A.6), (A.11) and (A.14) imply that

$$\|\varphi\|_{L^{q\xi}(\Omega)}^\xi \leq (\operatorname{meas}(\operatorname{supp}\varphi))^{\frac{1}{q}} \prod_{i=1}^n \left(\alpha_i \left\| |\varphi|^{\frac{n(\xi-1)}{p_i}}\,\partial_i\varphi\right\|_{L^{p_i}(\operatorname{supp}\varphi)}\right)^{\frac{1}{n}}. \tag{A.15}$$

Using (A.12) and (A.15), we reach (A.2). This ends the proof of Lemma A.1. □

B Auxiliary derivations

A crucial tool in the proof of Lemma 3.3(2) is the anisotropic Sobolev inequality (B.1) of Lemma B.1 below. The main step in proving it is an inequality of Moser–Trudinger-type ([36, 44]), whose derivation relies essentially on Theorem 1 in Cianchi [14].

Lemma B.1. *Let Ω be a bounded open set in $\mathbb{R}^n$ with $n \geq 2$. If (1.2) and $p = n$ hold, then there exists a positive constant C, depending only on n and $\overrightarrow{p}$, such that*

$$\|u\|_{L^{\frac{qn}{n-1}}(\Omega)}^n \leq C q^{n-1} \left(\operatorname{meas}(\Omega) + \sum_{i=1}^n \int_\Omega |\partial_i u|^{p_i}\,dx\right)^{1+\frac{n-1}{q}} \tag{B.1}$$

for all $q \geq 1$ and all $u \in W_0^{1,\overrightarrow{p}}(\Omega)$.

Proof. Define $A(\xi) := \max\{\sum_{i=1}^n |\xi_i|, \sum_{i=1}^n |\xi_i|^{p_i}\}$ for every $\xi = (\xi_1,\ldots,\xi_n) \in \mathbb{R}^n$.

We split the proof into two steps:

Step 1. *There exists a constant $L = L(n, \overrightarrow{p}) > 0$ such that for every $u \in W_0^{1,\overrightarrow{p}}(\Omega)$*

$$\int_\Omega e^{\frac{|u(x)|^{n/(n-1)}}{L(\int_\Omega A(\nabla u)\,dx)^{1/(n-1)}}}\,dx \leq e^n \left(\operatorname{meas}(\Omega) + \int_\Omega A(\nabla u)\,dx\right). \tag{B.2}$$

For ease of reference, we shall use the notation

$$[A \leq t] := \{\xi \in \mathbb{R}^n : A(\xi) \leq t\} \quad \text{for every } t > 0.$$

Note that $A : \mathbb{R}^n \to [0,\infty)$ is a convex function satisfying (1.1) and (1.2) in [14], namely:

1. $A(0)=0$ and $A(\xi)=A(-\xi)$ for every $\xi\in\mathbb{R}^n$;
2. For every $t>0$, the set $[A\leq t]$ is compact, whose interior contains 0.

CLAIM: *There exists a positive constant C_1, depending only on n and $\overrightarrow{p}$, such that*

$$\operatorname{meas}\left([A\leq t]\right)\leq C_1\min\{t,t^n\}\quad\text{for every } t>0. \tag{B.3}$$

We prove this assertion. We define D_n and $F_{n,\overrightarrow{p}}$ as follows

$$D_n:=\operatorname{meas}\left\{\xi\in\mathbb{R}^n:\ \sum_{i=1}^n|\xi_i|\leq 1\right\},\quad F_{n,\overrightarrow{p}}:=\operatorname{meas}\left\{\xi\in\mathbb{R}^n:\ \sum_{i=1}^n|\xi_i|^{p_i}\leq 1\right\}.$$

Clearly, D_n and $F_{n,\overrightarrow{p}}$ are finite. For any $t\in(0,1]$, we have

$$\operatorname{meas}\left([A\leq t]\right)=\operatorname{meas}\left\{\xi\in\mathbb{R}^n:\ \sum_{i=1}^n|\xi_i|\leq t\right\}=D_nt^n. \tag{B.4}$$

Hence, (B.3) holds for every $t\in(0,1]$. To conclude (B.3), we show that

$$\operatorname{meas}\left([A\leq t]\right)\leq E_{n,\overrightarrow{p}}+F_{n,\overrightarrow{p}}\,t\quad\text{for every } t>1, \tag{B.5}$$

where we define $E_{n,\overrightarrow{p}}$ by

$$E_{n,\overrightarrow{p}}:=\operatorname{meas}\left\{\xi\in\mathbb{R}^n:\ A(\xi)=\sum_{i=1}^n|\xi_i|\right\}. \tag{B.6}$$

We show that $E_{n,\overrightarrow{p}}<\infty$. If we assume that there exists a sequence $(\xi_k)_{k\geq1}$ in $E_{n,\overrightarrow{p}}$ with $\lim_{k\to\infty}|\xi_k|=\infty$, then $\lim_{k\to\infty}\max_{i=1,\dots,n}|\xi_{ki}|=\infty$, where $\xi_k=(\xi_{k1},\dots,\xi_{kn})$ for every $k\geq1$. Hence, there exists $k_1\geq1$ such that $\max_{i=1,\dots,n}|\xi_{ki}|^{p_i-1}>n$ for every $k>k_1$, which gives that

$$\sum_{i=1}^n|\xi_{ki}|^{p_i}\geq\max_{i=1,\dots,n}|\xi_{ki}|^{p_i}>n\max_{i=1,\dots,n}|\xi_{ki}|\geq\sum_{i=1}^n|\xi_{ki}|\quad\text{for all } k>k_1.$$

This contradicts that $\xi_k\in E_{n,\overrightarrow{p}}$ for all $k\geq1$. We thus conclude that $E_{n,\overrightarrow{p}}<\infty$.

Since $p=n$, we observe that for every $t>0$, it holds

$$\operatorname{meas}\left\{\xi\in\mathbb{R}^n:\ \sum_{i=1}^n|\xi_i|^{p_i}\leq t\right\}=F_{n,\overrightarrow{p}}\,t^{\sum_{i=1}^n1/p_i}=F_{n,\overrightarrow{p}}\,t. \tag{B.7}$$

By the definition of $A(\xi)$, we have that $\operatorname{meas}\left([A\leq t]\right)$ is bounded from above by

$$\operatorname{meas}\left\{\xi\in\mathbb{R}^n:\ A(\xi)=\sum_{i=1}^n|\xi_i|\leq t\right\}+\operatorname{meas}\left\{\xi\in\mathbb{R}^n:\ \sum_{i=1}^n|\xi_i|^{p_i}\leq t\right\}.$$

Hence, using (B.6) and (B.7), we conclude (B.5). This proves the claim of (B.3).

Let ω_n denote the measure of the n-dimensional unit ball. We define $A_*(s)$ by

$$A_*(s) := \sup\{t > 0 : \quad \text{meas}\,([A \le t]) < \omega_n s^n\} \quad \text{for } s \ge 0.$$

From (B.4), we infer that $A_*(s) = (\omega_n/D_n)^{1/n} s$ for every $0 \le s \le (D_n/\omega_n)^{1/n}$. Consequently, A_* satisfies (1.3) in [14], that is

$$\int_{0^+} \left(\frac{t}{A_*(t)}\right)^{\frac{1}{n-1}} dt < \infty.$$

Using (B.3), we conclude that for every $s > 0$, it holds

$$A_*(s) \ge g(s), \quad \text{where } g(s) := \sup\{t > 0 : \ C_1 \min\{t, t^n\} < \omega_n s^n\}. \tag{B.8}$$

Following Cianchi [14], we define $H : [0, \infty) \to [0, \infty)$ by

$$H(r) := \left(\int_0^r \left(\frac{t}{A_*(t)}\right)^{\frac{1}{n-1}} dt\right)^{\frac{n-1}{n}} \quad \text{for } r \ge 0. \tag{B.9}$$

Let B be given by $B = A_* \circ H^{-1}$, where H^{-1} denotes the inverse of H. By Theorem 1 in [14], there exists a positive constant K, depending only on n, such that

$$\int_{\mathbb{R}^n} B\left(\frac{|u(x)|}{K\left(\int_{\mathbb{R}^n} A(\nabla u)\, dy\right)^{1/n}}\right) dx \le \int_{\mathbb{R}^n} A(\nabla u)\, dx \tag{B.10}$$

for every real-valued weakly differentiable function u on $\mathbb{R}^n$ decaying to 0 at infinity.

We next show that if $C_2 := (C_1/\omega_n)^{1/n}$, then

$$B(\tau) \ge e^{n[(\tau/C_2)^{\frac{n}{n-1}} - 1]} \quad \text{if } \tau \ge C_2. \tag{B.11}$$

Indeed, by the definition of g in (B.8), we have $g(s) = \max\{s/C_2, (s/C_2)^n\} = (s/C_2)^n$ if $s \ge C_2$ and $g(s) = s/C_2$ if $0 < s < C_2$. This, jointly with (B.9), implies that

$$H(r) \le C_2 \left[\ln(e\, r/C_2)\right]^{\frac{n-1}{n}} \quad \text{if } r \ge C_2.$$

Hence, we find that $H^{-1}(\tau) \ge C_2 e^{(\tau/C_2)^{\frac{n}{n-1}} - 1} \ge C_2$ if $\tau \ge C_2$. Using (B.8), we get that

$$B(\tau) = A_*(H^{-1}(\tau)) \ge g(H^{-1}(\tau)) = (H^{-1}(\tau)/C_2)^n \ge e^{n[(\tau/C_2)^{\frac{n}{n-1}} - 1]} \quad \text{if } \tau \ge C_2.$$

This proves (B.11). Let $u \in W_0^{1,\vec{p}}(\Omega)$ and K as in (B.10). We define v as follows

$$v(x) := \frac{u(x)}{C_2 K \left(\int_\Omega A(\nabla u)\, dy\right)^{1/n}} \quad \text{for } x \in \Omega.$$

From (B.10) and (B.11), we deduce that

$$\int_{\{x\in\Omega:\ |v(x)|\geq 1\}} e^{n\left[|v(x)|^{\frac{n}{n-1}}-1\right]}\,dx \leq \int_\Omega A(\nabla u)\,dx.$$

Consequently, we have

$$\int_\Omega e^{n\left[|v(x)|^{\frac{n}{n-1}}-1\right]}\,dx \leq \operatorname{meas}(\Omega) + \int_\Omega A(\nabla u)\,dx,$$

which proves (B.2) with the constant $L = L(n, \overrightarrow{p})$ given by $L = (1/n)(C_2K)^{n/(n-1)}$.

Step 2. *Proof of* (B.1) *concluded.*

Set $M_0 := \max\{\sum_{i=1}^n |\xi_i| \ :\ \xi \in E_{n,\overrightarrow{p}}\}$, where $E_{n,\overrightarrow{p}}$ is given by (B.6). Using that $A(\xi) = \max\{\sum_{i=1}^n |\xi_i|, \sum_{i=1}^n |\xi_i|^{p_i}\}$, we get an upper bound estimate for $\int_\Omega A(\nabla u)\,dx$:

$$\begin{aligned}\int_\Omega A(\nabla u)\,dx &\leq \int_{\{x\in\Omega:\ \sum_{i=1}^n |\partial_i u| > \sum_{i=1}^n |\partial_i u|^{p_i}\}} \sum_{i=1}^n |\partial_i u|\,dx + \int_\Omega \sum_{i=1}^n |\partial_i u|^{p_i}\,dx \\ &\leq M_0 \operatorname{meas}(\Omega) + \sum_{i=1}^n \int_\Omega |\partial_i u|^{p_i}\,dx.\end{aligned} \tag{B.12}$$

For $u \in W_0^{1,\overrightarrow{p}}(\Omega)$, we define $\mathcal{P}_u$ as follows

$$\mathcal{P}_u := \operatorname{meas}(\Omega) + \sum_{i=1}^n \int_\Omega |\partial_i u|^{p_i}\,dx. \tag{B.13}$$

Using (B.12) in (B.2), we find positive constants M and C_3, depending only n and $\overrightarrow{p}$, such that for every positive integer k, we have

$$\frac{1}{k!}\int_\Omega \frac{|u(x)|^{\frac{kn}{n-1}}}{M^k\left(\mathcal{P}_u\right)^{\frac{k}{n-1}}}\,dx \leq \int_\Omega e^{(1/M)|u(x)|^{\frac{n}{n-1}}(\mathcal{P}_u)^{-\frac{1}{n-1}}}\,dx \leq C_3\,\mathcal{P}_u.$$

Consequently, we arrive at

$$\|u\|^n_{L^{\frac{kn}{n-1}}(\Omega)} \leq C_3^{\frac{n-1}{k}} M^{n-1}(k!)^{\frac{n-1}{k}} \left(\mathcal{P}_u\right)^{1+\frac{n-1}{k}}.$$

Hence, by Stirling's formula, there exists a positive constant $C_4 = C_4(n, \overrightarrow{p})$, such that

$$\|u\|^n_{L^{\frac{kn}{n-1}}(\Omega)} \leq C_4 k^{n-1} \left(\mathcal{P}_u\right)^{1+\frac{n-1}{k}} \tag{B.14}$$

for every positive integer k. Let $q \geq 1$ be arbitrary. We define $\theta := \lfloor q\rfloor(\lfloor q\rfloor + 1 - q)/q$, where $\lfloor q\rfloor$ denotes the integer part of q. Since $1/q = \theta/\lfloor q\rfloor + (1-\theta)/(\lfloor q\rfloor + 1)$, by using an interpolation inequality and (B.14), we conclude that

$$\begin{aligned}\|u\|^n_{L^{\frac{qn}{n-1}}(\Omega)} &\leq \|u\|^{n\theta}_{L^{\frac{\lfloor q\rfloor n}{n-1}}(\Omega)} \|u\|^{n(1-\theta)}_{L^{\frac{(\lfloor q\rfloor+1)n}{n-1}}(\Omega)} \\ &\leq C_4 \lfloor q\rfloor^{(n-1)\theta}(\lfloor q\rfloor + 1)^{(n-1)(1-\theta)} \left(\mathcal{P}_u\right)^{1+\frac{n-1}{q}} \leq C q^{n-1} \left(\mathcal{P}_u\right)^{1+\frac{n-1}{q}}\end{aligned}$$

for some positive constant $C = C(n, \overrightarrow{p})$. This, jointly with (B.13), proves (B.1). □

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References

[1] Antontsev, S. N., Díaz, J. I., Shmarev, S. (2002). *Energy Methods for Free Boundary Problems. Applications to Nonlinear PDEs and Fluid Mechanics.* In: Progress in Nonlinear Differential Equations and Their Applications. vol. 48, Boston: Birkhäuser.

[2] Antontsev, S., Shmarev, S. (2006). Elliptic equations and systems with nonstandard growth conditions: existence, uniqueness and localization properties of solutions. *Nonlinear Anal.* 65:728–761.

[3] Armstrong, S. N., Sirakov, B., Smart, C. K. (2011). Fundamental solutions of homogeneous fully nonlinear elliptic equations. *Comm. Pure Appl. Math.* 64:737–777.

[4] Bendahmane, M., Karlsen, K. H. (2006). Renormalized solutions of an anisotropic reaction-diffusion-advection system with L^1 data. *Commun. Pure Appl. Anal.* 5:733–762.

[5] Bendahmane, M., Karlsen, K. H. (2006). Anisotropic nonlinear elliptic systems with measure data and anisotropic harmonic maps into spheres. *Electron. J. Differential Equations*, no. 46, 30 pp.

[6] Bendahmane, M., Langlais, M., Saad, M. (2003). On some anisotropic reaction-diffusion systems with L^1-data modeling the propagation of an epidemic disease. *Nonlinear Anal.* 54:617–636.

[7] Bénilan, P., Boccardo, L., Gallouët, T., Gariepy, R., Pierre, M., Vázquez, J. L. (1995). An L^1-theory of existence and uniqueness of solutions of nonlinear elliptic equations. *Ann. Sc. Norm. Super. Pisa Cl. Sci.* (4) 22:241–273.

[8] Boccardo, L., Gallouët, T. (1989). Nonlinear elliptic and parabolic equations involving measure data. *J. Funct. Anal.* 87:149–169.

[9] Boccardo, L., Gallouët, T. (1992). Nonlinear elliptic equations with right-hand side measures. *Comm. Part. Diff. Eqs.* 17:641–655.

[10] Boccardo, L., Gallouët, T., Marcellini, P. (1996). Anisotropic equations in L^1. *Diff. Int. Eqs.* 9:209–212.

[11] Boccardo, L., Marcellini, P., Sbordone, C. (1990). L^∞-regularity for variational problems with sharp nonstandard growth conditions. *Boll. Un. Mat. Ital. A* (7) 4:219–225.

[12] Brandolini, B., Chiacchio, F., Cîrstea, F. C., Trombetti, C. (2013). Local behaviour of singular solutions for nonlinear elliptic equations in divergence form. *Calc. Var. Part. Diff. Eqs.* 48:367–393.

[13] Brezis, H. (2011). *Functional analysis, Sobolev spaces and partial differential equations.* Universitext. New York: Springer.

[14] Cianchi, A. (2000). A fully anisotropic Sobolev inequality. *Pacific J. Math.* 196:283–295.

[15] Cianchi, A. (2007). Symmetrization in anisotropic elliptic problems. *Comm. Part. Diff. Eqs.* 32:693–717.

[16] Cîrstea, F. C. (2014). A complete classification of the isolated singularities for nonlinear elliptic equations with inverse square potentials. *Mem. Amer. Math. Soc.* 227, no. 1068.

[17] Cupini, G., Marcellini, P., Mascolo, E. (2012). Local boundedness of solutions to quasilinear elliptic systems. *Manuscripta Math.* 137:287–315.

[18] Dall'Acqua, A., Sweers, G. (2004). Estimates for Green function and Poisson kernels of higher-order Dirichlet boundary value problems. *J. Diff. Eqs.* 205:466–487.

[19] Dall'Aglio, A. (1996). Approximated solutions of equations with L^1 data. Application to the H-convergence of quasi-linear parabolic equations. *Ann. Mat. Pura Appl.* (4) 170:207–240.

[20] Druet, O., Hebey, E., Robert, F. (2004). *Blow-up theory for elliptic PDEs in Riemannian Geometry.* Mathematical Notes, vol. 45. Princeton, NJ: Princeton University Press.

[21] El Hamidi, A., Rakotoson, J. M. (2007). Extremal functions for the anisotropic Sobolev inequalities. *Ann. Inst. H. Poincaré Anal. Non Linéaire* 24:741–756.

[22] El Hamidi, A., Vétois, J. (2009). Sharp Sobolev asymptotics for critical anisotropic equations. *Arch. Ration. Mech. Anal.* 192:1–36.

[23] Felmer, P. L., Quaas, A. (2009). Fundamental solutions and two properties of elliptic maximal and minimal operators. *Trans. Amer. Math. Soc.* 361:5721–5736.

[24] Fragalà, I., Gazzola, F., Kawohl, B. (2004). Existence and nonexistence results for anisotropic quasilinear elliptic equations. *Ann. Inst. H. Poincaré Anal. Non Linéaire* 21:715–734.

[25] Fusco, N., Sbordone, C. (1993). Some remarks on the regularity of minima of anisotropic integrals. *Comm. Part. Diff. Eqs.* 18:153–167.

[26] Gilbarg, D., Trudinger, N. S. (1983). *Elliptic Partial Differential Equations of Second Order.* 2nd ed., Grundlehren der Mathematischen Wissenschaften, vol. 224. Berlin: Springer-Verlag.

[27] Grafakos, L. (2008). *Classical Fourier Analysis.* 2nd ed., Graduate Texts in Mathematics, vol. 249. New York: Springer.

[28] Grunau, H.-C., Robert, F. (2010). Positivity and almost positivity of biharmonic Green's functions under Dirichlet boundary conditions. *Arch. Ration. Mech. Anal.* 195:865–898.

[29] Han, Q., Lin, F. (2011). *Elliptic Partial Differential Equations.* 2nd ed., Courant Lecture Notes in Mathematics, vol. 1. New York University, New York: Courant Institute of Mathematical Sciences; Providence, RI: American Mathematical Society.

[30] Kichenassamy, S., Véron, L. (1986). Singular solutions of the p-Laplace equation. *Math. Ann.* 275:599–615.

[31] Krasovskiĭ, Ju. P. (1967). Isolation of the singularity in Green's function. *Izv. Akad. Nauk SSSR Ser. Mat.* 31:977–1010 (in Russian); [English transl.], *Math. USSR* 1:935–966.

[32] Labutin, D. A. (2001). Isolated singularities for fully nonlinear elliptic equations. *J. Diff. Eqs.* 177:49–76.

[33] Lieberman, G. M. (2005). Gradient estimates for anisotropic elliptic equations. *Adv. Diff. Eqs.* 10:767–812.

[34] Lions, J.-L. (1969). *Quelques méthodes de résolution des problèmes aux limites non linéaires.* Dunod, Paris: Gauthier-Villars (in French).

[35] Mingione, G. (2011). Nonlinear measure data problems. *Milan J. Math.* 79:429–496.

[36] Moser, J. (1970/71). A sharp form of an inequality by N. Trudinger. *Indiana Univ. Math. J.* 20:1077–1092.

[37] Namlyeyeva, Y. V., Shishkov, A. E., Skrypnik, I. I. (2006). Isolated singularities of solutions of quasilinear anisotropic elliptic equations. *Adv. Nonl. Stud.* 6:617–641.

[38] Nikol'skiĭ, S. M. (1961). On imbedding, continuation and approximation theorems for differentiable functions of several variables. *Uspehi Mat. Nauk* 16:63–114 (1961) (in Russian); [English transl.], *Russian Mathematical Surveys* 16:55–104.

[39] Rákosník, J. (1979). Some remarks to anisotropic Sobolev spaces. I. *Beiträge Anal.* No. 13, 55–68.

[40] Serrin, J. (1964). Local behavior of solutions of quasi-linear equations. *Acta Math.* 111:247–302.

[41] Serrin, J. (1965). Isolated singularities of solutions of quasi-linear equations. *Acta Math.* 113:219–240.

[42] Stroffolini, B. (1991). Global boundedness of solutions of anisotropic variational problems. *Boll. Un. Mat. Ital. A* (7) 5:345–352.

[43] Troisi, M. (1969). Teoremi di inclusione per spazi di Sobolev non isotropi. *Ricerche Mat.* 18:3–24 (in Italian).

[44] Trudinger, N. S. (1967). On imbeddings into Orlicz spaces and some applications. *J. Math. Mech.* 17:473–483.

[45] Trudinger, N. S., Wang, X.-J. (2002). Hessian measures. III. *J. Funct. Anal.* 193:1–23.

[46] Trudinger, N. S., Wang, X.-J. (2009). Quasilinear elliptic equations with signed measure data. *Discrete Contin. Dyn. Syst.* 23:477–494.

[47] Véron, L. (1996). *Singularities of Solutions of Second Order Quasilinear Equations.* Pitman Research Notes in Mathematics Series, vol. 353. Harlow, UK: Longman.

[48] Vétois, J. (2012). Strong maximum principles for anisotropic elliptic and parabolic equations. *Adv. Nonl. Stud.* 12:101–114.