SIGN-CHANGING BLOW-UP FOR SCALAR CURVATURE TYPE EQUATIONS

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ABSTRACT. Given (M, g) a compact Riemannian manifold of dimension $n \geq 3$, we are interested in the existence of blowing-up sign-changing families $(u_\varepsilon)_{\varepsilon > 0} \in C^{2,\theta}(M)$, $\theta \in (0, 1)$, of solutions to

$$\Delta_g u_\varepsilon + h u_\varepsilon = |u_\varepsilon|^{\frac{4}{n-2} - \varepsilon} u_\varepsilon \text{ in } M \,,$$

where $\Delta_g := -\operatorname{div}_g(\nabla)$ and $h \in C^{0,\theta}(M)$ is a potential. Assuming the existence of a nondegenerate solution to the limiting equation (which is a generic assumption), we prove that such families exist in two main cases: in small dimension $n \in \{3, 4, 5, 6\}$ for any potential h or in dimension $3 \leq n \leq 9$ when $h \equiv \frac{n-2}{4(n-1)} \operatorname{Scal}_g$. These examples yield a complete panorama of the compactness/noncompactness of critical elliptic equations of scalar curvature type on compact manifolds. The changing of the sign is necessary due to the compactness results of Druet [11] and Khuri–Marques–Schoen [19].

1. INTRODUCTION

Let (M, g) be a smooth compact Riemannian manifold of dimension $n \geq 3$. Given $\theta \in (0, 1)$, we consider solutions $u \in C^{2,\theta}(M)$ to the equation

$$(1) \qquad \Delta_g u + h u = |u|^{2^\star - 2} u \quad \text{in } M \,,$$

where $h \in C^{0,\theta}(M)$, $\Delta_g := -\operatorname{div}_g(\nabla)$ is the Laplace-Beltrami operator, and $2^\star := \frac{2n}{n-2}$. When $h \equiv \frac{n-2}{4(n-1)} \operatorname{Scal}_g$ (Scal_g being the scalar curvature of (M, g)), (1) is the Yamabe equation and rewrites

$$(2) \qquad \Delta_g u + c_n \operatorname{Scal}_g u = |u|^{2^\star - 2} u \quad \text{in } M \,,$$

where $c_n := \frac{n-2}{4(n-1)}$. The conformal invariance of the Yamabe equation induces a dynamic that makes equations (1) and (2) unstable. Taking inspiration from the terminology introduced by R. Schoen [32], we say that equation (1) is compact (resp. positively compact) if for any family $(q_\varepsilon)_\varepsilon \in (2, 2^\star]$ such that $q_\varepsilon \to 2^\star$ when $\varepsilon \to 0$ and for any family of functions (resp. positive functions) $(u_\varepsilon)_\varepsilon \in C^{2,\theta}(M)$ of solutions to

$$(3) \qquad \Delta_g u_\varepsilon + h u_\varepsilon = |u_\varepsilon|^{q_\varepsilon - 2} u_\varepsilon \quad \text{in } M$$

for $\varepsilon > 0$ small, then a uniform bound on the Dirichlet energy $(\|\nabla u_\varepsilon\|_2)_\varepsilon$ implies the relative compactness of $(u_\varepsilon)_\varepsilon$ in $C^2(M)$, and therefore the convergence of a subfamily of $(u_\varepsilon)_\varepsilon$ in $C^2(M)$. Otherwise, we say that equation (1) is noncompact (resp. non positively-compact). A basic example of non compact equation is (2)

on the canonical sphere $(\mathbb{S}^n, \mathrm{can})$: we refer to the second part of this section for (positive) compactness results for equations like (1).

We say that a family $(u_\varepsilon)_{\varepsilon>0} \in C^{2,\theta}(M)$ blows-up when $\varepsilon \to 0$ if $\lim_{\varepsilon\to 0} \|u_\varepsilon\|_\infty = +\infty$. It is now well-known (see Struwe [34] for a description in Sobolev spaces and Druet–Hebey–Robert [14] for a description in C^0) that noncompactness is described by bubbles. In the present paper, we investigate the existence of families $(u_\varepsilon)_\varepsilon \in C^{2,\theta}(M)$ of sign-changing blowing-up solutions to the equation

$$(4) \qquad\qquad \Delta_g u_\varepsilon + h u_\varepsilon = |u_\varepsilon|^{2^\star - 2 - \varepsilon} u_\varepsilon \quad \text{in } M, \, \varepsilon > 0.$$

In the sequel, we say that a blowing-up family $(u_\varepsilon)_\varepsilon \in C^{2,\theta}(M)$ is of type $(u_0 - B)$ if there exists $u_0 \in C^{2,\theta}(M)$ and a bubble $(B_\varepsilon)_\varepsilon$ (see definition (6) below) such that

$$(5) \qquad\qquad u_\varepsilon = u_0 - B_\varepsilon + \mathrm{o}(1),$$

where $\lim_{\varepsilon\to 0} \mathrm{o}(1) = 0$ in $H_1^2(M)$, the completion of $C^\infty(M)$ for the norm $u \mapsto \|u\|_2 + \|\nabla u\|_2$. Our first result is the following:

Theorem 1.1 (dimensions $3 \le n \le 6$ and arbitrary potential). *Let (M, g) be a smooth compact Riemannian manifold of dimension $3 \le n \le 6$ and let $h \in C^{0,\theta}(M)$ ($\theta \in (0, 1)$) be such that $\Delta_g + h$ is coercive. Assume that there exists a nondegenerate solution $u_0 \in C^{2,\theta}(M)$ to equation (1). In case $n = 6$, assume in addition that $c_n \operatorname{Scal}_g - h < 2u_0$ in M. Then for $\varepsilon > 0$ small, equation (4) admits a sign-changing solution u_ε of type $(u_0 - B)$. In particular, the family $(u_\varepsilon)_{\varepsilon>0}$ blows up as $\varepsilon \to 0$ and (1) is noncompact.*

In full generality, it is not possible to construct positive blowing-up solutions to (4). Indeed, in addition to the assumptions of Theorem 1.1, if we assume that $h < c_n \operatorname{Scal}_g$, then (1) is positively compact (Druet [11] and the discussion below), and therefore any family of blowing-up solutions to (4) must be sign-changing. In the early reference [10], Ding proved the existence of infinitely many nonequivalent solutions to (2) on the canonical sphere, highlighting the diversity of the behavior of solutions to (1) depending on whether they are positive or negative.

The nondegeneracy assumption in Theorem 1.1 is necessary. We refer to Proposition 3.1 in Section 3 for the proof of necessity. However, the nondegeneracy assumption of Theorem 1.1 is generic in the sense that any degenerate solution to (1) can be approached by a solution of a slight perturbation of (1). We refer to Proposition 3.2 of Section 3 for the precise genericity statement.

The above theorem outlines a role of the geometry in dimension $n = 6$. In higher dimension $n \ge 7$, the geometry of (M, g) is more present. When the potential is strictly below the scalar curvature (that is $h < c_n \operatorname{Scal}_g$), equation (1) is compact for $n \ge 7$, at least in the locally conformally flat case (Vétois [36]). Conversely, if $h(x_0) > c_n \operatorname{Scal}_g(x_0)$ for some $x_0 \in M$, then under some additional nondegeneracy assumption, equation (1) is non-compact when $n \ge 7$ (see Pistoia–Vétois [29] for general results). Our second result is in the case $h \equiv c_n \operatorname{Scal}_g$:

Theorem 1.2 (dimensions $3 \le n \le 9$ and $h \equiv c_n \operatorname{Scal}_g$). *Let (M, g) be a smooth compact Riemannian manifold of dimension $3 \le n \le 9$ with positive Yamabe invariant. Assume that there exists a nondegenerate positive solution $u_0 > 0$ to the Yamabe equation (2). Assume that $h \equiv c_n \operatorname{Scal}_g$. Then for $\varepsilon > 0$ small, equation (4) admits a sign-changing solution u_ε of type $(u_0 - B)$. In particular, the family $(u_\varepsilon)_{\varepsilon>0}$ blows up as $\varepsilon \to 0$ and (2) is noncompact.*

Constructing positive blowing-up solutions is not possible in this context. Indeed, for $3 \leq n \leq 9$ and except for the canonical sphere, the scalar curvature equation (2) is positively compact (see Li–Zhu [25], Druet [12], Marques [26], Li–Zhang [22, 23], Khuri–Marques–Schoen [19], and the discussion below). We refer also to Druet–Hebey [13] and Druet–Hebey–Vétois [15] for the extension of compactness issues to stability issues.

The restriction of the dimensions in Theorem 1.2 is due to the geometry of the manifold. We refer to Subsection 2.1 in Section 2 for the extension of Theorem 1.2 to dimension $n = 10$ in general and in any dimension in the locally conformally flat case.

Here again, it is natural to ask about the nondegeneracy assumption of a solution to the limit equation in Theorem 1.2: actually, it is both a necessary and a generic assumption. Concerning necessity, on the standard sphere (where all positive solutions to (2) are degenerate), it is not possible to construct blowing-up solutions of type $(u_0 - B)$, see Proposition 3.1 in Section 3. However, it is proved in Khuri–Marques–Schoen [19] that the nondegeneracy assumption is generic for the Yamabe equation (2), at least in dimensions $n \leq 24$, see Proposition 3.3 in Section 3.

Here is a brief overview of the positive compactness results known so far for equations like (1).

In 1987, Schoen [32] adressed the question of positive compactness of equation (2) for manifolds non conformally diffeomorphic to the canonical sphere $(\mathbb{S}^n, \text{can})$ (say aspherical manifolds). The known results are the following: positive compactness holds for aspherical locally conformally flat manifolds (Schoen [32, 33]) and for arbitrary aspherical manifolds of dimension $3 \leq n \leq 24$ (Li–Zhu [25] $(n = 3)$, Druet [12] $(n \leq 5)$, Marques [26] $(n \leq 7)$, Li–Zhang [22–24] $(n \leq 11)$, Kuhri–Marques–Schoen [19] $(n \leq 24)$). But positive compactness does not hold in general in dimension $n \geq 25$ (There are blowing-up examples by Brendle [5] and Brendle–Marques [6]). Combining these results with Theorem 1.1, we get that equation (2) is positively compact, but not compact, at least when $3 \leq n \leq 9$.

When $h \not\equiv c_n \text{Scal}_g$, the situation is different. When $h < c_n \text{Scal}_g$, Druet [11] proved that (3) is positively compact in dimension $n \geq 3$ (see also Li–Zhu [25] and Druet–Hebey–Vétois [15] for $n = 3$). Conversely, in dimension $n \geq 4$, Micheletti-Pistoia–Vétois [27] and Pistoia–Vétois [29] proved that if h is above $c_n \text{Scal}_g$ somewhere, then, under some some nondegeneracy assumption, equation (1) is not positively compact. On the canonical sphere, there are blowing-up positive solutions with arbitrarily high energy when $h \equiv Cte > c_n \text{Scal}_{\text{can}}$ (Chen–Wei–Yan [7] for $n \geq 5$). We refer to Esposito–Pistoia–Vétois [16] for blowing-up positive solutions in case of a potential h depending on ε and approaching $c_n \text{Scal}_g$, and to Hebey–Wei [18] for the construction of multi-peak solutions on the three-sphere with a potential approaching constants arbitrarily larger than the scalar curvature. Here again, combining Druet [11] and Theorem 1.2 yields the following: when $h < c_n \text{Scal}_g$ and $3 \leq n \leq 5$, equation (1) is positively compact, but not compact.

The proofs of Theorems 1.1 and 1.2 rely on a Lyapunov–Schmidt reduction. Over the past two decades, there has been intensive developments in Lyapunov–Schmidt reductions applied to critical elliptic equations. In addition to the references in the geometric context of a Riemannian manifold cited above, an early reference

for single-bubble solutions is Rey [30]. Possible references on the construction of blowing-up solutions to equations like (4) by glueing a fixed function to bubbles are del Pino–Musso–Pacard–Pistoia [8,9] and Guo–Li–Wei [17] (for the Yamabe equation on the canonical sphere) and Wei–Yan [37] (for a Lazer-McKenna type problem). The list of constributions above does not pretend to exhaustivity: we refer to the references of the above papers and also to the monograph [1] by Ambrosetti–Malchiodi for further bibliographic complements. Our paper is organized as follows. In Section 2, we discuss extensions and generalizations of the above theorems. In Section 3, we discuss the nondegeneracy assumption. The finite dimensional reduction is performed in Section 4. The reduced problem is studied in Section 5. Theorems 1.1 and 1.2 are proved in Section 6. The proof of the error estimate is postponed to Section 7.

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2. Miscellaneous extensions

2.1. **About the critical dimension $n = 10$ in Theorem 1.2.** As mentioned in the introduction, the method developed here fails to produce blowing-up solutions to (4) in higher dimension. Indeed, in dimensions $n \geq 7$, a term involving the Weyl tensor appear in the Taylor expansion (75) of the Lyapunov–Schmidt functional. In dimension $n < 10$, this term is dominated by the contribution of u_0. In dimension $n = 10$, there is a competition between the Weyl tensor and u_0, and one gets the following result:

Theorem 2.1 (dimension $n = 10$ and $h \equiv c_n \operatorname{Scal}_g$). *Let (M, g) be a smooth compact Riemannian manifold of dimension $n = 10$ with positive Yamabe invariant. Assume that there exists a nondegenerate positive solution to the Yamabe equation (2). Assume that $h \equiv c_n \operatorname{Scal}_g$ and that $u_0 > \frac{5}{567} | \operatorname{Weyl}_g |_g^2$. Then for $\varepsilon > 0$ small, equation (4) admits a sign-changing solution u_ε of type $(u_0 - B)$. In particular, the family $(u_\varepsilon)_{\varepsilon > 0}$ blows up as $\varepsilon \to 0$.*

In dimension $n > 10$, the Weyl tensor dominates the contribution of u_0, and it is not possible to produce blowing-up solutions in general (see the explicit Taylor expansion (75) in Section 6). However, in the locally conformally flat case, that is when the Weyl tensor vanishes (at least in dimension larger than four), one gets the following result:

Theorem 2.2 (the locally conformally flat case in any dimension). *Let (M, g) be a smooth compact Riemannian manifold of dimension $n \geq 3$ with positive Yamabe invariant. Assume that there exists a nondegenerate positive solution to the Yamabe equation (2). Assume that (M, g) is locally conformally flat and that $h \equiv c_n \operatorname{Scal}_g$. Then for $\varepsilon > 0$ small, equation (4) admits a sign-changing solution u_ε of type $(u_0 - B)$. In particular, the family $(u_\varepsilon)_{\varepsilon > 0}$ blows up as $\varepsilon \to 0$.*

Examples of manifolds and metrics satisfying the hypothesis of Theorem 2.2 are in Proposition 3.4. As stated in Theorems 2.1 and 2.2, the solutions we construct change sign. Here again, it is natural to ask if there exist positive blowing-up solutions to (4) under the assumptions of Theorems 2.1 and 2.2. The answer is negative. Indeed, it follows from positive compactness theorems (Khuri–Marques–Schoen [19] and Schoen [32,33]) that positive blowing-up solutions to equation (4)

do not exist in the locally conformally flat and aspherical case. A consequence of the above results is that the Yamabe equation (2) is positively compact, but not compact in the context of Theorems 2.1 and 2.2.

2.2. Positive blowing-up solutions in dimension $n = 6$.

In this subsection, we focus on positive solutions to (4). A direct offshot of the techniques developed for the proof of Theorem 1.1 yields the following result:

Theorem 2.3 (**positive solutions in dimension** $n = 6$). *Let (M, g) be a smooth compact Riemannian manifold of dimension $n = 6$ and let $h \in C^{0,\theta}(M)$ ($\theta \in (0, 1)$) be such that $\Delta_g + h$ is coercive. Assume that there exists a nondegenerate solution $u_0 \in C^{2,\theta}(M)$ to equation (1) and that*

$$h - c_6 \operatorname{Scal}_g > 2u_0 > 0 \quad in \ M \,.$$

Then for $\varepsilon > 0$ small, equation (4) admits a positive solution $u_\varepsilon > 0$ such that $u_\varepsilon = u_0 + B_\varepsilon + o(1)$, where $(B_\varepsilon)_\varepsilon$ is a bubble and $\lim_{\varepsilon \to 0} o(1) = 0$ in $H_1^2(M)$.

This result is a complement to a specific 6-dimensional result: Druet ([11] and private communication) showed that blow-up for positive solutions to (4) with bounded energy necessarily occurs at points $x \in M$ such that $(h - c_6 \operatorname{Scal}_g)(x) \geq 2u_0(x)$. Dimension six is critical when considering positive blowing-up solutions with nontrivial weak limit $u_0 > 0$. More precisely, the blow-up analysis shows that there is balance between the contributions of u_0 and $h - c_n \operatorname{Scal}_g$: one of the terms dominates the other when $n \neq 6$, and they compete at the same growth when $n = 6$. We refer to the Taylor expansion (45) and to [11] to outline this phenomenon. We refer to Druet–Hebey [13] for an extensive discussion on dimension six.

2.3. Prescription of the blow-up point.

The above theorems show the existence of blowing-up families of solutions, but the blow-up point (that is the point where the bubble is centered) is not prescribed. The only information obtained from the construction is that blow-up occurs at a minimum point of u_0 (for Theorems 1.1 and 1.2 when $n \neq 6$) or of $u_0 - (c_n \operatorname{Scal}_g - h)/2$ (for Theorem 1.1 when $n = 6$). Prescribing the location of the blow-up point of the bubbles requires additional informations. We define $\Phi : M \to \mathbb{R}$ as follows:

$$\Phi := \begin{cases} u_0 + \frac{1}{2}(h - c_n \operatorname{Scal}_g)\mathbf{1}_{n=6} & \text{in Theorems 1.1, 1.2 and 2.2} \\ u_0 - \frac{5}{567}|\operatorname{Weyl}_g|_g^2 & \text{in Theorem 2.1} \\ \frac{1}{2}(h - c_6 \operatorname{Scal}_g) - u_0 & \text{in Theorem 2.3.} \end{cases}$$

Our prescription result is the following:

Theorem 2.4 (**Prescription of the blow-up point**). *In addition to the hypothesis in Theorems 1.1, 1.2, 2.1, 2.2, and 2.3, assume that there exists $\xi_0 \in M$ which is a strict local minimum point of Φ with $\Phi(\xi_0) > 0$. Then the conclusions of the above theorems hold with the extra information that the bubbles are centered at a family $(\xi_\varepsilon)_\varepsilon \in M$ such that $\lim_{\varepsilon \to 0} \xi_\varepsilon = \xi_0$.*

In case $h \in C^1(M)$ and there exists $\xi_0 \in M$ which is a C^1−stable critical point of Φ with $\Phi(\xi_0) > 0$, the same conclusion holds with the convergence (45) holding in C^1.

2.4. **Comments and remarks.** A natural extension of our results is the con-
struction of multi-peak changing-sign solutions glued on a fixed function u_0. Since
the results are local in nature, a reasonable guess is that our techniques allow to
construct $k \geq 2$ peaks at k distinct strict local minima of u_0 if such minima exist.
Another natural but more difficult guess is that there are bubble-towers centered at
a C^1-stable critical point. The additional difficulty here is that the accumulation
of bubbles imposes to find a critical point around a saddle-type functional, which
requires to prove C^1-convergence for expansions like (45).

3. Discussion on the degenerate case

In the sequel, we say that $(B_\varepsilon)_\varepsilon$ is a bubble if there exists a family $(x_\varepsilon)_\varepsilon \in M$
and a family $(\mu_\varepsilon)_\varepsilon \in \mathbb{R}_{>0}$ such that $\lim_{\varepsilon \to 0} \mu_\varepsilon = 0$ and

$$(6) \qquad B_\varepsilon(x) := \left(\frac{\sqrt{n(n-2)}\mu_\varepsilon}{\mu_\varepsilon^2 + d_g(x, x_\varepsilon)^2} \right)^{\frac{n-2}{2}} \quad \text{for all } x \in M.$$

In this situation, we say that the bubble is centered at $(x_\varepsilon)_\varepsilon$. We say that a solution
$u_0 \in C^{2,\theta}(M)$ to

$$(7) \qquad \Delta_g u_0 + h u_0 = u_0^{2^\star - 1} \quad \text{in } M$$

is nondegenerate if the linearization of the equation has a trivial kernel, that is

$$(8) \qquad K_{h,u_0} := \left\{ \varphi \in C^{2,\theta}(M) / \ \Delta_g \varphi + h\varphi = (2^\star - 1)|u_0|^{2^\star - 2}\varphi \right\} = \{0\}.$$

Theorems 1.1 and 1.2 require the assumption that u_0 is a nondegenerate solution
to (7). In this section, we prove that this is a necessary assumption, and that it is
generic.

3.1. **The conformal geometric equation and necessity of the nondegen-
eracy assumption.** Let (M, g) be a compact Riemannian manifold of dimension
$n \geq 3$ with positive Yamabe invariant. Up to a conformal change of metric, it fol-
lows from the resolution of the Yamabe problem that we can assume that the scalar
curvature Scal_g is a positive constant, and we consider $u_{0,g} := (c_n \text{Scal}_g)^{1/(2^\star - 2)}$
the only positive constant solution to the Yamabe equation

$$(9) \qquad \Delta_g u_{0,g} + c_n \text{Scal}_g \, u_{0,g} = u_{0,g}^{2^\star - 1} \quad \text{in } M.$$

As is easily checked, in this situation,

$$K_{c_n \text{Scal}_g, u_{0,g}} = \left\{ \varphi \in C^2(M) / \ \Delta_g \varphi = \frac{\text{Scal}_g}{n-1} \varphi \right\},$$

where the kernel is defined in (8). Therefore

$$u_{0,g} \text{ is a nondegenerate solution to (9)} \Leftrightarrow \frac{\text{Scal}_g}{n-1} \notin \text{Spec}\,(\Delta_g),$$

where $\text{Spec}\,(\Delta_g)$ is the nonnegative spectrum of Δ_g. We define the Yamabe invari-
ant by

$$(10) \qquad \mu_{[g]}(M) := \inf_{g' \in [g]} \frac{\int_M \text{Scal}_{g'} \, dv_{g'}}{\text{Vol}_{g'}(M)^{\frac{n-2}{n}}},$$

where $[g]$ is the conformal class of g and dv_g is the Riemannian element of volume.
The Yamabe invariant $\mu_{[g]}(M)$ is positive iff the operator $\Delta_{g'} + c_n \text{Scal}_{g'}$ is coercive

for all $g' \in [g]$. It is well known that if g is a Yamabe metric (that is a minimizer of the Yamabe functional (10)), one has that $\frac{\mathrm{Scal}_g}{n-1} \leq \lambda_1(\Delta_g)$, the first nonzero eigenvalue of Δ_g. Note that equality is achieved on the canonical sphere $(\mathbb{S}^n, \mathrm{can})$. More generally, any positive solution to the Yamabe equation on the canonical sphere is a Yamabe metric and is degenerate.

The following result shows that the conclusion of Theorems 1.1 and 1.2 do not hold on the standard sphere (where positive solutions to the Yamabe equations are all degenerate):

Proposition 3.1. *There does not exist any family of functions* $(u_\varepsilon)_\varepsilon \in C^{2,\theta}(\mathbb{S}^n)$ *of type* $(u_0 - B)$ *to the equation*

$$(11) \qquad \Delta_{\mathrm{can}} u_\varepsilon + c_n \, \mathrm{Scal}_{\mathrm{can}} \, u_\varepsilon = |u_\varepsilon|^{2^\star - 2 - \varepsilon} u_\varepsilon \quad \text{in } M$$

for all $\varepsilon \in (0, 2^\star)$.

Proof. We argue by contradiction and assume the existence of a family $(u_\varepsilon)_\varepsilon \in C^{2,\theta}(\mathbb{S}^n)$ of the form (5) of solutions to equation (11). Multiplying (11) by the bubble B_ε and integrating by parts yield $\mu_\varepsilon^\varepsilon \to 1$ when $\varepsilon \to 0$. Fix $\phi \in \Lambda_1(\mathbb{S}^n, \mathrm{can})$, the set of eigenfunctions of $\lambda_1(\Delta_{\mathrm{can}}) = n$, the first nonzero eigenvalue of Δ_{can} on $\mathbb{S}^n$: indeed, see for instance Berger–Gauduchon–Mazet [3], we have that $\Lambda_1(\mathbb{S}^n, \mathrm{can}) = \{l_{|\mathbb{S}^n} \, / \, l : \mathbb{R}^{n+1} \to \mathbb{R} \text{ linear}\}$. It follows from Kazdan–Warner [20] that

$$\int_{\mathbb{S}^n} \Delta_{\mathrm{can}} u_\varepsilon \langle \nabla\phi, \nabla u_\varepsilon \rangle_{\mathrm{can}} \, dv_{\mathrm{can}} = \frac{n-2}{2n} \int_{\mathbb{S}^n} \Delta_{\mathrm{can}} \phi |\nabla u_\varepsilon|_{\mathrm{can}}^2 \, dv_{\mathrm{can}} \,.$$

Using equation (11) and integrating by parts yields

$$\varepsilon \int_{\mathbb{S}^n} \phi |u_\varepsilon|^{2^\star - \varepsilon} \, dv_{\mathrm{can}} = 0 \,.$$

Letting $\varepsilon \to 0$ and using (6), (5), and (15) yields, up to a subsequence,

$$\int_{\mathbb{S}^n} \phi u_0^{2^\star} \, dv_{\mathrm{can}} + \left(\int_{\mathbb{R}^n} U^{2^\star} \, dx \right) \phi(x_0) = 0 \,,$$

where $x_0 \in \mathbb{S}^n$. Passing to the weak limit in (11) when $\varepsilon \to 0$ implies that u_0 is a positive solutions to the Yamabe equation on $\mathbb{S}^n$. It then follows from Obata [28] that $\int_{\mathbb{S}^n} u_0^{2^\star} \, dv_{\mathrm{can}} = \int_{\mathbb{R}^n} U^{2^\star} \, dx$. Taking $\phi \in \Lambda_1(\mathbb{S}^n, \mathrm{can})$ such that $\min \phi = \phi(-x_0) \neq 0$ in the above equation yields a contradiction since $\phi \not\equiv 0$. This ends the proof of Proposition 3.1. $\qquad\square$

3.2. **Genericity of the nondegeneracy assumption.** The following proposition shows that the nondegeneracy hypothesis of Theorem 1.1 is generic:

Proposition 3.2. *Let* $h \in C^{0,\theta}(M)$ *and let* $u_0 \in C^{2,\theta}(M)$ *be a positive solution to* $\Delta_g u_0 + h u_0 = u_0^{2^\star - 1}$. *Fix* $\nu > 0$. *Then there exist* $\tilde{h}_\nu \in C^{0,\theta}(M)$ *and* $\tilde{u}_{0,\nu} \in C^{2,\theta}(M)$ *such that* $\|h - \tilde{h}_\nu\|_{C^{0,\theta}} + \|u_0 - \tilde{u}_{0,\nu}\|_{C^{2,\theta}} < \nu$ *and* $\tilde{u}_{0,\nu} > 0$ *is a nondegenerate solution to* $\Delta_g \tilde{u}_{0,\nu} + \tilde{h}_\nu \tilde{u}_{0,\nu} = \tilde{u}_{0,\nu}^{2^\star - 1}$ *in* M.

Proof. We define

$$\mu_\eta := \inf_{u \in H_1^2(M) \backslash \{0\}} \frac{\int_M (|\nabla u|_g^2 + (h - (2^\star - 1)u_0^{2^\star - 2} - \eta)u^2) \, dv_g}{\left(\int_M |u|^{2^\star} \, dv_g \right)^{\frac{2}{2^\star}}}$$

for all $\eta \geq 0$. Testing the functional on u_0 yields $\mu_\eta < 0$ for all $\eta \geq 0$. As is easily checked, $\lim_{\eta \to 0} \mu_\eta = \mu_0 < 0$. Standard variational arguments yield the existence of a positive minimizer $w_\eta \in C^{2,\theta}(M)$ for μ_η such that $\Delta_g w_\eta + (h - (2^\star - 1)u_0^{2^\star - 2} - \eta)w_\eta = -(2^\star - 2)w_\eta^{2^\star - 1}$ in M for all $\eta > 0$; moreover, the family $(w_\eta)_{\eta \geq 0}$ is relatively compact in $C^2(M)$. Since u_0 is the only positive solution to the equation $\Delta_g v + (h - (2^\star - 1)u_0^{2^\star - 2})v = -(2^\star - 2)v^{2^\star - 1}$ (let w be the quotient of two positive solutions and estimate $\Delta_g w$ at extremal points of w), one gets that $\lim_{\eta \to 0} w_\eta = u_0$ in $C^2(M)$, and then $C^{2,\theta}(M)$ by elliptic regularity. One defines $h_\eta := h - (2^\star - 1)(u_0^{2^\star - 2} - w_\eta^{2^\star - 2}) - \eta$. Then $\Delta_g w_\eta + h_\eta w_\eta = w_\eta^{2^\star - 1}$ in M and spectral theory yields the existence of $\eta_0 > 0$ such that $K_{h_\eta, w_\eta} = \{0\}$ for all $\eta \in (0, \eta_0)$. The conclusion of the proposition follows from taking $\tilde{u}_{0,\nu} := w_\eta$ and $\tilde{h}_\nu := h_\eta$ for $\eta > 0$ small enough. $\square$

We now focus on the geometric case, that is Theorem 1.2. We adopt the terminology of Khuri–Marques–Schoen [19]: given M a compact manifold, g_0 a background Riemannian metric on M, and ω a volume form on M, to each class $c \in \mathcal{C} := \{$Conformal classes of Riemannian metrics on $M\}$, we associate the unique metric $g \in c$ for which the Riemannian $n-$volume form is ω. The $C^{k,\theta}-$distance between two classes is defined as the $C^{k,\theta}$-distance between their representatives via this analogy. The following result holds:

Proposition 3.3 (Khuri–Marques–Schoen [19]). *There exists $\mathcal{O} \subset \mathcal{C}$ an open dense set such that for all $c \in \mathcal{O}$, there exists exactly a finite nonzero number of metrics $g \in c$ (up to homothetic transformations) such that the constant positive function $u_{0,g}$ is nondegenerate.*

In other words, up to a perturbation in the conformal class, the hypothesis of Theorem 1.2 holds.

3.3. A family of nondegenerate geometric solutions in the locally conformally flat case.
We exhibit here a situation in which the nondegeneracy assumption is satisfied for the geometric equation in the locally conformally flat case (see Theorem 2.2). For all $k \geq 1$ and $t > 0$, $(\mathbb{S}^k(t), \mathrm{can})$ is the canonical sphere ot radius t in $\mathbb{R}^{k+1}$:

Proposition 3.4. *Let $M_r := \mathbb{S}^1(r) \times \mathbb{S}^{n-1}$ be endowed with its canonical product metric g_r, where $r > 0$. Then (M_r, g_r) is locally conformally flat with positive constant scalar curvature. Moreover, for any $r \notin \{\frac{i}{\sqrt{n-2}} / i \in \mathbb{Z}_{>0}\}$, the positive constant solution to the Yamabe equation is nondegenerate.*

Proof. Recall that on the canonical sphere $\mathbb{S}^k(t)$, the spectrum of the Laplacian is $\{\frac{i(k+i-1)}{t^2} / i \in \mathbb{Z}_{\geq 0}\}$ (see [3]). Then the spectrum of Δ_{g_r} is $\{\frac{i^2}{r^2} + j(n - 2 + j) / i, j \in \mathbb{Z}_{\geq 0}\}$. Independently, the scalar curvature is $\mathrm{Scal}_{g_r} := (n-1)(n-2)$. As a consequence,

$$\frac{\mathrm{Scal}_{g_r}}{n-1} \notin \mathrm{Spec}(\Delta_{g_r}) \Leftrightarrow r \notin \left\{ \frac{i}{\sqrt{n-2}} / i \in \mathbb{Z}_{>0} \right\}.$$

In addition, it is standard that the product of a one-dimensional circle with a space form is locally conformally flat. $\square$

4. Finite dimensional reduction

Since the operator $\Delta_g + h$ is coercive, the Sobolev space $H_1^2(M)$ is endowed with the scalar product $\langle \cdot, \cdot \rangle_h$ defined by

$$(12) \qquad \langle u, v \rangle_h = \int_M \langle \nabla u, \nabla v \rangle_g \, dv_g + \int_M huv dv_g$$

for all $u, v \in H_1^2(M)$. We let $\| \cdot \|_h$ be the norm induced by $\langle \cdot, \cdot \rangle_h$: this norm is equivalent to the standard norm on $H_1^2(M)$. We let $i^* : L^{\frac{2n}{n+2}}(M) \to H_1^2(M)$ be such that for any w in $L^{\frac{2n}{n+2}}(M)$, the function $u = i^*(w)$ in $H_1^2(M)$ is the unique solution of the equation $\Delta_g u + hu = w$ in M. We then rewrite equation (4) as

$$(13) \qquad u = i^*(f_\varepsilon(u)), \quad u \in H_1^2(M),$$

where $f_\varepsilon(u) := |u|^{2^*-2-\varepsilon} u$.

In case (M, g) is locally conformally flat, it follows from the compactness of M that there exists $r_0 \in (0, i_g(M))$ (where $i_g(M) > 0$ is the injectivity radius of (M, g)) such that for any point ξ in M, there exists $\Lambda_\xi \in C^\infty(M)$ such that the conformal metric $g_\xi = \Lambda_\xi^{4/(n-2)} g$ is flat in $B_\xi(r_0)$ and $i_{g_\xi}(M) > r_0$. As is easily seen, the functions Λ_ξ can be chosen smooth with respect to ξ and such that $\Lambda_\xi(\xi) = 1$. If the manifold is not locally conformally flat, then we let $\Lambda_\xi(x) = 1$ for all points x and ξ in M, and we fix $r_0 \in (0, i_g(M))$ arbitrarily. We let χ be a smooth cutoff function such that $0 \leq \chi \leq 1$ in $\mathbb{R}$, $\chi = 1$ in $[-r_0/2, r_0/2]$, and $\chi = 0$ in $\mathbb{R} \setminus (-r_0, r_0)$. For any positive real number δ and any point ξ in M, we define the function $W_{\delta,\xi}$ on M by

$$(14) \qquad W_{\delta,\xi}(x) := \chi\left(d_{g_\xi}(x, \xi)\right) \Lambda_\xi(x) \delta^{\frac{2-n}{2}} U\left(\delta^{-1} \exp_\xi^{-1}(x)\right),$$

where d_{g_ξ} is the geodesic distance on M associated with the metric g_ξ, the exponential map is taken with respect to the same metric g_ξ and

$$(15) \qquad U(x) := \left(\frac{\sqrt{n(n-2)}}{1 + |x|^2}\right)^{\frac{n-2}{2}}$$

for all $x \in \mathbb{R}^n$. For any positive real number δ, the function $U_\delta(x) = \delta^{\frac{2-n}{2}} U\left(\delta^{-1} x\right)$ satisfies the equation $\Delta_{\mathrm{Eucl}} U_\delta = U_\delta^{2^*-1}$ where Δ_{Eucl} is the Laplace operator associated with the Euclidean metric. Moreover, by Bianchi–Egnell [4], any solution in $v \in D_1^2(\mathbb{R}^n)$ (the completion of $C_c^\infty(\mathbb{R}^n)$ for the norm $\|u\|_{D_1^2} := \|\nabla u\|_2$) of the linear equation $\Delta_{\mathrm{Eucl}} v = (2^* - 1) U^{2^*-2} v$ is a linear combination of the functions

$$(16) \qquad V_0(x) := \frac{|x|^2 - 1}{\left(1 + |x|^2\right)^{\frac{n}{2}}} \quad \text{and} \quad V_i(x) := \frac{x_i}{\left(1 + |x|^2\right)^{\frac{n}{2}}}$$

for all $i = 1, \dots, n$ and $x \in \mathbb{R}^n$. For any positive real number δ and any point (ξ, ω) in TM, we define the functions $Z_{\delta,\xi}$ and $Z_{\delta,\xi,\omega}$ in M by

$$(17) \qquad Z_{\delta,\xi}(x) := \chi\left(d_{g_\xi}(x,\xi)\right) \Lambda_\xi(x) \delta^{\frac{n-2}{2}} \frac{d_{g_\xi}(x,\xi)^2 - \delta^2}{\left(\delta^2 + d_{g_\xi}(x,\xi)^2\right)^{\frac{n}{2}}},$$

$$(18) \qquad Z_{\delta,\xi,\omega}(x) := \chi\left(d_{g_\xi}(x,\xi)\right) \Lambda_\xi(x) \delta^{\frac{n}{2}} \frac{\left\langle \exp_\xi^{-1} x, \omega \right\rangle_{g_\xi}}{\left(\delta^2 + d_{g_\xi}(x,\xi)^2\right)^{\frac{n}{2}}}$$

for all $x \in M$. We then let $\Pi_{\delta,\xi}$ and $\Pi_{\delta,\xi}^\perp$ be the projections of the Sobolev space $H_1^2(M)$ onto the respective closed subspaces

$$K_{\delta,\xi} := \{\, \lambda Z_{\delta,\xi} + Z_{\delta,\xi,\omega} \, / \, \lambda \in \mathbb{R} \text{ and } \omega \in T_\xi M \}$$

and
(19)
$$K_{\delta,\xi}^\perp := \left\{ \phi \in H_1^2(M) \, / \, \langle \phi, Z_{\delta,\xi} \rangle_h = 0 \text{ and } \langle \phi, Z_{\delta,\xi,\omega} \rangle_h = 0 \quad \text{for all } \omega \in T_\xi M \right\},$$

where the scalar product $\langle \cdot, \cdot \rangle_h$ is as in (12).

Recall that $u_0 \in C^{2,\theta}(M)$ is a nondegenerate positive solution to equation (7). We construct solutions of type $(u_0 - B)$ to equations (4), or equivalently (13), like

$$u_\varepsilon := u_0 - W_{\delta_\varepsilon(t_\varepsilon),\xi_\varepsilon} + \phi_{\delta_\varepsilon(t_\varepsilon),\xi_\varepsilon}, \quad \text{with } \delta_\varepsilon(t_\varepsilon) := t_\varepsilon \varepsilon^{\frac{2}{n-2}},$$

where $W_{\delta_\varepsilon(t_\varepsilon),\xi_\varepsilon}$ is as in (14), $\phi_{\delta_\varepsilon(t_\varepsilon),\xi_\varepsilon}$ is a function in $K_{\delta_\varepsilon(t_\varepsilon),\xi_\varepsilon}^\perp$, and $t_\varepsilon > 0$. As easily checked, if $\phi_{\delta_\varepsilon(t_\varepsilon),\xi_\varepsilon} \to 0$ in $H_1^2(M)$ when $\varepsilon \to 0$, then (u_ε) is of type $u_0 - B$. We rewrite equation (4) as the couple of equations

$$(20) \qquad \Pi_{\delta_\varepsilon(t),\xi}\left(u_0 - W_{\delta_\varepsilon(t),\xi} + \phi_{\delta_\varepsilon(t),\xi} - i^*\left(f_\varepsilon\left(u_0 - W_{\delta_\varepsilon(t),\xi} + \phi_{\delta_\varepsilon(t),\xi}\right)\right)\right) = 0$$

and

$$(21) \qquad \Pi_{\delta_\varepsilon(t),\xi}^\perp\left(u_0 - W_{\delta_\varepsilon(t),\xi} + \phi_{\delta_\varepsilon(t),\xi} - i^*\left(f_\varepsilon\left(u_0 - W_{\delta_\varepsilon(t),\xi} + \phi_{\delta_\varepsilon(t),\xi}\right)\right)\right) = 0.$$

We begin with solving equation (21) in Proposition 4.1 below:

Proposition 4.1. *Let $u_0 \in C^{2,\theta}(M)$ be a positive nondegenerate solution to (7). Given two positive real numbers $a < b$, there exists a positive constant $C_{a,b}$ such that for ε small, for any real number t in $[a, b]$, and any point ξ in M, there exists a unique function $\phi_{\delta_\varepsilon(t),\xi}$ in $K_{\delta_\varepsilon(t),\xi}^\perp$ which solves equation (21) and satisfies*
(22)

$$\left\| \phi_{\delta_\varepsilon(t),\xi} \right\|_h \leq C_{a,b} \begin{cases} \varepsilon \left| \ln \varepsilon \right| & \text{if } 3 \leq n \leq 6 \\ \varepsilon^{\frac{4}{n-2}} & \text{if } n \geq 7 \\ \varepsilon^{\frac{n+2}{2(n-2)}} & \text{if } n \geq 7, \ h \equiv c_n \operatorname{Scal}_g, \text{ and } (M, g) \text{ is loc. conf. flat.} \end{cases}$$

Moreover, $\phi_{\delta_\varepsilon(t),\xi}$ is continuously differentiable with respect to t and ξ.

The sequel of this section is devoted to the proof of Proposition 4.1. For ε small, for any positive real number δ, and any point ξ in M, we let the map $L_{\varepsilon,\delta,\xi} : K_{\delta,\xi}^\perp \to K_{\delta,\xi}^\perp$ be defined by

$$(23) \qquad L_{\varepsilon,\delta,\xi}(\phi) := \Pi_{\delta,\xi}^\perp\left(\phi - i^*\left(f_\varepsilon'\left(u_0 - W_{\delta,\xi}\right)\phi\right)\right)$$

for all $\phi \in K_{\delta,\xi}^{\perp}$, where $u_0 \in C^{2,\theta}(M)$ is a nondegenerate positive solution to (7), $W_{\delta,\xi}$ is as in (14) and $K_{\delta,\xi}^{\perp}$ is as in (19). Clearly, we get that $L_{\varepsilon,\delta,\xi}$ is linear and continuous. In Lemma 4.2 below, we prove the invertibility of $L_{\varepsilon,\delta,\xi}$ for δ and ε small.

Lemma 4.2. *Given two positive real numbers $a < b$, for ε small, the map $L_{\varepsilon,\delta_\varepsilon(t),\xi}$ is invertible for all real numbers t in $[a,b]$ and all points ξ in M, where $\delta_\varepsilon(t) = t\varepsilon^{2/(n-2)}$ and $L_{\varepsilon,\delta_\varepsilon(t),\xi}$ is as in (23). Moreover, there exists a positive constant $C_{a,b}$ such that for ε small, for any real number $t \in [a,b]$, any point $\xi \in M$, and any function $\phi \in K_{\delta_\varepsilon(t),\xi}^{\perp}$, there holds*

$$(24) \qquad \left\|L_{\varepsilon,\delta_\varepsilon(t),\xi}(\phi)\right\|_h \geq C_{a,b} \left\|\phi\right\|_h .$$

In particular, the inverse map $L_{\varepsilon,\delta_\varepsilon(t),\xi}^{-1}$ is continuous.

Proof. We prove (24). We proceed by contradiction. We assume that there exist two sequences of positive real numbers $(\varepsilon_\alpha)_\alpha$ and $(t_\alpha)_\alpha$ such that $\varepsilon_\alpha \to 0$ for all $\alpha \to +\infty$ and $a \leq t_\alpha \leq b$, a sequence of points $(\xi_\alpha)_\alpha$ in M, and a sequence of functions $(\phi_\alpha)_\alpha$ such that

$$(25) \qquad \phi_\alpha \in K_{\delta_{\varepsilon_\alpha}(t_\alpha),\xi_\alpha}^{\perp} , \quad \left\|\phi_\alpha\right\|_h = 1 , \quad \text{and} \quad \left\|L_{\varepsilon_\alpha,\delta_{\varepsilon_\alpha}(t_\alpha),\xi_\alpha}(\phi_\alpha)\right\|_h \longrightarrow 0$$

as $\alpha \to +\infty$. We define $W_\alpha := W_{\delta_{\varepsilon_\alpha}(t_\alpha),\xi_\alpha}$. First, we claim that

$$(26) \qquad \left\|\phi_\alpha - i^*\left(f'_{\varepsilon_\alpha}(u_0 - W_\alpha)\phi_\alpha\right)\right\|_h \longrightarrow 0$$

as $\alpha \to +\infty$. Passing if necessary to a subsequence, we may assume that all the points ξ_α belong to a small open subset Ω in M on which there exists a smooth orthonormal frame. Thanks to this frame, we identify the tangent space $T_\xi M$ with $\mathbb{R}^n$ for all points ξ in Ω, so that $\exp_\xi$ is in fact the composition of the standard exponential map with a linear isometry $\Psi_\xi : \mathbb{R}^n \to T_\xi M$ which is smooth with respect to ξ. We define

$$(27) \qquad Z_{0,\alpha} := Z_{\delta_{\varepsilon_\alpha}(t_\alpha),\xi_\alpha} \quad \text{and} \quad Z_{i,\alpha} = Z_{\delta_{\varepsilon_\alpha}(t_\alpha),\xi_\alpha,e_i}$$

for all $i = 1,\ldots,n$, where e_i is the i-th vector in the canonical basis of $\mathbb{R}^n$ and the functions $Z_{\delta_{\varepsilon_\alpha}(t_\alpha),\xi_\alpha}$ and $Z_{\delta_{\varepsilon_\alpha}(t_\alpha),\xi_\alpha,e_i}$ are as in (17)–(18). For any α, by definition of $L_{\varepsilon,\delta_{\varepsilon_\alpha}(t_\alpha),\xi_\alpha}$, we get that

$$(28) \qquad \phi_\alpha - i^*\left(f'_{\varepsilon_\alpha}(u_0 - W_\alpha)\phi_\alpha\right) - L_{\varepsilon_\alpha,\delta_{\varepsilon_\alpha}(t_\alpha),\xi_\alpha}(\phi_\alpha) = \sum_{i=0}^{n} \lambda_{i,\alpha} Z_{i,\alpha}$$

for some real numbers $\lambda_{i,\alpha}$, where the functions $Z_{i,\alpha}$ are as in (27). Taking into account (25) and (28), one sees that in order to get (26), it suffices to prove that $\lambda_{i,\alpha} \to 0$ as $\alpha \to +\infty$ for all $i = 0,\ldots,n$. As is easily checked, for any $i,j = 0,\ldots,n$, there holds

$$(29) \qquad \langle Z_{i,\alpha}, Z_{j,\alpha} \rangle_h \longrightarrow \|\nabla V_i\|_2^2 \delta_{ij}$$

as $\alpha \to +\infty$ where the function V_i is as in (16) and the real numbers δ_{ij} are the Kronecker symbols. By (28), (29), and since the functions ϕ_α and $L_{\varepsilon_\alpha,\delta_{\varepsilon_\alpha}(t_\alpha),\xi_\alpha}(\phi_\alpha)$ belong to $K_{\delta_{\varepsilon_\alpha}(t_\alpha),\xi_\alpha}^{\perp}$, for any $i = 0,\ldots,n$, we get that

$$(30) \qquad \int_M f'_{\varepsilon_\alpha}(u_0 - W_\alpha) Z_{i,\alpha}\phi_\alpha dv_g = -\lambda_{i,\alpha}\|\nabla V_i\|_2^2 + o\left(\sum_{j=0}^{n}|\lambda_{j,\alpha}|\right)$$

as $\alpha \to +\infty$. As is easily checked, we get that

$$(31) \qquad \int_M f'_{\varepsilon_\alpha}(u_0 - W_\alpha) Z_{i,\alpha} \phi_\alpha dv_g = \int_M f'_{\varepsilon_\alpha}(W_\alpha) Z_{i,\alpha} \phi_\alpha dv_g + \mathrm{o}(1)$$

as $\alpha \to +\infty$. We find

$$(32) \qquad \frac{\int_M f'_{\varepsilon_\alpha}(W_\alpha) Z_{i,\alpha} \phi_\alpha dv_g}{2^\star - 1 - \varepsilon_\alpha} = \delta_{\varepsilon_\alpha}(t_\alpha)^{\frac{n-2}{2} \varepsilon_\alpha} \int_{\mathbb{R}^n} \chi_\alpha \Lambda_\alpha^{2^\star - 1 - \epsilon_\alpha} U^{2^\star - 2 - \varepsilon_\alpha} V_i \widetilde{\phi}_\alpha dv_{\widetilde{g}_\alpha}$$

as $\alpha \to +\infty$, where the functions U and V_i are as in (15) and (16), the cutoff function χ is as in Section 4 and

$$\chi_\alpha := \chi \left(\delta_{\varepsilon_\alpha}(t_\alpha) |x| \right)^{2^\star - 2 - \varepsilon_\alpha},$$
$$\Lambda_\alpha := \Lambda_{\xi_\alpha}(\exp_{\xi_\alpha}(\delta_{\varepsilon_\alpha}(t_\alpha) x)),$$
$$(33) \qquad \widetilde{\phi}_\alpha(x) := \delta_{\varepsilon_\alpha}(t_\alpha)^{\frac{n-2}{2}} \chi \left(\delta_{\varepsilon_\alpha}(t_\alpha) |x| \right) \phi_\alpha \left(\exp_{\xi_\alpha}(\delta_{\varepsilon_\alpha}(t_\alpha) x) \right),$$
$$(34) \qquad \widetilde{g}_\alpha(x) := \exp_{\xi_\alpha}^* g \left(\delta_{\varepsilon_\alpha}(t_\alpha) x \right)$$

for any $\alpha \to +\infty$ and $x \in \mathbb{R}^n$ small enough. In the definitions above, the exponential map is taken with respect to the metric g_{ξ_α}. Since $(\phi_\alpha)_\alpha$ is bounded in $H_1^2(M)$, we get that $(\widetilde{\phi}_\alpha)_\alpha$ is bounded in $D^{1,2}(\mathbb{R}^n)$. Passing to a subsequence, we may assume that $(\widetilde{\phi}_\alpha)_\alpha$ converges weakly to some function $\tilde{\phi}$ in $D^{1,2}(\mathbb{R}^n)$. Passing to the limit into (32) yields

$$(35) \qquad \int_M f'_{\varepsilon_\alpha}(W_\alpha) Z_{i,\alpha} \phi_\alpha dv_g \longrightarrow (2^\star - 1) \int_{\mathbb{R}^n} U^{2^\star - 2} V_i \tilde{\phi} dx$$

as $\alpha \to +\infty$. Since the function V_i satisfies the equation $\Delta_{\mathrm{Eucl}} V_i = (2^\star - 1) U^{2^\star - 2} V_i$ in $\mathbb{R}^n$, and since, for any α, the function ϕ_α belongs to $K_{\delta_{\varepsilon_\alpha}(t_\alpha), \xi_\alpha}^\perp$, passing to the limit as $\alpha \to +\infty$ into the equation $\langle Z_{i,\alpha}, \phi_\alpha \rangle_h = 0$, we get that

$$(36) \qquad \int_{\mathbb{R}^n} \left\langle \nabla V_i, \nabla \tilde{\phi} \right\rangle dx = (2^\star - 1) \int_{\mathbb{R}^n} U^{2^\star - 2} V_i \tilde{\phi} dx = 0.$$

By (30), (31), (35), and (36), we get that

$$\lambda_{i,\alpha} = \mathrm{o}(1) + \mathrm{o}\left(\sum_{j=0}^n |\lambda_{j,\alpha}| \right)$$

as $\alpha \to +\infty$. It follows that $\lambda_{i,\alpha} \to 0$ as $\alpha \to +\infty$ for all $i = 0, \ldots, n$. The claim (26) then follows from (25) and (28).

For any sequence $(\varphi_\alpha)_\alpha$ in $H_1^2(M)$, and by (26), we get that

$$(37) \qquad \left| \langle \phi_\alpha, \varphi_\alpha \rangle_h - \int_M f'_{\varepsilon_\alpha}(u_0 - W_\alpha) \phi_\alpha \varphi_\alpha dv_g \right|$$
$$= \left| \left\langle \phi_\alpha - i^* \left(f'_{\varepsilon_\alpha}(u_0 - W_\alpha) \phi_\alpha \right), \varphi_\alpha \right\rangle_h \right|$$
$$\leq \left\| \phi_\alpha - i^* \left(f'_{\varepsilon_\alpha}(u_0 - W_\alpha) \phi_\alpha \right) \right\|_h \|\varphi_\alpha\|_h = \mathrm{o}\left(\|\varphi_\alpha\|_h \right)$$

as $\alpha \to +\infty$.

We claim that $\phi_\alpha \rightharpoonup 0$ weakly in $H_1^2(M)$ when $\alpha \to +\infty$. We prove the claim. Since (ϕ_α) is bounded in $H_1^2(M)$, up to a subsequence, there exists $\phi \in H_1^2(M)$

such that $(\phi_\alpha) \rightharpoonup \phi$ weakly in $H_1^2(M)$ when $\alpha \to +\infty$. Then for any $\varphi \in H_1^2(M)$, taking $\varphi_\alpha \equiv \varphi$ in (37) and letting $\alpha \to +\infty$ yields

$$\langle \phi, \varphi \rangle_h = \int_M (2^\star - 1) u_0^{2^\star - 2} \phi\varphi \, dv_g$$

for all $\varphi \in H_1^2(M)$, and then $\Delta_g \phi + h\phi = (2^\star - 1) u_0^{2^\star - 2} \phi$, which implies $\phi \equiv 0$ since u_0 is nondegenerate. This proves the claim.

We claim that $\tilde{\phi}_\alpha \rightharpoonup 0$ weakly in $D_1^2(\mathbb{R}^n)$ when $\alpha \to +\infty$, where $\tilde{\phi}_\alpha$ has been defined in (33). We prove the claim. Given a smooth function φ with compact support in $\mathbb{R}^n$, we define

$$\varphi_\alpha(x) := \chi\left(d_{g_{\xi_\alpha}}(x, \xi_\alpha)\right) \delta_{\varepsilon_\alpha}(t_\alpha)^{\frac{2-n}{2}} \varphi\left(\delta_{\varepsilon_\alpha}(t_\alpha)^{-1} \exp_{\xi_\alpha}^{-1}(x)\right)$$

for all $x \in M$. It follows from (37) together with a change of variable that

$$(38) \quad \int_{\mathbb{R}^n} \Lambda_\alpha^{-2} \langle \nabla\tilde{\phi}_\alpha, \nabla\varphi \rangle_{\tilde{g}_\alpha} dv_{\tilde{g}_\alpha} + \delta_{\varepsilon_\alpha}(t_\alpha)^2 \int_{\mathbb{R}^n} \Lambda_\alpha^{-2^\star} h\left(\exp_{\xi_\alpha}(\delta_{\varepsilon_\alpha}(t_\alpha) x)\right) \tilde{\phi}_\alpha \varphi dv_{\tilde{g}_\alpha}$$

$$= \delta_{\varepsilon_\alpha}(t_\alpha)^2 \int_{\mathbb{R}^n} \Lambda_\alpha^{-2^\star} f'_{\varepsilon_\alpha}\left(u_{0,\alpha}(x) - W_\alpha\left(\exp_{\xi_\alpha}(\delta_{\varepsilon_\alpha}(t_\alpha) x)\right)\right) \tilde{\phi}_\alpha \varphi dv_{\tilde{g}_\alpha} + o(1)$$

as $\alpha \to +\infty$, where $u_{0,\alpha}(\cdot) := u_0\left(\exp_{\xi_\alpha}(\delta_{\varepsilon_\alpha}(t_\alpha) \cdot)\right)$, $\tilde{\phi}_\alpha$ and $\tilde{g}_\alpha$ are as in (33) and (34). One easily checks that

$$\Lambda_\alpha^{-2^\star} \delta_{\varepsilon_\alpha}(t_\alpha)^2 f'_{\varepsilon_\alpha}\left(u_0\left(\exp_{\xi_\alpha}(\delta_{\varepsilon_\alpha}(t_\alpha) \cdot)\right) - W_\alpha\left(\exp_{\xi_\alpha}(\delta_{\varepsilon_\alpha}(t_\alpha) \cdot)\right)\right)$$

goes to $(2^* - 1) U^{2^* - 2}$ as $\alpha \to +\infty$ in $C_{loc}^0(\mathbb{R}^n)$. Moreover, since $\left(\tilde{\phi}_\alpha\right)_\alpha$ converges weakly to $\tilde{\phi}$ in $D_1^2(\mathbb{R}^n)$, passing to the limit into (38) as $\alpha \to +\infty$ yields

$$(39) \quad \int_{\mathbb{R}^n} \left\langle \nabla\tilde{\phi}, \nabla\varphi \right\rangle dx = (2^* - 1) \int_{\mathbb{R}^n} U^{2^* - 2} \tilde{\phi}\varphi dx.$$

Since (39) holds for all smooth functions φ with compact support in $\mathbb{R}^n$, we get that the function $\tilde{\phi}$ satisfies the equation $\Delta_{\mathrm{Eucl}}\tilde{\phi} = (2^* - 1) U^{2^* - 2}\tilde{\phi}$ in $\mathbb{R}^n$. By Bianchi–Egnell [4], it follows that $\tilde{\phi} = \sum_{i=0}^n \lambda_i V_i$ for some real numbers λ_i. It then follows from the orthogonality condition (36) that $\tilde{\phi} \equiv 0$ is identically zero. This proves the claim.

Letting $\varphi_\alpha := \phi_\alpha$ and using (37) together with a change of variable, we get

$$\|\phi_\alpha\|_h^2 = (2^\star - 1 - \varepsilon_\alpha) \int_M |u_0 - W_\alpha|^{2^\star - 2 - \varepsilon_\alpha} \phi_\alpha^2 \, dv_g + o(1)$$

$$\leq C \int_M \phi_\alpha^2 \, dv_g + C \int_M |W_\alpha|^{2^\star - 2 - \varepsilon_\alpha} \phi_\alpha^2 \, dv_g + o(1)$$

$$\leq C \int_M \phi_\alpha^2 \, dv_g + C \int_M |U|^{2^\star - 2 - \varepsilon_\alpha} \tilde{\phi}_\alpha^2 \, dv_{\tilde{g}_\alpha} + o(1)$$

as $\alpha \to +\infty$, where $\tilde{\phi}_\alpha$ and $\tilde{g}_\alpha$ are as in (33) and (34). Since $\phi_\alpha \to 0$ strongly in $L^2(M)$, $\left(\tilde{\phi}_\alpha^2\right)_\alpha$ is bounded in $L^{\frac{n}{n-2}}(\mathbb{R}^n)$ and converges almost everywhere to 0, standard elliptic theory yields $\phi_\alpha \to 0$ as $\alpha \to +\infty$ in $H_1^2(M)$. This is a contradiction with (25). This ends the proof of (24).

The invertibility of $L_{\varepsilon, \delta_\varepsilon(t), \xi}$ follows from the Fredholm alternative. This ends the proof of Lemma 4.2. □

Now, we prove Proposition 4.1 by using Lemma 4.2 together with the error estimate in Section 7.

Proof of Proposition 4.1. We let a and b be two positive real numbers such that $a < b$. For ε small, for any real number t in $[a,b]$, and any point ξ in M, equation (21) is equivalent to

$$(40) \qquad L_{\varepsilon,\delta_\varepsilon(t),\xi}(\phi) = N_{\varepsilon,\delta_\varepsilon(t),\xi}(\phi) + R_{\varepsilon,\delta_\varepsilon(t),\xi},$$

where $\delta_\varepsilon(t) = t\varepsilon^{2/(n-2)}$, $L_{\varepsilon,\delta_\varepsilon(t),\xi}$ is as in (23), and

$$(41) \qquad N_{\varepsilon,\delta_\varepsilon(t),\xi}(\phi) := \Pi^{\perp}_{\delta_\varepsilon(t),\xi}\big(i^*\big(f_\varepsilon\big(u_0 - W_{\delta_\varepsilon(t),\xi} + \phi\big) \\ - f_\varepsilon\big(u_0 - W_{\delta_\varepsilon(t),\xi}\big) - f'_\varepsilon\big(u_0 - W_{\delta_\varepsilon(t),\xi}\big)\phi\big)\big)$$

and

$$(42) \qquad R_{\varepsilon,\delta_\varepsilon(t),\xi} := \Pi^{\perp}_{\delta_\varepsilon(t),\xi}\big(i^*\big(f_\varepsilon\big(u_0 - W_{\delta_\varepsilon(t),\xi}\big)\big) - u_0 + W_{\delta_\varepsilon(t),\xi}\big).$$

By Lemma 4.2, for ε small, we get that the map $L_{\varepsilon,\delta_\varepsilon(t),\xi}$ is invertible for all real numbers t in $[a,b]$ and all points ξ in M. We then let the map $T_{\varepsilon,\delta_\varepsilon(t),\xi} : K^{\perp}_{\delta_\varepsilon(t),\xi} \to K^{\perp}_{\delta_\varepsilon(t),\xi}$ be defined for all $\phi \in K^{\perp}_{\delta_\varepsilon(t),\xi}$ by

$$T_{\varepsilon,\delta_\varepsilon(t),\xi}(\phi) := L^{-1}_{\varepsilon,\delta_\varepsilon(t),\xi}\left(N_{\varepsilon,\delta_\varepsilon(t),\xi}(\phi) + R_{\varepsilon,\delta_\varepsilon(t),\xi}\right),$$

where $N_{\varepsilon,\delta_\varepsilon(t),\xi}(\phi)$ and $R_{\varepsilon,\delta_\varepsilon(t),\xi}$ are as in (41) and (42). For any positive real number Ξ, we let $B_{\varepsilon,\delta_\varepsilon(t),\xi}(\Xi)$ be the closed ball defined by

$$\overline{B}_{\varepsilon,\delta_\varepsilon(t),\xi}(\Xi) := \left\{\phi \in K^{\perp}_{\delta_\varepsilon(t),\xi} \,/\, \|\phi\|_h \le \Xi\nu_\varepsilon\right\},$$

where $\nu_\varepsilon > 0$ is the error obtained in Lemma 7.1 of Section 7, namely

$$(43) \quad \nu_\varepsilon := \begin{cases} \varepsilon|\ln\varepsilon| & \text{if } n \le 6 \\ \varepsilon^{\frac{4}{n-2}} & \text{if } n \ge 7 \\ \varepsilon^{\frac{n+2}{2(n-2)}} & \text{if } n \ge 7, \ h \equiv c_n\,\mathrm{Scal}_g, \text{ and } (M,g) \text{ loc. conformally flat.} \end{cases}$$

We fix $\theta_0 \in (0, \min\{1, 2^\star - 2\})$, so that $u \mapsto f_\varepsilon(u)$ is locally in C^{1,θ_0} on $H^2_1(M)$ uniformly with respect to $\varepsilon > 0$ small. By Lemma 4.2 and by continuity of i^*, for ε small, for any real number t in $[a,b]$, any point ξ in M, and any functions ϕ, ϕ_1, and ϕ_2 in $H^2_1(M)$, we get that

$$\left\|T_{\varepsilon,\delta_\varepsilon(t),\xi}(\phi_1) - T_{\varepsilon,\delta_\varepsilon(t),\xi}(\phi_2)\right\|_h \le C \cdot \left(\max\{\|\phi_1\|_h^{\theta_0}, \|\phi_2\|_h^{\theta_0}\}\right) \cdot \|\phi_1 - \phi_2\|_h$$

for some positive constant C independent of Ξ, ε, t, ξ, ϕ, ϕ_1, and ϕ_2, where ν_ε is as in (43). By Lemma 7.1, we have that $\left\|T_{\varepsilon,\delta_\varepsilon(t),\xi}(0)\right\| \le C\nu_\varepsilon$. We then get that for $\Xi > 0$ large enough, and then for ε small, for any real number t in $[a,b]$, and any point ξ in M, then the map $T_{\varepsilon,\delta_\varepsilon(t),\xi}$ is a contraction map from the closed ball $\overline{B}_{\varepsilon,\delta_\varepsilon(t),\xi}(\Xi)$ into itself . We then get that the map $T_{\varepsilon,\delta_\varepsilon(t),\xi}$ admits a unique fixed point $\phi_{\delta_\varepsilon(t),\xi}$ in the ball $\overline{B}_{\varepsilon,\delta_\varepsilon(t),\xi}(\Xi)$. In other words, the function $\phi_{\delta_\varepsilon(t),\xi}$ is the unique solution of equation (40), or equivalently (21), which satisfies (22) with $C_{a,b} = \Xi$.

The continuous differentiability of $(t,\xi) \mapsto \phi_{\delta_\varepsilon(t),\xi}$ on $(a,b) \times M$ is standard. This ends the proof of Proposition 4.1. $\qquad\qquad\square$

5. The reduced problem

For ε small, we introduce the functional J_ε defined on $H_1^2(M)$ by

$$J_\varepsilon(u) := \frac{1}{2}\int_M |\nabla u|_g^2\, dv_g + \frac{1}{2}\int_M hu^2 dv_g - \int_M F_\varepsilon(u)\, dv_g,$$

where $F_\varepsilon(u) := \int_0^u f_\varepsilon(s)\, ds$. The critical points of J_ε are the solutions of equation (13). For any positive real number t and any point ξ in M, we define

(44) $$\mathcal{J}_\varepsilon(t,\xi) := J_\varepsilon\left(u_0 - W_{\delta_\varepsilon(t),\xi} + \phi_{\delta_\varepsilon(t),\xi}\right),$$

where $W_{\delta_\varepsilon(t),\xi}$ is as in (14) and $\phi_{\delta_\varepsilon(t),\xi}$ is given by Proposition 4.1. We solve equation (20) in Proposition 5.1 below:

Proposition 5.1. *Let $u_0 \in C^{2,\theta}(M)$ be a positive nondegenerate solution to (7). Assume that either $\{h \in C^{0,\theta}(M)$ and $3 \le n \le 6\}$ or $\{h \in C^2(M)$ and $3 \le n \le 9\}$ or $\{(M,g)$ is locally conformally flat and $h \equiv c_n \operatorname{Scal}_g\}$. Then*

(45)
$$\mathcal{J}_\varepsilon(t,\xi) = c_1(n,u_0) + c_2(n,u_0)\varepsilon + c_3(n)\varepsilon\ln\varepsilon + c_4(n)\varepsilon\ln\frac{1}{t} + c_5(n)\left(\varepsilon t^{\frac{n-2}{2}} u_0(\xi)\right.$$

$$\left. + \frac{n^{\frac{n-2}{4}}(n-2)^{\frac{n-6}{4}}(n-1)\omega_n \varepsilon^{\frac{4}{n-2}}t^2}{2^{n-1}(n-4)\omega_{n-1}} \cdot (h(\xi) - c_n\operatorname{Scal}_g(\xi))\mathbf{1}_{n\ge 6}\right) + \mathrm{o}(\varepsilon)$$

as $\varepsilon \to 0$, uniformly with respect to t in compact subsets of $\mathbb{R}_{>0}$ and with respect to the point ξ in M, ω_n (resp. ω_{n-1}) is the volume of the unit n-sphere (resp. $(n-1)$-sphere), $c_i(n,u_0)$ $(i=1,2)$ are positive constants depending only on n, u_0, and the manifold, $c_i(n)$ $(i=3,4,5)$ depend only on n, and $c_4(n), c_5(n) > 0$. Moreover, given two positive real numbers $a < b$, for ε small, if $(t_\varepsilon, \xi_\varepsilon) \in (a,b) \times M$ is a critical point of $\mathcal{J}_\varepsilon$, then the function $u_0 - W_{\delta_\varepsilon(t_\varepsilon),\xi_\varepsilon} + \phi_{\delta_\varepsilon(t_\varepsilon),\xi_\varepsilon}$ is a solution to equation (13), or equivalently (4).

This section is devoted to the proof of Proposition 5.1. We define the optimal Sobolev constant K_n by

(46) $$\frac{1}{K_n} := \inf_{u \in D_1^2(\mathbb{R}^n)\setminus\{0\}} \frac{\|\nabla u\|_2}{\|u\|_{2^\star}} = \sqrt{\frac{n(n-2)\omega_n^{2/n}}{4}},$$

where ω_n is the volume of the unit n-sphere: see Aubin [2], Talenti [35], Rodemich [31]. The infimum in (46) is achieved by the function U defined in (15).

Lemma 5.2. *Let $u_0 \in C^2(M)$ be a positive solution to (14). Assume that either $\{h \in C^{0,\theta}(M)$ and $3 \le n \le 6\}$ or $\{h \in C^2(M)$ and $3 \le n \le 9\}$ or $\{(M,g)$ is locally conformally flat and $h \equiv c_n \operatorname{Scal}_g\}$. Then*

(47) $$J_\varepsilon\left(u_0 - W_{\delta_\varepsilon(t),\xi}\right) = \frac{1}{n}\int_M u_0^{2^*} dv_g + \frac{\varepsilon}{2^*}\int_M u_0^{2^*}\left(\ln u_0 - \frac{1}{2^*}\right) dv_g$$

$$+ \frac{K_n^{-n}}{n}\left(1 - \beta_n\varepsilon - \frac{n-2}{2}\varepsilon\ln\varepsilon - \frac{(n-2)^2}{4}\varepsilon\ln t + \frac{2^n\omega_{n-1}\varepsilon t^{\frac{n-2}{2}} u_0(\xi)}{(n(n-2))^{\frac{n-2}{4}}\omega_n}\right.$$

$$\left. + \frac{2(n-1)\varepsilon^{\frac{4}{n-2}}t^2}{(n-2)(n-4)}(h(\xi) - c_n\operatorname{Scal}_g(\xi))\mathbf{1}_{n\ge 6} + \mathrm{o}(\varepsilon)\right)$$

as $\varepsilon \to 0$, *uniformly with respect to t in compact subsets of $\mathbb{R}_{>0}$ and with respect to the point ξ in M, where ω_n (resp. ω_{n-1}) is the volume of the unit n-sphere (resp. $(n-1)$-sphere), K_n is as in (46), and*
$$(48)$$
$$\beta_n = 2^{n-3}(n-2)^2 \frac{\omega_{n-1}}{\omega_n} \int_0^{+\infty} \frac{r^{\frac{n-2}{2}} \ln(1+r)}{(1+r)^n} dr + \frac{(n-2)^2}{4n} \left(1 - n \ln \sqrt{n(n-2)}\right).$$

Proof. All our estimates in this proof are uniform with respect to t in compact subsets of $\mathbb{R}_{>0}$, with respect to the point ξ in M, and with respect to ε in $(0, \varepsilon_0)$ for some fixed positive real number ε_0. Expanding $J_\varepsilon(u_0 - W_{\delta_\varepsilon(t),\xi})$, using that u_0 is a solution to (7) and rough estimates yield

$$(49) \quad J_\varepsilon\left(u_0 - W_{\delta_\varepsilon(t),\xi}\right) = \frac{1}{n} \int_M u_0^{2^*} dv_g + \frac{\varepsilon}{2^*} \int_M u_0^{2^*} \left(\ln u_0 - \frac{1}{2^*}\right) dv_g$$
$$+ I_{1,\varepsilon,t,\xi} + I_{2,\varepsilon,t,\xi} - I_{3,\varepsilon,t,\xi} + \mathrm{O}\left(\varepsilon^2\right)$$

when $\varepsilon \to 0$ where

$$I_{1,\varepsilon,t,\xi} := \frac{1}{2} \int_M \left|\nabla W_{\delta_\varepsilon(t),\xi}\right|_g^2 dv_g + \frac{1}{2} \int_M h W_{\delta_\varepsilon(t),\xi}^2 dv_g - \frac{1}{2^* - \varepsilon} \int_M W_{\delta_\varepsilon(t),\xi}^{2^* - \varepsilon} dv_g,$$

$$I_{2,\varepsilon,t,\xi} := \int_M u_0 W_{\delta_\varepsilon(t),\xi}^{2^* - 1 - \varepsilon} dv_g,$$

$$I_{3,\varepsilon,t,\xi} := \int_M \left(F_\varepsilon\left(u_0 - W_{\delta_\varepsilon(t),\xi}\right) - F_\varepsilon(u_0) - F_\varepsilon\left(W_{\delta_\varepsilon(t),\xi}\right)\right.$$
$$\left. + f_\varepsilon(u_0) W_{\delta_\varepsilon(t),\xi} + f_\varepsilon\left(W_{\delta_\varepsilon(t),\xi}\right) u_0\right) dv_g.$$

We estimate these terms separately.

Step 1: Estimate of $I_{1,\varepsilon,t,\xi}$ in the locally conformally case when $h \equiv c_n \operatorname{Scal}_g$.
In case $h \equiv c_n \operatorname{Scal}_g$ and the manifold is locally conformally flat, the conformal change of metric $g_\xi = \Lambda_\xi^{4/(n-2)} g$ yields

$$\frac{1}{2} \int_M \left|\nabla W_{\delta_\varepsilon(t),\xi}\right|_g^2 dv_g + \frac{1}{2} \int_M h W_{\delta_\varepsilon(t),\xi}^2 dv_g - \frac{1}{2^* - \varepsilon} \int_M W_{\delta_\varepsilon(t),\xi}^{2^* - \varepsilon} dv_g$$
$$= \frac{1}{2} \int_M \left|\nabla \widetilde{W}_{\delta_\varepsilon(t),\xi}\right|_{g_\xi}^2 dv_{g_\xi} - \frac{1}{2^* - \varepsilon} \int_M \Lambda_\xi^{-\varepsilon} \widetilde{W}_{\delta_\varepsilon(t),\xi}^{2^* - \varepsilon} dv_{g_\xi},$$

where $\widetilde{W}_{\delta_\varepsilon(t),\xi} = W_{\delta_\varepsilon(t),\xi}/\Lambda_\xi$. In this case, since the metric g_ξ is flat in $B_\xi(r_0)$, we find

$$\int_M \left|\nabla \widetilde{W}_{\delta_\varepsilon(t),\xi}\right|_{g_\xi}^2 dv_{g_\xi} = \int_{\mathbb{R}^n} |\nabla U|^2 dx + \mathrm{O}\left(\delta_\varepsilon(t)^{n-2}\right) = K_n^{-n} + \mathrm{O}\left(\varepsilon^2\right)$$

when $\varepsilon \to 0$. Moreover, since g_ξ is flat around ξ, we get that

$$
\frac{1}{2^* - \varepsilon} \int_M \Lambda_\xi^{-\varepsilon} \widetilde{W}_{\delta_\varepsilon(t),\xi}^{2^*-\varepsilon} dv_{g_\xi}
$$

$$
= \frac{(n(n-2))^{\frac{n-2}{4}(2^*-\varepsilon)}}{2^* - \varepsilon} \omega_{n-1} \delta_\varepsilon(t)^{\frac{n-2}{2}\varepsilon} \int_0^{\frac{r_0}{2\delta_\varepsilon(t)}} \frac{r^{n-1}dr}{(1+r^2)^{\frac{n-2}{2}(2^*-\varepsilon)}} + O(\delta_\varepsilon(t)^n)
$$

$$
+ O(\varepsilon \delta_\varepsilon(t))
$$

$$
= \frac{n-2}{2n} K_n^{-n} \left(1 + \frac{2\beta_n}{n-2}\varepsilon + \varepsilon \ln \varepsilon + \frac{n-2}{2}\varepsilon \ln t \right) + O\left(\varepsilon^2 |\ln \varepsilon|^2 \right) + O(\varepsilon \delta_\varepsilon(t)),
$$

where K_n is as in (46), and β_n is as in (48). Therefore, we get that

$$
(50) \qquad I_{1,\varepsilon,t,\xi} = \frac{K_n^{-n}}{n} \left(1 - \beta_n \varepsilon - \frac{n-2}{2}\varepsilon \ln \varepsilon - \frac{(n-2)^2}{4}\varepsilon \ln t \right) + o(\varepsilon)
$$

when $\varepsilon \to 0$ uniformly for all $\xi \in M$ and t in a compact of $(0, +\infty)$ when $h \equiv c_n \operatorname{Scal}_g$ and (M, g) is locally conformally flat.

Step 2: Estimate of $I_{1,\varepsilon,t,\xi}$ in the general case.
Cartan's expansion of the metric in geodesic normal coordinates yields for any $\alpha, \beta = 1, \ldots, n$ and for y close to 0, there holds

$$
(51) \qquad \sqrt{|g(\exp_\xi y)|} = 1 - \frac{1}{6} \sum_{\mu,\nu=1}^n R_{\mu\nu}(\xi) y^\mu y^\nu + P_3(y) + O\left(|y|^4\right),
$$

where the function $|g|$ is the determinant of the metric, the functions $R_{\mu\nu}$ are the components of the Ricci curvature tensor in geodesic normal coordinates associated with the map $\exp_\xi$ and $P_3(y)$ is a homogenous polynomial of degree three. Using (51) together with expression of the gradient of a radially symmetrical function in geodesic normal chart, we get that

$$
(52) \quad \int_M \left| \nabla W_{\delta_\varepsilon(t),\xi} \right|_g^2 dv_g = n^{\frac{n-2}{2}} (n-2)^{\frac{n+2}{2}} \omega_{n-1}
$$

$$
\times \int_0^{\frac{r_0}{2\delta_\varepsilon(t)}} \frac{r^{n+1}}{(1+r^2)^n} \left(1 - \frac{1}{6n} \operatorname{Scal}_g(\xi) \delta_\varepsilon(t)^2 r^2 + O\left(\delta_\varepsilon(t)^4 r^4\right) \right) dr
$$

$$
+ O\left(\delta_\varepsilon(t)^{n-2}\right)
$$

$$
= \begin{cases}
K_3^{-3} + O\left(\varepsilon^2\right) & \text{if } n = 3 \\[2mm]
K_4^{-4} \left(1 + \frac{1}{4} \operatorname{Scal}_g(\xi) t^2 \varepsilon^2 \ln \varepsilon \right) + O\left(\varepsilon^2\right) & \text{if } n = 4 \\[2mm]
K_n^{-n} \left(1 - \frac{n+2}{6n(n-4)} \operatorname{Scal}_g(\xi) t^2 \varepsilon^{\frac{4}{n-2}} \right) + O\left(\varepsilon^{\frac{8}{n-2}} + \varepsilon^2 |\ln \varepsilon|\right) & \text{if } n \geq 5
\end{cases}
$$

when $\varepsilon \to 0$. Taylor's expansion at ξ yields on the one hand

$$(53) \quad \frac{1}{2^* - \varepsilon} \int_M W^{2^* - \varepsilon}_{\delta_\varepsilon(t), \xi} dv_g = \frac{(n(n-2))^{\frac{n-2}{4}(2^* - \varepsilon)}}{2^* - \varepsilon} \omega_{n-1} \delta_\varepsilon(t)^{\frac{n-2}{2}\varepsilon}$$

$$\times \int_0^{\frac{r_0}{2\delta_\varepsilon(t)}} \frac{r^{n-1}}{(1+r^2)^{\frac{n-2}{2}(2^* - \varepsilon)}} \left(1 - \frac{1}{6n} \mathrm{Scal}_g(\xi) \delta_\varepsilon(t)^2 r^2 \right.$$

$$\left. + \mathrm{O}\left(\delta_\varepsilon(t)^4 r^4 \right) \right) dr + \mathrm{O}\left(\delta_\varepsilon(t)^n \right)$$

$$= \frac{n-2}{2n} K_n^{-n} \left(1 + \frac{2\beta_n}{n-2}\varepsilon + \varepsilon \ln \varepsilon + \frac{n-2}{2}\varepsilon \ln t - \frac{1}{6(n-2)} \mathrm{Scal}_g(\xi) t^2 \varepsilon^{\frac{4}{n-2}} \right)$$

$$+ \mathrm{O}\left(\varepsilon^{\frac{8}{n-2}} + \varepsilon^2 |\ln \varepsilon|^2 \right)$$

when $\varepsilon \to 0$. On the other hand,

$$(54) \quad \int_M h W^2_{\delta_\varepsilon(t), \xi} dv_g = n^{\frac{n-2}{2}} (n-2)^{\frac{n-2}{2}} \delta_\varepsilon(t)^2$$

$$\times \int_{B_{\frac{r_0}{2\delta_\varepsilon(t)}}(0)} \frac{h(\exp_\xi(\delta_\varepsilon(t)x))}{(1+|x|^2)^{n-2}} (1 + \mathrm{O}(\delta_\varepsilon(t)^2 |x|^2)) \, dx + \mathrm{O}\left(\delta_\varepsilon(t)^{n-2} \right)$$

$$= \begin{cases} \mathrm{O}\left(\varepsilon^2 \right) & \text{if } n = 3 \\ -\dfrac{3}{2} K_4^{-4} h(\xi) t^2 \varepsilon^2 \ln \varepsilon + \mathrm{O}\left(\varepsilon^2 \right) & \text{if } n = 4 \\ \dfrac{4(n-1)}{n(n-2)(n-4)} K_n^{-n} h(\xi) t^2 \varepsilon^{\frac{4}{n-2}} + \mathrm{O}(R_\varepsilon) & \text{if } n \geq 5 \end{cases}$$

when $\varepsilon \to 0$, where

$$R_\varepsilon := \begin{cases} \varepsilon^{\frac{8}{n-2}} + \varepsilon^2 |\ln \varepsilon| & \text{if } h \in C^2(M) \\ \varepsilon^{\frac{2(2+\theta)}{n-2}} & \text{if } h \in C^{0,\theta}(M). \end{cases}$$

Plugging together (52), (54), and (53) yields

$$(55) \quad I_{1,\varepsilon,t,\xi} = \frac{K_n^{-n}}{n} \left(1 - \beta_n \varepsilon - \frac{n-2}{2}\varepsilon \ln \varepsilon - \frac{(n-2)^2}{4}\varepsilon \ln t \right.$$

$$\left. + \frac{2(n-1)\varepsilon^{\frac{4}{n-2}} t^2}{(n-2)(n-4)} \left(h(\xi) - c_n \mathrm{Scal}_g(\xi) \right) \mathbf{1}_{n \geq 6} \right) + \mathrm{o}(\varepsilon) + \mathrm{O}(R_\varepsilon)$$

when $\varepsilon \to 0$.

Step 3: Estimate of $I_{2,\varepsilon,t,\xi}$.

$$(56) \quad \int_M u_0 W^{2^* - 1 - \varepsilon}_{\delta_\varepsilon(t), \xi} dv_g = (n(n-2))^{\frac{n-2}{4}(2^* - 1 - \varepsilon)} \omega_{n-1} u_0(\xi) \delta_\varepsilon(t)^{\frac{n-2}{2}(1+\varepsilon)}$$

$$\times \int_0^{\frac{r_0}{2\delta_\varepsilon(t)}} \frac{r^{n-1}}{(1+r^2)^{\frac{n-2}{2}(2^* - 1 - \varepsilon)}} \left(1 + \mathrm{O}\left(\delta_\varepsilon(t)^2 r^2 \right) \right) dr + \mathrm{O}\left(\delta_\varepsilon(t)^{\frac{n+2}{2}} \right)$$

$$= \frac{2^n \omega_{n-1} K_n^{-n} u_0(\xi) t^{\frac{n-2}{2}}\varepsilon}{n^{\frac{n+2}{4}} (n-2)^{\frac{n-2}{4}} \omega_n} + \mathrm{O}\left(\varepsilon^{\frac{n+2}{n-2}} |\ln \varepsilon| + \varepsilon^2 |\ln \varepsilon| \right)$$

when $\varepsilon \to 0$, where K_n is as in (46), and β_n is as in (48). We have used here that $\int_{\mathbb{R}^n} U^{2^\star-1}\,dx = \lim_{R \to +\infty} \int_{B_R(0)} \Delta_{\mathrm{Eucl}} U\,dx$ and integrated by parts.

Step 4: Estimate of $I_{3,\varepsilon,t,\xi}$.
We have that

$$(57) \quad \left| \int_M \left(F_\varepsilon \left(u_0 - W_{\delta_\varepsilon(t),\xi} \right) - F_\varepsilon \left(u_0 \right) - F_\varepsilon \left(W_{\delta_\varepsilon(t),\xi} \right) \right. \right.$$

$$\left. \left. + f_\varepsilon \left(u_0 \right) W_{\delta_\varepsilon(t),\xi} + f_\varepsilon \left(W_{\delta_\varepsilon(t),\xi} \right) u_0 \right) dv_g \right|$$

$$\leq \int_{B_\xi(\sqrt{\delta_\varepsilon(t)})} \left| F_\varepsilon \left(u_0 - W_{\delta_\varepsilon(t),\xi} \right) - F_\varepsilon \left(W_{\delta_\varepsilon(t),\xi} \right) + f_\varepsilon \left(W_{\delta_\varepsilon(t),\xi} \right) u_0 \right| dv_g$$

$$+ \int_{M \setminus B_\xi(\sqrt{\delta_\varepsilon(t)})} \left| F_\varepsilon \left(u_0 - W_{\delta_\varepsilon(t),\xi} \right) - F_\varepsilon \left(u_0 \right) + f_\varepsilon \left(u_0 \right) W_{\delta_\varepsilon(t),\xi} \right| dv_g$$

$$+ \int_{B_\xi(\sqrt{\delta_\varepsilon(t)})} \left| F_\varepsilon \left(u_0 \right) \right| dv_g + \int_{B_\xi(\sqrt{\delta_\varepsilon(t)})} \left| f_\varepsilon \left(u_0 \right) W_{\delta_\varepsilon(t),\xi} \right| dv_g$$

$$+ \int_{M \setminus B_\xi(\sqrt{\delta_\varepsilon(t)})} \left| F_\varepsilon \left(W_{\delta_\varepsilon(t),\xi} \right) \right| dv_g + \int_{M \setminus B_\xi(\sqrt{\delta_\varepsilon(t)})} \left| f_\varepsilon \left(W_{\delta_\varepsilon(t),\xi} \right) u_0 \right| dv_g \,.$$

As is easily checked, Taylor expansions of $F(u_0 - W_{\delta_\varepsilon(t),\xi})$ yield

$$\int_{B_\xi(\sqrt{\delta_\varepsilon(t)})} \left| F_\varepsilon \left(u_0 - W_{\delta_\varepsilon(t),\xi} \right) - F_\varepsilon \left(W_{\delta_\varepsilon(t),\xi} \right) + f_\varepsilon \left(W_{\delta_\varepsilon(t),\xi} \right) u_0 \right| dv_g$$

$$(58) \qquad = \mathrm{O} \left(\int_{B_\xi(\sqrt{\delta_\varepsilon(t)})} u_0^2 W_{\delta_\varepsilon(t),\xi}^{2^\star-2-\varepsilon} dv_g \right),$$

$$\int_{M \setminus B_\xi(\sqrt{\delta_\varepsilon(t)})} \left| F_\varepsilon \left(u_0 - W_{\delta_\varepsilon(t),\xi} \right) - F_\varepsilon \left(u_0 \right) + f_\varepsilon \left(u_0 \right) W_{\delta_\varepsilon(t),\xi} \right| dv_g$$

$$(59) \qquad = \mathrm{O} \left(\int_{M \setminus B_\xi(\sqrt{\delta_\varepsilon(t)})} u_0^{2^\star-2-\varepsilon} W_{\delta_\varepsilon(t),\xi}^2 dv_g \right).$$

Bounding u_0 and $W_{\delta_\varepsilon(t),\xi}$ pointwisely roughly from above in (58) and (59) and plugging this in (57) yields

$$(60) \qquad I_{3,\varepsilon,t,\xi} = \begin{cases} \mathrm{O}\left(\varepsilon^2\right) & \text{if } n = 3 \\ \mathrm{O}\left(\varepsilon^2 \left|\ln \varepsilon\right|\right) & \text{if } n = 4 \\ \mathrm{O}\left(\varepsilon^{\frac{n}{n-2}}\right) & \text{if } n \geq 5 \end{cases}$$

when $\varepsilon \to 0$.

Step 5: End of proof of Lemma 5.2.
The asymptotic expansion (47) follows from (49), (50), (55), (56) and (60).   □

In Lemma 5.3 below, we show that the first order terms in the asymptotic expansion of $\mathcal{J}_\varepsilon(t,\xi)$, defined in (44), are the same as for $J_\varepsilon\left(u_0 - W_{\delta_\varepsilon(t),\xi}\right)$.

Lemma 5.3. *Assume that either $\{3 \leq n \leq 9\}$ or $\{(M,g)$ is locally conformally flat and $h \equiv c_n \operatorname{Scal}_g\}$. Then*

$$(61) \qquad \mathcal{J}_\varepsilon(t,\xi) = J_\varepsilon\left(u_0 - W_{\delta_\varepsilon(t),\xi}\right) + \mathrm{o}\left(\varepsilon\right)$$

as $\varepsilon \to 0$, *uniformly with respect to t in compact subsets of $\mathbb{R}_{>0}$ and with respect to the point ξ in M.*

Proof. All the estimates in this proof are uniform with respect to t on compact subsets of $\mathbb{R}_{>0}$, with respect to the point ξ in M, and with respect to ε in $(0, \varepsilon_0)$ for some fixed positive real number ε_0. We get that

$$(62) \quad \mathcal{J}_\varepsilon\left(t, \xi\right) - J_\varepsilon\left(u_0 - W_{\delta_\varepsilon(t), \xi}\right)$$
$$= \left\langle u_0 - W_{\delta_\varepsilon(t), \xi} - i^*\left(f_\varepsilon\left(u_0 - W_{\delta_\varepsilon(t), \xi}\right)\right), \phi_{\delta_\varepsilon(t), \xi}\right\rangle_h + \mathrm{O}\left(\left\|\phi_{\delta_\varepsilon(t), \xi}\right\|_h^2\right)$$

when $\varepsilon \to 0$. Proposition 4.1 and Lemma 7.1 yield

$$(63) \quad \left\langle u_0 - W_{\delta_\varepsilon(t), \xi} - i^*\left(f_\varepsilon\left(u_0 - W_{\delta_\varepsilon(t), \xi}\right)\right), \phi_{\delta_\varepsilon(t), \xi}\right\rangle_h + \mathrm{O}\left(\left\|\phi_{\delta_\varepsilon(t), \xi}\right\|_h^2\right)$$
$$= \begin{cases} \mathrm{O}\left(\varepsilon^2 \left|\ln \varepsilon\right|^2\right) & \text{if } n \le 6 \\ \mathrm{O}\left(\varepsilon^{\frac{8}{n-2}}\right) & \text{if } n \ge 7 \\ \mathrm{O}\left(\varepsilon^{\frac{n+2}{n-2}}\right) & \text{if } n \ge 7, \, h \equiv c_n \, \mathrm{Scal}_g, \, (M, g) \text{ loc. conformally flat} \end{cases}$$

when $\varepsilon \to 0$. Finally, (61) follows from (62), and (63). $\qquad \square$

The asymptotic expansion (45) follows from Lemmas 5.2 and 5.3. Now, we prove the second part of Proposition 5.1.

End of proof of Proposition 5.1. Given two positive real numbers $a < b$, it remains to prove that for ε small, if $(t_\varepsilon, \xi_\varepsilon) \in [a, b] \times M$ is a critical point of $\mathcal{J}_\varepsilon$, then the function $u_0 - W_{\delta_\varepsilon(t_\varepsilon), \xi_\varepsilon} + \phi_{\delta_\varepsilon(t_\varepsilon), \xi_\varepsilon}$ is a solution of equation (13). In order to prove this claim, we consider a sequence of points $(\xi_\alpha)_\alpha$ in M and two sequences of positive real numbers $(\varepsilon_\alpha)_\alpha$ and $(t_\alpha)_\alpha$ such that $\varepsilon_\alpha \to 0$ as $\alpha \to +\infty$, $a \le t_\alpha \le b$, and (t_α, ξ_α) is a critical point of $\mathcal{J}_{\varepsilon_\alpha}$ for all α. It is enough to show that for α large, the function $u_0 - W_{\delta_{\varepsilon_\alpha}(t_\alpha), \xi_\alpha} + \phi_{\delta_{\varepsilon_\alpha}(t_\alpha), \xi_\alpha}$ is a solution of equation (13). As in the proof of Lemma 4.2, up to a subsequence, we identify the tangent space with $\mathbb{R}^n$ around the ξ_α's. We define

$$(64) \qquad Z_{0, \delta_{\varepsilon_\alpha}(t_\alpha), \xi_\alpha} := Z_{\delta_{\varepsilon_\alpha}(t_\alpha), \xi_\alpha} \quad \text{and} \quad Z_{i, \delta_{\varepsilon_\alpha}(t_\alpha), \xi_\alpha} := Z_{\delta_{\varepsilon_\alpha}(t_\alpha), \xi_\alpha, e_i}$$

for all $i = 1, \dots, n$, where e_i is the i-th vector in the canonical basis of $\mathbb{R}^n$ and the functions $Z_{\delta_{\varepsilon_\alpha}(t_\alpha), \xi_\alpha}$ and $Z_{\delta_{\varepsilon_\alpha}(t_\alpha), \xi_\alpha, e_i}$ are as in (17) and (18). By Proposition 4.1, we get that

$$(65) \qquad DJ_{\varepsilon_\alpha}\left(u_0 - W_{\delta_{\varepsilon_\alpha}(t_\alpha), \xi_\alpha} + \phi_{\delta_{\varepsilon_\alpha}(t_\alpha), \xi_\alpha}\right) = \sum_{i=0}^n \lambda_{i, \alpha} \left\langle Z_{i, \delta_{\varepsilon_\alpha}(t_\alpha), \xi_\alpha}, \cdot \right\rangle_h$$

for some real numbers $\lambda_{i, \alpha}$, where the functions $Z_{i, \delta_{\varepsilon_\alpha}(t_\alpha), \xi_\alpha}$ are as in (64). It follows from (65) that

$$(66) \quad \frac{\partial \mathcal{J}_{\varepsilon_\alpha}}{\partial t}\left(t_\alpha, \xi_\alpha\right) = \sum_{i=0}^n \lambda_{i, \alpha} \langle Z_{i, \delta_{\varepsilon_\alpha}(t_\alpha), \xi_\alpha}, \left. \frac{d}{dt}\left(-W_{\delta_{\varepsilon_\alpha}(t), \xi_\alpha} + \phi_{\delta_{\varepsilon_\alpha}(t), \xi_\alpha}\right)\right|_{t=t_\alpha} \rangle_h.$$

On the one hand, we find

$$(67) \qquad \left. \frac{d}{dt} W_{\delta_{\varepsilon_\alpha}(t), \xi_\alpha}\right|_{t=t_\alpha} = \frac{n^{\frac{n-2}{4}} (n-2)^{\frac{n+2}{4}}}{2t_\alpha} Z_{0, \delta_{\varepsilon_\alpha}(t_\alpha), \xi_\alpha}.$$

On the other hand, for any $i = 0, \ldots, n$ and any α, since the function $\phi_{\delta_{\varepsilon_\alpha}(t_\alpha),\xi_\alpha}$ belongs to $K_{\delta_{\varepsilon_\alpha}(t_\alpha),\xi_\alpha}^\perp$, differentiating $\left\langle Z_{i,\delta_{\varepsilon_\alpha}(t),\xi_\alpha}, \phi_{\delta_{\varepsilon_\alpha}(t),\xi_\alpha} \right\rangle_h = 0$ with respect to t yields

$$(68) \quad \left\langle Z_{i,\delta_{\varepsilon_\alpha}(t_\alpha),\xi_\alpha}, \left. \frac{d}{dt} \phi_{\delta_{\varepsilon_\alpha}(t),\xi_\alpha} \right|_{t=t_\alpha} \right\rangle_h = -\left\langle \left. \frac{d}{dt} Z_{i,\delta_{\varepsilon_\alpha}(t),\xi_\alpha} \right|_{t=t_\alpha}, \phi_{\delta_{\varepsilon_\alpha}(t_\alpha),\xi_\alpha} \right\rangle_h.$$

Moreover, one easily checks

$$(69) \quad \left\| \left. \frac{d}{dt} Z_{i,\delta_{\varepsilon_\alpha}(t),\xi_\alpha} \right|_{t=t_\alpha} \right\|_h = \mathrm{O}\,(1)$$

as $\alpha \to +\infty$. Proposition 4.1, (68), (69), (29), (66), and (67) yield

$$(70) \quad \frac{\partial \mathcal{J}_{\varepsilon_\alpha}}{\partial t}(t_\alpha, \xi_\alpha) = -\frac{n^{\frac{n-2}{4}} (n-2)^{\frac{n+2}{4}}}{2t_\alpha} \lambda_{0,\alpha} \|\nabla V_0\|_2^2 + \mathrm{o}\left(\sum_{i=0}^n |\lambda_{i,\alpha}|\right)$$

as $\alpha \to +\infty$, where the function V_0 is as in (16). For any $i = 1, \ldots, n$, by (65), we get that

$$\left. \frac{d}{dy_i} \mathcal{J}_{\varepsilon_\alpha}\left(t_\alpha, \exp_{\xi_\alpha} y\right) \right|_{y=0}$$

$$= \sum_{j=0}^n \lambda_{j,\alpha} \left\langle Z_{j,\delta_{\varepsilon_\alpha}(t_\alpha),\xi_\alpha}, \left. \frac{d}{dy_i}\left(-W_{\delta_{\varepsilon_\alpha}(t_\alpha),\exp_{\xi_\alpha} y} + \phi_{\delta_{\varepsilon_\alpha}(t_\alpha),\exp_{\xi_\alpha} y}\right)\right|_{y=0} \right\rangle_h,$$

where the exponential map is taken with respect to the metric g_{ξ_α}. On the one hand, direct computations yield

$$\left. \frac{d}{dy_i} W_{\delta_{\varepsilon_\alpha}(t_\alpha),\exp_{\xi_\alpha} y} \right|_{y=0} = \frac{n^{\frac{n-2}{4}} (n-2)^{\frac{n+2}{4}}}{\delta_{\varepsilon_\alpha}(t)} \left(Z_{i,\delta_{\varepsilon_\alpha}(t_\alpha),\xi_\alpha} + R_{i,\delta_{\varepsilon_\alpha}(t_\alpha),\xi_\alpha}\right),$$

where $R_{i,\delta_{\varepsilon_\alpha}(t_\alpha),\xi_\alpha} \to 0$ as $\alpha \to +\infty$ in $H_1^2\,(M)$. For any $i = 1, \ldots, n$, $j = 0, \ldots, n$, and any α, since the function $\phi_{\delta_{\varepsilon_\alpha}(t_\alpha),\xi_\alpha} \in K_{\delta_{\varepsilon_\alpha}(t_\alpha),\xi_\alpha}^\perp$, differentiating the equation $\left\langle Z_{j,\delta_{\varepsilon_\alpha}(t_\alpha),\exp_{\xi_\alpha} y}, \phi_{\delta_{\varepsilon_\alpha}(t_\alpha),\exp_{\xi_\alpha} y} \right\rangle_h = 0$ with respect to y_i at 0 yields

$$\left\langle Z_{j,\delta_{\varepsilon_\alpha}(t_\alpha),\xi_\alpha}, \frac{d}{dy_i} \phi_{\delta_{\varepsilon_\alpha}(t_\alpha),\exp_{\xi_\alpha} y} \right\rangle_h = -\left\langle \frac{d}{dy_i} Z_{j,\delta_{\varepsilon_\alpha}(t_\alpha),\exp_{\xi_\alpha} y}, \phi_{\delta_{\varepsilon_\alpha}(t_\alpha),\xi_\alpha} \right\rangle_h.$$

Moreover, one easily checks

$$\left\| \left. \frac{d}{dy_i} Z_{j,\delta_{\varepsilon_\alpha}(t_\alpha),\exp_{\xi_\alpha} y} \right|_{y=0} \right\|_h = \mathrm{O}\left(\frac{1}{\delta_{\varepsilon_\alpha}(t)}\right)$$

as $\alpha \to +\infty$. Similarly to the derivative in the $t-$direction, we get that
$$(71)$$

$$\delta_{\varepsilon_\alpha}(t_\alpha) \left. \frac{d}{dy_i} \mathcal{J}_{\varepsilon_\alpha}\left(t_\alpha, \exp_{\xi_\alpha} y\right) \right|_{y=0} = -n^{\frac{n-2}{4}} (n-2)^{\frac{n+2}{4}} \lambda_{i,\alpha} \|\nabla V_i\|_2 + \mathrm{o}\left(\sum_{j=0}^n |\lambda_{j,\alpha}|\right)$$

as $\alpha \to +\infty$, where the function V_i is as in (16). If (t_α, ξ_α) is a critical point of $\mathcal{J}_{\varepsilon_\alpha}$ for all α, then it follows from (70) and (71) that for any $i = 0, \ldots, n$, there $\lambda_{i,\alpha} = 0$ for all $i = 0, \ldots, n$. By (65), if follows that for α large, the function $u_0 - W_{\delta_{\varepsilon_\alpha}(t_\alpha),\xi_\alpha} + \phi_{\delta_{\varepsilon_\alpha}(t_\alpha),\xi_\alpha}$ is a critical point of the functional J_{ε_α}, and therefore a solution of equation (13). This ends the proof of Proposition 5.1. $\quad\square$

6. Proof of the theorems

Proof of Theorems 1.2 and 2.2. We let $\mathcal{G}$ be the function defined on $\mathbb{R}_{>0} \times M$ by

$$(72) \qquad \mathcal{G}(t, \xi) := c_4(n) \ln \frac{1}{t} + c_5(n) t^{\frac{n-2}{2}} u_0(\xi),$$

where $c_4(n)$ and $c_5(n)$ are as in (45). Since u_0 is positive and M is compact, we get

$$(73) \qquad \lim_{t \to 0} \mathcal{G}(t, \xi) = +\infty \quad \text{and} \quad \lim_{t \to +\infty} \mathcal{G}(t, \xi) = +\infty$$

uniformly with respect to $\xi \in M$. Since $h \equiv c_n \operatorname{Scal}_g$ and either $\{3 \leq n \leq 9\}$ or $\{(M, g)$ is locally conformally flat$\}$, it follows from Proposition 5.1 that

$$(74) \qquad \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \left(\mathcal{J}_\varepsilon(t, \xi) - c_1(n, u_0) - c_2(n, u_0)\varepsilon - c_3(n)\varepsilon \ln \varepsilon \right) = \mathcal{G}(t, \xi)$$

uniformly with respect to t in compact subsets of $\mathbb{R}_{>0}$ and with respect to the point ξ in M. For ε small, by (73), (74), and by continuity of $\mathcal{J}_\varepsilon$ and $\mathcal{G}$, we get the existence of a family of points $(t_\varepsilon, \xi_\varepsilon)$ which realize the minimum values of the functions $\mathcal{J}_\varepsilon$ in $(a, b) \times M$ for some positive real numbers $a < b$ independent of ε. By Proposition 5.1, it follows that for ε small, the function $u_\varepsilon = u_0 - W_{\delta_\varepsilon(t_\varepsilon), \xi_\varepsilon} + \phi_{\delta_\varepsilon(t_\varepsilon), \xi_\varepsilon}$ is a solution of equation (2), where $W_{\delta_\varepsilon(t), \xi}$ is as in (14) and $\phi_{\delta_\varepsilon(t), \xi}$ is given by Proposition 4.1.

We get that $\lim_{\varepsilon \to 0} u_\varepsilon = u_0$ in $H^2_{1, loc.}(M \setminus \{\xi_0\})$ where $\xi_0 := \lim_{\varepsilon \to 0} \xi_\varepsilon$ (up to a subsequence): it then follows from standard elliptic theory that $\lim_{\varepsilon \to 0} u_\varepsilon = u_0$ in $C^2_{loc}(M \setminus \{\xi_0\})$. Independently, $\lim_{\varepsilon \to 0} \delta_\varepsilon(t_\varepsilon)^{\frac{n-2}{2}} u_\varepsilon(\exp_{\xi_\varepsilon} \cdot) = -U$ in $H^2_{1, loc.}(\mathbb{R}^n)$, and still by elliptic theory, one then gets the convergence in $C^2_{loc}(\mathbb{R}^n)$. This proves that $(u_\varepsilon)_{\varepsilon > 0}$ changes sign and blows-up when $\varepsilon \to 0$. This ends the proof of Theorems 1.2 and 2.2. $\qquad\square$

Proof of Theorems 1.1 and 2.3. In dimensions $3 \leq n \leq 5$, the proof of Theorem 1.1 is similar to the proof of Theorem 1.2. The specificity of dimension $n = 6$, is that the function $\mathcal{G}$ in (72) is replaced by

$$\mathcal{G}(t, \xi) := c_4(6) \ln \frac{1}{t} + c_5(6) \left(u_0(\xi) + \frac{1}{2} (h(\xi) - c_6 \operatorname{Scal}_g(\xi)) \right) t^2,$$

where $c_4(6), c_5(6) > 0$ are as in (45): therefore (73) holds with the hypothesis of Theorem 1.1 and the proof of Theorem 1.1 goes as for Theorem 1.2. We focus on Theorem 2.3. In dimension $n = 6$, computations similar to (45) yield

$$J_\varepsilon^+ \left(u_0 + W_{\delta_\varepsilon(t), \xi} \right) = c_1(6, u_0) + c_2(6, u_0)\varepsilon + c_3(6)\varepsilon \ln \varepsilon$$
$$+ \left(c_4(6) \ln \frac{1}{t} + c_5(6) \left(\frac{1}{2} (h(\xi) - c_6 \operatorname{Scal}_g(\xi)) - u_0(\xi) \right) t^2 \right) \varepsilon + \mathrm{o}(\varepsilon)$$

as $\varepsilon \to 0$, where $J_\varepsilon^+(u) := \frac{1}{2} \int_M |\nabla u|_g^2 \, dv_g + \frac{1}{2} \int_M h u^2 dv_g - \frac{1}{2^\star - \varepsilon} \int_M u_+^{2^\star - \varepsilon} dv_g$. The proof then is similar to the proof of Theorem 1.1. $\qquad\square$

Proof of Theorem 2.1. The introduction of another type of model for blow-up is required here. It follows from Lee–Parker [21] that for any $\xi \in M$, there exists $\Lambda_\xi \in C^\infty(M)$ positive such that $g_\xi := \Lambda_\xi^{\frac{4}{n-2}} g$ satisfies $dv_{g_\xi} = (1 + O(d_{g_\xi}(\xi, \cdot)^n)) \, dx$ in a geodesic normal chart. An immediate consequence is that $\operatorname{Scal}_{g_\xi}(\xi) = |\nabla \operatorname{Scal}_{g_\xi}(\xi)| = 0$ and $\Delta_{g_\xi} \operatorname{Scal}_{g_\xi}(\xi) = \frac{1}{6} |\operatorname{Weyl}_g(\xi)|_g^2$. Moreover, we can assume that $(\xi, x) \mapsto \Lambda_\xi(x)$

is C^∞ and $\nabla \Lambda_\xi(\xi) = 0$. We define $W_{\delta,\xi}$ in (14) with the function Λ_ξ above. When $h \equiv c_n \operatorname{Scal}_g$, the conformal law of change of metric yields the Taylor expansion

(75)

$$J_\varepsilon \left(u_0 - W_{\delta_\varepsilon(t),\xi} \right) = c_1(n, u_0) + c_2(n, u_0)\varepsilon + c_3(n)\varepsilon \ln \varepsilon + \frac{K_n^{-n}}{n} \left(\frac{(n-2)^2}{4} \varepsilon \ln \frac{1}{t} \right.$$

$$+ \frac{2^n \omega_{n-1}}{\omega_n (n(n-2))^{(n-2)/4}} u(\xi) \varepsilon t^{\frac{n-2}{2}} - \left. \frac{|\operatorname{Weyl}_g(\xi)|_g^2}{24(n-4)(n-6)} \varepsilon^{\frac{8}{n-2}} t^4 \right) + \operatorname{o}\left(\varepsilon + \varepsilon^{\frac{8}{n-2}} \right)$$

when $\varepsilon \to 0$ for $n \geq 7$. When $n < 10$, the term involving the Weyl tensor is neglictible. When $n = 10$, it competes with the one involving u_0: arguing as in the proofs above, we get the existence of a blowing-up family when $u_0 > \frac{5}{567} |\operatorname{Weyl}_g|_g^2$, which proves Theorem 2.1 since the additional terms involving $\phi_{\delta_\varepsilon(t),\xi}$ are neglictible when $n \leq 17$. When $n > 10$, the Weyl tensor dominates but the negative sign does not allow to construct a critical point for the reduced functional. □

Proof of Theorem 2.4. If $\xi_0 \in M$ is a strict minimizer of Φ on $\overline{B}_{\xi_0}(\nu_0) \subset M$ with $\nu_0 > 0$, the arguments above extend by minimizing $\mathcal{G}$ on $(0, +\infty) \times B_{\xi_0}(\nu_0)$. □

7. Error estimate

This section is devoted to the error estimate used in previous sections. All notations refer to Section 4. The estimate is as follows:

Lemma 7.1. *Given two positive real numbers $a < b$, there exists a positive constant $C'_{a,b}$ such that for ε small, for any real number t in $[a, b]$, and any point ξ in M, there holds*

(76) $\left\| i^* \left(f_\varepsilon \left(u_0 - W_{\delta_\varepsilon(t),\xi} \right) \right) - u_0 + W_{\delta_\varepsilon(t),\xi} \right\|_h$

$$\leq C'_{a,b} \begin{cases} \varepsilon |\ln \varepsilon| & \text{if } n \leq 6 \\ \varepsilon^{\frac{4}{n-2}} & \text{if } n \geq 7 \\ \varepsilon^{\frac{n+2}{2(n-2)}} & \text{if } n \geq 7, \ h \equiv c_n \operatorname{Scal}_g, \ \text{and } (M, g) \ \text{loc. conformally flat,} \end{cases}$$

where $\delta_\varepsilon(t) = t\varepsilon^{2/(n-2)}$ and $W_{\delta_\varepsilon(t),\xi}$ is as in (14).

Proof. All our estimates in this proof are uniform with respect to t in $[a, b]$, ξ in M, and ε in $(0, \varepsilon_0)$ for some fixed positive real number ε_0. The continuity of i^* yields

$$\left\| i^* \left(f_\varepsilon \left(u_0 - W_{\delta_\varepsilon(t),\xi} \right) \right) - u_0 + W_{\delta_\varepsilon(t),\xi} \right\|_h$$

$$= \operatorname{O}\left(\left\| f_\varepsilon \left(u_0 - W_{\delta_\varepsilon(t),\xi} \right) - (\Delta_g + h) \left(u_0 - W_{\delta_\varepsilon(t),\xi} \right) \right\|_{\frac{2n}{n+2}} \right).$$

It follows that

(77) $\left\| i^* \left(f_\varepsilon \left(u_0 - W_{\delta_\varepsilon(t),\xi} \right) \right) - u_0 + W_{\delta_\varepsilon(t),\xi} \right\|_h = \operatorname{O}\left(\tilde{I}_{1,\varepsilon,t,\xi} + \tilde{I}_{2,\varepsilon,t,\xi} + \tilde{I}_{3,\varepsilon,t,\xi} \right),$

where

$$\tilde{I}_{1,\varepsilon,t,\xi} := \left\| f_\varepsilon \left(u_0 - W_{\delta_\varepsilon(t),\xi} \right) - f_\varepsilon \left(u_0 \right) + f_\varepsilon \left(W_{\delta_\varepsilon(t),\xi} \right) \right\|_{\frac{2n}{n+2}},$$

$$\tilde{I}_{2,\varepsilon,t,\xi} := \left\| f_\varepsilon \left(u_0 \right) - \Delta_g u_0 - h u_0 \right\|_{\frac{2n}{n+2}},$$

$$\tilde{I}_{3,\varepsilon,t,\xi} := \left\| f_\varepsilon \left(W_{\delta_\varepsilon(t),\xi} \right) - \Delta_g W_{\delta_\varepsilon(t),\xi} - h W_{\delta_\varepsilon(t),\xi} \right\|_{\frac{2n}{n+2}}.$$

We estimate these terms separately.

Step 1: Estimate of $\tilde{I}_{1,\varepsilon,t,\xi}$.
We get

$$\tilde{I}_{1,\varepsilon,t,\xi} \leq \left\| \left(f_\varepsilon \left(u_0 - W_{\delta_\varepsilon(t),\xi} \right) + f_\varepsilon \left(W_{\delta_\varepsilon(t),\xi} \right) \right) \mathbf{1}_{B_\xi\left(\sqrt{\delta_\varepsilon(t)}\right)} \right\|_{\frac{2n}{n+2}}$$

$$+ \left\| \left(f_\varepsilon \left(u_0 - W_{\delta_\varepsilon(t),\xi} \right) - f_\varepsilon \left(u_0 \right) \right) \mathbf{1}_{M \setminus B_\xi\left(\sqrt{\delta_\varepsilon(t)}\right)} \right\|_{\frac{2n}{n+2}}$$

$$+ \left\| f_\varepsilon \left(W_{\delta_\varepsilon(t),\xi} \right) \mathbf{1}_{M \setminus B_\xi\left(\sqrt{\delta_\varepsilon(t)}\right)} \right\|_{\frac{2n}{n+2}} + \left\| f_\varepsilon \left(u_0 \right) \mathbf{1}_{B_\xi\left(\sqrt{\delta_\varepsilon(t)}\right)} \right\|_{\frac{2n}{n+2}} .$$

As is easily checked, Taylor's expansion for $f_\varepsilon \left(u_0 - W_{\delta_\varepsilon(t),\xi} \right)$ yields

$$\left\| \left(f_\varepsilon \left(u_0 - W_{\delta_\varepsilon(t),\xi} \right) + f_\varepsilon \left(W_{\delta_\varepsilon(t),\xi} \right) \right) \mathbf{1}_{B_\xi\left(\sqrt{\delta_\varepsilon(t)}\right)} \right\|_{\frac{2n}{n+2}}$$

$$\leq C \left(\left\| u_0 W_{\delta_\varepsilon(t),\xi}^{2^*-2-\varepsilon} \mathbf{1}_{B_\xi\left(\sqrt{\delta_\varepsilon(t)}\right)} \right\|_{\frac{2n}{n+2}} + \left\| u_0^{2^*-1-\varepsilon} \mathbf{1}_{B_\xi\left(\sqrt{\delta_\varepsilon(t)}\right)} \right\|_{\frac{2n}{n+2}} \right)$$

and

$$\left\| \left(f_\varepsilon \left(u_0 - W_{\delta_\varepsilon(t),\xi} \right) - f_\varepsilon \left(u_0 \right) \right) \mathbf{1}_{M \setminus B_\xi\left(\sqrt{\delta_\varepsilon(t)}\right)} \right\|_{\frac{2n}{n+2}}$$

$$\leq C \left(\left\| u_0^{2^*-2-\varepsilon} W_{\delta_\varepsilon(t),\xi} \mathbf{1}_{M \setminus B_\xi\left(\sqrt{\delta_\varepsilon(t)}\right)} \right\|_{\frac{2n}{n+2}} + \left\| W_{\delta_\varepsilon(t),\xi}^{2^*-1-\varepsilon} \mathbf{1}_{M \setminus B_\xi\left(\sqrt{\delta_\varepsilon(t)}\right)} \right\|_{\frac{2n}{n+2}} \right) .$$

Estimating roughly these terms yields

$$(78) \qquad \tilde{I}_{1,\varepsilon,t,\xi} = \begin{cases} \mathrm{O}\left(\varepsilon\right) & \text{if } n \leq 5 \\ \mathrm{O}\left(\varepsilon \left|\ln \varepsilon\right|^{\frac{2}{3}}\right) & \text{if } n = 6 \\ \mathrm{O}\left(\varepsilon^{\frac{n+2}{2(n-2)}}\right) & \text{if } n \geq 7 \end{cases}$$

when $\varepsilon \to 0$.

Step 2: Estimate of $\tilde{I}_{2,\varepsilon,t,\xi}$.
Since u_0 is a solution of (7), we get that

$$(79) \qquad \tilde{I}_{2,\varepsilon,t,\xi} = \left\| f_\varepsilon \left(u_0 \right) - f_0 \left(u_0 \right) \right\|_{\frac{2n}{n+2}} = \mathrm{O}\left(\varepsilon\right) .$$

Step 3: Estimate of $\tilde{I}_{3,\varepsilon,t,\xi}$.
We define $\chi_\xi \left(\cdot \right) = \chi \left(d_{g_\xi} \left(\cdot, \xi \right) \right)$, $U_{\delta,\xi} \left(\cdot \right) = \delta^{\frac{2-n}{2}} U(\delta^{-1} \exp_\xi^{-1}(\cdot))$, where the function U is as in (15) and the exponential map is taken with respect to the metric g_ξ.

Step 3.1: Estimate of $I_{3,\varepsilon,t,\xi}$ when (M,g) is locally conformally flat and $h \equiv c_n \operatorname{Scal}_g$.
Since $g_\xi = \Lambda_\xi^{4/(n-2)} g$ is flat, we get that

$$f_\varepsilon \left(W_{\delta_\varepsilon(t),\xi} \right) - \Delta_g W_{\delta_\varepsilon(t),\xi} - h W_{\delta_\varepsilon(t),\xi} = \Lambda_\xi^{2^*-1} \left(\Lambda_\xi^{-\varepsilon} f_\varepsilon \left(\widetilde{W}_{\delta_\varepsilon(t),\xi} \right) - \Delta_{g_\xi} \widetilde{W}_{\delta_\varepsilon(t),\xi} \right),$$

where $\widetilde{W}_{\delta_\varepsilon(t),\xi} = W_{\delta_\varepsilon(t),\xi}/\Lambda_\xi$. In this case, since the metric g_ξ is flat in $B_\xi \left(r_0 \right)$ and since the function U is a solution of the equation $\Delta_{\mathrm{Eucl}} U = U^{2^*-1}$ in $\mathbb{R}^n$, we get

that

$$(80) \quad I_{3,\varepsilon,t,\xi} \leq \left\| (\chi_\xi \Lambda_\xi)^{2^*-1-\varepsilon} \left(U_{\delta_\varepsilon(t),\xi}^{2^*-1-\varepsilon} - U_{\delta_\varepsilon(t),\xi}^{2^*-1} \right) \right\|_{\frac{2n}{n+2}}$$

$$+ \left\| \left(\chi_\xi^{2^*-1-\varepsilon} \Lambda_\xi^{-\varepsilon} - \chi_\xi \right) \Lambda_\xi^{2^*-1} U_{\delta_\varepsilon(t),\xi}^{2^*-1} \right\|_{\frac{2n}{n+2}} + \left\| \Lambda_\xi^{2^*-1} U_{\delta_\varepsilon(t),\xi} \Delta_{g_\xi} \chi_\xi \right\|_{\frac{2n}{n+2}}$$

$$+ 2 \left\| \Lambda_\xi^{2^*-1} \left\langle \nabla \chi_\xi, \nabla U_{\delta_\varepsilon(t),\xi} \right\rangle_{g_\xi} \right\|_{\frac{2n}{n+2}}.$$

Step 3.2: Estimate of $I_{3,\varepsilon,t,\xi}$ in the general case.
In general, we get that

$$(81)$$

$$I_{3,\varepsilon,t,\xi} \leq \left\| \chi_\xi^{2^*-1-\varepsilon} \left(U_{\delta_\varepsilon(t),\xi}^{2^*-1-\varepsilon} - U_{\delta_\varepsilon(t),\xi}^{2^*-1} \right) \right\|_{\frac{2n}{n+2}} + \left\| \left(\chi_\xi^{2^*-1-\varepsilon} - \chi_\xi \right) U_{\delta_\varepsilon(t),\xi}^{2^*-1} \right\|_{\frac{2n}{n+2}}$$

$$+ \left\| \chi_\xi \left(U_{\delta_\varepsilon(t),\xi}^{2^*-1} - \Delta_g U_{\delta_\varepsilon(t),\xi} \right) \right\|_{\frac{2n}{n+2}} + \left\| U_{\delta_\varepsilon(t),\xi} \Delta_g \chi_\xi \right\|_{\frac{2n}{n+2}}$$

$$+ 2 \left\| \left\langle \nabla \chi_\xi, \nabla U_{\delta_\varepsilon(t),\xi} \right\rangle_g \right\|_{\frac{2n}{n+2}} + \left\| h \chi_\xi U_{\delta_\varepsilon(t),\xi} \right\|_{\frac{2n}{n+2}}.$$

Step 3.3: Estimates of the terms in (80) *and* (81).
Since $\chi_\xi \equiv 1$ on $B_\xi(r_0/2)$ and $\chi_\xi \equiv 0$ on $M \setminus B_\xi(r_0)$, we get that

$$(82) \quad \int_M \left| (\chi_\xi \Lambda_\xi)^{2^*-1-\varepsilon} \left(U_{\delta_\varepsilon(t),\xi}^{2^*-1-\varepsilon} - U_{\delta_\varepsilon(t),\xi}^{2^*-1} \right) \right|^{\frac{2n}{n+2}} dv_g = O\left(\varepsilon^{\frac{2n}{n+2}} |\ln \varepsilon|^{\frac{2n}{n+2}} \right),$$

$$(83) \quad \int_M \left| \left(\chi_\xi^{2^*-1-\varepsilon} \Lambda_\xi^{-\varepsilon} - \chi_\xi \right) \Lambda_\xi^{2^*-1} U_{\delta_\varepsilon(t),\xi}^{2^*-1} \right|^{\frac{2n}{n+2}} dv_g = O\left(\varepsilon^{\frac{2n}{n+2}} \right),$$

$$(84) \quad \int_M \left| \Lambda_\xi^{2^*-1} U_{\delta_\varepsilon(t),\xi} \Delta_g \chi_\xi \right|^{\frac{2n}{n+2}} = O\left(\varepsilon^{\frac{2n}{n+2}} \right)$$

when $\varepsilon \to 0$. A rough L^∞ upper bound for $|\nabla U_{\delta_\varepsilon(t),\xi}|$ on $M \setminus B_{r_0/2}(\xi)$ yields

$$(85) \quad \int_M \left| \left\langle \Lambda_\xi^{2^*-1} \nabla \chi_\xi, \nabla U_{\delta_\varepsilon(t),\xi} \right\rangle_g \right|^{\frac{2n}{n+2}} dv_g = O\left(\varepsilon^{\frac{2n}{n+2}} \right).$$

Since $\Delta_{\mathrm{Eucl}} U = U^{2^*-1}$, we get in the chart $\exp_\xi$ that

$$\Delta_g U_{\delta_\varepsilon(t),\xi} - U_{\delta_\varepsilon(t),\xi}^{2^*-1} = -(g^{ij} - \delta^{ij}) \partial_{ij} U_{\delta_\varepsilon(t),\xi} + g^{ij} \Gamma_{ij}^k \partial_k U_{\delta_\varepsilon(t),\xi},$$

where the g^{ij}'s are the coordinate of the metric $g = g_\xi$ and $\Gamma_{\alpha\beta}^\gamma$'s are the Christoffel symbols of the metric g in the normal chart $\exp_\xi$. Cartan's expansion of the metric yields $|g^{ij}(x) - \delta^{ij}| \leq C|x|^2$ and $|\Gamma_{ij}^k(x)| \leq C|x|$ around 0, and therefore

$$\left| U_{\delta_\varepsilon(t),\xi}^{2^*-1} - \Delta_g U_{\delta_\varepsilon(t),\xi} \right| \leq C|x|^2 |\nabla^2 U_{\delta_\varepsilon(t),\xi}| + C|x| \cdot |\nabla U_{\delta_\varepsilon(t),\xi}|$$

via the chart $\exp_\xi$. Therefore, we get that

$$(86) \quad \int_M \left| \chi_\xi \left(U_{\delta_\varepsilon(t),\xi}^{2^*-1} - \Delta_g U_{\delta_\varepsilon(t),\xi} \right) \right|^{\frac{2n}{n+2}} dv_g = \begin{cases} O\left(\varepsilon^{\frac{2n}{n+2}} \right) & \text{if } n \leq 5 \\ O\left(\varepsilon^{\frac{3}{2}} |\ln \varepsilon| \right) & \text{if } n = 6 \\ O\left(\varepsilon^{\frac{8n}{(n+2)(n-2)}} \right) & \text{if } n \geq 7. \end{cases}$$

It remains to compute

$$
(87) \qquad \int_M \left| h\chi_\xi U_{\delta_\varepsilon(t),\xi} \right|^{\frac{2n}{n+2}} dv_g = \begin{cases} \mathrm{O}\left(\varepsilon^{\frac{2n}{n+2}}\right) & \text{if } n \leq 5 \\[2mm] \mathrm{O}\left(\varepsilon^{\frac{3}{2}} \left|\ln \varepsilon\right|\right) & \text{if } n = 6 \\[2mm] \mathrm{O}\left(\varepsilon^{\frac{8n}{(n+2)(n-2)}}\right) & \text{if } n \geq 7 \end{cases}
$$

when $\varepsilon \to 0$.

Step 3.4: End of estimate of $\tilde{I}_{3,\varepsilon,t,\xi}$.
By (81)–(87), we get

$$
(88) \qquad \left\| f_\varepsilon \left(W_{\delta_\varepsilon(t),\xi} \right) - \Delta_g W_{\delta_\varepsilon(t),\xi} - h W_{\delta_\varepsilon(t),\xi} \right\|_{\frac{2n}{n+2}} = \begin{cases} \mathrm{O}\left(\varepsilon \left|\ln \varepsilon\right|\right) & \text{if } n \leq 6 \\[2mm] \mathrm{O}\left(\varepsilon^{\frac{4}{n-2}}\right) & \text{if } n \geq 7. \end{cases}
$$

In case $h \equiv c_n \, \mathrm{Scal}_g$ and the manifold is locally conformally flat, by (80), (82)–(85), we get

$$
(89) \qquad \left\| f_\varepsilon \left(W_{\delta_\varepsilon(t),\xi} \right) - \Delta_g W_{\delta_\varepsilon(t),\xi} - h W_{\delta_\varepsilon(t),\xi} \right\|_{\frac{2n}{n+2}} = \mathrm{O}\left(\varepsilon \left|\ln \varepsilon\right|\right)
$$

when $\varepsilon \to 0$.

Step 4: End of proof of (76).
Finally, (76) follows from (77), (78), (79), (88), and (89). $\qquad\square$

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