MULTIPLE SOLUTIONS FOR CRITICAL ELLIPTIC SYSTEMS IN POTENTIAL FORM

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Abstract. We discuss and prove existence of multiple solutions for critical elliptic systems in potential form on compact Riemannian manifolds.

1. Introduction

Let (M, g) be a smooth, compact Riemannian n-manifold, $n \geq 3$. Let also $p \geq 1$ be a natural number and $M_p^s(\mathbb{R})$ be the vector space of all symmetric $p \times p$ real matrices. Namely, $M_p^s(\mathbb{R})$ is the vector space of $p \times p$ real matrices $S = (S_{ij})$ which are such that $S_{ij} = S_{ji}$ for all i, j. For $A : M \to M_p^s(\mathbb{R})$ smooth, $A = (A_{ij})$, we consider vector valued equations like

$$\Delta_g^p \mathcal{U} + A(x)\mathcal{U} = \frac{1}{2^*} D_{\mathcal{U}} \left|\mathcal{U}\right|^{2^*}, \tag{1.1}$$

where $\mathcal{U} : M \to \mathbb{R}^p$ is a map, referred to as a *p-map* in order to underline the fact that the target space is $\mathbb{R}^p$, Δ_g^p is the Laplace–Beltrami operator acting on p-maps, $2^* = 2n/(n-2)$, and $D_{\mathcal{U}}$ is the derivation operator with respect to $\mathcal{U}$. Writing $\mathcal{U} = (u_1, \ldots, u_p)$, we get $\left|\mathcal{U}\right|^{2^*} = \sum_{i=1}^p \left|u_i\right|^{2^*}$, $\frac{1}{2^*} D_{\mathcal{U}} \left|\mathcal{U}\right|^{2^*} = \left(\left|u_i\right|^{2^*-2} u_i\right)_i$, and $\Delta_g^p \mathcal{U} = \left(\Delta_g u_i\right)_i$, where $\Delta_g = -\operatorname{div}_g \nabla$ is the Laplace–Beltrami operator for functions. Another way in which we can write (1.1) is like in the form of the following elliptic system

$$\Delta_g u_i + \sum_{j=1}^p A_{ij}(x) u_j = \left|u_i\right|^{2^*-2} u_i , \tag{1.2}$$

where the equations have to be satisfied in M, and for all $i = 1, \ldots, p$. We say that the system is of order p, and refer to it as a p-system in potential form because of the nature of the nonlinearity. The system has a variational structure. It is also critical from the Sobolev viewpoint since, if H_1^2 is the Sobolev space of functions in L^2 with one derivative in L^2, then 2^* is the critical Sobolev exponent for the embeddings of H_1^2 into Lebesgue spaces. In case $p = 1$, (1.1)–(1.2) reduces to Yamabe-type equations, and we regard (1.1)–(1.2) as a natural extension of such equations to weakly coupled systems. We introduce the Sobolev space $H_{1,p}^2(M)$ of all p-maps whose components belong to $H_1^2(M)$, and we say that a p-map $\mathcal{U}$ in $H_{1,p}^2(M)$ is a solution of (1.1)–(1.2) if its components u_i solve (1.2) weakly for $i = 1, \ldots, n$. By regularity theory, see Hebey [27], the components of any weak solution belong to $C^{2,\theta}(M)$ for all real numbers θ in $(0, 1)$. We define the energy of a solution $\mathcal{U}$ of (1.1)–(1.2) by

$$\mathcal{E}(\mathcal{U}) = \sum_{i=1}^p \int_M \left|u_i\right|^{2^*} dv_g , \tag{1.3}$$

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where the u_i's are the components of $\mathcal{U}$, and dv_g is the volume element of the manifold (M,g). We also define the functional $I_{A,g}$ acting on $H^2_{1,p}(M)$ by

$$I_{A,g}(\mathcal{U}) = \frac{1}{2}\int_M \sum_{i=1}^p |\nabla u_i|_g^2 \, dv_g + \frac{1}{2}\int_M \sum_{i,j=1}^p A_{ij} u_i u_j dv_g - \frac{1}{2^*}\sum_{i=1}^p \int_M |u_i|^{2^*} dv_g \,. \tag{1.4}$$

Critical points of $I_{A,g}$ are solutions of the system (1.1)–(1.2). We set

$$\mu_{A,g} = \inf_{\mathcal{U}\in\mathcal{N}} I_{A,g}(\mathcal{U}) \,, \tag{1.5}$$

where $\mathcal{N}$ is the Nehari manifold of the functional $I_{A,g}$ defined as the set of p-maps $\mathcal{U}$ in $H^2_{1,p}(M)\setminus\{0\}$ such that $DI_{A,g}(\mathcal{U}).\mathcal{U} = 0$. We say that the operator $\Delta^p_g + A$ is *coercive* on $H^2_{1,p}(M)$ if its energy controls the $H^2_{1,p}$-norm. A precise definition is given in Section 2. When $\Delta^p_g + A$ is coercive on $H^2_{1,p}(M)$, the lower bound $\mu_{A,g}$ is positive. Following standard terminology we say that a map $A : M \to M^s_p(\mathbb{R})$ is *cooperative* if its off-diagonal components are nonnegative. In other words, A is said to be cooperative if there holds $A_{ij} \geq 0$ in M for all distinct indices i and j. Still following standard terminology, we say that (1.1)–(1.2) is *fully coupled* if the index set $\{1,\dots,p\}$ does not split into two disjoint subsets $\{i_1,\dots,i_k\}$ and $\{j_1,\dots,j_{k'}\}$, $k+k'=p$, such that there holds $A_{i_\alpha j_\beta} \equiv 0$ in M for all $\alpha = 1,\dots,k$ and $\beta = 1,\dots,k'$. When (1.1)–(1.2) is not fully coupled, permuting if necessary the equations, A may be written in diagonal blocks and the p-system may split into two independent systems. A p-map is said to be *positive* if its components are all positive. In what follows we associate each solution of equation (1.1)–(1.2) with its opposite one, and call that a pair of solutions. A pair $(\mathcal{U}, -\mathcal{U})$ is said to be positive if either $\mathcal{U}$ or $-\mathcal{U}$ is positive. We let K_n be the sharp constant for the embedding of $\dot{H}^2_1(\mathbb{R}^n)$ into $L^{2^*}(\mathbb{R}^n)$. Then, as is well known,

$$K_n = \sqrt{\frac{4}{n(n-2)\,\omega_n^{2/n}}} \,, \tag{1.6}$$

where ω_n is the volume of the unit n-sphere. When $\Delta^p_g + A$ is coercive on $H^2_{1,p}(M)$, the p-map

$$\mathcal{W}_{\mathcal{U}} = \left(\frac{2\left(I_{A,g}(\mathcal{U}) + \frac{1}{2^*}\mathcal{E}(\mathcal{U})\right)}{\mathcal{E}(\mathcal{U})}\right)^{(n-2)/4} \mathcal{U} \tag{1.7}$$

belongs to $\mathcal{N}$ for all $\mathcal{U} \in H^2_{1,p}(M)\setminus\{0\}$, where $I_{A,g}$ is as in (1.4), and $\mathcal{E}$ is the energy function as in (1.3). In particular, for any $\mathcal{U} \in H^2_{1,p}(M)\setminus\{0\}$, we get by (1.5) that $\mu_{A,g} \leq I_{A,g}(\mathcal{W}_{\mathcal{U}})$, where $\mathcal{W}_{\mathcal{U}}$ is as in (1.7), and it follows that

$$\mu_{A,g} \leq \frac{1}{n}\inf_{\mathcal{U}\in\mathcal{H}} \left(\frac{2\left(I_{A,g}(\mathcal{U}) + \frac{1}{2^*}\mathcal{E}(\mathcal{U})\right)}{\mathcal{E}(\mathcal{U})^{2/2^*}}\right)^{n/2} \leq \frac{1}{n}K_n^{-n} \tag{1.8}$$

for all (M,g) and all A such that $\Delta^p_g + A$ is coercive, where $\mathcal{H} = H^2_{1,p}(M)\setminus\{0\}$. The second inequality in (1.8) follows from standard developments on the Yamabe problem, as in Aubin [4], by testing p-maps with components all zero, except one which we choose to be like minimizers for the embedding of $\dot{H}^2_1(\mathbb{R}^n)$ into $L^{2^*}(\mathbb{R}^n)$. The first result we prove is as follows.

Theorem 1.1. *Let (M,g) be a smooth, compact Riemannian manifold of dimension $n \geq 3$, let $p \geq 1$ be a natural number, and let A be a smooth map from M to $M_p^s(\mathbb{R})$ such that the operator $\Delta_g^p + A$ is coercive on $H_{1,p}^2(M)$. Assume that for some $k \geq 1$, there exists an odd, continuous map $\Phi : \mathbb{R}^{k+1} \to H_{1,p}^2(M)$ such that there hold $I_{A,g} \circ \Phi < 2K_n^{-n}/n$ and $I_{A,g} \circ \Phi(z) \to -\infty$ as $|z| \to +\infty$. Then (1.1)–(1.2) admits at least $k/2$ pairs of nonzero solutions with energy less than $2K_n^{-n}$. If moreover there holds $\mu_{A,g} < K_n^{-n}/n$, $(-A)$ is cooperative, and (1.1)–(1.2) is fully coupled, then (1.1)–(1.2) admits at least $(k+1)/2$ pairs of nonzero solutions with energy less than $2K_n^{-n}$, and one of these pairs is positive.*

Theorem 1.1 reduces the question of the existence of multiple solutions of (1.1)–(1.2) to the proof of the existence of an odd, continuous map $\Phi : \mathbb{R}^{k+1} \to H_{1,p}^2(M)$ such that there hold $I_{A,g} \circ \Phi < 2K_n^{-n}/n$ and $I_{A,g} \circ \Phi(z) \to -\infty$ as $|z| \to +\infty$. Following the very nice construction in Clapp–Weth [10], we prove the existence of such a map with $k = n+1$, see Section 7, as soon as we can prove the existence of a two-parameters family of test functions $U_{x,\varepsilon}$, $x \in \Omega$ and $\varepsilon > 0$, $U_{x,\varepsilon}$ depending continuously on x, such that

(i) $I_{A,g}(\mathcal{W}_{\mathcal{U}_{x,\varepsilon}}) < K^{-n}/n$ uniformly in x as $\varepsilon \to 0$,
(ii) $\operatorname{Supp}\mathcal{U}_{x,\varepsilon} \subset B_x(\varepsilon)$ for all x and all $\varepsilon > 0$,

where $\mathcal{W}_{\mathcal{U}_{x,\varepsilon}}$ is as in (1.7), Ω is an open subset of M, $\operatorname{Supp}\mathcal{U}_{x,\varepsilon}$ stands for the support of $\mathcal{U}_{x,\varepsilon}$, and $B_x(\varepsilon)$ is the ball in M of radius ε centered at x. With such a test function reduction, which extends classical existence conditions of Aubin's type [4], Theorem 1.1 provides several examples of systems like (1.1)–(1.2) with multiplicity of solutions. In particular, the following result holds true. We let $h_g = \frac{n-2}{4(n-1)} \operatorname{Scal}_g$, where Scal_g is the scalar curvature of g, so that h_g is the factor of the linear term in the n-dimensional Yamabe equation.

Theorem 1.2. *Let (M,g) be a smooth, compact Riemannian manifold of dimension $n \geq 4$, let $p \geq 1$ be a natural number, and let A be a smooth map from M to $M_p^s(\mathbb{R})$ such that the operator $\Delta_g^p + A$ is coercive on $H_{1,p}^2(M)$. If some diagonal component of A is less than the function h_g at some point in M (resp. equal to the function h_g in a nonempty, open subset of M in which the metric g is not conformally flat and if $n \geq 6$) then (1.1)–(1.2) admits at least $(n+1)/2$ pairs of nonzero solutions with energy less than $2K_n^{-n}$. If moreover $(-A)$ is cooperative and (1.1)–(1.2) is fully coupled, then (1.1)–(1.2) admits at least $(n+2)/2$ pairs of nonzero solutions with energy less than $2K_n^{-n}$, and one of these pairs is positive.*

Theorem 1.2 is proved by using test functions with no coupling, acting on the sole diagonal coefficient they are concerned with. A major difficulty lies in property (ii) which requires that the test functions we use should have their support shrinking to points as the dilatation parameter ε goes to zero. This difficulty is of a new type in test functions computations for manifolds. Another consequence of the shrinking property (ii) is that Schoen's global argument [41] developed for the Yamabe problem [45] cannot be used in the critical case where the diagonal components of A are equal to h_g and the manifold is locally conformally flat. From the local viewpoint, such a manifold looks like the sphere. In particular, by conformal invariance, when the diagonal components of A are equal to h_g, strict inequalities like in property (i) are unreachable by test functions with small support acting on a single diagonal component of A. We overcome this difficulty in Theorem 1.3 below, when $n \geq 7$, by using our system structure and test functions with coupling, acting on different coefficients of the matrix.

Theorem 1.3. *Let (M, g) be a smooth, compact Riemannian manifold of dimension $n \geq 7$, let $p \geq 2$ be a natural number, and let A be a smooth map from M to $M_p^s(\mathbb{R})$ such that the operator $\Delta_g^p + A$ is coercive on $H_{1,p}^2(M)$. If some diagonal component $A_{i_0 i_0}$ of A is equal to h_g around some point x_0 in M, if the metric g is conformally flat around x_0, and if there exists $j_0 \neq i_0$ such that $A_{i_0 j_0}(x_0) \neq 0$, then (1.1)–(1.2) admits at least $(n+1)/2$ pairs of nonzero solutions with energy less than $2K_n^{-n}$. If moreover $(-A)$ is cooperative and (1.1)–(1.2) is fully coupled, then (1.1)–(1.2) admits at least $(n+2)/2$ pairs of nonzero solutions with energy less than $2K_n^{-n}$, and one of these pairs is positive.*

Without any pretention to exhaustivity, possible references on elliptic systems are Amster–De Nápoli–Mariani [1], Angenent–van der Vorst [2, 3], Clément–Manásevich–Mitidieri [11], de Figueiredo [19], de Figueiredo–Ding [20], de Figueiredo–Felmer [21], de Figueiredo–Sirakov [22], Druet–Hebey [14], El Hamidi [18], Giaquinta–Martinazzi [24], Hebey [27–29], Hulshof–Mitidieri–van der Vorst [30], Jost–Lin–Wang [31], Jost–Wang [32], Mancini–Mitidieri [34], Mitidieri–Sweers [36], Montenegro [37], Pompino [39], Qing [40], and Sweers [43]. We also mention the reference Vétois [44] for closely related developments.

2. Preliminary material

In the following, we let (M, g) be a smooth, compact Riemannian manifold of dimension $n \geq 3$ and A be a smooth map from M to $M_p^s(\mathbb{R})$. We first set some notations. We define a scalar product on $H_{1,p}^2(M)$ by

$$\langle \mathcal{U}, \mathcal{V} \rangle_{H_{1,p}^2(M)} = \sum_{i=1}^{p} \left(\int_M \langle \nabla u_i, \nabla v_i \rangle_g \, dv_g + \Lambda \int_M u_i v_i dv_g \right), \tag{2.1}$$

where $\mathcal{U} = (u_1, \dots, u_p)$ and $\mathcal{V} = (v_1, \dots, v_p)$, and where Λ is a positive constant to be chosen large later on. Then the operator $\Delta_g^p + A$ is coercive on $H_{1,p}^2(M)$ if there exists $\Lambda_0 > 0$ such that there holds

$$\sum_{i=1}^{p} \int_M |\nabla u_i|_g^2 \, dv_g + \sum_{i,j=1}^{p} \int_M A_{ij} u_i u_j dv_g \geq \Lambda_0 \, \|\mathcal{U}\|_{H_{1,p}^2(M)}^2$$

for all p-maps $\mathcal{U} = (u_1, \dots, u_p)$ in $H_{1,p}^2(M)$, where $\|\cdot\|_{H_{1,p}^2(M)}$ is the norm associated to $\langle \cdot, \cdot \rangle_{H_{1,p}^2(M)}$. For instance, if A is positive definite at all points in M, then the operator $\Delta_g^p + A$ is coercive on $H_{1,p}^2(M)$. Given a positive real number δ and a subset C of $H_{1,p}^2(M)$, we let $\mathcal{B}_\delta(C)$ stand for the neighborhood of C formed by all p-maps in $H_{1,p}^2(M)$ at a distance from C less than or equal to δ. Given a real number c, we set $I_{A,g}^c = I_{A,g}^{-1}((-\infty, c])$. We let $\nabla I_{A,g}$ stand for the operator acting on $H_{1,p}^2(M)$ satisfying

$$\langle \nabla I_{A,g}(\mathcal{U}), \mathcal{V} \rangle_{H_{1,p}^2(M)} = DI_{A,g}(\mathcal{U}).\mathcal{V}$$

for all p-maps $\mathcal{U}$ and $\mathcal{V}$ in $H_{1,p}^2(M)$, where $\langle \cdot, \cdot \rangle_{H_{1,p}^2(M)}$ is as in (2.1). We define the operators $\mathfrak{L}_1 : L_p^2(M) \to H_{1,p}^2(M)$ and $\mathfrak{L}_2 : L_p^{2^*}(M) \to H_{1,p}^2(M)$ by

$$\Delta_g^p \mathfrak{L}_1(\mathcal{U}) + \Lambda \operatorname{Id}_p \mathfrak{L}_1(\mathcal{U}) = (\Lambda \operatorname{Id}_p - A)\,\mathcal{U}, \tag{2.2}$$

$$\text{resp. } \Delta_g^p \mathfrak{L}_2(\mathcal{U}) + \Lambda \operatorname{Id}_p \mathfrak{L}_2(\mathcal{U}) = \frac{1}{2^*} D_{\mathcal{U}} |\mathcal{U}|^{2^*}, \tag{2.3}$$

where Id_p is the identity matrix in $M_p^s(\mathbb{R})$, and $L_p^q(M)$ for $q \geq 1$ is the set of p-maps with components all in L^q. Then there holds

$$\nabla I_{A,g}(\mathcal{U}) = \mathcal{U} - \mathfrak{L}_1(\mathcal{U}) - \mathfrak{L}_2(\mathcal{U}) \tag{2.4}$$

for all $\mathcal{U} \in H^2_{1,p}(M)$. As one can check, $\mathfrak{L}_1$ and $\mathfrak{L}_2$ are locally Lipschitz when acting in $H^2_{1,p}(M)$. In what follows, we let $\varphi_{A,g}$ stand for the flow defined by

$$\begin{cases} \dfrac{\partial \varphi_{A,g}}{\partial t}(t,\mathcal{U}) = -\nabla I_{A,g}(\varphi_{A,g}(t,\mathcal{U})) & \text{if } 0 \le t < T(\mathcal{U}), \\ \varphi_{A,g}(0,\mathcal{U}) = \mathcal{U}, \end{cases}$$

where for any p-map $\mathcal{U}$ in $H^2_{1,p}(M)$, $T(\mathcal{U})$ is the maximal existence time for the trajectory $t \to \varphi_{A,g}(t,\mathcal{U})$. By construction, for any p-map $\mathcal{U}$ in $H^2_{1,p}(M)$, and for any positive time t, there holds

$$\frac{\partial (I_{A,g} \circ \varphi_{A,g})}{\partial t}(t,\mathcal{U}) = -\|\nabla I_{A,g}(\varphi_{A,g}(t,\mathcal{U}))\|^2_{H^2_{1,p}(M)}. \tag{2.5}$$

A subset D of $H^2_{1,p}(M)$ is said to be strictly positively invariant for the flow $\varphi_{A,g}$ if for any $\mathcal{U}$ in D and any time t in $(0,T(\mathcal{U}))$, the p-map $\varphi_{A,g}(t,\mathcal{U})$ belongs to the interior of D. As an example, one can easily see by using (2.5) that the set $I^c_{A,g}$ is strictly positively invariant for the flow $\varphi_{A,g}$ for all non-critical values c. Independently, we say that a subset D of $H^2_{1,p}(M)$ is symmetric if there holds $D = -D$. The following deformation lemma is used in several places in the proof of Theorem 1.1.

Lemma 2.1. *Let (M,g) be a smooth, compact Riemannian manifold of dimension $n \ge 3$, let $p \ge 1$ be a natural number, let A be a smooth map from M to $M^s_p(\mathbb{R})$, and let D be a symmetric, closed subset of $H^2_{1,p}(M)$ which we assume to be strictly positively invariant for the flow $\varphi_{A,g}$. Let $c \in \mathbb{R}$, $\delta, \varepsilon \in \mathbb{R}^+$, and let a symmetric subset C of $H^2_{1,p}(M)$ be such that for any p-map $\mathcal{U}$ in $I^{-1}_{A,g}([c-\varepsilon, c+\varepsilon]) \cap \mathcal{B}_\delta(C)$, there holds*

$$\|\nabla I_{A,g}(\mathcal{U})\|_{H^2_{1,p}(M)} \ge \frac{2\varepsilon}{\delta}. \tag{2.6}$$

Then there exists an odd, continuous map $\nu : (I^{c+\varepsilon}_{A,g} \cap C) \cup D \to I^{c-\varepsilon}_{A,g} \cup D$ such that $\nu \equiv \mathrm{id}$ in the set D.

Proof. First, we claim that for any $\mathcal{U} \in I^{c+\varepsilon}_{A,g} \cap C$, the trajectory $t \to \varphi_{A,g}(t,\mathcal{U})$ cannot stay in the set $I^{-1}_{A,g}((c-\varepsilon, c+\varepsilon])$ for all positive times $t \in (0,T(\mathcal{U}))$. By (2.5), this implies that for any p-map $\mathcal{U} \in I^{c+\varepsilon}_{A,g} \cap C$, there exists $t_0 > 0$ such that $\varphi_{A,g}(t,\mathcal{U})$ belongs to $I^{c-\varepsilon}_{A,g}$ for all $t \ge t_0$. We prove this claim by contradiction. Thus, we assume that there exists a p-map $\mathcal{U} \in I^{c+\varepsilon}_{A,g} \cap C$ such that $\varphi_{A,g}(t,\mathcal{U})$ belongs to the set $I^{-1}_{A,g}((c-\varepsilon, c+\varepsilon])$ for all $t \in (0,T(\mathcal{U}))$. As long as $\varphi_{A,g}(t,\mathcal{U}) \in \mathcal{B}_\delta(C)$, by (2.6), we get

$$\begin{aligned} \|\varphi_{A,g}(t,\mathcal{U}) - \mathcal{U}\|_{H^2_{1,p}(M)} &\le \int_0^t \left\| \frac{\partial \varphi}{\partial t}(s,\mathcal{U}) \right\|_{H^2_{1,p}(M)} ds \\ &\le \frac{\delta}{2\varepsilon} \int_0^t \|\nabla I_{A,g}(\varphi_{A,g}(t,\mathcal{U}))\|^2_{H^2_{1,p}(M)} ds \\ &= -\frac{\delta}{2\varepsilon} \int_0^t \frac{\partial (I_{A,g} \circ \varphi_{A,g})}{\partial t}(s,\mathcal{U})\, ds \\ &= \frac{\delta}{2\varepsilon}\left(I_{A,g}(\mathcal{U}) - I_{A,g}(\varphi_{A,g}(t,\mathcal{U}))\right). \end{aligned} \tag{2.7}$$

In particular, by (2.7), the trajectory $t \to \varphi_{A,g}(t,\mathcal{U})$ stays in the ball $\mathcal{B}_\delta(\mathcal{U})$ as long as it stays in $\mathcal{B}_\delta(C)$. By (2.7), we also get

$$t \le \left(\frac{\delta}{2\varepsilon}\right)^2 \left(I_{A,g}(\mathcal{U}) - I_{A,g}(\varphi_{A,g}(t,\mathcal{U}))\right) \le \frac{\delta^2}{2\varepsilon}. \tag{2.8}$$

In particular, since the trajectory $t \to \varphi_{A,g}(t,\mathcal{U})$ stays in the ball $\mathcal{B}_\delta(\mathcal{U})$ as long as it stays in $\mathcal{B}_\delta(C)$, and by (2.8), the standard extension theorem for solutions of ordinary differential equations gives that $t \to \varphi_{A,g}(t,\mathcal{U})$ cannot stay in $\mathcal{B}_\delta(C)$ for all positive times. Let $t_1 > 0$ be the first positive time that the trajectory intersects $\partial\mathcal{B}_\delta(C)$. By (2.7) with $t = t_1$, we get

$$I_{A,g}(\varphi_{A,g}(t,\mathcal{U})) \le I_{A,g}(\mathcal{U}) - 2\varepsilon \le c - \varepsilon\,,$$

and this is in contradiction with the assumption that the trajectory $t \to \varphi_{A,g}(t,\mathcal{U})$ belongs to $I_{A,g}^{-1}((c-\varepsilon, c+\varepsilon])$ for all $t \in (0, T(\mathcal{U}))$. This proves the above claim that for any p-map $\mathcal{U} \in I_{A,g}^{c+\varepsilon} \cap C$, the trajectory $t \to \varphi_{A,g}(t,\mathcal{U})$ cannot stay in the set $I_{A,g}^{-1}((c-\varepsilon, c+\varepsilon])$ for all positive times $t \in (0, T(\mathcal{U}))$. In particular, as already mentioned, this implies that for any p-map $\mathcal{U} \in I_{A,g}^{c+\varepsilon} \cap C$, there exists $t_0 > 0$ such that $\varphi_{A,g}(t,\mathcal{U})$ belongs to $I_{A,g}^{c-\varepsilon}$ for all $t \ge t_0$. By the positive invariance of D we then get that for any p-map $\mathcal{U} \in (I_{A,g}^{c+\varepsilon} \cap C) \cup D$, there exists a nonnegative time $\tau(\mathcal{U})$ from which the trajectory $t \to \varphi_{A,g}(t,\mathcal{U})$ belongs to $I_{A,g}^{c-\varepsilon} \cup D$, and $\tau(\mathcal{U}) = 0$ if $\mathcal{U} \in D$. The function $\tau : (I_{A,g}^{c+\varepsilon} \cap C) \cup D \to \mathbb{R}^+$ is clearly even. Now we claim that τ is also a continuous function. The lower semicontinuity of τ is straightforward to check. We prove the upper semicontinuity of τ in what follows. Let $\mathcal{U} \in (I_{A,g}^{c+\varepsilon} \cap C) \cup D$. By definition of τ, we get $\tau(\mathcal{U}) \in \partial(I_{A,g}^{c-\varepsilon} \cup D)$. If $\varphi_{A,g}(\tau(\mathcal{U}),\mathcal{U}) \in \partial D$, then the upper semicontinuity of τ at $\mathcal{U}$ follows from the strict positive invariance of D. If $\varphi_{A,g}(\tau(\mathcal{U}),\mathcal{U}) \in \partial I_{A,g}^{c-\varepsilon}$, by (2.7) with $t = \tau(\mathcal{U})$, then we get $\varphi_{A,g}(\tau(\mathcal{U}),\mathcal{U}) \in \mathcal{B}_\delta(C)$, and the upper semicontinuity of τ at $\mathcal{U}$ then follows from (2.5) and (2.6). Letting ν be given by $\nu(\mathcal{U}) = \varphi_{A,g}(\tau(\mathcal{U}),\mathcal{U})$, this ends the proof of the lemma. ☐

3. The H_1^2-theory for blow-up

The H_1^2-theory for the blow-up of Palais–Smale sequences, together with the above deformation Lemma 2.1, is an essential ingredient in the proof of Theorem 1.1. Following standard terminology, a sequence $(\mathcal{U}_\alpha)_\alpha$ in $H_{1,p}^2(M)$ is said to be a Palais–Smale sequence for the functional $I_{A,g}$ if the sequence $(I_{A,g}(\mathcal{U}_\alpha))_\alpha$ is bounded and if there holds $DI_{A,g}(\mathcal{U}_\alpha) \to 0$ in $H_{1,p}^2(M)'$ as $\alpha \to +\infty$. When $I_{A,g}(\mathcal{U}_\alpha)$ converges to a real number c as $\alpha \to +\infty$, the sequence $(\mathcal{U}_\alpha)_\alpha$ is said to be a Palais–Smale sequence for the functional $I_{A,g}$ at level c. Bounded sequences in $H_{1,p}^2(M)$ of solutions of equation (1.1) are Palais–Smale sequences for the functional $I_{A,g}$. The H_1^2-theory we briefly discuss in this section provides a structure equation for Palais–Smale sequences which describes their asymptotic behavior in $H_{1,p}^2(M)$ as $\alpha \to +\infty$. In what follows, we let η be a smooth cutoff function on the Euclidean space centered at 0 with small support around 0. By small, we mean, for instance, that the support of η is included in $B_0(i_g)$, where i_g is the injectivity radius of the manifold (M,g), and $B_0(i_g)$ is the Euclidean ball of center 0 and radius i_g. Given a converging sequence $(x_\alpha)_\alpha$ of points in M, and a sequence $(\mu_\alpha)_\alpha$ of positive real numbers converging to 0, we define the bubble in M of centers x_α and weights μ_α as the sequence $(B_\alpha)_\alpha$ of functions defined in M by

$$B_\alpha(x) = \mu_\alpha^{-(n-2)/2}\eta_\alpha(x)\,u\left(\mu_\alpha^{-1}\exp_{x_\alpha}^{-1}(x)\right), \tag{3.1}$$

where $\eta_\alpha = \eta \circ \exp_{x_\alpha}^{-1}$, and u is a nontrivial solution in $\dot{H}_1^2(\mathbb{R}^n)$ of the equation

$$\Delta_\delta u = |u|^{2^*-2}\,u\,, \tag{3.2}$$

where δ is the Euclidean metric in $\mathbb{R}^n$. As is easily checked, $(B_\alpha)_\alpha$ converges to 0 weakly in $H_1^2(M)$ and strongly in $L^2(M)$, but the H_1^2-norm of the functions B_α converges to $\|\nabla u\|_{L^2(\mathbb{R}^n)}$

as $\alpha \to +\infty$. We also get

$$\int_M |B_\alpha|^{2^*} dv_g = \int_{\mathbb{R}^n} |u|^{2^*} dx + \mathrm{o}(1) \tag{3.3}$$

when $(B_\alpha)_\alpha$ is given by (3.1), where $\mathrm{o}(1) \to 0$ as $\alpha \to +\infty$. By extension, we define a p-bubble in M as a sequence $(\mathcal{B}_\alpha)_\alpha$ of p-maps with components all zero, except one which is a bubble $(B_\alpha)_\alpha$. In other words, $(\mathcal{B}_\alpha)_\alpha$ is a p-bubble if there exists $i = 1, \dots, p$ and a bubble $(B_\alpha)_\alpha$ such that $\mathcal{B}^i_\alpha = B_\alpha$ and $\mathcal{B}^j_\alpha = 0$ for all α and all $j \neq i$. The following result was proved by Struwe [42] for scalar equations like (3.2) in bounded domains of the Euclidean space. The result easily extends to systems and manifolds as shown in Hebey [27]. We state the result with no proof and refer to [27] for more details.

Lemma 3.1. *Let (M, g) be a smooth, compact Riemannian manifold of dimension $n \geq 3$, let $p \geq 1$ be a natural number, and let A be a smooth map from M to $M^s_p(\mathbb{R})$. For any Palais–Smale sequence $(\mathcal{U}_\alpha)_\alpha$ for the functional $I_{A,g}$, there exist a solution $\mathcal{U}_\infty$ of (1.1)–(1.2), a natural number k, and p-bubbles $(\mathcal{B}^1_\alpha)_\alpha, \dots, (\mathcal{B}^k_\alpha)_\alpha$ such that, up to a subsequence, there hold*

$$\mathcal{U}_\alpha = \mathcal{U}_\infty + \sum_{i=1}^k \mathcal{B}^i_\alpha + \mathcal{R}_\alpha \tag{3.4}$$

and

$$I_{A,g}(\mathcal{U}_\alpha) = I_{A,g}(\mathcal{U}_\infty) + \frac{1}{n} \sum_{i=1}^k \mathcal{E}\left(\mathcal{B}^i_\alpha\right) + o(1)$$

for all α, where $\mathcal{R}_\alpha \to 0$ in $H^2_{1,p}(M)$ as $\alpha \to +\infty$, $I_{A,g}$ is as in (1.4), $\mathcal{E}(\mathcal{B}^i_\alpha)$ is the energy of the p-bubble $(\mathcal{B}^i_\alpha)_\alpha$, $\mathcal{E}$ is as in (1.3), and $o(1) \to 0$ as $\alpha \to +\infty$.

As a remark, see Ding [13] for more details, there exist solutions of (3.2) with arbitrarily high energies. However, all positive solutions have the same minimal energy and are classified. More precisely, nonnegative solutions of (3.2) have been classified by Caffarelli–Gidas–Spruck [6] and Obata [38]. They are all of the form

$$u_{\mu,x_0}(x) = \left(\frac{\mu}{\mu^2 + \frac{|x-x_0|^2}{n(n-2)}}\right)^{(n-2)/2}, \tag{3.5}$$

where μ is a nonnegative real number and x_0 is a point in the Euclidean space. The u_{μ,x_0}'s are extremal functions for the sharp Euclidean Sobolev inequality, and one can easily compute

$$\int_{\mathbb{R}^n} |\nabla u_{\mu,x_0}|^2 dx = \int_{\mathbb{R}^n} |u_{\mu,x_0}|^{2^*} dx = K_n^{-n},$$

where K_n is the sharp constant given by (1.6). On the other hand, if $u \in \dot{H}^2_1(\mathbb{R}^n)$ is a changing sign solution of (3.2), then, decomposing u into its positive and negative parts, we get

$$\int_{\mathbb{R}^n} |\nabla u|^2 dx = \int_{\mathbb{R}^n} |u|^{2^*} dx > 2K_n^{-n}.$$

In other words, coming back to (3.3), we get that the energy of a constant sign bubble is precisely K_n^{-n}, while the energy of a changing sign bubble is greater than $2K_n^{-n}$. We also get that if, in (3.4), one of the $(\mathcal{B}^1_\alpha)_\alpha$'s come from a constant sign bubble $(B_\alpha)_\alpha$, then, changing if necessary the x_α's and μ_α's, u in (3.1) can be chosen so that there holds $u = \pm u_{1,0}$, where $u_{1,0}$ is as in (3.5).

4. Relative equivariant Lusternik–Schnirelmann category

We very briefly discuss the notion of relative equivariant Lusternik–Schnirelmann category, and the properties of relative equivariant Lusternik–Schnirelmann category we need in the sequel. More developments can be found in Bartsch–Clapp [5] and Clapp–Puppe [8, 9]. Let B and D be two symmetric, closed subsets of a Banach space E such that D is included in B. By definition, the equivariant Lusternik–Schnirelmann category of B relatively to D, denoted by $\gamma_D(B)$, is the smallest natural number k such that there exist

(i) symmetric, open subsets $U_0, \dots, U_k$ of E,
(ii) odd, continuous maps $\chi_i : U_i \to \{-1, +1\}$, $i = 1, \dots, k$,
(iii) an odd, continuous map $\chi_0 : U_0 \to D$,

with the properties that the U_i's cover B, $D \subset U_0$, and $\chi_0 \equiv \mathrm{id}$ in the set D. If no such natural number exists then we set $\gamma_D(B) = +\infty$. When D is empty, the equivariant Lusternik–Schnirelmann category of B relatively to $D = \emptyset$ is defined as the smallest natural number k such that there exist k symmetric, open subsets $U_1, \dots, U_k$ of E, and k odd, continuous maps $\chi_i : U_i \to \{-1, +1\}$, $i = 1, \dots, k$, with the property that the U_i's cover B. The equivariant Lusternik–Schnirelmann category of B relatively to $D = \emptyset$ reduces to the Krasnosel′skiĭ genus $\gamma(B)$ of B, defined as the smallest natural number $k \geq 1$ such that there exists an odd, continuous map $\chi : B \to \mathbb{R}^k \setminus \{0\}$. In other words, we claim that if $D = \emptyset$, then $\gamma(B) = \gamma_\emptyset(B)$. In order to prove this claim, we first assume that $\gamma(B) < +\infty$ and we let $k \geq 1$ be such that there exists an odd, continuous map $\chi : B \to \mathbb{R}^k \setminus \{0\}$. Letting $U_i = \chi^{-1}(U_i')$, where U_i' is the subset of $\mathbb{R}^k \setminus \{0\}$ consisting of the points $(x_1, \dots, x_k)$ such that $x_i \neq 0$, and letting $\chi_i : U_i \to \{-1, +1\}$ be defined by $\chi_i(u) = \operatorname{sign}(\chi(u)_i)$, where sign is the sign function, we get $\gamma_\emptyset(B) \leq k$. In particular, $\gamma_\emptyset(B) \leq \gamma(B)$. Conversely, we assume that $\gamma_\emptyset(B) < +\infty$ and let k be a natural number such that there exist k symmetric, open subsets $U_1, \dots, U_k$ of E which cover B, and k odd, continuous maps $\chi_i : U_i \to \{-1, +1\}$ for $i = 1, \dots, k$. We let $(\eta_i)_i$ be a partition of unity of B subordinated to the covering $(U_i)_i$. Without loss of generality, we may assume that the η_i's are even functions. Then, we define $\chi : B \to \mathbb{R}^k \setminus \{0\}$ by letting $\chi(u) = \sum_{i=1}^k \eta_i(u) \chi_i(u) e_i$, where $(e_1, \dots, e_k)$ is a basis of $\mathbb{R}^k$. As is easily checked, χ is odd, continuous, and nowhere vanishing. It follows that $\gamma(B) \leq k$, and thus that $\gamma(B) \leq \gamma_\emptyset(B)$. This proves the above claim that $\gamma(B) = \gamma_\emptyset(B)$. We now state two properties of the relative equivariant Lusternik–Schnirelmann category that we repeatedly use in the proof of Theorem 1.1. We let B, C, and D be three symmetric, closed subsets of E. The two following properties hold true:

(A1) If $D \subset B \cap C$ and there exists an odd, continuous map $\nu : B \to C$ such that $\nu \equiv \mathrm{id}$ in D, then $\gamma_D(B) \leq \gamma_D(C)$.
(A2) If $D \subset B$, then $\gamma_D(B \cup C) \leq \gamma_D(B) + \gamma(C)$.

In particular, it follows from (A1) that if $D \subset B \subset C$, then $\gamma_D(B) \leq \gamma_D(C)$. These two properties (A1) and (A2) follow in a straightforward manner from the definition of relative equivariant Lusternik–Schnirelmann category. Concerning (A1), we may assume that $\gamma_D(C) < +\infty$. Let the U_i's and χ_i's be given by (i)-(iii) for the relative equivariant Lusternik–Schnirelmann category $\gamma_D(C)$. We let $V_0 = U_0$ and $\chi_0' = \chi_0$. We let also $V_i = \nu^{-1}(U_i)$ and $\chi_i' = \chi_i \circ \nu$ for $i = 1, \dots, \gamma_D(C)$. The V_i's are symmetric, open subsets of E, and the χ_i''s are odd, continuous maps from the V_i's to $\{-1, +1\}$ when $i \geq 1$. Noting that the V_i's cover B, we get $\gamma_D(B) \leq \gamma_D(C)$. This proves (A1). As for (A2), we may assume that $\gamma_D(B) < +\infty$ and $\gamma(C) < +\infty$. Let the U_i's and χ_i's be given by (i)-(iii) for the relative equivariant Lusternik–Schnirelmann category $\gamma_D(B)$, and let the V_j's and χ_j''s be given by the definition of the Krasnosel′skiĭ genus $\gamma(C) = \gamma_\emptyset(C)$. The union of these two families (U_i, χ_i) and $\left(V_j, \chi_j'\right)$ gives a family $(W_m, \widetilde{\chi}_m)$ consisting of $\gamma_D(B) + \gamma(C) + 1$ elements which satisfies (i)-(iii) for $B \cup C$ and D. In particular, $\gamma_D(B \cup C) \le \gamma_D(B) + \gamma(C)$, and this proves (A2).

5. Proof of Theorem 1.1 - Part 1

We prove the first part of Theorem 1.1 in this section, and the second part in the following section, following arguments from Clapp–Weth [10]. Related references are Ekeland–Ghoussoub [16,17], and Ghoussoub [23]. We let A be a smooth map from M to $M_p^s(\mathbb{R})$ such that the operator $\Delta_g^p + A$ is coercive on $H_{1,p}^2(M)$. We assume that for some $k \ge 1$, there exists $\Phi : \mathbb{R}^{k+1} \to H_{1,p}^2(M)$ an odd, continuous map such that there hold $I_{A,g} \circ \Phi < 2K_n^{-n}/n$ and $I_{A,g} \circ \Phi(y) \to -\infty$ as $|y| \to +\infty$, where $I_{A,g}$ is as in (1.4), and K_n is the sharp constant as in (1.6). We aim in proving that (1.1)–(1.2) possesses at least $k/2$ pairs of nonzero solutions with energy less than $2K_n^{-n}$.

By the coercivity of the operator $\Delta_g^p + A$ on $H_{1,p}^2(M)$, and by the Sobolev embedding theorem, there exist $\Lambda_1 \Lambda_2 > 0$ such that

$$I_{A,g}(\mathcal{U}) \ge \Lambda_1 \|\mathcal{U}\|^2_{H^2_{1,p}(M)} \left(1 - \Lambda_2 \|\mathcal{U}\|^{2^*-2}_{H^2_{1,p}(M)}\right) \tag{5.1}$$

for all $\mathcal{U} \in H_{1,p}^2(M)$. In particular, by (5.1), we get that there exist $c_0 \in (0, \mu_{A,g})$ and $r_0 > 0$ such that for any $\mathcal{U} \in H_{1,p}^2(M)$, there holds

$$\|\mathcal{U}\|_{H^2_{1,p}(M)} = r_0 \Longrightarrow I_{A,g}(\mathcal{U}) \ge 2c_0\,. \tag{5.2}$$

Since $c_0 < \mu_{A,g}$, it follows from the definition of $\mu_{A,g}$ in (1.5) that c_0 is not a critical value of $I_{A,g}$. Thus, $I_{A,g}^{c_0}$ is strictly positive invariant for the flow $\varphi_{A,g}$. We prove the first part of Theorem 1.1 in several steps. A preliminary step, which easily follows from the strict positive invariance of $I_{A,g}^{c_0}$, is as follows.

Step 5.1. *For any $c \in (0, \mu_{A,g})$, $\gamma_{I_{A,g}^c}(I_{A,g}^c) = 0$, where $\mu_{A,g}$ is as in (1.5), and $\gamma_{I_{A,g}^c}(I_{A,g}^c)$ is the equivariant Lusternik–Schnirelmann category of $I_{A,g}^c$ relatively to $I_{A,g}^c$. In particular, $\gamma_{I_{A,g}^{c_0}}(I_{A,g}^{c_0}) = 0$, where c_0 is as in (5.2).*

Proof. We let $c \in (0, \mu_{A,g})$ and define U_0 to be the set of all p-maps $\mathcal{U} \in H_{1,p}^2(M)$ with the property that there exists a time $\tau(\mathcal{U}) \in [0, T(\mathcal{U}))$ from which the trajectory $t \to \varphi_{A,g}(t, \mathcal{U})$ belongs to $I_{A,g}^c$, where $T(\mathcal{U})$ is the maximal existence time for the trajectory $t \to \varphi_{A,g}(t, \mathcal{U})$. Let $\mathcal{U}_0 \in U_0$. Then, by the definition of $\tau(\mathcal{U}_0)$ and by the strict positive invariance of $I_{A,g}^c$, we get that for any $\varepsilon_0 > 0$ sufficiently small, $\tau(\mathcal{U}_0) + \varepsilon_0 < T(\mathcal{U}_0)$ and $\varphi_{A,g}(\tau(\mathcal{U}_0) + \varepsilon_0, \mathcal{U}_0)$ belongs to the interior of $I_{A,g}^c$. For $\mathcal{U} \in H_{1,p}^2(M)$ sufficiently close to $\mathcal{U}_0$ we then get that $\tau(\mathcal{U}_0) + \varepsilon_0 < T(\mathcal{U})$ and that $\varphi_{A,g}(\tau(\mathcal{U}_0) + \varepsilon_0, \mathcal{U})$ also belongs to the interior of $I_{A,g}^c$. This implies that $\mathcal{U} \in U_0$ and that $\tau(\mathcal{U}) \le \tau(\mathcal{U}_0) + \varepsilon_0$ when $\mathcal{U} \in H_{1,p}^2(M)$ is sufficiently close to $\mathcal{U}_0$. In particular, U_0 is an open subset of $H_{1,p}^2(M)$ and $\tau : U_0 \to \mathbb{R}$ is upper semicontinuous. The lower semicontinuity of τ is a straightforward consequence of the fact that $I_{A,g}^c$ is closed. By the definition of the equivariant Lusternik–Schnirelmann category in Section 4, this ends the proof of Step 5.1. □

In what follows, given a natural number $\beta \ge 1$, we define c_β by

$$c_\beta = \inf\left\{c > c_0;\ \gamma_{I_{A,g}^{c_0}}\left(I_{A,g}^c\right) \ge \beta\right\}, \tag{5.3}$$

where $\gamma_{I_{A,g}^{c_0}}\left(I_{A,g}^c\right)$ is the equivariant Lusternik–Schnirelmann category of $I_{A,g}^c$ relatively to $I_{A,g}^{c_0}$, and we adopt the convention that $\inf \emptyset = +\infty$. A preliminary straightforward remark is that the sequence $(c_\beta)_\beta$ is nondecreasing. Another preliminary step is as follows.

Step 5.2. *For any $\beta \geq 1$, if c_β in (5.3) is finite, then there exists a Palais–Smale sequence for the functional $I_{A,g}$ at level c_β.*

Proof. Let $\beta \geq 1$ be such that $c_\beta < +\infty$. It suffices to prove that for any $\varepsilon > 0$, there exists $\mathcal{U}$ in $I_{A,g}^{-1}([c_\beta - \varepsilon, c_\beta + \varepsilon])$ such that there holds $\|\nabla I_{A,g}(\mathcal{U})\|_{H_{1,p}^2(M)} < \varepsilon$. We proceed by contradiction, and assume that there exists $\varepsilon_0 > 0$ such that for any $\mathcal{U}$ in $I_{A,g}^{-1}([c_\beta - \varepsilon_0, c_\beta + \varepsilon_0])$, there holds $\|\nabla I_{A,g}(\mathcal{U})\|_{H_{1,p}^2(M)} \geq \varepsilon_0$. By Lemma 2.1 with $C = H_{1,p}^2(M)$ and $D = I_{A,g}^{c_0}$, there exists an odd, continuous map

$$\nu : I_{A,g}^{c_\beta+\varepsilon_0} \longrightarrow I_{A,g}^{\max(c_\beta-\varepsilon_0,c_0)}$$

such that $\nu \equiv \mathrm{id}$ in the set $I_{A,g}^{c_0}$. By (A1) in Section 4 we then get

$$\gamma_{I_{A,g}^{c_0}}\left(I_{A,g}^{c_\beta+\varepsilon_0}\right) \leq \gamma_{I_{A,g}^{c_0}}\left(I_{A,g}^{\max(c_\beta-\varepsilon_0,c_0)}\right). \tag{5.4}$$

Since we also get by (A1) in Section 4 that $\gamma_{I_{A,g}^{c_0}}(I_{A,g}^{c}) \leq \gamma_{I_{A,g}^{c_0}}(I_{A,g}^{c_\beta+\varepsilon_0})$ for any $c_\beta \leq c \leq c_\beta + \varepsilon_0$, (5.4) contradicts the definition of c_β in (5.3) if $c_0 < c_\beta$, and thus if $\max(c_\beta - \varepsilon_0, c_0) < c_\beta$. If this is not the case, $\max(c_\beta - \varepsilon_0, c_0) = c_0$ and we conclude to a contradiction with Step 5.1. This ends the proof of Step 5.2. ☐

Let $\beta \geq 1$ be such that $c_\beta < +\infty$. By Step 5.2, there exists a Palais–Smale sequence $(\mathcal{U}_\alpha)_\alpha$ for the functional $I_{A,g}$ at level c_β. By Lemma 3.1, we then get all the possible decompositions of the sequence $(\mathcal{U}_\alpha)_\alpha$ according to the value of c_β. In case $c_\beta < K_n^{-n}/n$, a subsequence of $(\mathcal{U}_\alpha)_\alpha$ converges in $H_{1,p}^2(M)$ to a nontrivial critical point of the functional $I_{A,g}$. In particular, $c_0 < c_\beta$, and since $c_0 < K_n^{-n}/n$, we always get $c_0 < c_\beta$. In case $K_n^{-n}/n \leq c_\beta < 2K_n^{-n}/n$, there is at most one constant sign p-bubble in the decomposition (3.4) of $(\mathcal{U}_\alpha)_\alpha$, where a p-bubble is said to be of constant sign if the bubble from which the p-bubble is defined comes in (3.1) with a nonnegative or nonpositive solution u of (3.2). In particular, either c_β or $c_\beta - K_n^{-n}/n$ is a critical level of the functional $I_{A,g}$. In what follows, for any real number c, we let K_c be the set of all critical points of the functional $I_{A,g}$ at level c, namely

$$K_c = \left\{\mathcal{U} \in H_{1,p}^2(M);\ I_{A,g}(\mathcal{U}) = c \quad \text{and} \quad \nabla I_{A,g}(\mathcal{U}) = 0\right\}. \tag{5.5}$$

Step 5.3 in the proof of the first part of Theorem 1.1 is as follows.

Step 5.3. *Let η be a smooth cutoff function as in Section 3. For any positive real number θ and for $i = 1, \dots, p$, define*

$$U_\theta^i = \mathcal{B}_{2\theta}\left(K_{c_\beta - K_n^{-n}/n} + P_\theta^i\right),$$

where P_θ^i is the set consisting of the p-maps $\mathcal{U} = (u_1, \dots, u_p)$ such that $u_j = 0$ for all $j \neq i$, and $u_i = (\eta u_{\mu,0}) \circ \exp_{x_0}^{-1}$ for some $0 < \mu \leq \theta$ and $x_0 \in M$, where $u_{\mu,0}$ is as in (3.5). Let also $U_\theta = \bigcup_{i=1}^p U_\theta^i$. When $\theta > 0$ is sufficiently small, the sets U_θ and $-U_\theta$ are disjoint.

Proof. By contradiction, we assume that for any natural number $\alpha \geq 1$, the intersection of $U_{1/\alpha}$ with $-U_{1/\alpha}$ is not empty. Passing if necessary to a subsequence, we may assume that there exist two indices i_1 and i_2 such that for any α, the intersection of $U_{1/\alpha}^{i_1}$ with $-U_{1/\alpha}^{i_2}$ is not empty and thus that there exist sequences of p-maps $\mathcal{U}_\alpha^1$ and $\mathcal{U}_\alpha^2$ in $K_{c_\beta - K_n^{-n}/n}$, $\mathcal{B}_\alpha^1$ in $P_{1/\alpha}^{i_1}$, and $\mathcal{B}_\alpha^2$ in $P_{1/\alpha}^{i_2}$, such that

$$\left(\mathcal{U}_\alpha + \mathcal{B}_\alpha^1\right) - \left(\mathcal{V}_\alpha - \mathcal{B}_\alpha^2\right) \longrightarrow 0 \tag{5.6}$$

in $H_{1,p}^2(M)$ as $\alpha \to +\infty$. Passing if necessary to a subsequence, $(\mathcal{B}_\alpha^1)_{\alpha\in\mathbb{N}}$ and $(\mathcal{B}_\alpha^2)_{\alpha\in\mathbb{N}}$ are two nonnegative p-bubbles. Taking into account that p-bubbles converge weakly to 0 and that sequences in $K_{c_\beta - K_n^{-n}/n}$ are compact in $H_{1,p}^2(M)$, since by assumption $c_\beta - K_n^{-n}/n < K_n^{-n}/n$, it follows that up to a subsequence, $(\mathcal{U}_\alpha^1)_{\alpha\in\mathbb{N}}$ and $(\mathcal{U}_\alpha^2)_{\alpha\in\mathbb{N}}$ converge to the same limit in $H_{1,p}^2(M)$. This leads to a contradiction since by (5.6), the p-bubbles $(\mathcal{B}_\alpha^1)_{\alpha\in\mathbb{N}}$ and $(\mathcal{B}_\alpha^2)_{\alpha\in\mathbb{N}}$ would converge up to a subsequence to 0 in $H_{1,p}^2(M)$. We assumed here that the set $K_{c_\beta - K_n^{-n}/n}$ is not empty but the proof goes similarly, and is even easier if $K_{c_\beta - K_n^{-n}/n} = \emptyset$. Step 5.3 is proved. □

In what follows we let $\sharp B$ be the cardinal number of a set, with the convention that $\sharp B = 0$ when $B = \emptyset$ and $\sharp B = +\infty$ when B is infinite. Step 5.4 in the proof of the first part of Theorem 1.1 is as follows.

Step 5.4. *If there exists β such that $c_\beta = c_{\beta+1} < 2K_n^{-n}/n$, then $\sharp K_{c_\beta} = +\infty$ and the functional $I_{A,g}$ thus has infinitely many critical points at level c_β, where c_β is as in* (5.3)*, and K_{c_β} is as in* (5.5)*.*

Proof. We proceed by contradiction, and assume that $\sharp K_{c_\beta} < +\infty$. If we also get $\sharp K_{c_\beta} > 0$, and thus if K_{c_β} is not empty, then there holds $\gamma(K_{c_\beta}) = 1$ and there also holds $\gamma(\mathcal{B}_{2\theta}(K_{c_\beta})) = \gamma(K_{c_\beta}) = 1$ for $\theta > 0$ sufficiently small. First we assume that $c_\beta < K_n^{-n}/n$. In that case, by Step 5.2 and by the discussion after the proof of Step 5.2, K_{c_β} is not empty and Palais–Smale sequences for the functional $I_{A,g}$ at level c_β are compact in $H_{1,p}^2(M)$. We let $\theta > 0$ sufficiently small be such that $\gamma(\mathcal{B}_{2\theta}(K_{c_\beta})) = 1$. By the compactness of Palais–Smale sequences for the functional $I_{A,g}$ at level c_β, there exists $\varepsilon \in (0, c_\beta - c_0)$ such that for any p-map $\mathcal{U}$ in $\overline{I_{A,g}^{-1}([c_\beta - \varepsilon, c_\beta + \varepsilon]) \backslash \mathcal{B}_\theta(K_{c_\beta})}$, there holds

$$\|\nabla I_{A,g}(\mathcal{U})\|_{H_{1,p}^2(M)} \geq \frac{2\varepsilon}{\theta}\,.$$

Thus, we can apply Lemma 2.1 with $C = \overline{H_{1,p}^2(M) \backslash \mathcal{B}_{2\theta}(K_{c_\beta})}$, $D = I_{A,g}^{c_0}$, and $\delta = \theta$. This yields an odd, continuous map

$$\nu : \overline{I_{A,g}^{c_\beta+\varepsilon} \backslash \mathcal{B}_{2\theta}(K_{c_\beta})} \cup I_{A,g}^{c_0} \longrightarrow I_{A,g}^{c_\beta - \varepsilon} \tag{5.7}$$

such that $\nu \equiv \mathrm{id}$ in the set $I_{A,g}^{c_0}$. By the definition of c_β and by the properties (A1) and (A2) of the relative equivariant Lusternik–Schnirelmann category listed in Section 4, it follows from (5.7) that

$$\begin{aligned} \beta + 1 &\leq \gamma_{I_{A,g}^{c_0}}\left(I_{A,g}^{c_\beta+\varepsilon}\right) \\ &\leq \gamma_{I_{A,g}^{c_0}}\left(\overline{I_{A,g}^{c_\beta+\varepsilon} \backslash \mathcal{B}_{2\theta}(K_{c_\beta})} \cup I_{A,g}^{c_0}\right) + \gamma\left(\mathcal{B}_{2\theta}(K_{c_\beta})\right) \\ &\leq \gamma_{I_{A,g}^{c_0}}\left(I_{A,g}^{c_\beta-\varepsilon}\right) + \gamma\left(\mathcal{B}_{2\theta}(K_{c_\beta})\right) \\ &< \beta + \gamma\left(\mathcal{B}_{2\theta}(K_{c_\beta})\right). \end{aligned} \tag{5.8}$$

Clearly, (5.8) is in contradiction with $\gamma(\mathcal{B}_{2\theta}(K_{c_\beta})) = 1$, and so we are left with the remaining case where $c_\beta \geq K_n^{-n}/n$. Given $\theta' > 0$ we adopt here the convention that $\mathcal{B}_{\theta'}(K_{c_\beta}) = \emptyset$ when the set K_{c_β} is empty. We let U_θ be as in Step 5.3 and choose $\theta > 0$ sufficiently small such that U_θ and $-U_\theta$ are disjoint. Without any loss of generality, since $K_{c_\beta - K_n^{-n}/n}$ is compact and $\sharp K_{c_\beta} < +\infty$, we may also choose $\theta > 0$ sufficiently small such that $\mathcal{B}_{2\theta}(K_{c_\beta})$ and U_θ are disjoint, and such that $\mathcal{B}_{2\theta}(K_{c_\beta})$ and $-U_\theta$ are disjoint. For $\theta' > 0$, we define

$$Z_{\theta'} = \mathcal{B}_{\theta'}(K_{c_\beta}) \cup U_{\theta'/2} \cup \left(-U_{\theta'/2}\right).$$

As is easily checked from the definition of the Krasnosel′skiĭ genus, $\gamma(Z_{2\theta}) = 1$. Now we proceed as above. Since Palais–Smale sequences for the functional $I_{A,g}$ at level c_β have at most one constant sign p-bubble in their decomposition, there exists a real number $\varepsilon \in (0, c_\beta - c_0)$ such that for any p-map $\mathcal{U}$ in $\overline{I_{A,g}^{-1}([c_\beta - \varepsilon, c_\beta + \varepsilon]) \backslash Z_\theta}$, there holds

$$\|\nabla I_{A,g}(\mathcal{U})\|_{H^2_{1,p}(M)} \ge \frac{2\varepsilon}{\theta}\,.$$

Thus, we can apply Lemma 2.1 with $C = \overline{H^2_{1,p}(M) \backslash Z_{2\theta}}$, $D = I^{c_0}_{A,g}$, and $\delta = \theta$. This yields an odd, continuous map

$$\nu : \overline{I^{c_\beta+\varepsilon}_{A,g} \backslash Z_{2\theta}} \cup I^{c_0}_{A,g} \longrightarrow I^{c_\beta-\varepsilon}_{A,g} \tag{5.9}$$

such that $\nu \equiv \mathrm{id}$ in the set $I^{c_0}_{A,g}$. By the definition of c_β and by the properties (A1) and (A2) of the relative equivariant Lusternik–Schnirelmann category listed in Section 4, it follows from (5.9) that

$$\begin{aligned} \beta + 1 &\le \gamma_{I^{c_0}_{A,g}}\left(I^{c_\beta+\varepsilon}_{A,g}\right) \\ &\le \gamma_{I^{c_0}_{A,g}}\left(\overline{I^{c_\beta+\varepsilon}_{A,g} \backslash Z_{2\theta}} \cup I^{c_0}_{A,g}\right) + \gamma(Z_{2\theta}) \\ &\le \gamma_{I^{c_0}_{A,g}}\left(I^{c_\beta-\varepsilon}_{A,g}\right) + \gamma(Z_{2\theta}) \\ &< \beta + \gamma(Z_{2\theta})\,. \end{aligned} \tag{5.10}$$

Clearly, (5.10) is in contradiction with $\gamma(Z_{2\theta}) = 1$. In the two possible cases where $c_\beta < K_n^{-n}/n$ and $c_\beta \in [K_n^{-n}/n, 2K_n^{-n}/n)$, we get a contradiction. This ends the proof of Step 5.4. □

Step 5.5 is the last step in the proof of the first part of Theorem 1.1. It states as follows.

Step 5.5. *For $k \ge 1$ as in Theorem 1.1, there holds $c_{k+1} < 2K_n^{-n}/n$, where c_{k+1} is as in* (5.3).

Proof. We let $k \ge 1$ and Φ be as in Theorem 1.1. Then Φ is an odd, continuous map from $\mathbb{R}^{k+1}$ to $H^2_{1,p}(M)$ such that $I_{A,g} \circ \Phi < 2K_n^{-n}/n$ and $I_{A,g} \circ \Phi(y) \to -\infty$ as $|y| \to +\infty$. We let

$$\widetilde{k} = \gamma_{I^{c_0}_{A,g}}\left(I^{\sup(I_{A,g}\circ\Phi)}_{A,g}\right), \tag{5.11}$$

and prove that $\widetilde{k} \ge k+1$. Since $I_{A,g} \circ \Phi < 2K_n^{-n}/n$ and $I_{A,g} \circ \Phi$ is negative outside large balls, Step 5.5 obviously follows from this inequality. Without loss of generality, we may assume that $\widetilde{k}$ is finite. We let $U_0, \dots, U_{\widetilde{k}}$ and $\chi_0, \dots, \chi_{\widetilde{k}}$ be given by the definition in Section 4 of the relative equivariant Lusternik–Schnirelmann category $\widetilde{k}$ in (5.11). In particular, $I^{c_0}_{A,g} \subset U_0$ and $\chi_0 : U_0 \to I^{c_0}_{A,g}$ is odd, continuous, and such that $\chi_0 = \mathrm{id}$ in $I^{c_0}_{A,g}$. Changing U_0 if necessary and using Dugundji's [15] extension of Tietze's theorem, we may regard χ_0 as the restriction of an odd, continuous map, still denoted χ_0, defined from the whole Sobolev space $H^2_{1,p}(M)$ to itself. We set

$$\mathcal{O} = (\chi_0 \circ \Phi)^{-1}(B_0(r_0))\,, \tag{5.12}$$

where r_0 is as in (5.2). Clearly, $\mathcal{O}$ is symmetric, open, $0 \in \mathcal{O}$, and, since $I_{A,g} \circ \Phi(y) \to -\infty$ as $|y| \to +\infty$, we also get that $\mathcal{O}$ is bounded. By our choice of r_0 in (5.2), $I^{c_0}_{A,g} \cap \partial B_0(r_0) = \emptyset$, and we thus get that the set $\Phi^{-1}(U_0)$ does not intersect $\partial\mathcal{O}$. In particular, the boundary of $\mathcal{O}$ is covered by the sets $\partial\mathcal{O} \cap \Phi^{-1}(U_i)$, $i = 1, \dots, \widetilde{k}$. Let $(\eta_j)_j$, where $\eta_j : \partial\mathcal{O} \to \mathbb{R}$ and $j = 1, \dots, l$, be a partition of unity subordinated to this covering. Without loss of generality, we may assume that $\mathrm{Supp}\,\eta_j \subset \partial\mathcal{O} \cap \Phi^{-1}(U_j)$ for all j and that $l \leq \widetilde{k}$. We may also assume that the η_j's consist of even functions. We define a map $\chi : \partial\mathcal{O} \to \mathbb{R}^l$ by

$$\chi(y) = \sum_{i=1}^{l} \eta_i(y)\,\chi_i(\Phi(y))\,e_i\,, \tag{5.13}$$

where e_i is the i-th vector in the canonical basis of $\mathbb{R}^l$. The map χ as defined in (5.13) is odd, continuous, and nowhere vanishing. Since $\mathcal{O}$ in (5.12) is symmetric, bounded, open, and contains 0, it follows from the Borsuk–Ulam theorem (see, for instance, Kavian [33]) that l is greater than or equal to $k+1$. But $l \leq \widetilde{k}$ and we thus get $\widetilde{k} \geq k+1$. This ends the proof of Step 5.5. □

By Steps 5.2, 5.4, and 5.5, we are now in position to prove the first part of Theorem 1.1. The proof goes as follows.

Proof of the first part of Theorem 1.1. We let the c_β's be as in (5.3), and $k \geq 1$ be as in Theorem 1.1. By Steps 5.4 and 5.5, we may assume that the finite sequence $(c_1, \ldots, c_{k+1})$ is increasing and strictly bounded from above by $2K_n^{-n}/n$. We let $c_{k+2} = 2K_n^{-n}/n$, and $l \in \mathbb{N}$ be such that $c_l < K_n^{-n}/n \leq c_{l+1}$. If $l \geq 1$ then, by Step 5.2 and the discussion following Step 5.2, c_β is a critical level of the functional $I_{A,g}$ for $\beta = 1, \ldots, l$. Moreover, c_β comes with a nontrivial solution of equation (1.1), and we get that there are l distinct nonzero critical levels of the functional $I_{A,g}$ which are less than or equal to K_n^{-n}/n. For $\beta = l+1, \ldots, k+1$, see again Step 5.2 and the discussion following Step 5.2, either c_β or $c_\beta - K_n^{-n}/n$ is a critical level of the functional $I_{A,g}$. It follows that we also get the existence of at least $k-l$ distinct critical levels of $I_{A,g}$ in $(0, 2K_n^{-n}/n)$. We finally conclude that there exist at least $\frac{l+(k-l)}{2} = \frac{k}{2}$ distinct critical levels of $I_{A,g}$ in $(0, 2K_n^{-n}/n)$. This ends the proof of the first part of Theorem 1.1. □

6. Proof of Theorem 1.1 - Part 2

We prove the second part of Theorem 1.1 in this section. We let A be a smooth map from M to $M_p^s(\mathbb{R})$ such that the operator $\Delta_g^p + A$ is coercive on $H_{1,p}^2(M)$. We assume that for some $k \geq 1$, there exists $\Phi : \mathbb{R}^{k+1} \to H_{1,p}^2(M)$ an odd, continuous map such tha $I_{A,g} \circ \Phi < 2K_n^{-n}/n$ and $I_{A,g} \circ \Phi(y) \to -\infty$ as $|y| \to +\infty$, where $I_{A,g}$ is as in (1.4), and K_n is the sharp constant as in (1.6). We also assume that $\mu_{A,g} < K_n^{-n}/n$, $(-A)$ is cooperative, and (1.1)–(1.2) is fully coupled, where $\mu_{A,g}$ is as in (1.5). Then, we aim in proving that (1.1)–(1.2) possesses at least $(k+1)/2$ pairs of nonzero solutions with energy less than $2K_n^{-n}$. If u is a function in $H_1^2(M)$, we let $u^+ = \max(u, 0)$ and $u^- = \max(-u, 0)$. For a p-map $\mathcal{U} = (u_1, \ldots, u_p)$ in $H_{1,p}^2(M)$, we let $\mathcal{U}^+ = (u_1^+, \ldots, u_p^+)$ and $\mathcal{U}^- = (u_1^-, \ldots, u_p^-)$. We let also $\mathcal{P}$ be the set of all p-maps in $H_{1,p}^2(M)$ with nonnegative components. Here again, we proceed in several steps. A first step is as follows.

Step 6.1. *Let Λ be the positive constant appearing in the definition of the scalar product* (2.1). *If Λ is large enough then for sufficiently small positive real numbers δ, the sets $\mathcal{B}_\delta(\mathcal{P})$ and $\mathcal{B}_\delta(-\mathcal{P})$ are strictly positively invariant for the flow $\varphi_{A,g}$.*

Proof. Since $\nabla I_{A,g}$ is odd, we may restrict ourselves to considering the sole sets $\mathcal{B}_\delta(\mathcal{P})$. We write $\nabla I_{A,g}$ as in (2.4) so that $\nabla I_{A,g} = \mathrm{id} - \mathfrak{L}_1 - \mathfrak{L}_2$, where $\mathfrak{L}_1$ and $\mathfrak{L}_2$ are as in (2.2) and (2.3). First, we claim that if the constant Λ is large enough then for sufficiently small positive real numbers δ, there exists ν in $(0,1)$ such that for any p-map $\mathcal{U}$ in $\mathcal{B}_\delta(\mathcal{P})$, there holds

$$d(\mathfrak{L}_1(\mathcal{U}) + \mathfrak{L}_2(\mathcal{U}), \mathcal{P}) \leq \nu d(\mathcal{U}, \mathcal{P})\,, \tag{6.1}$$

where d is the distance on the Sobolev space $H_{1,p}^2(M)$. In what follows, we set $\mathcal{U} = (u_1, \dots, u_p)$, $\mathfrak{L}_1(\mathcal{U}) = (F_1(\mathcal{U}), \dots, F_p(\mathcal{U}))$, $\mathfrak{L}_2(\mathcal{U}) = (G_1(\mathcal{U}), \dots, G_p(\mathcal{U}))$. We begin with estimating $d(\mathfrak{L}_1(\mathcal{U}), \mathcal{P})$ for all p-maps $\mathcal{U}$ in $H_{1,p}^2(M)$. We assume that the constant Λ is greater than or equal to any diagonal component of A so that all the components of $\Lambda \operatorname{Id}_p - A$ are nonnegative. By the maximum principle, we then get that $\mathfrak{L}_1$ sends $\mathcal{P}$ to itself. By using (2.2) and the coercivity of the operator $\Delta_g^p + A$, we get that if Λ is chosen large enough, then

$$\|\mathfrak{L}_1(\mathcal{U})\|_{H_{1,p}^2(M)}^2 = \sum_{i,j=1}^p \int_M (\Lambda\delta_{ij} - A_{ij}) u_i F_j(\mathcal{U})\, dv_g$$
$$\le (1-\varepsilon)\left(\|\mathcal{U}\|_{H_{1,p}^2(M)}^2 + \|\mathfrak{L}_1(\mathcal{U})\|_{H_{1,p}^2(M)}^2 \right)$$

for some constant $\varepsilon \in (0,1)$. It follows that

$$\|\mathfrak{L}_1(\mathcal{U})\|_{H_{1,p}^2(M)} \le \sqrt{1-\varepsilon}\, \|\mathcal{U}\|_{H_{1,p}^2(M)}\,. \tag{6.2}$$

We let $\mathcal{V}$ be the orthogonal projection of $\mathcal{U}$ on the closed convex set $\mathcal{P}$. Applying (6.2) to the p-map $\mathcal{U} - \mathcal{V}$, since there holds $\mathfrak{L}_1(\mathcal{P}) \subset \mathcal{P}$, and since $\mathfrak{L}_1$ is a linear operator, we get

$$d(\mathfrak{L}_1(\mathcal{U}), \mathcal{P}) \le \|\mathfrak{L}_1(\mathcal{U}) - \mathfrak{L}_1(\mathcal{V})\|_{H_{1,p}^2(M)} \le \sqrt{1-\varepsilon}\, d(\mathcal{U}, \mathcal{P})\,. \tag{6.3}$$

Now, we estimate $d(\mathfrak{L}_2(\mathcal{U}), \mathcal{P})$. Here again, by the maximum principle, $\mathfrak{L}_2$ sends $\mathcal{P}$ to itself. Multiplying (2.3) by the p-map $-\mathfrak{L}_2(\mathcal{U})^-$, summing the p equations we get with this process, and integrating by parts on M yield

$$\left\|\mathfrak{L}_2(\mathcal{U})^-\right\|_{H_{1,p}^2(M)}^2 = -\sum_{i=1}^p \int_M |u_i|^{2^*-2} u_i G_i(\mathcal{U})^-\, dv_g \le \sum_{i=1}^p \int_M \left|u_i^-\right|^{2^*-2} u_i^- G_i(\mathcal{U})^-\, dv_g\,,$$

and by Hölder's inequality we then get

$$\left\|\mathfrak{L}_2(\mathcal{U})^-\right\|_{H_{1,p}^2(M)}^2 \le \sum_{i=1}^p \left\|u_i^-\right\|_{L^{2^*}(M)}^{2^*-1} \left\|G_i(u)^-\right\|_{L^{2^*}(M)}\,. \tag{6.4}$$

We also clearly get

$$\left\|u_i^-\right\|_{L^{2^*}(M)} = \min_{v \in H_1^2(M)^+} \|u_i - v\|_{L^{2^*}(M)} \tag{6.5}$$

for all $i = 1, \dots, p$, where $H_1^2(M)^+$ stands for the set of the nonnegative functions in $H_1^2(M)$. Combining (6.4) and (6.5), thanks to the Sobolev embedding theorem, we then get that there exists $C_\Lambda > 0$, independent of $\mathcal{U}$, such that

$$\left\|\mathfrak{L}_2(\mathcal{U})^-\right\|_{H_{1,p}^2(M)} \le C_\Lambda d(\mathcal{U}, \mathcal{P})^{2^*-1}\,. \tag{6.6}$$

Summing (6.3) with (6.6) yields

$$d(\mathfrak{L}_1(\mathcal{U}) + \mathfrak{L}_2(\mathcal{U}), \mathcal{P}) \le \sqrt{1-\varepsilon}\, d(\mathcal{U}, \mathcal{P}) + C d(\mathcal{U}, \mathcal{P})^{2^*-1}\,. \tag{6.7}$$

Then, it easily follows from (6.7) that for $\delta > 0$ sufficiently small, there exists ν in $(0,1)$ such that (6.1) holds true for all $\mathcal{U}$ in $\mathcal{B}_\delta(\mathcal{P})$. This proves the above claim, and now that we get (6.1), we fix $\delta > 0$ sufficiently small and write with (2.4) and (6.1) that for any $\lambda \in (0,1]$ and any $\mathcal{U} \in \mathcal{B}_\delta(\mathcal{P})$, there holds

$$d(\mathcal{U} - \lambda \nabla I_{A,g}(\mathcal{U}), \mathcal{P}) \le d((1-\lambda)\mathcal{U}, \mathcal{P}) + d(\lambda(F(\mathcal{U}) + G(\mathcal{U})), \mathcal{P}) < d(\mathcal{U}, \mathcal{P})\,.$$

We then get that $d(\mathcal{U} - \lambda \nabla I_{A,g}(\mathcal{U}), \mathcal{B}_\delta(\mathcal{P})) = 0$ for all $\lambda \in (0,1]$ and all p-maps $\mathcal{U}$ in $\mathcal{B}_\delta(\mathcal{P})$. Since $\mathcal{B}_\delta(\mathcal{P})$ is closed, convex, and its interior is nonempty, it follows from Deimling [12, Theorem 5.2] that $\mathcal{B}_\delta(\mathcal{P})$ is positively invariant in the sense that for any p-map $\mathcal{U}$ in $\mathcal{B}_\delta(\mathcal{P})$, the trajectory $t \to \varphi_{A,g}(t,\mathcal{U})$ stays in the set $\mathcal{B}_\delta(\mathcal{P})$ for all positive times. It remains to exhibit a contradiction in case such a trajectory intersects $\partial\mathcal{B}_\delta(\mathcal{P})$ for some positive time t_0. If such a $t_0 > 0$ exists, then by Mazur's separation theorem (see, for instance, Megginson [35]), there exists a continuous linear form ℓ on $H^2_{1,p}(M)$ such that $\ell(\varphi_{A,g}(t_0,\mathcal{U})) < \ell(\mathrm{int}(\mathcal{B}_\delta(\mathcal{P})))$, where $\mathrm{int}(\mathcal{B}_\delta(\mathcal{P}))$ is the interior of $\mathcal{B}_\delta(\mathcal{P})$. By (6.1), the operator $\mathfrak{L}_1 + \mathfrak{L}_2$ sends $\mathcal{B}_\delta(\mathcal{P})$ to its interior $\mathrm{int}(\mathcal{B}_\delta(\mathcal{P}))$. Thus we can write

$$\frac{\partial(\ell\circ\varphi_{A,g})}{\partial t}(t_0,\mathcal{U}) = \ell((\mathfrak{L}_1+\mathfrak{L}_2)(\varphi_{A,g}(t_0,\mathcal{U}))) - \ell(\varphi_{A,g}(t_0,\mathcal{U})) > 0\,.$$

Hence, for sufficiently small $\varepsilon > 0$, there holds $\ell(\varphi_{A,g}(t_0-\varepsilon,\mathcal{U})) < \ell(\varphi_{A,g}(t_0,\mathcal{U}))$, and we get $\varphi_{A,g}(t_0-\varepsilon,\mathcal{U})$ does not belong to $\mathcal{B}_\delta(\mathcal{P})$. This contradicts the positive invariance of the set $\mathcal{B}_\delta(\mathcal{P})$, and ends the proof of Step 6.1. □

Henceforth, we assume that Λ is sufficiently large and that δ is sufficiently small such that the sets $\mathcal{B}_\delta(\mathcal{P})$ and $\mathcal{B}_\delta(-\mathcal{P})$ are strictly positively invariant for the flow $\varphi_{A,g}$. We set $\mathcal{D}_\delta = \mathcal{B}_\delta(\mathcal{P}\cup(-\mathcal{P}))$. Since 0 is the only critical point of the functional $I_{A,g}$ at level 0, it follows from (2.5) and Step 6.1 that $I^0_{A,g}\cup\mathcal{D}_\delta$ is strictly positively invariant for the flow $\varphi_{A,g}$. Mimicking the proof of Step 5.1 in Section 5, we get that the following step holds true.

Step 6.2. *There holds $\gamma_{I^0_{A,g}\cup\mathcal{D}_\delta}\left(I^0_{A,g}\cup\mathcal{D}_\delta\right) = 0$, where $\gamma_{I^0_{A,g}\cup\mathcal{D}_\delta}\left(I^0_{A,g}\cup\mathcal{D}_\delta\right)$ is the equivariant Lusternik–Schnirelmann category of $I^0_{A,g}\cup\mathcal{D}_\delta$ relatively to $I^0_{A,g}\cup\mathcal{D}_\delta$, and $D_\delta = \mathcal{B}_\delta(\mathcal{P}\cup(-\mathcal{P}))$ is as above.*

In what follows, given a natural number $\beta \geq 1$, we define $\widetilde{c}_\beta$ by

$$\widetilde{c}_\beta = \inf\left\{c > 0;\ \gamma_{I^0_{A,g}\cup\mathcal{D}_\delta}\left(I^c_{A,g}\cup\mathcal{D}_\delta\right) \geq \beta\right\}, \tag{6.8}$$

where $\gamma_{I^0_{A,g}\cup\mathcal{D}_\delta}\left(I^c_{A,g}\cup\mathcal{D}_\delta\right)$ is the equivariant Lusternik–Schnirelmann category of $I^c_{A,g}\cup\mathcal{D}_\delta$ relatively to $I^0_{A,g}\cup\mathcal{D}_\delta$, and we adopt the convention that $\inf\emptyset = +\infty$. Here again, the sequence $(\widetilde{c}_\beta)_\beta$ is nondecreasing. Following Step 5.2 in Section 5, we now claim that the following step holds true.

Step 6.3. *For any $\beta \geq 1$, if $\widetilde{c}_\beta$ in (6.8) is finite, then there exists a Palais–Smale sequence $(\mathcal{U}_\alpha)_\alpha$ for the functional $I_{A,g}$ at level $\widetilde{c}_\beta$. Moreover, $\mathcal{U}_\alpha \in \overline{H^2_{1,p}(M)\backslash\mathcal{D}_{\delta/2}}$ for all α.*

Proof. In order to prove Step 6.3, it suffices to prove that for any $\varepsilon > 0$, there exists $\mathcal{U}$ in $I^{-1}_{A,g}([\widetilde{c}_\beta-\varepsilon,\widetilde{c}_\beta+\varepsilon])\cap\overline{H^2_{1,p}(M)\backslash\mathcal{D}_{\delta/2}}$ such that

$$\|\nabla I_{A,g}(\mathcal{U})\|_{H^2_{1,p}(M)} < \frac{4\varepsilon}{\delta}\,.$$

We proceed by contradiction and assume that there exists $\varepsilon_0 > 0$ such that for any p-map $\mathcal{U}$ in $I^{-1}_{A,g}([\widetilde{c}_\beta-\varepsilon_0,\widetilde{c}_\beta+\varepsilon_0])\cap\overline{H^2_{1,p}(M)\backslash\mathcal{D}_{\delta/2}}$, there holds $\|\nabla I_{A,g}(\mathcal{U})\|_{H^2_{1,p}(M)} \geq 4\varepsilon_0/\delta$. Since there holds

$$B_{\delta/2}(\overline{H^2_{1,p}(M)\backslash\mathcal{D}_\delta}) \subset \overline{H^2_{1,p}(M)\backslash\mathcal{D}_{\delta/2}}\,,$$

we may apply Lemma 2.1 with $C = \overline{H^2_{1,p}(M)\backslash\mathcal{D}_\delta}$, $D = I^0_{A,g}\cup\mathcal{D}_\delta$, and $\delta/2$ instead of δ. In particular, we get the existence of an odd, continuous map

$$\nu : I^{\widetilde{c}_\beta+\varepsilon_0}_{A,g}\cup\mathcal{D}_\delta \longrightarrow I^{\max(\widetilde{c}_\beta-\varepsilon_0,0)}_{A,g}\cup\mathcal{D}_\delta$$

such that $\nu \equiv \mathrm{id}$ in the set $I^0_{A,g}\cup\mathcal{D}_\delta$. Then, by (A1) in Section 4, we get

$$\gamma_{I^0_{A,g}\cup\mathcal{D}_\delta}\left(I^{\widetilde{c}_\beta+\varepsilon_0}_{A,g}\cup\mathcal{D}_\delta\right) \leq \gamma_{I^0_{A,g}\cup\mathcal{D}_\delta}\left(I^{\max(\widetilde{c}_\beta-\varepsilon_0,0)}_{A,g}\cup\mathcal{D}_\delta\right), \tag{6.9}$$

and the contradiction with the definition of $\widetilde{c}_\beta$ follows from (6.9) if $\widetilde{c}_\beta > 0$. If $\widetilde{c}_\beta = 0$, the contradiction follows from (6.9) and Step 6.2. This ends the proof of Step 6.3. □

Let $\beta \geq 1$ be such that $\widetilde{c}_\beta$ is finite, and let $(\mathcal{U}_\alpha)_\alpha$ be the Palais–Smale sequence for $I_{A,g}$ at level $\widetilde{c}_\beta$ we get from Step 6.3. We get $d(\mathcal{U}_\alpha, \mathcal{P} \cup (-\mathcal{P})) \geq \delta/2$ for all α. By Lemma 3.1, since 0 is the only critical point of the functional $I_{A,g}$ at level 0, we then get that $\widetilde{c}_\beta$ cannot be equal to 0. In particular, $\widetilde{c}_\beta > 0$. Assuming that $\widetilde{c}_\beta \leq K_n^{-n}/n$, we also get that there exists a subsequence of $(\mathcal{U}_\alpha)_\alpha$ converging to a nontrivial changing sign critical point of the functional $I_{A,g}$. By a changing sign p-map $\mathcal{U}$, we mean that $\mathcal{U} \notin (-\mathcal{P}) \cup \mathcal{P}$. If we assume that $K_n^{-n}/n < \widetilde{c}_\beta < 2K_n^{-n}/n$, then there is at most one constant sign p-bubble in the decomposition of $(\mathcal{U}_\alpha)_\alpha$. Thus, either $\widetilde{c}_\beta$ or $\widetilde{c}_\beta - K_n^{-n}/n$ is a critical level of the functional $I_{A,g}$. Step 6.4 in the proof of the second part of Theorem 1.1 is as follows.

Step 6.4. *If there exists β such that $\widetilde{c}_\beta = \widetilde{c}_{\beta+1} < 2K_n^{-n}/n$, then $\sharp K_{c_\beta} = +\infty$ and the functional $I_{A,g}$ thus has infinitely many critical points at level $\widetilde{c}_\beta$, where $\widetilde{c}_\beta$ is as in* (6.8)*, and K_{c_β} is as in* (5.5)*.*

The proof of Step 6.4 goes as for the proof of Step 5.4. When applying Lemma 2.1 we just set $D = I^0_{A,g} \cup \mathcal{D}_\delta$ instead of $D = I^{c_0}_{A,g}$ as done in the proof of Step 5.4. We omit the proof of Step 6.4 here. Step 6.5 in the proof of the second part of Theorem 1.1 is as follows.

Step 6.5. *For $k \geq 1$ as in Theorem 1.1, and for $\delta > 0$ in* (6.8) *sufficiently small, there holds $\widetilde{c}_k < 2K_n^{-n}/n$, where $\widetilde{c}_k$ is as in* (6.8)*.*

Proof. We let $k \geq 1$ and Φ be as in Theorem 1.1. Then, Φ is an odd, continuous map from $\mathbb{R}^{k+1}$ to $H^2_{1,p}(M)$ such that $I_{A,g} \circ \Phi < 2K_n^{-n}/n$ and $I_{A,g} \circ \Phi(y) \to -\infty$ as $|y| \to +\infty$. We set

$$\widetilde{k} = \gamma_{I^0_{A,g} \cup \mathcal{D}_\delta}\left(I^{\sup\left(I_{A,g}\circ\Phi\right)}_{A,g} \cup \mathcal{D}_\delta\right).$$

It suffices to prove that $\widetilde{k}$ is greater than or equal to k. We may assume that $\widetilde{k}$ is finite. By the definition of $\widetilde{k}$, there exist $\widetilde{k}+1$ symmetric, open subsets $U_0, \dots, U_{\widetilde{k}}$ of $H^2_{1,p}(M)$ which cover $I^{\sup\left(I_{A,g}\circ\Phi\right)}_{A,g} \cup \mathcal{D}_\delta$ and such that $\left(I^0_{A,g} \cup \mathcal{D}_\delta\right) \subset U_0$, and there exist $\widetilde{k}+1$ odd, continuous maps $\chi_0 : U_0 \to I^0_{A,g} \cup \mathcal{D}_\delta$ and $\chi_i : U_i \to \{-1, 1\}$, $i = 1, \dots, \widetilde{k}$, such that $\chi_0 \equiv \mathrm{id}$ in the set $I^0_{A,g} \cup \mathcal{D}_\delta$. Changing U_0, if necessary, and using Dugundji's [15] extension of Tietze's theorem as in Step 5.5, we may regard χ_0 as the restriction of an odd, continuous map, still denoted χ_0, defined from the whole Sobolev space $H^2_{1,p}(M)$ to itself. Now, we claim that there exists an odd, continuous map $\chi : \mathcal{N} \cap \mathcal{D}_\delta \to \{-1, 1\}$, where $\mathcal{N}$ is the Nehari manifold of $I_{A,g}$ as defined in the introduction. In order to prove this claim, we set

$$\mathcal{E} = \left\{\mathcal{U} \in \mathcal{N};\ \mathcal{U}^+ \in \mathcal{N} \quad \text{and} \quad \mathcal{U}^- \in \mathcal{N}\right\},$$

where $\mathcal{U}^+ = \left(u_1^+, \dots, u_p^+\right)$ and $\mathcal{U}^- = \left(u_1^-, \dots, u_p^-\right)$ if $\mathcal{U} = (u_1, \dots, u_p)$. The distance between the sets $\mathcal{E}$ and $\mathcal{P} \cup (-\mathcal{P})$ is positive. Indeed, by the continuity of the embedding of $H^2_1(M)$ into $L^{2^*}(M)$, we can write that there exists $C > 0$ such that for any p-maps $\mathcal{U} = (u_1, \dots, u_p)$ in $\mathcal{E}$ and $\mathcal{V} = (v_1, \dots, v_p)$ in $\mathcal{P}$, there holds

$$\|\mathcal{U} \pm \mathcal{V}\|^{2^*}_{H^2_{1,p}(M)} \geq C \sum_{i=1}^p \int_M |u_i \pm v_i|^{2^*} dv_g \geq C \sum_{i=1}^p \int_M \left|u_i^\pm\right|^{2^*} dv_g = nCI_{A,g}\left(\mathcal{U}^\pm\right) \geq nC\mu_{A,g}\,.$$

Decreasing $\delta > 0$, if necessary, we may now assume that the sets $\mathcal{E}$ and $\mathcal{D}_\delta$ are disjoint. As in Castro–Cossio–Neuberger [7, Lemma 2.5] we get that the set $\mathcal{N} \backslash \mathcal{E}$ consists in two connected components, namely $\{\mathcal{U} \in \mathcal{N};\ \mathcal{U} \in \mathcal{P} \text{ or } DI_{A,g}(\mathcal{U}^+).\mathcal{U}^+ < 0\}$ and its symmetric. It follows that $\mathcal{N} \cap \mathcal{D}_\delta$ also consists in two connected components and we get that there exists an odd, continuous map $\chi : \mathcal{N} \cap \mathcal{D}_\delta \to \{-1, 1\}$. This proves the above claim. Now, we let $\mathcal{O}$ be the inverse image by the map $\chi_0 \circ \Phi$ of the connected component $\mathcal{C}$ of $H^2_{1,p}(M) \backslash \mathcal{N}$ which contains 0, and let $K : H^2_{1,p}(M) \backslash \{0\} \to \mathbb{R}^+$ be such that $\mathcal{W}_\mathcal{U} = K(\mathcal{U})\mathcal{U}$ for all $\mathcal{U} \in H^2_{1,p}(M) \backslash \{0\}$, where $\mathcal{W}_\mathcal{U}$ is as in (1.7). We get that there holds $K(\mathcal{U}) < 1$ when $\mathcal{U} \in I^0_{A,g}$ and, by the Sobolev embedding theorem, that there holds $K(\mathcal{U}) > 1$ when $\mathcal{U}$ is sufficiently close to 0 in $H^2_{1,p}(M)$. In particular, since $\mathcal{W}_\mathcal{U} = \mathcal{U}$ if and only if $\mathcal{U} \in \mathcal{N}$, we get $\left(I^0_{A,g} \backslash \{0\}\right) \cap \mathcal{C} = \emptyset$. Since there holds $I_{A,g} \circ \Phi(y) \to -\infty$ as $|y| \to +\infty$, it follows that $\mathcal{O}$ is a symmetric, bounded, open neighborhood of 0. The boundary of $\mathcal{O}$ is covered by the sets $\partial\mathcal{O} \cap \Phi^{-1}(U_i)$, $i = 0, \dots, \widetilde{k}$. Let $\widetilde{\chi}_i = \chi_i$ when $i = 1, \dots, \widetilde{k}$, and $\widetilde{\chi}_0 = \chi \circ \chi_0$, where χ is the map we constructed above. Let also $\left(\eta_{i_j}\right)_j$, where $\eta_{i_j} : \partial\mathcal{O} \to \mathbb{R}$ and $j = 0, \dots, l$, be a partition of unity subordinated to this covering. Without loss of generality we may assume that $\operatorname{Supp} \eta_{i_j} \subset \partial\mathcal{O} \cap \Phi^{-1}\left(U_{i_j}\right)$ for all j and that $l \le \widetilde{k}$. We may also assume that the η_j's consist of even functions. We define a map $\chi : \partial\mathcal{O} \to \mathbb{R}^{l+1}$ by

$$\chi(y) = \sum_{j=0}^{l} \eta_{i_j}(y)\, \widetilde{\chi}_{i_j} \circ \Phi(y)\, e_j\,,$$

where e_j is the j-th vector in the canonical basis of $\mathbb{R}^{l+1}$. This map is odd, continuous, and nowhere vanishing. Since $\mathcal{O}$ is symmetric, bounded, open, and contains 0, it follows from the Borsuk–Ulam theorem (see, for instance, Kavian [33]) that $l + 1$ is greater than or equal to $k + 1$. But $l \le \widetilde{k}$ and we thus get that $\widetilde{k} \ge k$. This ends the proof of Step 6.5. □

By Steps 6.3, 6.4, and 6.5, we are now in position to prove the second part of Theorem 1.1. The proof goes as follows.

Proof of the second part of Theorem 1.1. We let the $\widetilde{c}_\beta$'s be as in (6.8), and let $k \ge 1$ be as in Theorem 1.1. By Steps 6.4 and 6.5, we may assume that the finite sequence $(\widetilde{c}_1, \dots, \widetilde{c}_k)$ is increasing and strictly bounded from above by $2K_n^{-n}/n$. By assumption, $\mu_{A,g} < K_n^{-n}/n$, where $\mu_{A,g}$ is as in (1.5). Then, since $(-A)$ is cooperative, and the system (1.1)–(1.2) is fully coupled, we get from Hebey [27] that $\mu_{A,g}$ is attained by a positive p-map which, up to a subsequence, is in turn a positive solution of (1.1)–(1.2). As is easily checked, $\mu_{A,g}$ can only be attained by p-maps in $\pm\mathcal{P}$ when $(-A)$ is cooperative. Since $\widetilde{c}_1$ is attained by a changing sign p-map when $\widetilde{c}_1 \le K_n^{-n}/n$, we get $\mu_{A,g} < \widetilde{c}_1$. In what follows, we let $\widetilde{c}_0 = \mu_{A,g}$, $\widetilde{c}_{k+1} = 2K_n^{-n}/n$, and $l \ge 0$ be such that $\widetilde{c}_l \le K_n^{-n}/n < \widetilde{c}_{l+1}$. Then there exist at least $l+1$ distinct nonzero critical levels of the functional $I_{A,g}$ which are less than or equal to K_n^{-n}/n. Since $k+1 > (k+1)/2$, we may assume in what follows $l < k$. For $\beta = l+1, \dots, k$, either $\widetilde{c}_\beta$ or $\widetilde{c}_\beta - K_n^{-n}/n$ is a critical level of the functional $I_{A,g}$. In particular, we get the existence of $k - l$ distinct critical levels of $I_{A,g}$ in $(0, 2K_n^{-n}/n)$. We finally conclude that there exist at least $\frac{l+1+(k-l)}{2} = \frac{k+1}{2}$ distinct critical levels of the functional $I_{A,g}$ in $(0, 2K_n^{-n}/n)$, and that $\widetilde{c}_0$ is attained by a positive p-map. This ends the proof of the second part of Theorem 1.1. □

7. Proof of Theorem 1.2

We let $i_0 = 1, \dots, p$ be given, $\delta \in (0,1)$, and $\eta : \mathbb{R} \to \mathbb{R}$ be a smooth cutoff function centered at 0 with support in $[-1, +1]$ such that $\eta \equiv 1$ in $[-\delta, +\delta]$. We let also $(\mu_\varepsilon)_\varepsilon$ be a sequence of positive real numbers such that $\mu_\varepsilon \to 0$ as $\varepsilon \to 0$, and, for $x \in M$, we define $u_\varepsilon = u_{\varepsilon,x}$,

$u_\varepsilon : M \to \mathbb{R}$, by

$$u_\varepsilon(y) = \frac{\eta\left(\frac{r}{\varepsilon}\right)}{\left(\mu_\varepsilon + r^2\right)^{(n-2)/2}}\,, \tag{7.1}$$

where $r = d_g(x,y)$ is the Riemannian distance from x to y. Then $\mathrm{Supp}\, u_\varepsilon \subset B_x(\varepsilon)$ for all $\varepsilon > 0$. We also define the p-map $\mathcal{U}_\varepsilon = \mathcal{U}_{\varepsilon,i_0,x}$, $\mathcal{U}_\varepsilon : M \to \mathbb{R}^p$, by

$$\mathcal{U}_\varepsilon = (0,\dots,0,u_\varepsilon,0,\dots,0)\,, \tag{7.2}$$

where u_ε is as in (7.1) and is placed at rank i_0 so that $u_\varepsilon^i = 0$ for all $i \neq i_0$ and $u_\varepsilon^{i_0} = u_\varepsilon$ if the u_ε^i's stand for the components of $\mathcal{U}_\varepsilon$. We compute expansions for $I_{A,g}\left(\mathcal{W}_{\mathcal{U}_\varepsilon}\right)$ in Lemmas 7.1 and 7.2 below. As a remark, there holds

$$I_{A,g}\left(\mathcal{W}_{\mathcal{U}_\varepsilon}\right) = \frac{1}{n}\left(\frac{\int_M\left(|\nabla u_\varepsilon|^2 + A_{i_0i_0}u_\varepsilon^2\right)dv_g}{\left(\int_M u_\varepsilon^{2^*}dv_g\right)^{2/2^*}}\right)^{n/2}.$$

The expansions for $I_{A,g}\left(\mathcal{W}_{\mathcal{U}_\varepsilon}\right)$ in Lemmas 7.1 and 7.2 are closely related to those of Aubin [4]. However, we face in our computations the difficulty that the supports of the $\mathcal{U}_\varepsilon$'s shrink to a point as $\varepsilon \to 0$. Because of this ε-shrinking of the supports of the $\mathcal{U}_\varepsilon$'s, we need to compute the ε-rate at which μ_ε should converge to zero. Lemmas 7.1 and 7.2 correspond to the two cases in Theorem 1.2 where

(i) $n \geq 4$ and $A_{i_0i_0}(x_0) < \frac{n-2}{4(n-1)}\mathrm{Scal}_g(x_0)$ for some i_0 and some x_0,

(ii) $n \geq 6$, $A_{i_0i_0} \equiv \frac{n-2}{4(n-1)}\mathrm{Scal}_g$ around some x_0 for some i_0, and $\mathrm{Weyl}_g(x_0) \not\equiv 0$,

where Scal_g is the scalar curvature of g, and Weyl_g is the Weyl curvature of g.

Lemma 7.1. *Let $i_0 = 1,\dots,p$ be given. Assume $\mu_\varepsilon^{\varepsilon^2} = O(\varepsilon)$ as $\varepsilon \to 0$ when $n = 4$, and $\mu_\varepsilon = O\left(\varepsilon^\theta\right)$ as $\varepsilon \to 0$ when $n > 4$, where $\theta > \frac{2(n-2)}{n-4}$. Then for any $x \in M$, there holds*

$$I_{A,g}\left(\mathcal{W}_{\mathcal{U}_\varepsilon}\right) = \frac{K_4^{-4}}{4} + \frac{K_4^{-4}}{16}\left(\mathrm{Scal}_g(x) - 6A_{i_0i_0}(x)\right)\mu_\varepsilon\ln\mu_\varepsilon + o\left(\mu_\varepsilon\ln\mu_\varepsilon\right) \tag{7.3}$$

when $n = 4$, and

$$I_{A,g}\left(\mathcal{W}_{\mathcal{U}_\varepsilon}\right) = \frac{K_n^{-n}}{n} - \frac{K_n^{-n}}{2n(n-4)}\left(\mathrm{Scal}_g(x) - \frac{4(n-1)}{n-2}A_{i_0i_0}(x)\right)\mu_\varepsilon + o\left(\mu_\varepsilon\right) \tag{7.4}$$

when $n > 4$, where K_n is as in (1.6)*, $I_{A,g}$ is as in* (1.4)*, $\mathcal{W}_\mathcal{U}$ for a p-map $\mathcal{U}$ is defined in* (1.7)*, Scal_g is the scalar curvature of g, and $\mathcal{U}_\varepsilon = \mathcal{U}_{\varepsilon,i_0,x}$ is as in* (7.1)–(7.2)*. Furthermore,* (7.3) *and* (7.4) *are uniform in x.*

Proof. We assume that $\mu_\varepsilon^{\varepsilon^2} = \mathrm{O}(\varepsilon)$ as $\varepsilon \to 0$ when $n = 4$, and that $\mu_\varepsilon = \mathrm{O}\left(\varepsilon^\theta\right)$ as $\varepsilon \to 0$ when $n > 4$, where $\theta > \frac{2(n-2)}{n-4}$. For any $x \in M$, there holds

$$\frac{1}{\omega_{n-1}r^{n-1}}\int_{\partial B_x(r)}\sqrt{|g|}\,d\sigma = 1 - \frac{1}{6n}\mathrm{Scal}_g(x)r^2 + \mathrm{O}\left(r^4\right) \tag{7.5}$$

as $r \to 0$, where $|g|$ is the determinant of the components of g in geodesic normal coordinates. By standard properties of the exponential map, the rest $\mathrm{O}\left(r^4\right)$ in (7.5) can be made uniform with respect to x. We set $I_p^q = \int_0^{+\infty}(1+r)^{-p}r^q dr$ for all positive real numbers p and q such that $p - q > 1$. When $n = 4$, thanks to (7.5), we compute

$$\int_M |\nabla u_\varepsilon|^2 dv_g = \frac{2\omega_3}{\mu_\varepsilon}\left(I_4^2 + \frac{\mathrm{Scal}_g(x)}{24}\mu_\varepsilon\ln\mu_\varepsilon + \mathrm{o}\left(\mu_\varepsilon\ln\mu_\varepsilon\right)\right) \tag{7.6}$$

and

$$\int_M A_{i_0i_0} u_\varepsilon^2 dv_g = -\frac{\omega_3 A_{i_0i_0}(x)}{2} \ln \mu_\varepsilon + \mathrm{o}(\ln \mu_\varepsilon) \tag{7.7}$$

as $\varepsilon \to 0$, uniformly with respect to x, where $u_\varepsilon = u_{\varepsilon,x}$ is as in (7.1). Similarly, when $n > 4$, still thanks to (7.5), we compute

$$\int_M |\nabla u_\varepsilon|^2 dv_g = \frac{(n-2)^2 \omega_{n-1} I_n^{n/2}}{2\mu_\varepsilon^{(n-2)/2}} \left(1 - \frac{(n+2)\operatorname{Scal}_g(x)}{6n(n-4)} \mu_\varepsilon + \mathrm{o}(\mu_\varepsilon)\right) \tag{7.8}$$

and

$$\int_M A_{i_0i_0} u_\varepsilon^2 dv_g = \frac{2(n-1)(n-2)\omega_{n-1} I_n^{n/2} A_{i_0i_0}(x)}{n(n-4)\mu_\varepsilon^{(n-4)/2}} + \mathrm{o}\left(\mu_\varepsilon^{(4-n)/2}\right) \tag{7.9}$$

as $\varepsilon \to 0$, uniformly with respect to x. Thanks to (7.5), we also compute that when $n \geq 4$, there holds

$$\int_M u_\varepsilon^{2^*} dv_g = \frac{\omega_{n-1} I_n^{(n-2)/2}}{2\mu_\varepsilon^{n/2}} \left(1 - \frac{\operatorname{Scal}_g(x)}{6(n-2)} \mu_\varepsilon + \mathrm{o}(\mu_\varepsilon)\right) \tag{7.10}$$

as $\varepsilon \to 0$, uniformly with respect to x. We get

$$\frac{n-2}{n} I_n^{n/2} = I_n^{(n-2)/2} = \frac{\omega_n}{2^{n-1}\omega_{n-1}} \tag{7.11}$$

and

$$\frac{(n-2)^2 \omega_{n-1}}{2} I_n^{n/2} = \frac{1}{K_n^2} \left(\frac{(n-2)\omega_{n-1} I_n^{n/2}}{2n}\right)^{2/2^*}, \tag{7.12}$$

and combining (7.6)–(7.12), we get that (7.3) and (7.4) hold true as $\varepsilon \to 0$, uniformly with respect to x. This ends the proof of Lemma 7.1. ☐

Lemma 7.2. *Let $x_0 \in M$ and $i_0 = 1, \dots, p$ be such that $A_{i_0i_0} \equiv h_g$ in an open neighborhood of x_0, where h_g is as in Theorem 1.2. Assume $\mu_\varepsilon^{\varepsilon^4} = O(\varepsilon)$ as $\varepsilon \to 0$ when $n = 6$, and $\mu_\varepsilon = O(\varepsilon^\theta)$ as $\varepsilon \to 0$ when $n \geq 7$, where $\theta > \frac{2(n-2)}{n-6}$. Then there exists an open neighborhood Ω of x_0 such that for any $x \in \Omega$, there holds*

$$I_{A,g}(\mathcal{W}_{\mathcal{U}_\varepsilon}) = \frac{K_6^{-6}}{6} + \frac{K_6^{-6}}{25920} \left(45 \left|\operatorname{Weyl}_g(x)\right|^2 - 75 \left|\operatorname{E}_g(x)\right|^2 - 2\operatorname{Scal}_g(x)^2\right) \mu_\varepsilon^2 \ln \mu_\varepsilon + o\left(\mu_\varepsilon^2 \ln \mu_\varepsilon\right) \tag{7.13}$$

when $n = 6$, and

$$I_{A,g}(\mathcal{W}_{\mathcal{U}_\varepsilon}) = \frac{K_n^{-n}}{n} - \frac{3\left|\operatorname{Weyl}_g(x)\right|^2 - \alpha_n \left|\operatorname{E}_g(x)\right|^2 + \beta_n \operatorname{Scal}_g(x)^2}{72n(n-4)(n-6)K_n^n} \mu_\varepsilon^2 + o\left(\mu_\varepsilon^2\right) \tag{7.14}$$

when $n > 6$, where K_n is as in (1.6), $I_{A,g}$ is as in (1.4), $\mathcal{W}_{\mathcal{U}}$ for a p-map $\mathcal{U}$ is defined in (1.7), Weyl_g is the Weyl curvature of g, E_g is the traceless part of the Ricci curvature of g, Scal_g is the scalar curvature of g, $\alpha_n = 4(2n-7)/(n-2)$, $\beta_n = -2(n+2)/(n(n-1)(n-2))$, and $\mathcal{U}_\varepsilon = \mathcal{U}_{\varepsilon,i_0,x}$ is as in (7.1)–(7.2). Furthermore, (7.3) and (7.4) are uniform in x.

Proof. We assume that $\mu_\varepsilon^{\varepsilon^4} = \mathrm{O}(\varepsilon)$ as $\varepsilon \to 0$ when $n = 6$, and that $\mu_\varepsilon = \mathrm{O}(\varepsilon^\theta)$ as $\varepsilon \to 0$ when $n \geq 7$, where $\theta > \frac{2(n-2)}{n-6}$. For any $x \in M$, there holds

$$\frac{1}{\omega_{n-1} r^{n-1}} \int_{\partial B_x(r)} A_{i_0i_0} \sqrt{|g|} d\sigma = A_{i_0i_0}(x) - \frac{\Delta A_{i_0}(x)}{2n} r^2 + \mathrm{O}\left(r^4\right) \tag{7.15}$$

and

$$\frac{1}{\omega_{n-1}r^{n-1}}\int_{\partial B_x(r)}\sqrt{|g|}d\sigma = 1-\frac{1}{6n}\operatorname{Scal}_g(x)r^2+A_g(x)r^4+\operatorname{O}\left(r^5\right) \tag{7.16}$$

as $r\to 0$, where $|g|$ is the determinant of the components of g in geodesic normal coordinates,

$$\begin{aligned}\Lambda_{i_0}(x) &= \Delta_g A_{i_0i_0}(x)+\frac{1}{3}A_{i_0i_0}(x)\operatorname{Scal}_g(x)\,,\\ A_g(x) &= \frac{18\Delta_g\operatorname{Scal}_g(x)+8\left|\operatorname{Ric}_g(x)\right|^2-3\left|\operatorname{Rm}_g(x)\right|^2+5\operatorname{Scal}_g(x)^2}{360n(n+2)}\,,\end{aligned}$$

and Rm_g, Ric_g, and Scal_g are respectively the Riemann curvature, the Ricci curvature, and the scalar curvature of g. By standard properties of the exponential map, the rests $\operatorname{O}(r^4)$ and $\operatorname{O}(r^5)$ in (7.15) can be made uniform with respect to x. We let $I_p^q=\int_0^{+\infty}(1+r)^{-p}r^q dr$, $p-q>1$, be as in the proof of Lemma 7.1. When $n=6$, thanks to (7.15), we compute

$$\int_M|\nabla u_\varepsilon|^2dv_g=\frac{3\omega_6}{8\mu_\varepsilon^2}\left(1-\frac{\operatorname{Scal}_g(x)}{9}\mu_\varepsilon-20A_g(x)\mu_\varepsilon^2\ln\mu_\varepsilon+\operatorname{o}\left(\mu_\varepsilon^2\ln\mu_\varepsilon\right)\right) \tag{7.17}$$

and

$$\int_M A_{i_0i_0}u_\varepsilon^2dv_g=\frac{1}{8\mu_\varepsilon}\left(\frac{5\omega_6}{4}A_{i_0i_0}(x)+\frac{\omega_5\Lambda_{i_0}(x)}{3}\mu_\varepsilon\ln\mu_\varepsilon+\operatorname{o}\left(\mu_\varepsilon\ln\mu_\varepsilon\right)\right) \tag{7.18}$$

as $\varepsilon\to 0$, uniformly with respect to x, where $u_\varepsilon=u_{\varepsilon,x}$ is as in (7.1). Similarly, when $n>6$, still thanks to (7.15), we compute

$$\begin{aligned}\int_M|\nabla u_\varepsilon|^2dv_g=&\frac{(n-2)^2\omega_{n-1}I_n^{n/2}}{2\mu_\varepsilon^{(n-2)/2}}\\ &\times\left(1-\frac{(n+2)\operatorname{Scal}_g(x)}{6n(n-4)}\mu_\varepsilon+\frac{(n+2)(n+4)A_g(x)}{(n-4)(n-6)}\mu_\varepsilon^2+\operatorname{o}\left(\mu_\varepsilon^2\right)\right)\end{aligned} \tag{7.19}$$

and

$$\int_M A_{i_0i_0}u_\varepsilon^2dv_g=\frac{\omega_n}{2^{n-1}\mu_\varepsilon^{(n-4)/2}}\left(\frac{2(n-1)A_{i_0i_0}(x)}{n-4}-\frac{(n-1)\Lambda_{i_0}(x)}{(n-4)(n-6)}\mu_\varepsilon+\operatorname{o}\left(\mu_\varepsilon\right)\right) \tag{7.20}$$

as $\varepsilon\to 0$, uniformly with respect to x. Thanks to (7.15), we also compute that when $n\geq 6$, there holds

$$\int_M u_\varepsilon^{2^*}dv_g=\frac{\omega_n}{2^n\mu_\varepsilon^{n/2}}\left(1-\frac{\operatorname{Scal}_g(x)}{6(n-2)}\mu_\varepsilon+\frac{n(n+2)A_g(x)}{(n-2)(n-4)}\mu_\varepsilon^2+\operatorname{o}\left(\mu_\varepsilon^2\right)\right) \tag{7.21}$$

as $\varepsilon\to 0$, uniformly with respect to x. Assume that $A_{i_0i_0}\equiv h_g$ in $B_{x_0}(2\delta)$ for some $x_0\in M$ and some $\delta>0$, combining (7.17)–(7.21), and thanks to (7.11) and (7.12), we get that (7.13) and (7.14) hold true as $\varepsilon\to 0$, uniformly with respect to x in $B_{x_0}(\delta)$. This ends the proof of Lemma 7.2. □

By Lemmas 7.1 and 7.2, we are now in position to prove Theorem 1.2. By these lemmas, see (7.23) below, $\mu_{A,g}<\frac{1}{n}K_n^{-n}$ when (i) or (ii) hold true. By Theorem 1.1 the proof of Theorem 1.2 then reduces to proving that there exists an odd, continuous map $\Phi:\mathbb{R}^{n+2}\to H_{1,p}^2(M)$ such that $I_{A,g}\circ\Phi<2K_n^{-n}/n$ and $I_{A,g}\circ\Phi(y)\to-\infty$ as $|y|\to+\infty$. Preliminary remarks are that

$$I_{A,g}\left(\mathcal{W}_{\mathcal{U}}\right)=\max_{t\geq 0}I_{A,g}\left(t\mathcal{U}\right), \tag{7.22}$$

and $\mathcal{W}_{\lambda\mathcal{U}}=\mathcal{W}_{\mathcal{U}}$ for all $\lambda>0$ and $\mathcal{U}\in H_{1,p}^2(M)\backslash\{0\}$. The proof of Theorem 1.2 goes as follows.

Proof of Theorem 1.2. We assume that (i) or (ii) holds true, and let $r_0 \in (0, i_g/3)$ be sufficiently small so that either $A_{i_0 i_0} < \lambda_n \operatorname{Scal}_g$ in $B_{x_0}(4r_0)$ and $n \geq 4$, or $A_{i_0 i_0} \equiv \lambda_n \operatorname{Scal}_g$ in $B_{x_0}(4r_0)$, $\operatorname{Weyl}_g \not\equiv 0$ at any point in $B_{x_0}(4r_0)$ and $n \geq 6$, where $\lambda_n = (n-2)/4(n-1)$. By conformal invariance of the conformal Laplacian, if $\widetilde{g} = \varphi^{4/(n-2)} g$ is a conformal metric to g, and $A_{i_0 i_0} \equiv \lambda_n \operatorname{Scal}_g$ in $B_{x_0}(4r_0)$, then

$$I_{A,g}\left(\mathcal{W}_{\varphi \mathcal{U}_\varepsilon}\right) = I_{\lambda_n \operatorname{Scal}_{\widetilde{g}} Id_p, \widetilde{g}}\left(\mathcal{W}_{\mathcal{U}_\varepsilon}\right)$$

for all $\varepsilon \in (0, \theta^{-1} r_0)$ and all $x \in B_{x_0}(3r_0)$, where $\mathcal{U}_\varepsilon = \mathcal{U}_{\varepsilon, i_0, x}$ is as in (7.1)–(7.2), but now defined with respect to $\widetilde{g}$, $\theta \geq 1$ depends only on g and $\widetilde{g}$, $\operatorname{Scal}_{\widetilde{g}}$ is the scalar curvature of $\widetilde{g}$, and Id_p is the identity $p \times p$ matrix. As is easily checked, we can choose $\widetilde{g}$ such that $\operatorname{Ric}_{\widetilde{g}}(x_0) \equiv 0$, where $\operatorname{Ric}_{\widetilde{g}}$ is the Ricci curvature of $\widetilde{g}$. We then get $\alpha_n \left|\mathrm{E}_g(x_0)\right|^2 - \beta_n \operatorname{Scal}_g(x_0)^2 = 0$, where α_n and β_n are as in Lemma 7.2. In particular, the Weyl curvature becomes the leading term in the expansions (7.13)–(7.14) of Lemma 7.2. By Lemmas 7.1 and 7.2, we then get that for $r_0 > 0$ sufficiently small, there exists $\varepsilon_0 \in (0, r_0)$ such that

$$I_{A,g}\left(\mathcal{W}_{\widetilde{\mathcal{U}}_\varepsilon}\right) < \frac{1}{n} K_n^{-n} \tag{7.23}$$

for all $x \in \overline{B_{x_0}(2r_0)}$ and all $\varepsilon \in (0, \varepsilon_0]$, where $\widetilde{\mathcal{U}}_\varepsilon = \varphi \mathcal{U}_\varepsilon$, $\varphi > 0$ is a smooth, positive function, and $\mathcal{U}_\varepsilon = \mathcal{U}_{\varepsilon, i_0, x}$ is as in (7.1)–Th2Eq2 with r in (7.1) being the distance with respect to $\widetilde{g}$. Here, $\varphi \equiv 1$ in case (i) holds true and φ comes from the conformal change of metric $\widetilde{g} = \varphi^{4/(n-2)} g$ in case (ii) holds true, where $\widetilde{g}$ is chosen so that $\operatorname{Ric}_{\widetilde{g}}(x_0) \equiv 0$. *A priori*, $\widetilde{\mathcal{U}}_\varepsilon$ has its support in the closure of the $\widetilde{g}$-ball $\widetilde{B}_x(\varepsilon)$. Decreasing ε if necessary, letting $\widetilde{\mathcal{U}}_\varepsilon = \varphi \mathcal{U}_{\theta\varepsilon}$ for $\theta > 0$ sufficiently small, we may assume that $\widetilde{\mathcal{U}}_\varepsilon$ has its support in the closure of the g-ball $B_x(\varepsilon)$, and thus that $\operatorname{Supp} \widetilde{\mathcal{U}}_\varepsilon \subset \overline{B_x(\varepsilon)}$ for all x and all ε. Now we claim that there exist a real number ε_1 in $(0, \varepsilon_0)$ and a smooth cutoff function v such that $v \equiv 1$ in $B_{x_0}(\varepsilon_1)$, $v \equiv 0$ in $M \backslash B_{x_0}(\varepsilon_0)$, and such that

$$I_{A,g}\left(\mathcal{W}_{(1-v)\widetilde{\mathcal{U}}_{\varepsilon_0}}\right) < \frac{1}{n} K_n^{-n} \tag{7.24}$$

for all points x in the ball $\overline{B_{x_0}(2r_0)}$, where $\widetilde{\mathcal{U}}_{\varepsilon_0}$ is as in (7.23). In order to prove this claim, we first note that by standard properties of the capacities of balls,

$$\operatorname{cap}_2\left(B_{x_0}(\varepsilon), B_{x_0}(\varepsilon_0)\right) = \inf_{u \in \mathcal{H}_{\varepsilon,\varepsilon_0}} \int_{B_{x_0}(\varepsilon_0) \backslash B_{x_0}(\varepsilon)} |\nabla u|_g^2 \, dv_g \longrightarrow 0 \tag{7.25}$$

as $\varepsilon \to 0$, where $\mathcal{H}_{\varepsilon,\varepsilon_0}$ is the set of all Lipschitz functions such that $u \equiv 1$ in $B_{x_0}(\varepsilon)$ and u has compact support in $B_{x_0}(\varepsilon_0)$. We refer to Grigor′yan [25] for more details on the notion of capacity. For $\varepsilon_0 > 0$ sufficiently small, we also get (see, for instance, Hebey [26]) that the Poincaré inequality holds true in the set $\mathcal{H}_{\varepsilon,\varepsilon_0}$. In other words, we also get that there exists a positive constant C such that for any function u in $\mathcal{H}_{\varepsilon,\varepsilon_0}$, there holds $\|u\|_{L^2(M)} \leq C \|\nabla u\|_{L^2(M)}$. The existence of a real number ε_1 and a smooth cutoff function v such that $v \equiv 1$ in $B_{x_0}(\varepsilon_1)$, $v \equiv 0$ in $M \backslash B_{x_0}(\varepsilon_0)$, and such that (7.24) holds true for all points x in the ball $\overline{B_{x_0}(2r_0)}$ then follows from (7.23) and (7.25) by the continuity of the functionals in $I_{A,g}$. This proves (7.24). Without loss of generality, we may assume that there exists $\Lambda > 1$ such that

$$\frac{1}{\Lambda} |y - x| \leq d_g\left(\exp_{x_0}(x), \exp_{x_0}(y)\right) \tag{7.26}$$

for all points x and y in the Euclidean ball $B_0(r_0)$, where $\exp_{x_0}$ is the exponential map at x_0. We also may assume that $2\Lambda\varepsilon_0 < r_0$. For any natural number $k > 0$, we let $\mathbb{B}^k$ be the unit ball in $\mathbb{R}^k$ and $\mathbb{S}^k$ be the unit sphere in $\mathbb{R}^{k+1}$. We also let $\mathcal{N}$ be the Nehari manifold of $I_{A,g}$ as in (1.5). We now define $\varPhi_1, \varPhi_2 : \overline{\mathbb{B}^n} \to \mathcal{N}$ by

$$\begin{cases} \varPhi_1(x) = \mathcal{W}\left(x_0, \varepsilon_1\right) & \text{if } |x| \le \dfrac{1}{2}\,, \\ \varPhi_1(x) = \mathcal{W}\left(x_1(x), \varepsilon(x)\right) & \text{if } \dfrac{1}{2} < |x| \le 1\,, \\ \varPhi_2(x) = \hat{\mathcal{W}}\left(x_2(x), \varepsilon_0\right) & \text{if } |x| \le \dfrac{1}{2}\,, \\ \varPhi_2(x) = \mathcal{W}\left(x_3(x), \varepsilon_0\right) & \text{if } \dfrac{1}{2} < |x| \le 1\,, \end{cases} \tag{7.27}$$

where $\mathcal{W}\left(x,\varepsilon\right) = \mathcal{W}_{\widetilde{\mathcal{U}}(x,\varepsilon)}$, $\hat{\mathcal{W}}\left(x,\varepsilon\right) = \mathcal{W}_{(1-v)\widetilde{\mathcal{U}}(x,\varepsilon)}$, $\widetilde{\mathcal{U}}(x,\varepsilon) = \widetilde{\mathcal{U}}_{\varepsilon,i_0,x}$ is as in (7.23), v and ε_1 are as in (7.24), $\varepsilon\left(x\right)$ is given by $\varepsilon\left(x\right) = 2\left(\varepsilon_0 - \varepsilon_1\right)|x| + 2\varepsilon_1 - \varepsilon_0$, and for $\varLambda > 1$ as in (7.26), $x_1\left(x\right)$, $x_2\left(x\right)$, and $x_3\left(x\right)$ are given by

$$\begin{aligned} x_1\left(x\right) &= \exp_{x_0}\left(-2\varLambda\varepsilon_0\left(2 - \frac{1}{|x|}\right)x\right), \\ x_2\left(x\right) &= \exp_{x_0}\left(4\varLambda\varepsilon_0 x\right), \quad \text{and} \quad x_3\left(x\right) = \exp_{x_0}\left(2\varLambda\varepsilon_0\frac{x}{|x|}\right). \end{aligned} \tag{7.28}$$

For any point x such that $1/2 \le |x| \le 1$, we get that $\varepsilon\left(x\right)$ belongs to $\left[\varepsilon_1, \varepsilon_0\right]$ and that the points $x_i\left(x\right)$ in (7.28) belong to $\overline{B_{x_0}\left(2r_0\right)}$ for $i = 1, 2, 3$ since $\varLambda\varepsilon_0 < r_0/2$. For any point x such that $|x| = 1/2$, there hold $x_1\left(x\right) = x_0$ and $\varepsilon\left(x\right) = \varepsilon_1$. It follows that $\varPhi_1$ in (7.27) is continuous. Similarly, for any point x such that $|x| = 1/2$, there hold $x_2\left(x\right) = x_3\left(x\right)$ and $v \equiv 0$ in $B_{x_2(x)}\left(\varepsilon_0\right)$ since $d_g\left(x_0, x_2\left(x\right)\right) = 2\varLambda\varepsilon_0$, $v \equiv 0$ in $M \backslash B_{x_0}\left(\varepsilon_0\right)$, and $\varLambda > 1$. It follows that $\varPhi_2$ in (7.27) is also continuous. There hold $\operatorname{Supp}\varPhi_1\left(x\right) \subset \overline{B_{x_0}\left(\varepsilon_1\right)}$ and $\operatorname{Supp}\varPhi_2\left(x\right) \subset \overline{B_{x_2(x)}\left(\varepsilon_0\right) \backslash B_{x_0}\left(\varepsilon_1\right)}$ in case $|x| \le 1/2$, and there hold $\operatorname{Supp}\varPhi_1\left(x\right) \subset \overline{B_{x_1(x)}\left(\varepsilon\left(x\right)\right)}$ and $\operatorname{Supp}\varPhi_2\left(x\right) \subset \overline{B_{x_3(x)}\left(\varepsilon_0\right)}$ in case $|x| \ge 1/2$. By (7.26), we get $d_g\left(x_1(x), x_3(x)\right) \ge 4\varepsilon_0|x|$. It follows that for any point $x \in \overline{\mathbb{B}^n}$, there holds

$$\operatorname{Supp}\varPhi_1\left(x\right) \cap \operatorname{Supp}\varPhi_2\left(x\right) = \emptyset\,. \tag{7.29}$$

We also easily get $\varPhi_1\left(x\right) = \varPhi_2\left(-x\right)$ for all x in $\mathbb{S}^{n-1}$. We then define the map $\varPhi_0 : \mathbb{S}^n \to \mathcal{N}$ by

$$\varPhi_0\left(x_1, \dots, x_{n+1}\right) = \begin{cases} \varPhi_1\left(x_1, \dots, x_n\right) & \text{if } x_{n+1} \ge 0 \\ \varPhi_2\left(-x_1, \dots, -x_n\right) & \text{otherwise,} \end{cases} \tag{7.30}$$

where $\varPhi_1$ and $\varPhi_2$ are as in (7.27). It is easily checked that the map $\varPhi_0$ is also continuous. We now introduce the map

$$\widetilde{\varPhi} : \left(\mathbb{S}^n \times (-1,1)\right) \cup \left(\mathbb{B}^{n+1} \times \{-1,1\}\right) \to H^2_{1,p}\left(M\right) \setminus \{0\}$$

defined by

$$\widetilde{\varPhi}\left(x,t\right) = \begin{cases} (1+t)\,\varPhi_0\left(x\right) - (1-t)\,\varPhi_0\left(-x\right) & \text{if } x \in \mathbb{S}^n\,, \\ 2\,|x|\,\varPhi_0\left(\dfrac{x}{|x|}\right) + (1-|x|)\,\mathcal{W}\left(y_0, \varepsilon_0\right) & \text{if } t = 1\,, \\ -\,2\,|x|\,\varPhi_0\left(-\dfrac{x}{|x|}\right) - (1-|x|)\,\mathcal{W}\left(y_0, \varepsilon_0\right) & \text{if } t = -1\,, \end{cases}$$

where $y_0 = \exp_{x_0}\left(2r_0\theta_0\right)$ for some θ_0 in $\mathbb{S}^n$, and $\varPhi_0$ is as in (7.30). For any $x \in \overline{\mathbb{B}^n}$, the supports of the p-maps $\varPhi_1$ and $\varPhi_2$ in (7.27) are subsets of $B_{x_0}\left((2\varLambda+1)\varepsilon_0\right)$, and since $d_g(x_0, y_0) = 2r_0$, and $r_0 > (\Lambda + 1)\varepsilon_0$, we get

$$\operatorname{Supp} \mathcal{W}(y_0, \varepsilon_0) \cap \operatorname{Supp} \Phi_1(x) = \emptyset \tag{7.31}$$

and

$$\operatorname{Supp} \mathcal{W}(y_0, \varepsilon_0) \cap \operatorname{Supp} \Phi_2(x) = \emptyset \tag{7.32}$$

for all $x \in \overline{\mathbb{B}^n}$. By (7.29), (7.31), and (7.32), the supports of the p-maps $\Phi_1(x)$, $\Phi_2(x)$ and $\mathcal{W}(y_0, \varepsilon_0)$ are mutually disjoint for all points x in $\overline{\mathbb{B}^n}$. In particular, the p-map $\widetilde{\Phi}$ takes its values in $H^2_{1,p}(M) \backslash \{0\}$. It is easily checked that $\widetilde{\Phi}$ is odd and continuous. Taking into account that the domain of definition of the map $\widetilde{\Phi}$ is precisely the boundary of the set $\mathbb{B}^{n+1} \times (-1, 1)$, we may define the radial extension of $\widetilde{\Phi}$ as the map $\Phi : \mathbb{R}^{n+2} \to H^2_{1,p}(M) \backslash \{0\}$ given by $\Phi(tx) = t\widetilde{\Phi}(x)$ for all positive real numbers t and all points x in $\partial(\mathbb{B}^{n+1} \times (-1, 1))$. Then the map Φ is odd and continuous. By (7.22), (7.23), and (7.24), we get that there holds $I_{A,g} \circ \Phi < 2K_n^{-n}/n$. As is easily checked, $\mathcal{E}(\mathcal{U}) \geq n\mu_{A,g} > 0$ for all $\mathcal{U} \in \mathcal{N}$. It follows that

$$\max_{\mathcal{U} \in \mathcal{N}} I_{A,g}(t\mathcal{U}) \longrightarrow -\infty$$

as $t \to +\infty$. We then get that there holds $I_{A,g} \circ \Phi(y) \to -\infty$ as $|y| \to +\infty$. This ends the proof of Theorem 1.2. □

8. Proof of Theorem 1.3

Let $x_0 \in M$, $r_0 > 0$, and $i_0 \in \{1, \ldots, p\}$ be such that g is conformally flat in $B_{x_0}(4r_0)$ and $A_{i_0 i_0} = \lambda_n \operatorname{Scal}_g$ in $B_{x_0}(4r_0)$, where $\lambda_n = (n-2)/4(n-1)$, and Scal_g is the scalar curvature of g. We also assume that $n > 6$. Decreasing r_0 if necessary, there exists $\widetilde{g} = \varphi^{4/(n-2)} g$ a conformal metric to g such that $\widetilde{g}$ is flat in $B_{x_0}(4r_0)$. We define $u_\varepsilon = u_{\varepsilon,x}$ as in (7.1) with r being the distance with respect to $\widetilde{g}$. Let $K \geq 1$ be such that $K^{-1} d_g \leq d_{\widetilde{g}} \leq K d_g$, where d_g and $d_{\widetilde{g}}$ are the distances with respect to g and $\widetilde{g}$. Since $\widetilde{g}$ is flat in $B_{x_0}(4r_0)$, we easily compute that for any $x \in B_{x_0}(3r_0)$, any $\varepsilon \in (0, K^{-1} r_0)$, and any smooth function h in M, there holds

$$\int_M |\nabla u_\varepsilon|^2 dv_{\widetilde{g}} = \frac{n(n-2)\omega_n}{2^n \mu_\varepsilon^{(n-2)/2}} \left(1 + \mathrm{O}\left(\varepsilon^{2-n} \mu_\varepsilon^{(n-2)/2}\right)\right), \tag{8.1}$$

$$\int_M h u_\varepsilon^2 dv_{\widetilde{g}} = \frac{(n-1)\omega_n h(x)}{2^{n-2}(n-4)\mu_\varepsilon^{(n-2)/2}} \left(\mu_\varepsilon + \mathrm{O}\left(\varepsilon^{4-n} \mu_\varepsilon^{(n-4)/2}\right)\right), \tag{8.2}$$

$$\int_M u_\varepsilon^{2^*} dv_{\widetilde{g}} = \frac{\omega_n}{2^n \mu_\varepsilon^{n/2}} \left(1 + \mathrm{O}\left(\varepsilon^{-n} \mu_\varepsilon^{n/2}\right)\right) \tag{8.3}$$

uniformly with respect to x. Assume that $\mu_\varepsilon = \mathrm{O}\left(\varepsilon^{2\theta}\right)$ for some $\theta > \theta_n$, where $\theta_n = \frac{n-2}{n-6}$. Then, by (8.1), we get that there exists $\kappa = \frac{n-6}{2\theta}(\theta - \theta_n)$, $\kappa > 0$, such that there holds

$$\int_M |\nabla u_\varepsilon|^2 dv_{\widetilde{g}} = \frac{n(n-2)\omega_n}{2^n \mu_\varepsilon^{(n-2)/2}} \left(1 + \mathrm{O}\left(\mu_\varepsilon^{2+\kappa}\right)\right), \tag{8.4}$$

$$\int_M h u_\varepsilon^2 dv_{\widetilde{g}} = \frac{(n-1)\omega_n h(x)}{2^{n-2}(n-4)\mu_\varepsilon^{(n-2)/2}} \left(\mu_\varepsilon + \mathrm{O}\left(\mu_\varepsilon^2\right)\right), \tag{8.5}$$

$$\int_M u_\varepsilon^{2^*} dv_{\widetilde{g}} = \frac{\omega_n}{2^n \mu_\varepsilon^{n/2}} \left(1 + \mathrm{O}\left(\mu_\varepsilon^{2+\kappa}\right)\right) \tag{8.6}$$

uniformly with respect to x. Now we assume that there exists $i_1 \neq i_0$, for instance $i_1 > i_0$, such that $A_{i_0 i_1}(x_0) \neq 0$. We define $\mathcal{U}_\varepsilon = \mathcal{U}_{\varepsilon, i_0, i_1, x}$ by

$$\mathcal{U}_\varepsilon = (0, \ldots, 0, u_\varepsilon, 0, \ldots, 0, \alpha\nu_\varepsilon u_\varepsilon, 0, \ldots, 0), \tag{8.7}$$

where u_ε is as above, the two nonzero components u_ε and $\alpha\nu_\varepsilon u_\varepsilon$ of $\mathcal{U}_\varepsilon$ are placed at ranks i_0 and i_1, $\alpha = -\operatorname{sign}(A_{i_0i_1}(x_0))$ is equal to -1 if $A_{i_0i_1}(x_0)$ is positive and 1 if $A_{i_0i_1}(x_0)$ is negative, and the ν_ε's, to be chosen later on, are positive and such that $\nu_\varepsilon \to 0$ as $\varepsilon \to 0$. By conformal invariance of the conformal Laplacian and since $\widetilde{g} = \varphi^{4/(n-2)}g$ is flat in $B_{x_0}(4r_0)$, we get

$$I_{A,g}\left(\mathcal{W}_{\varphi\mathcal{U}_\varepsilon}\right) = I_{\widetilde{A},\widetilde{g}}\left(\mathcal{W}_{\mathcal{U}_\varepsilon}\right) \tag{8.8}$$

for all $x \in B_{x_0}(3r_0)$ and all $\varepsilon \in (0, K^{-1}r_0)$, where

$$\widetilde{A} = \varphi^{2-2^*}\left(A - \lambda_n \operatorname{Scal}_g Id_p\right) \tag{8.9}$$

and Id_p is the identity $p \times p$ matrix. Let $\widetilde{A}_{ij}$ be the components of $\widetilde{A}$ in (8.9). Since $\widetilde{A}_{i_0i_0} \equiv 0$ in $B_{x_0}(4r_0)$, we compute

$$\begin{aligned}\int_M \Big(|\nabla\mathcal{U}_\varepsilon|^2 + \widetilde{A}(\mathcal{U}_\varepsilon,\mathcal{U}_\varepsilon)\Big)\, dv_{\widetilde{g}} \\ = \left(1+\nu_\varepsilon^2\right)\int_M |\nabla u_\varepsilon|^2\, dv_{\widetilde{g}} + 2\alpha\nu_\varepsilon \int_M \frac{A_{i_0i_1}}{\varphi^{2^*-2}} u_\varepsilon^2 dv_{\widetilde{g}} + \nu_\varepsilon^2 \int_M \widetilde{A}_{i_1i_1} u_\varepsilon^2 dv_{\widetilde{g}}\,,\end{aligned}$$

and thus, by (8.4), we get

$$\begin{aligned}\int_M \Big(|\nabla\mathcal{U}_\varepsilon|^2 + \widetilde{A}(\mathcal{U}_\varepsilon,\mathcal{U}_\varepsilon)\Big)\, dv_{\widetilde{g}} = \frac{n(n-2)\omega_n}{2^n\mu_\varepsilon^{(n-2)/2}} \\ \times\left(1 + \frac{8(n-1)\alpha A_{i_0i_1}(x)}{n(n-2)(n-4)\varphi^{2^*-2}(x)}\mu_\varepsilon\nu_\varepsilon + \mathrm{O}\left(\mu_\varepsilon^{2+\kappa}\right) + \mathrm{O}\left(\nu_\varepsilon^2\right) + \mathrm{O}\left(\mu_\varepsilon^2\nu_\varepsilon\right)\right).\end{aligned} \tag{8.10}$$

We now set $\nu_\varepsilon = \mu_\varepsilon^{1+\theta'}$ for some $\theta' \in (0,\kappa)$. Then the $\mathrm{O}\left(\mu_\varepsilon^2\mu_\varepsilon^\kappa\right)$'s, $\mathrm{O}\left(\nu_\varepsilon^2\right)$'s, and $\mathrm{O}\left(\nu_\varepsilon\mu_\varepsilon^2\right)$'s are like $\mathrm{o}\left(\mu_\varepsilon\nu_\varepsilon\right)$. In particular, we get with (8.4), (8.8), and (8.10) that

$$I_{A,g}\left(\mathcal{W}_{\varphi\mathcal{U}_\varepsilon}\right) = \frac{1}{n}K_n^{-n}\left(1 + \frac{8(n-1)\alpha A_{i_0i_1}(x)}{n(n-2)(n-4)\varphi^{2^*-2}(x)}\mu_\varepsilon\nu_\varepsilon + \mathrm{o}\left(\mu_\varepsilon\nu_\varepsilon\right)\right) \tag{8.11}$$

for all $x \in B_{x_0}(3r_0)$ and all $\varepsilon \in (0, K^{-1}r_0)$, the expansion being uniform with respect to x. Since $\alpha = -\operatorname{sign}(A_{i_0i_1}(x_0))$, it follows from (8.11) that for $r_0 > 0$ sufficiently small, there exists $\varepsilon_0 \in (0, r_0)$ such that

$$I_{A,g}\left(\mathcal{W}_{\widetilde{\mathcal{U}}_\varepsilon}\right) < \frac{1}{n}K_n^{-n} \tag{8.12}$$

for all $x \in \overline{B_{x_0}(2r_0)}$ and all $\varepsilon \in (0, \varepsilon_0]$, where $\widetilde{\mathcal{U}}_\varepsilon = \varphi\mathcal{U}_\varepsilon$, $\varphi > 0$ is a smooth, positive function, and $\mathcal{U}_\varepsilon = \mathcal{U}_{\varepsilon,i_0,i_1,x}$ is as in (8.7). In particular, by (8.12) we are back to estimates like in (7.23), and we may then continue as in the proof of Theorem 1.2 in Section 7. This ends the proof of Theorem 1.3.

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