ASYMPTOTIC STABLILITY, CONVEXITY, AND LIPSCHITZ REGULARITY OF DOMAINS IN THE ANISOTROPIC REGIME

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ABSTRACT. Anisotropic operators appear in several branches of applied sciences and, in particular, in physics. They involve directional derivatives with distinct weights which create distortions in the ambient space. Anisotropic rescaling comes with the notion of asymptotically stable domains. We prove two results, one of geometric nature, the other one of analytic nature, which both guarantee that a given domain is asymptotically stable. We also discuss specific examples.

1. INTRODUCTION

Anisotropic operators appear in several places in the literature. Recent references can be found in physics [9–11, 17, 18], in biology [6, 7], and in image processing (see, for instance, the monograph by Weickert [34]). By definition, anisotropic operators involve directional derivatives with distinct weights. A model of such operators is the anisotropic Laplace operator. In dimension $n \geq 2$, given $\overrightarrow{p} = (p_1, \dots, p_n)$ with $p_i > 1$ for $i = 1, \dots, n$, the anisotropic Laplace operator $\Delta_{\overrightarrow{p}}$ is defined by

$$\Delta_{\overrightarrow{p}} u = \sum_{i=1}^{n} \frac{\partial}{\partial x_i} \nabla_{x_i}^{p_i} u \,,$$

where $\nabla_{x_i}^{p_i} u = |\partial u/\partial x_i|^{p_i - 2} \, \partial u/\partial x_i$. We let p be an exponent greater than p_i for $i = 1, \dots, n$, and we introduce a natural notion of nonlinear anisotropic equations associated with $\Delta_{\overrightarrow{p}}$ and p. On a domain Ω of the Euclidean space $\mathbb{R}^n$, taking zero Dirichlet boundary condition, such equations are written as

$$\begin{cases} -\Delta_{\overrightarrow{p}} u = f(\cdot, u) & \text{in } \Omega \,, \\ u = 0 & \text{on } \partial\Omega \,, \end{cases} \tag{1.1}$$

where $f(\cdot, u)$ stands for an arbitrary nonlinearity satisfying $f(\cdot, u) \sim \lambda |u|^{p-2} u$ as $|u| \to +\infty$ for some positive real number λ. Anisotropic equations like (1.1) have been investigated by Antontsev–Shmarev [2–4], Fragalà–Gazzola–Kawohl [15], Fragalà–Gazzola–Lieberman [16], El Hamidi–Rakotoson [12, 13], El Hamidi–Vétois [14], Lieberman [21, 22], Mihăilescu–Pucci–Rădulescu [24, 25], and Vétois [31–33]. Time evolution versions of these equations appear in several branches of applied sciences. They emerge, for instance, from the mathematical description of the dynamics of fluids in anisotropic media when the conductivities of the media are different in different directions. We refer to the extensive books by Antontsev–Díaz–Shmarev [1] and Bear [5] for discussions in this direction. They also appear in biology as a model for the propagation of epidemic diseases in heterogeneous domains. We refer to Bendahmane–Karlsen [6] and Bendahmane–Langlais–Saad [7] for the mathematical description of this model. Anisotropic Sobolev spaces in connection with (1.1) can be defined.

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Possible references on the theory of such anisotropic Sobolev spaces are Besov [8], Kruzhkov–Kolodii [19], Kruzhkov–Korolev [20], Lu [23], Nikol′skiĭ [26], Rákosník [27,28], and Troisi [30]. Note that in our case, because of the nature of the questions we investigate, (1.1) can be thought as being subcritical, critical, or even supercritical with respect to Sobolev embeddings.

Together with the nonlinear equation (1.1) comes a rescaling invariance rule. For any $a = (a_1, \dots, a_n)$ in $\mathbb{R}^n$ and any $\mu > 0$, we define the affine transformation $\tau_{\mu,a}^{\overrightarrow{p}} : \mathbb{R}^n \to \mathbb{R}^n$ by

$$\tau_{\mu,a}^{\overrightarrow{p}}(x_1, \dots, x_n) = \left(\mu^{\frac{p_1-p}{p_1}}(x_1 - a_1), \dots, \mu^{\frac{p_n-p}{p_n}}(x_n - a_n)\right). \tag{1.2}$$

Then, as is easily checked, (1.2) provides a general rescaling invariance rule associated with equation (1.1). In particular, u solves (1.1) in Ω if and only if $\mu u \circ (\tau_{\mu,a}^{\overrightarrow{p}})^{-1}$ solves (1.1) in $\tau_{\mu,a}^{\overrightarrow{p}}(\Omega)$ when $f(\cdot, u) = |u|^{p-2} u$, where

$$\left(\tau_{\mu,a}^{\overrightarrow{p}}\right)^{-1}(x_1, \dots, x_n) = \left(a_1 + \mu^{\frac{p-p_1}{p_1}} x_1, \dots, a_n + \mu^{\frac{p-p_n}{p_n}} x_n\right).$$

The affine transformation (1.2) clearly distorts the ambient space as $\mu \to 0$ when the weights are different in different directions, namely when the p_i's are not all equal. Domains may become quite odd under its effect (see Figure 1 below), and analysis on the resulting limit sets may become impossible. In contrast, in the isotropic case $p_i = p_j$ for $i, j = 1, \dots, n$, when starting from a smooth bounded domain, the resulting limit sets are either the whole space $\mathbb{R}^n$ or halfspaces which, of course, have nothing odd. An important notion associated with the distortion in (1.2) is that of asymptotically $\overrightarrow{p}$-stable domains. For instance, see El Hamidi–Vétois [14], this notion turns out to be fundamentally associated with the question of proving bubble tree decompositions for equations like (1.1). Asymptotically $\overrightarrow{p}$-stable domains are domains which, in the limit, after blow-up, still satisfy the segment property. The limit domain may be odd (see Figure 2 below) but, at least, it preserves extension properties of Sobolev spaces. A domain U is said to satisfy the segment property if for any point a on ∂U, there exist a neighborhood X_a of a and a nonzero vector σ_a such that there holds $X_a \cap \overline{U} + t\sigma_a \subset U$ for all t in $(0,1)$. By convention, the empty set satisfies the segment property. The precise definition of an asymptotically $\overrightarrow{p}$-stable domain is as follows.

Definition 1.1. *An open subset Ω of $\mathbb{R}^n$ is said to be asymptotically $\overrightarrow{p}$-stable if for any sequence $(\mu_\alpha)_\alpha$ of positive real numbers converging to 0 and for any sequence $(x_\alpha)_\alpha$ in $\mathbb{R}^n$, the sets $\Omega_\alpha = \tau_{\mu_\alpha,x_\alpha}^{\overrightarrow{p}}(\Omega)$, where $\tau_{\mu_\alpha,x_\alpha}^{\overrightarrow{p}}$ is as in (1.2), converge, up to a subsequence, to an open subset U of $\mathbb{R}^n$ satisfying the segment property as $\alpha \to +\infty$ in the sense that the two following properties hold true:*

(i) *any compact subset of U is included in Ω_α for α large,*
(ii) *for any compact subset K of $\mathbb{R}^n$, there holds $|K \cap \Omega_\alpha \backslash U| \to 0$ as $\alpha \to +\infty$.*

Limits in the sense of (i)–(ii) are unique up to sets of measure zero. Uniqueness, without subtracting sets of measure zero, is recovered when requiring in addition that the limit domain should satisfy the segment property. Important questions which come with this notion of asymptotic $\overrightarrow{p}$-stability are whether or not we can give geometric conditions on a domain which ensure its asymptotic $\overrightarrow{p}$-stability, and whether or not we can give regularity conditions for a domain to be asymptotically $\overrightarrow{p}$-stable. An important related question (see, for instance, the analysis in El Hamidi–Vétois [14]) is whether or not we can characterize the limit sets we obtain after blow-up. We answer these questions by proving that convex domains are always asymptotically $\overrightarrow{p}$-stable and that $\overrightarrow{p}$-Lipschitz domains, in a sense to be defined, are also always asymptotically $\overrightarrow{p}$-stable. In the first case, we get a purely geometric condition for asymptotic $\overrightarrow{p}$-stability. In the second case, we get an analytic regularity condition involving only the boundary of the domain. In both situations, we also get informations on the limit sets. We illustrate the sharpness of our results by discussing the case of ellipsoidal disks and annuli. Our main result, as stated in Theorems 2.1 and 4.3 below, is as follows. The notion of $\overrightarrow{p}$-Lipschitz regularity is defined in Section 4. When no anisotropy is involved, namely when $p_i = p_j$ for $i, j = 1, \dots, n$, a domain is $\overrightarrow{p}$-Lipschitz if and only if it is Lipschitz, namely if its boundary is locally the graph of a Lipschitz continuous function.

Theorem 1.2. *Any open, convex subset of $\mathbb{R}^n$ is asymptotically $\overrightarrow{p}$-stable, and the limit domains U in Definition 1.1 can be chosen to be convex. Any $\overrightarrow{p}$-Lipschitz domain is asymptotically $\overrightarrow{p}$-stable, and we can choose the limit domains U in Definition 1.1 to be either the empty set, the whole space $\mathbb{R}^n$, or delimited by the graph of a locally Lipschitz continuous function.*

We prove Theorem 1.2 in Sections 2 and 4 below, and we discuss the sharp examples of ellipsoidal annuli and disks in Section 3 and at the end of Section 4.

Figure 1 below illustrates what can go wrong with a domain which is not asymptotically $\overrightarrow{p}$-stable. The domain, even though regular at the origin, gets torn through the transformations $\tau^{\overrightarrow{p}}_{\mu_\alpha, x_\alpha}$ as $\alpha \to +\infty$. Here, $\mu_\alpha \to 0$ as $\alpha \to +\infty$ and $x_\alpha = (0, 0, 0)$ for all α. The domain converges to $(\mathbb{R}^+ \times \mathbb{R}^- \times \mathbb{R}) \cup (\mathbb{R}^- \times \mathbb{R}^+ \times \mathbb{R})$ in the sense of Definition 1.1. The split limit set does not satisfy the segment property.

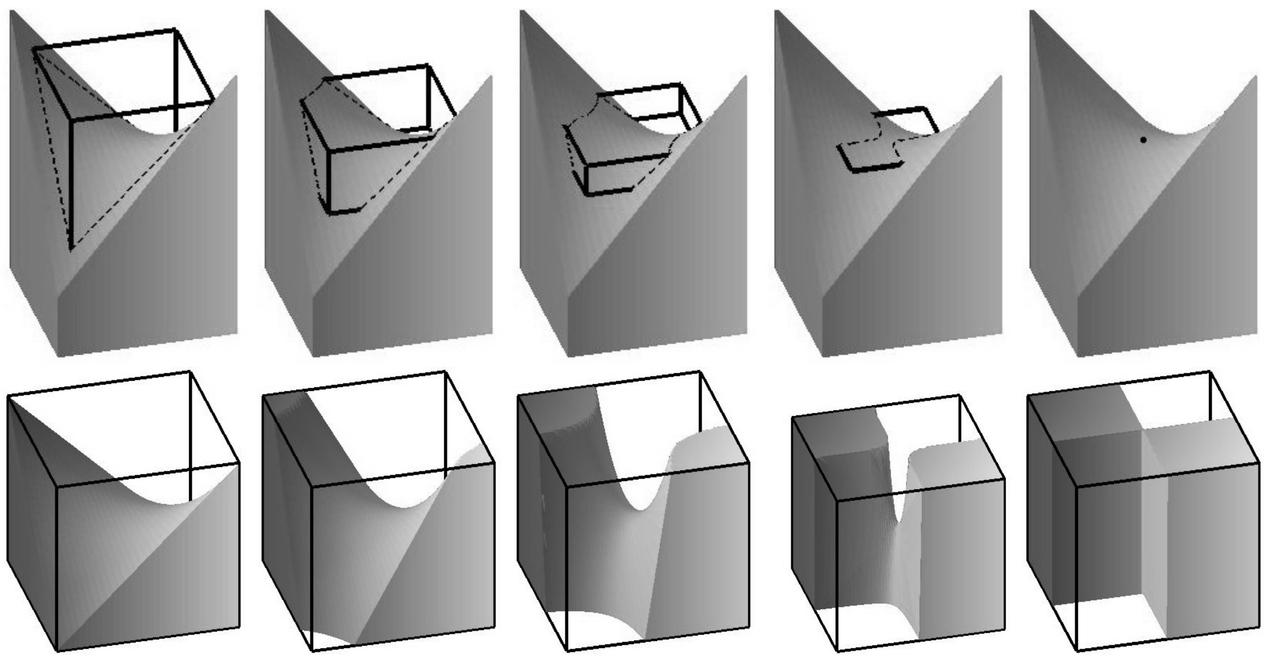

FIGURE 1. Rescaling of $\{x_3 + x_1 x_2 < 0\}$($n = 3, p = 11, p_1 = p_2 = 10, p_3 = 1.1$). The first line describes the scale in the rescaling. The second line describes the deformation of the domain.

2. Convex domains

We prove here that convex domains, in the classical sense, are asymptotically $\overrightarrow{p}$-stable. Convexity provides a simple geometric criterium which guarantees asymptotic $\overrightarrow{p}$-stability.

Theorem 2.1. *Any open, convex subset of $\mathbb{R}^n$ is asymptotically $\overrightarrow{p}$-stable. Furthermore, the limit domains U in Definition 1.1 can be chosen to be convex.*

Proof. We let Ω be an open, convex subset of $\mathbb{R}^n$, $(\mu_\alpha)_\alpha$ be a sequence of positive real numbers converging to 0, $(x_\alpha)_\alpha$ be a sequence in $\mathbb{R}^n$, and $\Omega_\alpha = \tau^{\overrightarrow{p}}_{\mu_\alpha,x_\alpha}(\Omega)$ for all α. Since the transformations $\tau^{\overrightarrow{p}}_{\mu_\alpha,x_\alpha}$ are affine, the domains Ω_α remain convex. Passing if necessary to a subsequence, we may assume that there exist positive real numbers C_0 and R_0 such that the open ball $\mathcal{B}_0$ of center 0 and radius R_0 satisfies $|\mathcal{B}_0 \cap \Omega_\alpha| \geq C_0$ for all α. Indeed, if not the case, one can easily get that the domains Ω_α converge to the empty set as $\alpha \to +\infty$ in the sense of Definition 1.1. By Steinhagen's theorem [29], any bounded, convex subset C of $\mathbb{R}^n$ lies between a pair of parallel hyperplanes at distance $A_n R_C$ apart from each other, where A_n is a constant independent of C, and where R_C stands for the upper bound for the radii of balls included in C. Since there holds $|\mathcal{B}_0 \cap \Omega_\alpha| \geq C_0$ for all α, it follows that, up to a subsequence, there exists a sequence of balls $(\mathcal{B}'_\alpha)_\alpha$ of the same radius such that for any α, there holds $\mathcal{B}'_\alpha \subset \mathcal{B}_0 \cap \Omega_\alpha$. Passing if necessary again to a subsequence, we may assume that there exists an open ball $\mathcal{B}'$ of center x' and radius R' such that for any α, there holds $\mathcal{B}' \subset \mathcal{B}'_\alpha \subset \Omega_\alpha$. For any point a in $\mathbb{R}^n$, we let ξ_a be a Cartesian coordinate transformation satisfying $\xi_a(0) = a$ and $\xi_a(0, \dots, 0, |x' - a|) = x'$. We also let $\mathcal{B}'_0$ be the open $(n-1)$-ball of center 0 and radius $R'/2$. For any α, since the domain Ω_α is convex and since there holds $\mathcal{B}' \subset \Omega_\alpha$, we easily get that either the set $X_a = \xi_a(\mathcal{B}'_0 \times (-\infty, |x' - a|))$ is included in Ω_α, or there exists a convex function φ^α_a on $\overline{\mathcal{B}'_0}$ such that there holds

$$\overline{X_a} \cap \Omega_\alpha = \xi_a\left(\left\{(x_1, \dots, x_n) \in \overline{X_a}\,; \quad x_n > \varphi^\alpha_a(x_1, \dots, x_{n-1})\right\}\right).$$

In this last case, we can prove that the function φ^α_a is Lipschitz continuous with Lipschitz constant equal to $2\left(|x' - a| - \inf \varphi^\alpha_a\right)/R'$. By Arzela-Ascoli theorem, it follows that, up to a subsequence, one of the three following situations occur, either $X_a \subset \Omega_\alpha$ for all α, or $\inf \varphi^\alpha_a \to -\infty$ as $\alpha \to +\infty$, or the sequence $(\varphi^\alpha_a)_\alpha$ converges uniformly to a function φ_a. Iterating the above construction and by a diagonal extraction argument, working with balls $B_0(R_\gamma)$ and letting $R_\gamma \to +\infty$, passing if necessary to a subsequence, we then get that there exist three sequences $(a_\gamma)_\gamma$, $(b_\gamma)_\gamma$, and $(b'_\gamma)_\gamma$ of points in $\mathbb{R}^n$ such that

$$\mathbb{R}^n = \bigcup_{\gamma=0}^{+\infty} \left(X_{a_\gamma} \cup X_{b_\gamma} \cup X_{b'_\gamma}\right), \tag{2.1}$$

and such that for any γ, there hold

$$\begin{aligned} X_{a_\gamma} \subset \Omega_\alpha\,, \quad X_{b_\gamma} \cap \Omega_\alpha &= \xi_{b_\gamma}(\{\,(x_1, \dots, x_n) \in X_{b_\gamma}\,; \quad x_n > \varphi^\alpha_{b_\gamma}(x_1, \dots, x_{n-1})\,\}), \\ \text{and} \quad X_{b'_\gamma} \cap \Omega_\alpha &= \xi_{b'_\gamma}(\{\,(x_1, \dots, x_n) \in X_{b'_\gamma}\,; \quad x_n > \varphi^\alpha_{b'_\gamma}(x_1, \dots, x_{n-1})\,\}) \end{aligned} \tag{2.2}$$

for α large and for some convex functions $\varphi^\alpha_{b_\gamma}$ on $\mathcal{B}'_0$ satisfying $\inf \varphi^\alpha_{b_\gamma} \to -\infty$ as $\alpha \to +\infty$, and some convex functions $\varphi^\alpha_{b'_\gamma}$ on $\mathcal{B}'_0$ converging uniformly to a function φ_a. We let U be the lower limit of the domains Ω_α as $\alpha \to +\infty$, namely

$$U = \bigcup_{\alpha_0=0}^{+\infty} \operatorname{int}\left(\bigcap_{\alpha=\alpha_0}^{+\infty} \Omega_\alpha\right),$$

where $\operatorname{int}(E)$ is the interior of a set E. In particular, U is convex. By (2.2), for any γ, we get

$$\begin{aligned} &X_{a_\gamma} \subset U\,, \quad X_{b_\gamma} \subset U\,, \quad \text{and} \\ &X_{b'_\gamma} \cap U = \xi_{b'_\gamma}(\{\,(x_1, \dots, x_n) \in X_{b'_\gamma}\,; \quad x_n > \varphi_{b'_\gamma}(x_1, \dots, x_{n-1})\,\}). \end{aligned} \tag{2.3}$$

If $K \subset U$ is compact, it is easily checked that there holds $K \subset \Omega_\alpha$ for α large. Now, we let K

be a compact subset of $\mathbb{R}^n$, and I, J, and J' be three finite index sets such that

$$K \subset \left(\bigcup_{\gamma\in I} X_{a_\gamma}\right) \cup \left(\bigcup_{\gamma\in J} X_{b_\gamma}\right) \cup \left(\bigcup_{\gamma\in J'} X_{b'_\gamma}\right). \tag{2.4}$$

By (2.1)–(2.4), we get

$$|K \cap \Omega_\alpha \backslash U| \le \sum_{\gamma\in J'} \left|X_{b'_\gamma} \cap \Omega_\alpha \backslash U\right| \longrightarrow 0$$

as $\alpha \to +\infty$. In particular, we have proved that Ω_α converges to the open set U as $\alpha \to +\infty$ in the sense of Definition 1.1. This ends the proof of Theorem 2.1. ☐

By Theorem 2.1, any convex domain is asymptotically $\overrightarrow{p}$-stable. In the isotropic regime, when starting from a smooth bounded domain, the limit sets would be either the empty set, the whole space $\mathbb{R}^n$, or halfspaces. When anisotropy is involved, several different types of limit sets can be obtained. Figure 2 below describes the rescaled evolution in the very simple situation of a disk when there is strong anisotropy (the p_i's are far from each other). The centers of the rescalings in Figure 2 belong to the interior of the domain and converge to the boundary. The disk converges to a strip, which, needless to mention, is geometrically quite far from what we would get when no anisotropy (or even small anisotropy) is involved.

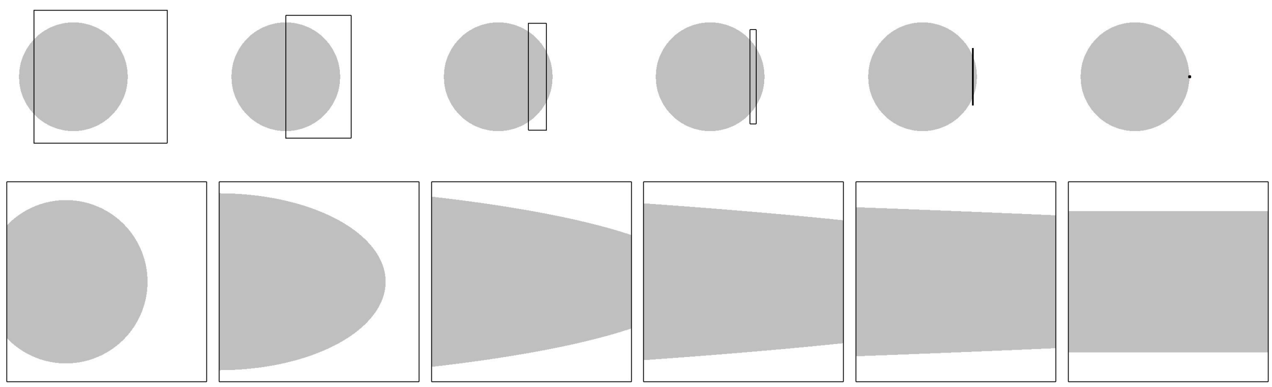

FIGURE 2. Rescaling of a disk with strong anisotropy ($n = 2$, $p = 5$, $p_1 = 1.1$, $p_2 = 2$). The first line describes the scale in the rescaling. The second line describes the deformation of the domain.

3. Ellipsoidal annuli

In this section, we discuss asymptotic $\overrightarrow{p}$-stability in the particular case of ellipsoidal annuli. Even though very regular, such domains are not asymptotically $\overrightarrow{p}$-stable when strong anisotropy (in a quantified sense) is involved. On the other hand, ellipsoidal disks are always asymptotically $\overrightarrow{p}$-stable by Theorem 2.1. Given $\overrightarrow{a} = (a_1, \dots, a_n)$ in $(\mathbb{R}_+^*)^n$, we let here $\mathcal{E}\left(\overrightarrow{a}\right)$ be the ellipsoidal disk consisting of the points $(y_1, \dots, y_n)$ in $\mathbb{R}^n$ such that $\sum_{i=1}^n a_i y_i^2 < 1$.

Proposition 3.1. *Given $\overrightarrow{a} = (a_1, \dots, a_n)$ and $\overrightarrow{b} = (b_1, \dots, b_n)$ in $(\mathbb{R}_+^*)^n$ satisfying $b_i < a_i$ for $i = 1, \dots, n$, the ellipsoidal annulus $\mathcal{E}(\overrightarrow{b})\backslash\overline{\mathcal{E}(\overrightarrow{a})}$ is asymptotically $\overrightarrow{p}$-stable if and only if*

$$\frac{p_+}{p_-} + \frac{p_+}{p} \le 2\,, \tag{3.1}$$

where $p_- = \min(p_1, \dots, p_n)$ and $p_+ = \max(p_1, \dots, p_n)$.

As a remark, (3.1) is automatically satisfied in the isotropic case.

Proof. First, we assume that (3.1) holds true. We let $(\mu_\alpha)_\alpha$ be a sequence of positive real numbers converging to 0, $(x_\alpha)_\alpha$ be a sequence in $\mathbb{R}^n$. For $\overrightarrow{c} = \overrightarrow{a}$ or $\overrightarrow{c} = \overrightarrow{b}$, we let $\varphi_{\overrightarrow{c}} : \mathbb{R}^n \to \mathbb{R}$ be the function defined by

$$\varphi_{\overrightarrow{c}}(y_1, \dots, y_n) = \sum_{i=1}^n c_i y_i^2 - 1 .$$

For any α and any point $y = (y_1, \dots, y_n)$ in $\mathbb{R}^n$, we get

$$\varphi_{\overrightarrow{c}} \circ \left(\tau^{\overrightarrow{p}}_{\mu_\alpha, x_\alpha}\right)^{-1}(y) = \sum_{i=1}^n c_i \mu_\alpha^{2\frac{p-p_i}{p_i}} y_i^2 + 2\sum_{i=1}^n c_i x_\alpha^i \mu_\alpha^{\frac{p-p_i}{p_i}} y_i + \sum_{i=1}^n c_i \left(x_\alpha^i\right)^2 - 1 , \tag{3.2}$$

where $x_\alpha = (x_\alpha^1, \dots, x_\alpha^n)$. Passing if necessary to a subsequence, we may assume that there exist $l_{\overrightarrow{a}}$ and $l_{\overrightarrow{b}}$ in $[0, +\infty]$ such that $\sum_{i=1}^n a_i (x_\alpha^i)^2 \to l_{\overrightarrow{a}}$ and $\sum_{i=1}^n a_i (x_\alpha^i)^2 \to l_{\overrightarrow{b}}$ as $\alpha \to +\infty$. One can easily check that the sets

$$\mathcal{F}_\alpha = \tau^{\overrightarrow{p}}_{\mu_\alpha, x_\alpha}(\mathcal{E}(\overrightarrow{b}) \backslash \overline{\mathcal{E}(\overrightarrow{a})})$$

converge, in the sense of Definition 1.1, as $\alpha \to +\infty$, to $\mathbb{R}^n$ when $l_{\overrightarrow{a}} > 1$ and $l_{\overrightarrow{b}} < 1$, and to $\emptyset$ when $l_{\overrightarrow{a}} < 1$ or $l_{\overrightarrow{b}} > 1$. In case $l_{\overrightarrow{b}} = 1$, up to a subsequence, it follows from (3.2) that there exist a sequence $(\nu_\alpha)_\alpha$ of positive real numbers converging to 0, some real numbers $d_i \geq 0$ and c_i, $i = 1, \dots, n$, not all zero, such that

$$\varphi_{\overrightarrow{b}} \circ \left(\tau^{\overrightarrow{p}}_{\mu_\alpha, x_\alpha}\right)^{-1}(y) = \left(\sum_{i=1}^n d_i y_i^2 + \sum_{i=1}^n c_i y_i + c_0\right) \nu_\alpha + \mathrm{o}_\alpha(\nu_\alpha)$$

as $\alpha \to +\infty$, uniformly in any compact subset of $\mathbb{R}^n$. One can then easily check that the sets $\mathcal{F}_\alpha$ converge to the domain

$$U = \left\{(y_1, \dots, y_n) \in \mathbb{R}^n ; \quad \sum_{i=1}^n d_i y_i^2 + \sum_{i=1}^n c_i y_i + c_0 < 0\right\}$$

as $\alpha \to +\infty$ in the sense of Definition 1.1. Clearly, U satisfies the segment property when not empty. Now we consider the remaining case $l_{\overrightarrow{a}} = 1$. We let i_0 be such that $x_\alpha^{i_0}$ converges to a positive real number as $\alpha \to +\infty$. By (3.1), we can write

$$\mu_\alpha^{2\frac{p-p_i}{p_i}} = \mathrm{O}_\alpha\left(x_\alpha^{i_0} \mu_\alpha^{\frac{p-p_{i_0}}{p_{i_0}}}\right) \tag{3.3}$$

as $\alpha \to +\infty$ for $i = 1, \dots, n$. As above, passing if necessary to a subsequence, we may assume that there exist a sequence $(\nu_\alpha)_\alpha$ of positive real numbers converging to 0, some real numbers $d_i \geq 0$ and c_i, $i = 1, \dots, n$, not all zero, such that

$$\varphi_{\overrightarrow{a}} \circ \left(\tau^{\overrightarrow{p}}_{\mu_\alpha, x_\alpha}\right)^{-1}(y) = \left(\sum_{i=1}^n d_i y_i^2 + \sum_{i=1}^n c_i y_i + c_0\right) \nu_\alpha + \mathrm{o}_\alpha(\nu_\alpha)$$

By (3.3), if $c_{i_0} = 0$, then $d_i = 0$ for all i. It easily follows that $\mathcal{F}_\alpha$ converges to

$$U = \left\{(y_1, \dots, y_n) \in \mathbb{R}^n ; \quad \sum_{i=1}^n d_i y_i^2 + \sum_{i=1}^n c_i y_i + c_0 > 0\right\} \tag{3.4}$$

as $\alpha \to +\infty$ in the sense of Definition 1.1. The domain U is either empty, or it satisfies the segment property. We have proved that if (3.1) holds true, then the ellipsoidal annulus

$\mathcal{E}(\overrightarrow{b})\backslash\overline{\mathcal{E}(\overrightarrow{a})}$ is asymptotically $\overrightarrow{p}$-stable. In order to get the converse, we let $(\mu_\alpha)_\alpha$ be a sequence of positive real numbers converging to 0, i_0 be an index such that $p_{i_0} = p_-$, and $x_0 = (x_0^1, \ldots, x_0^n)$ be the point given by

$$x_0^i = \begin{cases} \dfrac{1}{\sqrt{a_{i_0}}} & \text{if } i = i_0 \\ 0 & \text{otherwise.} \end{cases}$$

We define

$$I_0 = \Big\{ i \in \{1, \ldots, n\}\,;\quad \frac{p_i}{p_-} + \frac{p_i}{p} = 2 \Big\}$$

and

$$I_1 = \Big\{ i \in \{1, \ldots, n\}\,;\quad \frac{p_i}{p_-} + \frac{p_i}{p} > 2 \Big\}.$$

We let $\hat{U} = \hat{U}_1 \cup \hat{U}_2$, where

$$\hat{U}_1 = \Big\{ (y_1, \ldots, y_n) \in \mathbb{R}^n\,;\quad \sum_{i \in I_1} a_i y_i^2 > 0 \Big\}$$

and

$$\hat{U}_2 = \Big\{ (y_1, \ldots, y_n) \in \mathbb{R}^n\,;\quad \sum_{i \in I_1} a_i y_i^2 = 0 \quad \text{and} \quad \sum_{i \in I_0} a_i y_i^2 + 2\sqrt{a_{i_0}} y_{i_0} > 0 \Big\}.$$

We let also

$$\widetilde{U} = \Big\{ (y_1, \ldots, y_n) \in \mathbb{R}^n\,;\quad \sum_{i \in I_1} a_i y_i^2 = 0 \quad \text{and} \quad \sum_{i \in I_0} a_i y_i^2 + 2\sqrt{a_{i_0}} y_{i_0} = 0 \Big\}.$$

Clearly, the sets $\mathcal{F}_\alpha^0 = \tau_{\mu_\alpha, x_0}^{\overrightarrow{p}}(\mathcal{E}(\overrightarrow{b})\backslash\overline{\mathcal{E}(\overrightarrow{a})})$ converges to $\hat{U}_1$ as $\alpha \to +\infty$ in the sense of Definition 1.1. Now we let U be an open subset of $\mathbb{R}^n$ which is the limit of the sets $\mathcal{F}_\alpha^0$ as $\alpha \to +\infty$ in the sense of Definition 1.1. By (3.2), we get that U is included in $\hat{U} \cup \widetilde{U}$ and thus in $\hat{U}$ since the interior of the set $\hat{U} \cup \widetilde{U}$ is precisely $\hat{U}$. It follows that $U = \hat{U}\backslash E$ for some subset E of $\hat{U}$ satisfying $|E| = 0$. As is easily checked, such U's never satisfy the segment property when the set I_1 is not empty, namely when (3.1) does not hold true. This ends the proof of Proposition 3.1. □

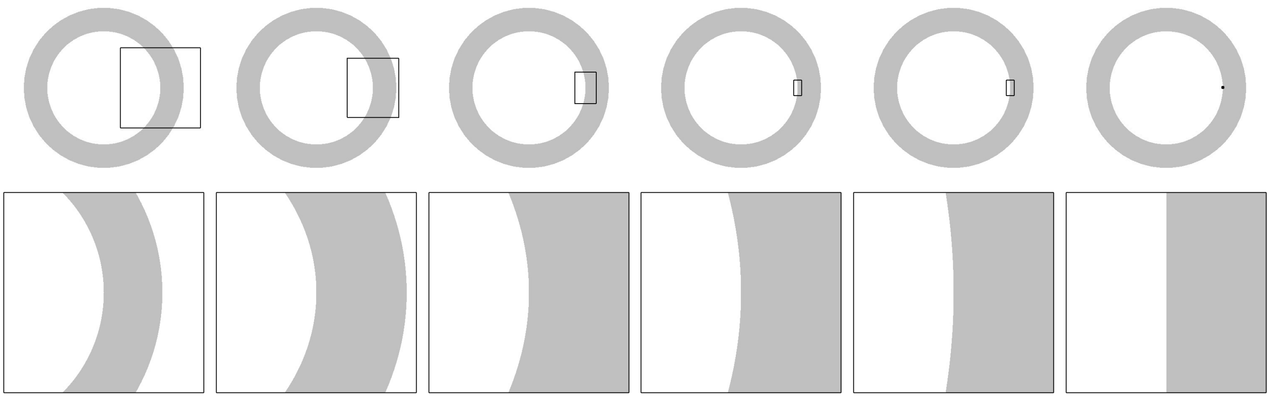

FIGURE 3. Rescaling of an annulus with small anisotropy ($n = 2$, $p = 12$, $p_1 = 1.5$, $p_2 = 2$). The first line describes the scale in the rescaling. The second line describes the deformation of the domain.

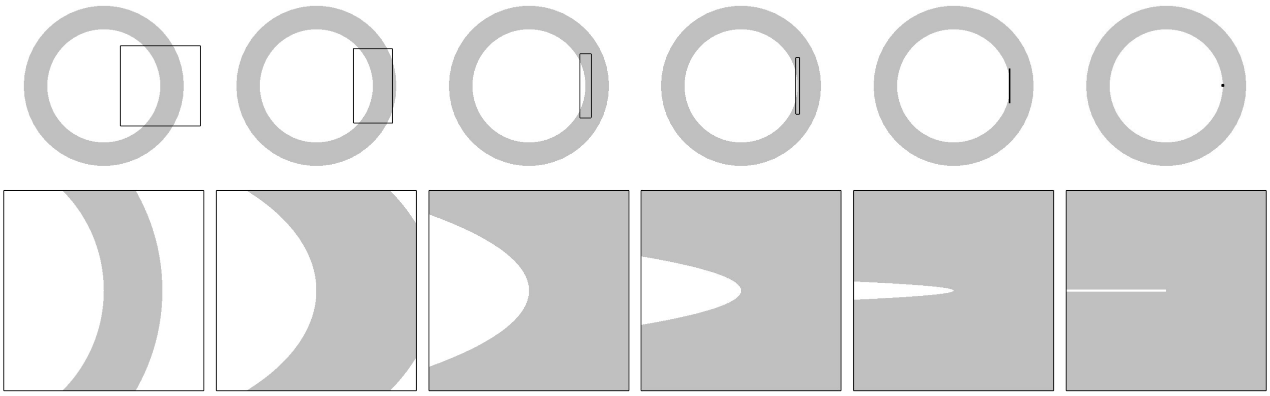

Figure 4. Rescaling of an annulus with strong anisotropy ($n = 2$, $p = 5$, $p_1 = 1.1$, $p_2 = 2$). The first line describes the scale in the rescaling. The second line describes the deformation of the domain.

Figures 3 and 4 above describe two opposite situations in the case of an annulus ($a_i = a_j$ and $b_i = b_j$ for $i, j = 1, \ldots, n$). In Figure 3, there is small anisotropy (p_+ is close to p_-) and the domain behaves in the same way as in the isotropic case. The limit domain is a halfspace. In Figure 4, there is strong anisotropy (p_+ is far from p_-). The domain bends on itself and converges to the whole plane minus a half-line, a domain which does not satisfy the segment property.

Ellipsoidal disks are always asymptotically $\overrightarrow{p}$-stable by Theorem 2.1. The interior boundary in ellipsoidal annuli is the boundary which creates problems.

4. Anisotropic Lipschitz regularity

In this section, we define the class of anisotropic $\overrightarrow{p}$-Lipschitz domains and prove first that anisotropic $\overrightarrow{p}$-Lipschitz domains are exactly Lipschitz domains in the isotropic regime, and then that $\overrightarrow{p}$-Lipschitz domains are always asymptotically $\overrightarrow{p}$-stable. A main feature of the $\overrightarrow{p}$-Lipschitz regularity we define is that it involves only the boundary of the domain. First we fix some notations. For any positive real number μ and any point $a = (a_1, \ldots, a_n)$ in $\mathbb{R}^n$, we define

$$\mathcal{P}_a^{\overrightarrow{p}}(\mu) = \Big\{(x_1, \ldots, x_n) \in \mathbb{R}^n\,; \quad |x_i - a_i| < \frac{1}{2}\mu^{\frac{p-p_i}{p_i}} \quad \forall i \in \{1, \ldots, n\}\Big\}. \tag{4.1}$$

In particular, $\mathcal{P}_a^{\overrightarrow{p}}(1)$ stands for the cube centered at the point a with an edge length of 1. As an easy remark on such domains we get that for any point a in $\mathbb{R}^n$, and for any positive real number μ, there holds $\tau_{\mu,a}^{\overrightarrow{p}}(\mathcal{P}_a^{\overrightarrow{p}}(\mu)) = \mathcal{P}_0^{\overrightarrow{p}}(1)$. We now define $\overrightarrow{p}$-Lipschitz regularity.

Definition 4.1. *An open subset Ω of $\mathbb{R}^n$ is said to be a $\overrightarrow{p}$-Lipschitz domain if there holds $\partial\overline{\Omega} = \partial\Omega$ and if for any sequence $(\mu_\alpha)_\alpha$ of positive real numbers converging to 0, and for any sequence $(x_\alpha)_\alpha$ on $\partial\Omega$, there exists a unit vector σ such that, up to a subsequence, there holds*

$$\liminf_{\alpha\to+\infty} \inf_{y,z\in\mathcal{P}_{x_\alpha}^{\overrightarrow{p}}(R\mu_\alpha)\cap\partial\Omega} \left| \frac{\tau_{\mu_\alpha,y}^{\overrightarrow{p}}(z)}{\left|\tau_{\mu_\alpha,y}^{\overrightarrow{p}}(z)\right|} - \sigma \right| > 0 \tag{4.2}$$

for all positive real numbers R, where $\mathcal{P}_{x_\alpha}^{\overrightarrow{p}}(R\mu_\alpha)$ is as in (4.1).

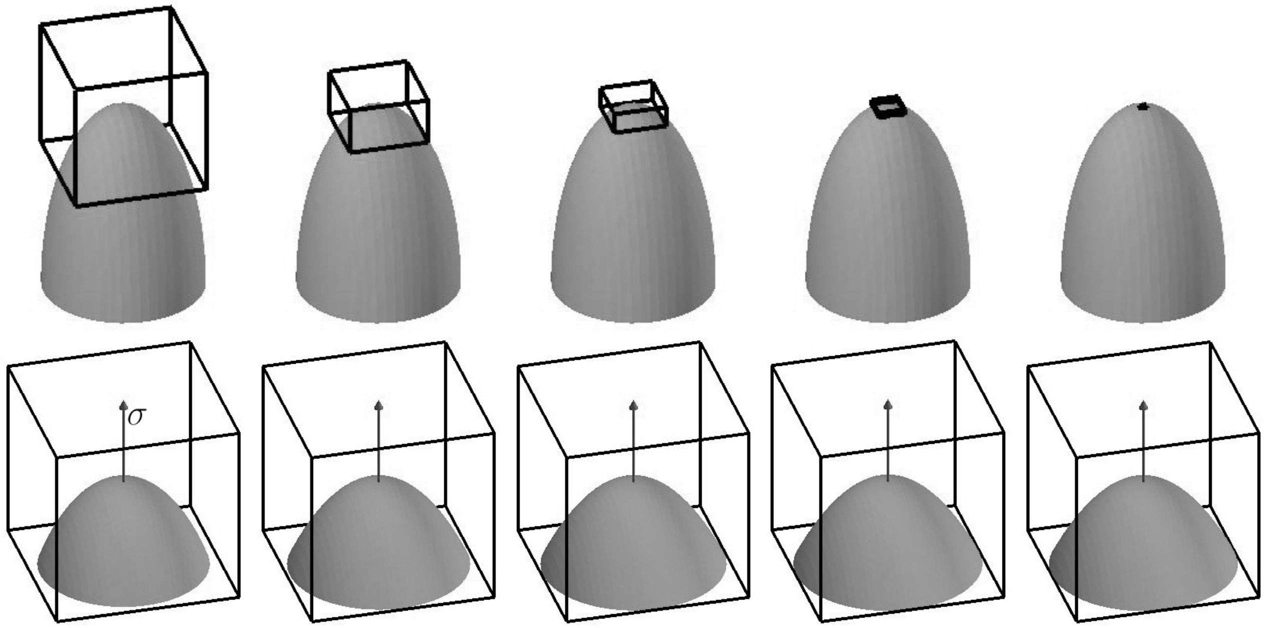

FIGURE 5. Rescaling of the domain $\{6x_1^2 + 6x_2^2 + x_3^2 < 1\}$ with constant centers ($n = 3$, $p = 4$, $p_1 = p_2 = 2$, $p_3 = 4/3$). The first line describes the scale in the rescaling. The second line describes the deformation of the domain.

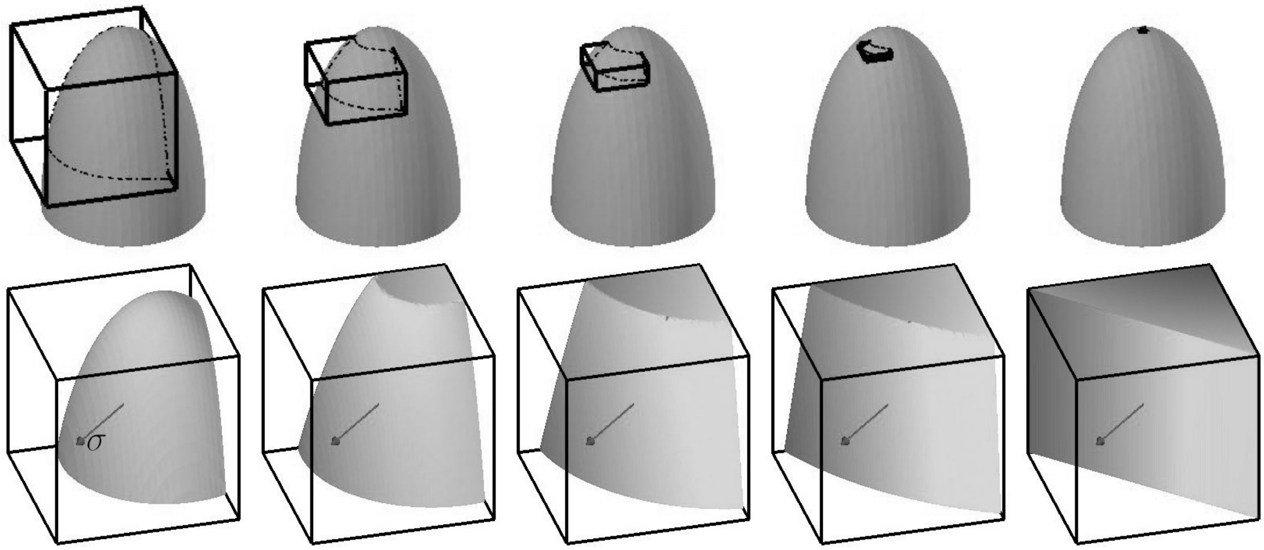

FIGURE 6. Rescaling of the domain $\{6x_1^2 + 6x_2^2 + x_3^2 < 1\}$ with moving centers ($n = 3$, $p = 4$, $p_1 = p_2 = 2$, $p_3 = 4/3$). The first line describes the scale in the rescaling. The second line describes the deformation of the domain.

The vector σ in Definition 4.1 depends on the sequences $(x_\alpha)_\alpha$ and $(\mu_\alpha)_\alpha$. We illustrate this dependency in Figures 5 and 6 above in the case of an ellipsoidal disk. In Figure 5, the disk is rescaled with respect to the constant centers $x_\alpha = (0, 0, 1)$, and converges to a domain delimited by a paraboloid. The limit domain satisfies (4.2) with $\sigma = (0, 0, 1)$, and this is the only possible choice for σ. In Figure 6, even though the centers of the rescalings still converge to the point $(0, 0, 1)$, the domain converges to a halfspace, and (4.2) holds true with any unit vector σ not coplanar with the boundary of the limit domain. In particular, for such x_α's and μ_α's, we cannot take $\sigma = (0,0,1)$. The centers $x_\alpha = (x_\alpha^1, x_\alpha^2, x_\alpha^3)$ of the rescalings in Figure 6 belong to the boundary of the ellipsoidal disk, and they are chosen so that there hold $x_\alpha^1 + x_\alpha^2 = 0$ for all α and $\mu_\alpha = \mathrm{o}(x_\alpha^1)$ as $\alpha \to \infty$.

In the classical isotropic regime, an open subset Ω of $\mathbb{R}^n$ is said to be Lipschitz if for any point a on $\partial\Omega$, there exist a Cartesian coordinate system $(\xi_1, \dots, \xi_n)$ of $\mathbb{R}^n$, a Lipschitz continuous function $\varphi_a : \mathbb{R}^{n-1} \to \mathbb{R}$ and an open neighborhood X_a of a such that the set $X_a \cap \Omega$ consists of the points $(\xi_1, \dots, \xi_n)$ in X_a such that there holds $\xi_n < \varphi_a(\xi_1, \dots, \xi_{n-1})$. First, we prove in Proposition 4.2 below that $\overrightarrow{p}$-Lipschitz domains with bounded boundary are precisely Lipschitz domains in the isotropic case $p_i = p_j$ for $i, j = 1, \dots, n$.

Proposition 4.2. *In case there holds $p_i = p_j$ for all $i, j = 1, \dots, n$, any open subset of $\mathbb{R}^n$ with bounded boundary is $\overrightarrow{p}$-Lipschitz if and only if it is Lipschitz.*

Proof. We let Ω be an open subset of $\mathbb{R}^n$ with bounded boundary. We first assume that Ω is Lipschitz. Clearly, there holds $\partial\overline{\Omega} = \partial\Omega$. We let $(\mu_\alpha)_\alpha$ be a sequence of positive real numbers converging to 0 and $(x_\alpha)_\alpha$ be a sequence on $\partial\Omega$. Since $\partial\Omega$ is bounded, passing if necessary to a subsequence, we may assume that $(x_\alpha)_\alpha$ converges to a point a on $\partial\Omega$. We let X_a be an open neighborhood of a, $(\xi_1, \dots, \xi_n)$ be a Cartesian coordinate system of $\mathbb{R}^n$, and $\varphi_a : \mathbb{R}^{n-1} \to \mathbb{R}$ be a Lipschitz continuous function such that the set $X_a \cap \Omega$ consists of the points $(\xi_1, \dots, \xi_n)$ in X_a satisfying $\xi_n < \varphi_a(\xi_1, \dots, \xi_{n-1})$. We then set $\sigma = (0, \dots, 0, 1)$ in the new coordinate system. It easily follows from the Lipschitz continuity of the function φ that there holds

$$\inf_{y,z \in X_a \cap \partial\Omega} \left| \frac{y-z}{|y-z|} - \sigma \right| > 0\,. \tag{4.3}$$

Since for any positive real number R, there holds $\mathcal{P}_{x_\alpha}^{\overrightarrow{p}}(R\mu_\alpha) \subset X_a$ for α large, we then get that (4.2) holds true. In order to get the converse, we now assume that Ω is $\overrightarrow{p}$-Lipschitz. For any point a on $\partial\Omega$, by (4.2) with $x_\alpha = a$ for all α, we get that there exist an open neighborhood X_a of a and a unit vector σ such that (4.3) holds true. Up to a Cartesian change of coordinate system $(\xi_1, \dots, \xi_n)$ of $\mathbb{R}^n$, we may assume that $a = 0$ and $\sigma = (0, \dots, 0, 1)$. Up to a restriction of the set X_a, we may assume moreover that $X_a = [-\delta, \delta]^{n-1} \times [-\varepsilon, \varepsilon]$ for some positive real numbers δ and ε. Plugging $z = 0$ into (4.3), we get that there exists a positive constant C such that there holds $\sum_{i=1}^{n-1} \xi_i^2 > C\xi_n^2$ for all points $(\xi_1, \dots, \xi_n)$ on $X_a \cap \partial\Omega$. Hence, decreasing δ if necessary so that $\delta \le \varepsilon\sqrt{C/(n-1)}$, we get

$$\left([-\delta, \delta]^{n-1} \times \{-\varepsilon, \varepsilon\}\right) \cap \partial\Omega = \emptyset\,. \tag{4.4}$$

Independently, by (4.3), we get that for any ξ in $[-\delta, \delta]^{n-1}$, the segment $\{\xi\} \times [-\varepsilon, \varepsilon]$ cannot intersect $\partial\Omega$ at more than one point. Taking into account that there holds $\partial\overline{\Omega} = \partial\Omega$, it follows that the set $[-\delta, \delta]^{n-1} \times \{-\varepsilon, \varepsilon\}$ cannot be included neither in Ω nor in $\mathbb{R}^n \backslash \overline{\Omega}$. Hence, by (4.4), changing if necessary ξ_n into $-\xi_n$, we may assume that the set $[-\delta, \delta]^{n-1} \times \{-\varepsilon\}$ is included in Ω and that the set $[-\delta, \delta]^{n-1} \times \{\varepsilon\}$ is included in $\mathbb{R}^n \backslash \overline{\Omega}$. We then let $\varphi_a : [-\delta, \delta]^{n-1} \to [-\varepsilon, \varepsilon]$ be such that for any ξ in $[-\delta, \delta]^{n-1}$, $(\xi, \varphi_a(\xi))$ is the intersection point of the segment $\{\xi\} \times [-\varepsilon, \varepsilon]$ with the boundary of the domain Ω. In particular, the set $X_a \cap \Omega$ consists of the points $(\xi_1, \dots, \xi_n)$ in X_a satisfying $\xi_n < \varphi_a(\xi_1, \dots, \xi_{n-1})$. It easily follows from (4.3) that the function φ_a is Lipschitz continuous. Since the above holds true for all points a on $\partial\Omega$, we get that the domain Ω is Lipschitz. This ends the proof of Proposition 4.2. $\square$

Now we prove that, in the general anisotropic case, $\overrightarrow{p}$-Lipschitz domains always are asymptotically $\overrightarrow{p}$-stable.

Theorem 4.3. *Any $\overrightarrow{p}$-Lipschitz domain is asymptotically $\overrightarrow{p}$-stable. Furthermore, we can choose the limit domains U in Definition 1.1 to be either the empty set, the whole space $\mathbb{R}^n$, or delimited by the graph of a locally Lipschitz continuous function.*

Proof. We let Ω be a $\overrightarrow{p}$-Lipschitz domain, $(\mu_\alpha)_\alpha$ be a sequence of positive real numbers converging to 0, $(x_\alpha)_\alpha$ be a sequence in $\mathbb{R}^n$, and $\Omega_\alpha = \tau^{\overrightarrow{p}}_{\mu_\alpha,x_\alpha}(\Omega)$ for all α. In case for any positive real number R, up to a subsequence, the domain $\mathcal{P}^{\overrightarrow{p}}_{x_\alpha}(R\mu_\alpha)$ remains included in $\mathbb{R}^n\backslash\Omega$, one can easily get that the domains Ω_α converge to the empty set as $\alpha \to +\infty$ in the sense of Definition 1.1. Analogously, in case for any positive real number R, up to a subsequence, the domain $\mathcal{P}^{\overrightarrow{p}}_{x_\alpha}(R\mu_\alpha)$ remains included in Ω, we get that the domains Ω_α converge to $\mathbb{R}^n$ as $\alpha \to +\infty$ in the sense of Definition 1.1. Hence, we may assume that there exist a positive real number R_0 and a sequence of points $\widetilde{x}_\alpha$ on $\mathcal{P}^{\overrightarrow{p}}_{x_\alpha}(R_0\mu_\alpha) \cap \partial\Omega$. We then let $(R_\beta)_\beta$ and $(\widetilde{R}_\beta)_\beta$ be two increasing sequences of real numbers converging to $+\infty$, satisfying $R_\beta \geq R_0$ and $\widetilde{R}_\beta > 0$ for all $\beta > 0$, and such that the open ball $\mathcal{B}_\beta$ of center 0 and radius $\widetilde{R}_\beta$ is included in the set $\mathcal{P}^{\overrightarrow{p}}_0(R_\beta)$. Since the domain Ω is $\overrightarrow{p}$-Lipschitz, taking into account that there holds $\mathcal{P}^{\overrightarrow{p}}_{x_\alpha}(R_\beta\mu_\alpha) \subset \mathcal{P}^{\overrightarrow{p}}_{\widetilde{x}_\alpha}(CR_\beta\mu_\alpha)$ for some positive constant C independent of α and β, we get that there exists a unit vector σ such that, up to a subsequence, there holds

$$\liminf_{\alpha\to+\infty} \inf_{y,z\in\mathcal{B}_\beta\cap\partial\Omega_\alpha} \left| \frac{y-z}{|y-z|} - \sigma \right| > 0\,. \tag{4.5}$$

Up to a Cartesian change of coordinate system $(\xi_1,\ldots,\xi_n)$ of $\mathbb{R}^n$, we may take $\sigma = (0,\ldots,0,1)$. For any β, it follows from (4.5) that there exists a positive constant C_β such that there holds $\sum_{i=1}^{n-1}(\xi_i - \xi_i')^2 > C_\beta(\xi_n - \xi_n')^2$ for all points $(\xi_1,\ldots,\xi_n)$ and $(\xi_1',\ldots,\xi_n')$ on $\mathcal{B}_\beta \cap \partial\Omega_\alpha$. We then let δ_β and ε_β be two real numbers in $(0,(\widetilde{R}_{\beta+1} - \widetilde{R}_\beta)/\sqrt{n})$ satisfying $\delta_\beta < \varepsilon_\beta\sqrt{C_{\beta+1}/(n-1)}$. We set $A_0 = A \cap \mathcal{B}_0$ and $A_\beta = A \cap \mathcal{B}_\beta \backslash \mathcal{B}_{\beta-1}$ for all $\beta > 0$, where A is the set of all points a in $\mathbb{R}^n$ such that for any positive real number ε, there exists $\alpha \geq 1/\varepsilon$ such that there holds $d(a,\partial\Omega_\alpha) < \varepsilon$. For any point a in A_β, it follows from our choice of the real numbers δ_β and ε_β that, up to a subsequence, there holds

$$\left(\{a\} + [-\delta_\beta,\delta_\beta]^{n-1} \times \{-\varepsilon_\beta,\varepsilon_\beta\}\right) \cap \partial\Omega_\alpha = \emptyset$$

for all α. We then set $X_a = \{a\} + (-\delta_\beta,\delta_\beta)^{n-1} \times (-\varepsilon_\beta,\varepsilon_\beta)$, and in the same way as in the proof of Proposition 4.2, we get that there exists a sequence of Lipschitz equicontinuous functions

$$\varphi_a^\alpha : \prod_{i=1}^{n-1} [a_i - \delta_\beta, a_i + \delta_\beta] \longrightarrow [a_n - \varepsilon_\beta, a_n + \varepsilon_\beta]\,,$$

where $a = (a_1,\ldots,a_n)$, such that, up to a subsequence, for any α, either $\overline{X_a} \cap \Omega_\alpha$ or $\overline{X_a}\backslash\overline{\Omega_\alpha}$ consists of the points $(\xi_1,\ldots,\xi_n)$ in X_a satisfying $\xi_n < \varphi_a^\alpha(\xi_1,\ldots,\xi_{n-1})$. By Arzela-Ascoli theorem, the sequence $(\varphi_a^\alpha)_\alpha$ converges, up to a subsequence, uniformly to a Lipschitz continuous function φ_a. Since A is covered by the distinct sets A_β, iterating the above construction and using a diagonal extraction argument, up to a subsequence, we then get that there exist two sequences $(a_\gamma)_\gamma$ and $(a_\gamma')_\gamma$ of points in A such that

$$A \subset \bigcup_{\gamma=0}^{+\infty} \left(X_{a_\gamma} \cup X_{a_\gamma'}\right),$$

and such that for any γ, there hold

$$X_{a_\gamma} \cap \Omega_\alpha = \left\{ (\xi_1,\ldots,\xi_n) \in X_{a_\gamma}\,;\quad \xi_n < \varphi_{a_\gamma}^\alpha(\xi_1,\ldots,\xi_{n-1}) \right\} \tag{4.6}$$

and

$$X_{a'_\gamma} \cap \Omega_\alpha = \{ (\xi_1, \dots, \xi_n) \in X_{a'_\gamma}\,; \quad \xi_n > \varphi^\alpha_{a'_\gamma} (\xi_1, \dots, \xi_{n-1}) \}. \tag{4.7}$$

for α large, where the sequences $(\varphi^\alpha_{a_\gamma})_\alpha$ and $(\varphi^\alpha_{a'_\gamma})_\alpha$ converge uniformly for all γ. Since A is closed, we also get that for any point b in $\mathbb{R}^n \backslash A$, there exists an open connected neighborhood X_b of b strictly included in $\mathbb{R}^n \backslash A$, and thus either in Ω_α or in $\mathbb{R}^n \backslash \Omega_\alpha$, up to a subsequence, for all α. By a diagonal extraction argument, it follows that there exist two sequences $(b_\gamma)_\gamma$ and $(b'_\gamma)_\gamma$ of points in $\mathbb{R}^n \backslash A$ such that

$$\mathbb{R}^n = \bigcup_{\gamma=0}^{+\infty} \left(X_{a_\gamma} \cup X_{a'_\gamma} \cup X_{b_\gamma} \cup X_{b'_\gamma} \right), \tag{4.8}$$

and such that for any γ, there hold $X_{b_\gamma} \subset \Omega_\alpha$ and $X_{b'_\gamma} \subset \mathbb{R}^n \backslash \Omega_\alpha$ for α large. We let U be the lower limit of the domains Ω_α as $\alpha \to +\infty$, namely

$$U = \bigcup_{\alpha_0=0}^{+\infty} \operatorname{int} \left(\bigcap_{\alpha=\alpha_0}^{+\infty} \Omega_\alpha \right),$$

where $\operatorname{int}(E)$ stands for the interior of a set E. By (4.6), (4.7), (4.8), and since the sequences $(\varphi^\alpha_{a_\gamma})_\alpha$ and $(\varphi^\alpha_{a'_\gamma})_\alpha$ converge uniformly to functions φ_{a_γ} and $\varphi_{a'_\gamma}$ for all γ, we get $\partial U = A$ and

$$\begin{aligned} & X_{a_\gamma} \cap U = \left\{ (\xi_1, \dots, \xi_n) \in X_{a_\gamma}\,; \quad \xi_n < \varphi_{a_\gamma} (\xi_1, \dots, \xi_{n-1}) \right\}, \\ & X_{a'_\gamma} \cap U = \left\{ (\xi_1, \dots, \xi_n) \in X_{a'_\gamma}\,; \quad \xi_n > \varphi_{a'_\gamma} (\xi_1, \dots, \xi_{n-1}) \right\}, \\ & X_{b_\gamma} \subset U\,, \quad \text{and} \quad X_{b'_\gamma} \subset \mathbb{R}^n \backslash \overline{U}. \end{aligned} \tag{4.9}$$

It follows from (4.8) and (4.9) that the domain U is Lipschitz. In order to prove that the domains Ω_α converge to U as $\alpha \to +\infty$ in the sense of Definition 1.1, we let K be a compact subset of $\mathbb{R}^n$, and I, I', J, J' be four finite index sets such that

$$K \subset \left(\bigcup_{\gamma \in I} X_{a_\gamma} \right) \cup \left(\bigcup_{\gamma \in I'} X_{a'_\gamma} \right) \cup \left(\bigcup_{\gamma \in J} X_{b_\gamma} \right) \cup \left(\bigcup_{\gamma \in J'} X_{b'_\gamma} \right). \tag{4.10}$$

By (4.6)–(4.10), we get

$$|K \cap \Omega_\alpha \backslash U| \le \sum_{i \in I} |X_{a_i} \cap \Omega_\alpha \backslash U| + \sum_{i \in I'} \left| X_{a'_i} \cap \Omega_\alpha \backslash U \right| \longrightarrow 0$$

as $\alpha \to +\infty$. Independently, if $K \subset U$ is compact, then it is easily checked that $K \subset \Omega_\alpha$ for α large. We have proved that the sets Ω_α converge to the Lipschitz domain U as $\alpha \to +\infty$ in the sense of Definition 1.1. It remains to show that the domain U is either the empty set, the whole space $\mathbb{R}^n$, or delimited by the graph of a locally Lipschitz continuous function. We let D be the set of all points $(\xi_1, \dots, \xi_{n-1})$ in $\mathbb{R}^{n-1}$ such that the line $(\xi_1, \dots, \xi_{n-1}) \times \mathbb{R}$ intersects ∂U. By (4.8) and (4.9), we get that the set D is open. By (4.5) and since $\partial U = A$, we also get that for any point $(\xi_1, \dots, \xi_{n-1})$ in D, there exists only one real number ξ_n such that $(\xi_1, \dots, \xi_n)$ belongs to ∂U. It follows that there exists a function $\varphi : D \to \mathbb{R}$ such that

$$\partial U = \left\{ (\xi_1, \dots, \xi_n) \in D\,; \quad \xi_n = \varphi (\xi_1, \dots, \xi_{n-1}) \right\}.$$

By (4.9), φ is locally Lipschitz continuous. This ends the proof of Theorem 4.3. □

We illustrate the notion of $\overrightarrow{p}$-Lipschitz regularity on ellipsoidal disks and annuli. Contrary to the notion of asymptotic $\overrightarrow{p}$-stability, $\overrightarrow{p}$-Lipschitz regularity does not distinguish these two types of domains since they have common boundaries. We prove in Proposition 4.4 below that ellipsoidal disks and annuli are $\overrightarrow{p}$-Lipschitz if and only if small anisotropy is involved, the quantification for small anisotropy being precisely the quantification which, when not correct, makes that ellipsoidal annuli are not asymptotically $\overrightarrow{p}$-stable (ellipsoidal disks are always asymptotically $\overrightarrow{p}$-stable by Theorem 2.1). The notations in Proposition 4.4 below are those of Proposition 3.1.

Proposition 4.4. *Ellipsoidal disks like $\mathcal{E}(\overrightarrow{a})$ and annuli like $\mathcal{E}(\overrightarrow{b})\backslash\overline{\mathcal{E}(\overrightarrow{a})}$ are $\overrightarrow{p}$-Lipschitz if and only if there holds*

$$\frac{p_+}{p_-}+\frac{p_+}{p}\le 2\,, \tag{4.11}$$

where $p_-=\min(p_1,\dots,p_n)$ and $p_+=\max(p_1,\dots,p_n)$.

Proof. If (4.11) does not hold true, then we know from Proposition 3.1 that ellipsoidal annuli are not asymptotically $\overrightarrow{p}$-stable, and thus neither $\overrightarrow{p}$-Lipschitz by Theorem 4.3. Since ellipsoidal disks have common boundaries with ellipsoidal annuli, they are neither $\overrightarrow{p}$-Lipschitz in case (4.11) does not hold true. The proof of Proposition 4.4 then reduces to the proof that ellipsoidal disks are $\overrightarrow{p}$-Lipschitz in case (4.11) holds true. Clearly, for any $\overrightarrow{a}$ in $\left(\mathbb{R}_+^*\right)^n$, we get $\partial\overline{\mathcal{E}\left(\overrightarrow{a}\right)}=\partial\mathcal{E}\left(\overrightarrow{a}\right)$. We let $\left(\mu_\alpha\right)_\alpha$ be a sequence of positive real numbers converging to 0, $\left(x_\alpha\right)_\alpha$ be a sequence on $\partial\mathcal{E}\left(\overrightarrow{a}\right)$. Up to a subsequence, we may assume that there exists an index i_0 such that

$$x_\alpha^i\mu_\alpha^{\frac{p-p_i}{p_i}}=\mathrm{O}\left(x_\alpha^{i_0}\mu_\alpha^{\frac{p-p_{i_0}}{p_{i_0}}}\right) \tag{4.12}$$

as $\alpha\to+\infty$ for $i=1,\dots,n$, where $x_\alpha=(x_\alpha^1,\dots,x_\alpha^n)$, and such that for any index i satisfying $p_i<p_{i_0}$, there holds

$$x_\alpha^i\mu_\alpha^{\frac{p-p_i}{p_i}}=\mathrm{o}\left(x_\alpha^{i_0}\mu_\alpha^{\frac{p-p_{i_0}}{p_{i_0}}}\right) \tag{4.13}$$

as $\alpha\to+\infty$. Since x_α belongs to $\partial\mathcal{E}\left(\overrightarrow{a}\right)$, we may assume moreover that there exists an index i_1 such that there holds $x_\alpha^{i_1}\ge C>0$ for all α. By (4.11) and (4.12), it follows that

$$\mu_\alpha^{2\frac{p-p_i}{p_i}}=\mathrm{O}\left(\mu_\alpha^{\frac{p-p_{i_1}}{p_{i_1}}}\right)=\mathrm{O}\left(x_\alpha^{i_1}\mu_\alpha^{\frac{p-p_{i_1}}{p_{i_1}}}\right)=\mathrm{O}\left(x_\alpha^{i_0}\mu_\alpha^{\frac{p-p_{i_0}}{p_{i_0}}}\right) \tag{4.14}$$

as $\alpha\to+\infty$ for $i=1,\dots,n$.We then claim that there holds

$$\mu_\alpha^{\frac{p-p_{i_0}}{p_{i_0}}}=\mathrm{o}\left(x_\alpha^{i_0}\right) \tag{4.15}$$

as $\alpha\to+\infty$. In order to prove this claim, we distinguish three cases. In case $p_{i_0}<p_+$, (4.15) follows from (4.14) with $p_i=p_+$. In case $p_{i_1}=p_{i_0}=p_+$, by (4.12), we get that there holds $x_\alpha^{i_0}\ge C>0$ for all α, and thus (4.15) holds true. In case $p_{i_1}<p_{i_0}=p_+$, (4.15) follows from (4.13) with $p_i=p_{i_1}$. By (4.12) and (4.14), we get that for any positive real number R, there exists a positive constant C_R such that, up to a subsequence, there holds

$$C_R\sum_{i=1}^n a_i\left(2\left|x_\alpha^i\right|\mu_\alpha^{\frac{p-p_i}{p_i}}+R^{\frac{p-p_i}{p_i}}\mu_\alpha^{2\frac{p-p_i}{p_i}}\right)\le\left|x_\alpha^{i_0}\right|\mu_\alpha^{\frac{p-p_{i_0}}{p_{i_0}}} \tag{4.16}$$

for all α. We prove that the ellipsoidal disk $\mathcal{E}\left(\overrightarrow{a}\right)$ is $\overrightarrow{p}$-Lipschitz by showing that for any positive real number R, there holds

$$\liminf_{\alpha\to+\infty}\inf_{y,z\in\mathcal{P}_{x_\alpha}^{\overrightarrow{p}}(R\mu_\alpha)\cap\partial\mathcal{E}\left(\overrightarrow{a}\right)}\left|\frac{\tau_{\mu_\alpha,y}^{\overrightarrow{p}}\left(z\right)}{\left|\tau_{\mu_\alpha,y}^{\overrightarrow{p}}\left(z\right)\right|}-e_{i_0}\right|\ge 2a_{i_0}C_R\,,$$

where e_{i_0} is the i_0-th vector in the canonical basis of $\mathbb{R}^n$. We proceed by contradiction and we assume that there exist two positive real numbers ε and R and two sequences of points $y_\alpha = (y_\alpha^1, \dots, y_\alpha^n)$ and $z_\alpha = (z_\alpha^1, \dots, z_\alpha^n)$ on $\mathcal{P}_{x_\alpha}^{\overrightarrow{p}}(R\mu_\alpha) \cap \partial\mathcal{E}(\overrightarrow{a})$ such that there holds

$$\left| \frac{\tau_{\mu_\alpha, y_\alpha}^{\overrightarrow{p}}(z_\alpha)}{\left|\tau_{\mu_\alpha, y_\alpha}^{\overrightarrow{p}}(z_\alpha)\right|} - e_{i_0} \right| \le 2a_{i_0}(C_R - \varepsilon) \,. \tag{4.17}$$

For any α, we set

$$\xi_\alpha^i = \frac{\mu_\alpha^{\frac{p_i - p}{p_i}} (z_\alpha^i - y_\alpha^i)}{\left|\tau_{\mu_\alpha, y}^{\overrightarrow{p}}(z)\right|}$$

for $i = 1, \dots, n$, and we compute

$$\begin{aligned} 0 = \frac{\sum_{i=1}^n a_i (z_\alpha^i)^2 - \sum_{i=1}^n a_i (y_\alpha^i)^2}{\left|\tau_{\mu_\alpha, y_\alpha}^{\overrightarrow{p}}(z_\alpha)\right|} &= \sum_{i=1}^n a_i \xi_\alpha^i \mu_\alpha^{\frac{p-p_i}{p_i}} \left(z_\alpha^i + y_\alpha^i\right) \\ &= \sum_{i=1}^n a_i \xi_\alpha^i \left(2x_\alpha^i \mu_\alpha^{\frac{p-p_i}{p_i}} + R^{\frac{p-p_i}{p_i}} \eta_\alpha^i \mu_\alpha^{2\frac{p-p_i}{p_i}}\right), \end{aligned}$$

where $|\eta_\alpha^i| < 1$ for all α and i. By (4.15), (4.16), and (4.17), it follows that

$$\begin{aligned} \left|x_\alpha^{i_0}\right| \mu_\alpha^{\frac{p-p_{i_0}}{p_{i_0}}} &\le (C_R - \varepsilon) \sum_{i=1}^n a_i \left(2\left|x_\alpha^i\right| \mu_\alpha^{\frac{p-p_i}{p_i}} + R^{\frac{p-p_i}{p_i}} \mu_\alpha^{2\frac{p-p_i}{p_i}}\right) + \mathrm{O}\left(\mu_\alpha^{2\frac{p-p_{i_0}}{p_{i_0}}}\right) \\ &\le \frac{C_R - \varepsilon}{C_R} \left|x_\alpha^{i_0}\right| \mu_\alpha^{\frac{p-p_{i_0}}{p_{i_0}}} + \mathrm{o}\left(\left|x_\alpha^{i_0}\right| \mu_\alpha^{\frac{p-p_{i_0}}{p_{i_0}}}\right) \end{aligned}$$

as $\alpha \to +\infty$, and thus there is a contradiction. We have proved that the ellipsoidal disk $\mathcal{E}(\overrightarrow{a})$ is $\overrightarrow{p}$-Lipschitz in case (4.11) holds true. This ends the proof of Proposition 4.4. □

The two figures below, Figures 7 and 8, illustrate Proposition 4.4 in the case of a disk ($a_i = a_j$ for $i, j = 1, \dots, n$). In Figure 7, there is small anisotropy (the strict inequality in (4.11) holds true) and the domain behaves like in the isotropic case. The limit domain is a halfspace. In Figure 8 the inequality in (4.11) is an equality and we are in the border case of Proposition 4.4. The limit domain is delimited by a parabola.

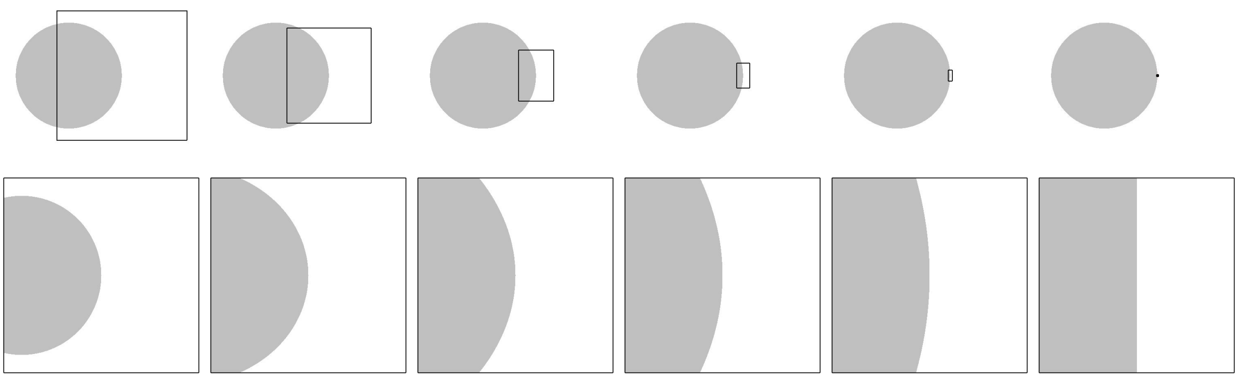

FIGURE 7. Rescaling of a disk with small anisotropy ($n = 2$, $p = 12$, $p_1 = 1.5$, $p_2 = 2$). The first line describes the scale in the rescaling. The second line describes the deformation of the domain.

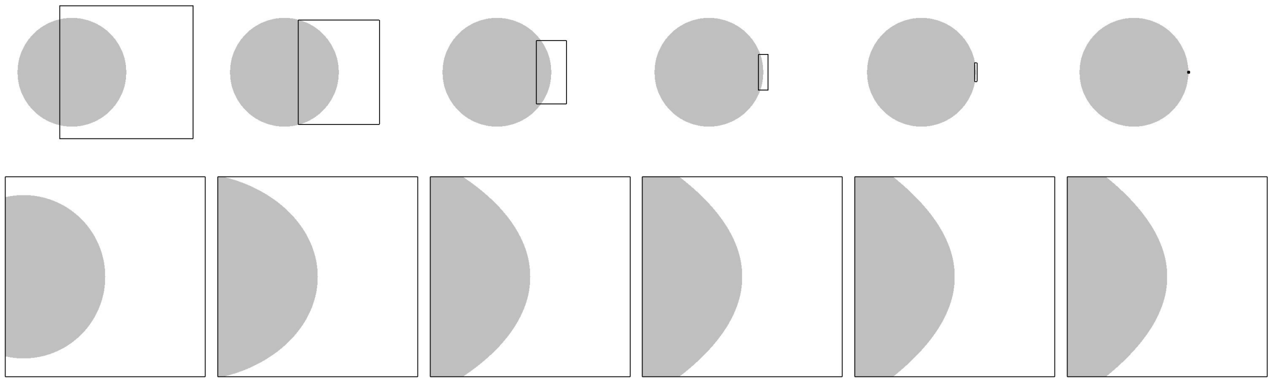

FIGURE 8. Rescaling of a disk: the limit case ($n = 2$, $p = 6$, $p_1 = 1.2$, $p_2 = 2$). The first line describes the scale in the rescaling. The second line describes the deformation of the domain.

In addition to $\overrightarrow{p}$-Lipschitz regularity for domains, we can also define a notion of $\overrightarrow{p}$-regular domains. The notion was introduced in El Hamidi–Vétois [14]. Ellipsoidal disks and annuli, see [14], are $\overrightarrow{p}$-regular if and only if the inequality in (4.11) is strict. For $\overrightarrow{p}$-regular domains, the limit sets turn out to be exactly like in the isotropic case, namely either the empty set, a halfspace, or the whole space $\mathbb{R}^n$. It can be proved that $\overrightarrow{p}$-regular domains are always $\overrightarrow{p}$-Lipschitz.

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