SHARP SOBOLEV ASYMPTOTICS FOR CRITICAL ANISOTROPIC EQUATIONS

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Abstract. We investigate blow-up theory and prove sharp Sobolev asymptotics for a general class of anisotropic critical equations in bounded domains of the Euclidean space.

1. Introduction and statement of the results

We consider in this paper critical anisotropic equations in bounded domains of the Euclidean space. Anisotropic operators appear in several places in the literature. Recent references can be found in physics [13, 17, 18, 23, 24], in biology [10, 11], and in image processing (see, for instance, the monograph by Weickert [50]). By definition, anisotropic operators involve directional derivatives with distinct weights. Given an open subset Ω of $\mathbb{R}^n$, $n \geq 2$, and $\overrightarrow{p} = (p_1, \dots, p_n)$, we let $D^{1,\overrightarrow{p}}(\Omega)$ be the Sobolev space defined as the completion of the vector space of all smooth functions with compact support in Ω with respect to the norm

$$\|u\|_{D^{1,\overrightarrow{p}}(\Omega)} = \sum_{i=1}^{n} \left\| \frac{\partial u}{\partial x_i} \right\|_{L^{p_i}(\Omega)}.$$

We let also p^* be the corresponding critical exponent for the embeddings of the anisotropic Sobolev space $D^{1,\overrightarrow{p}}(\Omega)$ into Lebesgue spaces. We assume that the exponents p_i satisfy

$$\sum_{i=1}^{n} \frac{1}{p_i} > 1 \quad \text{and} \quad 1 < p_i < \frac{n}{\sum_{j=1}^{n} \frac{1}{p_j} - 1} \quad \text{for } i = 1, \dots, n\,. \tag{1.1}$$

Then p^* is given by

$$p^* = \frac{n}{\sum_{j=1}^{n} \frac{1}{p_j} - 1}\,, \tag{1.2}$$

and there is a continuous embedding of $D^{1,\overrightarrow{p}}(\Omega)$ into $L^r(\Omega)$ for all $r \leq p^*$ which turns out to be compact only when $r < p^*$. Possible references on the theory of anisotropic Sobolev spaces are Besov [12], Kruzhkov–Kolodiĭ [28], Kruzhkov–Korolev [29], Lu [34], Nikol′skiĭ [37], Rákosník [41, 42], and Troisi [49]. In what follows, we let f be a Caratheodory function in $\Omega \times \mathbb{R}$ satisfying the conditions

$$f(\cdot, 0) = 0 \quad \text{and} \quad |f(\cdot, u)| \leq C\left(|u|^{q-1} + 1\right) \quad \text{a.e. in } \Omega \tag{1.3}$$

for some real number q in $(1, p^*)$ and for some positive constant C independent of u. We consider the following critical anisotropic equation with zero Dirichlet boundary condition

$$\begin{cases} -\displaystyle\sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(\left|\frac{\partial u}{\partial x_i}\right|^{p_i-2} \frac{\partial u}{\partial x_i}\right) = f(\cdot, u) + |u|^{p^*-2} u & \text{in } \Omega\,, \\ u = 0 & \text{on } \partial\Omega\,. \end{cases} \tag{1.4}$$

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Equations like (1.4) have received much attention in recent years. They have been investigated by Antontsev–Shmarev [6–8], Fragalà–Gazzola–Kawohl [21], Fragalà–Gazzola–Lieberman [22], El Hamidi–Rakotoson [19, 20], Lieberman [30, 31], and Mihăilescu–Pucci–Rădulescu [35, 36]. Time evolution versions of these equations appear in several branches of applied sciences. They have strong physical background. They appear, for instance, when dealing with the dynamics of fluids in the context of anisotropic media when the conductivities of the media are different in different directions. We refer to the extensive books by Antontsev–Díaz–Shmarev [5] and Bear [9] for discussions in this direction. They also appear in biology, see, for instance, Bendahmane–Karlsen [10] for a mathematical discussion, as a model for the propagation of epidemic diseases in heterogeneous domains.

We aim here in describing the asymptotic behavior in the energy space of Palais–Smale sequences associated with equation (1.4). Such a description is well-known in the isotropic regime, where, by definition, $p_+ = p_-$ if we let $p_+ = \max(p_1, \dots, p_n)$ and $p_- = \min(p_1, \dots, p_n)$ stand for the maximum and minimum values of the anisotropic configuration. In particular, for smooth, bounded domains in the isotropic regime, the geometry of the domain play no role in the description. The situation is different when anisotropy is involved. As we shall see below, the boundary of Ω and the geometry of the domain turn out to play a crucial role through the action of the anisotropic blow-up transformation rule described by (1.5). The anisotropic affine transformation (1.5) when $\mu \to 0$ distorts the ambient space, and $\partial\Omega$ may develop cusp points in the limit. Because of this distortion, we are led to introduce geometric properties of Ω such as the property of being asymptotically $\overrightarrow{p}$-stable or strongly asymptotically $\overrightarrow{p}$-stable. Roughly speaking, asymptotically $\overrightarrow{p}$-stable domains are domains which, in the limit, after blow-up, turn out to satisfy the segment property. The limit domain may still be odd but, at least, it preserves extension properties of Sobolev spaces. Strongly asymptotically $\overrightarrow{p}$-stable domains are domains which, in the limit, after blow-up, turn out to be, as it is in the isotropic regime for bounded, smooth domains, either the empty set, the whole space $\mathbb{R}^n$, or a halfspace. These geometric notions of asymptotic stability are investigated in Section 2. Among other results, we prove in Section 2 that ellipsoidal disks are always asymptotically $\overrightarrow{p}$-stable, that ellipsoidal annuli are asymptotically $\overrightarrow{p}$-stable if and only if $(p_+/p_-) + (p_+/p^\star) \le 2$, and that both ellipsoidal disks and annuli are strongly asymptotically $\overrightarrow{p}$-stable if and only if $(p_+/p_-) + (p_+/p^\star) < 2$.

Needless to mention, bubble tree decompositions and the analysis of asymptotic behaviors in energy spaces have numerous applications in the isotropic regime. They quickly turned out to be key points in the use of topological arguments such as Lusternik–Schnirelmann equivariant categories. They also turned out to be key points in the analysis of ruling out bubbling and proving compactness of solutions of critical equations. Possible references in book form on these subjects are Druet–Hebey–Robert [16], Ghoussoub [25], and Struwe [48]. Our Theorem 1.2 below provides such bubble tree decompositions and analysis of asymptotic behaviors in the more involved anisotropic regime. Our result should be seen as a key step in the development of topological and renormalization arguments for critical anisotropic equations.

Before stating our result, let us fix some notations. In order to enlarge our viewpoint, we let $(r_\alpha)_\alpha$ be a sequence of real numbers in $(1, p^*]$ converging to p^*, and for any α, we define the functional I_α on $D^{1,\overrightarrow{p}}(\Omega)$ by

$$I_\alpha(u) = \sum_{i=1}^{n} \frac{1}{p_i} \int_\Omega \left| \frac{\partial u}{\partial x_i} \right|^{p_i} dx - \int_\Omega F(x, u)\, dx - \frac{1}{r_\alpha} \int_\Omega |u|^{r_\alpha} dx\,,$$

where $F(x,u) = \int_0^u f(x,s)\,ds$. We also define the functional I_∞ by

$$I_\infty(u) = \sum_{i=1}^n \frac{1}{p_i} \int_\Omega \left|\frac{\partial u}{\partial x_i}\right|^{p_i} dx - \int_\Omega F(x,u)\,dx - \frac{1}{p^*} \int_\Omega |u|^{p^*} dx\,.$$

We say that a sequence $(u_\alpha)_\alpha$ in $D^{1,\overrightarrow{p}}(\Omega)$ is Palais–Smale for the functionals $(I_\alpha)_\alpha$ if the sequence $(I_\alpha(u_\alpha))_\alpha$ is bounded and if there holds $DI_\alpha(u_\alpha) \to 0$ in $D^{1,\overrightarrow{p}}(\Omega)'$ as $\alpha \to +\infty$. The classical mountain pass lemma, as expressed in Ambrosetti–Rabinowitz [4], provides the existence of such objects. Needless to say, a bounded sequence $(u_\alpha)_\alpha$ in $D^{1,\overrightarrow{p}}(\Omega)$ of solutions of the equations associated with the functionals $(I_\alpha)_\alpha$ is Palais–Smale for the functionals $(I_\alpha)_\alpha$. Some existence results for subcritical anisotropic problems can be found in Fragalà–Gazzola–Kawohl [21].

For any point $a = (a_1, \dots, a_n)$ in $\mathbb{R}^n$ and for any positive real number μ, we define the anisotropic affine transformation $\tau^{\overrightarrow{p}}_{\mu,a} : \mathbb{R}^n \to \mathbb{R}^n$ by

$$\tau^{\overrightarrow{p}}_{\mu,a}(x_1, \dots, x_n) = \Big(\mu^{\frac{p_1 - p^*}{p_1}}(x_1 - a_1), \dots, \mu^{\frac{p_n - p^*}{p_n}}(x_n - a_n)\Big). \tag{1.5}$$

We then introduce the notion of a bubble associated with equation (1.4).

Definition 1.1. *Let $(\mu_\alpha)_\alpha$ be a sequence of positive real numbers converging to 0, $(x_\alpha)_\alpha$ be a converging sequence in $\mathbb{R}^n$, λ be a positive real number, U be an open subset of $\mathbb{R}^n$, and u be a nontrivial solution in $D^{1,\overrightarrow{p}}(U)$ of the equation*

$$\begin{cases} -\displaystyle\sum_{i=1}^n \frac{\partial}{\partial x_i}\left(\left|\frac{\partial u}{\partial x_i}\right|^{p_i - 2} \frac{\partial u}{\partial x_i}\right) = \lambda |u|^{p^* - 2} u & \text{in } U\,, \\ u = 0 & \text{on } \partial U\,. \end{cases} \tag{1.6}$$

We call $\overrightarrow{p}$-bubble of centers $(x_\alpha)_\alpha$, weights $(\mu_\alpha)_\alpha$, multiplier λ, domain U, and shape function u, the sequence $(B_\alpha)_\alpha$ of functions defined by

$$B_\alpha = \frac{1}{\mu_\alpha} u \circ \tau^{\overrightarrow{p}}_{\mu_\alpha, x_\alpha}$$

for all α, where $\tau^{\overrightarrow{p}}_{\mu_\alpha, x_\alpha}$ is as in (1.5). In case $U = \mathbb{R}^n$, we say that the $\overrightarrow{p}$-bubble $(B_\alpha)_\alpha$ is global. In case $u \ge 0$, we say that the $\overrightarrow{p}$-bubble $(B_\alpha)_\alpha$ is nonnegative.

If a $\overrightarrow{p}$-bubble $(B_\alpha)_\alpha$ is not global, then we extend the B_α's and u by 0 outside of their domains of definition so as to regard them as functions in $D^{1,\overrightarrow{p}}(\mathbb{R}^n)$. We also define the energy $E(B_\alpha)$ of the $\overrightarrow{p}$-bubble $(B_\alpha)_\alpha$ by

$$E(B_\alpha) = \sum_{i=1}^n \frac{1}{p_i} \int_{\mathbb{R}^n} \left|\frac{\partial u}{\partial x_i}\right|^{p_i} dx - \frac{\lambda}{p^*} \int_{\mathbb{R}^n} |u|^{p^*} dx\,.$$

Taking into account equation (1.6), we compute

$$E(B_\alpha) = \sum_{i=1}^n \frac{p^* - p_i}{p_i p^*} \int_{\mathbb{R}^n} \left|\frac{\partial u}{\partial x_i}\right|^{p_i} dx > 0\,.$$

Moreover, it is easily checked that a $\overrightarrow{p}$-bubble $(B_\alpha)_\alpha$ converges weakly to 0 in $D^{1,\overrightarrow{p}}(\mathbb{R}^n)$ and $L^{p^*}(\mathbb{R}^n)$ as $\alpha \to +\infty$ while there hold $\|B_\alpha\|_{D^{1,\overrightarrow{p}}(\mathbb{R}^n)} = \|u\|_{D^{1,\overrightarrow{p}}(\mathbb{R}^n)}$ and $\|B_\alpha\|_{L^{p^*}(\mathbb{R}^n)} = \|u\|_{L^{p^*}(\mathbb{R}^n)}$ for all α.

As already mentioned, we are concerned in this paper with the asymptotic behavior in $D^{1,\overrightarrow{p}}(\Omega)$ of Palais–Smale sequences $(u_\alpha)_\alpha$ for the functionals $(I_\alpha)_\alpha$. Unless otherwise stated, we extend the u_α's by 0 so as to regard them as functions in $D^{1,\overrightarrow{p}}(\mathbb{R}^n)$. Theorem 1.2 below generalizes to the anisotropic case the compactness result proved by Struwe [47] in the case of the Laplace operator. Related references to Struwe [47] are Brézis–Coron [14], Lions [32, 33], Sacks–Uhlenbeck [44], Schoen [46], and Wente [51]. We refer also to Alves [2, 3], Hebey [26], Hebey-Robert [27], Robert [43], Saintier [45], and Yan [52]. The definition of an asymptotically $\overrightarrow{p}$-stable domain is postponed to the next section.

Theorem 1.2. *Let Ω be an asymptotically $\overrightarrow{p}$-stable bounded domain of $\mathbb{R}^n$, $n \geq 2$, the exponents p_i satisfy* (1.1)*, and f be a Caratheodory function in $\Omega \times \mathbb{R}$ satisfying* (1.3)*. Let also $(r_\alpha)_\alpha$ be a sequence of real numbers in $(1, p^*]$ converging to p^*. For any Palais–Smale sequence $(u_\alpha)_\alpha$ for the functionals $(I_\alpha)_\alpha$, there exist a weak solution u_∞ of problem* (1.4)*, a natural number k, and for $j = 1, \dots, k$, a $\overrightarrow{p}$-bubble $(B_\alpha^j)_\alpha$ of weights $(\mu_\alpha^j)_\alpha$ and multiplier λ_j satisfying $(\mu_\alpha^j)^{p^*-r_\alpha} \to \lambda_j$ as $\alpha \to +\infty$, such that, up to a subsequence,*

$$u_\alpha = u_\infty + \sum_{j=1}^{k} B_\alpha^j + R_\alpha \tag{1.7}$$

for all α, where $R_\alpha \to 0$ in $D^{1,\overrightarrow{p}}(\mathbb{R}^n)$ as $\alpha \to +\infty$. Moreover, there holds

$$I_\alpha(u_\alpha) = I_\infty(u_\infty) + \sum_{j=1}^{k} E\left(B_\alpha^j\right) + \mathrm{o}(1) \tag{1.8}$$

as $\alpha \to +\infty$. If in addition the functions u_α are nonnegative, then so are u_∞ and the $\overrightarrow{p}$-bubbles $(B_\alpha^1)_\alpha, \dots, (B_\alpha^k)_\alpha$.

Concerning the multipliers λ_j of the $\overrightarrow{p}$-bubbles in Theorem 1.2, we get $\lambda_j \leq 1$ since there holds $(\mu_\alpha^j)^{p^*-r_\alpha} \leq 1$ for α large and for $j = 1, \dots, k$. Moreover, in the model case where there holds $r_\alpha = p^*$ for all α, we get $\lambda_j = 1$ for $j = 1, \dots, k$.

There are several situations where we do know that the solution u_∞ in Theorem 1.2 is identically zero. For instance, by Fragalà–Gazzola–Kawohl [21], we get that there does not exist any nontrivial, nonnegative weak solution of problem (1.4) on a smooth, bounded domain Ω when there hold $f \equiv 0$ and

$$\sum_{i=1}^{n} \frac{p^* - p_i}{p_i} (x_i - a_i)\, \nu_i(x) > 0$$

for all points x on $\partial\Omega$, where $(\nu_1(x), \dots, \nu_n(x))$ is the outward unit normal vector to $\partial\Omega$ at x, and when

$$p_+ < \frac{n+2}{n} p_-\,,$$

where $p_+ = \max p_i$ and $p_- = \min p_i$. Moreover, as recently shown by Fragalà–Gazzola–Lieberman [22], the last assumption can be removed in case Ω is convex.

Our next result comes to complete Theorem 1.2. We state it as follows. Here again, we refer the reader to the next section for the definition of strongly asymptotically $\overrightarrow{p}$-stable domains.

Theorem 1.3. *If the domain Ω is strongly asymptotically $\overrightarrow{p}$-stable and if $p_- \geq 2$, then the $\overrightarrow{p}$-bubbles we get in Theorem 1.2 are global.*

Final remarks in this introduction concern the regularity of the shape functions of global $\overrightarrow{p}$-bubbles, namely the regularity of the nontrivial weak solutions of the first equation in (1.6) when posed in the whole space $\mathbb{R}^n$. By El Hamidi–Rakotoson [20], we know that there exists at least one nontrivial, nonnegative weak solution of the equation, and it is shown that this solution belongs to $L^r(\mathbb{R}^n)$ for all r in $[p^*, +\infty]$. We even get by Lieberman [30, 31] that the gradient of this solution belongs to $L^\infty(\mathbb{R}^n)$. For instance, in the isotropic case $p_i = p$ for $i = 1, \dots, n$, for any $\mu \geq 0$ and for any point $a = (a_1, \dots, a_n)$ in $\mathbb{R}^n$, the function

$$\mathcal{U}^p_{\mu,a}(x) = \left(\frac{\mu}{\mu^p + C_{n,p} \lambda^{\frac{1}{p-1}} \sum_{i=1}^n |x_i - a_i|^{\frac{p}{p-1}}} \right)^{\frac{n-p}{p}},$$

where $C_{n,p} = (p-1)n^{-1/(p-1)}/(n-p)$, is a weak solution of the first equation in (1.6) when posed in the whole of $\mathbb{R}^n$. As a remark, it can be noted that the $\mathcal{U}^p_{\mu,a}$'s all have the same energy depending only on n and p.

After discussing our geometric hypotheses on the domain Ω in Section 2, we describe the proof of Theorem 1.2 in Section 3, leaving the main lemma to Section 4 and the proof of Theorem 1.3 to Section 5.

2. The anisotropic geometry of the domain

In this section, we define and comment the geometric notions of asymptotically $\overrightarrow{p}$-stable and strongly asymptotically $\overrightarrow{p}$-stable domains which can be found in the statements of Theorems 1.2 and 1.3. The general definition of asymptotically $\overrightarrow{p}$-stable domains and strongly asymptotically $\overrightarrow{p}$-stable domains is given in Definition 2.1 below and illustrated on ellipsoidal disks and annuli in Proposition 2.2. Sufficiently regular domains, in the isotropic sense, are shown to be strongly asymptotically $\overrightarrow{p}$-stable in Proposition 2.4, see also the remark following the proposition. Anisotropic regularity provides a simple criteria for asymptotic $\overrightarrow{p}$-stability. We recall that an open subset U of $\mathbb{R}^n$ is said to satisfy the segment property if for any point a on ∂U there exist a neighborhood X_a of a and a nonzero vector σ_a such that $X_a \cap \overline{U} + t\sigma_a \subset U$ for all real numbers t in $(0, 1)$. By convention, the empty set satisfies the segment property.

Definition 2.1. *An open subset Ω of $\mathbb{R}^n$ is said to be asymptotically $\overrightarrow{p}$-stable if for any sequence $(\mu_\alpha)_\alpha$ of positive real numbers converging to 0 and for any sequence $(x_\alpha)_\alpha$ in $\mathbb{R}^n$, the sets $\Omega_\alpha = \tau^{\overrightarrow{p}}_{\mu_\alpha, x_\alpha}(\Omega)$, where $\tau^{\overrightarrow{p}}_{\mu_\alpha, x_\alpha}$ is as in (1.5), converge, up to a subsequence, to an open subset U of $\mathbb{R}^n$ satisfying the segment property as $\alpha \to +\infty$ in the sense that the two following properties hold true:*

(i) *any compact subset of U is included in Ω_α for α large,*
(ii) *for any compact $K \subset \mathbb{R}^n$, there holds $|K \cap \Omega_\alpha \backslash U| \to 0$ as $\alpha \to +\infty$.*

Moreover, Ω is said to be strongly asymptotically $\overrightarrow{p}$-stable if we can choose U to be either the empty set, the whole space $\mathbb{R}^n$, or a halfspace.

Adapting classical arguments, as developed, for instance, in Adams–Fournier [1], we get that any nonempty open subset U of $\mathbb{R}^n$ satisfying the segment property is such that the anisotropic Sobolev space $D^{1,\overrightarrow{p}}(U)$ consists of the restrictions to U of functions in $D^{1,\overrightarrow{p}}(\mathbb{R}^n)$ with support included in $\overline{U}$. Asymptotic $\overrightarrow{p}$-stability and strong asymptotic $\overrightarrow{p}$-stability are subtle notions. Figures 1 and 2 describe two opposite situations in the case of an annulus. In Figure 1, there is small anisotropy (p_+ is close to p_-), and the domain behaves in the same way as in the isotropic case, namely it converges to a halfspace. In Figure 2, there is strong anisotropy (p_+ is far from p_-), and the domain bends on itself and converges to the whole plane minus a half-line.

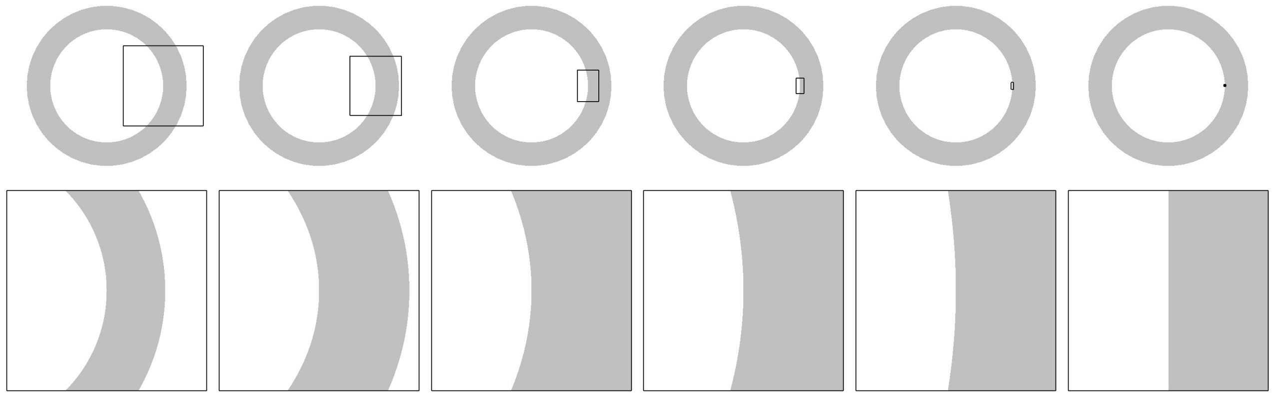

FIGURE 1. Rescaling of an annulus with small anisotropy ($n = 2$, $p_1 = 1.5$, $p_2 = 2$). The first line describes the scale in the rescaling. The second line describes the deformation of the domain.

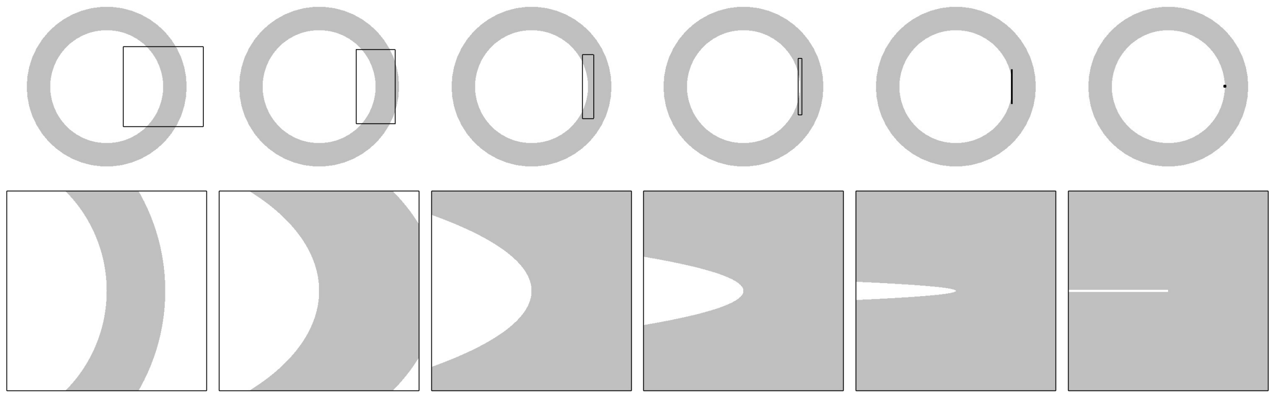

FIGURE 2. Rescaling of an annulus with strong anisotropy ($n = 2$, $p_1 = 1.1$, $p_2 = 9$). The first line describes the scale in the rescaling. The second line describes the deformation of the domain.

Proposition 2.2 below illustrates asymptotic $\overrightarrow{p}$-stability on ellipsoidal disks and annuli. The cases described in Figures 1 and 2 are contained in the $\mathcal{E}(\overrightarrow{b})\backslash\overline{\mathcal{E}(\overrightarrow{a})}$-part of the proposition. Ellipsoidal disks are always asymptotically $\overrightarrow{p}$-stable. The interior boundary in ellipsoidal annuli is the boundary which creates problems.

Proposition 2.2. *Let $\overrightarrow{p} = (p_1, \dots, p_n)$ satisfy (1.1). Given $\overrightarrow{a} = (a_1, \dots, a_n)$ in $(\mathbb{R}_+^*)^n$, we let $\mathcal{E}(\overrightarrow{a})$ be the ellipsoidal disk consisting of the points $(y_1, \dots, y_n)$ in $\mathbb{R}^n$ such that $\sum_{i=1}^n a_i y_i^2 < 1$. For any $\overrightarrow{a}$ in $(\mathbb{R}_+^*)^n$, the ellipsoidal disk $\mathcal{E}(\overrightarrow{a})$ is asymptotically $\overrightarrow{p}$-stable. On the other hand, for any $\overrightarrow{a} = (a_1, \dots, a_n)$ and $\overrightarrow{b} = (b_1, \dots, b_n)$ in $(\mathbb{R}_+^*)^n$ satisfying $b_i < a_i$ for $i = 1, \dots, n$, the ellipsoidal annulus $\mathcal{E}(\overrightarrow{b})\backslash\overline{\mathcal{E}(\overrightarrow{a})}$ is asymptotically $\overrightarrow{p}$-stable if and only if there holds*

$$\frac{p_+}{p_-} + \frac{p_+}{p^*} \le 2\,, \tag{2.1}$$

where $p_+ = \max p_i$, $p_- = \min p_i$, and where p^* is as in (1.2). At last, both $\mathcal{E}(\overrightarrow{a})$ and $\mathcal{E}(\overrightarrow{b})\backslash\overline{\mathcal{E}(\overrightarrow{a})}$ are strongly asymptotically $\overrightarrow{p}$-stable if and only if the inequality in (2.1) is strict.

As a remark, the strict inequality in (2.1) is automatically satisfied in the isotropic case.

Proof. We start with the proof that the ellipsoidal disk $\mathcal{E}(\overrightarrow{a})$ is asymptotically $\overrightarrow{p}$-stable. We let $(\mu_\alpha)_\alpha$ be a sequence of positive real numbers converging to 0, $(x_\alpha)_\alpha$ be a sequence in $\mathbb{R}^n$, and $\varphi_{\overrightarrow{a}} : \mathbb{R}^n \to \mathbb{R}$ be the function defined by

$$\varphi_{\overrightarrow{a}}(y_1, \dots, y_n) = \sum_{i=1}^n a_i y_i^2 - 1 \,.$$

For any α and any point $y = (y_1, \dots, y_n)$ in $\mathbb{R}^n$, we get

$$\varphi_{\overrightarrow{a}} \circ \left(\tau^{\overrightarrow{p}}_{\mu_\alpha, x_\alpha}\right)^{-1}(y) = \sum_{i=1}^n a_i \mu_\alpha^{2\frac{p^*-p_i}{p_i}} y_i^2 + 2\sum_{i=1}^n a_i x_\alpha^i \mu_\alpha^{\frac{p^*-p_i}{p_i}} y_i + \sum_{i=1}^n a_i \left(x_\alpha^i\right)^2 - 1 \,, \tag{2.2}$$

where $x_\alpha = (x_\alpha^1, \dots, x_\alpha^n)$. Passing if necessary to a subsequence, we may assume that there exists l in $[0, +\infty]$ such that there holds $\sum_{i=1}^n a_i (x_\alpha^i)^2 \to l$ as $\alpha \to +\infty$. One can easily check that the sets $\mathcal{E}_\alpha = \tau^{\overrightarrow{p}}_{\mu_\alpha, x_\alpha}(\mathcal{E}(\overrightarrow{a}))$ converge to the whole space $\mathbb{R}^n$ when $l < 1$ and to the empty set when $l > 1$ in the sense of Definition 2.1. In case $l = 1$, up to a subsequence, it follows from (2.2) that there exist a sequence $(\nu_\alpha)_\alpha$ of positive real numbers converging to 0, some real numbers $d_i \geq 0$ and c_i, $i = 1, \dots, n$, not all zero, such that

$$\varphi_{\overrightarrow{a}} \circ \left(\tau^{\overrightarrow{p}}_{\mu_\alpha, x_\alpha}\right)^{-1}(y) = \left(\sum_{i=1}^n d_i y_i^2 + \sum_{i=1}^n c_i y_i + c_0\right) \nu_\alpha + \mathrm{o}(\nu_\alpha) \tag{2.3}$$

as $\alpha \to +\infty$, uniformly in any compact subset of $\mathbb{R}^n$. One can then easily check that the sets $\mathcal{E}_\alpha$ converge to the domain

$$U = \left\{(y_1, \dots, y_n) \in \mathbb{R}^n;\ \sum_{i=1}^n d_i y_i^2 + \sum_{i=1}^n c_i y_i + c_0 < 0\right\} \tag{2.4}$$

as $\alpha \to +\infty$ in the sense of Definition 2.1. Clearly, U satisfies the segment property when not empty. This ends the proof of the asymptotic $\overrightarrow{p}$-stability of the ellipsoidal disk $\mathcal{E}(\overrightarrow{a})$. Now we prove that the ellipsoidal annulus $\mathcal{E}(\overrightarrow{b})\backslash\overline{\mathcal{E}(\overrightarrow{a})}$ is asymptotically $\overrightarrow{p}$-stable if and only if (2.1) holds true. First, we assume that (2.1) holds true, and let $(\mu_\alpha)_\alpha$ be a sequence of positive real numbers converging to 0, $(x_\alpha)_\alpha$ be a sequence in $\mathbb{R}^n$, and $\mathcal{F}_\alpha = \tau^{\overrightarrow{p}}_{\mu_\alpha, x_\alpha}(\mathcal{E}(\overrightarrow{b})\backslash\overline{\mathcal{E}(\overrightarrow{a})})$. Thanks to the above arguments, we may assume moreover that there holds $\sum_{i=1}^n a_i (x_\alpha^i)^2 \to 1$ as $\alpha \to +\infty$. In particular, there exists an index i_0 such that $x_\alpha^{i_0}$ converges to a positive real number as $\alpha \to +\infty$. By (2.1), we can write

$$\mu_\alpha^{2\frac{p^*-p_i}{p_i}} = \mathrm{O}\left(x_\alpha^{i_0} \mu_\alpha^{\frac{p^*-p_{i_0}}{p_{i_0}}}\right) \tag{2.5}$$

as $\alpha \to +\infty$ for $i = 1, \dots, n$. In the same way as in the proof of the asymptotic $\overrightarrow{p}$-stability of the ellipsoidal disk $\mathcal{E}(\overrightarrow{a})$, passing if necessary to a subsequence, we may assume that there exist a sequence $(\nu_\alpha)_\alpha$ of positive real numbers converging to 0, some real numbers $d_i \geq 0$ and c_i, $i = 1, \dots, n$, not all zero, such that (2.3) holds true. By (2.5), if $c_{i_0} = 0$, then $d_i = 0$ for $i = 1, \dots, n$. It easily follows that the sets $\mathcal{F}_\alpha$ converge to the domain

$$U = \left\{(y_1, \dots, y_n) \in \mathbb{R}^n;\ \sum_{i=1}^n d_i y_i^2 + \sum_{i=1}^n c_i y_i + c_0 > 0\right\} \tag{2.6}$$

as $\alpha \to +\infty$ in the sense of Definition 2.1. The domain U is either empty, or it satisfies the segment property. We have proved that if (2.1) holds true, then the ellipsoidal annulus $\mathcal{E}(\overrightarrow{b})\backslash\overline{\mathcal{E}(\overrightarrow{a})}$ is asymptotically $\overrightarrow{p}$-stable. In order to get the converse, we let $(\mu_\alpha)_\alpha$ be a sequence of positive real numbers converging to 0, i_0 be an index such that $p_{i_0} = p_-$, and $x_0 = (x_0^1, \dots, x_0^n)$ be the point given by

$$x_0^i = \begin{cases} \dfrac{1}{\sqrt{a_{i_0}}} & \text{if } i = i_0 \\ 0 & \text{otherwise.} \end{cases}$$

We define

$$I_0 = \Big\{ i \in \{1, \dots, n\}\,;\ \frac{p_i}{p_-} + \frac{p_i}{p^*} = 2 \Big\}$$

and

$$I_1 = \Big\{ i \in \{1, \dots, n\}\,;\ \frac{p_i}{p_-} + \frac{p_i}{p^*} > 2 \Big\}.$$

We let $\hat{U} = \hat{U}_1 \cup \hat{U}_2$, where

$$\hat{U}_1 = \Big\{ (y_1, \dots, y_n) \in \mathbb{R}^n;\ \sum_{i \in I_1} a_i y_i^2 > 0 \Big\}$$

and

$$\hat{U}_2 = \Big\{ (y_1, \dots, y_n) \in \mathbb{R}^n;\ \sum_{i \in I_1} a_i y_i^2 = 0 \quad \text{and} \quad \sum_{i \in I_0} a_i y_i^2 + 2\sqrt{a_{i_0}}\, y_{i_0} > 0 \Big\}.$$

We let also

$$\widetilde{U} = \Big\{ (y_1, \dots, y_n) \in \mathbb{R}^n;\ \sum_{i \in I_1} a_i y_i^2 = 0 \quad \text{and} \quad \sum_{i \in I_0} a_i y_i^2 + 2\sqrt{a_{i_0}}\, y_{i_0} = 0 \Big\}.$$

Clearly, $\mathcal{F}_\alpha^0 = \tau_{\mu_\alpha, x_0}^{\vec{p}} (\mathcal{E}(\vec{b}) \backslash \overline{\mathcal{E}(\vec{a})})$ converge to the domain $\hat{U}_1$ as $\alpha \to +\infty$ in the sense of Definition 2.1. Now we let U be an open subset of $\mathbb{R}^n$ which is the limit of the sets $\mathcal{F}_\alpha^0$ as $\alpha \to +\infty$ in the sense of Definition 2.1. By (2.2), we get that U is included in $\hat{U} \cup \widetilde{U}$ and thus in $\hat{U}$ since the interior of the set $\hat{U} \cup \widetilde{U}$ is precisely $\hat{U}$. It follows that there holds $U = \hat{U} \backslash E$ for some subset E of $\hat{U}$ satisfying $|E| = 0$. As is easily checked, such U's never satisfy the segment property when the set I_1 is not empty, namely when (2.1) does not hold true. This ends the proof that the ellipsoidal annulus $\mathcal{E}(\vec{b}) \backslash \overline{\mathcal{E}(\vec{a})}$ is asymptotically $\vec{p}$-stable if and only if (2.1) holds true. One can easily see in the above proofs that the d_i's all are zero in case the inequality in (2.1) is strict, and thus that none of the sets U in (2.4) and (2.6) is either the empty set, the whole space $\mathbb{R}^n$, or a halfspace. It follows that if the inequality in (2.1) is strict, then both domains $\mathcal{E}(\vec{a})$ and $\mathcal{E}(\vec{b}) \backslash \overline{\mathcal{E}(\vec{a})}$ are strongly asymptotically $\vec{p}$-stable. In order to get the converse, we now assume that the strict inequality in (2.1) does not hold true, namely that the index set $I_0 \cup I_1$ is not empty. Letting $\hat{U}$ be as in the above, we first get that none of the sets of the form $U = \hat{U} \backslash E$, where $|E| = 0$, is either empty, the whole space $\mathbb{R}^n$, or a halfspace, and thus that the ellipsoidal annulus $\mathcal{E}(\vec{b}) \backslash \overline{\mathcal{E}(\vec{a})}$ is not strongly asymptotically $\vec{p}$-stable. Then, we consider the ellipsoidal disk $\mathcal{E}(\vec{a})$. We let $(\mu_\alpha)_\alpha$ be a sequence in $(0, 1)$ converging to 0, i_0 and i_1 be two indices such that $p_{i_0} = p_-$ and $p_{i_1} = p_+$, and $x_\alpha = (x_\alpha^1, \dots, x_\alpha^n)$ be the point given by

$$x_\alpha^i = \begin{cases} \sqrt{\dfrac{1-\mu_\alpha^{2\frac{p^*-p_+}{p_+}}}{a_{i_0}}} & \text{if } i = i_0\,, \\ \dfrac{\mu_\alpha^{\frac{p^*-p_+}{p_+}}}{\sqrt{a_{i_1}}} & \text{if } i = i_1\,, \\ 0 & \text{otherwise} \end{cases}$$

for all α. As is easily seen, the only possible limits of the sets $\tau_{\mu_\alpha,x_\alpha}^{\overrightarrow{p}}(\mathcal{E}(\overrightarrow{a}))$ as $\alpha \to +\infty$ in the sense of Definition 2.1 are of the form

$$U = \Big\{(y_1, \dots, y_n) \in \mathbb{R}^n \backslash E;\ \sum_{i \in I_2} a_i y_i^2 + 2\sqrt{a_{i_0}} y_{i_0} + 2\sqrt{a_{i_1}} y_{i_1} < 0\Big\}$$

in case $(p_+/p_-) + (p_+/p^\star) = 2$, and

$$U = \Big\{(y_1, \dots, y_n) \in \mathbb{R}^n \backslash E;\ \sum_{i \in I_2} a_i y_i^2 + 2\sqrt{a_{i_1}} y_{i_1} < 0\Big\}$$

in case $(p_+/p_-) + (p_+/p^\star) > 2$, where $|E| = 0$, and where I_2 is the set of all indices i such that $p_i = p_+$. None of such U's is the empty set, the whole space $\mathbb{R}^n$, or a halfspace, thus the ellipsoidal disk $\mathcal{E}(\overrightarrow{a})$ is not strongly asymptotically $\overrightarrow{p}$-stable when the strict inequality in (2.1) does not hold true. This ends the proof of Proposition (2.2). $\square$

In order to illustrate the notion of strongly asymptotically $\overrightarrow{p}$-stable domains, we introduce the following anisotropic class of regularity. This class is of importance for Theorem 1.3.

Definition 2.3. *We say that an open subset Ω of $\mathbb{R}^n$ is a $\overrightarrow{p}$-regular domain if for any point a on $\partial\Omega$, there exist an index i_0, a neighborhood X_a of a, and a function $\varphi_a : \mathbb{R}^{n-1} \to \mathbb{R}$ of class C^1 such that the set $X_a \cap \Omega$ consists of the points $(x_1, \dots, x_n)$ in X_a satisfying*

$$\varepsilon x_{i_0} < \varphi_a(x_1, \dots, x_{i_0-1}, x_{i_0+1}, \dots, x_n)\,,$$

where $\varepsilon = \pm 1$, and such that for any $i, j \neq i_0$ satisfying $p_j \geq p_i \geq p_{i_0}$, there holds

$$\left(\frac{\partial \varphi_a}{\partial x_i}(x + \mu e_j) - \frac{\partial \varphi_a}{\partial x_i}(x)\right) |\mu|^{\frac{p_j p^*(p_{i_0}-p_i)}{p_i p_{i_0}(p^*-p_j)}} \longrightarrow 0\,, \tag{2.7}$$

as $\mu \to 0$, uniformly in x, where e_j stands for the j-th vector in the canonical basis of $\mathbb{R}^n$.

As is easily checked, when the strict inequality in (2.1) holds true, domains of classical Hölder regularity $C^{1,\gamma}$ are $\overrightarrow{p}$-regular as soon as

$$\gamma > \frac{p^*(p_+ - p_-)}{p_-(p^* - p_+)}\,.$$

In the isotropic case $p_i = p_j$ for $i, j = 1, \dots, n$, $\overrightarrow{p}$-regular domains coincide with the class C^1. In the general case, we get the following result.

Proposition 2.4. *Any $\overrightarrow{p}$-regular bounded domain is strongly asymptotically $\overrightarrow{p}$-stable.*

Proof. Let Ω be a $\overrightarrow{p}$-regular bounded domain of $\mathbb{R}^n$, $(\mu_\alpha)_\alpha$ be a sequence of positive real numbers converging to 0, $(x_\alpha)_\alpha$ be a sequence in $\mathbb{R}^n$, and $\Omega_\alpha = \tau_{\mu_\alpha,x_\alpha}^{\overrightarrow{p}}(\Omega)$ for all α. If x_α keeps far from $\partial\Omega$ as $\alpha \to +\infty$, one can easily get that, up to a subsequence, either there holds $x_\alpha \in \Omega$ and $\Omega_\alpha \to \mathbb{R}^n$, or there holds $x_\alpha \in \mathbb{R}^n \backslash \Omega$ and $\Omega_\alpha \to \emptyset$. Therefore, passing if necessary to a subsequence, we may assume that the points x_α keep close to $\partial\Omega$, and then converge to a point a on $\partial\Omega$ as $\alpha\to+\infty$ since $\partial\Omega$ is bounded. We let i_0, X_a and φ_a be as in Definition 2.3, and we define a function $\widetilde{\varphi}_a$ by

$$\widetilde{\varphi}_a\left(x_1,\dots,x_n\right)=\varepsilon x_{i_0}-\varphi_a\left(x_1,\dots,x_{i_0-1},x_{i_0+1},\dots,x_n\right),$$

where $\varepsilon=\pm1$. By (2.7), we get

$$\widetilde{\varphi}_a\circ\left(\tau^{\overrightarrow{p}}_{\mu_\alpha,x_\alpha}\right)^{-1}\left(y_1,\dots,y_n\right)=\widetilde{\varphi}_a\left(x_\alpha\right)+\sum_{i\in I_0}\frac{\partial\widetilde{\varphi}_a}{\partial x_i}\left(x_\alpha\right)y_i\mu_\alpha^{\frac{p^*-p_i}{p_i}}+\mathrm{o}\left(\mu_\alpha^{\frac{p^*-p_{i_0}}{p_{i_0}}}\right)$$

as $\alpha\to+\infty$, uniformly in any compact subset of $\mathbb{R}^n$, where I_0 is the set of all indices i such that $p_i\ge p_{i_0}$. It follows that there exist a sequence $\left(\nu_\alpha\right)_\alpha$ of positive real numbers converging to 0 and some real numbers c_0 and c_i, $i\in I_0$, not all zero, such that, up to a subsequence, there holds

$$\widetilde{\varphi}_a\circ\left(\tau^{\overrightarrow{p}}_{\mu_\alpha,x_\alpha}\right)^{-1}\left(y_1,\dots,y_n\right)=\left(\sum_{i\in I_0}c_iy_i+c_0\right)\nu_\alpha+\mathrm{o}\left(\nu_\alpha\right)$$

as $\alpha\to+\infty$, uniformly in any compact subset of $\mathbb{R}^n$. One can then easily check that the sets Ω_α converge to U as $\alpha\to+\infty$ in the sense of Definition 2.1, where

$$U=\Big\{\left(y_1,\dots,y_n\right)\in\mathbb{R}^n;\ \sum_{i\in I_0}c_iy_i+c_0<0\Big\}.$$

In particular, the set U is either the empty set, the whole space $\mathbb{R}^n$,, or a halfspace. This ends the proof of Proposition 2.4. $\square$

As an easy consequence of Propositions 2.2 and 2.4, the ellipsoidal disks and annuli defined in Proposition 2.2 are $\overrightarrow{p}$-regular if and only if the strict inequality in (2.1) holds true.

3. The decomposition of Palais–Smale sequences

In the following, we let Ω be an asymptotically $\overrightarrow{p}$-stable bounded domain of $\mathbb{R}^n$ and f be a Caratheodory function in $\Omega\times\mathbb{R}$ satisfying (1.3). We let also $\left(r_\alpha\right)_\alpha$ be a sequence of real numbers in $(1,p^*]$ converging to p^*. We prove Theorem 1.2 in this section assuming that Lemma 3.1 below holds true. The difficulties associated to the anisotropic regime are almost all contained in this lemma. With this induction-type lemma, the other arguments are rather standard and only few changes with respect to the isotropic regime are required. For this reason, we make the section short. More details can be found in Struwe [47] for the isotropic linear case, and in Saintier [45] for the isotropic nonlinear case. We prove the induction Lemma 3.1 in Section 4.

Lemma 3.1. *Under the conditions in Theorem 1.2, for any Palais–Smale sequence $\left(u_\alpha\right)_\alpha$ for the functionals $\left(I_\alpha\right)_\alpha$ converging weakly but not strongly in $D^{1,\overrightarrow{p}}\left(\Omega\right)$ to 0, there exist a Palais–Smale sequence $\left(v_\alpha\right)_\alpha$ for the functionals $\left(I_\alpha\right)_\alpha$ and a $\overrightarrow{p}$-bubble $\left(B_\alpha\right)_\alpha$ of weights $\left(\mu_\alpha\right)_\alpha$ and multiplier λ satisfying $\mu_\alpha^{p^*-r_\alpha}\to\lambda$ as $\alpha\to+\infty$, such that, up to a subsequence,*

$$v_\alpha=u_\alpha-B_\alpha+R_\alpha$$

for all α, where $R_\alpha\to0$ in $D^{1,\overrightarrow{p}}\left(\mathbb{R}^n\right)$ as $\alpha\to+\infty$. Moreover, up to a subsequence,

$$I_\alpha\left(v_\alpha\right)=I_\alpha\left(u_\alpha\right)-E\left(B_\alpha\right)+\mathrm{o}\left(1\right)\tag{3.1}$$

as $\alpha\to+\infty$. If in addition the functions u_α are nonnegative, then so are the functions v_α and the $\overrightarrow{p}$-bubble $\left(B_\alpha\right)_\alpha$.

Assuming Lemma 3.1, we prove Theorem 1.2.

Proof of Theorem 1.2. As a preliminary remark, there exists a positive constant $\Lambda = \Lambda(n, p_i)$ such that for any λ in $(0,1]$ and for any $\overrightarrow{p}$-bubble $(B_\alpha)_\alpha$ of multiplier λ, there holds

$$E(B_\alpha) \geq \Lambda \lambda^{-p_-/(p^*-p_-)}. \tag{3.2}$$

In order to see this, we let u be the shape function of a $\overrightarrow{p}$-bubble $(B_\alpha)_\alpha$ of multiplier λ in $(0,1]$. On the one hand, see for instance El Hamidi–Rakotoson [20], the anisotropic Sobolev inequality yields a positive constant C_1 independent of u such that there holds

$$C_1 \min_{1 \leq i \leq n} \left(\int_{\mathbb{R}^n} |u|^{p^*} dx \right)^{p_i/p^*} \leq E(B_\alpha).$$

On the other hand, we get

$$\lambda \int_{\mathbb{R}^n} |u|^{p^*} dx = \sum_{i=1}^n \int_{\mathbb{R}^n} \left| \frac{\partial u}{\partial x_i} \right|^{p_i} dx \geq C_2 E(B_\alpha),$$

where C_2 is a positive constant independent of u. It follows that

$$C_1 \min_{1 \leq i \leq n} \left(\frac{C_2}{\lambda} E(B_\alpha) \right)^{p_i/p^*} \leq E(B_\alpha),$$

and we get that there exists a positive constant Λ independent of u and λ such that (3.2) holds true. Another general remark is that any Palais–Smale sequence $(u_\alpha)_\alpha$ for the functionals $(I_\alpha)_\alpha$ is bounded in $D^{1,\overrightarrow{p}}(\Omega)$. In order to see this, we fix a real number p in (p_+, p^*) and a real number q in $(1, p^*)$ such that the growth condition (1.3) holds true. We then compute

$$\begin{aligned}
&\sum_{i=1}^n \frac{p - p_i}{p_i p} \int_\Omega \left| \frac{\partial u_\alpha}{\partial x_i} \right|^{p_i} dx + \frac{r_\alpha - p}{r_\alpha p} \int_\Omega |u_\alpha|^{r_\alpha} dx \\
&= I_\alpha(u_\alpha) - \frac{1}{p} DI_\alpha(u_\alpha).u_\alpha + \int_\Omega F(x, u_\alpha) dx - \frac{1}{p} \int_\Omega f(x, u_\alpha) u_\alpha dx \\
&= \mathrm{O}(1) + \mathrm{o}\left(\|u_\alpha\|_{D^{1,\overrightarrow{p}}(\Omega)} \right) + \mathrm{O}\left(\int_\Omega |u_\alpha|^q dx \right) \\
&= \mathrm{O}(1) + \mathrm{o}\left(\sum_{i=1}^n \left\| \frac{\partial u_\alpha}{\partial x_i} \right\|_{L^{p_i}(\Omega)} \right) + \mathrm{O}\left(\|u_\alpha\|^q_{L^{r_\alpha}(\Omega)} \right)
\end{aligned}$$

as $\alpha \to +\infty$. It easily follows that the sequence $(u_\alpha)_\alpha$ is bounded in $D^{1,\overrightarrow{p}}(\Omega)$. Now, we start with the induction procedure of the proof. We let $(u_\alpha)_\alpha$ be a Palais–Smale sequence for the functionals $(I_\alpha)_\alpha$. By the above, we get that the sequence $(u_\alpha)_\alpha$ is bounded in $D^{1,\overrightarrow{p}}(\Omega)$, and thus converges weakly, up to a subsequence, to a function u_∞ in $D^{1,\overrightarrow{p}}(\Omega)$. Proceeding, mutatis mutandis, in the same way as in Saintier [45], we get that u_∞ is a weak solution of problem (1.4). By an easy adaptation of the argument in Brézis–Lieb [15], we also get that, up to a subsequence, $(u^0_\alpha)_\alpha$, where $u^0_\alpha = u_\alpha - u_\infty$, is Palais–Smale for the functionals $(I_\alpha)_\alpha$, and satisfies $I_\alpha(u^0_\alpha) = I_\alpha(u_\alpha) - I_\alpha(u_\infty) + \mathrm{o}(1)$ as $\alpha \to +\infty$. If the sequence $(u_\alpha)_\alpha$ converges strongly, up to a subsequence, to the function u_∞ in $D^{1,\overrightarrow{p}}(\Omega)$, then there is nothing more to say. Otherwise, we apply Lemma 3.1 to the sequence $(u^0_\alpha)_\alpha$ in order to construct a $\overrightarrow{p}$-bubble $(B^1_\alpha)_\alpha$ and a sequence $(R^1_\alpha)_\alpha$ converging strongly to 0 in $D^{1,\overrightarrow{p}}(\mathbb{R}^n)$ such that, up to a subsequence, $(u^1_\alpha)_\alpha$, where $u^1_\alpha = u^0_\alpha - B^1_\alpha + R^1_\alpha$, is Palais–Smale for the functionals $(I_\alpha)_\alpha$ and satisfies $I_\alpha(u^1_\alpha) = I_\alpha(u^0_\alpha) - E(B^1_\alpha) + \mathrm{o}(1)$ as $\alpha \to +\infty$. We iterate Lemma 3.1 as long as we do not obtain a sequence converging strongly, up to a subsequence, to 0 in $D^{1,\overrightarrow{p}}(\Omega)$. We then get $\overrightarrow{p}$-bubbles $(B^\gamma_\alpha)_\alpha$ and sequences $(R^\gamma_\alpha)_\alpha$ converging strongly to 0 in $D^{1,\overrightarrow{p}}(\mathbb{R}^n)$ such that for all γ, up to a subsequence, $(u_\alpha^\gamma)_\alpha$, where $u_\alpha^\gamma = u_\alpha^{\gamma-1} - B_\alpha^\gamma + R_\alpha^\gamma$, is Palais–Smale for the functionals $(I_\alpha)_\alpha$ and satisfies $I_\alpha(u_\alpha^\gamma) = I_\alpha(u_\alpha^{\gamma-1}) - E(B_\alpha^\gamma) + \mathrm{o}(1)$ as $\alpha \to +\infty$. Up to a subsequence, by summing, we get

$$u_\alpha^\gamma = u_\alpha - u_\infty - \sum_{j=1}^{\gamma} B_\alpha^j + \sum_{j=1}^{\gamma} R_\alpha^j \tag{3.3}$$

and

$$I_\alpha\left(u_\alpha^\gamma\right) = I_\alpha\left(u_\alpha\right) - I_\alpha\left(u_\infty\right) - \sum_{j=1}^{\gamma} E\left(B_\alpha^j\right) + \mathrm{o}\left(1\right) \tag{3.4}$$

as $\alpha \to +\infty$. Since the sequence $(u_\alpha)_\alpha$ is Palais–Smale for the functionals $(I_\alpha)_\alpha$, there exists a constant C independent of α and γ such that, up to a subsequence,

$$I_\alpha\left(u_\alpha^\gamma\right) \le C - \sum_{j=1}^{\gamma} E\left(B_\alpha^j\right) \le C - \gamma\Lambda\,,$$

where Λ is as in (3.2), and by the growth condition (1.3), increasing if necessary the constant C, it follows that

$$\sum_{i=1}^{n} \frac{r_\alpha - p_i}{p_i r_\alpha} \int_{\mathbb{R}^n} \left|\frac{\partial u_\alpha^\gamma}{\partial x_i}\right|^{p_i} dx = I_\alpha\left(u_\alpha^\gamma\right) - \frac{1}{r_\alpha} DI_\alpha\left(u_\alpha^\gamma\right).u_\alpha^\gamma \\ + \int_\Omega F\left(x, u_\alpha^\gamma\right) dx - \frac{1}{r_\alpha}\int_\Omega f\left(x, u_\alpha^\gamma\right) u_\alpha^\gamma dx \le C - \gamma\Lambda$$

for all α. In particular, there is a contradiction when γ is large enough, and the induction stops after some index k in the sense that the sequence $(u_\alpha^k)_\alpha$ converges strongly, up to a subsequence, to 0 in $D^{1,\overrightarrow{p}}\left(\Omega\right)$. Then (1.7) (resp. (1.8)) follows from (3.3) (resp. (3.4)). Now we assume that the functions u_α are nonnegative. Since the sequence $(u_\alpha)_\alpha$ converges, up to a subsequence, almost everywhere to u_∞ in Ω, it follows that u_∞ is nonnegative. For any α, we set

$$u_\alpha^0 = \max\left(u_\alpha - u_\infty, 0\right) = u_\alpha - u_\infty + R_\alpha\,,$$

where $R_\alpha = \max\left(u_\infty - u_\alpha, 0\right)$. We write

$$\sum_{i=1}^{n} \int_\Omega \left|\frac{\partial R_\alpha}{\partial x_i}\right|^{p_i} dx = DI_\alpha\left(u_\infty - u_\alpha\right).R_\alpha + \int_\Omega f\left(x, u_\alpha - u_\infty\right) R_\alpha dx + \int_\Omega R_\alpha^{r_\alpha} dx \\ \le DI_\alpha\left(u_\infty - u_\alpha\right).R_\alpha + C\int_\Omega R_\alpha dx + C\int_\Omega R_\alpha^q dx + \int_\Omega R_\alpha^{r_\alpha} dx\,,$$

where q is a real number in $(1, p^*)$ such that the growth condition (1.3) holds true. Taking into account that the sequence $(R_\alpha)_\alpha$ converges, up to a subsequence, almost everywhere to 0 in Ω and that there holds $R_\alpha^q \le u_\infty^q + 1$ and $R_\alpha^{r_\alpha} \le u_\infty^{p^*} + 1$ for all α, we then get

$$\sum_{i=1}^{n} \int_\Omega \left|\frac{\partial R_\alpha}{\partial x_i}\right|^{p_i} dx = \mathrm{o}\left(\left\|R_\alpha\right\|_{D^{1,\overrightarrow{p}}(\Omega)}\right) + \mathrm{o}\left(1\right)$$

as $\alpha \to +\infty$. It easily follows that the sequence $(R_\alpha)_\alpha$ converges to 0 in $D^{1,\overrightarrow{p}}\left(\Omega\right)$ as $\alpha \to +\infty$, and thus we get both that $I_\alpha(u_\alpha^0) = I_\alpha(u_\alpha) - I_\alpha(u_\infty) + \mathrm{o}\left(1\right)$ and that the sequence $(u_\alpha^0)_\alpha$ is Palais–Smale for the functionals $(I_\alpha)_\alpha$. Since the functions u_α^0 are nonnegative, by Lemma 3.1, we may follow the above procedure with nonnegative functions $u_\alpha^1, \dots, u_\alpha^k$ and get that (1.7) and (1.8) hold true with nonnegative $\overrightarrow{p}$-bubbles. This ends the proof of Theorem 1.2. ☐

4. Proof of the induction lemma 3.1

In the following, we let Ω be an asymptotically $\overrightarrow{p}$-stable bounded domain of $\mathbb{R}^n$ and f be a Caratheodory function in $\Omega \times \mathbb{R}$ satisfying (1.3). We let also $(r_\alpha)_\alpha$ be a sequence of real numbers in $(1, p^*]$ converging to p^*. We aim to prove Lemma 3.1 in this section. We let $(u_\alpha)_\alpha$ be a Palais–Smale sequence for the functionals $(I_\alpha)_\alpha$ converging weakly but not strongly to 0 in $D^{1,\overrightarrow{p}}(\Omega)$. For any real number r in $[1, p^*)$, by the compactness of the embedding of $D^{1,\overrightarrow{p}}(\Omega)$ into $L^r(\Omega)$, we get that the sequence $(u_\alpha)_\alpha$ converges strongly to 0 in $L^r(\Omega)$. By the growth condition (1.3), it easily follows that the sequence $(u_\alpha)_\alpha$ remains Palais–Smale for the functionals

$$J_\alpha = \sum_{i=1}^{n} \frac{1}{p_i} \int_\Omega \left|\frac{\partial u}{\partial x_i}\right|^{p_i} dx - \frac{1}{r_\alpha} \int_\Omega |u|^{r_\alpha} dx\,.$$

Since $(u_\alpha)_\alpha$ does not converge strongly to 0 in $D^{1,\overrightarrow{p}}(\Omega)$, passing if necessary to a subsequence, we may assume that there holds

$$\liminf_{\alpha \to +\infty} \sum_{i=1}^{n} \int_\Omega \left|\frac{\partial u_\alpha}{\partial x_i}\right|^{p_i} dx > 0\,.$$

Taking into account that by Hölder's inequality, we get

$$\sum_{i=1}^{n} \int_\Omega \left|\frac{\partial u_\alpha}{\partial x_i}\right|^{p_i} dx = \int_\Omega |u_\alpha|^{r_\alpha} dx + \mathrm{o}(1) \le |\Omega|^{\frac{p^*-r_\alpha}{p^*}} \left(\int_\Omega |u_\alpha|^{p^*} dx\right)^{\frac{r_\alpha}{p^*}} + \mathrm{o}(1) \tag{4.1}$$

as $\alpha \to +\infty$, it follows that

$$\liminf_{\alpha \to +\infty} \int_\Omega |u_\alpha|^{p^*} dx > 0\,. \tag{4.2}$$

In the sequel, we extend the u_α's by 0 outside of the domain Ω. For any α, we then define the concentration function $Q_\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ by

$$Q_\alpha(s) = \max_{y \in \overline{\Omega}} \int_{\mathcal{P}_y^{\overrightarrow{p}}(s)} |u_\alpha|^{p^*} dx\,,$$

where for any positive real number s and for any point $y = (y_1, \dots, y_n)$ in $\mathbb{R}^n$, the domain $\mathcal{P}_y^{\overrightarrow{p}}(s)$ is defined by

$$\mathcal{P}_y^{\overrightarrow{p}}(s) = \left\{(x_1, \dots, x_n) \in \mathbb{R}^n;\ |x_i - y_i| < \frac{1}{2} s^{\frac{p^*-p_i}{p_i}} \quad \forall i \in \{1, \dots, n\}\right\}.$$

In particular, $\mathcal{P}_y^{\overrightarrow{p}}(1)$ stands for the cube centered at y with an edge length of 1. As easy remarks about these domains, we get that for any point y in $\mathbb{R}^n$ and for any positive real number s, there hold $|\mathcal{P}_y^{\overrightarrow{p}}(s)| = s^{p^*}$ and $\tau_{s,y}^{\overrightarrow{p}}(\mathcal{P}_y^{\overrightarrow{p}}(s)) = \mathcal{P}_0^{\overrightarrow{p}}(1)$, where $\tau_{s,y}^{\overrightarrow{p}}$ is as in (1.5). The functions Q_α are continuous, and by (4.2), passing if necessary to a subsequence, we may assume that there exist two positive real numbers s_0 and Λ_0 such that there holds $Q_\alpha(s_0) > \Lambda_0$ for all α. It follows that there exists a sequence $(\mu_\alpha)_\alpha$ of real numbers in $(0, s_0)$ such that there holds $Q_\alpha(\mu_\alpha) = \Lambda_0$ for all α. We let x_α be the point in $\overline{\Omega}$ for which $Q_\alpha(\mu_\alpha)$ is reached. Up to a subsequence, we may assume that the sequence $(x_\alpha)_\alpha$ converges. We claim that if the constant Λ_0 is chosen small enough, then the sequence $(\mu_\alpha)_\alpha$ converges to 0. Indeed, if not the case, then for any $\varepsilon > 0$, there exists $s_\varepsilon > 0$ such that for any point y in $\overline{\Omega}$ and for any α, there holds

$$\int_{\mathcal{P}_y^{\overrightarrow{p}}(s_\varepsilon)} |u_\alpha|^{p^*} dx \le \varepsilon\,.$$

From the concentration-compactness principle of Lions [32, 33] (see El Hamidi–Rakotoson [20] for a proof in the anisotropic case), it follows that the sequence $(u_\alpha)_\alpha$ converges in fact strongly to 0 in $L^{p^*}(\Omega)$ which together with (4.1) contradicts its non-convergence in $D^{1,\overrightarrow{p}}(\Omega)$. Then, since the domain Ω is asymptotically $\overrightarrow{p}$-stable, up to a subsequence, we may assume that there exists an open subset U satisfying the segment property when not empty, such that the sets $\Omega_\alpha = \tau^{\overrightarrow{p}}_{\mu_\alpha,x_\alpha}(\Omega)$ converge to U as $\alpha \to +\infty$ in the sense of Definition 2.1. For any α, we define the function $\widetilde{u}_\alpha$ on Ω_α by

$$\widetilde{u}_\alpha = \mu_\alpha u_\alpha \circ \left(\tau^{\overrightarrow{p}}_{\mu_\alpha,x_\alpha}\right)^{-1}, \tag{4.3}$$

where $\tau^{\overrightarrow{p}}_{\mu_\alpha,x_\alpha}$ is as in (1.5). As a first easy remark, $\|\widetilde{u}_\alpha\|_{D^{1,\overrightarrow{p}}(\mathbb{R}^n)} = \|u_\alpha\|_{D^{1,\overrightarrow{p}}(\mathbb{R}^n)}$ for all α. It follows that the sequence $(\widetilde{u}_\alpha)_\alpha$ is bounded in $D^{1,\overrightarrow{p}}(\mathbb{R}^n)$. In particular, passing if necessary to a subsequence, we may assume that $(\widetilde{u}_\alpha)_\alpha$ converges weakly to a function $\widetilde{u}$ in $D^{1,\overrightarrow{p}}(\mathbb{R}^n)$. Furthermore, by the compact embeddings in Rákosník [41], we can also assume that the sequence $(\widetilde{u}_\alpha)_\alpha$ converges strongly to the function $\widetilde{u}$ in $L^r(\Omega')$ for all real numbers r in $[1, p^*)$ and for all bounded domains Ω' of $\mathbb{R}^n$, and thus, up to a subsequence, almost everywhere in $\mathbb{R}^n$. Since $\Omega_\alpha = \tau^{\overrightarrow{p}}_{\mu_\alpha,x_\alpha}(\Omega)$ converge to U as $\alpha \to +\infty$ in the sense of Definition 2.1, we get that the support of $\widetilde{u}$ is included in $\overline{U}$. In particular, see Section 2, the function $\widetilde{u}$ belongs to the anisotropic Sobolev space $D^{1,\overrightarrow{p}}(U)$. If the set U is empty, then $\widetilde{u}$ is identically zero. For any α, we then define

$$B_\alpha = \frac{1}{\mu_\alpha}\widetilde{u} \circ \tau^{\overrightarrow{p}}_{\mu_\alpha,x_\alpha}. \tag{4.4}$$

We state five preliminary steps. The first one is as follows.

Step 4.1. *If the constant Λ_0 is small enough, then the sequence $(\widetilde{u}_\alpha)_\alpha$ defined in* (4.3) *converges strongly to the function $\widetilde{u}$ in $L^{p^*}(\Omega')$ for all bounded domains Ω' of $\mathbb{R}^n$, where $\widetilde{u}$ is as in* (4.4).

Proof. Since they keep bounded in $L^{p_i/(p_i-1)}(\mathbb{R}^n)$, passing if necessary to a subsequence, we may assume that the functions $|\partial\widetilde{u}_\alpha/\partial x_i|^{p_i-2}\,\partial\widetilde{u}_\alpha/\partial x_i$ converge weakly to a function ψ_i in $L^{p_i/(p_i-1)}(\mathbb{R}^n)$ as $\alpha \to +\infty$, for $i = 1, \dots, n$. Since the sequence $(\widetilde{u}_\alpha)_\alpha$ converges weakly to the function $\widetilde{u}$ in $D^{1,\overrightarrow{p}}(\mathbb{R}^n)$, by the anisotropic version of the concentration-compactness principle in El Hamidi–Rakotoson [20], up to a subsequence, we may assume moreover that there exist a positive constant $\Lambda = \Lambda(n, p_i)$ and an at most countable index set K of distinct points y_k in $\mathbb{R}^n$ and positive real numbers ω_k, $k \in K$ such that there hold

$$|\widetilde{u}_\alpha|^{p^*} \longrightarrow |\widetilde{u}|^{p^*} + \sum_{k\in K}\omega_k\delta_{y_k} \tag{4.5}$$

and

$$\sum_{i=1}^n \left|\frac{\partial\widetilde{u}_\alpha}{\partial x_i}\right|^{p_i} \longrightarrow \nu \ge \sum_{i=1}^n \left|\frac{\partial\widetilde{u}}{\partial x_i}\right|^{p_i} + \Lambda\sum_{k\in K}\omega_k^{\frac{p_+}{p^*}}\delta_{y_k} \tag{4.6}$$

as $\alpha \to +\infty$ in the sense of measures, where δ_{y_k} stands for the Dirac mass at the point y_k. We have to prove that if the constant Λ_0 is small enough, then the index set K is in fact empty. As a preliminary estimate, we show that for any function φ in $D^{1,\overrightarrow{p}}(U)$, there holds

$$\left|\sum_{i=1}^n \int_{\mathbb{R}^n}\psi_i\frac{\partial\varphi}{\partial x_i}dx\right| \le \int_{\mathbb{R}^n}|\widetilde{u}|^{p^*-1}|\varphi|\,dx\,. \tag{4.7}$$

By an easy density argument, it suffices to prove (4.7) for all smooth functions φ with compact support in U. We set $\varphi_\alpha = \varphi \circ \tau^{\overrightarrow{p}}_{\mu_\alpha,x_\alpha}$ for all α. Since Ω_α converges to U as $\alpha \to +\infty$ in the sense of Definition 2.1, we get that the support of the function $\widetilde{\varphi}_\alpha$ is included in Ω for α large. Since the sequence $(u_\alpha)_\alpha$ is Palais–Smale for the functionals $(J_\alpha)_\alpha$, taking into account that there holds $\|\widetilde{\varphi}_\alpha\|_{D^{1,\vec{p}}(\mathbb{R}^n)} = \|\varphi\|_{D^{1,\vec{p}}(\mathbb{R}^n)}$ for all α, it follows that $DJ_\alpha(u_\alpha).\widetilde{\varphi}_\alpha \to 0$ as $\alpha \to +\infty$. Direct computations give

$$\sum_{i=1}^n \int_{\mathbb{R}^n} \left|\frac{\partial \widetilde{u}_\alpha}{\partial x_i}\right|^{p_i-2} \frac{\partial \widetilde{u}_\alpha}{\partial x_i}\frac{\partial \varphi}{\partial x_i} dx = \mu_\alpha^{p^*-r_\alpha} \int_{\mathbb{R}^n} |\widetilde{u}_\alpha|^{r_\alpha-2} \widetilde{u}_\alpha \varphi dx + \mathrm{o}(1)$$

as $\alpha \to +\infty$, and thus

$$\left|\sum_{i=1}^n \int_{\mathbb{R}^n} \left|\frac{\partial \widetilde{u}_\alpha}{\partial x_i}\right|^{p_i-2} \frac{\partial \widetilde{u}_\alpha}{\partial x_i}\frac{\partial \varphi}{\partial x_i} dx\right| \le \int_{\mathbb{R}^n} |\widetilde{u}_\alpha|^{r_\alpha-1} |\varphi| dx + \mathrm{o}(1). \tag{4.8}$$

The functions $|\widetilde{u}_\alpha|^{r_\alpha-1}$ converge, up to a subsequence, almost everywhere to $|\widetilde{u}|^{p^*-1}$ in $\mathbb{R}^n$ as $\alpha \to +\infty$ and keep bounded in $L^{p^*/(p^*-1)}(\mathbb{R}^n)$. By standard integration theory, it follows that

$$\int_{\mathbb{R}^n} |\widetilde{u}_\alpha|^{r_\alpha-1} |\varphi| dx \longrightarrow \int_{\mathbb{R}^n} |\widetilde{u}|^{p^*-1} |\varphi| dx \tag{4.9}$$

as $\alpha \to +\infty$. By (4.9), passing to the limit into (4.8) as $\alpha \to +\infty$ yields (4.7). For any α and any nonnegative smooth function φ with compact support in $\mathbb{R}^n$, we then compute

$$\begin{aligned}\sum_{i=1}^n &\left(\int_{\mathbb{R}^n} \left|\frac{\partial \widetilde{u}_\alpha}{\partial x_i}\right|^{p_i} \varphi dx + \int_{\mathbb{R}^n} \left|\frac{\partial \widetilde{u}_\alpha}{\partial x_i}\right|^{p_i-2} \frac{\partial \widetilde{u}_\alpha}{\partial x_i}\widetilde{u}_\alpha\frac{\partial \varphi}{\partial x_i} dx\right)\\ &= DJ_\alpha(u_\alpha).(u_\alpha\varphi_\alpha) + \mu_\alpha^{p^*-r_\alpha}\int_{\mathbb{R}^n} |\widetilde{u}_\alpha|^{r_\alpha}\varphi dx\\ &\le DJ_\alpha(u_\alpha).(u_\alpha\varphi_\alpha) + \left(\int_{\mathbb{R}^n} |\widetilde{u}_\alpha|^{p^*}\varphi dx + \int_{\mathbb{R}^n}\varphi dx\right),\end{aligned} \tag{4.10}$$

where $\varphi_\alpha = \varphi \circ \tau^{\vec{p}}_{\mu_\alpha,x_\alpha}$. Since $(u_\alpha)_\alpha$ is Palais–Smale for the functionals $(J_\alpha)_\alpha$, taking into account that $\|u_\alpha\varphi_\alpha\|_{D^{1,\vec{p}}(\Omega)} = \|\widetilde{u}_\alpha\varphi\|_{D^{1,\vec{p}}(\mathbb{R}^n)}$ for all α, we get $DJ_\alpha(u_\alpha).(u_\alpha\varphi_\alpha) \to 0$ as $\alpha \to +\infty$. Moreover, since the sequence $(\widetilde{u}_\alpha)_\alpha$ converges to $\widetilde{u}$ in $L^{p_i}(\operatorname{Supp}\varphi)$, we get

$$\int_{\mathbb{R}^n} \left|\frac{\partial \widetilde{u}_\alpha}{\partial x_i}\right|^{p_i-2} \frac{\partial \widetilde{u}_\alpha}{\partial x_i}\widetilde{u}_\alpha\frac{\partial \varphi}{\partial x_i} dx \longrightarrow \int_{\mathbb{R}^n} \psi_i\widetilde{u}\frac{\partial \varphi}{\partial x_i} dx \tag{4.11}$$

as $\alpha \to +\infty$. By (4.5), (4.6), and (4.11), passing to the limit into (4.10) as $\alpha \to +\infty$ yields

$$\begin{aligned}\sum_{i=1}^n \int_{\mathbb{R}^n} \left|\frac{\partial \widetilde{u}}{\partial x_i}\right|^{p_i} \varphi dx + \Lambda \sum_{k\in K} \omega_k^{\frac{p_+}{p^*}}\varphi(y_k) + \sum_{i=1}^n \int_{\mathbb{R}^n} \psi_i\widetilde{u}\frac{\partial \varphi}{\partial x_i} dx&\\ \le \int_{\mathbb{R}^n} |\widetilde{u}|^{p^*}\varphi dx + \sum_{k\in K}\omega_k\varphi(y_k) + \int_{\mathbb{R}^n}\varphi dx&.\end{aligned}$$

By (4.7), it follows that

$$\begin{aligned}\sum_{i=1}^n \int_{\mathbb{R}^n} \left|\frac{\partial \widetilde{u}}{\partial x_i}\right|^{p_i} \varphi dx + \Lambda \sum_{k\in K} \omega_k^{\frac{p_+}{p^*}}\varphi(y_k) - \sum_{i=1}^n \int_{\mathbb{R}^n} \psi_i\frac{\partial \widetilde{u}}{\partial x_i}\varphi dx&\\ \le 2\int_{\mathbb{R}^n} |\widetilde{u}|^{p^*}\varphi dx + \sum_{k\in K}\omega_k\varphi(y_k) + \int_{\mathbb{R}^n}\varphi dx&.\end{aligned} \tag{4.12}$$

We let η be a nonnegative, smooth cutoff function on $\mathbb{R}^n$ with compact support and such that $\eta(0) = 1$. For any k in K, plugging $\varphi(x) = \eta(\beta(x-y_k))$ into (4.12) for all natural number β and passing to the limit as $\beta \to +\infty$ yield $\omega_k \geq \Lambda \omega_k^{p_+/p^*}$, and thus $\omega_k \geq \Lambda^{p^*/(p^*-p_+)}$. Independently, by (4.5), we get

$$\omega_k \leq \lim_{\alpha \to +\infty} \int_{\mathcal{P}_{y_k}^{\overrightarrow{p}}(1)} |\widetilde{u}_\alpha|^{p^*} dx \leq \lim_{\alpha \to +\infty} Q_\alpha(\mu_\alpha) = \Lambda_0 \,.$$

If Λ_0 is small enough so that there holds $\Lambda_0 < \Lambda^{p^*/(p^*-p_+)}$, then there is a contradiction, and the set K is empty. This ends the proof of Step 4.1. □

We assume in what follows that the constant Λ_0 is small enough so that Step 4.1 can be applied. As a first consequence of the former step, we get

$$\int_{\mathcal{P}_0^{\overrightarrow{p}}(1)} |\widetilde{u}|^{p^*} dx = \lim_{\alpha \to +\infty} \int_{\mathcal{P}_0^{\overrightarrow{p}}(1)} |\widetilde{u}_\alpha|^{p^*} dx = \lim_{\alpha \to +\infty} \int_{\mathcal{P}_{x_\alpha}^{\overrightarrow{p}}(\mu_\alpha)} |u_\alpha|^{p^*} dx = \Lambda_0 \,,$$

thus the function $\widetilde{u}$ is not identically zero. In particular, the set U is not empty.

The second step in the proof of Lemma 3.1 states as follows.

Step 4.2. *Up to a subsequence, for $i = 1, \dots, n$, the functions $\partial \widetilde{u}_\alpha / \partial x_i$ converge almost everywhere to $\partial \widetilde{u} / \partial x_i$ in $\mathbb{R}^n$ as $\alpha \to +\infty$.*

Proof. We let φ be a nonnegative, smooth function with compact support in U, and for any α, we let $\varphi_\alpha = \varphi \circ \tau_{\mu_\alpha, x_\alpha}^{\overrightarrow{p}}$. Since the sets Ω_α converge to U as $\alpha \to +\infty$ in the sense of Definition 2.1, we get that the support of the function φ_α is included in Ω for α large. Since the sequence $(u_\alpha)_\alpha$ is Palais–Smale for the functionals $(J_\alpha)_\alpha$, we can write that $DJ_\alpha(u_\alpha).((u_\alpha - B_\alpha)\varphi_\alpha) \to 0$ as $\alpha \to +\infty$, where B_α is as in (4.4). Direct computations then yield

$$\sum_{i=1}^n \int_{\mathbb{R}^n} \left| \frac{\partial \widetilde{u}_\alpha}{\partial x_i} \right|^{p_i - 2} \frac{\partial \widetilde{u}_\alpha}{\partial x_i} \left(\frac{\partial \widetilde{u}_\alpha}{\partial x_i} - \frac{\partial \widetilde{u}}{\partial x_i} \right) \varphi dx = \mu_\alpha^{p^* - r_\alpha} \int_{\mathbb{R}^n} |\widetilde{u}_\alpha|^{r_\alpha - 2} \widetilde{u}_\alpha (\widetilde{u}_\alpha - \widetilde{u}) \varphi dx$$
$$- \sum_{i=1}^n \int_{\mathbb{R}^n} \left| \frac{\partial \widetilde{u}_\alpha}{\partial x_i} \right|^{p_i - 2} \frac{\partial \widetilde{u}_\alpha}{\partial x_i} (\widetilde{u}_\alpha - \widetilde{u}) \frac{\partial \varphi}{\partial x_i} dx + \mathrm{o}(1) \,. \quad (4.13)$$

as $\alpha \to +\infty$. Hölder's inequality and Step 4.1 gives

$$\left| \int_{\mathbb{R}^n} \left| \frac{\partial \widetilde{u}_\alpha}{\partial x_i} \right|^{p_i - 2} \frac{\partial \widetilde{u}_\alpha}{\partial x_i} (\widetilde{u}_\alpha - \widetilde{u}) \frac{\partial \varphi}{\partial x_i} dx \right|$$
$$\leq \left\| \frac{\partial \widetilde{u}_\alpha}{\partial x_i} \right\|_{L^{p_i}(\mathbb{R}^n)}^{p_i - 1} \left\| \frac{\partial \varphi}{\partial x_i} \right\|_{L^{\frac{p_i p^*}{p^* - p_i}}(\mathbb{R}^n)} \|\widetilde{u}_\alpha - \widetilde{u}\|_{L^{p^*}(\operatorname{Supp}\varphi)} \longrightarrow 0 \quad (4.14)$$

and

$$\left| \int_{\mathbb{R}^n} |\widetilde{u}_\alpha|^{r_\alpha - 2} \widetilde{u}_\alpha (\widetilde{u}_\alpha - \widetilde{u}) \varphi dx \right| \leq \|\widetilde{u}_\alpha\|_{L^{p^*}(\mathbb{R}^n)}^{r_\alpha - 1} \|\varphi\|_{L^{\frac{p^*}{p^* - r_\alpha}}(\mathbb{R}^n)} \|\widetilde{u}_\alpha - \widetilde{u}\|_{L^{p^*}(\operatorname{Supp}\varphi)} \longrightarrow 0 \quad (4.15)$$

as $\alpha \to +\infty$. By (4.13), (4.14), and (4.15), we get

$$\sum_{i=1}^n \int_{\mathbb{R}^n} \left| \frac{\partial \widetilde{u}_\alpha}{\partial x_i} \right|^{p_i - 2} \frac{\partial \widetilde{u}_\alpha}{\partial x_i} \left(\frac{\partial \widetilde{u}_\alpha}{\partial x_i} - \frac{\partial \widetilde{u}}{\partial x_i} \right) \varphi dx \longrightarrow 0 \quad (4.16)$$

as $\alpha \to +\infty$. Independently, since the sequence $(\widetilde{u}_\alpha)_\alpha$ converges weakly to the function $\widetilde{u}$ in $D^{1,\overrightarrow{p}}(\mathbb{R}^n)$, there holds

$$\int_{\mathbb{R}^n} \left| \frac{\partial \widetilde{u}}{\partial x_i} \right|^{p_i - 2} \frac{\partial \widetilde{u}}{\partial x_i} \frac{\partial \widetilde{u}_\alpha}{\partial x_i} \varphi dx \longrightarrow \int_{\mathbb{R}^n} \left| \frac{\partial \widetilde{u}}{\partial x_i} \right|^{p_i} \varphi dx \quad (4.17)$$

as $\alpha \to +\infty$, for $i = 1, \dots, n$. By (4.16) and (4.17), we get

$$\sum_{i=1}^{n} \int_{\mathbb{R}^n} \left(\left| \frac{\partial \widetilde{u}_\alpha}{\partial x_i} \right|^{p_i - 2} \frac{\partial \widetilde{u}_\alpha}{\partial x_i} - \left| \frac{\partial \widetilde{u}}{\partial x_i} \right|^{p_i - 2} \frac{\partial \widetilde{u}}{\partial x_i} \right) \left(\frac{\partial \widetilde{u}_\alpha}{\partial x_i} - \frac{\partial \widetilde{u}}{\partial x_i} \right) \varphi dx \longrightarrow 0$$

as $\alpha \to +\infty$. Since this estimate holds true for all nonnegative smooth functions φ with compact support in U, it easily follows that

$$\int_{\Omega'} \left(\left| \frac{\partial \widetilde{u}_\alpha}{\partial x_i} \right|^{p_i - 2} \frac{\partial \widetilde{u}_\alpha}{\partial x_i} - \left| \frac{\partial \widetilde{u}}{\partial x_i} \right|^{p_i - 2} \frac{\partial \widetilde{u}}{\partial x_i} \right) \left(\frac{\partial \widetilde{u}_\alpha}{\partial x_i} - \frac{\partial \widetilde{u}}{\partial x_i} \right) dx \longrightarrow 0$$

as $\alpha \to +\infty$, for $i = 1, \dots, n$ and for all bounded domains Ω' strictly included in U. In particular, up to a subsequence,

$$\left(\left| \frac{\partial \widetilde{u}_\alpha}{\partial x_i} \right|^{p_i - 2} \frac{\partial \widetilde{u}_\alpha}{\partial x_i} - \left| \frac{\partial \widetilde{u}}{\partial x_i} \right|^{p_i - 2} \frac{\partial \widetilde{u}}{\partial x_i} \right) \left(\frac{\partial \widetilde{u}_\alpha}{\partial x_i} - \frac{\partial \widetilde{u}}{\partial x_i} \right) \longrightarrow 0 \quad \text{a.e. in } U$$

as $\alpha \to +\infty$. As an easy consequence, for $i = 1, \dots, n$, the functions $\partial \widetilde{u}_\alpha / \partial x_i$ converge, up to a subsequence, almost everywhere to $\partial \widetilde{u} / \partial x_i$ in U as $\alpha \to +\infty$. Independently, since the sets Ω_α converge to U as $\alpha \to +\infty$ in the sense of Definition 2.1, we get that for almost every point x in $\mathbb{R}^n \backslash U$, there holds $\partial \widetilde{u}_\alpha / \partial x_i (x) = \partial \widetilde{u} / \partial x_i (x) = 0$ for α large. This ends the proof of Step 4.2. □

The next step in the proof of Lemma 3.1 is as follows.

Step 4.3. *There exists a positive real number λ such that $\mu_\alpha^{p^* - r_\alpha} \to \lambda$ as $\alpha \to +\infty$, and such that the function $\widetilde{u}$ is a weak solution of the problem*

$$\begin{cases} -\displaystyle\sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left(\left| \frac{\partial \widetilde{u}}{\partial x_i} \right|^{p_i - 2} \frac{\partial \widetilde{u}}{\partial x_i} \right) = \lambda \left| \widetilde{u} \right|^{p^* - 2} \widetilde{u} & \text{in } U\,, \\ \widetilde{u} = 0 & \text{on } \partial U\,. \end{cases}$$

Proof. For any smooth function φ with compact support in U and for α large enough so that the support of φ is included in the domain Ω_α, in the same way as in Step 4.1, we get

$$\sum_{i=1}^{n} \int_{\mathbb{R}^n} \left| \frac{\partial \widetilde{u}_\alpha}{\partial x_i} \right|^{p_i - 2} \frac{\partial \widetilde{u}_\alpha}{\partial x_i} \frac{\partial \varphi}{\partial x_i} dx = \mu_\alpha^{p^* - r_\alpha} \int_{\mathbb{R}^n} \left| \widetilde{u}_\alpha \right|^{r_\alpha - 2} \widetilde{u}_\alpha \varphi dx + \mathrm{o}(1)\,. \tag{4.18}$$

By Step 4.2 (resp. Step 4.1), $\partial \widetilde{u}_\alpha / \partial x_i$ (resp. $\widetilde{u}_\alpha$) converges almost everywhere to the function $\partial \widetilde{u} / \partial x_i$ (resp. $\widetilde{u}$) in U as $\alpha \to +\infty$. Moreover, $\left| \partial \widetilde{u}_\alpha / \partial x_i \right|^{p_i - 2} \partial \widetilde{u}_\alpha / \partial x_i$ (resp. $\left| \widetilde{u}_\alpha \right|^{r_\alpha - 2} \widetilde{u}_\alpha$) keep bounded in $L^{p_i/(p_i - 1)}(U)$ (resp. $L^{p^*/(p^* - 1)}(U)$). By standard integration theory, it follows that

$$\int_U \left| \frac{\partial \widetilde{u}_\alpha}{\partial x_i} \right|^{p_i - 2} \frac{\partial \widetilde{u}_\alpha}{\partial x_i} \frac{\partial \varphi}{\partial x_i} dx \longrightarrow \int_U \left| \frac{\partial \widetilde{u}}{\partial x_i} \right|^{p_i - 2} \frac{\partial \widetilde{u}}{\partial x_i} \frac{\partial \varphi}{\partial x_i} dx \tag{4.19}$$

and

$$\int_U \left| \widetilde{u}_\alpha \right|^{r_\alpha - 2} \widetilde{u}_\alpha \varphi dx \longrightarrow \int_U \left| \widetilde{u} \right|^{p^* - 2} \widetilde{u} \varphi dx \tag{4.20}$$

as $\alpha \to +\infty$. Since (4.18) holds true for all smooth functions φ with compact support in U, by (4.19) and (4.20), passing to the limit as $\alpha \to +\infty$ yields the existence of a nonnegative real number λ such that there holds $\mu_\alpha^{p^*-r_\alpha} \to \lambda$ as $\alpha \to +\infty$ and such that the function $\widetilde{u}$ is a weak solution of the problem

$$\begin{cases} -\displaystyle\sum_{i=1}^n \frac{\partial}{\partial x_i}\left(\left|\frac{\partial u}{\partial x_i}\right|^{p_i-2}\frac{\partial u}{\partial x_i}\right) = \lambda\,|u|^{p^*-2}\,u & \text{in } U\,,\\ u = 0 & \text{on } \partial U\,. \end{cases}$$

In particular, there holds $\lambda > 0$ since the function $\widetilde{u}$ is not identically zero. This ends the proof of Step 4.3. ☐

The fourth step in the proof of Lemma 3.1 states as follows.

Step 4.4. *There exist smooth functions $\widetilde{b}_\alpha$ with compact support in the sets Ω_α such that*

(i) $\widetilde{b}_\alpha \longrightarrow \widetilde{u}$ *in* $D^{1,\overrightarrow{p}}(\mathbb{R}^n)$,

(ii) $|\widetilde{b}_\alpha|^{r_\alpha-1-\varepsilon}\widetilde{b}_\alpha \longrightarrow |\widetilde{u}|^{p^*-1-\varepsilon}\,\widetilde{u}$ *in* $L^{\frac{p^*}{p^*-\varepsilon}}(\mathbb{R}^n)$ *for all* ε *in* $[0,p^*)$.

Proof. Since the function $\widetilde{u}$ belongs to $D^{1,\overrightarrow{p}}(U)$ and since the sets Ω_α converge to U as $\alpha \to +\infty$ in the sense of Definition 2.1, we get that there exist smooth functions $\widetilde{b}_\alpha$ with compact support in Ω_α converging to $\widetilde{u}$ as $\alpha \to +\infty$ in $D^{1,\overrightarrow{p}}(\mathbb{R}^n)$, and thus also in $L^{p^*}(\mathbb{R}^n)$ by the continuity of the embedding of $D^{1,\overrightarrow{p}}(\mathbb{R}^n)$ into $L^{p^*}(\mathbb{R}^n)$. In order to prove the second property, we let ε be a real number in $[0,p^*)$, and for any positive real number R, by standard integration theory, we easily check

$$\left\||\widetilde{b}_\alpha|^{r_\alpha-1-\varepsilon}\widetilde{b}_\alpha - |\widetilde{u}|^{p^*-1-\varepsilon}\,\widetilde{u}\right\|_{L^{\frac{p^*}{p^*-\varepsilon}}\left(\mathcal{P}_0^{\overrightarrow{p}}(R)\right)} \longrightarrow 0$$

as $\alpha \to +\infty$, while on the other hand, we compute

$$\begin{aligned}
&\left\||\widetilde{b}_\alpha|^{r_\alpha-1-\varepsilon}\widetilde{b}_\alpha - |\widetilde{u}|^{p^*-1-\varepsilon}\,\widetilde{u}\right\|_{L^{\frac{p^*}{p^*-\varepsilon}}\left(\mathbb{R}^n\backslash\mathcal{P}_0^{\overrightarrow{p}}(R)\right)}\\
&\le \left\|\widetilde{b}_\alpha\right\|^{r_\alpha-\varepsilon}_{L^{\frac{p^*(r_\alpha-\varepsilon)}{p^*-\varepsilon}}\left(\Omega_\alpha\backslash\mathcal{P}_0^{\overrightarrow{p}}(R)\right)} + \|\widetilde{u}\|_{L^{p^*}\left(\mathbb{R}^n\backslash\mathcal{P}_0^{\overrightarrow{p}}(R)\right)}\\
&\le \left(\frac{|\Omega|}{\mu_\alpha^{p^*}}\right)^{\frac{p^*-r_\alpha}{p^*}} \left\|\widetilde{b}_\alpha\right\|^{r_\alpha-\varepsilon}_{L^{p^*}\left(\mathbb{R}^n\backslash\mathcal{P}_0^{\overrightarrow{p}}(R)\right)} + \|\widetilde{u}\|_{L^{p^*}\left(\mathbb{R}^n\backslash\mathcal{P}_0^{\overrightarrow{p}}(R)\right)}\\
&\longrightarrow \frac{1}{\lambda}\,\|\widetilde{u}\|^{p^*-\varepsilon}_{L^{p^*}\left(\mathbb{R}^n\backslash\mathcal{P}_0^{\overrightarrow{p}}(R)\right)} + \|\widetilde{u}\|_{L^{p^*}\left(\mathbb{R}^n\backslash\mathcal{P}_0^{\overrightarrow{p}}(R)\right)}
\end{aligned}$$

as $\alpha \to +\infty$. Summing these two estimates and passing to the limit as $R \to +\infty$ yield (ii). This ends the proof of Step 4.4. ☐

For any α, we define the function

$$v_\alpha = u_\alpha - b_\alpha = u_\alpha - B_\alpha + R_\alpha,$$

where $b_\alpha = \mu_\alpha^{-1}\widetilde{b}_\alpha \circ \tau^{\overrightarrow{p}}_{\mu_\alpha,x_\alpha}$, $R_\alpha = B_\alpha - b_\alpha$, and where B_α is as in (4.4). Since the support of $\widetilde{b}_\alpha$ is included in Ω_α, the function v_α belongs to $D^{1,\overrightarrow{p}}(\Omega)$ for all α. Moreover, by an easy change of variable and by the former step, we get

$$\|R_\alpha\|_{D^{1,\overrightarrow{p}}(\mathbb{R}^n)} = \|\widetilde{b}_\alpha - \widetilde{u}\|_{D^{1,\overrightarrow{p}}(\mathbb{R}^n)} \longrightarrow 0$$

as $\alpha \to +\infty$.

The fifth step in the proof of Lemma 3.1 is as follows.

Step 4.5. *There holds* $DI_\alpha(v_\alpha) \to 0$ *and* $DJ_\alpha(v_\alpha) \to 0$ *in* $D^{1,\overrightarrow{p}}(\Omega)'$ *as* $\alpha \to +\infty$.

Proof. By the growth condition (1.3), one can easily see that it suffices to show that there holds $DJ_\alpha(v_\alpha) \to 0$ in $D^{1,\overrightarrow{p}}(\Omega)'$ as $\alpha \to +\infty$. We let φ be a function in $D^{1,\overrightarrow{p}}(\Omega)$, and we set $\varphi_\alpha = \mu_\alpha \varphi \circ (\tau^{\overrightarrow{p}}_{\mu_\alpha, x_\alpha})^{-1}$ for all α. We first prove that

$$DJ_\alpha(b_\alpha).\varphi = \mathrm{o}\left(\|\varphi\|_{D^{1,\overrightarrow{p}}(\Omega)}\right) \tag{4.21}$$

as $\alpha \to +\infty$. By renormalizing and using the density of $C_c^\infty(\Omega)$ in $D^{1,\overrightarrow{p}}(\Omega)$, we may assume without loss of generality that $\varphi \in C_c^\infty(\Omega)$ and $\|\varphi\|_{D^{1,\overrightarrow{p}}(\Omega)} = 1$. By an easy change of variable, we get

$$DJ_\alpha(b_\alpha).\varphi = \sum_{i=1}^n \int_{\mathbb{R}^n} \left|\frac{\partial \widetilde{b}_\alpha}{\partial x_i}\right|^{p_i-2} \frac{\partial \widetilde{b}_\alpha}{\partial x_i}\frac{\partial \varphi_\alpha}{\partial x_i} dx - \mu_\alpha^{p^*-r_\alpha} \int_{\mathbb{R}^n} |\widetilde{b}_\alpha|^{r_\alpha-2}\widetilde{b}_\alpha \varphi_\alpha dx \tag{4.22}$$

for all α. By observing that $(\varphi_\alpha)_\alpha$ is bounded, hence weakly compact, in $D^{1,\overrightarrow{p}}(\Omega)$ and using Step 4.3 together with the fact that Ω_α converges to U as $\alpha \to \infty$ in the sense of Definition 2.1, we obtain

$$\sum_{i=1}^n \int_{\mathbb{R}^n} \left|\frac{\partial \widetilde{u}}{\partial x_i}\right|^{p_i-2} \frac{\partial \widetilde{u}}{\partial x_i}\frac{\partial \varphi_\alpha}{\partial x_i} dx = \lambda \int_{\mathbb{R}^n} |\widetilde{u}|^{p^*-2}\widetilde{u}\varphi_\alpha dx + \mathrm{o}(1)$$

as $\alpha \to \infty$. It follows that

$$\begin{aligned} DJ_\alpha(b_\alpha).\varphi = &\sum_{i=1}^n \int_{\mathbb{R}^n} \left(\left|\frac{\partial \widetilde{b}_\alpha}{\partial x_i}\right|^{p_i-2} \frac{\partial \widetilde{b}_\alpha}{\partial x_i} - \left|\frac{\partial \widetilde{u}}{\partial x_i}\right|^{p_i-2} \frac{\partial \widetilde{u}}{\partial x_i}\right)\frac{\partial \varphi_\alpha}{\partial x_i} dx \\ &- \int_{\mathbb{R}^n} \left(\mu_\alpha^{p^*-r_\alpha}|\widetilde{b}_\alpha|^{r_\alpha-2}\widetilde{b}_\alpha - \lambda|\widetilde{u}|^{p^*-2}\widetilde{u}\right)\varphi_\alpha dx + \mathrm{o}(1)\end{aligned}$$

as $\alpha \to \infty$. Since $\|\partial\varphi_\alpha/\partial x_i\|_{L^{p_i}(\mathbb{R}^n)} = \|\partial\varphi/\partial x_i\|_{L^{p_i}(\Omega)}$ for $i = 1, \dots, n$ and $\|\varphi_\alpha\|_{L^{p^*}(\mathbb{R}^n)} = \|\varphi\|_{L^{p^*}(\Omega)}$, by Hölder's inequality, we then get

$$\begin{aligned} |DJ_\alpha(b_\alpha).\varphi| \le &\sum_{i=1}^n \left\|\left|\frac{\partial \widetilde{b}_\alpha}{\partial x_i}\right|^{p_i-2} \frac{\partial \widetilde{b}_\alpha}{\partial x_i} - \left|\frac{\partial \widetilde{u}}{\partial x_i}\right|^{p_i-2} \frac{\partial \widetilde{u}}{\partial x_i}\right\|_{L^{\frac{p_i}{p_i-1}}(\mathbb{R}^n)} \left\|\frac{\partial\varphi}{\partial x_i}\right\|_{L^{p_i}(\Omega)} \\ &+ \left\|\mu_\alpha^{p^*-r_\alpha}|\widetilde{b}_\alpha|^{r_\alpha-2}\widetilde{b}_\alpha - \lambda|\widetilde{u}|^{p^*-2}\widetilde{u}\right\|_{L^{\frac{p^*}{p^*-1}}(\mathbb{R}^n)} \|\varphi\|_{L^{p^*}(\Omega)} + \mathrm{o}(1)\end{aligned}$$

as $\alpha \to \infty$. By Step 4.4 and by the continuity of the embedding of $D^{1,\overrightarrow{p}}(\Omega)$ into $L^{p^*}(\Omega)$, it follows that (4.21) holds true. For any α, by the same change of variable as in (4.22), we write

$$DJ_\alpha(v_\alpha).\varphi = \sum_{i=1}^n \int_{\mathbb{R}^n} \psi_\alpha^i \frac{\partial\varphi_\alpha}{\partial x_i} dx - \mu_\alpha^{p^*-r_\alpha}\int_{\mathbb{R}^n} \psi_\alpha\varphi_\alpha dx + DJ_\alpha(u_\alpha).\varphi - DJ_\alpha(b_\alpha).\varphi\,,$$

where

$$\psi_\alpha^i = \left|\frac{\partial(\widetilde{u}_\alpha - \widetilde{b}_\alpha)}{\partial x_i}\right|^{p_i-2}\frac{\partial(\widetilde{u}_\alpha - \widetilde{b}_\alpha)}{\partial x_i} - \left|\frac{\partial \widetilde{u}_\alpha}{\partial x_i}\right|^{p_i-2}\frac{\partial \widetilde{u}_\alpha}{\partial x_i} + \left|\frac{\partial \widetilde{b}_\alpha}{\partial x_i}\right|^{p_i-2}\frac{\partial \widetilde{b}_\alpha}{\partial x_i}$$

and

$$\psi_\alpha = |\widetilde{u}_\alpha - \widetilde{b}_\alpha|^{r_\alpha-2}(\widetilde{u}_\alpha - \widetilde{b}_\alpha) - |\widetilde{u}_\alpha|^{p^*-2}\widetilde{u}_\alpha + |\widetilde{b}_\alpha|^{r_\alpha-2}\widetilde{b}_\alpha\,.$$

By (4.21) and since $(u_\alpha)_\alpha$ is Palais–Smale for the functionals $(J_\alpha)_\alpha$, it follows that

$$DJ_\alpha(v_\alpha).\varphi = \sum_{i=1}^n \int_{\mathbb{R}^n} \psi_\alpha^i \frac{\partial\varphi_\alpha}{\partial x_i} dx - \mu_\alpha^{p^*-r_\alpha}\int_{\mathbb{R}^n} \psi_\alpha\varphi_\alpha dx + \mathrm{o}\left(\|\varphi\|_{D^{1,\overrightarrow{p}}(\Omega)}\right)$$

as $\alpha \to +\infty$. By Hölder's inequality, we then get

$$\left|DJ_\alpha\left(v_\alpha\right).\varphi\right| \le \sum_{i=1}^n \left\|\psi_\alpha^i\right\|_{L^{\frac{p_i}{p_i-1}}(\mathbb{R}^n)} \left\|\frac{\partial\varphi}{\partial x_i}\right\|_{L^{p_i}(\Omega)} + \left\|\psi_\alpha\right\|_{L^{\frac{p^*}{p^*-1}}(\mathbb{R}^n)} \left\|\varphi\right\|_{L^{p^*}(\Omega)} + \mathrm{o}\left(\left\|\varphi\right\|_{D^{1,\vec{p}}(\Omega)}\right)$$

as $\alpha \to +\infty$. It remains to show that for $i = 1, \dots, n$, there hold

$$\left\|\psi_\alpha^i\right\|_{L^{\frac{p_i}{p_i-1}}(\mathbb{R}^n)} \longrightarrow 0 \quad \text{and} \quad \left\|\psi_\alpha\right\|_{L^{\frac{p^*}{p^*-1}}(\mathbb{R}^n)} \longrightarrow 0 \tag{4.23}$$

as $\alpha \to +\infty$. As is easily checked, given two real numbers q_1 and q_2 such that $1 < q_1 \le q_2$, there exists a positive constant C such that for any real number q in $[q_1, q_2]$ and for small $\varepsilon > 0$, there holds

$$\left||x+y|^{q-2}(x+y) - |x|^{q-2}x - |y|^{q-2}y\right| \le C\left(|x|^\varepsilon |y|^{q-1-\varepsilon} + |x|^{q-1-\varepsilon}|y|^\varepsilon\right)$$

for all real numbers x and y. It follows that there exists a positive constant C independent of α and i such that for small $\varepsilon > 0$, there hold

$$\left|\psi_\alpha^i\right| \le C\left(\left|\frac{\partial(\widetilde{u}_\alpha - \widetilde{b}_\alpha)}{\partial x_i}\right|^\varepsilon \left|\frac{\partial\widetilde{b}_\alpha}{\partial x_i}\right|^{p_i-1-\varepsilon} + \left|\frac{\partial(\widetilde{u}_\alpha - \widetilde{b}_\alpha)}{\partial x_i}\right|^{p_i-1-\varepsilon} \left|\frac{\partial\widetilde{b}_\alpha}{\partial x_i}\right|^\varepsilon\right) \tag{4.24}$$

and

$$\left|\psi_\alpha\right| \le C\left(|\widetilde{u}_\alpha - \widetilde{b}_\alpha|^\varepsilon |\widetilde{b}_\alpha|^{r_\alpha-1-\varepsilon} + |\widetilde{u}_\alpha - \widetilde{b}_\alpha|^{r_\alpha-1-\varepsilon}|\widetilde{b}_\alpha|^\varepsilon\right). \tag{4.25}$$

By Steps 4.1, 4.2, and 4.4, we get that the functions $|\partial(\widetilde{u}_\alpha - \widetilde{b}_\alpha)/\partial x_i|^{\frac{\varepsilon p_i}{p_i-1}}$ (resp. $|\widetilde{u}_\alpha - \widetilde{b}_\alpha|^{\frac{\varepsilon p^*}{p^*-1}}$) converge almost everywhere to 0 in $\mathbb{R}^n$ as $\alpha \to +\infty$. Moreover, they keep bounded in $L^{\frac{p_i-1}{\varepsilon}}(\mathbb{R}^n)$ (resp. $L^{\frac{p^*-1}{\varepsilon}}(\mathbb{R}^n)$). Standard integration theory then yields that the functions $|\partial(\widetilde{u}_\alpha - \widetilde{b}_\alpha)/\partial x_i|^{\frac{\varepsilon p_i}{p_i-1}}$ (resp. $|\widetilde{u}_\alpha - \widetilde{b}_\alpha|^{\frac{\varepsilon p^*}{p^*-1}}$) converge weakly to 0 in $L^{\frac{p_i-1}{\varepsilon}}(\mathbb{R}^n)$ (resp. $L^{\frac{p^*-1}{\varepsilon}}(\mathbb{R}^n)$) as $\alpha \to +\infty$. Step 4.4 also yields that the functions $|\partial\widetilde{b}_\alpha/\partial x_i|^{\frac{p_i(p_i-1-\varepsilon)}{p_i-1}}$ (resp. $|\widetilde{b}_\alpha|^{\frac{p^*(r_\alpha-1-\varepsilon)}{p^*-1}}$) converge strongly to $|\partial\widetilde{u}/\partial x_i|^{\frac{p_i(p_i-1-\varepsilon)}{p_i-1}}$ (resp. $|\widetilde{u}|^{\frac{p^*(p^*-1-\varepsilon)}{p^*-1}}$) in $L^{\frac{p_i-1}{p_i-1-\varepsilon}}(\mathbb{R}^n)$ (resp. $L^{\frac{p^*-1}{p^*-1-\varepsilon}}(\mathbb{R}^n)$) as $\alpha \to +\infty$. It follows that

$$\int_{\mathbb{R}^n} \left|\frac{\partial(\widetilde{u}_\alpha - \widetilde{b}_\alpha)}{\partial x_i}\right|^{\frac{\varepsilon p_i}{p_i-1}} \left|\frac{\partial\widetilde{b}_\alpha}{\partial x_i}\right|^{\frac{p_i(p_i-1-\varepsilon)}{p_i-1}} dx \longrightarrow 0 \tag{4.26}$$

and

$$\int_{\mathbb{R}^n} |\widetilde{u}_\alpha - \widetilde{b}_\alpha|^{\frac{\varepsilon p^*}{p^*-1}} |\widetilde{b}_\alpha|^{\frac{p^*(r_\alpha-1-\varepsilon)}{p^*-1}} dx \longrightarrow 0 \tag{4.27}$$

as $\alpha \to +\infty$. Analogously, we get

$$\int_{\mathbb{R}^n} \left|\frac{\partial(\widetilde{u}_\alpha - \widetilde{b}_\alpha)}{\partial x_i}\right|^{\frac{p_i(p_i-1-\varepsilon)}{p_i-1}} \left|\frac{\partial\widetilde{b}_\alpha}{\partial x_i}\right|^{\frac{\varepsilon p_i}{p_i-1}} dx \longrightarrow 0 \tag{4.28}$$

and

$$\int_{\mathbb{R}^n} |\widetilde{u}_\alpha - \widetilde{b}_\alpha|^{\frac{p^*(r_\alpha-1-\varepsilon)}{p^*-1}} |\widetilde{b}_\alpha|^{\frac{\varepsilon p^*}{p^*-1}} dx \longrightarrow 0 \tag{4.29}$$

as $\alpha \to +\infty$. Finally, (4.23) follows from the estimates (4.24) to (4.29). This ends the proof of Step 4.5. □

We are now in position to prove that the estimate (3.1) holds true.

Proof of (3.1). By the growth condition (1.3), one can easily see that it suffices to prove that there hold

$$\int_{\Omega}\left|\frac{\partial v_\alpha}{\partial x_i}\right|^{p_i} dx=\int_{\Omega}\left|\frac{\partial u_\alpha}{\partial x_i}\right|^{p_i} dx-\int_{\mathbb{R}^n}\left|\frac{\partial \widetilde{u}}{\partial x_i}\right|^{p_i} dx+\mathrm{o}(1) \tag{4.30}$$

and

$$\int_{\Omega}|v_\alpha|^{r_\alpha} dx=\int_{\Omega}|u_\alpha|^{r_\alpha} dx-\lambda\int_{\mathbb{R}^n}|\widetilde{u}|^{p^*} dx+\mathrm{o}(1) \tag{4.31}$$

as $\alpha\to+\infty$. As is easily checked, given two positive real numbers q_1 and q_2 such that $q_1\le q_2$, there exists a positive constant C such that for any real number q in $[q_1,q_2]$ and for small $\varepsilon>0$, there holds

$$||x+y|^q-|x|^q-|y|^q|\le C\left(|x|^\varepsilon|y|^{q-\varepsilon}+|x|^{q-\varepsilon}|y|^\varepsilon\right)$$

for all real numbers x and y. It follows that there exists a positive constant C independent of α and i such that for small $\varepsilon>0$, there hold

$$\begin{aligned}&\left|\left|\frac{\partial(\widetilde{u}_\alpha-\widetilde{b}_\alpha)}{\partial x_i}\right|^{p_i}-\left|\frac{\partial\widetilde{u}_\alpha}{\partial x_i}\right|^{p_i}+\left|\frac{\partial\widetilde{b}_\alpha}{\partial x_i}\right|^{p_i}\right|\\&\qquad\le C\left(\left|\frac{\partial(\widetilde{u}_\alpha-\widetilde{b}_\alpha)}{\partial x_i}\right|^{\varepsilon}\left|\frac{\partial\widetilde{b}_\alpha}{\partial x_i}\right|^{p_i-\varepsilon}+\left|\frac{\partial(\widetilde{u}_\alpha-\widetilde{b}_\alpha)}{\partial x_i}\right|^{p_i-\varepsilon}\left|\frac{\partial\widetilde{b}_\alpha}{\partial x_i}\right|^{\varepsilon}\right)\end{aligned} \tag{4.32}$$

and

$$\left||\widetilde{u}_\alpha-\widetilde{b}_\alpha|^{r_\alpha}-|\widetilde{u}_\alpha|^{r_\alpha}+|\widetilde{b}_\alpha|^{r_\alpha}\right|\le C\left(|\widetilde{u}_\alpha-\widetilde{b}_\alpha|^\varepsilon|\widetilde{b}_\alpha|^{r_\alpha-\varepsilon}+|\widetilde{u}_\alpha-\widetilde{b}_\alpha|^{r_\alpha-\varepsilon}|\widetilde{b}_\alpha|^\varepsilon\right). \tag{4.33}$$

In the same way as in the proof of Step 4.5, we then get that both right members in (4.32) and (4.33) converge to 0 in $L^1(\mathbb{R}^n)$ as $\alpha\to+\infty$. By Step 4.4, it follows that

$$\int_{\mathbb{R}^n}\left|\frac{\partial(\widetilde{u}_\alpha-\widetilde{b}_\alpha)}{\partial x_i}\right|^{p_i} dx=\int_{\mathbb{R}^n}\left|\frac{\partial\widetilde{u}_\alpha}{\partial x_i}\right|^{p_i} dx-\int_{\mathbb{R}^n}\left|\frac{\partial\widetilde{u}}{\partial x_i}\right|^{p_i} dx+\mathrm{o}(1)$$

and

$$\int_{\mathbb{R}^n}|\widetilde{u}_\alpha-\widetilde{b}_\alpha|^{r_\alpha}dx=\int_{\mathbb{R}^n}|\widetilde{u}_\alpha|^{r_\alpha} dx-\int_{\mathbb{R}^n}|\widetilde{u}|^{p^*} dx+\mathrm{o}(1)$$

as $\alpha\to+\infty$. Direct computations yield (4.30) and (4.31). This ends the proof of (3.1). ☐

By Step 4.5 and by (3.1), we get that the sequence $(v_\alpha)_\alpha$ is Palais–Smale for the functionals $(I_\alpha)_\alpha$. In order to end the proof of Lemma 3.1, we treat the case where the functions u_α are nonnegative. In this case, since by Step 4.1, the sequence $(\widetilde{u}_\alpha)_\alpha$ converges, up to a subsequence, almost everywhere to $\widetilde{u}$ in $\mathbb{R}^n$ as $\alpha\to+\infty$, it follows that $\widetilde{u}$ is nonnegative. We consider the functions $(v_\alpha)_+=\max(v_\alpha,0)$ instead of v_α, and we write

$$(v_\alpha)_+=u_\alpha-B_\alpha+R_\alpha$$

where $R_\alpha=B_\alpha-b_\alpha+(v_\alpha)_-$ and $(v_\alpha)_-=\max(-v_\alpha,0)$. As is easily seen, we only have to prove that $(v_\alpha)_-$ converges to 0 in $D^{1,\overrightarrow{p}}(\Omega)$ as $\alpha\to+\infty$. We write

$$\sum_{i=1}^n\int_{\Omega}\left|\frac{\partial(v_\alpha)_-}{\partial x_i}\right|^{p_i} dx=\int_{\Omega}(v_\alpha)_-^{r_\alpha} dx+DJ_\alpha(v_\alpha).(v_\alpha)_-$$

By an easy change of variable and by Step 4.5, it follows that

$$\sum_{i=1}^n\int_{\Omega}\left|\frac{\partial(v_\alpha)_-}{\partial x_i}\right|^{p_i} dx=\mu_\alpha^{p^*-r_\alpha}\int_{\mathbb{R}^n}(\widetilde{u}_\alpha-\widetilde{b}_\alpha)_-^{r_\alpha}dx+\mathrm{o}\left(\left\|(v_\alpha)_-\right\|_{D^{1,\overrightarrow{p}}(\Omega)}\right) \tag{4.34}$$

as $\alpha \to +\infty$. For any positive real number R, by Hölder's inequality and by Steps 4.1 and 4.4, we get

$$\int_{\mathcal{P}_0^{\overrightarrow{p}}(R)} (\widetilde{u}_\alpha - \widetilde{b}_\alpha)_-^{r_\alpha} dx \le R^{r_\alpha (r_\alpha - p^*)} \|\widetilde{u}_\alpha - \widetilde{b}_\alpha\|^{r_\alpha}_{L^{p^*}\left(\mathcal{P}_0^{\overrightarrow{p}}(R)\right)} \longrightarrow 0$$

as $\alpha \to +\infty$, while Step 4.4 also yields

$$\int_{\mathbb{R}^n \backslash \mathcal{P}_0^{\overrightarrow{p}}(R)} (\widetilde{u}_\alpha - \widetilde{b}_\alpha)_-^{r_\alpha} dx \le \int_{\mathbb{R}^n \backslash \mathcal{P}_0^{\overrightarrow{p}}(R)} \left|\widetilde{b}_\alpha\right|^{r_\alpha} dx \longrightarrow \int_{\mathbb{R}^n \backslash \mathcal{P}_0^{\overrightarrow{p}}(R)} \widetilde{u}^{p^*} dx\,.$$

Summing these two estimates and passing to the limit as $R \to +\infty$ gives

$$\int_{\mathbb{R}^n} (\widetilde{u}_\alpha - \widetilde{b}_\alpha)_-^{r_\alpha} dx \longrightarrow 0 \tag{4.35}$$

as $\alpha \to +\infty$. It easily follows from (4.34) and (4.35) that there holds $(v_\alpha)_- \to 0$ in $D^{1,\overrightarrow{p}}(\Omega)$ as $\alpha \to +\infty$. This ends the proof of Lemma 3.1.

5. Proof of Theorem 1.3

In this section, we assume that the domain Ω is strongly asymptotically $\overrightarrow{p}$-stable and that $p_- \ge 2$, and we prove Theorem 1.3 by using a sequence of approximating problems inspired by Otani [38] and Fragalà–Gazzola–Kawhol [21]. We let $\widetilde{u}$ be one of the shape functions of the $\overrightarrow{p}$-bubbles we get in Theorem 1.2. By Section 3, $\widetilde{u}$ is a nontrivial weak solution of the problem

$$\begin{cases} -\displaystyle\sum_{i=1}^n \frac{\partial}{\partial x_i}\left(\left|\frac{\partial \widetilde{u}}{\partial x_i}\right|^{p_i-2} \frac{\partial \widetilde{u}}{\partial x_i}\right) = \lambda \left|\widetilde{u}\right|^{p^*-2} \widetilde{u} & \text{in } U\,, \\ \widetilde{u} = 0 & \text{on } \partial U\,, \end{cases}$$

where λ is a positive real number, and where U is either $\mathbb{R}^n$ or a halfspace since the domain Ω is strongly asymptotically $\overrightarrow{p}$-stable. In case U is a halfspace, we have to prove that if we extend the function $\widetilde{u}$ by 0 outside of U, then it is still a weak solution of the equation

$$-\sum_{i=1}^n \frac{\partial}{\partial x_i}\left(\left|\frac{\partial \widetilde{u}}{\partial x_i}\right|^{p_i-2} \frac{\partial \widetilde{u}}{\partial x_i}\right) = \lambda \left|\widetilde{u}\right|^{p^*-2} \widetilde{u} \quad \text{in } \mathbb{R}^n.$$

Up to a translation, we may assume that the boundary of the halfspace U contains the point 0. We let $\nu = (\nu_1, \dots, \nu_n)$ be the outward unit normal vector to ∂U at 0, and i_0 be an index satisfying $\nu_{i_0} \neq 0$. We define $\widetilde{\nu} = (\widetilde{\nu}_1, \dots, \widetilde{\nu}_n)$ by

$$\widetilde{\nu}_i = \begin{cases} 1/\nu_{i_0} & \text{if } i = i_0 \\ 0 & \text{otherwise.} \end{cases}$$

In particular, we get $\langle \nu, \widetilde{\nu}\rangle = 1$. We then easily construct a smooth, convex, bounded, domain W included in U such that 0 belongs to the interior of the complementary of the set $U\backslash W$ and such that for any point x on ∂W, if $\langle x, \widetilde{\nu}\rangle = 0$, then $\langle \nu(x), \widetilde{\nu}\rangle > 0$, where $\nu(x)$ is the outward unit normal vector to ∂W at x. We let $(R_\alpha)_\alpha$ be a sequence of positive real numbers converging to $+\infty$, and we set $W_\alpha = R_\alpha W$ for all α. As is easily seen, for any bounded subset Ω' of $\mathbb{R}^n$, there holds $\Omega' \cap (U\backslash W_\alpha) = \emptyset$ for α large. Independently, we can prove that the function $\widetilde{u}$ belongs to $L^\infty(U)$. This result is stated in El Hamidi–Rakotoson [20] for nonnegative solutions of the anisotropic equation on the whole Euclidean Space, but the proof still works in our case. We then set $g = \lambda \left|\widetilde{u}\right|^{p^*-2} \widetilde{u}$. For any α, we easily get a sequence $\left(g_\alpha^\beta\right)_\beta$ in $C_0^\infty(W_\alpha)$ bounded in $L^\infty(W_\alpha)$ and converging to the function g in $L^r(W_\alpha)$, for r in $[1, +\infty)$. We let $(\varepsilon_\beta)_\beta$ be a sequence of positive real numbers converging to 0 as $\beta \to +\infty$.

For any α and β, by Fragalà-Gazzola–Lieberman [22], since $p_- \geq 2$ and since the domain W_α is convex and bounded, there exists a unique solution w_α^β in $C^{2,\gamma}(W_\alpha)$ for γ in $(0,1)$ of the problem

$$\begin{cases} -\sum_{i=1}^n \frac{\partial}{\partial x_i}\left(\left(\left|\frac{\partial w_\alpha^\beta}{\partial x_i}\right|^{p_i-2} + \varepsilon_\beta\left(1+|\nabla w_\alpha^\beta|^2\right)^{\frac{p_--2}{2}}\right)\frac{\partial w_\alpha^\beta}{\partial x_i}\right) & \text{in } W_\alpha, \\ \qquad\qquad + \lambda |w_\alpha^\beta|^{p^*-2} w_\alpha^\beta = 2 g_\alpha^\beta & \\ w_\alpha^\beta = 0 & \text{on } \partial W_\alpha. \end{cases} \tag{5.1}$$

We state three preliminary steps. The first one is as follows. We refer to Fragalà–Gazzola–Kawhol [21] for its proof.

Step 5.1. *For any α, up to a subsequence, $\left(w_\alpha^\beta\right)_\beta$ is bounded in $L^\infty(W_\alpha)$, and converges in $D^{1,\overrightarrow{p}}(W_\alpha)$ and in $L^r(W_\alpha)$ for r in $[1,+\infty)$ to a weak solution w_α of the problem*

$$\begin{cases} -\sum_{i=1}^n \frac{\partial}{\partial x_i}\left(\left|\frac{\partial w_\alpha}{\partial x_i}\right|^{p_i-2}\frac{\partial w_\alpha}{\partial x_i}\right) + \lambda |w_\alpha|^{p^*-2} w_\alpha = 2g & \text{in } W_\alpha\,, \\ w_\alpha = 0 & \text{on } \partial W_\alpha\,. \end{cases} \tag{5.2}$$

We then extend the w_α's by 0 outside of the domain W_α so as to regard them as functions in $D^{1,\overrightarrow{p}}(U)$. Our second step states as follows.

Step 5.2. *The sequence $(w_\alpha)_\alpha$ converges to the function $\widetilde{u}$ in $D^{1,\overrightarrow{p}}(U)$.*

Proof. For any α, since w_α is a weak solution of problem (5.2), we get

$$\sum_{i=1}^n \int_U \left|\frac{\partial w_\alpha}{\partial x_i}\right|^{p_i} dx + \lambda \int_U |w_\alpha|^{p^*} dx = 2\int_U g w_\alpha dx\,, \tag{5.3}$$

and Hölder's inequality yields

$$\sum_{i=1}^n \int_U \left|\frac{\partial w_\alpha}{\partial x_i}\right|^{p_i} dx + \lambda \int_U |w_\alpha|^{p^*} dx \leq 2\lambda \left(\int_U |\widetilde{u}|^{p^*} dx\right)^{\frac{p^*-1}{p^*}} \left(\int_U |w_\alpha|^{p^*} dx\right)^{\frac{1}{p^*}}.$$

It easily follows that the sequence $(w_\alpha)_\alpha$ is bounded in $D^{1,\overrightarrow{p}}(U)$, and thus converges, up to a subsequence, weakly to a function w in $D^{1,\overrightarrow{p}}(U)$. For any smooth function φ with compact support in U and for α large enough so that the support of φ is included in W_α, there holds

$$\sum_{i=1}^n \int_U \left|\frac{\partial w_\alpha}{\partial x_i}\right|^{p_i-2} \frac{\partial w_\alpha}{\partial x_i}\frac{\partial(\varphi - w_\alpha)}{\partial x_i} dx = \int_U \left(2g - \lambda |w_\alpha|^{p^*-2} w_\alpha\right)(\varphi - w_\alpha)\, dx\,,$$

and by Young's inequality, it follows that

$$\sum_{i=1}^n \frac{1}{p_i}\left(\int_U \left|\frac{\partial \varphi}{\partial x_i}\right|^{p_i} dx - \int_U \left|\frac{\partial w_\alpha}{\partial x_i}\right|^{p_i} dx\right) \geq \int_U \left(2g - \lambda |w_\alpha|^{p^*-2} w_\alpha\right)(\varphi - w_\alpha)\, dx\,. \tag{5.4}$$

By the continuity of the embedding of $D^{1,\overrightarrow{p}}(U)$ into $L^{p^*}(U)$, we get that the functions $|w_\alpha|^{p^*-2} w_\alpha$ keep bounded in $L^{p^*/(p^*-1)}(U)$, and thus by the compact embedding theorem in Rákosník [41], one can easily check that they converge, up to a subsequence, almost everywhere to the function $|w|^{p^*-2} w$ in U as $\alpha \to +\infty$. By standard integration theory, it follows that the functions $|w_\alpha|^{p^*-2} w_\alpha$ converge, up to a subsequence, weakly to $|w|^{p^*-2} w$ in

$L^{p^*/(p^*-1)}(U)$ as $\alpha \to +\infty$. Passing to the limit, up to a subsequence, into (5.4) as $\alpha \to +\infty$ then gives

$$\sum_{i=1}^{n} \frac{1}{p_i} \left(\int_U \left| \frac{\partial \varphi}{\partial x_i} \right|^{p_i} dx - \int_U \left| \frac{\partial w}{\partial x_i} \right|^{p_i} dx \right) \ge \int_U \left(2g - \lambda |w|^{p^*-2} w \right) (\varphi - w) \, dx \, . \tag{5.5}$$

By an easy density argument, we get that (5.5) holds true for all functions φ in $D^{1,\overrightarrow{p}}(U)$. In particular, plugging $\varphi = (1-t) w + t\widetilde{u}$ in (5.5) and passing to the limit as $t \to 0$ yield

$$\sum_{i=1}^{n} \int_U \left| \frac{\partial w}{\partial x_i} \right|^{p_i-2} \frac{\partial w}{\partial x_i} \left(\frac{\partial \widetilde{u}}{\partial x_i} - \frac{\partial w}{\partial x_i} \right) dx \ge \int_U \left(2g - \lambda |w|^{p^*-2} w \right) (\widetilde{u} - w) \, dx \, .$$

Taking into account that there holds

$$\sum_{i=1}^{n} \int_U \left| \frac{\partial \widetilde{u}}{\partial x_i} \right|^{p_i-2} \frac{\partial \widetilde{u}}{\partial x_i} \left(\frac{\partial \widetilde{u}}{\partial x_i} - \frac{\partial w}{\partial x_i} \right) dx = \lambda \int_U |\widetilde{u}|^{p^*-2} \widetilde{u} \, (\widetilde{u} - w) \, dx \, ,$$

it follows that

$$\begin{aligned} \sum_{i=1}^{n} \int_U \left(\left| \frac{\partial \widetilde{u}}{\partial x_i} \right|^{p_i-2} \frac{\partial \widetilde{u}}{\partial x_i} - \left| \frac{\partial w}{\partial x_i} \right|^{p_i-2} \frac{\partial w}{\partial x_i} \right) \left(\frac{\partial \widetilde{u}}{\partial x_i} - \frac{\partial w}{\partial x_i} \right) dx & \\ \le \lambda \int_U \left(|w|^{p^*-2} w - |\widetilde{u}|^{p^*-2} \widetilde{u} \right) (\widetilde{u} - w) \, dx \, . & \end{aligned} \tag{5.6}$$

The left hand side in (5.6) is nonpositive while the right hand side is nonnegative, thus they both are equal to 0. In particular, there holds $w = \widetilde{u}$. Passing to the limit, up to a subsequence, into (5.3) then gives

$$\sum_{i=1}^{n} \int_U \left| \frac{\partial w_\alpha}{\partial x_i} \right|^{p_i} dx + \lambda \int_U |w_\alpha|^{p^*} dx \longrightarrow 2\lambda \int_U |\widetilde{u}|^{p^*} dx = \sum_{i=1}^{n} \int_U \left| \frac{\partial \widetilde{u}}{\partial x_i} \right|^{p_i} dx + \lambda \int_U |\widetilde{u}|^{p^*} dx$$

as $\alpha \to +\infty$. Therefore, up to a subsequence, $(w_\alpha)_\alpha$ converges in fact strongly in $D^{1,\overrightarrow{p}}(U)$ to the function $\widetilde{u}$. This ends the proof of Step 5.2. ☐

Our third and last step is as follows.

Step 5.3. *For any bounded subset Ω' of $\mathbb{R}^n$ and for $i = 1, \dots, n$, there holds*

$$\lim_{\alpha \to +\infty} \limsup_{\beta \to +\infty} \int_{\Omega' \cap \partial U} \left| \frac{\partial w_\alpha^\beta}{\partial x_i} \right|^{p_i} d\sigma = 0$$

where $d\sigma$ is the volume element on ∂U.

Proof. Given a smooth function $h : \mathbb{R}^n \to \mathbb{R}^n$, for any α and β, a generalization of the Pohožaev identity [39] stated in Pucci–Serrin [40] yields

$$\begin{aligned} &\sum_{i=1}^{n} \frac{p_i - 1}{p_i} \int_{\partial W_\alpha} \left| \frac{\partial w_\alpha^\beta}{\partial x_i} \right|^{p_i} \langle h, \nu \rangle \, d\sigma \\ &\quad = \frac{\varepsilon_\beta}{p_-} \int_{\partial W_\alpha} \left(1 + \left(\frac{\partial w_\alpha^\beta}{\partial \nu} \right)^2 \right)^{\frac{p_- - 2}{2}} \left(1 - (p_- - 1) \left(\frac{\partial w_\alpha^\beta}{\partial \nu} \right)^2 \right) \langle h, \nu \rangle \, d\sigma \\ &\quad + \sum_{i=1}^{n} \int_{W_\alpha} \left(\left| \frac{\partial w_\alpha^\beta}{\partial x_i} \right|^{p_i-2} \frac{\partial w_\alpha^\beta}{\partial x_i} + \varepsilon_\beta \left(1 + \left| \nabla w_\alpha^\beta \right|^2 \right)^{\frac{p_- - 2}{2}} \frac{\partial w_\alpha^\beta}{\partial x_i} \right) \left\langle \nabla w_\alpha^\beta, \frac{\partial h}{\partial x_i} \right\rangle dx \end{aligned}$$

$$- \int_{W_\alpha} \left(\sum_{i=1}^{n} \frac{1}{p_i} \left| \frac{\partial w_\alpha^\beta}{\partial x_i} \right|^{p_i} + \frac{\varepsilon_\beta}{p_-} \left(1 + \left| \nabla w_\alpha^\beta \right|^2 \right)^{\frac{p_-}{2}} + \frac{\lambda}{p^*} \left| w_\alpha^\beta \right|^{p^*} - 2 g_\alpha^\beta w_\alpha^\beta \right) \operatorname{div} h dx$$

$$+ 2 \int_{W_\alpha} w_\alpha^\beta \left\langle \nabla g_\alpha^\beta, h \right\rangle dx \,, \quad (5.7)$$

where ν is the outward unit normal vector to ∂W_α, and where $d\sigma$ is the volume element on ∂W_α. Since there holds $w_\alpha^\beta \equiv 0$ on ∂W_α, the divergence theorem gives

$$\int_{W_\alpha} g_\alpha^\beta w_\alpha^\beta \operatorname{div} h dx + \int_{W_\alpha} w_\alpha^\beta \left\langle \nabla g_\alpha^\beta, h \right\rangle dx = - \int_{W_\alpha} g_\alpha^\beta \left\langle \nabla w_\alpha^\beta, h \right\rangle dx \,. \quad (5.8)$$

We now choose some appropriate functions h in order to prove Step 5.3. We let η be a smooth cutoff function such that $0 \le \eta \le 1$ in $\mathbb{R}$, $\eta \equiv 1$ in $[-1, 1]$, and $\eta \equiv 0$ out of $[-2, 2]$. For any positive real number R, plugging $h(x) = \eta\left(\langle x, \widetilde{\nu} \rangle / R\right) \widetilde{\nu}$ into (5.7) and (5.8) then yields

$$\sum_{i=1}^{n} \frac{p_i - 1}{p_i} \int_{\partial W_\alpha} \left| \frac{\partial w_\alpha^\beta}{\partial x_i} \right|^{p_i} \eta \left(\frac{\langle x, \widetilde{\nu} \rangle}{R} \right) \langle \nu, \widetilde{\nu} \rangle \, d\sigma$$

$$= \frac{\varepsilon_\beta}{p_-} \int_{\partial W_\alpha} \left(1 + \left(\frac{\partial w_\alpha^\beta}{\partial \nu} \right)^2 \right)^{\frac{p_- - 2}{2}} \left(1 - (p_- - 1) \left(\frac{\partial w_\alpha^\beta}{\partial \nu} \right)^2 \right) \eta \left(\frac{\langle x, \widetilde{\nu} \rangle}{R} \right) \langle \nu, \widetilde{\nu} \rangle \, d\sigma$$

$$+ \frac{\widetilde{\nu}_{i_0}^2}{R} \int_{W_\alpha} \left(\left| \frac{\partial w_\alpha^\beta}{\partial x_{i_0}} \right|^{p_{i_0}} + \varepsilon_\beta \left(1 + \left| \nabla w_\alpha^\beta \right|^2 \right)^{\frac{p_- - 2}{2}} \left| \frac{\partial w_\alpha^\beta}{\partial x_{i_0}} \right|^2 \right.$$

$$\left. - \sum_{i=1}^{n} \frac{1}{p_i} \left| \frac{\partial w_\alpha^\beta}{\partial x_i} \right|^{p_i} - \frac{\varepsilon_\beta}{p_-} \left(1 + \left| \nabla w_\alpha^\beta \right|^2 \right)^{\frac{p_-}{2}} - \frac{\lambda}{p^*} \left| w_\alpha^\beta \right|^{p^*} \right) \eta' \left(\frac{\langle x, \widetilde{\nu} \rangle}{R} \right) dx$$

$$- 2 \widetilde{\nu}_{i_0} \int_{W_\alpha} g_\alpha^\beta \frac{\partial w_\alpha^\beta}{\partial x_{i_0}} \eta \left(\frac{\langle x, \widetilde{\nu} \rangle}{R} \right) dx \,. \quad (5.9)$$

Taking into account that for any x on ∂W, if $\langle x, \widetilde{\nu} \rangle = 0$, then $\langle \nu(x), \widetilde{\nu} \rangle > 0$, one can easily get that for α large, for any x on ∂W_α, if $-2R < \langle x, \widetilde{\nu} \rangle < 2R$, then $\langle \nu(x), \widetilde{\nu} \rangle > 0$. Hence, we get that there holds $\eta\left(\langle x, \widetilde{\nu} \rangle / R\right) \langle \nu(x), \widetilde{\nu} \rangle \ge 0$ on ∂W_α for α large. For $i = 1, \dots, n$, since $\langle \nu, \widetilde{\nu} \rangle = 1$ on ∂U, it follows that

$$\int_{\partial U} \left| \frac{\partial w_\alpha^\beta}{\partial x_i} \right|^{p_i} \eta \left(\frac{\langle x, \widetilde{\nu} \rangle}{R} \right) d\sigma \le \int_{\partial W_\alpha} \left| \frac{\partial w_\alpha^\beta}{\partial x_i} \right|^{p_i} \eta \left(\frac{\langle x, \widetilde{\nu} \rangle}{R} \right) \langle \nu, \widetilde{\nu} \rangle \, d\sigma \,. \quad (5.10)$$

Noting that the function $s \mapsto (1 + s^2)^{(p_- - 2)/2} \left(1 - (p_- - 1) s^2\right)$ is bounded from above on $\mathbb{R}$, we also get

$$\limsup_{\beta \to +\infty} \frac{\varepsilon_\beta}{p_-} \int_{\partial W_\alpha} \left(1 + \left(\frac{\partial w_\alpha^\beta}{\partial \nu} \right)^2 \right)^{\frac{p_- - 2}{2}} \left(1 - (p_- - 1) \left(\frac{\partial w_\alpha^\beta}{\partial \nu} \right)^2 \right) \eta \left(\frac{\langle x, \widetilde{\nu} \rangle}{R} \right) \langle \nu, \widetilde{\nu} \rangle \, d\sigma \le 0 \quad (5.11)$$

Since the sequence $\left(g_\alpha^\beta\right)_\beta$ converges to the function g in $L^r(W_\alpha)$ for r in $[1, +\infty)$ and since the sequence $\left(w_\alpha^\beta\right)_\beta$ converges to the function w_α in $D^{1,\overrightarrow{p}}(W_\alpha)$ and in $L^r(W_\alpha)$ for r in $[1, +\infty)$, by (5.10) and (5.11), passing to the upper limit into (5.9) as $\beta \to +\infty$ yields

$$\limsup_{\beta \to +\infty} \sum_{i=1}^{n} \frac{p_i - 1}{p_i} \int_{\partial U} \left| \frac{\partial w_\alpha^\beta}{\partial x_i} \right|^{p_i} \eta \left(\frac{\langle x, \widetilde{\nu} \rangle}{R} \right) d\sigma \le \frac{\widetilde{\nu}_{i_0}^2}{R} \int_{W_\alpha} \Bigg(\left| \frac{\partial w_\alpha}{\partial x_{i_0}} \right|^{p_{i_0}} - \sum_{i=1}^{n} \frac{1}{p_i} \left| \frac{\partial w_\alpha}{\partial x_i} \right|^{p_i} - \frac{\lambda}{p^*} \left| w_\alpha \right|^{p^*} \Bigg) \eta' \left(\frac{\langle x, \widetilde{\nu} \rangle}{R} \right) dx - 2 \widetilde{\nu}_{i_0} \int_{W_\alpha} g \frac{\partial w_\alpha}{\partial x_{i_0}} \eta \left(\frac{\langle x, \widetilde{\nu} \rangle}{R} \right) dx \,.$$

Since by Step 5.2, the sequence $(w_\alpha)_\alpha$ converges to the function $\widetilde{u}$ in $D^{1,\overrightarrow{p}}(U)$ and thus in $L^{p^*}(U)$ by the continuity of the embedding of $D^{1,\overrightarrow{p}}(U)$ into $L^{p^*}(U)$, passing to the upper limit as $\alpha \to +\infty$ gives

$$\limsup_{\alpha \to +\infty} \limsup_{\beta \to +\infty} \sum_{i=1}^{n} \frac{p_i - 1}{p_i} \int_{\partial U} \left| \frac{\partial w_\alpha^\beta}{\partial x_i} \right|^{p_i} \eta \left(\frac{\langle x, \widetilde{\nu} \rangle}{R} \right) d\sigma \le \frac{\widetilde{\nu}_{i_0}^2}{R} \int_{U} \Bigg(\left| \frac{\partial \widetilde{u}}{\partial x_{i_0}} \right|^{p_{i_0}} - \sum_{i=1}^{n} \frac{1}{p_i} \left| \frac{\partial \widetilde{u}}{\partial x_i} \right|^{p_i} - \frac{\lambda}{p^*} \left| \widetilde{u} \right|^{p^*} \Bigg) \eta' \left(\frac{\langle x, \widetilde{\nu} \rangle}{R} \right) dx - 2 \widetilde{\nu}_{i_0} \int_{U} g \frac{\partial \widetilde{u}}{\partial x_{i_0}} \eta \left(\frac{\langle x, \widetilde{\nu} \rangle}{R} \right) dx \,.$$

Then, since the function g belongs to $L^{p_i/(p_i-1)}(U)$ for $i = 1, \dots, n$, passing to the upper limit as $R \to +\infty$ yields

$$\limsup_{R \to +\infty} \limsup_{\alpha \to +\infty} \limsup_{\beta \to +\infty} \sum_{i=1}^{n} \frac{p_i - 1}{p_i} \int_{\partial U} \left| \frac{\partial w_\alpha^\beta}{\partial x_i} \right|^{p_i} \eta \left(\frac{\langle x, \widetilde{\nu} \rangle}{R} \right) d\sigma \le -2 \widetilde{\nu}_{i_0} \int_{U} g \frac{\partial \widetilde{u}}{\partial x_{i_0}} dx \,.$$

Given any bounded subset Ω' of $\mathbb{R}^n$, it follows that

$$\limsup_{\alpha \to +\infty} \limsup_{\beta \to +\infty} \sum_{i=1}^{n} \frac{p_i - 1}{p_i} \int_{\Omega' \cap \partial U} \left| \frac{\partial w_\alpha^\beta}{\partial x_i} \right|^{p_i} d\sigma \le -2 \widetilde{\nu}_{i_0} \int_{U} g \frac{\partial \widetilde{u}}{\partial x_{i_0}} dx \,.$$

It remains to show that there holds

$$\int_{U} g \frac{\partial \widetilde{u}}{\partial x_{i_0}} dx = 0 \,. \tag{5.12}$$

For any positive real number R, we let Ω_R be a smooth, bounded domain of $\mathbb{R}^n$ satisfying $B_0(2R) \cap U \subset \Omega_R \subset U$. We then get

$$\int_{\Omega_R} g \frac{\partial \widetilde{u}}{\partial x_{i_0}} \eta \left(\frac{|x|}{R} \right) dx = - \frac{\lambda}{R p^*} \int_{\Omega_R} |\widetilde{u}|^{p^*} \frac{x_{i_0}}{|x|} \eta' \left(\frac{|x|}{R} \right) dx \,,$$

and passing to the limit as $R \to +\infty$ finally yields (5.12). This ends the proof of Step 5.3. □

Thanks to Steps 5.1 to 5.3, we are now in position to prove Theorem 1.3.

Proof of Theorem 1.3. For any smooth function φ with compact support in $\mathbb{R}^n$, for α large enough so that $\operatorname{Supp} \varphi \cap \partial W_\alpha \backslash \partial U = \emptyset$, and for any β, multiplying equation (5.1) by φ and integrating by parts on W_α then give

$$\sum_{i=1}^n \int_{\partial U} \left|\frac{\partial w_\alpha^\beta}{\partial x_i}\right|^{p_i-2} \frac{\partial w_\alpha^\beta}{\partial x_i} \varphi \nu_i d\sigma + \varepsilon_\beta \int_{\partial U} \left(1 + \left(\frac{\partial w_\alpha^\beta}{\partial \nu}\right)^2\right)^{\frac{p_- -2}{2}} \frac{\partial w_\alpha^\beta}{\partial \nu} \varphi d\sigma$$
$$= \sum_{i=1}^n \int_{W_\alpha} \left|\frac{\partial w_\alpha^\beta}{\partial x_i}\right|^{p_i-2} \frac{\partial w_\alpha^\beta}{\partial x_i} \frac{\partial \varphi}{\partial x_i} dx + \varepsilon_\beta \int_{W_\alpha} \left(1 + \left|\nabla w_\alpha^\beta\right|^2\right)^{\frac{p_- -2}{2}} \left\langle \nabla w_\alpha^\beta, \nabla \varphi \right\rangle dx$$
$$+ \lambda \int_{W_\alpha} \left|w_\alpha^\beta\right|^{p^*-2} w_\alpha^\beta \varphi dx - 2 \int_{W_\alpha} g_\alpha^\beta \varphi dx\,.$$

Since the sequence $\left(g_\alpha^\beta\right)_\beta$ converges to the function g in $L^1(W_\alpha)$ and since by Step 5.1 the sequence $\left(w_\alpha^\beta\right)_\beta$ converges to the function w_α in $D^{1,\vec{p}}(W_\alpha)$ and in $L^r(W_\alpha)$ for r in $[1, +\infty)$, passing to the limit as $\beta \to +\infty$ yields

$$\sum_{i=1}^n \int_{\partial U} \left|\frac{\partial w_\alpha^\beta}{\partial x_i}\right|^{p_i-2} \frac{\partial w_\alpha^\beta}{\partial x_i} \varphi \nu_i d\sigma + \varepsilon_\beta \int_{\partial U} \left(1 + \left(\frac{\partial w_\alpha^\beta}{\partial \nu}\right)^2\right)^{\frac{p_- -2}{2}} \frac{\partial w_\alpha^\beta}{\partial \nu} \varphi d\sigma$$
$$\to \sum_{i=1}^n \int_{W_\alpha} \left|\frac{\partial w_\alpha}{\partial x_i}\right|^{p_i-2} \frac{\partial w_\alpha}{\partial x_i} \frac{\partial \varphi}{\partial x_i} dx + \lambda \int_{W_\alpha} \left|w_\alpha\right|^{p^*-2} w_\alpha \varphi dx - 2 \int_{W_\alpha} g \varphi dx \tag{5.13}$$

as $\beta \to +\infty$. By Step 5.3, one can easily check

$$\lim_{\alpha \to +\infty} \limsup_{\beta \to +\infty} \left| \sum_{i=1}^n \int_{\partial U} \left|\frac{\partial w_\alpha^\beta}{\partial x_i}\right|^{p_i-2} \frac{\partial w_\alpha^\beta}{\partial x_i} \varphi \nu_i d\sigma \right| = 0 \tag{5.14}$$

and

$$\lim_{\alpha \to +\infty} \limsup_{\beta \to +\infty} \left| \int_{\partial U} \left(1 + \left(\frac{\partial w_\alpha^\beta}{\partial \nu}\right)^2\right)^{\frac{p_- -2}{2}} \frac{\partial w_\alpha^\beta}{\partial \nu} \varphi d\sigma \right| = 0\,. \tag{5.15}$$

By (5.14), (5.15), and since by step 5.2, $(w_\alpha)_\alpha$ converges to the function $\widetilde{u}$ in $D^{1,\vec{p}}(U)$ and thus in $L^{p^*}(U)$ by the continuity of the embedding of $D^{1,\vec{p}}(U)$ into $L^{p^*}(U)$, passing to the limit into (5.13) as $\alpha \to +\infty$ gives

$$\sum_{i=1}^n \int_{\mathbb{R}^n} \left|\frac{\partial \widetilde{u}}{\partial x_i}\right|^{p_i-2} \frac{\partial \widetilde{u}}{\partial x_i} \frac{\partial \varphi}{\partial x_i} dx = \lambda \int_{\mathbb{R}^n} \left|\widetilde{u}\right|^{p^*-2} \widetilde{u} \varphi dx\,.$$

It follows from an easy density argument that this estimates holds true for all functions φ in $D^{1,\vec{p}}(\mathbb{R}^n)$. This ends the proof of Theorem 1.3. □

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