BLOWING-UP SOLUTIONS FOR SECOND-ORDER CRITICAL ELLIPTIC EQUATIONS: THE IMPACT OF THE SCALAR CURVATURE

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Abstract. Given a closed manifold (M^n, g), $n \geq 3$, Olivier Druet [7] proved that a necessary condition for the existence of energy-bounded blowing-up solutions to perturbations of the equation

$$\Delta_g u + h_0 u = u^{\frac{n+2}{n-2}},\ u > 0 \text{ in } M$$

is that $h_0 \in C^1(M)$ touches the Scalar curvature somewhere when $n \geq 4$ (the condition is different for $n = 6$). In this paper, we prove that Druet's condition is also sufficient provided we add its natural differentiable version. For $n \geq 6$, our arguments are local. For the low dimensions $n \in \{4, 5\}$, our proof requires the introduction of a suitable mass that is defined only where Druet's condition holds. This mass carries global information both on h_0 and (M, g).

1. Introduction and main results

Let (M, g) be a compact Riemannian manifold of dimension $n \geq 3$ without boundary and $h_0 \in C^p(M)$, $1 \leq p \leq \infty$. We consider the equation

$$\Delta_g u + h_0 u = u^{2^\star - 1},\ u > 0 \text{ in } M, \tag{1}$$

where $\Delta_g := -\operatorname{div}_g(\nabla)$ is the Laplace–Beltrami operator and $2^\star := \frac{2n}{n-2}$. We investigate the existence of families $(h_\epsilon)_{\epsilon>0} \in C^p(M)$ and $(u_\epsilon)_{\epsilon>0} \in C^2(M)$ satisfying

$$\Delta_g u_\epsilon + h_\epsilon u_\epsilon = u_\epsilon^{2^\star - 1},\ u_\epsilon > 0 \text{ in } M \text{ for all } \epsilon > 0, \tag{2}$$

and such that $h_\epsilon \to h_0$ in $C^p(M)$ and $\max_M u_\epsilon \to \infty$ as $\epsilon \to 0$. We say that $(u_\epsilon)_\epsilon$ *blows up* at some point $\xi_0 \in M$ as $\epsilon \to 0$ if for all $r > 0$, $\lim_{\epsilon\to 0} \max_{B_r(\xi_0)} u_\epsilon = +\infty$. Druet [7, 9] obtained the following necessary condition for blow-up:

Theorem 1.1 (Druet [7, 9])**.** *Let (M, g) be a compact Riemannian manifold of dimension $n \geq 4$. Let $h_0 \in C^1(M)$ be such that $\Delta_g + h_0$ is coercive. Assume that there exist families $(h_\epsilon)_{\epsilon>0} \in C^1(M)$ and $(u_\epsilon)_{\epsilon>0} \in C^2(M)$ satisfying (2) and such that $h_\epsilon \to h_0$ strongly in $C^1(M)$ and $u_\epsilon \rightharpoonup u_0$ weakly in $L^{2^\star}(M)$. Assume that $(u_\epsilon)_\epsilon$ blows-up. Then there exists $\xi_0 \in M$ such that $(u_\epsilon)_\epsilon$ blows up at ξ_0 and*

$$(h_0 - c_n \operatorname{Scal}_g)(\xi_0) = 0 \text{ if } n \neq 6 \text{ and } (h_0 - c_n \operatorname{Scal}_g - 2u_0)(\xi_0) = 0 \text{ if } n = 6. \tag{3}$$

Furthermore, if $n \in \{4, 5\}$, then $u_0 \equiv 0$.

Here $c_n := \frac{n-2}{4(n-1)}$ and Scal_g is the Scalar curvature of (M, g). This result does not hold in dimension $n = 3$. Indeed, Hebey–Wei [15] constructed examples of blowing-up solutions to (2) on the standard sphere $(\mathbb{S}^3, g_0)$, which are bounded in $L^{2^\star}(\mathbb{S}^3)$ but do not satisfy (3).

Date: February 28th, 2020.

This paper is concerned with the converse of Theorem 1.1 in dimensions $n \geq 4$. For the sake of clarity, we state separately our results in the cases $u_0 \equiv 0$ in dimension $n \geq 4$ (Theorem 1.2) and $u_0 > 0$ in dimension $n \geq 6$ (Theorem 1.3):

Theorem 1.2. *Let (M, g) be a compact Riemannian manifold of dimension $n \geq 4$. Let $h_0 \in C^p(M)$, $1 \leq p \leq \infty$, be such that $\Delta_g + h_0$ is coercive. Assume that there exists a point $\xi_0 \in M$ such that*

$$(h_0 - c_n \operatorname{Scal}_g)(\xi_0) = |\nabla (h_0 - c_n \operatorname{Scal}_g)(\xi_0)| = 0. \tag{4}$$

Then there exist families $(h_\epsilon)_{\epsilon>0} \in C^p(M)$ and $(u_\epsilon)_{\epsilon>0} \in C^2(M)$ satisfying (2) and such that $h_\epsilon \to h_0$ strongly in $C^p(M)$, $u_\epsilon \rightharpoonup 0$ weakly in $L^{2^\star}(M)$ and $(u_\epsilon)_\epsilon$ blows up at ξ_0.

For convenience, for every $h_0, u_0 \in C^0(M)$, we define

$$\varphi_{h_0} := h_0 - c_n \operatorname{Scal}_g \text{ and } \varphi_{h_0,u_0} := \begin{cases} h_0 - c_n \operatorname{Scal}_g & \text{if } n \neq 6 \\ h_0 - 2u_0 - c_n \operatorname{Scal}_g & \text{if } n = 6. \end{cases} \tag{5}$$

Theorem 1.3. *Let (M, g) be a compact Riemannian manifold of dimension $n \geq 6$. Let $h_0 \in C^p(M)$, $1 \leq p \leq \infty$, be such that $\Delta_g + h_0$ is coercive. Assume that there exist a solution $u_0 \in C^2(M)$ of (1) and a point $\xi_0 \in M$ such that*

$$\varphi_{h_0,u_0}(\xi_0) = |\nabla \varphi_{h_0,u_0}(\xi_0)| = 0. \tag{6}$$

Then there exist families $(h_\epsilon)_{\epsilon>0} \in C^p(M)$ and $(u_\epsilon)_{\epsilon>0} \in C^2(M)$ satisfying (2) and such that $h_\epsilon \to h_0$ strongly in $C^p(M)$, $u_\epsilon \rightharpoonup u_0$ weakly in $L^{2^\star}(M)$ and $(u_\epsilon)_\epsilon$ blows up at ξ_0.

Compared with Theorem 1.1, we have assumed here that condition (3) is also satisfied at order 1. However, this stronger condition is actually expected to be necessary for the existence of blowing-up solutions (see Theorem 14.1 in the last section of this paper and the discussion in Druet [9, Section 2.5]).

We refer to Section 2 for examples of functions h_0 and u_0 satisfying the assumptions of Theorem 1.3. Recently, Premoselli–Thizy [23] obtained a beautiful example of blowing-up solutions showing that in dimension $n \in \{4, 5\}$, condition (4) may not be satisfied at all blow-up points.

When $h_0 \equiv c_n \operatorname{Scal}_g$, that is when (1) is the Yamabe equation, several examples of blowing-up solutions have been obtained. In the perturbative case, that is when $h_\epsilon \not\equiv c_n \operatorname{Scal}_g$, examples of blowing-up solutions have been obtained by Druet–Hebey [10], Esposito–Pistoia–Vétois [12], Morabito–Pistoia–Vaira [22], Pistoia–Vaira [24] and Robert–Vétois [27]. In the nonpertubative case $h_\epsilon \equiv c_n \operatorname{Scal}_g$, we refer to Brendle [3] and Brendle–Marques [4] regarding the non-compactness of Yamabe metrics. When solutions blow-up not only pointwise but also in energy, the function φ_{h_0} may not vanish (see Chen–Wei–Yan [5] for $n \geq 5$ and Vétois–Wang [32] for $n = 4$).

When there does not exist any blowing-up solutions to the equations (2), then equation (1) is *stable*. We refer to the survey of Druet [9] and the book of Hebey [14] for exhaustive studies of the various concepts of stability. Stability also arises in the Lin–Ni–Takagi problem (see for instance del Pino–Musso–Roman–Wei [6] for a recent reference on this topic). In Geometry, stability is linked to the problem of compactness of the Yamabe equation (see Schoen [29, 30], Li–Zhu [20], Druet [8], Marques [21], Li–Zhang [18, 19], Khuri–Marques–Schoen [16]).

Let us give some general considerations about the proofs. Theorem 1.1 yields *local* information on blow-up points. It is essentially the consequence of the concentration of the L^2–norm of the solutions at one of the blow-up points when $n \geq 4$. However, in our construction, the problem may be *both local and global.* Indeed, we reduce the problem to finding critical points of a functional defined on a finite-dimensional space. The first term in the asymptotic expansion of the reduced functional is local. This is due to the L^2–concentration of the standard bubble in the definition of our ansatz. The second term in the expansion plays a decisive role for obtaining critical points. For the high dimensions $n \geq 6$, this term is also local (see e.g. (54)). However, for $n \in \{4, 5\}$, the second term is global and we are then compelled to introduce a suitable notion of mass, which carries global information on h_0 and (M, g), and to add a corrective term to the standard bubble (see (100)) in order to obtain a reasonable expansion (see e.g. (113)). Unlike the case where $n = 3$ or $h_0 \equiv c_n \operatorname{Scal}_g$, the mass is not defined at all points in the manifold, but only at the points where the condition (6) is satisfied.

More precisely, Theorems 1.2 and 1.3 are consequences of Theorems 1.4 and 1.5 below. The latter are the core results of our paper. In these theorems, we fix a linear perturbation $h_\epsilon = h_0 + \epsilon f$ for some function $f \in C^p(M)$. Furthermore, we specify the behavior of the blowing-up solutions that we obtain. We let $H_1^2(M)$ be the completion of $C^\infty(M)$ for the norm $\|u\|_{H_1^2} := \|\nabla u\|_2 + \|u\|_2$. We say that $(u_\epsilon)_\epsilon$ blows up with one bubble at some point $\xi_0 \in M$ if $u_\epsilon = u_0 + U_{\delta_\epsilon,\xi_\epsilon} + \mathrm{o}(1)$ as $\epsilon \to 0$ in $H_1^2(M)$, where $u_0 \in H_1^2(M)$ is such that $u_\epsilon \rightharpoonup u_0$ weakly in $H_1^2(M)$, $U_{\delta_\epsilon,\xi_\epsilon}$ is as in (24), $(\delta_\epsilon, \xi_\epsilon) \to (0, \xi_0)$ and $o(1) \to 0$ strongly in $H_1^2(M)$ as $\epsilon \to 0$.

Our first result deals with the case where $u_0 \equiv 0$ in dimension $n \geq 4$:

Theorem 1.4. *Let (M, g) be a compact Riemannian manifold of dimension $n \geq 4$. Let $h_0 \in C^p(M)$, $p \geq 2$, be such that $\Delta_g + h_0$ is coercive. Assume that there exists a point $\xi_0 \in M$ satisfying (4). Assume in addition that ξ_0 is a nondegenerate critical point of $h_0 - c_n \operatorname{Scal}_g$ and*

$$(7) \qquad K_{h_0}(\xi_0) := \left\{ \begin{array}{ll} m_{h_0}(\xi_0) & \text{if } n = 4, 5 \\ \Delta_g \left(h_0 - c_n \operatorname{Scal}_g\right)(\xi_0) + \dfrac{c_n}{6} |\operatorname{Weyl}_g(\xi_0)|_g^2 & \text{if } n \geq 6 \end{array} \right\} \neq 0,$$

where $m_{h_0}(\xi_0)$ is the mass of $\Delta_g + h_0$ at the point ξ_0 (see Proposition-Definition 8.1), and Weyl_g is the Weyl curvature tensor of the manifold. We fix a function $f \in C^p(M)$ such that $f(\xi_0) \times K_{h_0}(\xi_0) > 0$. Then for small $\epsilon > 0$, there exists $u_\epsilon \in C^2(M)$ satisfying

$$(8) \qquad \Delta_g u_\epsilon + (h_0 + \epsilon f) u_\epsilon = u_\epsilon^{2^\star - 1} \text{ in } M, \;\; u_\epsilon > 0,$$

and such that $u_\epsilon \rightharpoonup 0$ weakly in $L^{2^\star}(M)$ and $(u_\epsilon)_\epsilon$ blows up with one bubble at ξ_0.

The definition of $K_{h_0}(\xi_0)$ outlines the major difference between high- and low-dimensions that was mentioned above: for $n \geq 6$, it is a local quantity, but for $n \in \{4, 5\}$, it carries global information (see Section 8 for more discussions).

Next we deal with the case where $u_0 > 0$ in dimension $n \geq 6$:

Theorem 1.5. *Let (M, g) be a compact Riemannian manifold of dimension $n \geq 6$. Let $h_0 \in C^p(M)$, $p \geq 2$, be such that $\Delta_g + h_0$ is coercive. Assume that there exist a nondegenerate solution $u_0 \in C^2(M)$ to equation (1) and $\xi_0 \in M$ satisfying (6).*

Assume in addition that ξ_0 is a nondegenerate critical point of φ_{h_0,u_0} and
(9)
$$K_{h_0,u_0}(\xi_0) := \left\{\begin{array}{ll} \Delta_g\varphi_{h_0,u_0}(\xi_0) + \frac{c_6}{6}|\operatorname{Weyl}_g(\xi_0)|_g^2 & \text{if } n=6 \\ u_0(\xi_0) & \text{if } 7\le n\le 9 \\ 672u_0(\xi_0) + \Delta_g\varphi_{h_0,u_0}(\xi_0) + \frac{c_{10}}{6}|\operatorname{Weyl}_g(\xi_0)|_g^2 & \text{if } n=10 \\ \Delta_g\varphi_{h_0,u_0}(\xi_0) + \frac{c_n}{6}|\operatorname{Weyl}_g(\xi_0)|_g^2 & \text{if } n\ge 11 \end{array}\right\} \neq 0.$$

We fix a function $f \in C^p(M)$ such that

$$K_{h_0,u_0}(\xi_0) \times \left\{\begin{array}{ll} \left[f + 2(\Delta_g + h_0 - 2u_0)^{-1}(fu_0)\right](\xi_0) & \text{if } n=6 \\ f(\xi_0) & \text{if } n>6 \end{array}\right\} > 0. \tag{10}$$

Then for small $\epsilon > 0$, there exists $u_\epsilon \in C^2(M)$ satisfying (8) and such that $u_\epsilon \rightharpoonup u_0$ weakly in $L^{2^\star}(M)$ and $(u_\epsilon)_\epsilon$ blows up with one bubble at ξ_0.

The paper is organized as follows. In Section 2, we discuss the question of existence of functions h_0 and u_0 satisfying the assumptions of Theorem 1.3. In Section 3, we introduce our notations and discuss the general setting of the problem. In Section 4, we establish a general C^1-estimate on the energy functional, which holds in all dimensions. In Sections 5, 6 and 7, we then compute a C^1-asymptotic expansion of the energy functional in the case where $n \ge 6$, which we divide in the following subcases: $[n \ge 6$ and $u_0 \equiv 0]$ in Section 5, $[n \ge 7$ and $u_0 > 0]$ in Section 6 and $[n = 6$ and $u_0 > 0]$ in Section 7. In Section 8, we discuss the specific setting of dimensions $n \in \{4, 5\}$ and we define the mass of $\Delta_g + h_0$ in this case. In Section 9, we then deal with the C^1-asymptotic expansion of the energy functional when $n \in \{4, 5\}$. In Sections 10, 11, 12 and 13, we complete the proofs of Theorems 1.4, 1.5, 1.2 and 1.3, respectively. Finally, in Section 14, we deal with the necessity of condition (4) on the gradient

2. Existence results for h_0 and u_0

This short section deals with two results which provide conditions for the existence of functions h_0 and u_0 satisfying the assumptions of Theorem 1.3 with prescribed φ_{h_0,u_0} and ξ_0. The first result is a straightforward consequence of classical works on the Yamabe equation:

Theorem 2.1. *(Aubin [1], Schoen [28], Trudinger [31]) Assume that $n \ge 3$. Then there exists $\epsilon_0 \ge 0$ depending only on n and (M, g) such that $\epsilon_0 > 0$ if (M, g) is not conformally diffeomorphic to the standard sphere, $\epsilon_0 = 0$ otherwise, and for every $\varphi_0 \in C^1(M)$ such that*

$$\varphi_0 \le \epsilon_0 \text{ and } \lambda_1(\Delta_g + h_0) > 0, \text{ where } h_0 := \varphi_0 + c_n \operatorname{Scal}_g,$$

there exists a solution $u_0 \in C^2(M)$ of the equation (1). In particular, if $n \ne 6$ and $\varphi_0(\xi_0) = |\nabla\varphi_0(\xi_0)| = 0$ at some point $\xi_0 \in M$, then h_0 satisfies (6).

It remains to deal with the case where $n = 6$. In this case, we obtain the following:

Proposition 2.1. *Assume that $n = 6$. Let $\varphi_0 \in C^p(M)$, $1 \le p \le \infty$, be such that*

$$\lambda_1(\Delta_g + \varphi_0 + c_n \operatorname{Scal}_g) < 0. \tag{11}$$

Then there exists $h_0 \in C^p(M)$ *such that the equation* (1) *admits a solution* $u_0 \in C^2(M)$ *satisfying* $h_0 - c_n \operatorname{Scal}_g - 2u_0 \equiv \varphi_0$. *In particular, if* $\varphi_0(\xi_0) = |\nabla\varphi_0(\xi_0)| = 0$ *at some point* $\xi_0 \in M$, *then* (h_0, u_0) *satisfy* (6).

Proof of Proposition 2.1. Note that $2^\star - 1 = 2$ when $n = 6$. In this case, we can rewrite the equation (1) as

$$\Delta_g u + (h_0 - 2u)u = -u^2, \ u > 0 \text{ in } M. \tag{12}$$

Using (11) together with a standard variational method, we obtain that there exists a solution $u_0 \in C^{p+1}(M) \subset C^2(M)$ of the equation (12) with $h_0 := \varphi_0 + c_n \operatorname{Scal}_g + 2u_0 \in C^p(M)$. This ends the proof of Proposition 2.1. □

3. Notations and general setting

We follow the notations and definitions of Robert–Vétois [26].

3.1. **Euclidean setting.** We define

$$U_{1,0}(x) := \left(\frac{\sqrt{n(n-2)}}{1+|x|^2}\right)^{\frac{n-2}{2}} \text{ for all } x \in \mathbb{R}^n, \tag{13}$$

so that $U_{1,0}$ is a positive solution to the equation

$$\Delta_{\text{Eucl}} U = U^{2^\star - 1} \text{ in } \mathbb{R}^n,$$

where Eucl stands for the Euclidean metric. For every $\delta > 0$ and $\xi \in \mathbb{R}^n$, we define

$$U_{\delta,\xi}(x) := \delta^{-\frac{n-2}{2}} U\left(\delta^{-1}(x-\xi)\right) = \left(\frac{\sqrt{n(n-2)}\delta}{\delta^2 + |x-\xi|^2}\right)^{\frac{n-2}{2}} \text{ for all } x \in \mathbb{R}^n. \tag{14}$$

We define

$$Z_0 := (\partial_\delta U_{\delta,\xi})_{|(1,0)} \text{ and } Z_i := (\partial_{\xi_i} U_{\delta,\xi})_{|(1,0)} \text{ for all } i = 1, \dots, n. \tag{15}$$

As one checks,

$$Z_0 = -\frac{n-2}{2} U - (x, \nabla U) = \sqrt{n(n-2)}^{\frac{n-2}{2}} \frac{n-2}{2} \frac{|x|^2 - 1}{(1+|x|^2)^{\frac{n}{2}}} \tag{16}$$

and

$$Z_i = -\partial_{x_i} U = \sqrt{n(n-2)}^{\frac{n-2}{2}} (n-2) \frac{x_i}{(1+|x|^2)^{\frac{n}{2}}} \text{ for all } i = 1, \dots, n. \tag{17}$$

We denote $p = (p_0, p_1, \dots, p_n) := (\delta, \xi) \in (0, \infty) \times \mathbb{R}^n$. Straightforward computations yield

$$\partial_{p_i} U_{\delta,\xi} = \delta^{-1} (Z_i)_{\delta,\xi} := \delta^{-1} \delta^{-\frac{n-2}{2}} Z_i\left(\delta^{-1}(x-\xi)\right) \text{ for all } i = 0, \dots, n, \tag{18}$$

$$\partial_\delta U_{\delta,\xi} = \sqrt{n(n-2)}^{\frac{n-2}{2}} \frac{n-2}{2} \delta^{\frac{n-2}{2}-1} \frac{|x-\xi|^2 - \delta^2}{(\delta^2 + |x-\xi|^2)^{n/2}} \tag{19}$$

and

$$\partial_{\xi_i} U_{\delta,\xi} = \sqrt{n(n-2)}^{\frac{n-2}{2}} (n-2) \delta^{\frac{n-2}{2}} \frac{(x-\xi)_i}{(\delta^2 + |x-\xi|^2)^{n/2}} \text{ for all } i = 1, \dots, n. \tag{20}$$

It follows from Rey [25] (see also Bianchi–Egnell [2]) that

$$\{\phi \in D_1^2(\mathbb{R}^n) : \ \Delta_{\text{Eucl}} \phi = (2^\star - 1) U^{2^\star - 2} \phi \text{ in } \mathbb{R}^n\} = \operatorname{Span}\{Z_i\}_{i=0,\dots,n},$$

where $D_1^2(\mathbb{R}^n)$ is the completion of $C_c^\infty(\mathbb{R}^n)$ for $u \mapsto \|\nabla u\|_2$.

3.2. **Riemannian setting.** We fix $N > n-2$ to be chosen large later. It follows from Lee–Parker [17] that there exists a function $\Lambda \in C^\infty(M \times M)$ such that, defining $\Lambda_\xi := \Lambda(\xi, \cdot)$, we have

$$\Lambda_\xi > 0,\ \Lambda_\xi(\xi) = 1 \text{ and } \nabla\Lambda_\xi(\xi) = 0 \text{ for all } \xi \in M \tag{21}$$

and, defining the metric $g_\xi := \Lambda_\xi^{2^\star-2} g$ conformal to g, we have

$$\operatorname{Scal}_{g_\xi}(\xi) = 0,\ \nabla \operatorname{Scal}_{g_\xi}(\xi) = 0,\ \Delta_g \operatorname{Scal}_{g_\xi}(\xi) = \frac{1}{6}|\operatorname{Weyl}_g(\xi)|_g^2 \tag{22}$$

and

$$dv_{g_\xi}(x) = (1 + \mathrm{O}(|x|^N))\, dx \text{ via the chart } \exp_\xi^{g_\xi} \text{ around } 0, \tag{23}$$

where dx is the Euclidean volume element, dv_{g_ξ} is the Riemannian volume element of (M, g_ξ) and $\exp_\xi^{g_\xi}$ is the exponential chart at ξ with respect to the metric g_ξ. The compactness of M yields the existence of $r_0 > 0$ such that the injectivity radius of the metric g_ξ satisfies $i_{g_\xi}(M) \geq 3r_0$ for all $\xi \in M$. We let $\chi \in C^\infty(\mathbb{R})$ be such that $\chi(t) = 1$ for all $t \leq r_0$, $\chi(t) = 0$ for all $t \geq 2r_0$ and $0 \leq \chi \leq 1$. For every $\delta > 0$ and $\xi \in M$, we then define the bubble as

$$\begin{aligned} U_{\delta,\xi}(x) :&= \chi(d_{g_\xi}(x,\xi))\Lambda_\xi(x)\delta^{-\frac{n-2}{2}} U_{1,0}(\delta^{-1}(\exp_\xi^{g_\xi})^{-1}(x)) \\ &= \chi(d_{g_\xi}(x,\xi))\Lambda_\xi(x)\left(\frac{\delta\sqrt{n(n-2)}}{\delta^2 + d_{g_\xi}(x,\xi)^2}\right)^{\frac{n-2}{2}}, \end{aligned} \tag{24}$$

where $d_{g_\xi}(x,\xi)$ is the geodesic distance between x and ξ with respect to the metric g_ξ. Since there will never be ambiguity, to avoid unnecessary heavy notations, we will keep the notations $U_{\delta,\xi}$ as (14) when $p = (\delta,\xi) \in (0,\infty) \times \mathbb{R}^n$, and as (24) when $p = (\delta,\xi) \in (0,\infty) \times M$. Finally, for every $p = (\delta,\xi) \in (0, \infty \times M$, we define

$$K_{\delta,\xi} := \operatorname{Span}\{(Z_i)_{\delta,\xi}\}_{i=0,\dots,n},$$

where

$$(Z_i)_{\delta,\xi}(x) := \chi(d_{g_\xi}(x,\xi))\Lambda_\xi(x)\delta^{-\frac{n-2}{2}} Z_i(\delta^{-1}(\exp_\xi^{g_\xi})^{-1}(x))$$

for all $x \in M$ and $i = 0, \dots, n$.

3.3. **General reduction theorem.** For every $1 \leq q \leq \infty$, we let $\|\cdot\|_q$ be the usual norm of $L^q(M)$. For every $h \in C^0(M)$, we define

$$J_h(u) := \frac{1}{2}\int_M \left(|\nabla u|_g^2 + hu^2\right) dv_g - \frac{1}{2^\star}\int_M u_+^{2^\star}\, dv_g \text{ for all } u \in H_1^2(M),$$

where $u_+ := \max(u, 0)$. The space $H_1^2(M)$ is endowed with the bilinear form $\langle\cdot,\cdot\rangle_h$, where

$$\langle u, v\rangle_h := \int_M (\nabla u \nabla v + huv)\, dv_g \text{ for all } u, v \in H_1^2(M).$$

If $\Delta_g + h_0$ is coercive and $\|h - h_0\|_\infty$ is small enough, then $\langle\cdot,\cdot\rangle_h$ is positive definite and $(H_1^2(M), \langle\cdot,\rangle_h)$ is a Hilbert space. We then have that $J_h \in C^1(H_1^2(M))$ and

$$J_h'(u)[\phi] = \int_M (\nabla u \nabla\phi + hu\phi)\, dv_g - \int_M u_+^{2^\star-1}\phi\, dv_g = \langle u, \phi\rangle_h - \int_M u_+^{2^\star-1}\phi\, dv_g$$

for all $u,\phi\in H_1^2(M)$. We let $(\delta,\xi)\to B_{h,\delta,\xi}=B_h(\delta,\xi)$ be a function in $C^1((0,\infty)\times M,H_1^2(M))$ such that for every $\delta>0$, there exists $\epsilon(\delta)>0$ independent of h and ξ such that

$$\|B_{h,\delta,\xi}\|_{H_1^2}+\delta\|\partial_p B_{h,\delta,\xi}\|_{H_1^2}<\epsilon(\delta) \text{ for all } p=(\delta,\xi)\in(0,\infty)\times M \tag{25}$$

and $\epsilon(\delta)\to 0$ as $\delta\to 0$. The function $B_{h,\delta,\xi}$ will be fixed later. We also let $\tilde u_0\in C^2(M)$. We define

$$W_{h,\tilde u_0,\delta,\xi}:=\tilde u_0+U_{\delta,\xi}+B_{h,\delta,\xi}.$$

We fix a point $\xi_0\in M$ and a function $h_0\in C^0(M)$ such that Δ_g+h_0 is coercive. We let $u_0\in C^2(M)$ be a solution of the equation

$$\Delta_g u_0+h_0u_0=u_0^{2^\star-1},\ u_0\ge 0 \text{ in } M.$$

It follows from the strong maximum principle that either $u_0\equiv 0$ or $u_0>0$. We assume that u_0 is *nondegenerate*, that is, for every $\phi\in H_1^2(M)$,

$$\Delta_g\phi+h_0\phi=(2^\star-1)u_0^{2^\star-2}\phi \iff \phi\equiv 0.$$

It then follows from Robert–Vétois [26] that there exist $\epsilon_0>0$, $U_0\subset M$ a small open neighborhood of ξ_0 and $\Phi_{h,\tilde u_0}\in C^1((0,\epsilon_0)\times U_0,H_1^2(M))$ such that, when $\|h-h_0\|_\infty<\epsilon_0$ and $\|\tilde u_0-u_0\|_{C^2}<\epsilon_0$, we have

$$\Pi_{K_{\delta,\xi}^\perp}(W_{h,\tilde u_0,\delta,\xi}+\Phi_{h,\tilde u_0,\delta,\xi}-(\Delta_g+h)^{-1}((W_{h,\tilde u_0,\delta,\xi}+\Phi_{h,\tilde u_0,\delta,\xi})_+^{2^\star-1}))=0 \tag{26}$$

and

$$\|\Phi_{h,\tilde u_0,\delta,\xi}\|_{H_1^2}\le C\|W_{h,\tilde u_0,\delta,\xi}-(\Delta_g+h)^{-1}((W_{h,\tilde u_0,\delta,\xi})_+^{2^\star-1})\|_{H_1^2}\le C\left\|R_{\delta,\xi}\right\|_{\frac{2n}{n+2}} \tag{27}$$

for all $(\delta,\xi)\in(0,\epsilon_0)\times U_0$, where $C>0$ does not depend on $(h,\tilde u_0,\delta,\xi)$, $\Phi_{h,\tilde u_0,\delta,\xi}:=\Phi_{h,\tilde u_0}(\delta,\xi)$, $\Pi_{K_{\delta,\xi}^\perp}$ is the orthogonal projection of $H_1^2(M)$ onto $K_{\delta,\xi}^\perp$ (here, the orthogonality is taken with respect to $\langle\cdot,\cdot\rangle_h$) and

$$R_{\delta,\xi}:=(\Delta_g+h)W_{h,\tilde u_0,\delta,\xi}-(W_{h,\tilde u_0,\delta,\xi})_+^{2^\star-1}. \tag{28}$$

Furthermore, for every $(\delta_0,\xi_0)\in(0,\epsilon_0)\times U_0$, we have

$$\begin{aligned}&J_h'(W_{h,\tilde u_0,\delta_0,\xi_0}+\Phi_{h,\tilde u_0,\delta_0,\xi_0})=0\\ &\qquad\iff (\delta_0,\xi_0) \text{ is a critical point of } (\delta,\xi)\mapsto J_h(W_{h,\tilde u_0,\delta,\xi}+\Phi_{h,\tilde u_0,\delta,\xi}).\end{aligned} \tag{29}$$

It follows from Robert–Vétois [26] that

$$J_h(W_{h,\tilde u_0,\delta,\xi}+\Phi_{h,\tilde u_0,\delta,\xi})=J_h\left(W_{h,\tilde u_0,\delta,\xi}\right)+\mathrm{O}(\|\Phi_{h,\tilde u_0,\delta,\xi}\|_{H_1^2}^2) \tag{30}$$

uniformly with respect to $(\delta,\xi)\in(0,\epsilon_0)\times U_0$ and $(h,\tilde u_0)$ such that $\|h-h_0\|_\infty<\epsilon_0$ and $\|\tilde u_0-u_0\|_{C^2}<\epsilon_0$.

Conventions:

- To avoid unnecessarily heavy notations, we will often drop the indices $(h,,\tilde{u}_0,\delta,\xi)$, so that $U:=U_{\delta,\xi}$, $B:=B_{h,\delta,\xi}$, $W:=W_{h,\tilde{u}_0,\delta,\xi}$, $\Phi:=\Phi_{h,\tilde{u}_0,\delta,\xi}$, etc. The differentiation with respect to the variable (δ,ξ) will always be denoted by ∂_p, and the differentiation with respect to $x\in M$ (or $\mathbb{R}^n$) by ∂_x. For example,

$$\partial_{x_i}\partial_{p_j}W=\begin{cases}\dfrac{\partial^2 W_{h,\tilde{u}_0,\delta,\xi}(x)}{\partial x_i\partial\delta} & \text{if } j=0\\ \dfrac{\partial^2 W_{h,\tilde{u}_0,\delta,\xi}(x)}{\partial x_i\partial\xi_j} & \text{if } j\geq 1.\end{cases}$$

- For every $\xi\in U_0$, we identify the tangent space $T_\xi M$ with $\mathbb{R}^n$. Indeed, assuming that the neighborhood U_0 is small enough, it follows from the Gram–Schmidt orthonormalization procedure that there exists an orthonormal frame with respect to the metric g_ξ, which is smooth with respect to the point ξ. Such a frame provides a smooth family of linear isometries $(\psi_\xi)_{\xi\in U_0}$, $\psi_\xi:\mathbb{R}^n\to T_\xi M$, which allow to identify $T_\xi M$ with $\mathbb{R}^n$. In particular, in this paper, the chart $\exp_\xi^{g_\xi}$ will denote the composition of the usual exponential chart with the isometry ψ_ξ.
- Throughout the paper, C will denote a positive constant such that
  - C depends on n, (M,g), $\xi_0\in M$, the functions $h_0,u_0\in C^2(M)$ and a constant $A>0$ such that $\|h_0\|_{C^2}<A$ and $\lambda_1(\Delta_g+h_0)>1/A$. In the case where $u_0>0$, we also assume that $\|u_0\|_{C^2}<A$ and $u_0>1/A$.
  - C does not depend on $x\in M$ (or $x\in\mathbb{R}^n$, depending on the context), ξ in the neighborhood U_0, $\delta>0$ small and $h\in C^2(M)$ such that $\|h\|_{C^2}<A$ and $\lambda_1(\Delta_g+h)>1/A$. In the case where $u_0>0$, C is also independent of $\tilde{u}_0\in C^2(M)$ such that $\|\tilde{u}_0\|_{C^2}<A$ and $\tilde{u}_0>1/A$.

  The value of C might change from line to line, and even in the same line.
- For every $f,g\in\mathbb{R}$, the notations $f=\mathrm{O}(g)$ and $f=\mathrm{o}(g)$ will stand for $|f|\leq C|g|$ and $|f|\leq C\epsilon(h,\delta,\xi)|g|$, respectively, where $\epsilon(h,\delta,\xi)\to 0$ as $h\to h_0$ in $C^2(M)$, $\delta\to 0$ and $\xi\to\xi_0$.

4. C^1-estimates for the energy functional

For every $\delta>0$ and $\xi\in U_0$, we define

$$\tilde{U}_{\delta,\xi}(x):=\left(\frac{\delta\sqrt{n(n-2)}}{\delta^2+d_{g_\xi}(x,\xi)^2}\right)^{\frac{n-2}{2}}\quad\text{for all } x\in M. \tag{31}$$

Our first result is the differentiable version of (30).

Proposition 4.1. *In addition to the assumptions of Section 3, we assume that*

$$|B_{h,\delta,\xi}(x)|+\delta|\partial_p B_{h,\delta,\xi}(x)|\leq C(U_{\delta,\xi}(x)+\delta\tilde{U}_{\delta,\xi}(x))\ \text{for all } x\in M. \tag{32}$$

We then have

$$\partial_p J_h(W+\Phi)=\partial_p J_h(W)+\mathrm{O}(\delta^{-1}\|\Phi\|_{H_1^2}(\|R\|_{\frac{2n}{n+2}}+\delta\|\partial_p R\|_{\frac{2n}{n+2}}+\|\Phi\|_{H_1^2}))+\mathrm{O}(\mathbf{1}_{n\geq 7}\delta^{-1}\|\Phi\|_{H_1^2}^{2^\star-1}), \tag{33}$$

where $R=R_{\delta,\xi}$ is as in (28).

Proof of Proposition 4.1. It follows from (26) that there exist real numbers $\lambda_j := \lambda_j(\delta, \xi)$ such that

$$W + \Phi - (\Delta_g + h)^{-1}(W + \Phi)_+^{2^\star - 1} = \sum_{j=0}^{n} \lambda_j Z_j.$$

This can be written as

$$J_h'(W + \Phi) = \sum_{j=0}^{n} \lambda_j \langle Z_j, \cdot \rangle_h. \tag{34}$$

We fix $i \in \{0, \dots, n\}$. We obtain

$$\begin{aligned}
\partial_{p_i} J_h(W + \Phi) &= J_h'(W + \Phi)[\partial_{p_i} W + \partial_{p_i} \Phi] \\
&= J_h'(W)[\partial_{p_i} W] + (J_h'(W + \Phi) - J_h'(W))\, [\partial_{p_i} W] + J_h'(W + \Phi)[\partial_{p_i} \Phi] \\
&= J_h'(W)[\partial_{p_i} W] + (J_h'(W + \Phi) - J_h'(W))\, [\partial_{p_i} W] + \sum_{j=0}^{n} \lambda_j \langle Z_j, \partial_{p_i} \Phi \rangle_h \\
&= \partial_{p_i} J_h(W) + (J_h'(W + \Phi) - J_h'(W))\, [\partial_{p_i} W] - \sum_{j=0}^{n} \lambda_j \langle \partial_{p_i} Z_j, \Phi \rangle_h,
\end{aligned} \tag{35}$$

where, for the last line, we have used that $\langle (Z_i)_{\delta,\xi}, \Phi_{h,\tilde{u}_0,\delta,\xi} \rangle_h = 0$ for all (δ, ξ) since $\Phi_{h,\tilde{u}_0,\delta,\xi} \in K_{\delta,\xi}^{\perp}$. We estimate separately the two last terms in the right-hand side of (35). As regards the first of these two term, we have

$$\begin{aligned}
&(J_h'(W + \Phi) - J_h'(W))\, [\partial_{p_i} W] \\
&= \int_M (\nabla \Phi \nabla \partial_{p_i} W + h \Phi \partial_{p_i} W) - \int_M ((W + \Phi)_+^{2^\star - 1} - W_+^{2^\star - 1}) \partial_{p_i} W \, dv_g \\
&= \int_M \Phi((\Delta_g + h) \partial_{p_i} W - (2^\star - 1) W_+^{2^\star - 1} \partial_{p_i} W) \, dv_g \\
&\quad - \int_M ((W + \Phi)_+^{2^\star - 1} - W_+^{2^\star - 1} - (2^\star - 1) W_+^{2^\star - 1} \Phi) \partial_{p_i} W \, dv_g.
\end{aligned} \tag{36}$$

With the definition (28), Hölder's and Sobolev's inequalities, we obtain

$$\begin{aligned}
&\int_M \Phi((\Delta_g + h) \partial_{p_i} W - (2^\star - 1) W_+^{2^\star - 1} \partial_{p_i} W) \, dv_g \\
&\qquad = \int_M \Phi \partial_{p_i} R \, dv_g = \mathrm{O}(\|\Phi\|_{2^\star} \|\partial_{p_i} R\|_{\frac{2n}{n+2}}) = \mathrm{O}(\|\Phi\|_{H_1^2} \|\partial_{p_i} R\|_{\frac{2n}{n+2}}).
\end{aligned} \tag{37}$$

In the sequel, we will need the following lemma:

Lemma 4.1. *We have*

$$U_{\delta,\xi}(x) + \delta |\partial_p U_{\delta,\xi}(x)| \le C \tilde{U}_{\delta,\xi}(x) \tag{38}$$

for all $(\delta, \xi) \in (0, \epsilon_0) \times U_0$ and $x \in M$.

Proof of Lemma 38. Most of the proof is easy computations. The only delicate point is to prove that $|\partial_\xi d_{g_\xi}(x, \xi)^2| \le C d_{g_\xi}(x, \xi)$ for all $x \in M$ and $\xi \in U_0$. We define $F(x, \xi) := d_{g_\xi}(x, \xi)^2$ and $G(\xi, y) := \exp_\xi^{g_\xi}(y)$. Proving the desired inequality amounts to proving that $(\partial_\xi F(x, \xi))_{|\xi = x} = 0$ for all $x \in M$. Note that $F(G(\xi, y), \xi) = |y|^2$ for small $y \in \mathbb{R}^n$. Differentiating this equality with respect to ξ yields a relation between $\partial_x F$ and $\partial_\xi F$, and the requested inequality follows. □

End of proof of Proposition 4.1. Using Lemma 4.1, the assumption (32) on $B_{h,\delta,\xi}$, and that $\partial_{p_i}\tilde{u}_0 = 0$, we obtain

$$\left|\int_M ((W+\Phi)_+^{2^\star-1} - W_+^{2^\star-1} - (2^\star-1)W_+^{2^\star-2}\Phi)\partial_{p_i} W\,dv_g\right| \\ \leq C\delta^{-1}\int_M |(W+\Phi)_+^{2^\star-1} - W_+^{2^\star-1} - (2^\star-1)W_+^{2^\star-2}\Phi|\tilde{U}\,dv_g.$$

We split the integral in two. First

$$\int_{|W|\leq 2|\Phi|} |(W+\Phi)_+^{2^\star-1} - W_+^{2^\star-1} - (2^\star-1)W_+^{2^\star-2}\Phi|\tilde{U}\,dv_g \\ \leq C\int_M |\Phi|^{2^\star-1}\tilde{U}\,dv_g \leq C\|\Phi\|_{2^\star}^{2^\star-1}\|\tilde{U}\|_{2^\star} \leq C\|\Phi\|_{H_1^2}^{2^\star-1}.$$

As regards the other part, looking carefully at the signs of the different terms, we obtain

$$\int_{|\Phi|\leq|W|/2} |(W+\Phi)_+^{2^\star-1} - W_+^{2^\star-1} - (2^\star-1)W_+^{2^\star-2}\Phi|\tilde{U}\,dv_g \\ = \int_{|\Phi|\leq|W|/2} |W|^{2^\star-1}\left|\left(1+\frac{\Phi}{W}\right)^{2^\star-1} - 1 - (2^\star-1)\frac{\Phi}{W}\right|\tilde{U}\,dv_g \\ \leq C\int_{|\Phi|\leq|W|/2} |W|^{2^\star-1}\left(\frac{\Phi}{W}\right)^2\tilde{U}\,dv_g = C\int_{|\Phi|\leq|W|/2}|W|^{2^\star-3}\Phi^2\tilde{U}\,dv_g.$$

In case $n\leq 6$, that is $2^\star\geq 3$, we obtain

$$\int_{|\Phi|\leq|W|/2}|W|^{2^\star-3}|\Phi|^2\tilde{U}\,dv_g \leq \int_M \tilde{U}^{2^\star-2}|\Phi|^2 dv_g \leq C\|\tilde{U}\|_{2^\star}^{2^\star-2}\|\Phi\|_{2^\star}^2 \leq C\|\Phi\|_{H_1^2}^2.$$

In case $n\geq 7$, that is $2^\star<3$, arguing as above, we obtain

$$\int_{|\Phi|\leq|W|/2}|W|^{2^\star-3}\Phi^2\tilde{U}\,dv_g \leq C\int_M|\Phi|^{2^\star-1}\tilde{U}\,dv_g \leq C\|\Phi\|_{H_1^2}^{2^\star-1}.$$

Plugging these estimates together yields

$$\left|\int_M ((W+\Phi)_+^{2^\star-1} - W_+^{2^\star-1} - (2^\star-1)W_+^{2^\star-2}\Phi)\partial_{p_i} W\,dv_g\right| \\ \leq C\delta^{-1}(\|\Phi\|_{H_1^2}^2 + \mathbf{1}_{n\geq 7}\|\Phi\|_{H_1^2}^{2^\star-1}). \tag{39}$$

As regards the last term in the right-hand side of (35), arguing as in the proof of Lemma 4.1, we obtain $\|\partial_{p_i}Z_j\|_{H_1^2}\leq C/\delta$ for all $i,j=0,\dots,n$. Therefore, we obtain

$$\left|\sum_{j=0}^n \lambda_j\langle\partial_{p_i}Z_j,\Phi\rangle_h\right| \leq C\delta^{-1}\Lambda\|\Phi\|_{H_1^2},\ \text{ where } \Lambda := \sum_{j=0}^n|\lambda_j|. \tag{40}$$

It follows from (34) that

$$J_h'(W+\Phi)[Z_i] = \sum_{j=0}^n \lambda_j\langle Z_i, Z_j\rangle_h$$

for all $i = 0, \dots, n$. Since $\langle Z_i, Z_j \rangle_h \to 0$ if $i \neq j$ and $\to 1$ if $i = j$ as $\delta \to 0$ and uniformly with respect to $\xi \in U_0$, we obtain

$$\Lambda \leq C \sum_{i=0}^{n} |J_h'(W + \Phi)[Z_i]| \,.$$

For every $i = 0, \dots, n$, using that $\langle \Phi, Z_i \rangle_h = 0$ and $\|W\|_{2^\star} + \|Z_i\|_{2^\star} \leq C$, we obtain

$$\begin{aligned} |J_h'(W + \Phi)[Z_i]| &\leq |J_h'(W)[Z_i]| + \left| \langle \Phi, Z_i \rangle_h - \int_M ((W + \Phi)_+^{2^\star-1} - W_+^{2^\star-1}) Z_i \, dv_g \right| \\ &\leq \left| \int_M R Z_i \, dv_g \right| + C \int_M (|W|^{2^\star-2} |\Phi| + |\Phi|^{2^\star-1}) |Z_i| \, dv_g \\ &\leq C \|R\|_{\frac{2n}{n+2}} + C(\|\Phi\|_{2^\star} + \|\Phi\|_{2^\star}^{2^\star-1}) \leq C \|R\|_{\frac{2n}{n+2}} + C \|\Phi\|_{2^\star}. \end{aligned}$$

Therefore,

$$\Lambda \leq C \|R\|_{\frac{2n}{n+2}} + C \|\Phi\|_{2^\star}. \tag{41}$$

Plugging (36), (37), (39), (40) and (41) into (35) yields (33). This proves Proposition 4.1. ☐

5. Energy and remainder estimates: the case $n \geq 6$ and $u_0 \equiv \tilde{u}_0 \equiv 0$

In this section, we consider the case where $n \geq 6$ and $u_0 \equiv \tilde{u}_0 \equiv 0$. In this case, we set $B_{h,\delta,\xi} \equiv 0$. Then $W_{h,\tilde{u}_0,\delta,\xi} = W_{\delta,\xi} \equiv U_{\delta,\xi}$ and the assumptions of Proposition 4.1 are satisfied. We prove the following estimates for $R = R_{\delta,\xi}$:

Proposition 5.1. *Assume that $n \geq 6$ and $u_0 \equiv \tilde{u}_0 \equiv 0$. Then*

$$\|R\|_{\frac{2n}{n+2}} + \delta \|\partial_p R\|_{\frac{2n}{n+2}} \leq C \begin{cases} \delta^2 + D_{h,\xi} \delta^2 \left(\ln(1/\delta)\right)^{2/3} & \text{if } n = 6 \\ \delta^{\frac{n-2}{2}} + D_{h,\xi} \delta^2 & \text{if } 7 \leq n \leq 9 \\ \delta^4 \left(\ln(1/\delta)\right)^{3/5} + D_{h,\xi} \delta^2 & \text{if } n = 10 \\ \delta^4 + D_{h,\xi} \delta^2 & \text{if } n \geq 11, \end{cases} \tag{42}$$

where

$$D_{h,\xi} := \|h - h_0\|_\infty + d_g(\xi, \xi_0)^2. \tag{43}$$

Proof of Proposition 5.1. Let $L_g := \Delta_g + c_n \operatorname{Scal}_g$ be the conformal Laplacian. For a metric $g' = w^{4/(n-2)} g$ conformal to g ($w \in C^\infty(M)$ is positive), the conformal invariance law gives that

$$L_{g'} \phi = w^{-(2^\star-1)} L_g(w\phi) \text{ for all } \phi \in C^\infty(M). \tag{44}$$

Therefore, we have

$$\begin{aligned} R = (\Delta_g + h) U - U^{2^\star-1} &= L_g U - U^{2^\star-1} + \varphi_h U \\ &= \Lambda_\xi^{2^\star-1} (L_{g_\xi}(\Lambda_\xi^{-1} U) - (\Lambda_\xi^{-1} U)^{2^\star-1}) + \varphi_h U \\ &= \Lambda_\xi^{2^\star-1} (\Delta_{g_\xi}(\Lambda_\xi^{-1} U) - (\Lambda_\xi^{-1} U)^{2^\star-1}) + \hat{h}_\xi U, \end{aligned}$$

where φ_h is as in (5) and

$$\hat{h}_\xi := \varphi_h + c_n \Lambda_\xi^{2^\star-2} \operatorname{Scal}_{g_\xi} . \tag{45}$$

Via the exponential chart, using the radial symmetry of $U_{\delta,0}:\mathbb{R}^n\to\mathbb{R}$, we obtain that around 0,

(46)
$$\Delta_{g_\xi}(\Lambda_\xi^{-1}U)-(\Lambda_\xi^{-1}U)^{2^\star-1}=\Delta_{\text{Eucl}}U_{\delta,0}+\frac{\partial_r\sqrt{|g_\xi|}}{\sqrt{|g_\xi|}}\partial_rU_{\delta,0}-U_{\delta,0}^{2^\star-1}=\frac{\partial_r\sqrt{|g_\xi|}}{\sqrt{|g_\xi|}}\partial_rU_{\delta,0}.$$

It then follows from (23) that

$$R(x)=\hat{h}_\xi(x)U(x)+\delta^{\frac{n-2}{2}}\Theta_{\delta,\xi}(x),\ \text{ where } |\Theta_{\delta,\xi}(x)|+|\partial_p\Theta_{\delta,\xi}(x)|\le C \tag{47}$$

for all $(\delta,\xi)\in(0,\infty)\times U_0$ and $x\in M$. Note that these estimates are a consequence of (46) when x is close to ξ, and they are straightforward when x is far from ξ. Using Lemma 4.1, we then obtain

$$|R(x)|+\delta|\partial_\delta R(x)|\le C\delta^{\frac{n-2}{2}}+C|\hat{h}_\xi(x)|\tilde{U}_{\delta,\xi}(x) \tag{48}$$

and

$$\delta|\partial_\xi R(x)|\le C\delta^{\frac{n-2}{2}}+C|\tilde{h}_\xi(x)|\tilde{U}_{\delta,\xi}(x)+C\delta|\partial_p\tilde{h}_\xi(x)|\tilde{U}_{\delta,\xi}(x). \tag{49}$$

Since (6) and (22) hold, we have

$$|\hat{h}_\xi(x)|\le CD_{h,\xi}+Cd_{g_\xi}(x,\xi)^2 \text{ and } |\partial_\xi\hat{h}_\xi(x)|\le Cd_{g_\xi}(x,\xi). \tag{50}$$

It is a straightforward computation that for every $\alpha>0$ and $p\ge1$, we have

$$\|d_{g_\xi}(\cdot,\xi)^\alpha\tilde{U}_{\delta,\xi}\|_p\le C\begin{cases}\delta^{\frac{n-2}{2}} & \text{if } n>(n-2-\alpha)p\\ \delta^{\frac{n-2}{2}}\left(\ln(1/\delta)\right)^{1/p} & \text{if } n=(n-2-\alpha)p\\ \delta^{\frac{n}{p}+\alpha-\frac{n-2}{2}} & \text{if } n<(n-2-\alpha)p.\end{cases} \tag{51}$$

Plugging together (48), (49), (50) and (51), long but painless computations yield (42). This ends the proof of Proposition 5.1. □

Since $n\ge6$, that is $2^\star-1\le2$, we have $\|\Phi\|_{H_1^2}^2=\text{O}(\|\Phi\|_{H_1^2}^{2^\star-1})$. Plugging together (30), (27), (33) and (42), we obtain

$$J_h(W+\Phi)=J_h(W)+\text{O}\begin{pmatrix}\delta^4+D_{h,\xi}^2\delta^4\left(\ln(1/\delta)\right)^{4/3} & \text{if } n=6\\ \delta^{n-2}+D_{h,\xi}^2\delta^4 & \text{if } 7\le n\le9\\ \delta^8\left(\ln(1/\delta)\right)^{6/5}+D_{h,\xi}^2\delta^4 & \text{if } n=10\\ \delta^8+D_{h,\xi}^2\delta^4 & \text{if } n\ge11\end{pmatrix} \tag{52}$$

and

$$\begin{aligned}\partial_{p_i}J_h(W+\Phi)&=\partial_{p_i}J_h(W)\\ &\quad+\text{O}\left(\delta^{-1}\right)\begin{cases}\delta^4+D_{h,\xi}^2\delta^4\left(\ln(1/\delta)\right)^{4/3} & \text{if } n=6\\ (\delta^{\frac{n-2}{2}}+D_{h,\xi}\delta^2)^{2^\star-1} & \text{if } 7\le n\le9\\ (\delta^4\left(\ln(1/\delta)\right)^{3/5}+D_{h,\xi}\delta^2)^{2^\star-1} & \text{if } n=10\\ (\delta^4+D_{h,\xi}\delta^2)^{2^\star-1} & \text{if } n\ge11\end{cases}\end{aligned} \tag{53}$$

for all $i=0,\dots,n$. We now estimate $J_h(W+\Phi)$:

Proposition 5.2. *Assume that $n \geq 6$ and $u_0 \equiv \tilde{u}_0 \equiv 0$. Then*

$$J_h(W+\Phi) = \frac{1}{n}\int_{\mathbb{R}^n} U_{1,0}^{2^\star}\,dx + \frac{1}{2}\varphi_h(\xi)\delta^2\int_{\mathbb{R}^n} U_{1,0}^2\,dx \tag{54}$$
$$-\frac{1}{4n}\begin{cases} 24^2\omega_5 K_{h_0}(\xi_0)\delta^4\ln(1/\delta) + \mathrm{O}(\delta^4(\mathrm{o}(\ln(1/\delta) + D_{h,\xi}^2(\ln(1/\delta))^{4/3})) & \text{if } n=6 \\ K_{h_0}(\xi_0)\delta^4\int_{\mathbb{R}^n}|x|^2U_{1,0}^2\,dx + \mathrm{o}(\delta^4) & \text{if } n\geq 7\end{cases}$$

as $\delta \to 0$, $\xi \to \xi_0$ and $h \to h_0$ in $C^2(M)$, where $K_{h_0}(\xi_0)$ is as in (7).

Proof of Proposition 5.2. Integrating by parts, we obtain

$$\begin{aligned} J_h(U) &= \frac{1}{2}\int_M [(\Delta_g + h)U]\,U\,dv_g - \frac{1}{2^\star}\int_M U^{2^\star}\,dv_g \\ &= \frac{1}{2}\int_M [(\Delta_g+h)U - U^{2^\star-1}]U\,dv_g + \frac{1}{n}\int_M U^{2^\star}\,dv_g. \end{aligned} \tag{55}$$

It follows from (47) that

$$\int_M (\Delta_g U + hU - U^{2^\star-1})U\,dv_g = \int_M \hat{h}_\xi U^2 dv_g + \mathrm{O}(\delta^{n-2}). \tag{56}$$

Using the volume estimate (23), we obtain

$$\begin{aligned} \int_M U^{2^\star}\,dv_g = \int_M (\Lambda_\xi^{-1}U)^{2^\star}\,dv_{g_\xi} &= \int_{B_{r_0}(0)} U_{\delta,0}^{2^\star}(1+\mathrm{O}(|x|^N)\,dx + \mathrm{O}(\delta^n) \\ &= \int_{\mathbb{R}^n} U_{1,0}^{2^\star}\,dx + \mathrm{O}(\delta^n). \end{aligned} \tag{57}$$

Plugging (56) and (57) into (55), we obtain

$$J_h(U) = \frac{1}{2}\int_M \hat{h}_\xi U^2 dv_g + \frac{1}{n}\int_{\mathbb{R}^n} U_{1,0}^{2^\star}\,dx + \mathrm{O}(\delta^{n-2}).$$

With the change of metric, the definition of the bubble (24) and the property of the volume (23), we obtain

$$\int_M \hat{h}_\xi U^2 dv_g = \int_{B_{r_0}(\xi)} \hat{h}_\xi U^2 dv_g + \mathrm{O}(\delta^{n-2}) = \int_{B_{r_0}(0)} A_{h,\xi}U_{\delta,0}^2\,dx + \mathrm{O}(\delta^{n-2}), \tag{58}$$

where

$$A_{h,\xi}(x) := (\hat{h}_\xi \Lambda_\xi^{2-2^\star})(\exp_\xi^{g_\xi}(x)). \tag{59}$$

Using the radial symmetry of $U_{\delta,0}$ and since $h_0 \in C^2(M)$, we obtain

$$\begin{aligned} \int_{B_{r_0}(0)} A_{h,\xi}U_{\delta,0}^2\,dx &= \int_{B_{r_0}(0)} (A_{h,\xi}(0) + \partial_{x_\alpha}A_{h,\xi}(0)x^\alpha \\ &\quad + \frac{1}{2}\partial_{x_\alpha}\partial_{x_\beta}A_{h,\xi}(0)x^\alpha x^\beta + \mathrm{O}(\|h-h_0\|_{C^2}\,|x|^2 + \epsilon_{h_0,\xi}(x)|x|^2))U_{\delta,0}^2\,dx \\ &= A_{h,\xi}(0)\int_{B_{r_0}(0)} U_{\delta,0}^2\,dx - \frac{1}{2n}\Delta_{\mathrm{Eucl}}A_{h,\xi}(0)\int_{B_{r_0}(0)} |x|^2U_{\delta,0}^2\,dx \\ &\quad + \mathrm{O}\left(\int_{B_{r_0}(0)} (\|h-h_0\|_{C^2} + \epsilon_{h_0,\xi}(x))|x|^2U_{\delta,0}^2\,dx\right) + \mathrm{O}(\delta^{n-2}), \end{aligned} \tag{60}$$

where $\epsilon_{h_0,\xi}(x)\to 0$ as $x\to 0$, uniformly in $\xi\in U_0$. With a change of variable and Lebesgue convergence theorem, we obtain

$$\int_{B_{r_0}(0)} U_{\delta,0}^2\,dx=\delta^2\int_{\mathbb{R}^n} U_{1,0}^2\,dx+\mathrm{O}(\delta^{n-2}),\tag{61}$$

$$\int_{B_{r_0}(0)} |x|^2U_{\delta,0}^2\,dx=\begin{cases}24^2\omega_5\delta^4\ln(1/\delta)+\mathrm{O}(\delta^4) & \text{if } n=6\\ \delta^4\displaystyle\int_{\mathbb{R}^n}|x|^2U_{1,0}^2\,dx+\mathrm{O}(\delta^5) & \text{if } n\geq 7,\end{cases}\tag{62}$$

and

$$\int_{B_{r_0}(0)} \epsilon_{h_0,\xi}(x)|x|^2U_{\delta,0}^2\,dx=\mathrm{o}\begin{pmatrix}\delta^4\ln(1/\delta) & \text{if } n=6\\ \delta^4 & \text{if } n\geq 7\end{pmatrix}.\tag{63}$$

Furthermore, we have $A_{h,\xi}(0)=\varphi_h(\xi)$ and

$$\begin{aligned}\Delta_{\mathrm{Eucl}}A_{h,\xi}(0)&=\Delta_{g_\xi}(\hat{h}_\xi\Lambda_\xi^{2-2^\star})(\xi)=L_{g_\xi}(\varphi_h\Lambda_\xi^{2-2^\star})(\xi)+c_n\Delta_{g_\xi}\mathrm{Scal}_{g_\xi}(\xi)\\&=L_g(\varphi_h\Lambda_\xi^{3-2^\star})(\xi)+\frac{c_n}{6}|\mathrm{Weyl}_g(\xi)|_g^2\\&=L_g(\varphi_{h_0}\Lambda_\xi^{3-2^\star})(\xi)+\frac{c_n}{6}|\mathrm{Weyl}_g(\xi)|_g^2+\mathrm{O}(\|h-h_0\|_{C^2})\\&=K_{h_0}(\xi_0)+\mathrm{O}(\epsilon_{h_0}(\xi)+\|h-h_0\|_{C^2}),\end{aligned}\tag{64}$$

where $\epsilon_{h_0}(\xi)\to 0$ as $\xi\to\xi_0$. Therefore, plugging together these identities yields

$$\begin{aligned}J_h(U)=\frac{1}{n}\int_{\mathbb{R}^n}U_{1,0}^{2^\star}\,dx+\frac{1}{2}\varphi_h(\xi)\delta^2\int_{\mathbb{R}^n}U_{1,0}^2\,dx\\-\frac{1}{4n}\begin{cases}24^2\omega_5K_{h_0}(\xi_0)\delta^4\ln(1/\delta)+\mathrm{o}\left(\delta^4\ln(1/\delta)\right) & \text{if } n=6\\ K_{h_0}(\xi_0)\delta^4\displaystyle\int_{\mathbb{R}^n}|x|^2U_{1,0}^2\,dx+\mathrm{o}(\delta^4) & \text{if } n\geq 7.\end{cases}\end{aligned}\tag{65}$$

Plugging together (52) and (65), we obtain (54). This ends the proof of Proposition 5.2. ☐

We now estimate the derivatives of $J_h(W+\Phi)$:

Proposition 5.3. *Assume that $n\geq 6$ and $u_0\equiv\tilde{u}_0\equiv 0$. Then*

$$\begin{aligned}\partial_\delta J_h(W+\Phi)=\varphi_h(\xi)\delta\int_{\mathbb{R}^n}U_{1,0}^2dx\\-\begin{cases}24^2\omega_5K_{h_0}(\xi_0)\delta^3\ln(1/\delta)+\mathrm{o}\left(\delta^3\ln(1/\delta)\right)+\mathrm{O}(D_{h,\xi}^2\delta^3(\ln(1/\delta))^{4/3}) & \text{if } n=6\\ K_{h_0}(\xi_0)\delta^3\displaystyle\int_{\mathbb{R}^n}|x|^2U_{1,0}^2\,dx+\mathrm{o}(\delta^3)+\mathrm{O}(D_{h,\xi}^{2^\star-1}\delta^{\frac{n+6}{n-2}}) & \text{if } n\geq 7\end{cases}\end{aligned}\tag{66}$$

and

$$\begin{aligned}\partial_{\xi_i}J_h(W+\Phi)=\frac{1}{2}\partial_{\xi_i}\varphi_h(\xi)\delta^2\int_{\mathbb{R}^n}U_{1,0}^2\,dx\\+\mathrm{O}\begin{pmatrix}\mathrm{o}(\delta^3\ln(1/\delta))+\mathrm{O}(D_{h,\xi}^2\delta^3\left(\ln(1/\delta)\right)^{4/3}) & \text{if } n=6\\ \mathrm{o}(\delta^3)+\mathrm{O}(D_{h,\xi}^{2^\star-1}\delta^{\frac{n+6}{n-2}}) & \text{if } n\geq 7\end{pmatrix}\end{aligned}\tag{67}$$

for all $i=1,\dots,n$, as $\delta\to 0$, $\xi\to\xi_0$ and $h\to h_0$ in $C^2(M)$.

Proof of Proposition 5.3. We fix $i\in\{0,\dots,n\}$. Using (47) and (38) and arguing as in (58), we obtain

$$\begin{aligned}(68)\quad \partial_{p_i}J_h(U)=J_h'(U)[\partial_{p_i}U]&=\int_M(\Delta_g U+hU-U^{2^\star-1})\partial_{p_i}U\,dv_g\\&=\int_M R\partial_{p_i}U\,dv_g=\int_M\hat{h}_\xi U\partial_{p_i}U\,dv_g+\mathrm{O}\left(\delta^{\frac{n-2}{2}}\int_M|\partial_{p_i}U|\,dv_g\right)\\&=\int_M\hat{h}_\xi U\partial_{p_i}U\,dv_g+\mathrm{O}\left(\delta^{-1}\delta^{n-2}\right)\\&=\int_{B_{r_0}(0)}A_{h,\xi}U_{\delta,\xi}(\Lambda_\xi^{-1}\partial_{p_i}U)(\exp_\xi^{g_\xi}(x))\,dx+\mathrm{O}\left(\delta^{-1}\delta^{n-2}\right)\end{aligned}$$

As in (60), we write

$$\begin{aligned}(69)\quad A_{h,\xi}(x)=A_{h,\xi}(0)+\partial_{x_\alpha}A_{h,\xi}(0)x^\alpha+\frac{1}{2}\partial_{x_j}\partial_{x_k}A_{h,\xi}(0)x^jx^k\\+\mathrm{O}(\epsilon_{h_0,\xi}(x)|x|^2+\|h-h_0\|_{C^2}|x|^2)\end{aligned}$$

for all $x\in B_{r_0}(0)$, where $\epsilon_{h_0,\xi}(x)\to 0$ as $x\to 0$, uniformly in $\xi\in U_0$. With (38), (62) and (63), we obtain

$$\begin{aligned}(70)\quad &\left|\int_{B_{r_0}(0)}(\epsilon_{h_0,\xi}(x)+\|h-h_0\|_{C^2})|x|^2U_{\delta,0}(\Lambda_\xi^{-1}\partial_{p_i}U)(\exp_\xi^{g_\xi}(x))\,dx\right|\\&\le C\delta^{-1}\int_{B_{r_0}(0)}(\epsilon_{h_0,\xi}(x)+\|h-h_0\|_{C^2})|x|^2\tilde{U}_{\delta,0}^2\,dx=\mathrm{o}(\delta^{-1})\begin{cases}\delta^4\ln(1/\delta)&\text{if } n=6\\\delta^4&\text{if } n\ge 7.\end{cases}\end{aligned}$$

We write

$$\begin{aligned}\int_{B_{r_0}(0)}&A_{h,\xi}U_{\delta,0}(\Lambda_\xi^{-1}\partial_{p_i}U)(\exp_\xi^{g_\xi}(x))\,dx\\&=\int_{B_{r_0}(0)}A_{h,\xi}U_{\delta,0}\partial_{p_i}(\Lambda_\xi^{-1}U)(\exp_\xi^{g_\xi}(x))\,dx\\&\quad-\int_{B_{r_0}(0)}A_{h,\xi}U_{\delta,0}^2(\Lambda_\xi^{-1}\partial_{p_i}\Lambda_\xi^{-1})(\exp_\xi^{g_\xi}(x))\,dx.\end{aligned}$$

Since $\nabla\Lambda_\xi(\xi)=0$, we obtain

$$\begin{aligned}\int_{B_{r_0}(0)}&A_{h,\xi}U_{\delta,0}^2(\Lambda_\xi^{-1}\partial_{p_i}\Lambda_\xi^{-1})(\exp_\xi^{g_\xi}(x))\,dx\\&=\mathrm{O}\left(A_{\delta,\xi}(0)\int_{B_{r_0}(0)}|x|U_{\delta,0}^2\,dx\right)+\mathrm{O}\left(\int_{B_{r_0}(0)}|x|^2U_{\delta,0}^2\,dx\right).\end{aligned}$$

With the definition (59) of $A_{h,\xi}$ and the assumption (6) on h_0, it follows that

$$\begin{aligned}\int_{B_{r_0}(0)}&A_{h,\xi}U_{\delta,0}^2(\Lambda_\xi^{-1}\partial_{p_i}\Lambda_\xi^{-1})(\exp_\xi^{g_\xi}(x))\,dx\\&=\mathrm{O}\left(\delta^{-1}\delta^4\left(D_{h,\xi}+\begin{Bmatrix}\delta\ln(1/\delta)&\text{if } n=6\\\delta&\text{if } n\ge 7\end{Bmatrix}\right)\right).\end{aligned}$$

This estimate, the Taylor expansion (69) and the estimate (70) yield

$$
\begin{aligned}
(71)\quad \int_{B_{r_0}(0)} & A_{h,\xi}\Lambda_\xi^{-1}U_{\delta,0}(\Lambda_\xi^{-1}\partial_{p_i}U)(\exp_\xi^{g_\xi}(x))\,dx \\
&= A_{h,\xi}(0)\int_{B_{r_0}(0)} U_{\delta,0}\partial_{p_i}(\Lambda_\xi^{-1}U)(\exp_\xi^{g_\xi}(x))\,dx \\
&+ \partial_{x_\alpha}A_{h,\xi}(0)\int_{B_{r_0}(0)} x^\alpha U_{\delta,0}\partial_{p_i}(\Lambda_\xi^{-1}U)(\exp_\xi^{g_\xi}(x))\,dx \\
&+ \frac{1}{2}\partial_{x_j}\partial_{x_k}A_{h,\xi}(0)\int_{B_{r_0}(0)} x^j x^k U_{\delta,0}\partial_{p_i}(\Lambda_\xi^{-1}U)(\exp_\xi^{g_\xi}(x))\,dx \\
&+ \mathrm{o}(\delta^{-1})\begin{cases}\delta^4\ln(1/\delta) & \text{if } n=6\\ \delta^4 & \text{if } n\ge 7.\end{cases}
\end{aligned}
$$

We first deal with the case $i=0$, that is $\partial_{p_i}=\partial_{p_0}=\partial_\delta$. For every homogeneous polynomial Q on $\mathbb{R}^n$, it follows from (14) and (18) that

$$
\begin{aligned}
\int_{B_{r_0}(0)} & QU_{\delta,0}\partial_\delta(\Lambda_\xi^{-1}U)(\exp_\xi^{g_\xi}(x))\,dx \\
&= \int_{B_{r_0}(0)} Q\delta^{-1}\delta^{-\frac{n-2}{2}}U_{1,0}\left(x/\delta\right)\delta^{-\frac{n-2}{2}}Z_0\left(x/\delta\right)\,dx.
\end{aligned}
$$

The explicit expressions (13) and (15) of U and Z_0 and their radial symmetry then yield

$$
\int_{B_{r_0}(0)} U_{\delta,0}\partial_\delta(\Lambda_\xi^{-1}U)(\exp_\xi^{g_\xi}(x))\,dx = \delta^{-1}\delta^2\int_{\mathbb{R}^n}U_{1,0}Z_0\,dx + \mathrm{O}(\delta^{-1}\delta^{n-2}) \text{ for } n\ge 6,
$$

$$
\int_{B_{r_0}(0)} x^j U_{\delta,0}\partial_\delta(\Lambda_\xi^{-1}U)(\exp_\xi^{g_\xi}(x))\,dx = 0 \text{ for } n\ge 6,
$$

and

$$
\begin{aligned}
\int_{B_{r_0}(0)} & x^j x^k U_{\delta,0}\partial_\delta(\Lambda_\xi^{-1}U)(\exp_\xi^{g_\xi}(x))\,dx \\
&= \frac{\epsilon_{jk}}{n}\delta^{-1}\delta^4\begin{cases} c_6'\ln(1/\delta)+\mathrm{O}(\delta^{-1}\delta^4) & \text{if } n=6\\ \int_{\mathbb{R}^n}|x|^2U_{1,0}Z_0\,dx + \mathrm{O}(\delta^{-1}\delta^{n-2}) & \text{if } n\ge 7,\end{cases}
\end{aligned}
$$

where ϵ_{jk} is the Kronecker symbol and $c_6'>0$ is a constant that will be discussed later. Putting these estimates in (68), and (71), we obtain

$$
\begin{aligned}
\partial_\delta J_h(U) &= A_{h,\xi}(0)\delta^{-1}\delta^2\int_{\mathbb{R}^n}U_{1,0}Z_0\,dx \\
&- \frac{1}{2n}\delta^{-1}\delta^4\begin{cases} c_6'\Delta_{Eucl}A_{h,\xi}(0)\ln(1/\delta)+\mathrm{o}(\ln(1/\delta)) & \text{if } n=6\\ \Delta_{Eucl}A_{h,\xi}(0)\int_{\mathbb{R}^n}|x|^2U_{1,0}Z_0\,dx+\mathrm{o}(1) & \text{if } n\ge 7.\end{cases}
\end{aligned}
$$

For every $\delta>0$, we have

$$
\int_{\mathbb{R}^n}U_{\delta,0}^2\,dx = \delta^2\int_{\mathbb{R}^n}U_{1,0}^2\,dx \text{ for } n\ge 5
$$

and

$$\int_{\mathbb{R}^n} |x|^2 U_{\delta,0}^2\, dx = \delta^4 \int_{\mathbb{R}^n} |x|^2 U_{1,0}^2\, dx \text{ for } n \geq 7.$$

Differentiating these equalities with respect to δ at $\delta = 1$, we obtain

$$\int_{\mathbb{R}^n} U_{1,0} Z_0\, dx = \int_{\mathbb{R}^n} U_{1,0}^2 \text{ for } n \geq 5$$

and

$$\int_{\mathbb{R}^n} |x|^2 U_{1,0} Z_0\, dx = 2 \int_{\mathbb{R}^n} |x|^2 U_{1,0}^2 \text{ for } n \geq 7.$$

Therefore, with the computation (64) and the definition (7), we obtain

$$\begin{aligned} \partial_\delta J_h(U) = {} & \varphi_h(\xi)\delta^{-1}\delta^2 \int_{\mathbb{R}^n} U_{1,0}^2\, dx \\ & - \frac{1}{n}\delta^{-1}\delta^4 \begin{cases} c_6' K_{h_0}(\xi_0)\ln(1/\delta) + \mathrm{o}(\ln(1/\delta)) & \text{if } n = 6 \\ K_{h_0}(\xi_0) \int_{\mathbb{R}^n} |x|^2 U_{1,0}^2\, dx + \mathrm{o}(1) & \text{if } n \geq 7. \end{cases} \end{aligned} \tag{72}$$

Differentiating (65), we obtain $c_6'/2 = 24^2\omega_5$. Therefore, with (53), we obtain (66). We now deal with the case where $i \geq 1$, that is $\partial_{p_i} = \partial_{\xi_i}$. We first claim that

$$[\partial_{\xi_i}(\Lambda_\xi^{-1} U_{\delta,\xi})](\xi, \exp_\xi^{g_\xi}(x)) + [\partial_{x_i}(\Lambda_\xi^{-1} U_{\delta,\xi})](\xi, \exp_\xi^{g_\xi}(x)) = \mathrm{O}\left(\frac{\delta^{\frac{n-2}{2}}|x|^3}{(\delta^2+|x|^2)^{n/2}}\right), \tag{73}$$

where the differential for ξ is taken via the exponential chart. Before proving this claim, let us remark that it is trivial in the Euclidean context. Indeed, for every $\xi, x \in \mathbb{R}^n$ and $\delta > 0$, with the notation (14), we have

$$\partial_{\xi_i} U_{\delta,\xi}(x) = \partial_{\xi_i}(\delta^{-\frac{n-2}{2}} U(\delta^{-1}(x-\xi))) = -\partial_{x_i} U_{\delta,\xi}(x).$$

We now prove the claim (73). We fix $\xi \in U_0$. We define the path $\xi(t) := \exp_\xi^{g_\xi}(t\vec{e}_i)$ for small $t \in \mathbb{R}$, where $\vec{e}_i$ is the i-th vector in the canonical basis of $\mathbb{R}^n$. With (31), we obtain

$$\begin{aligned} [\partial_{x_i}(\Lambda_\xi^{-1} U_{\delta,\xi})](\xi, \exp_\xi^{g_\xi}(x)) &= \frac{d}{dt}\tilde{U}_{\delta,\xi}(\exp_\xi^{g_\xi}(x + t\vec{e}_i))_{|t=0} \\ &= -\frac{n-2}{2}\frac{\delta^{\frac{n-2}{2}}}{(\delta^2+|x|^2)^{n/2}} \cdot 2x_i \end{aligned} \tag{74}$$

and

$$\begin{aligned} [\partial_{\xi_i}(\Lambda_\xi^{-1} U_{\delta,\xi})](\xi, \exp_\xi^{g_\xi}(x)) &= \frac{d}{dt}\tilde{U}_{\delta,\xi(t)}(\exp_\xi^{g_\xi}(x))_{|t=0} \\ &= -\frac{n-2}{2}\frac{\delta^{\frac{n-2}{2}}}{(\delta^2+|x|^2)^{n/2}} \cdot \frac{d}{dt} d_{g_{\xi(t)}}^2(\xi(t), \exp_\xi^{g_\xi}(x)). \end{aligned} \tag{75}$$

It follows from Esposito–Pistoia–Vétois [12, Lemma A.2] that

$$\frac{d}{dt} d_{g_{\xi(t)}}^2(\xi(t), \exp_\xi^{g_\xi}(x)) + 2x_i = \mathrm{O}(|x|^3) \text{ as } x \to 0. \tag{76}$$

Putting together all these estimates yields (73). This proves the claim. With the definition (14), we obtain

$$\int_{B_{r_0}(0)} U_{\delta,0}\frac{\delta^{\frac{n-2}{2}}|x|^3}{(\delta^2+|x|^2)^{n/2}}\,dx = \mathrm{O}(\delta^3) \text{ for } n\geq 6,$$

$$\int_{B_{r_0}(0)} |x|U_{\delta,0}\frac{\delta^{\frac{n-2}{2}}|x|^3}{(\delta^2+|x|^2)^{n/2}}\,dx = \mathrm{O}\begin{pmatrix}\delta^4\ln(1/\delta) & \text{if } n=6\\ \delta^4 & \text{if } n\geq 7\end{pmatrix}$$

and

$$\int_{B_{r_0}(0)} |x|^2U_{\delta,0}\frac{\delta^{\frac{n-2}{2}}|x|^3}{(\delta^2+|x|^2)^{n/2}}\,dx = \mathrm{O}\begin{pmatrix}\delta^4 & \text{if } n=6\\ \delta^5\ln(1/\delta) & \text{if } n=7\\ \delta^5 & \text{if } n\geq 8.\end{pmatrix}.$$

Noting that $[\partial_{x_i}(\Lambda_\xi^{-1}U_{\delta,\xi})](\xi,\exp_\xi^{g_\xi}(x)) = \partial_{x_i}U_{\delta,0}$, we obtain by symmetry that

$$\int_{B_{r_0}(0)} \Lambda_\xi^{-1}U_{\delta,0}\partial_{x_i}(\Lambda_\xi^{-1}U)(\exp_\xi^{g_\xi}(x))\,dx = \int_{B_{r_0}(0)} U_{\delta,0}\partial_{x_i}U_{\delta,0}\,dx = 0$$

and similarly,

$$\int_{B_{r_0}(0)} x^jx^kU_{\delta,0}\partial_{x_i}(\Lambda_\xi^{-1}U)(\exp_\xi^{g_\xi}(x))\,dx = 0.$$

Integrating by parts, straightforward estimates yield

$$\begin{aligned}\int_{B_{r_0}(0)} x^jU_{\delta,0}\partial_{x_i}(\Lambda_\xi^{-1}U)(\exp_\xi^{g_\xi}(x))\,dx &= \int_{B_{r_0}(0)} x^jU_{\delta,0}\partial_{x_i}U_{\delta,0}\,dx\\ &= \frac{1}{2}\int_{B_{r_0}(0)} x^j\partial_{x_i}(U_{\delta,0}^2)\,dx = -\frac{\epsilon_{ij}}{2}\int_{B_{r_0}(0)} U_{\delta,0}^2\,dx + \frac{1}{2}\int_{\partial B_{r_0}(0)} x^j\vec{\nu}_iU_{\delta,0}^2\,d\sigma\\ &= -\frac{\epsilon_{ij}}{2}\delta^2\int_{\mathbb{R}^n} U_{1,0}^2\,dx + \mathrm{O}(\delta^{n-2}) \text{ for } n\geq 6,\end{aligned}$$

where $\vec{\nu} := (\vec{\nu}_1,\dots,\vec{\nu}_n)$ is the outward unit normal vector and $d\sigma$ is the volume element of $\partial B_{r_0}(0)$. Since $A_{h,\xi}(0) = \mathrm{O}(D_{h,\xi})$, plugging these estimates together with (68) and (71), we obtain

$$\partial_{\xi_i}J_h(U) = \frac{1}{2}\partial_{\xi_i}\varphi_h(\xi)\delta^2\int_{\mathbb{R}^n} U_{1,0}^2\,dx + \mathrm{o}\begin{pmatrix}\delta^3\ln(1/\delta) & \text{if } n=6\\ \delta^3 & \text{if } n\geq 7\end{pmatrix}. \tag{77}$$

With (53), we then obtain (67). This ends the proof of Proposition 5.3. □

Theorem 1.4 for $n\geq 6$ will be proved in Section 10.

6. Energy and remainder estimates: the case $n\geq 7$ and $u_0,\tilde{u}_0>0$

In this section, we assume that $u_0,\tilde{u}_0>0$ and $n\geq 7$, that is $2^\star-1<2$. As in the previous case, we set $B_{h,\delta,\xi}\equiv 0$, so that $W_{h,\tilde{u}_0,\delta,\xi} = W_{\tilde{u}_0,\delta,\xi}\equiv \tilde{u}_0+U_{\delta,\xi}$ and the assumptions of Proposition 4.1 are satisfied. We prove the following estimates for $R=R_{\delta,\xi}$:

Proposition 6.1. *Assume that $n\geq 7$ and $u_0,\tilde{u}_0>0$. Then*

$$\|R\|_{\frac{2n}{n+2}} \leq C\|\Delta_g\tilde{u}_0+h\tilde{u}_0-\tilde{u}_0^{2^\star-1}\|_\infty + C(D_{h,\xi}+\delta^2+\delta^{\frac{n-6}{2}})\delta^2 \text{ and } \|\partial_pR\|_{\frac{2n}{n+2}}\leq C\delta, \tag{78}$$

where $D_{h,\xi}$ is as in (43).

Proof of Proposition 6.1. We have

$$R = (\Delta_g \tilde{u}_0 + h\tilde{u}_0 - \tilde{u}_0^{2^\star-1}) + R^0 - ((\tilde{u}_0 + U)^{2^\star-1} - \tilde{u}_0^{2^\star-1} - U^{2^\star-1}), \tag{79}$$

where

$$R^0 := \Delta_g U + hU - U^{2^\star-1}.$$

Concerning the derivatives, given $i \in \{0, \dots, n\}$, we have

$$\begin{aligned} \partial_{p_i} R &= \Delta_g \partial_{p_i} U + h\partial_{p_i} U - (2^\star - 1)(\tilde{u}_0 + U)^{2^\star-2}\partial_{p_i} U \\ &= \partial_{p_i} R^0 - (2^\star - 1)((\tilde{u}_0 + U)^{2^\star-2} - U^{2^\star-2})\partial_{p_i} U. \end{aligned} \tag{80}$$

A straightforward estimate yields

$$|(\tilde{u}_0 + U)^{2^\star-1} - \tilde{u}_0^{2^\star-1} - U^{2^\star-1}| \leq C\mathbf{1}_{U\leq\tilde{u}_0}\tilde{u}_0^{2^\star-2}U + C\mathbf{1}_{\tilde{u}_0\leq U}\tilde{u}_0 U^{2^\star-2}.$$

With the expression (24), we obtain

$$\{U(x) \leq \tilde{u}_0(x) \Rightarrow d_{g_\xi}(x, \xi) \geq c_1\sqrt{\delta}\} \text{ and } \{U(x) \geq \tilde{u}_0(x) \Rightarrow d_{g_\xi}(x, \xi) \leq c_2\sqrt{\delta}\}$$

for all $x \in M$, where $c_1, c_2 > 0$ depend only on n, (M, g) and $A > 0$ such that $1/A < \tilde{u}_0 < A$ Therefore, with $r := d_{g_\xi}(x, \xi)$,

$$|(\tilde{u}_0 + U)^{2^\star-1} - \tilde{u}_0^{2^\star-1} - U^{2^\star-1}| \leq C\mathbf{1}_{r\geq c_1\sqrt{\delta}}U + C\mathbf{1}_{r\leq c_2\sqrt{\delta}}U^{2^\star-2}.$$

Since $U \leq C\delta^{\frac{n-2}{2}}(\delta^2 + r^2)^{1-n/2}$, we then obtain

$$\|(\tilde{u}_0 + U)^{2^\star-1} - \tilde{u}_0^{2^\star-1} - U^{2^\star-1}\|_{\frac{2n}{n+2}} \leq C\delta^{\frac{n+2}{4}} \text{ for } n \geq 7. \tag{81}$$

Since $0 < 2^\star - 2 < 1$, we have

$$|(\tilde{u}_0 + U)^{2^\star-2} - U^{2^\star-2}| \leq C.$$

Therefore, with (31) and (38), we obtain

$$\|((\tilde{u}_0 + U)^{2^\star-2} - U^{2^\star-2})\partial_{p_i} U\|_{\frac{2n}{n+2}} \leq C\delta^{-1}\|U\|_{\frac{2n}{n+2}} \leq C\delta^{-1}\delta^2 \text{ for } n \geq 7. \tag{82}$$

Merging the estimates (42), (79), (80), (81) and (82), we obtain (78). This ends the proof of Proposition 6.1. □

Plugging (78) and (78) together with (30), (27) and (33), we obtain

$$J_h(W + \Phi) = J_h(W) + \mathrm{O}(\|\Delta_g\tilde{u}_0 + h\tilde{u}_0 - \tilde{u}_0^{2^\star-1}\|_\infty^2 + D_{h,\xi}^2\delta^4 + \delta^8 + \delta^{n-2}) \tag{83}$$

and

$$\begin{aligned} &\partial_{p_i} J_h(W + \Phi) = \partial_{p_i} J_h(W) + \mathrm{O}(\|\Delta_g\tilde{u}_0 + h\tilde{u}_0 - \tilde{u}_0^{2^\star-1}\|_{\frac{2n}{n+2}}\delta \\ &+\|\Delta_g\tilde{u}_0+h\tilde{u}_0-\tilde{u}_0^{2^\star-1}\|_\infty^{2^\star-1}\delta^{-1}+(D_{h,\xi}+\delta^2+\delta^{\frac{n-6}{2}})^{2^\star-1}\delta^{\frac{n+6}{n-2}}+D_{h,\xi}\delta^3+\delta^5+\delta^{n/2}) \end{aligned} \tag{84}$$

for all $i = 0, \dots, n$. We now estimate $J_h(W + \Phi)$:

Proposition 6.2. *Assume that $n \geq 7$ and $u_0, \tilde{u}_0 > 0$. Then*

$$\begin{aligned} J_h(W + \Phi) &= J_h(\tilde{u}_0) + \frac{1}{n}\int_{\mathbb{R}^n} U_{1,0}^{2^\star}\,dx + \frac{1}{2}\varphi_h(\xi)\delta^2\int_{\mathbb{R}^n} U_{1,0}^2\,dx \\ &\quad - \frac{1}{4n}K_{h_0}(\xi_0)\delta^4\int_{\mathbb{R}^n}|x|^2U_{1,0}^2\,dx + \mathrm{o}(\delta^4) - u_0(\xi_0)\delta^{\frac{n-2}{2}}\int_{\mathbb{R}^n} U_{1,0}^{2^\star-1}dx \\ &+\mathrm{O}(\|\Delta_g\tilde{u}_0+h\tilde{u}_0-\tilde{u}_0^{2^\star-1}\|_\infty^2+\delta^{\frac{n-2}{2}}(\|\Delta_g\tilde{u}_0+h\tilde{u}_0-\tilde{u}_0^{2^\star-1}\|_\infty+\|\tilde{u}_0-u_0\|_\infty+\mathrm{o}(1))) \end{aligned} \tag{85}$$

as $\delta \to 0$, $\xi \to \xi_0$ and $h \to h_0$ in $C^2(M)$.

Proof of Proposition 6.2. We first write

$$J_h(\tilde{u}_0+U)=J_h(\tilde{u}_0)+J_h(U)-\int_M \tilde{u}_0 U^{2^\star-1}dv_g+\int_M(\Delta_g\tilde{u}_0+h\tilde{u}_0-\tilde{u}_0^{2^\star-1})U\,dv_g$$
$$-\frac{1}{2^\star}\int_M((\tilde{u}_0+U)^{2^\star}-\tilde{u}_0^{2^\star}-U^{2^\star}-2^\star\tilde{u}_0^{2^\star-1}U-2^\star\tilde{u}_0U^{2^\star-1})\,dv_g.$$

We fix $0<\theta<\frac{2}{n-2}<2^\star-2$. There exists $C>0$ such that

$$|(\tilde{u}_0+U)^{2^\star}-\tilde{u}_0^{2^\star}-U^{2^\star}-2^\star\tilde{u}_0^{2^\star-1}U-2^\star\tilde{u}_0U^{2^\star-1}|$$
$$\le C\mathbf{1}_{\tilde{u}_0\le U}\tilde{u}_0^{1+\theta}U^{2^\star-1-\theta}+C\mathbf{1}_{U\le\tilde{u}_0}\tilde{u}_0^{2^\star-1-\theta}U^{1+\theta}.$$

Using the definition (24) and arguing as in the proof of (81), we obtain

$$\left|\int_M((\tilde{u}_0+U)^{2^\star}-\tilde{u}_0^{2^\star}-U^{2^\star}-2^\star\tilde{u}_0^{2^\star-1}U-2^\star\tilde{u}_0U^{2^\star-1})\,dv_g\right|\le C\delta^{\frac{n-2}{2}+\frac{n-2}{2}\theta}.$$

Furthermore, we obtain

$$\left|\int_M(\Delta_g\tilde{u}_0+h\tilde{u}_0-\tilde{u}_0^{2^\star-1})U\,dv_g\right|\le C\|\Delta_g\tilde{u}_0+h\tilde{u}_0-\tilde{u}_0^{2^\star-1})\|_\infty\int_M U dv_g$$
$$\le C\|\Delta_g\tilde{u}_0+h\tilde{u}_0-\tilde{u}_0^{2^\star-1})\|_\infty\delta^{\frac{n-2}{2}}.$$

Using (24), that $\Lambda_\xi(x)=1+\mathrm{O}(d_g(x,\xi)^2)$ for all $x\in M$ and that $U_{\delta,0}$ is radially symmetrical, we obtain

$$\int_M\tilde{u}_0U^{2^\star-1}dv_g=\int_{B_{r_0}(0)}\tilde{u}_0(\exp_\xi^{g_\xi}(x))(1+\mathrm{O}(|x|^2))U_{\delta,0}^{2^\star-1}dx+\mathrm{O}(\delta^{\frac{n-2}{2}(2^\star-1)})$$
$$=\int_{B_{r_0}(0)}(\tilde{u}_0(\xi)+x^\alpha\partial_{x_\alpha}\tilde{u}_0(\exp_\xi^{g_\xi}(\xi))+\mathrm{O}(|x|^2))U_{\delta,0}^{2^\star-1}dx+\mathrm{O}(\delta^{\frac{n+2}{2}})$$
$$=\tilde{u}_0(\xi)\int_{B_{r_0}(0)}U_{\delta,0}^{2^\star-1}dx+\mathrm{O}\left(\int_{B_{r_0}(0)}|x|^2U_{\delta,0}^{2^\star-1}dx\right)+\mathrm{O}(\delta^{\frac{n+2}{2}})$$
$$=\tilde{u}_0(\xi)\delta^{\frac{n-2}{2}}\int_{B_{r_0/\delta}(0)}U_{1,0}^{2^\star-1}dx+\mathrm{O}\left(\delta^{\frac{n+2}{2}}\int_{B_{r_0/\delta}(0)}|x|^2U_{1,0}^{2^\star-1}dx\right)+\mathrm{O}(\delta^{\frac{n+2}{2}}).$$

Since $U_{1,0}\le C(1+|x|^2)^{1-n/2}$, we obtain

$$\int_{B_{r_0/\delta}(0)}U_{1,0}^{2^\star-1}dx=\int_{\mathbb{R}^n}U_{1,0}^{2^\star-1}dx+\mathrm{O}(\delta^2)$$

and

$$\int_{B_{r_0/\delta}(0)}|x|^2U_{1,0}^{2^\star-1}dx=\mathrm{O}\left(\ln(1/\delta)\right)\ \text{for } n\ge 7.$$

Therefore, plugging all these estimates together yields

$$\int_M\tilde{u}_0U^{2^\star-1}dv_g=\tilde{u}_0(\xi)\delta^{\frac{n-2}{2}}\int_{\mathbb{R}^n}U_{1,0}^{2^\star-1}dx+\mathrm{O}(\delta^{\frac{n+2}{2}}\ln(1/\delta)).$$

Consequently, we obtain that for every $0<\theta<\frac{2}{n-2}$,

$$J_h(\tilde{u}_0+U)=J_h(\tilde{u}_0)+J_h(U)-\tilde{u}_0(\xi)\delta^{\frac{n-2}{2}}\int_{\mathbb{R}^n}U_{1,0}^{2^\star-1}dx$$
$$+\int_M(\Delta_g\tilde{u}_0+h\tilde{u}_0-\tilde{u}_0^{2^\star-1})U\,dv_g+\mathrm{O}(\delta^{\frac{n-2}{2}+\frac{n-2}{2}\theta}).$$

Now, with the expansion (65), we obtain that for $n\geq 7$,

$$(86)\quad J_h(\tilde{u}_0+U)=J_h(\tilde{u}_0)+\frac{1}{n}\int_{\mathbb{R}^n}U_{1,0}^{2^\star}\,dx+\frac{1}{2}\varphi_h(\xi)\delta^2\int_{\mathbb{R}^n}U_{1,0}^2\,dx$$
$$-\frac{1}{4n}K_{h_0}(\xi_0)\delta^4\int_{\mathbb{R}^n}|x|^2U_{1,0}^2\,dx+\mathrm{o}(\delta^4)-u_0(\xi_0)\delta^{\frac{n-2}{2}}\int_{\mathbb{R}^n}U_{1,0}^{2^\star-1}dx$$
$$+\mathrm{O}(\delta^{\frac{n-2}{2}}(\|\Delta_g\tilde{u}_0+h\tilde{u}_0-\tilde{u}_0^{2^\star-1}\|_\infty+\|\tilde{u}_0-u_0\|_\infty+d_g(\xi,\xi_0)+\delta^{\frac{n-2}{2}\theta})).$$

Plugging together (83) and (86), we then obtain (85). This ends the proof of Proposition 6.2. ☐

We now estimate the derivatives of $J_h(W+\Phi)$:

Proposition 6.3. *Assume that $n\geq 7$ and $u_0,\tilde{u}_0>0$. Then*

$$(87)\quad \partial_\delta J_h(W+\Phi)=\varphi_h(\xi)\delta\int_{\mathbb{R}^n}U_{1,0}^2\,dx-\frac{1}{n}K_{h_0}(\xi_0)\delta^3\int_{\mathbb{R}^n}|x|^2U_{1,0}^2\,dx+\mathrm{o}(\delta^3)$$
$$-\frac{n-2}{2}u_0(\xi_0)\delta^{\frac{n-4}{2}}\int_{\mathbb{R}^n}U_{1,0}^{2^\star-1}dx+\mathrm{O}(\delta^{\frac{n-4}{2}}(\|\tilde{u}_0-u_0\|_\infty+\mathrm{o}(1))$$
$$+\|\Delta_g\tilde{u}_0+h\tilde{u}_0-\tilde{u}_0^{2^\star-1}\|_\infty\delta+\|\Delta_g\tilde{u}_0+h\tilde{u}_0-\tilde{u}_0^{2^\star-1}\|_\infty^{2^\star-1}\delta^{-1}+D_{h,\xi}^{2^\star-1}\delta^{\frac{n+6}{n-2}})$$

and

$$(88)\quad \partial_{\xi_i}J_h(W+\Phi)=\frac{1}{2}\partial_{\xi_i}\varphi_h(\xi)\delta^2\int_{\mathbb{R}^n}U_{1,0}^2\,dx+\mathrm{o}(\delta^3)$$
$$+\mathrm{O}(\delta^{\frac{n-4}{2}}(\|\tilde{u}_0-u_0\|_\infty+\mathrm{o}(1))+\|\Delta_g\tilde{u}_0+h\tilde{u}_0-\tilde{u}_0^{2^\star-1}\|_\infty\delta$$
$$+\|\Delta_g\tilde{u}_0+h\tilde{u}_0-\tilde{u}_0^{2^\star-1}\|_\infty^{2^\star-1}\delta^{-1}+D_{h,\xi}^{2^\star-1}\delta^{\frac{n+6}{n-2}})$$

for all $i=1,\dots,n$, as $\delta\to 0$, $\xi\to\xi_0$ and $h\to h_0$ in $C^2(M)$.

Proof of Proposition 6.3. We fix $i\in\{0,\dots,n\}$. We have

$$\partial_{p_i}J_h(\tilde{u}_0+U)=\int_M(\Delta_g\tilde{u}_0+h\tilde{u}_0-\tilde{u}_0^{2^\star-1})\partial_{p_i}U\,dv_g-(2^\star-1)\int_M\tilde{u}_0U^{2^\star-2}\partial_{p_i}U\,dv_g$$
$$+\partial_{p_i}J_h(U)-\int_M((\tilde{u}_0+U)^{2^\star-1}-U^{2^\star-1}-(2^\star-1)\tilde{u}_0U^{2^\star-2})\partial_{p_i}U\,dv_g.$$

There exists $C>0$ such that

$$|(\tilde{u}_0+U)^{2^\star-1}-\tilde{u}_0^{2^\star-1}-U^{2^\star-1}-(2^\star-1)\tilde{u}_0U^{2^\star-2}|$$
$$\leq C\mathbf{1}_{\tilde{u}_0\leq U}\tilde{u}_0^{2^\star-1}+C\mathbf{1}_{U\leq\tilde{u}_0}U^{2^\star-1}.$$

Since $|\partial_{p_i}U|\leq C\tilde{U}/\delta$ (see (38)), arguing as in the proof of (81), we obtain

$$\left|\int_M((\tilde{u}_0+U)^{2^\star-1}-U^{2^\star-1}-(2^\star-1)\tilde{u}_0U^{2^\star-2})\partial_{p_i}U\,dv_g\right|\leq C\delta^{\frac{n-2}{2}}.$$

Furthermore, we obtain

$$\left|\int_M (\Delta_g \tilde{u}_0 + h\tilde{u}_0 - \tilde{u}_0^{2^\star-1})\partial_{p_i} U\, dv_g\right| \leq C\|\Delta_g \tilde{u}_0 + h\tilde{u}_0 - \tilde{u}_0^{2^\star-1})\|_\infty \delta^{-1}\int_M \tilde{U} dv_g$$
$$\leq C\|\Delta_g \tilde{u}_0 + h\tilde{u}_0 - \tilde{u}_0^{2^\star-1})\|_\infty \delta^{-1}\delta^{\frac{n-2}{2}}.$$

Independently, using again (38), straightforward computations yield

$$\int_M \tilde{u}_0 U^{2^\star-2}\partial_{p_i} U\, dv_g = \int_M (u_0(\xi_0) + \mathrm{O}(\|\tilde{u}_0 - u_0\|_\infty + d_g(.,\xi_0))\, U^{2^\star-2}\partial_{p_i} U\, dv_g$$
$$= u_0(\xi_0)\int_M U^{2^\star-2}\partial_{p_i} U\, dv_g$$
$$+ \mathrm{O}\left(\delta^{-1}\int_M (\|\tilde{u}_0 - u_0\|_\infty + d_g(\xi,\xi_0) + d_g(.,\xi))\tilde{U}^{2^\star-1} dv_g\right)$$
$$= u_0(\xi_0)\int_M U^{2^\star-2}\partial_{p_i} U\, dv_g + \mathrm{O}(\delta^{-1}\delta^{\frac{n-2}{2}}(\|\tilde{u}_0 - u_0\|_\infty + d_g(\xi,\xi_0) + \delta)).$$

Arguing as in the proof of (71), we obtain

$$\int_M U^{2^\star-2}\partial_{p_i} U\, dv_g = \int_{B_{r_0}(0)} (\Lambda_\xi U)^{2^\star-2}\partial_{p_i}(\Lambda_\xi^{-1} U)(\exp_\xi^{g_\xi}(x))\, dx$$
$$+ \mathrm{O}\left(\delta^{-1}\int_{B_{r_0}(0)} |x|\tilde{U}^{2^\star-1} dx\right)$$
$$= \int_{B_{r_0}(0)} (\Lambda_\xi U)^{2^\star-2}\partial_{p_i}(\Lambda_\xi^{-1} U)(\exp_\xi^{g_\xi}(x))\, dx + \mathrm{O}(\delta^{\frac{n-2}{2}}).$$

We first deal with the case where $i=0$, that is $\partial_{p_i} = \partial_{p_0} = \partial_\delta$. With (18), we obtain

$$\int_{B_{r_0}(0)} (\Lambda_\xi U)^{2^\star-2}\partial_\delta(\Lambda_\xi^{-1} U)(\exp_\xi^{g_\xi}(x))\, dx = \int_{B_{r_0}(0)} U_{\delta,0}^{2^\star-2}\partial_\delta U_{\delta,0}\, dx$$
$$= \int_{B_{r_0}(0)} U_{\delta,0}^{2^\star-2}\partial_\delta U_{\delta,0} dx = \delta^{-1}\int_{B_{r_0}(0)} (\delta^{-\frac{n-2}{2}} U_{1,0}(\delta^{-1}x))^{2^\star-2}\delta^{-\frac{n-2}{2}} Z_0(\delta^{-1}x)\, dx$$
$$= \delta^{-1}\delta^{\frac{n-2}{2}}\int_{B_{r_0/\delta}(0)} U_{1,0}^{2^\star-2} Z_0\, dx.$$

Since $Z_0 \leq CU_{1,0}$, an asymptotic estimate yields

$$\int_{B_{r_0}(0)} (\Lambda_\xi U)^{2^\star-2}\partial_\delta(\Lambda_\xi^{-1} U)(\exp_\xi^{g_\xi}(x))\, dx = \delta^{-1}\delta^{\frac{n-2}{2}}\int_{\mathbb{R}^n} U_{1,0}^{2^\star-2} Z_0\, dx + \mathrm{O}(\delta^{\frac{n}{2}}).$$

Note that for every $\delta > 0$, we have

$$\int_{\mathbb{R}^n} U_{\delta,0}^{2^\star-1} dx = \delta^{\frac{n-2}{2}}\int_{\mathbb{R}^n} U_{1,0}^{2^\star-1} dx.$$

Differentiating this equality with respect to δ at 1, we obtain

$$(2^\star - 1)\int_{\mathbb{R}^n} U_{1,0}^{2^\star-2} Z_0\, dx = \frac{n-2}{2}\int_{\mathbb{R}^n} U_{1,0}^{2^\star-1} dx.$$

Therefore, we obtain

$$(2^\star - 1)\int_M U^{2^\star-2}\partial_\delta U\, dv_g = \frac{n-2}{2}\delta^{-1}\delta^{\frac{n-2}{2}}\int_{\mathbb{R}^n} U_{1,0}^{2^\star-1} dx + \mathrm{O}(\delta^{\frac{n-2}{2}}).$$

We now deal with the case $i\geq 1$, that is $\partial_{p_i}=\partial_{\xi_i}$. It follows from (75) and (76) that

$$\begin{aligned}&\int_{B_{r_0}(0)}(\Lambda_\xi U)^{2^\star-2}\partial_{\xi_i}(\Lambda_\xi^{-1}U)(\exp_\xi^{g_\xi}(x))\,dx\\&=\int_{B_{r_0}(0)}U_{\delta,0}^{2^\star-2}\left(-\frac{n-2}{2}\right)\frac{\delta^{\frac{n-2}{2}}}{(\delta^2+|x|^2)^{n/2}}\left(-2x_i+\mathrm{O}(|x|^3)\right)dx\\&=\mathrm{O}\left(\int_{B_{r_0}(0)}U_{\delta,0}^{2^\star-2}\frac{U_{\delta,0}}{\delta^2+|x|^2}|x|^3\,dx\right)=\mathrm{O}\left(\int_{B_{r_0}(0)}|x|U_{\delta,0}^{2^\star-1}dx\right)=\mathrm{O}(\delta^{\frac{n-2}{2}}).\end{aligned}$$

Putting these results together yields

$$\begin{aligned}&\partial_{\xi_i}J_h(\tilde u_0+U)=\partial_{\xi_i}J_h(U)-\frac{n-2}{2}\epsilon_{i,0}u_0(\xi_0)\delta^{-1}\delta^{\frac{n-2}{2}}\int_{\mathbb{R}^n}U_{1,0}^{2^\star-1}dx\\&+\mathrm{O}(\delta^{-1}\delta^{\frac{n-2}{2}}(\|\Delta_g\tilde u_0+h\tilde u_0-\tilde u_0^{2^\star-1}\|_\infty+\|\tilde u_0-u_0\|_\infty+d_g(\xi,\xi_0)+\delta))\end{aligned}$$

for all $i=0,\dots,n$. Using the estimates (72) and (77) for the derivatives of $J_h(U_{\delta,\xi})$, we obtain

$$\begin{aligned}\partial_\delta J_h(\tilde u_0+U)&=\varphi_h(\xi)\delta^{-1}\delta^2\int_{\mathbb{R}^n}U_{1,0}^2\,dx-4K_{h_0}(\xi_0)\delta^3\int_{\mathbb{R}^n}|x|^2U_{1,0}^2\,dx\\&\quad-\frac{n-2}{2}u_0(\xi_0)\delta^{\frac{n-4}{2}}\int_{\mathbb{R}^n}U_{1,0}^{2^\star-1}dx+\mathrm{o}(\delta^3)\\&\quad+\mathrm{O}(\delta^{\frac{n-4}{2}}(\|\Delta_g\tilde u_0+h\tilde u_0-\tilde u_0^{2^\star-1}\|_\infty+\|\tilde u_0-u_0\|_\infty+d_g(\xi,\xi_0)+\delta))\end{aligned}$$

and

$$\begin{aligned}\partial_{\xi_i}J_h(\tilde u_0+U)&=\frac{1}{2}\partial_{\xi_i}\varphi_h(\xi)\delta^2\int_{\mathbb{R}^n}U_{1,0}^2\,dx+\mathrm{o}(\delta^4)\\&\quad+\mathrm{O}(\delta^{\frac{n-4}{2}}(\|\Delta_g\tilde u_0+h\tilde u_0-\tilde u_0^{2^\star-1}\|_\infty+\|\tilde u_0-u_0\|_\infty+d_g(\xi,\xi_0)+\delta)).\end{aligned}$$

With (84), we then obtain (87) and (88). This ends the proof of Proposition 6.3. □

Theorem 1.5 for $n\geq 7$ will be proved in Section 11.

7. Energy and remainder estimates: the case $n=6$ and $u_0,\tilde u_0>0$

In this section, we assume that $u_0,\tilde u_0>0$ and $n=6$, that is $2^\star-1=2$. Here again, we set $B_{h,\delta,\xi}\equiv 0$, so that $W_{h,\tilde u_0,\delta,\xi}=W_{\tilde u_0,\delta,\xi}\equiv\tilde u_0+U_{\delta,\xi}$ and the assumptions of Proposition 4.1 are satisfied. The remark underlying this section is that

$$\Delta_g(u_0+U)+h(u_0+U)-(u_0+U)^2=\Delta_gU+(h-2u_0)U-U^2.$$

Therefore, to obtain a good approximation of the blowing-up solution, we will subtract a perturbation of $2u_0$ to the potential. We first estimate $R=R_{\delta,\xi}$:

Proposition 7.1. *Assume that $n=6$ and $u_0,\tilde u_0>0$. Then*

$$\|R\|_{3/2}+\delta\|\partial_pR\|_{3/2}\leq C\|\Delta_g\tilde u_0+h\tilde u_0-\tilde u_0^2\|_\infty+C\delta^2(1+\overline{D}_{h,\xi}\,(\ln(1/\delta))^{2/3}),\tag{89}$$

where

$$\overline{D}_{h,\xi}:=\|\bar h-\bar h_0\|_\infty+d_g(\xi,\xi_0)^2.\tag{90}$$

Proof of Proposition 7.1. Since $2^\star - 1 = 2$, we have

$$R = \Delta_g(\tilde{u}_0 + U) + h(\tilde{u}_0 + U) - (\tilde{u}_0 + U)^2$$
$$= \Delta_g \tilde{u}_0 + h\tilde{u}_0 - \tilde{u}_0^2 + \Delta_g U + (h - 2\tilde{u}_0)U - U^2$$

and

$$\partial_{p_i} R = \partial_{p_i} \left(\Delta_g U + (h - 2\tilde{u}_0)U - U^2\right)$$

for all $i = 0, \dots, n$. For convenience, we write

$$\bar{h} := h - 2\tilde{u}_0 \text{ and } \bar{h}_0 := h_0 - 2u_0.$$

The estimate (89) then follows from (42). This ends the proof of Proposition 7.1. □

We now estimate the derivatives of $J_h(W + \Phi)$:

Proposition 7.2. *Assume that $n = 6$ and $u_0, \tilde{u}_0 > 0$. Then*

$$J_h(W+\Phi) = J_h(\tilde{u}_0) + \frac{1}{n}\int_{\mathbb{R}^n} U_{1,0}^{2^\star}\, dx + \frac{1}{2}\varphi_{h,\tilde{u}_0}(\xi)\delta^2 \int_{\mathbb{R}^n} U_{1,0}^2\, dx$$
$$- 24\omega_5 K_{h_0,u_0}(\xi_0)\delta^4 \ln(1/\delta) + \mathrm{O}(\|\Delta_g \tilde{u}_0 + h\tilde{u}_0 - \tilde{u}_0^2\|_\infty^2 + \|\Delta_g \tilde{u}_0 + h\tilde{u}_0 - \tilde{u}_0^2\|_\infty \delta^2)$$
$$+ \mathrm{O}(\delta^4 \ln(1/\delta)(\mathrm{o}(1) + \overline{D}_{h,\xi}^2 (\ln(1/\delta))^{1/3})), \tag{91}$$

$$\partial_\delta J_h(W+\Phi) = \varphi_{h,\tilde{u}_0}(\xi)\delta \int_{\mathbb{R}^n} U_{1,0}^2\, dx - 96\omega_5 K_{h_0,u_0}(\xi_0)\delta^3 \ln(1/\delta)$$
$$+ \mathrm{O}(\|\Delta_g \tilde{u}_0 + h\tilde{u}_0 - \tilde{u}_0^2\|_\infty \delta + \|\Delta_g \tilde{u}_0 + h\tilde{u}_0 - \tilde{u}_0^2\|_\infty^2 \delta^{-1})$$
$$+ \mathrm{O}(\delta^3 \ln(1/\delta)(\mathrm{o}(1) + \overline{D}_{h,\xi}^2 (\ln(1/\delta))^{1/3})) \tag{92}$$

and

$$\partial_{\xi_i} J_h(W+\Phi) = \frac{1}{2}\partial_{\xi_i}\varphi_{h,\tilde{u}_0}(\xi)\delta^2 \int_{\mathbb{R}^n} U_{1,0}^2\, dx$$
$$+ \mathrm{O}(\|\Delta_g \tilde{u}_0 + h\tilde{u}_0 - \tilde{u}_0^2\|_\infty \delta + \|\Delta_g \tilde{u}_0 + h\tilde{u}_0 - \tilde{u}_0^2\|_\infty^2 \delta^{-1})$$
$$+ \mathrm{O}(\delta^3 \ln(1/\delta)(\mathrm{o}(1) + \overline{D}_{h,\xi}^2 (\ln(1/\delta))^{1/3})) \tag{93}$$

for all $i = 1, \dots, n$, as $\delta \to 0$, $\xi \to \xi_0$ and $h \to h_0$ in $C^2(M)$, where $\varphi_{h,\tilde{u}_0}$, $K_{h_0,u_0}(\xi_0)$ and $\overline{D}_{h,\xi}$ are as in (5), (9) and (90).

Proof of Proposition 7.2. As one checks, since $n = 6$ and $2^\star = 3$, we have

$$J_h(\tilde{u}_0 + U) = J_h(\tilde{u}_0) + J_{\bar{h}}(U) + \int_M \left(\Delta_g \tilde{u}_0 + h\tilde{u}_0 - \tilde{u}_0^2\right) U\, dv_g$$

and

$$\partial_{p_i} J_h(\tilde{u}_0 + U) = \partial_{p_i} J_{\bar{h}}(U) + \int_M \left(\Delta_g \tilde{u}_0 + h\tilde{u}_0 - \tilde{u}_0^2\right) \partial_{p_i} U\, dv_g$$

for all $i = 0, \dots, n$. Using the definition (24) and since $|\partial_{p_i} U| \leq C\tilde{U}/\delta$, we obtain

$$\left|\int_M (\Delta_g \tilde{u}_0 + h\tilde{u}_0 - \tilde{u}_0^2) U\, dv_g\right| \leq C\|\Delta_g \tilde{u}_0 + h\tilde{u}_0 - \tilde{u}_0^2)\|_\infty \int_M U dv_g$$
$$\leq C\|\Delta_g \tilde{u}_0 + h\tilde{u}_0 - \tilde{u}_0^2)\|_\infty \delta^2.$$

and

$$\left|\int_M (\Delta_g \tilde{u}_0 + h\tilde{u}_0 - \tilde{u}_0^2)\partial_{p_i} U \, dv_g\right| \le C\|\Delta_g \tilde{u}_0 + h\tilde{u}_0 - \tilde{u}_0^2)\|_\infty \delta^{-1} \int_M \tilde{U} dv_g$$
$$\le C\|\Delta_g \tilde{u}_0 + h\tilde{u}_0 - \tilde{u}_0^2)\|_\infty \delta^{-1}\delta^2.$$

Putting these estimates together with (6), (30), (33), (89), (65), (72) and (77), we obtain (91), (92) and (93). This ends the proof of Proposition 7.2. ☐

Theorem 1.5 for $n = 6$ will be proved in Section 11.

8. Setting and definition of the mass in dimensions $n = 3, 4, 5$

In this section, we assume that $n \le 5$. Our first lemma is a simple computation:

Lemma 8.1. *There exist two functions $(\xi, x) \mapsto f_i(\xi, x)$, $i = 1, 2$, defined and smooth on $M \times M$ such that for every function $f : \mathbb{R}^n \to \mathbb{R}$ that is radially symmetrical, we have*

$$(\Delta_g + h)(\chi(r)\Lambda_\xi(x) f(r)) = \Lambda_\xi(x)^{2^\star - 1}\chi \Delta_{\mathrm{Eucl}}(f(r)) + f_1(\xi, x) f'(r) + f_2(\xi, x) f(r)$$
$$+ \hat{h}_\xi \chi(x)\Lambda_\xi(x) f(r)$$

for all $x \in M \setminus \{\xi\}$, where $r := d_{g_\xi}(x, \xi)$ and $\hat{h}_\xi$ is as in (45). Furthermore, $f_i(\xi, x) = 0$ when $d_g(x, \xi) \ge r_0$ and there exists $C_N > 0$ such that

$$|f_1(\xi, x)(x)| \le C_N d_g(x, \xi)^{N-1} \text{ and } |f_2(\xi, x)| \le C_N d_g(x, \xi)^{N-2} \text{ for all } x, \xi \in M.$$

The proof of Lemma 8.1 follows the computations in (47). We leave the details to the reader.

We define

$$\Gamma_\xi(x) := \frac{\chi(d_{g_\xi}(x, \xi))\Lambda_\xi(x)}{(n-2)\omega_{n-1} d_{g_\xi}(x, \xi)^{n-2}}$$

for all $x \in M \setminus \{\xi\}$. It follows from Lemma 8.1 and the definition (14) that

$$\Delta_g U_{\delta,\xi} + h U_{\delta,\xi} = U_{\delta,\xi}^{2^\star - 1} + F_\delta(\xi, x)\delta^{\frac{n-2}{2}} + \hat{h}_\xi U_{\delta,\xi} \tag{94}$$

and

$$(\Delta_g + h)\Gamma_\xi = \frac{F_0(\xi, x)}{k_n} + \hat{h}_\xi \Gamma_\xi, \text{ where } k_n := (n-2)\omega_{n-1}\sqrt{n(n-2)}^{\frac{n-2}{2}}$$

and $(t, \xi, x) \to F_t(\xi, x)$ is of class C^p on $[0, \infty) \times M \times M$, with p being as large as we want provided we choose N large enough. This includes $t = 0$ and, therefore,

$$\lim_{t \to 0} F_t = F_0 \text{ in } C^p(M \times M). \tag{95}$$

For every $t \ge 0$, we define $\beta_{h,t,\xi} \in H_1^2(M)$ as the unique solution to

$$(\Delta_g + h)\beta_{h,t,\xi} = -\left(\frac{F_t(\xi, x)}{k_n} + \hat{h}_\xi \frac{\chi(d_{g_\xi}(\xi, x))\Lambda_\xi(x)}{(n-2)\omega_{n-1}(t^2 + d_{g_\xi}(\xi, x)^2)^{\frac{n-2}{2}}}\right) \tag{96}$$
$$= -\frac{F_t(\xi, x)}{k_n} - \hat{h}_\xi \begin{cases} \frac{U_{t,\xi}}{k_n t^{\frac{n-2}{2}}} & \text{if } t > 0 \\ \Gamma_\xi & \text{if } t = 0. \end{cases}$$

Since $N > n-2$ and $n \leq 5$, the right-hand-side is uniformly bounded in $L^q(M)$ for some $q > \frac{2n}{n+2}$, independently of $t \geq 0$, $\xi \in U_0$ and $h \in C^2(M)$ satisfying $\|h\|_\infty < A$ and $\lambda_1(\Delta_g + h) > 1/A$. Therefore, $\beta_{h,t,\xi}$ is well defined and we have

$$\|\beta_{h,t,\xi} - \beta_{h,0,\xi}\|_{H_1^2} = \mathrm{o}(1) \text{ as } t \to 0 \tag{97}$$

uniformly with respect to ξ and h. Furthermore, we have $\beta_{h,t,\xi} \in C^2(M)$ when $t > 0$. As one checks, with these definitions, we obtain that

$$G_{h,\xi} := \Gamma_\xi + \beta_{h,0,\xi}$$

is the Green's function of the operator $\Delta_g + h$ at the point ξ. We now define the *mass* of $\Delta_g + h$ at the point ξ:

Proposition-Definition 8.1. *Assume that $3 \leq n \leq 5$ and $N > n-2$. Let $h \in C^2(M)$ be such that $\Delta_g + h$ is coercive. In the case where $n \in \{4,5\}$, assume in addition that there exists $\xi \in M$ such that $\varphi_h(\xi) = |\nabla \varphi_h(\xi)| = 0$, where φ_h is as in (5). Then $\beta_{h,0,\xi} \in C^0(M)$. Furthermore, the number $\beta_{h,0,\xi}(\xi)$ does not depend on the choice of $N > n-2$ and g_ξ satisfying (21) and (23). We then define the mass of $\Delta_g + h$ at the point ξ as $m_h(\xi) := \beta_{h,0,\xi}(\xi)$.*

Proof of Proposition-Definition 8.1. As one checks, when $n = 3$, we have

$$\hat{h}_\xi(x)\Gamma_\xi(x) = \mathrm{O}(d_g(x,\xi)^{-1})$$

and when $n \in \{4,5\}$ and $\varphi_h(\xi) = |\nabla \varphi_h(\xi)| = 0$, we have

$$\hat{h}_\xi(x)\Gamma_\xi(x) = \mathrm{O}(d_g(x,\xi)^{4-n}).$$

Furthermore, we have

$$F_0(\xi,x) = \mathrm{O}(d_g(x,\xi)^{N-n}).$$

When $N > n-2$, this implies that $\beta_{h,0,\xi} \in C^0(M)$. The fact that the number $\beta_{h,0,\xi}(\xi)$ does not depend on the choice of N and g_ξ then follows from the uniqueness of conformal normal coordinates up to the action of $O(n)$ and the choice of the metric's one-jet at the point ξ (see Lee–Parker [17]). This ends the proof of Proposition-Definition 8.1. □

We now prove a differentiation result that will allow us to obtain Theorem 1.2:

Proposition 8.1. *Assume that $3 \leq n \leq 5$. Let $h \in C^2(M)$ be such that $\Delta_g + h$ is coercive. In the case where $n \in \{4,5\}$, assume that there exists $\xi \in M$ such that $\varphi_h(\xi) = |\nabla \varphi_h(\xi)| = 0$. Let $H \in C^2(M)$ be such that $H(\xi) = |\nabla H(\xi)| = 0$. Then $m_{h+\epsilon H}(\xi)$ is well defined for small $\epsilon \in \mathbb{R}$ and differentiable with respect to ϵ. Furthermore,*

$$\partial_\epsilon (m_{h+\epsilon H}(\xi))_{|0} = -\int_M H G_{h,\xi}^2 \, dv_g.$$

Proof of Proposition 8.1. In order to differentiate the mass with respect to the potential function h, it is convenient to write

$$G_{h,\xi} = G_{c_n \mathrm{Scal}_g,\xi} + \hat{\beta}_{h,\xi},$$

where $\hat{\beta}_{h,\xi} \in H_1^2(M)$ is the solution to

$$(\Delta_g + h)\hat{\beta}_{h,\xi} = -\varphi_h G_{c_n \mathrm{Scal}_g,\xi}. \tag{98}$$

Under the assumptions of the proposition, we have $\hat{\beta}_{h,\xi} \in C^0(M)$ and

$$\hat{\beta}_{h,\xi}(\xi) = -\int_M \varphi_h G_{c_n \operatorname{Scal}_g,\xi} G_{h,\xi}\, dv_g.$$

Furthermore, as one checks, we have

$$m_h(\xi) = m_{c_n \operatorname{Scal}_g}(\xi) - \hat{\beta}_{h,\xi}(\xi). \tag{99}$$

It follows from standard elliptic theory that $\hat{\beta}_{h+\epsilon H,\xi}$ is differentiable with respect to ϵ. Differentiating (98) then yields

$$(\Delta_g + h)\partial_\epsilon(\hat{\beta}_{h+\epsilon H,\xi})_{|0} + H\hat{\beta}_{h,\xi} = -HG_{c_n \operatorname{Scal}_g,\xi},$$

which gives

$$(\Delta_g + h)\partial_\epsilon(\hat{\beta}_{h+\epsilon H,\xi})_{|0} = -HG_{h,\xi}.$$

Therefore,

$$\partial_\epsilon(\hat{\beta}_{h+\epsilon H,\xi}(x))_{|0} = -\int_M G_{h,x} H G_{h,\xi}\, dv_g.$$

It then follows from (99) that

$$\partial_\epsilon(m_{h+\epsilon H}(\xi))_{|0} = -\int_M HG_{h,\xi}^2\, dv_g.$$

This ends the proof of Proposition 8.1. □

9. Energy and remainder estimates in dimensions $n = 3, 4, 5$

In this section, we assume that $n \leq 5$ and $u_0 \equiv \tilde{u}_0 \equiv 0$. When $n \in \{4, 5\}$, we assume in addition that the condition (4) is satisfied. We define

$$W_{h,\tilde{u}_0,\delta,\xi} = W_{h,\delta,\xi} := U_{\delta,\xi} + B_{h,\delta,\xi}, \text{ where } B_{h,\delta,\xi} := k_n \delta^{\frac{n-2}{2}} \beta_{h,\delta,\xi}. \tag{100}$$

In order to use the C^1-estimates of Proposition 4.1, our first step is to obtain estimates for $\beta_{h,\delta,\xi}$ and its derivatives in $H_1^2(M)$:

Proposition 9.1. *For $3 \leq n \leq 5$, let $B_{h,\delta,\xi}$ be as in (100). Then (25) holds.*

Proof of Proposition 9.1. It follows from (97) that

$$\|\beta_{h,\delta,\xi}\|_{H_1^2} \leq C.$$

Differentiating (96) with respect to ξ_i, $i = 1, \dots, n$, we obtain

$$(\Delta_g + h)(\partial_{\xi_i}\beta_{h,\delta,\xi}) = -\frac{1}{k_n}\left(\partial_{\xi_i} F_\delta(\xi, \cdot) + \partial_{\xi_i}\hat{h}_\xi \frac{U_{\delta,\xi}}{\delta^{\frac{n-2}{2}}} + \hat{h}_\xi \frac{\partial_{\xi_i} U_{\delta,\xi}}{\delta^{\frac{n-2}{2}}}\right).$$

It follows from (95) that

$$\|\partial_{\xi_i} F_\delta(\xi, \cdot)\|_\infty \leq C$$

With the definition (45) of $\hat{h}_\xi$, we obtain

$$\partial_{\xi_i}\hat{h}_\xi = \partial_{\xi_i}(c_n \operatorname{Scal}_{g_\xi} \Lambda_\xi^{2-2^\star}) = \mathrm{O}(d_g(\cdot, \xi)).$$

Therefore, with (14), we obtain

$$\left|\partial_{\xi_i}\hat{h}_\xi \frac{U_{\delta,\xi}}{\delta^{\frac{n-2}{2}}}\right| \leq C \frac{d_g(x, \xi)}{(\delta^2 + d_g(x, \xi)^2)^{\frac{n-2}{2}}}.$$

With (73) and (74), we obtain

$$|\delta^{-\frac{n-2}{2}}\partial_{\xi_i}U_{\delta,\xi}|\leq C\frac{1}{(\delta^2+d_g(x,\xi)^2)^{\frac{n-2}{2}}}+C\frac{d_g(x,\xi)}{(\delta^2+d_g(x,\xi)^2)^{n/2}}\,.$$

The definition (45) of $\hat{h}_\xi$ and the assumption $\varphi_{h_0}(\xi_0)=|\nabla\varphi_{h_0}(\xi_0)|=0$ yield

$$\hat{h}_\xi(x)=\mathrm{O}(d_g(x,\xi)^2+D_{h,\xi}),\tag{101}$$

where $D_{h,\xi}$ is as in (43). Putting together these inequalities yields

$$|(\Delta_g+h)(\partial_{\xi_i}\beta_{h,\delta,\xi})|\leq C+C\frac{d_g(x,\xi)}{(\delta^2+d_g(x,\xi)^2)^{\frac{n-2}{2}}}+CD_{h,\xi}\frac{\delta^2+d_g(x,\xi)}{(\delta^2+d_g(x,\xi)^2)^{n/2}}\,.\tag{102}$$

It then follows from standard elliptic theory and straightforward computations that

$$\|\partial_{\xi_i}\beta_{h,\delta,\xi}\|_{H_1^2}\leq C\begin{cases}1 & \text{if } n=3\\ (\ln(1/\delta))^{4/3} & \text{if } n=4\\ \delta^{-1/2} & \text{if } n=5.\end{cases}$$

Similarly, differentiating with respect to δ, we obtain

$$\begin{aligned}|(\Delta_g+h)(\partial_\delta\beta_{h,\delta,\xi})| &= \left|-\frac{1}{k_n}\left(\partial_\delta F_\delta(\xi,\cdot)+\hat{h}_\xi\partial_\delta(\delta^{-\frac{n-2}{2}}U_{\delta,\xi})\right)\right|\\ &\leq C+C\frac{\delta(d_g(x,\xi)^2+D_{h,\xi})}{(\delta^2+d_g(x,\xi)^2)^{n/2}}\end{aligned}\tag{103}$$

and, therefore, elliptic estimates and straightforward computations yield

$$\|\partial_\delta\beta_{h,\delta,\xi}\|_{\frac{2n}{n+2}}\leq C+C\left\|\frac{\delta}{(\delta^2+d_g(x,\xi)^2)^{n/2}}\right\|_{H_1^2}\leq C\begin{cases}1 & \text{if } n=3\\ \delta^{2-n/2} & \text{if } n=4,5.\end{cases}$$

With the definition (100), all these estimates yield (25). This ends the proof of Proposition 9.1. ☐

The sequel of the analysis requires a pointwise control for $\beta_{h,\delta,\xi}$ and its derivatives. This is the objective of the following proposition:

Proposition 9.2. *We have*

$$|\beta_{h,\delta,\xi}(x)|\leq C\begin{cases}1 & \text{if } n=3\\ 1+|\ln\left(\delta^2+d_g(x,\xi)^2\right)| & \text{if } n=4\\ \left(\delta^2+d_g(x,\xi)^2\right)^{-1/2} & \text{if } n=5,\end{cases}\tag{104}$$

$$|\partial_\delta\beta_{h,\delta,\xi}(x)|\leq C+CD_{h,\xi}\delta\ln(1/\delta)\left(\delta^2+d_g(x,\xi)^2\right)^{-\frac{n-2}{2}}\tag{105}$$

and

(106)

$$|\partial_{\xi_i}\beta_{h,\delta,\xi}(x)|\leq C+C\begin{cases}D_{h,\xi}\left|\ln(\delta^2+d_g(x,\xi)^2)\right| & \text{if } n=3\\ D_{h,\xi}\left(\delta^2+d_g(x,\xi)^2\right)^{-1/2} & \text{if } n=4\\ \left|\ln(\delta^2+d_g(x,\xi)^2)\right|+D_{h,\xi}(\delta^2+d_g(x,\xi)^2)^{-1} & \text{if } n=5\end{cases}$$

for all $i=1,\dots,n$, where $D_{h,\xi}$ is as in (43).

Proof of Proposition 9.2. These estimates will be consequences of Green's representation formula and Giraud's Lemma. More precisely, it follows from (96) that

$$\beta_{h,\delta,\xi}(x) = -\int_M G_{h,x}(y)\left(\frac{F_\delta(\xi,y)}{k_n} + \hat{h}_\xi \frac{\chi(d_{g_\xi}(y,\xi))\Lambda_\xi(y)}{(n-2)\omega_{n-1}(\delta^2 + d_{g_\xi}(y,\xi)^2)^{\frac{n-2}{2}}}\right) dv_g(y) \tag{107}$$

for all $x \in M$. With (95) and the standard estimates of the Green's function $0 < G_{h,x}(y) \le C d_g(x,y)^{2-n}$ for all $x, y \in M$, $x \ne y$, we obtain

$$|\beta_{h,\delta,\xi}(x)| \le C + C\int_M \frac{d_g(x,y)^{2-n}}{(\delta^2 + d_g(y,\xi)^2)^{\frac{n-2}{2}}}\, dv_g(y). \tag{108}$$

Recall Giraud's Lemma (see [11] for the present statement): For every α, β such that $0 < \alpha, \beta < n$ and $x, z \in M$, $x \ne z$, we have

$$\int_M d_g(x,y)^{\alpha-n} d_g(y,z)^{\beta-n} dv_g(z) \le C \begin{cases} d_g(x,z)^{\alpha+\beta-n} & \text{if } \alpha+\beta < n \\ 1 + |\ln d_g(x,z)| & \text{if } \alpha+\beta = n \\ 1 & \text{if } \alpha+\beta > n. \end{cases}$$

Therefore, (108) yields (104) when $d_g(x,\xi) \ge \delta$. When $d_g(x,\xi) \le \delta$, (108) yields

$$|\beta_{h,\delta,\xi}(x)| \le C + C\int_M \frac{d_g(x,y)^{2-n}}{(\delta^2 + d_g(y,x)^2)^{\frac{n-2}{2}}}\, dv_g(y),$$

which in this case also yields (104). To prove (106), we use (102) and the same method as for (104). The inequality (105) is a little more delicate. With (103) and Green's identity, we obtain

$$\begin{aligned} |\partial_\delta \beta_{h,\delta,\xi}(x)| &= \left|\int_M G_{h,x}(y)\,(\Delta_g + h)\,\partial_\delta \beta_{h,\delta,\xi}(y)\, dv_g(y)\right| \\ &\le C + C\int_M d_g(x,y)^{2-n} \frac{\delta(d_g(y,\xi)^2 + D_{h,\xi})}{(\delta^2 + d_g(y,\xi)^2)^{n/2}}\, dv_g(y). \end{aligned}$$

We then obtain

$$\begin{aligned} |\partial_\delta \beta_{h,\delta,\xi}(x)| \le C + C\delta \int_M & d_g(x,y)^{2-n} d_g(y,\xi)^{2-n} dv_g(y) \\ &+ C\delta D_{h,\xi} \int_M \frac{d_g(x,y)^{2-n}}{(\delta^2 + d_g(y,\xi)^2)^{n/2}}\, dv_g(y). \end{aligned}$$

We estimate the first two terms in the right-hand side by using Giraud's lemma as in the proof of (104). We split the integral of the third term as

$$\int_M \frac{d_g(x,y)^{2-n}}{(\delta^2 + d_g(y,\xi)^2)^{n/2}}\, dv_g(y) = \int_{\{d_g(x,y) < d_g(x,\xi)/2\}} + \int_{\{d_g(x,y) \ge d_g(x,\xi)/2\}}.$$

Since $d_g(y,\xi) > d_g(x,\xi)/2$ when $d_g(x,y) < d_g(x,\xi)/2$, we have

$$\begin{aligned} &\int_{\{d_g(x,y) < d_g(x,\xi)/2\}} \frac{d_g(x,y)^{2-n}}{(\delta^2 + d_g(y,\xi)^2)^{n/2}}\, dv_g(y) \\ &\qquad \le C d_g(x,\xi)^{-n} \int_{\{d_g(x,y) < d_g(x,\xi)/2\}} d_g(x,y)^{2-n} dv_g(y) \le C d_g(x,\xi)^{2-n}. \end{aligned}$$

As regards the second part of the integral, we have

$$\int_{\{d_g(x,y)\geq d_g(x,\xi)/2\}} \frac{d_g(x,y)^{2-n}}{(\delta^2+d_g(y,\xi)^2)^{n/2}}\,dv_g(y)$$
$$\leq C d_g(x,\xi)^{2-n}\int_M (\delta^2+d_g(y,\xi)^2)^{-n/2}dv_g(y)\leq C d_g(x,\xi)^{2-n}\ln(1/\delta).$$

This yields (105) when $d_g(x,\xi)>\delta$. Finally, we treat the case $d_g(x,\xi)\leq\delta$ in the same way as (106). This ends the proof of Proposition 9.2. □

It is a direct consequence of Proposition 9.2 that (25) is satisfied. Therefore Proposition 4.1 applies. It follows from (27), (30) and (33) that

$$J_h(W+\Phi)=J_h\left(W\right)+\mathrm{O}(\|R\|^2_{\frac{2n}{n+2}}) \tag{109}$$

and, since $n\leq 5$,

$$\partial_p J_h(W+\Phi)=\partial_p J_h(W)+\mathrm{O}(\delta^{-1}\|R\|_{\frac{2n}{n+2}}(\|R\|_{\frac{2n}{n+2}}+\delta\|\partial_p R\|_{\frac{2n}{n+2}})), \tag{110}$$

where $R=R_{\delta,\xi}$ is as in (28). We prove the following estimates for R:

Proposition 9.3. *We have*

$$\|R\|_{\frac{2n}{n+2}}+\delta\|\partial_p R\|_{\frac{2n}{n+2}}\leq C\begin{cases}\delta & \text{if } n=3\\ \delta^2\ln(1/\delta) & \text{if } n=4\\ D_{h,\xi}\delta^2\ln(1/\delta)+\delta^2 & \text{if } n=5.\end{cases} \tag{111}$$

Proof of Proposition 9.3. Note that since $n<6$, we have $2^\star>3$. The definitions (96), (100) and (100) combined with (94) yield

$$\begin{aligned} R&=(\Delta_g+h)U+(\Delta_g+h)B-(U+B)_+^{2^\star-1}=U^{2^\star-1}-(U+B)_+^{2^\star-1}\\ &=-(2^\star-1)U^{2^\star-2}B+\mathrm{O}(U^{2^\star-3}B^2+|B|^{2^\star-1}),\end{aligned} \tag{112}$$

where we have used that $U\geq 0$. Therefore,

$$\|R\|_{\frac{2n}{n+2}}\leq C\|U^{2^\star-2}B\|_{\frac{2n}{n+2}}+\||B|^{2^\star-1}\|_{\frac{2n}{n+2}}.$$

Since $B=k_n\delta^{\frac{n-2}{2}}\beta$, the pointwise estimate (104), the estimate $U\leq C\tilde{U}$ and the estimates (51) yield

$$\|R\|_{\frac{2n}{n+2}}\leq C\begin{cases}\delta & \text{if } n=3\\ \delta^2\ln(1/\delta) & \text{if } n=4\\ \delta^2 & \text{if } n=5.\end{cases}$$

We now deal with the gradient term. We fix $i\in\{0,\dots,n\}$. We have

$$\begin{aligned}\partial_{p_i}R&=\partial_{p_i}(U^{2^\star-1}-(U+B)_+^{2^\star-1})\\ &=-(2^\star-1)((U+B)_+^{2^\star-2}(\partial_{p_i}U+\partial_{p_i}B)-U^{2^\star-2}\partial_{p_i}U)\\ &=-(2^\star-1)(((U+B)_+^{2^\star-2}-U^{2^\star-2})\partial_{p_i}U+(U+B)_+^{2^\star-2}\partial_{p_i}B).\end{aligned}$$

Using that $2^\star>3$ together with (32) and (38), we obtain

$$\delta|\partial_{p_i}R|\leq C\tilde{U}^{2^\star-2}|B|+C\tilde{U}|B|^{2^\star-2}+C\delta|\partial_{p_i}B|\tilde{U}^{2^\star-2}.$$

Since $B = k_n \delta^{\frac{n-2}{2}} \beta$, using the estimates of β and its derivatives in Proposition 9.2 and the estimates (51), long but easy computations yield

$$\delta \|\partial_{p_i} R\|_{\frac{2n}{n+2}} \leq C \begin{cases} \delta & \text{if } n = 3 \\ \delta^2 \ln(1/\delta) & \text{if } n = 4 \\ D_{h,\xi} \delta^2 \ln(1/\delta) + \delta^2 & \text{if } n = 5. \end{cases}$$

Therefore, we obtain (111). This ends the proof of Proposition 9.3. □

With (111), the estimates (109) and (110) become

$$J_h(W + \Phi) = J_h(W) + O \begin{pmatrix} \delta^2 & \text{if } n = 3 \\ \delta^4 (\ln(1/\delta))^2 & \text{if } n = 4 \\ \delta^4 + D_{h,\xi}^2 \delta^4 (\ln(1/\delta))^2 & \text{if } n = 5 \end{pmatrix}$$

and

$$\partial_{p_i} J_h(W + \Phi) = \partial_{p_i} J_h(W) + \mathrm{O} \begin{pmatrix} \delta & \text{if } n = 3 \\ \delta^3 (\ln(1/\delta))^2 & \text{if } n = 4 \\ \delta^3 + D_{h,\xi}^2 \delta^3 (\ln(1/\delta))^2 & \text{if } n = 5 \end{pmatrix}.$$

We now estimate $J_h(W + \Phi)$:

Proposition 9.4. *We have*

$$(113) \quad J_h(W + \Phi) = \frac{1}{n} \int_{\mathbb{R}^n} U_{1,0}^{2^\star} \, dx + \frac{1}{2} \varphi_h(\xi) \begin{Bmatrix} 0 & \textit{if } n = 3 \\ 8\omega_{n-1} \delta^2 \ln(1/\delta) & \textit{if } n = 4 \\ \delta^2 \int_{\mathbb{R}^n} U_{1,0}^2 \, dx & \textit{if } n = 5 \end{Bmatrix} - \frac{k_n^2}{2} m_{h_0}(\xi_0) \delta^{n-2} + \mathrm{o}(\delta^{n-2})$$

as $\delta \to 0$, $\xi \to \xi_0$ and $h \to h_0$ in $C^2(M)$.

Proof of Proposition 9.4. We have

$$(114) \quad \begin{aligned} J_h(W) &= \frac{1}{2} \int_M \left(|\nabla W|^2 + h W^2 \right) dv_g - \frac{1}{2^\star} \int_M W_+^{2^\star} \, dv_g \\ &= \frac{1}{2} \int_M R W \, dv_g + \left(\frac{1}{2} - \frac{1}{2^\star} \right) \int_M W_+^{2^\star} \, dv_g. \end{aligned}$$

Using that $U \geq 0$, we obtain

$$(115) \quad W_+^{2^\star} = (U + B)_+^{2^\star} = U^{2^\star} + 2^\star B U^{2^\star - 1} + \mathrm{O}\left(B^2 U^{2^\star - 2} + |B|^{2^\star} \right).$$

Plugging (112) and (115) into (114), and using (32) and (38), we obtain

$$J_h(W) = \frac{1}{n} \int_M U^{2^\star} dv_g - \frac{1}{2} \int_M B U^{2^\star - 1} dv_g + \mathrm{O}\left(\int_M (\tilde{U}^{2^\star - 2} B^2 + \tilde{U} |B|^{2^\star - 1} + |B|^{2^\star}) \, dv_g \right).$$

Since $B = k_n \delta^{\frac{n-2}{2}} \beta$, the pointwise estimate (104), the definition (14) and (57) yield

$$J_h(W) = \frac{1}{n} \int_{\mathbb{R}^n} U_{1,0}^{2^\star} \, dx - \frac{1}{2} \int_M B U^{2^\star - 1} dv_g + O \begin{pmatrix} \delta^2 & \text{if } n = 3 \\ \delta^4 \left(\ln(1/\delta)\right)^3 & \text{if } n = 4 \\ \delta^4 & \text{if } n = 5 \end{pmatrix}. \tag{116}$$

The definitions (96) and (100) of β and B yield

$$\Delta_g B + hB = U^{2^\star - 1} - (\Delta_g U + hU) \text{ in } M. \tag{117}$$

Therefore, we obtain

$$\begin{aligned} \int_M B U^{2^\star-1} dv_g &= \int_M B(U^{2^\star-1} - (\Delta_g U + hU)) \, dv_g + \int_M B(\Delta_g U + hU) \, dv_g \\ &= \int_M \left(|\nabla B|^2 + hB^2\right) dv_g + \int_M (\Delta_g B + hB) U \, dv_g \\ &= \int_M \left(|\nabla B|^2 + hB^2\right) dv_g - \delta^{\frac{n-2}{2}} \int_M F_\delta(\xi, \cdot) U \, dv_g - \int_M \hat{h}_\xi U^2 dv_g. \end{aligned}$$

Since $B = k_n \delta^{\frac{n-2}{2}} \beta$, using (97) and (95) together with Lebesgue's convergence theorem, we obtain

$$\begin{aligned} \int_M B U^{2^\star-1} dv_g = \delta^{n-2} k_n^2 \bigg(&\int_M \left(|\nabla \beta_{h,0,\xi}|^2 + h \beta_{h,0,\xi}^2\right) dv_g \\ &- \frac{1}{k_n} \int_M F_0(\xi, \cdot) \Gamma_\xi \, dv_g \bigg) - \int_M \hat{h}_\xi U^2 dv_g + \mathrm{o}(\delta^{n-2}). \end{aligned} \tag{118}$$

Since $U(x)^2 \le C \delta^{n-2} d_g(\xi, x)^{4-2n}$, letting $\xi \to \xi_0$ and $h \to h_0$ in $C^2(M)$, integration theory yields

$$\int_M \hat{h}_\xi U^2 dv_g = \delta k_n^2 \int_M (\hat{h_0})_{\xi_0} \Gamma_{\xi_0}^2 \, dv_g + \mathrm{o}(\delta) \text{ when } n = 3.$$

We now assume that $n \in \{4, 5\}$. We write

$$\begin{aligned} \int_M \hat{h}_\xi U^2 dv_g = \hat{h}_\xi(\xi) \int_M U^2 dv_g + \partial_{\xi_i} \hat{h}_\xi(\xi) \int_M x^i U^2 dv_g \\ + \int_M (\hat{h}_\xi - \hat{h}_\xi(\xi) - \partial_{\xi_i} \hat{h}_\xi(\xi) x^i) U^2 dv_g, \end{aligned}$$

where the coordinates are taken with respect to the exponential chart at ξ. As one checks, there exists $C > 0$ such that

$$|\hat{h}_\xi - \hat{h}_\xi(\xi) - \partial_{\xi_i} \hat{h}_\xi(\xi) x^i| U^2 \le C \delta^{n-2} d_g(\xi, x)^{6-2n}$$

for all $x, \xi \in M$, $x \ne \xi$. Since $n < 6$ and ξ remains in a neighborhood of ξ (so that the exponential chart remains nicely bounded), integration theory then yields

$$\begin{aligned} \int_M (\hat{h}_\xi - \hat{h}_\xi(\xi) - \partial_{\xi_i} \hat{h}_\xi(\xi) x^i) U^2 dv_g = \delta^{n-2} k_n^2 \int_M (\hat{h}_\xi - \hat{h}_\xi(\xi) - \partial_{\xi_i} \hat{h}_\xi(\xi) x^i) \Gamma_\xi^2 dv_g \\ + \mathrm{o}(\delta^{n-2}). \end{aligned}$$

Furthermore, letting $\xi \to \xi_0$, $h \to h_0$ and using (4), we obtain

$$\int_M (\hat{h}_\xi - \hat{h}_\xi(\xi) - \partial_{\xi_i} \hat{h}_\xi(\xi) x^i) U^2 dv_g = \delta^{n-2} k_n^2 \int_M (\hat{h_0})_{\xi_0} \Gamma_{\xi_0}^2 \, dv_g + \mathrm{o}(\delta^{n-2}). \tag{119}$$

Via the exponential chart, using the radial symmetry of U, we obtain

$$\begin{aligned}\int_M x^i U^2 dv_g &= \sqrt{n(n-2)}^{n-2} \int_{B_{r_0}(0)} x^i \left(\frac{\delta}{\delta^2+|x|^2}\right)^{n-2} (1+\mathrm{O}(|x|))\, dx \\ &= \mathrm{O}\left(\int_{B_{r_0}(0)} |x|^2 \left(\frac{\delta}{\delta^2+|x|^2}\right)^{n-2} dx\right) = \mathrm{O}(\delta^{n-2})\end{aligned}$$

since $n < 6$. It then follows from (61), (62), (63) and the above estimates that

$$\int_M \hat{h}_\xi U^2 dv_g = \hat{h}_\xi(\xi) \begin{Bmatrix} 0 & \text{if } n=3 \\ 8\omega_{n-1}\delta^2 \ln(1/\delta) & \text{if } n=4 \\ \delta^2 \int_{\mathbb{R}^n} U_{1,0}^2\, dx & \text{if } n=5 \end{Bmatrix} + \delta^{n-2} k_n^2 \int_M \hat{(h_0)}_{\xi_0} \Gamma_{\xi_0}^2\, dv_g + \mathrm{o}(\delta^{n-2}).$$

Combining this estimate with (118), we obtain

$$\int_M BU^{2^\star-1} dv_g = -\hat{h}_\xi(\xi) \begin{Bmatrix} 0 & \text{if } n=3 \\ 8\omega_{n-1}\delta^2 \ln(1/\delta) & \text{if } n=4 \\ \delta^2 \int_{\mathbb{R}^n} U_{1,0}^2\, dx & \text{if } n=5 \end{Bmatrix} + \delta^{n-2} k_n^2 I_{h_0,\xi_0} + \mathrm{o}(\delta^{n-2}),$$

where

$$I_{h_0,\xi_0} := \int_M \left(|\nabla \beta_{h_0,0,\xi_0}|^2 + h_0 \beta_{h_0,0,\xi_0}^2\right) dv_g - \frac{1}{k_n} \int_M F_0(\xi,\cdot)\Gamma_{\xi_0}\, dv_g - \int_M \hat{(h_0)}_{\xi_0} \Gamma_{\xi_0}^2\, dv_g. \tag{120}$$

Integrating by parts and using the definition (96), we obtain

$$\begin{aligned} I_{h_0,\xi_0} &= \int_M \beta_{h_0,0,\xi_0} \left(\Delta_g \beta_{h_0,0,\xi_0} + h_0 \beta_{h_0,0,\xi_0}\right) dv_g \\ &\quad - \int_M \Gamma_{\xi_0} \left(\frac{1}{k_n} F_0(\xi,\cdot) + \hat{(h_0)}_{\xi_0} \Gamma_{\xi_0}\right) dv_g \\ &= \int_M \left(\beta_{h_0,0,\xi_0} + \Gamma_{\xi_0}\right) \left(\Delta_g \beta_{h_0,0,\xi_0} + h_0 \beta_{h_0,0,\xi_0}\right) dv_g \\ &= \int_M G_{h_0,\xi_0} (\Delta_g \beta_{h_0,0,\xi_0} + h_0 \beta_{h_0,0,\xi_0})\, dv_g. \end{aligned}$$

We now use (107) at the point ξ_0, which makes sense since $\beta_{h_0,0,\xi_0}$ is continuous on M. We then obtain

$$I_{h_0,\xi_0} = \beta_{h_0,0,\xi_0}(\xi_0) = m_{h_0}(\xi_0). \tag{121}$$

Putting these results together yields (113), which proves Proposition 9.4. □

We now estimate the derivatives of $J_h(W+\Phi)$:

Proposition 9.5. *We have*

$$\partial_\delta J_h(W+\Phi)=\varphi_h(\xi)\begin{Bmatrix} 0 & \text{if } n=3\\ 8\omega_{n-1}\delta\ln(1/\delta) & \text{if } n=4\\ \delta\displaystyle\int_{\mathbb{R}^n}U_{1,0}^2\,dx & \text{if } n=5\end{Bmatrix} - \frac{n-2}{2}k_n^2 m_{h_0}(\xi_0)\delta^{n-3}+\mathrm{o}(\delta^{n-3}) \tag{122}$$

and

$$\partial_{\xi_i} J_h(W+\Phi)=\frac{1}{2}\partial_{\xi_i}\varphi_h(\xi)\begin{Bmatrix} 0 & \text{if } n=3\\ 8\omega_{n-1}\delta^2\ln(1/\delta) & \text{if } n=4\\ \delta^2\displaystyle\int_{\mathbb{R}^n}U_{1,0}^2\,dx & \text{if } n=5\end{Bmatrix} + \mathrm{O}\begin{pmatrix} \delta & \text{if } n=3\\ \delta^2+D_{h,\xi}\delta^2\ln(1/\delta) & \text{if } n=4\\ \delta^3+D_{h,\xi}\delta^2 & \text{if } n=5\end{pmatrix} \tag{123}$$

for all $i=1,\dots,n$, as $\delta\to 0$, $\xi\to\xi_0$ and $h\to h_0$ in $C^2(M)$.

Proof of Proposition 9.5. We fix $i\in\{0,\dots,n\}$. With (112), (38) and (32), we obtain

$$\begin{aligned}
\partial_{p_i}J_h(W) &= J_h'(W)[\partial_{p_i}W]=\int_M(\Delta_g W+hW-W_+^{2^\star-1})\partial_{p_i}W\,dv_g=\int_M R\partial_{p_i}W\,dv_g\\
&=-(2^\star-1)\int_M U^{2^\star-2}B\partial_{p_i}W\,dv_g+\mathrm{O}\left(\int_M\left(U^{2^\star-3}B^2+|B|^{2^\star-1}\right)|\partial_{p_i}W|\,dv_g\right)\\
&=-(2^\star-1)\int_M U^{2^\star-2}B\partial_{p_i}W\,dv_g+\mathrm{O}\left(\delta^{-1}\int_M\left(\tilde U^{2^\star-2}B^2+\tilde U|B|^{2^\star-1}\right)dv_g\right).\\
&=-(2^\star-1)\int_M U^{2^\star-2}B\partial_{p_i}W\,dv_g+\mathrm{O}(\delta^{-1})\begin{cases}\delta^2 & \text{if } n=3\\ \delta^4\left(\ln(1/\delta)\right)^3 & \text{if } n=4\\ \delta^4 & \text{if } n=5.\end{cases}
\end{aligned}$$

The estimates (106) and (105) and the definition $B=k_n\delta^{\frac{n-2}{2}}\beta$ yield

$$\int_M U^{2^\star-2}B\partial_{p_i}B\,dv_g=\mathrm{O}(\delta^{-1})\begin{cases}\delta^2 & \text{if } n=3\\ \delta^4\left(\ln(1/\delta)\right)^3 & \text{if } n=4\\ \delta^4+\epsilon_{i0}D_{h,\xi}\delta^3\ln(1/\delta) & \text{if } n=5,\end{cases}$$

where ϵ_{i0} is the Kronecker symbol. Since $W=U+B$, we then obtain

$$\partial_{p_i}J_h(W)=-(2^\star-1)\int_M U^{2^\star-2}B\partial_{p_i}U\,dv_g +\mathrm{O}(\delta^{-1})\begin{cases}\delta^2 & \text{if } n=3\\ \delta^4\left(\ln(1/\delta)\right)^3 & \text{if } n=4\\ \delta^4+\epsilon_{i0}D_{h,\xi}\delta^3\ln(1/\delta) & \text{if } n=5,\end{cases}$$

Differentiating (117), we obtain

$$(\Delta_g+h)\partial_{p_i}B=(2^\star-1)U^{2^\star-2}\partial_{p_i}U-(\Delta_g+h)\partial_{p_i}U.$$

Multiplying by B and integrating by parts, we then obtain

$$\int_M \partial_{p_i}B(\Delta_g+h)B\,dv_g=(2^\star-1)\int_M U^{2^\star-2}B\partial_{p_i}U\,dv_g-\int_M \partial_{p_i}U(\Delta_g+h)B\,dv_g. \tag{124}$$

We begin with estimating the left-hand-side of (124). Using that $B=k_n\delta^{\frac{n-2}{2}}\beta$, we obtain

$$\int_M \partial_{p_i}B(\Delta_g+h)B\,dv_g=k_n^2\delta^{n-2}\int_M \partial_{p_i}\beta(\Delta_g+h)\beta\,dv_g + \epsilon_{i0}\frac{n-2}{2}k_n^2\delta^{n-2-1}\int_M \beta(\Delta_g\beta+h\beta)\,dv_g.$$

With (96) and the pointwise estimates (106) and (105), we obtain

$$\left|\int_M \partial_{p_i}\beta(\Delta_g+h)\beta\,dv_g\right|\le C\begin{cases}1 & \text{if } n=3,4\\ 1+D_{h,\xi}^2\ln(1/\delta) & \text{if } n=5.\end{cases}$$

Therefore, we obtain

$$\int_M \partial_{p_i}B(\Delta_g+h)B\,dv_g=\epsilon_{i0}\frac{n-2}{2}k_n^2\delta^{n-2-1}\int_M \beta(\Delta_g\beta+h\beta)\,dv_g + \mathrm{O}\begin{pmatrix}\delta^{n-2} & \text{if } n=3,4\\ \delta^3+D_{h,\xi}^2\delta^3\ln(1/\delta) & \text{if } n=5\end{pmatrix}. \tag{125}$$

We now deal with the second term in the right-hand-side of (124). We first consider the case where $i\ge 1$, so that $\partial_{p_i}=\partial_{\xi_i}$. In this case, it follows from (73) that $\partial_{\xi_i}U=-\partial_{x_i}U+\mathrm{O}(\tilde U)$. Then, using (96), we obtain

$$-\int_M \partial_{\xi_i}U(\Delta_g+h)B\,dv_g=\int_M \partial_{x_i}U(\Delta_g+h)B\,dv_g+\mathrm{O}\left(\int_M \tilde U(\delta^{\frac{n-2}{2}}+|\hat h_\xi|\tilde U)\,dv_g\right).$$

With (101), we obtain

$$\int_M \tilde U(\delta^{\frac{n-2}{2}}+|\hat h_\xi|\tilde U)\,dv_g\le C\begin{cases}\delta & \text{if } n=3\\ \delta^2+D_{h,\xi}\delta^2\ln(1/\delta) & \text{if } n=4\\ \delta^3+D_{h,\xi}\delta^2 & \text{if } n=5.\end{cases}$$

With (96) and since $\partial_{x_i}U=\mathrm{O}(\delta^{\frac{n-2}{2}}d_g(x,\xi)^{1-n})$ (see the definition (24)), we obtain

$$\int_M \partial_{x_i}U(\Delta_g+h)B\,dv_g=-\int_M \hat h_\xi U\partial_{x_i}U\,dv_g+\mathrm{O}(\delta^{n-2}).$$

Putting together the above estimates yields

$$-(2^\star-1)\int_M U^{2^\star-2}B\partial_{\xi_i}U\,dv_g=-\int_M \hat h_\xi U\partial_{x_i}U\,dv_g + \mathrm{O}\begin{pmatrix}\delta & \text{if } n=3\\ \delta^2+D_{h,\xi}\delta^2\ln(1/\delta) & \text{if } n=4\\ \delta^3+D_{h,\xi}\delta^2 & \text{if } n=5\end{pmatrix}.$$

Using the explicit expression (14) of U together with the facts that $\Lambda_\xi(\xi) = 1$, $\nabla\Lambda_\xi(\xi) = 0$ and $|x|\partial_{x_i} U = \mathrm{O}(\tilde{U})$, we obtain

$$\int_M \hat{h}_\xi U \partial_{x_i} U \, dv_g = \int_{B_{r_0}(0)} \hat{h}_\xi(\exp_\xi^{g_\xi}(x)) U_{\delta,0} \partial_{x_i} U_{\delta,0} (1 + \mathrm{O}(|x|^2)) \, dx$$
$$+ \mathrm{O}\left(\int_{B_{r_0}(0)} |\hat{h}_\xi(\exp_\xi^{g_\xi}(x))||x| \tilde{U}_{\delta,0}^2 \, dx\right) + \mathrm{O}(\delta^{n-2})$$

With a Taylor expansion of $\hat{h}_\xi$, using the radial symmetry of $U_{\delta,0}$ and the explicit expressions given in (20), we then obtain that there exists $c_4', c_5' > 0$ such that

$$\int_M \hat{h}_\xi U \partial_{x_i} U \, dv_g = -\partial_{\xi_i} \varphi_h(\xi) \left\{\begin{matrix} 0 & \text{if } n = 3 \\ c_4' \delta^2 \ln(1/\delta) & \text{if } n = 4 \\ c_5' \delta^2 & \text{if } n = 5 \end{matrix}\right\} + \mathrm{O}(\delta^{n-2})$$

and then

$$\partial_{\xi_i} J_h(W) = \partial_{\xi_i} \varphi_h(\xi) \left\{\begin{matrix} 0 & \text{if } n = 3 \\ c_4' \delta^2 \ln(1/\delta) & \text{if } n = 4 \\ c_5' \delta^2 & \text{if } n = 5 \end{matrix}\right\}$$
$$+ \mathrm{O}\begin{pmatrix} \delta & \text{if } n = 3 \\ \delta^2 + D_{h,\xi} \delta^2 \ln(1/\delta) & \text{if } n = 4 \\ \delta^3 + D_{h,\xi} \delta^2 & \text{if } n = 5. \end{pmatrix}.$$

We now consider the case where $i = 0$, so that $\partial_{p_i} = \partial_{p_0} = \partial_\delta$. In this case, we have

$$\begin{aligned} \int_M \partial_\delta U (\Delta_g + h) B \, dv_g &= -\int_M \hat{h}_\xi U \partial_\delta U \, dv_g - \delta^{\frac{n-2}{2}} \int_M F \partial_\delta U \, dv_g \\ &= -\int_M (\hat{h}_\xi(\xi) + x^i \partial_{\xi_i} \hat{h}_\xi(\xi)) U \partial_\delta U \, dv_g \\ &\quad - \int_M (\delta^{\frac{n-2}{2}} F + (\hat{h}_\xi - \hat{h}_\xi(\xi) - x^i \partial_{\xi i} \hat{h}_\xi(\xi)) U) \partial_\delta U \, dv_g, \end{aligned}$$

where the coordinates are taken with respect to the exponential chart at ξ. With (18), (16) and (19), arguing as in the proof of (119), we obtain

$$\begin{aligned} &\delta^{\frac{n-2}{2}} \int_M (F + (\hat{h}_\xi - \hat{h}_\xi(\xi) - x^i \partial_{\xi_i} \hat{h}_\xi(\xi)) \delta^{-\frac{n-2}{2}} U) \partial_\delta U \, dv_g \\ &\qquad = \frac{n-2}{2} k_n^2 \delta^{-1} \delta^{n-2} \int_M \left(\frac{F_{\xi,0}}{k_n} + \hat{h}_{\xi_0} \Gamma_{\xi_0}^h\right) \Gamma_{\xi_0}^h \, dv_g + \mathrm{o}(\delta^{-1} \delta^{n-2}). \end{aligned}$$

Using (61) and arguing as in the estimate of (60), we obtain that there exist $c_4'', c_5'' > 0$ such that

$$\int_M (\hat{h}_\xi(\xi) + x^i \partial_{\xi_i} \hat{h}_\xi(\xi)) U \partial_\delta U \, dv_g = \frac{\hat{h}_\xi(\xi)}{\delta} \int_{B_0(r_0)} U_{\delta,0} Z_{\delta,0} \, dx + \mathrm{o}(\delta^{-1}\delta^{n-2})$$
$$= \frac{\hat{h}_\xi(\xi)}{\delta} \begin{Bmatrix} 0 & \text{if } n=3 \\ c_4''\delta^2 \ln(1/\delta) & \text{if } n=4 \\ c_5''\delta^2 & \text{if } n=5 \end{Bmatrix} + \mathrm{o}(\delta^{-1}\delta^{n-2}).$$

Putting these estimates together yields

$$-(2^\star - 1)\int_M U^{2^\star-2} B \partial_{p_i} U \, dv_g = \frac{\hat{h}_\xi(\xi)}{\delta} \begin{Bmatrix} 0 & \text{if } n=3 \\ c_4''\delta^2 \ln(1/\delta) & \text{if } n=4 \\ c_5''\delta^2 & \text{if } n=5 \end{Bmatrix}$$
$$- \frac{n-2}{2} k_n^2 I_{h_0,\xi_0} \delta^{-1}\delta^{n-2} + \mathrm{o}(\delta^{-1}\delta^{n-2}),$$

where I_{h_0,ξ_0} is as in (120). Since $I_{h_0,\xi_0} = m_{h_0}(\xi_0)$ (see (121)), we obtain (122) and (123) up to the value of the constants. These values then follow from Proposition 9.4 together with the above estimates. This ends the proof of Proposition 9.4. □

Theorem 1.4 for $n \in \{4,5\}$ will be proved in Section 10.

10. Proof of Theorem 1.4

We let $h_0, f \in C^p(M)$, $p \geq 2$, and $\xi_0 \in M$ satisfy the assumptions of Theorem 1.4. For small $\epsilon > 0$ and $\tau \in \mathbb{R}^n$, we define

$$h_\epsilon := h_0 + \epsilon f \text{ and } \xi_\epsilon(\tau) := \exp_{\xi_0}^{g_{\xi_0}}(\sqrt{\epsilon}\tau). \tag{126}$$

We fix $R > 0$ and $0 < a < b$ to be chosen later.

10.1. Proof of Theorem 1.4 for $n \geq 6$.

In this case, we let $(\delta_\epsilon)_{\epsilon>0} > 0$ be such that $\delta_\epsilon \to 0$ as $\epsilon \to 0$. We define

$$\delta_\epsilon(t) := \delta_\epsilon t \text{ and } F_\epsilon(t,\tau) := J_{h_\epsilon}(U_{\delta_\epsilon(t),\xi_\epsilon(\tau)} + \Phi_{h_\epsilon,0,\delta_\epsilon(t),\xi_\epsilon(\tau)}) \tag{127}$$

for all $\tau \in \mathbb{R}^n$ such that $|\tau| < R$ and $t > 0$ such that $a < t < b$. Using the assumption $\varphi_{h_0}(\xi_0) = |\nabla \varphi_{h_0}(\xi_0)| = 0$, we obtain

$$\varphi_{h_\epsilon}(\xi_\epsilon(\tau)) = \frac{1}{2}\nabla^2 \varphi_{h_0}(\xi_0)[\tau,\tau]\epsilon + f(\xi_0)\epsilon + \mathrm{o}(\epsilon)$$

and

$$\nabla \varphi_{h_\epsilon}(\xi_\epsilon(\tau)) = \nabla^2 \varphi_{h_0}(\xi_0)[\tau,\cdot]\sqrt{\epsilon} + \mathrm{o}(\sqrt{\epsilon})$$

as $\epsilon \to 0$ uniformly with respect to $|\tau| < R$. We distinguish two cases:

Case $n \geq 7$. In this case, we set $\delta_\epsilon := \sqrt{\epsilon}$. It follows from (54) that

$$\lim_{\epsilon \to 0} \frac{F_\epsilon(t,\tau) - \frac{1}{n}\int_{\mathbb{R}^n} U_{1,0}^{2^\star} \, dx}{\epsilon^2} = E_0(t,\xi) \text{ in } C^0_{\mathrm{loc}}((0,\infty) \times \mathbb{R}^n), \tag{128}$$

where

$$E_0(t,\tau) := C_n \left(\frac{1}{2}\nabla^2 \varphi_{h_0}(\xi_0)[\tau,\tau] + f(\xi_0)\right) t^2 - D_n K_{h_0}(\xi_0) t^4,$$

with

$$C_n := \frac{1}{2}\int_{\mathbb{R}^n} U_{1,0}^2\, dx \text{ and } D_n := \frac{1}{4n}\int_{\mathbb{R}^n} |x|^2 U_{1,0}^2\, dx. \tag{129}$$

Furthermore, we have

$$\partial_t F_\epsilon(t,\tau) = \sqrt{\epsilon}\left(\partial_\delta J_{h_\epsilon}(U_{(\delta_\epsilon(t),\xi_\epsilon(\tau)} + \Phi_{(\delta_\epsilon(t),\xi_\epsilon(\tau))})\right)$$

and

$$\partial_{\tau_i} F_\epsilon(t,\tau) = \sqrt{\epsilon}\left(\partial_{\xi_i} J_{h_\epsilon}(U_{(\delta_\epsilon(t),\xi_\epsilon(\tau)} + \Phi_{(\delta_\epsilon(t),\xi_\epsilon(\tau))})\right).$$

Therefore, it follows from (66) and (67) that the limit in (128) actually holds in $C^1_{\text{loc}}((0,\infty)\times\mathbb{R}^n)$. Assuming that $f(\xi_0)\times K_{h_0}(\xi_0) > 0$, we can define

$$t_0 := \sqrt{\frac{C_n f(\xi_0)}{2D_n K_{h_0}(\xi_0)}}.$$

As one checks, $(t_0, 0)$ is a critical point of E_0. In addition, the Hessian matrix at the critical point $(t_0, 0)$ is

$$\nabla^2 E_0(t_0,0) = \begin{pmatrix} -8t_0^2 D_n K_{h_0}(\xi_0) & 0 \\ 0 & t_0^2 C_n \nabla^2\varphi_{h_0}(\xi_0) \end{pmatrix}.$$

Therefore, if ξ_0 is a nondegenerate critical point of φ_{h_0}, then $(t_0, 0)$ is a nondegenerate critical point of E_0. With the convergence in $C^1_{\text{loc}}((0,\infty)\times\mathbb{R}^n)$, we then obtain that there exists a critical point $(t_\epsilon,\tau_\epsilon)$ of F_ϵ such that $(t_\epsilon,\tau_\epsilon)\to(t_0,0)$ as $\epsilon\to 0$. It then follows from (29) that

$$u_\epsilon := U_{\delta_\epsilon(t_\epsilon),\xi_\epsilon(\tau_\epsilon)} + \Phi_{h_\epsilon,0,\delta_\epsilon(t_\epsilon),\xi_\epsilon(\tau_\epsilon)}$$

is a solution to (8). As one checks, $u_\epsilon \rightharpoonup 0$ weakly in $L^{2^\star}(M)$ and $(u_\epsilon)_\epsilon$ blows up with one bubble at ξ_0. This proves Theorem 1.4 for $n \geq 7$.

Case $n = 6$**.** In this case, we let $\delta_\epsilon > 0$ be such that

$$\delta_\epsilon^2 \ln(1/\delta_\epsilon) = \epsilon. \tag{130}$$

As one checks, $\delta_\epsilon \to 0$ as $\epsilon\to 0$. As in the previous case, we obtain

$$\lim_{\epsilon\to 0}\frac{F_\epsilon(t,\tau) - \frac{1}{n}\int_{\mathbb{R}^n} U_{1,0}^{2^\star}\, dx}{\epsilon\delta_\epsilon^2} = E_0(t,\xi) \text{ in } C^1_{\text{loc}}((0,\infty)\times\mathbb{R}^n),$$

where

$$E_0(t,\tau) := C_6\left(\frac{1}{2}\nabla^2\varphi_{h_0}(\xi_0)[\tau,\tau] + f(\xi_0)\right)t^2 - 24^2\omega_5 K_{h_0}(\xi_0)t^4$$

for all $t > 0$ and $\tau\in\mathbb{R}^n$. As in the previous case, E_0 has a nondegenerate critical point $(\tilde{t}_0, 0)$, which yields the existence of a critical point of F_ϵ and, therefore, a blowing-up solution to (8) satisfying the desired conditions. This proves Theorem 1.4 for $n = 6$.

10.2. **Proof of Theorem 1.4 for $n \in \{4,5\}$.** When $n \in \{4,5\}$, we define

$$F_\epsilon(t,\tau) := J_{h_\epsilon}(U_{\delta_\epsilon(t),\xi_\epsilon(\tau)} + B_{h_\epsilon,\delta_\epsilon(t),\xi_\epsilon(\tau)} + \Phi_{h_\epsilon,0,\delta_\epsilon(t),\xi_\epsilon(\tau)}),$$

where $\delta_\epsilon(t)$ will be chosen differently depending on the dimension.

Case $n = 5$. In this case, we set $\delta_\epsilon(t) := t\epsilon$. It follows from (113) that

$$\lim_{\epsilon\to 0} \frac{F_\epsilon(t,\tau) - \frac{1}{n}\int_{\mathbb{R}^n} U_{1,0}^{2^\star}\,dx}{\epsilon^3} = E_0(t,\xi) \text{ in } C^0_{\text{loc}}((0,\infty)\times\mathbb{R}^n),$$

where

$$E_0(t,\tau) := C_5\left(\frac{1}{2}\nabla^2\varphi_{h_0}(\xi_0)(\tau,\tau) + f(\xi_0)\right)t^2 - \frac{k_5^2}{2}m_{h_0}(\xi_0)t^3.$$

It follows from the C^1-estimates of Proposition 9.5 that the convergence holds in $C^1_{\text{loc}}((0,\infty)\times\mathbb{R}^n)$. Assuming that $f(\xi_0)\times m_{h_0}(\xi_0) > 0$, we then define

$$t_0 := \frac{4C_5 f(\xi_0)}{(n-2)k_5^2 m_{h_0}(\xi_0)}.$$

As in the previous cases, we obtain that $(t_0,0)$ is a nondegenerate critical point of E_0, which yields the existence of a critical point for F_ϵ and, therefore, a blowing-up solution to (8) satisfying the desired conditions. This proves Theorem 1.4 for $n = 5$.

Case $n = 4$. In this case, we set $\delta_\epsilon(t) := e^{-t/\epsilon}$. It follows from the C^1-estimates of Proposition 9.5 that

$$\lim_{\epsilon\to 0}\left(-\epsilon\delta_\epsilon(t)^{-2}\partial_t F_\epsilon(t,\tau), \delta_\epsilon(t)^{-2}\partial_\tau F_\epsilon(t,\tau)\right) = (\psi_0(t,\tau),\psi_1(t,\tau))$$

in $C^0_{\text{loc}}((0,\infty)\times\mathbb{R}^n)$, where

$$\psi_0(t,\tau) := C_4\left(\frac{1}{2}\nabla^2\varphi_{h_0}(\xi_0)(\tau,\tau) + f(\xi_0)\right)t - \frac{n-2}{2}k_n^2 m_{h_0}(\xi_0)$$

and

$$\psi_1(t,\tau) := \frac{1}{2}C_4\nabla^2\varphi_{h_0}(\xi_0)[\tau,\cdot]t.$$

As one checks, since ξ_0 is a nondegenerate critical point of φ_{h_0}, the function ψ has a unique zero point in $(0,\infty)\times\mathbb{R}^n$ which is of the form $(t_0,0)$ for some $t_0 > 0$. Furthermore, the nondegeneracy implies that the Jacobian determinant of ψ at $(t_0,0)$ is nonzero and, therefore, the degree of ψ at 0 is well-defined and nonzero. The invariance of the degree under uniform convergence then yields the existence of a critical point $(t_\epsilon,\tau_\epsilon)$ of F_ϵ such that $(t_\epsilon,\tau_\epsilon)\to(t_0,0)$ as $\epsilon\to 0$. It then follows from (29) that

$$u_\epsilon := U_{\delta_\epsilon(t),\xi_\epsilon(\tau)} + B_{h_\epsilon,\delta_\epsilon(t),\xi_\epsilon(\tau)} + \Phi_{h_\epsilon,0,\delta_\epsilon(t),\xi_\epsilon(\tau)}$$

is a critical point of J_{h_ϵ} that blows up at ξ_0 and converges weakly to 0 in $L^{2^\star}(M)$. This proves Theorem 1.4 for $n = 4$. □ □

11. Proof of Theorem 1.5

We let $h_0, f \in C^p(M)$, $p \geq 2$, $u_0 \in C^2(M)$ and $\xi_0 \in M$ satisfy the assumptions of Theorem 1.5. We let h_ϵ be as in (8). We let $\xi_\epsilon(\tau)$ and $\delta_\epsilon(t)$ be as in (126) and (127). Since u_0 is nondegenerate, the implicit function theorem yields the existence of $\epsilon_0' \in (0,\epsilon_0)$ and $(u_{0,\epsilon})_{0<\epsilon<\epsilon_0'} \in C^2(M)$ such that

$$\Delta_g u_{0,\epsilon} + h_\epsilon u_{0,\epsilon} = u_{0,\epsilon}^{2^\star-1},\ u_{0,\epsilon} > 0 \text{ in } M \tag{131}$$

and $(u_{0,\epsilon})_\epsilon$ is smooth with respect to ϵ, which implies in particular that

$$\|u_{0,\epsilon} - u_0\|_{C^2} \leq C\epsilon.$$

We fix $0 < a < b$ and $R > 0$ to be chosen later. We define

$$F_\epsilon(t,\tau) := J_{h_\epsilon}(u_{0,\epsilon} + U_{\delta_\epsilon(t),\xi_\epsilon(\tau)} + \Phi_{h_\epsilon,u_{0,\epsilon},\delta_\epsilon(\tau),\xi_\epsilon(\tau)})$$

for all $\tau \in \mathbb{R}^n$ such that $|\tau| < R$ and $t > 0$ such that $a < t < b$. With (85), we obtain that for $n \geq 7$,

$$F_\epsilon(t,\tau) = J_{h_\epsilon}(u_{0,\epsilon}) + \frac{1}{n}\int_{\mathbb{R}^n} U_{1,0}^{2^\star}\,dx + C_n\left(\frac{1}{2}\nabla^2\varphi_{h_0}(\xi_0)(\tau,\tau) + f(\xi_0)\right)t^2\epsilon\delta_\epsilon^2$$
$$+ \mathrm{o}(\epsilon\delta_\epsilon^2) - D_n K_{h_0}(\xi_0)t^4\delta_\epsilon^4 + \mathrm{o}(\delta_\epsilon^4) - B_n u_0(\xi_0)t^{\frac{n-2}{2}}\delta_\epsilon^{\frac{n-2}{2}} + \mathrm{o}(\delta_\epsilon^{\frac{n-2}{2}})$$

as $\epsilon \to 0$ uniformly with respect to $a < t < b$ and $|\tau| < R$, where C_n and D_n are as in (129) and

$$B_n := \int_{\mathbb{R}^n} U_{1,0}^{2^\star-1}dx.$$

We distinguish three cases:

Case $7 \leq n \leq 10$, that is $n \geq 7$ and $\frac{n-2}{2} \leq 4$. In this case, we set $\delta_\epsilon := \epsilon^{\frac{2}{n-6}}$, so that

$$\epsilon\delta_\epsilon^2 = \delta_\epsilon^{\frac{n-2}{2}}.$$

We then obtain

$$\lim_{\epsilon\to 0}\frac{F_\epsilon(t,\tau) - A_\epsilon}{\epsilon\delta_\epsilon^2} = E_0(t,\tau) \tag{132}$$

uniformly with respect to $a < t < b$ and $|\tau| < R$, where

$$A_\epsilon := J_{h_\epsilon}(u_{0,\epsilon}) + \frac{1}{n}\int_{\mathbb{R}^n} U_{1,0}^{2^\star}\,dx \tag{133}$$

and

$$E_0(t,\tau) := C_n\left(\frac{1}{2}\nabla^2\varphi_{h_0}(\xi_0)(\tau,\tau) + f(\xi_0)\right)t^2$$
$$- \left(B_n u_0(\xi_0) + \mathbf{1}_{n=10}D_n K_{h_0}(\xi_0)\right)t^{\frac{n-2}{2}}.$$

Moreover, the estimates (87) and (88) yield the convergence (132) in $C^1_{\mathrm{loc}}((0,\infty)\times\mathbb{R}^n)$. Straightforward changes of variable yield

$$\frac{B_{10}}{D_{10}} = 40\frac{\int_{\mathbb{R}^{10}} U_{1,0}^{3/2}dx}{\int_{\mathbb{R}^{10}} |x|^2 U_{1,0}^{3/2}dx} = 40\frac{\int_0^\infty \frac{r^9 dr}{(1+r^2)^6}}{\int_0^\infty \frac{r^{11}dr}{(1+r^2)^8}} = 40\frac{\int_0^\infty \frac{s^4 ds}{(1+s)^6}}{\int_0^\infty \frac{s^5 ds}{(1+s)^8}}.$$

Integrating by parts, we then obtain

$$\frac{B_{10}}{D_{10}} = \frac{40\times\frac{4}{5}\times\frac{3}{4}\times\frac{2}{3}\times\frac{1}{2}\int_0^\infty\frac{ds}{(1+s)^2}}{\frac{5}{7}\times\frac{4}{6}\times\frac{3}{5}\times\frac{2}{4}\times\frac{1}{3}\int_0^\infty\frac{ds}{(1+s)^3}} = \frac{40\times 6\times 7\int_0^\infty\frac{ds}{(1+s)^2}}{5\int_0^\infty\frac{ds}{(1+s)^3}} = \frac{40\times 6\times 7\times 2}{5}$$
$$= 672.$$

The assumption $K_{h_0,u_0}(\xi_0) \neq 0$ then gives $B_n u_0(\xi_0) + \mathbf{1}_{n=10}D_n K_{h_0}(\xi_0) \neq 0$ with same sign as $f(\xi_0)$. As in the proof of Theorem 1.4 for $n \geq 7$, we obtain that E_0 has a unique critical point in $(0,\infty)\times\mathbb{R}^n$, say $(t_0, 0)$, and this critical point is nondegenerate. Mimicking again the proof of Theorem 1.4 for $n \geq 7$, we obtain the existence of a critical point $(t_\epsilon, \tau_\epsilon)$ of F_ϵ such that $(t_\epsilon, \tau_\epsilon) \to (t_0, 0)$ as $\epsilon \to 0$. It then follows that

$$u_\epsilon := u_{0,\epsilon} + U_{\delta_\epsilon(t_\epsilon),\xi_\epsilon(\tau)} + \Phi_{h_\epsilon,u_{0,\epsilon},\delta_\epsilon(t_\epsilon),\xi_\epsilon(\tau)}$$

is a solution to (8). As one checks, $u_\epsilon \rightharpoonup 0$ weakly in $L^{2^\star}(M)$ and $(u_\epsilon)_\epsilon$ blows up with one bubble at ξ_0. This proves Theorem 1.5 for $7 \le n \le 10$.

Case $4 < \frac{n-2}{4}$, that is $n \ge 11$. In this case, we set $\delta_\epsilon := \sqrt{\epsilon}$, so that

$$\epsilon\delta_\epsilon^2 = \delta_\epsilon^4 \text{ and } \delta_\epsilon^{\frac{n-2}{2}} = \mathrm{o}(\delta_\epsilon^4) \text{ as } \epsilon \to 0.$$

We then obtain

$$\lim_{\epsilon\to 0} \frac{F_\epsilon(t,\tau) - A_\epsilon}{\epsilon\delta_\epsilon^2} = E_0(t,\tau) \text{ in } C^0_{\mathrm{loc}}((0,\infty)\times\mathbb{R}^n),$$

where A_ϵ is as in (133) and

$$E_0(t,\tau) := C_n\left(\frac{1}{2}\nabla^2\varphi_{h_0}(\xi_0)(\tau,\tau) + f(\xi_0)\right)t^2 - D_n K_{h_0}(\xi_0)t^4.$$

As in the previous case, we obtain that the convergence holds in $C^1_{\mathrm{loc}}((0,\infty)\times\mathbb{R}^n)$ and E_0 has a nondegenerate critical point in $(0,\infty)\times\mathbb{R}^n$, which yields the existence of a blowing-up solution $(u_\epsilon)_\epsilon$ to (8) satisfying the desired conditions. This proves Theorem 1.5 for $n \ge 11$.

Case n=6. Note that in this case, we have $2^\star - 1 = 2$. Differentiating (131) with respect to ϵ at 0, we obtain

$$(\Delta_g + h_0 - 2u_0)(\partial_\epsilon u_{0,\epsilon})_{|0} + f u_0 = 0 \text{ in } M.$$

Using that u_0 is nondegenerate, we then obtain

$$(\partial_\epsilon u_{0,\epsilon})_{|0} = -(\Delta_g + h_0 - 2u_0)^{-1}(f u_0).$$

It follows that

$$\varphi_{h_\epsilon,u_\epsilon} = h_\epsilon - 2u_{0,\epsilon} - c_n \,\mathrm{Scal}_g = \varphi_{h_0,u_0} + \tilde{f}\epsilon + \mathrm{o}(\epsilon) \text{ as } \epsilon \to 0,$$

where

$$\tilde{f} := f + 2(\Delta_g + h_0 - 2u_0)^{-1}(f u_0).$$

We let $\delta_\epsilon > 0$ be as in (130). With (91), we then obtain

$$\lim_{\epsilon\to 0} \frac{F_\epsilon(t,\tau) - A_\epsilon}{\epsilon\delta_\epsilon^2} = E_0(t,\tau) \text{ in } C^0_{\mathrm{loc}}((0,\infty)\times\mathbb{R}^n),$$

where A_ϵ is as in (133) and

$$E_0(t,\tau) := C_6\left(\frac{1}{2}\nabla^2\varphi_{h_0}(\xi_0)(\tau,\tau) + \tilde{f}(\xi_0)\right)t^2 - 24^2\omega_5 K_{h_0,u_0}(\xi_0)t^4.$$

As in the previous case, using (92) and (93), we obtain that the convergence holds in $C^1_{\mathrm{loc}}((0,\infty)\times\mathbb{R}^n)$. Furthermore, using (10), we obtain that E_0 has a nondegenerate critical point in $(0,\infty)\times\mathbb{R}^n$ and, therefore, that there exists a blowing-up solution to (8) satisfying the desired conditions. This proves Theorem 1.5 for $n = 6$. $\square$

12. Proof of Theorem 1.2

We let $h_0 \in C^p(M)$, $1 \leq p \leq \infty$, and $\xi_0 \in M$ be such that $\Delta_g + h_0$ is coercive and the condition (4) is satisfied. In the case where $p = 1$, a standard regularization argument give the existence of $(\hat{h}_\epsilon)_{\epsilon>0} \in C^2(M)$ such that $\hat{h}_\epsilon \to h_0$ in $C^1(M)$ as $\epsilon \to 0$. In the case where $p \geq 2$, we set $\hat{h}_\epsilon = h_0$. We then define

$$\tilde{h}_\epsilon := \hat{h}_\epsilon + f_\epsilon, \text{ where } f_\epsilon(x) := \chi(x)((h_0 - \hat{h}_\epsilon)(\xi_0) + \langle \nabla(h_0 - \hat{h}_\epsilon)(\xi_0), x\rangle + \lambda_\epsilon |x|^2),$$

where $\lambda_\epsilon > 0$, χ is a smooth cutoff function around 0 and the coordinates are taken with respect to the exponential chart at ξ_0. As one checks, for some suitable $\lambda_\epsilon \to 0$, we then have that $\tilde{h}_\epsilon \to h_0$ in $C^p(M)$, $\varphi_{\tilde{h}_\epsilon}(\xi_0) = \varphi_{h_0}(\xi_0) = 0$, $|\nabla \varphi_{\tilde{h}_\epsilon}(\xi_0)| = |\nabla \varphi_{h_0}(\xi_0)| = 0$ and for small $\epsilon > 0$, ξ_0 is a nondegenerate critical point of $\varphi_{\tilde{h}_\epsilon}$.

Assume first that $n \in \{4, 5\}$. Then the mass of $\tilde{h}_\epsilon$ is defined at ξ_0. As is easily seen, there exists $\psi : (0,1) \to (0,1)$ such that $\psi(\epsilon) \to 0$ as $\epsilon \to 0$ and either $\{m_{\tilde{h}_{\psi(\epsilon)}}(\xi_0) > 0 \text{ for all } \epsilon \in (0,1)\}$, $\{m_{\tilde{h}_{\psi(\epsilon)}}(\xi_0) < 0 \text{ for all } \epsilon \in (0,1)\}$ or $\{m_{\tilde{h}_{\psi(\epsilon)}}(\xi_0) = 0 \text{ for all } \epsilon \in (0,1)\}$. If $m_{h_{\psi(\epsilon)}}(\xi_0) = 0$ for all $\epsilon \in (0,1)$, then it follows from Proposition 8.1 that if we choose $\mu_\epsilon > 0$ small enough, then we obtain $m_{\check{h}_\epsilon}(\xi_0) < 0$ for small $\epsilon > 0$ with $\check{h}_\epsilon = \tilde{h}_{\psi(\epsilon)} + \mu_\epsilon \chi |\cdot|^2$. Therefore, in all cases, we can assume that $m_{\tilde{h}_\epsilon}(\xi_0) \neq 0$ for small $\epsilon > 0$, with a sign independent of ϵ.

Assume now that $n \geq 6$. With a similar argument, we can assume that, for small $\epsilon > 0$, $K_{\tilde{h}_\epsilon}(\xi_0) \neq 0$ with a sign independent of ϵ, where $K_{\tilde{h}_\epsilon}(\xi_0)$ is as in (7).

In all cases, we can now fix $f_0 \in C^\infty(M)$ such that $f_0(\xi_0) \times K_{\tilde{h}_\epsilon}(\xi_0) > 0$ for small $\epsilon > 0$. It then follows from Theorem 1.4 that there exist $\alpha_\epsilon > 0$ and a family $(\tilde{u}_{\epsilon,\alpha})_{0<\alpha<\alpha_\epsilon}$ of solutions to the equation

$$\Delta_g \tilde{u}_{\epsilon,\alpha} + (\tilde{h}_\epsilon + \alpha f_0)\tilde{u}_{\epsilon,\alpha} = \tilde{u}_{\epsilon,\alpha}^{2^\star - 1}, \ \tilde{u}_{\epsilon,\alpha} > 0 \text{ in } M$$

such that $\tilde{u}_{\epsilon,\alpha} \rightharpoonup 0$ weakly in $L^{2^\star}(M)$ and $(\tilde{u}_{\epsilon,\alpha})_\alpha$ blows up with one bubble at ξ_0 as $\alpha \to 0$. Therefore, we obtain that for every $\epsilon > 0$, there exists $\alpha'_\epsilon > 0$ such that

$$0 < \alpha'_\epsilon < \min(\epsilon, \alpha_\epsilon), \ \|\tilde{u}_{\epsilon,\alpha'_\epsilon}\|_2 < \epsilon, \ \left| \int_M |\tilde{u}_{\epsilon,\alpha'_\epsilon}|^{2^\star} dv_g - \int_{\mathbb{R}^n} U_{1,0}^{2^\star} dx \right| < \epsilon$$

and

$$\int_{M \setminus B_\epsilon(\xi_0)} |\tilde{u}_{\epsilon,\alpha'_\epsilon}|^{2^\star} dv_g < \epsilon\,.$$

We then define $u_\epsilon := \tilde{u}_{\epsilon,\alpha'_\epsilon}$, so that

$$\Delta_g u_\epsilon + h_\epsilon u_\epsilon = u_\epsilon^{2^\star - 1} \text{ in } M, \text{ where } h_\epsilon := \tilde{h}_\epsilon + \alpha'_\epsilon f_0 = h_0 + f_\epsilon + \alpha'_\epsilon f_0.$$

As one checks, $u_\epsilon \rightharpoonup 0$ weakly in $L^{2^\star}(M)$ and $(u_\epsilon)_\epsilon$ blows up with one bubble at ξ_0 as $\epsilon \to 0$. This proves Theorem 1.2. ☐

13. Proof of Theorem 1.3

We let $h_0 \in C^p(M)$, $1 \leq p \leq \infty$, $u_0 \in C^2(M)$ and $\xi_0 \in M$ be such that $\Delta_g + h_0$ is coercive, u_0 is a solution of (1) and the condition (6) is satisfied. We begin with proving the following:

Lemma 13.1. *There exists a neighborhood Ω_0 of ξ_0 and families $(\tilde{h}_\epsilon)_{\epsilon>0} \in C^p(M)$ and $(\tilde{u}_\epsilon)_{\epsilon>0} \in C^2(M)$ such that $\tilde{h}_\epsilon \to h_0$ in $C^p(M)$, $\tilde{u}_\epsilon \to u_0$ in $C^2(M)$ as $\epsilon \to 0$, $\tilde{h}_\epsilon \equiv h_0$ and $\tilde{u}_\epsilon \equiv u_0$ in Ω_0 and $\tilde{u}_\epsilon$ is a nondegenerate solution of*

$$\Delta_g \tilde{u}_\epsilon + \tilde{h}_\epsilon \tilde{u}_\epsilon = \tilde{u}_\epsilon^{2^\star-1},\ \tilde{u}_\epsilon > 0 \text{ in } M \text{ for all } \epsilon > 0. \tag{134}$$

Proof of Lemma 13.1. For all $v \in C^{p+2}(M)$ such that $v > -u_0$, we define

$$u(v) := u_0 + v \text{ and } h(v) := u(v)^{2^\star-2} - \frac{u_0^{2^\star-1} - h_0 u_0 + \Delta_g v}{u(v)} = u(v)^{2^\star-2} - \frac{\Delta_g u(v)}{u(v)},$$

so that

$$\Delta_g u(v) + h(v) u(v) = u(v)^{2^\star-1} \text{ in } M. \tag{135}$$

By elliptic regularity, we have $u_0 \in C^{p+1}(M)$. Since moreover $h_0 \in C^p(M)$ and $v \in C^{p+2}(M)$, we obtain that $u(v) \in C^{p+1}(M)$ and $h(v) \in C^p(M)$. Furthermore, we have that $h(v) \to h_0$ in $C^p(M)$ and $u(v) \to u_0$ in $C^2(M)$ as $v \to 0$ in $C^{p+2}(M)$. As is easily seen, to prove the lemma, it suffices to show that there exists a neighborhood Ω_0 of ξ_0 and a family $(v_\epsilon)_{\epsilon>0} \in C^{p+2}(M)$ such that $v_\epsilon \to 0$ in $C^{p+2}(M)$ as $\epsilon \to 0$, $v_\epsilon \equiv 0$ in Ω_0 and $u(v_\epsilon)$ is a nondegenerate solution of (135). Assume by contradiction that this is not true, that is for every neighborhood Ω of ξ_0, there exists a small neighborhood V_Ω of 0 in $C^{p+2}(M)$ such that for every $v \in V_\Omega$, if $v \equiv 0$ in Ω, then $u(v)$ is degenerate i.e. there exists $\phi(v) \in K_v \setminus \{0\}$, where

$$K_v := \{\phi \in H_1^2(M) :\ \Delta_g \phi + h(v)\phi = (2^\star - 1) u(v)^{2^\star-2} \phi \text{ in } M\}.$$

By renormalizing, we can assume that $\phi(v) \in \mathbb{S}_{K_v} := \{\phi \in K_v : \|\phi\|_{H_1^2} = 1\}$. Since $h(tv), u(tv) \to h_0, u_0$ in $C^0(M)$ as $t \to 0$, it then follows that there exists $\phi_v \in K_0$ and $(t_k)_{k\in\mathbb{N}} > 0$ such that $t_k \to 0$ and $\phi(t_k v) \rightharpoonup \phi_v$ weakly in $H_1^2(M)$. By compactness of the embedding $H_1^2(M) \hookrightarrow L^2(M)$, we obtain that $\phi(t_k v) \to \phi_v$ strongly in $L^2(M)$. By standard elliptic theory that we apply to the linear equation satisfied by $\phi(t_k v)$, we then obtain that $\phi(t_k v) \to \phi_v$ strongly in $H_1^2(M)$, so that in particular $\phi_v \in \mathbb{S}_{K_0}$. We then define

$$\psi_k(v) := \frac{\phi(t_k v) - \phi_v}{t_k}.$$

It is easy to check that $\psi_k(v)$ satisfies the equation

$$\Delta_g \psi_k(v) + h_0 \psi_k(v) = (2^\star - 1) u_0^{2^\star-2} \psi_k(v) + f_k(v)\phi(t_k v) \text{ in } M, \tag{136}$$

where

$$\begin{aligned} f_k(v) &:= \frac{1}{t_k}((2^\star - 1)(u(t_k v)^{2^\star-2} - u_0^{2^\star-2}) + h_0 - h(t_k v)) \\ &= \frac{1}{t_k}((2^\star - 2)(u(t_k v)^{2^\star-2} - u_0^{2^\star-2}) + t_k \frac{u_0 \Delta_g v - v \Delta_g u_0}{u_0 u(t_k v)}). \end{aligned}$$

A straightforward Taylor expansion gives

$$f_k(v) = (2^\star - 2)^2 u_0^{2^\star-3} v + u_0^{-1} \Delta_g v - u_0^{-2} v \Delta_g u_0 + \mathrm{o}(1) = u_0^{-1} L_0(v) + \mathrm{o}(1), \tag{137}$$

as $k \to \infty$, uniformly in $v \in V_\Omega$, where

$$L_0(v) := \Delta_g v + h_0 v - (1 - (2^\star - 2)^2) u_0^{2^\star-2} v. \tag{138}$$

It follows that

$$\|\Pi_{K_0^\perp}(\Psi_k(v))\|_{H_1^2} \le C \|f_k(v)\phi(t_k v)\|_{\frac{2n}{n+2}} \le C \|\phi(t_k v)\|_{\frac{2n}{n+2}} \le C \|\phi(t_k v)\|_{H_1^2} \le C,$$

where $\Pi_{K_0^\perp}$ is the orthogonal projection of H_1^2 onto $K_0^\perp$ and the letter C stands for positive constants independent of $k\in\mathbb{N}$ and $v\in V_\Omega$. Since $(\Pi_{K_0^\perp}(\Psi_k(v)))_k$ is bounded in $H_1^2(M)$, up to a subsequence, we may assume that there exists $\psi_v\in K_0^\perp$ such that $\Pi_{K_0^\perp}(\Psi_k(v))\rightharpoonup\psi_v$ weakly in $H_1^2(M)$. Passing to the limit in (136) and using (137), we then obtain that ψ_v satisfies the equation

$$\Delta_g\psi_v+h_0\psi_v=(2^\star-1)u_0^{2^\star-2}\psi_v+u_0^{-1}L_0(v)\phi_v \text{ in } M.$$

In particular, since $\phi_v\in K_0$, multiplying this equation by ϕ_v and integrating by parts yields

$$\int_M u_0^{-1}L_0(v)\phi_v^2\,dv_g=0. \tag{139}$$

We now construct v contradicting (139). For every $\alpha>0$, we choose $\Omega:=B_\alpha(\xi_0)$ and we consider the neighborhood $V_{B_\alpha(\xi_0)}$ of 0 in $C^{p+2}(M)$. We let $r_\alpha\in(0,\alpha)$ be such that $B_0(r_\alpha)\subset V_{B_\alpha(\xi_0)}$ and $\chi\in C^\infty(\mathbb{R})$ be such that $\chi(t)=0$ for $t\le 1$ and $\chi(t)=1$ for $t\ge 2$. We define

$$v_\alpha(x):=e^{-1/\alpha}r_\alpha\chi(d_g(x,\xi_0)/\alpha)u_0(x) \text{ for all } x\in M \text{ and } \alpha>0.$$

As one checks, for small $\alpha>0$,

$$v_\alpha\equiv 0 \text{ in } B_\alpha(\xi_0) \text{ and } v_\alpha\in B_0(r_\alpha)\subset V_{B_\alpha(\xi_0)}. \tag{140}$$

Therefore, $u(v_\alpha)$ is degenerate and the above analysis applies. Since $\|\phi_{v_\alpha}\|_{H_1^2}=1$, $\phi_{v_\alpha}\in K_0\subset C^2(M)$ and K_0 is of finite dimension, up to a subsequence, we can assume that there exists $\phi_0\in K_0$ such that

$$\lim_{\alpha\to 0}\phi_{v_\alpha}=\phi_0\ne 0 \text{ in } C^2(M). \tag{141}$$

Since L_0 is self-adjoint, it follows from (139) that

$$\int_M v_\alpha L_0(u_0^{-1}\phi_{v_\alpha}^2)\,dv_g=0 \text{ for all } \epsilon>0.$$

Since $e^{1/\alpha}r_\alpha^{-1}v_\alpha\to u_0$ in $L^2(M)$ as $\alpha\to 0$, passing to the limit in this equation and using (140) and (141), we obtain

$$\int_M u_0L_0(u_0^{-1}\phi_0^2)\,dv_g=0.$$

Integrating again by parts and noting that $L_0(u_0)=(2^\star-2)^2u_0^{2^\star-1}$, we then obtain

$$0=\int_M u_0^{-1}\phi_0^2L_0(u_0)\,dv_g=(2^\star-2)^2\int_M u_0^{2^\star-2}\phi_0^2\,dv_g,$$

which is a contradiction since $u_0>0$ and $\phi_0\not\equiv 0$. This ends the proof of Lemma 13.1. ☐

We can now end the proof of Theorem 1.3. We let Ω_0, $(\tilde h_\epsilon)_{\epsilon>0}$ and $(\tilde u_\epsilon)_{\epsilon>0}$ be given by Lemma 13.1. Since $\tilde h_\epsilon\equiv h_0$ and $\tilde u_\epsilon\equiv u_0$ in Ω_0, we obtain that $\varphi_{\tilde h_\epsilon,\tilde u_\epsilon}\equiv\varphi_{h_0,u_0}$ in Ω_0 and, therefore, $\varphi_{\tilde h_\epsilon,\tilde u_\epsilon}(\xi_0)=|\nabla\varphi_{\tilde h_\epsilon,\tilde u_\epsilon}(\xi_0)|=0$. For every $\epsilon>0$, we can then mimick the first part of the proof of Theorem 1.2 to construct a family $(\tilde h_{\epsilon,\alpha})_{\alpha>0}\in C^{\max(p,2)}(M)$ such that $\tilde h_{\epsilon,\alpha}\to\tilde h_\epsilon$ in $C^p(M)$ as $\alpha\to 0$, $\varphi_{\tilde h_{\epsilon,\alpha},\tilde u_\epsilon}(\xi_0)=0$, ξ_0 is a nondegenerate critical point of $\varphi_{\tilde h_{\epsilon,\alpha},\tilde u_\epsilon}$ and $K_{\tilde h_{\epsilon,\alpha},\tilde u_\epsilon}(\xi_0)\ne 0$. We now distinguish two cases:

Case $n \geq 7$**.** Note that in this case, we have $\varphi_{\tilde{h}_{\epsilon,\alpha},\tilde{u}_\epsilon} = \varphi_{\tilde{h}_\epsilon}$. Since $\tilde{u}_\epsilon$ is nondegenerate and $\tilde{h}_{\epsilon,\alpha} \to \tilde{h}_\epsilon$ in $C^1(M)$ as $\alpha \to \infty$, the implicit function theorem gives that for small $\alpha > 0$, there exists a nondegenerate solution $\tilde{u}_{\epsilon,\alpha} \in C^2(M)$ to the equation

$$\Delta_g \tilde{u}_{\epsilon,\alpha} + \tilde{h}_{\epsilon,\alpha}\tilde{u}_{\epsilon,\alpha} = \tilde{u}_{\epsilon,\alpha}^{2^\star-1},\ \tilde{u}_{\epsilon,\alpha} > 0 \text{ in } M$$

such that $\tilde{u}_{\epsilon,\alpha} \to \tilde{u}_\epsilon$ in $C^2(M)$ as $\alpha \to 0$. Applying Theorem 1.5, we then obtain that there exist $\beta_{\epsilon,\alpha} > 0$, $(\tilde{h}_{\epsilon,\alpha,\beta})_{0<\beta<\beta_{\epsilon,\alpha}} \in C^{\max(p,2)}(M)$ and $(\tilde{u}_{\epsilon,\alpha,\beta})_{0<\beta<\beta_{\epsilon,\alpha}} \in C^2(M)$ satisfying

$$\Delta_g \tilde{u}_{\epsilon,\alpha,\beta} + \tilde{h}_{\epsilon,\alpha,\beta}\tilde{u}_{\epsilon,\alpha,\beta} = \tilde{u}_{\epsilon,\alpha,\beta}^{2^\star-1} \text{ in } M,\ \tilde{u}_{\epsilon,\alpha,\beta} > 0 \text{ for all } 0<\beta<\beta_{\epsilon,\alpha}$$

and such that $\tilde{h}_{\epsilon,\alpha,\beta} \to \tilde{h}_{\epsilon,\alpha}$ in $C^{\max(p,2)}(M)$, $\tilde{u}_{\epsilon,\alpha,\beta} \rightharpoonup \tilde{u}_{\epsilon,\alpha}$ weakly in $L^{2^\star}(M)$ and $(\tilde{u}_{\epsilon,\alpha,\beta})_\beta$ blows up with one bubble at ξ_0 as $\beta \to 0$. Therefore, we obtain that for every $\epsilon > 0$, there exists $\alpha_\epsilon \in (0,\epsilon)$ and $\beta_\epsilon > 0$ such that

$$\|\tilde{h}_{\epsilon,\alpha_\epsilon} - \tilde{h}_\epsilon\|_{C^p} < \epsilon,\ \|\tilde{u}_{\epsilon,\alpha_\epsilon} - \tilde{u}_\epsilon\|_{C^2} < \epsilon,\ 0 < \beta_\epsilon < \min(\epsilon, \beta_{\epsilon,\alpha_\epsilon}),$$

$$\|\tilde{u}_{\epsilon,\alpha_\epsilon,\beta_\epsilon} - u_0\|_2 < \epsilon,\ \left|\int_M |\tilde{u}_{\epsilon,\alpha_\epsilon,\beta_\epsilon} - \tilde{u}_{\epsilon,\alpha_\epsilon}|^{2^\star} dv_g - \int_{\mathbb{R}^n} U_{1,0}^{2^\star}\, dx\right| < \epsilon$$

and

$$\int_{M\setminus B_\epsilon(\xi_0)} |\tilde{u}_{\epsilon,\alpha_\epsilon,\beta_\epsilon} - \tilde{u}_{\epsilon,\alpha_\epsilon}|^{2^\star} dv_g < \epsilon\,.$$

We then define $u_\epsilon := \tilde{u}_{\epsilon,\alpha_\epsilon,\beta_\epsilon}$, so that

$$\Delta_g u_\epsilon + h_\epsilon u_\alpha = u_\epsilon^{2^\star-1} \text{ in } M,\ \text{ where } h_\epsilon := \tilde{h}_{\epsilon,\alpha_\epsilon,\beta_\epsilon}.$$

As one checks, $h_\epsilon \to h_0$ in $C^p(M)$, $u_\epsilon \rightharpoonup u_0$ weakly in $L^{2^\star}(M)$ and $(u_\epsilon)_\epsilon$ blows up with one bubble at ξ_0 as $\epsilon \to 0$. This proves Theorem 1.3 for $n \geq 7$.

Case $n = 6$**.** In this case, we have $\varphi_{\tilde{h}_{\epsilon,\alpha},\tilde{u}_\epsilon} = \varphi_{\tilde{h}_{\epsilon,\alpha}} - 2\tilde{u}_\epsilon$. Furthermore, noting that $2^\star - 1 = 2$ when $n = 6$, we can rewrite the equation (134) as

$$\Delta_g \tilde{u}_\epsilon + (\tilde{h}_\epsilon - 2\tilde{u}_\epsilon)\tilde{u}_\epsilon = -\tilde{u}_\epsilon^2 \text{ in } M.$$

Since $\tilde{h}_{\epsilon,\alpha} - 2\tilde{u}_\epsilon \to \tilde{h}_\epsilon - 2\tilde{u}_\epsilon$ in $C^0(M)$ as $\alpha \to 0$, a standard minimization method gives that for small $\alpha > 0$, there exists a unique nondegenerate solution $\tilde{u}_{\epsilon,\alpha}$ to the equation

$$\Delta_g \tilde{u}_{\epsilon,\alpha} + (\tilde{h}_{\epsilon,\alpha} - 2\tilde{u}_\epsilon)\tilde{u}_{\epsilon,\alpha} = -\tilde{u}_{\epsilon,\alpha}^2,\ \tilde{u}_{\epsilon,\alpha} > 0 \text{ in } M.$$

As is easily seen, this equation can be rewritten as

$$\Delta_g \tilde{u}_{\epsilon,\alpha} + \mathring{h}_{\epsilon,\alpha}\tilde{u}_{\epsilon,\alpha} = \tilde{u}_{\epsilon,\alpha}^2,\ \tilde{u}_{\epsilon,\alpha} > 0 \text{ in } M,\ \text{ where } \mathring{h}_{\epsilon,\alpha} := \tilde{h}_{\epsilon,\alpha} - 2\tilde{u}_\epsilon + 2\tilde{u}_{\epsilon,\alpha}. \tag{142}$$

Since $\tilde{h}_{\epsilon,\alpha} \to \tilde{h}_\epsilon$ in $C^p(M)$ as $\alpha \to 0$, we obtain that $\mathring{h}_{\epsilon,\alpha} \to \tilde{h}_\epsilon$ in $C^p(M)$ and $\tilde{u}_{\epsilon,\alpha} \to \tilde{u}_\epsilon$ in $C^{p+1}(M)$ as $\alpha \to 0$. Furthermore, since $\tilde{u}_\epsilon$ is nondegenerate, we have that $\tilde{u}_{\epsilon,\alpha}$ is nondegenerate for small $\alpha > 0$. Similarly, since $K_{\tilde{h}_{\epsilon,\alpha},\tilde{u}_\epsilon}(\xi_0) \neq 0$, we obtain that $K_{\mathring{h}_{\epsilon,\alpha},\tilde{u}_{\epsilon,\alpha}}(\xi_0) \neq 0$ for small $\alpha > 0$. Furthermore, we have

$$\varphi_{\mathring{h}_{\epsilon,\alpha},\tilde{u}_{\epsilon,\alpha}} = \mathring{h}_{\epsilon,\alpha} - 2\tilde{u}_{\epsilon,\alpha} - c_n \operatorname{Scal}_g = \tilde{h}_{\epsilon,\alpha} - 2\tilde{u}_\epsilon - c_n \operatorname{Scal}_g = \varphi_{\tilde{h}_{\epsilon,\alpha},\tilde{u}_\epsilon}.$$

In view of the properties satisfied by $\tilde{h}_{\epsilon,\alpha}$, it follows that $\varphi_{\mathring{h}_{\epsilon,\alpha},\tilde{u}_{\epsilon,\alpha}}(\xi_0) = 0$ and ξ_0 is a nondegenerate critical point of $\varphi_{\mathring{h}_{\epsilon,\alpha},\tilde{u}_{\epsilon,\alpha}}$. Applying Theorem 1.5, we then obtain that there exist $\beta_{\epsilon,\alpha} > 0$, $(\tilde{h}_{\epsilon,\alpha,\beta})_{0<\beta<\beta_{\epsilon,\alpha}} \in C^{\max(p,2)}(M)$ and $(\tilde{u}_{\epsilon,\alpha,\beta})_{0<\beta<\beta_{\epsilon,\alpha}} \in C^2(M)$ satisfying

$$\Delta_g \tilde{u}_{\epsilon,\alpha,\beta} + \tilde{h}_{\epsilon,\alpha,\beta}\tilde{u}_{\epsilon,\alpha,\beta} = \tilde{u}_{\epsilon,\alpha,\beta}^{2^\star-1} \text{ in } M,\ \tilde{u}_{\epsilon,\alpha,\beta} > 0 \text{ for all } 0 < \beta < \beta_{\epsilon,\alpha}$$

and such that $\tilde{h}_{\epsilon,\alpha,\beta} \to \mathring{h}_{\epsilon,\alpha}$ in $C^{\max(p,2)}(M)$, $\tilde{u}_{\epsilon,\alpha,\beta} \rightharpoonup \tilde{u}_{\epsilon,\alpha}$ weakly in $L^{2^\star}(M)$ and $(\tilde{u}_{\epsilon,\alpha,\beta})_\beta$ blows up with one bubble at ξ_0 as $\beta \to 0$. Finally, as in the previous case, we obtain the existence of $\alpha_\epsilon > 0$ and $\beta_\epsilon > 0$ such that $u_\epsilon := \tilde{u}_{\epsilon,\alpha_\epsilon,\beta_\epsilon}$ satisfies the desired conditions. This proves Theorem 1.3 for $n = 6$. ☐

14. Necessity of the condition on the gradient

Theorem 14.1. *Let (M, g) be a compact Riemannian manifold of dimension $n \geq 4$. Let $h_0 \in C^1(M)$ be such that $\Delta_g + h_0$ is coercive. Assume that there exist families $(h_\epsilon)_{\epsilon>0} \in C^p(M)$ and $(u_\epsilon)_{\epsilon>0} \in C^2(M)$ satisfying (2) and such that $h_\epsilon \to h_0$ strongly in $C^1(M)$. Assume that (M, g) is locally conformally flat. If $(u_\epsilon)_\epsilon$ blows up with one bubble at some point $\xi_0 \in M$ and $u_\epsilon \rightharpoonup 0$ weakly as $\epsilon \to 0$, then (4) holds true.*

Proof of Theorem 14.1. Let φ_{h_0} be as in (5). The identity $\varphi_{h_0}(\xi_0) = 0$ is a consequence of the results of Druet [7, 9]. Since (M, g) is locally conformally flat, there exists $\Lambda \in C^\infty(M)$ positive such that $\hat{g} := \Lambda^{\frac{4}{n-2}} g$ is flat around ξ_0. Define

$$\hat{u}_\epsilon := \Lambda^{-1} u_\epsilon \text{ and } \hat{h}_\epsilon := (h_\epsilon - c_n \operatorname{Scal}_g)\,\Lambda^{2-2^\star} + c_n \operatorname{Scal}_{\hat{g}}\,.$$

The conformal law (44) yields

$$\Delta_{\hat{g}} \hat{u}_\epsilon + \hat{h}_\epsilon \hat{u}_\epsilon = \hat{u}_\epsilon^{2^\star-1},\ \hat{u}_\epsilon > 0 \text{ in } M. \tag{143}$$

As one checks, on $(M, \hat{g})$, $\hat{u}_\epsilon$ blows-up at ξ_0 in the sense that $\hat{u}_\epsilon = U_{\delta_\epsilon,\xi_\epsilon} + \mathrm{o}(1)$ as $\epsilon \to 0$ in $H_1^2(M)$, where $U_{\delta_\epsilon,\xi_\epsilon}$ is as in (24) (with respect to the metric $\hat{g}$) and $(\delta_\epsilon, \xi_\epsilon) \to (0, \xi_0)$ as $\epsilon \to 0$. It then follows from Druet–Hebey–Robert [11] that there exist $C, \epsilon_0 > 0$ such that for every $\epsilon \in (0, \epsilon_0)$,

$$\frac{1}{C}\left(\frac{\delta_\epsilon}{\delta_\epsilon^2 + d_{\hat{g}}\,(x, \xi_\epsilon)^2}\right)^{\frac{n-2}{2}} \leq \hat{u}_\epsilon(x) \leq C\left(\frac{\delta_\epsilon}{\delta_\epsilon^2 + d_{\hat{g}}\,(x, \xi_\epsilon)^2}\right)^{\frac{n-2}{2}} \tag{144}$$

for all $x \in M$ and, defining

$$U_\epsilon(x) := \delta_\epsilon^{\frac{n-2}{2}} \chi(x) \hat{u}_\epsilon(\xi_\epsilon + \delta_\epsilon x)$$

for all $x \in \mathbb{R}^n$, where χ is a cutoff function on a small ball centered at ξ_0, we have

$$\lim_{\epsilon\to 0} U_\epsilon = U_{1,0} = \left(\frac{\sqrt{n(n-2)}}{1 + |\cdot|^2}\right)^{\frac{n-2}{2}} \text{ in } C^2_{loc}(\mathbb{R}^n). \tag{145}$$

Without loss of generality, via a chart, we may assume that $\hat{g}$ is the Euclidean metric on $B_{2\nu}(\xi_0)$ for some $\nu > 0$. We fix $i \in \{1, ..., n\}$. Differentiating the Pohozaev identity for $\hat{u}_\epsilon$ on $B_\nu(\xi_\epsilon)$ (see for instance Ghoussoub–Robert [13, Proposition 7]) and integrating by parts, we obtain

$$\begin{aligned}(146)\quad &\frac{1}{2}\int_{B_\nu(\xi_\epsilon)}\partial_{x_i}\hat{h}_\epsilon\hat{u}_\epsilon^2\,dx\\ &=\int_{\partial B_\nu(\xi_\epsilon)}\left(\frac{x_i}{|x|}\left(\frac{|\nabla\hat{u}_\epsilon|^2+\hat{h}_\epsilon\hat{u}_\epsilon^2}{2}-\frac{\hat{u}_\epsilon^{2^\star}}{2^\star}\right)-\left\langle\frac{x}{|x|},\nabla\hat{u}_\epsilon\right\rangle\partial_{x_i}\hat{u}_\epsilon\right)d\sigma(x),\end{aligned}$$

where $d\sigma$ is the volume element on $\partial B_\nu(\xi_\epsilon)$. It follows from (144) that there exists $C(\nu)>0$ such that $\hat{u}_\epsilon(x)\le C(\nu)\delta_\epsilon^{\frac{n-2}{2}}$ for all $x\in M\setminus B_{\nu/4}(\xi_0)$ and $\epsilon\in(0,\epsilon_0)$. It then follows from (143) and standard elliptic theory that there exists $C_1>0$ such that $|\nabla\hat{u}_\epsilon(x)|\le C_1\delta_\epsilon^{\frac{n-2}{2}}$ for all $x\in M\setminus B_{\nu/2}(\xi_0)$ and $\epsilon\in(0,\epsilon_0)$. Plugging these inequalities into (146) yields

$$(147)\qquad \int_{B_\nu(\xi_\epsilon)}\partial_{x_i}\hat{h}_\epsilon\hat{u}_\epsilon^2\,dx=\mathrm{O}(\delta_\epsilon^{n-2})\text{ as }\epsilon\to 0.$$

On the other hand, with a change of variable, we obtain

$$\int_{B_\nu(\xi_\epsilon)}\partial_{x_i}\hat{h}_\epsilon\hat{u}_\epsilon^2\,dx=\delta_\epsilon^2\int_{B_{\nu/\delta_\epsilon}(0)}(\partial_{x_i}\hat{h}_\epsilon)(\xi_\epsilon+\delta_\epsilon x)U_\epsilon(x)^2\,dx.$$

The control (144) gives $U_\epsilon\le CU_{1,0}$. Therefore, when $n\ge 5$, Lebesgue's dominated convergence Theorem and (145) yield

$$\int_{B_\nu(\xi_\epsilon)}\partial_{x_i}\hat{h}_\epsilon\hat{u}_\epsilon^2\,dx=\delta_\epsilon^2\left(\partial_{x_i}\hat{h}_\epsilon(\xi_\epsilon)\int_{\mathbb{R}^n}U_{1,0}^2\,dx+\mathrm{o}(1)\right)\text{ as }\epsilon\to 0.$$

Combining this identity with (147), we obtain that $\partial_{x_i}(\varphi_{h_0}\Lambda^{2-2^\star})(\xi_0)=0$ when $n\ge 5$. Since $\Lambda>0$ and $\varphi_{h_0}(\xi_0)=0$, it follows that $\partial_{x_i}\varphi_{h_0}(\xi_0)=0$ when $n\ge 5$.

We now assume that $n=4$. With (144), we obtain

$$\int_{B_\nu(\xi_\epsilon)}|x-\xi_\epsilon|\hat{u}_\epsilon^2\,dx=\mathrm{O}(\delta_\epsilon^2).$$

Therefore, with (147), we obtain

$$(148)\qquad \partial_{x_i}\hat{h}_\epsilon(\xi_\epsilon)=\mathrm{O}\left(\delta_\epsilon^2\left(\int_{B_\nu(\xi_\epsilon)}\hat{u}_\epsilon^2\,dx\right)^{-1}\right).$$

With the lower bound in (144), we then obtain

$$(149)\qquad \int_{B_\nu(\xi_\epsilon)}\hat{u}_\epsilon^2\,dx\ge C\int_{B_\nu(\xi_\epsilon)}\left(\frac{\delta_\epsilon}{\delta_\epsilon^2+|x-\xi_\epsilon|^2}\right)^{n-2}dx\ge C\delta_\epsilon^2\ln(1/\delta_\epsilon).$$

It follows from (148) and (149) that $\partial_{x_i}\hat{h}_\epsilon(\xi_\epsilon)=\mathrm{o}(1)$ as $\epsilon\to 0$ and so again $\partial_{x_i}\varphi_{h_0}(\xi_0)=0$ when $n=4$.

In all cases, we thus obtain that $\nabla\varphi_{h_0}(\xi_0)=0$. This ends the proof of Theorem 14.1. □

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