INFINITELY MANY SOLUTIONS FOR CUBIC NONLINEAR SCHRÖDINGER EQUATIONS IN DIMENSION FOUR

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Abstract. We extend Chen, Wei, and Yan's constructions of families of solutions with unbounded energies ([5]) to the case of cubic nonlinear Schrödinger equations in the optimal dimension four.

1. Introduction and main results

In this note, we consider the cubic nonlinear Schrödinger equation

$$\Delta_g u + fu = u^3 \qquad \text{in } M \tag{1.1}$$

where (M, g) is a Riemannian manifold of dimension 4, $\Delta_g := -\operatorname{div}_g \nabla$ is the Laplace-Beltrami operator, and $f \in C^{0,\alpha}(M)$, $\alpha \in (0,1)$.

In case $(M, g) = (\mathbb{S}^4, g_0)$ where g_0 is the standard metric on the sphere $\mathbb{S}^4$, we obtain the following result:

Theorem 1.1. *Assume that $(M, g) = (\mathbb{S}^4, g_0)$ and $f > 2$ is constant. Then there exists a family of positive solutions $(u_\varepsilon)_{\varepsilon>0}$ to (1.1) such that $\|\nabla u_\varepsilon\|_{L^2(\mathbb{S}^4)} \to \infty$ as $\varepsilon \to 0$.*

Theorem 1.1 extends a result obtained by Chen, Wei, and Yan [5] in dimensions $n \geq 5$ for positive solutions of the equation

$$\Delta_g u + fu = u^{2^*-1} \qquad \text{in } M \tag{1.2}$$

where $2^* := 2n/(n-2)$. The dimension four is optimal for this result since Li and Zhu [11] obtained the existence of a priori bounds on the energy of positive solutions to (1.2) in dimension three.

It is also interesting to mention that in case $n \notin \{3, 6\}$ and $f > \frac{n(n-2)}{4}$ on $\mathbb{S}^n$ (or more generally $f > \frac{n-2}{4(n-1)} \operatorname{Scal}_g$ on a general closed manifold where Scal_g is the scalar curvature), Druet [6] obtained a compactness result for families of positive soutions $(u_\varepsilon)_{\varepsilon>0}$ of (1.2) with bounded energies, i.e. such that $\|\nabla u_\varepsilon\|_{L^2(M)} < C$ for some constant C independent of ε. The above Theorem 1.1 together with the result of Chen, Wei, and Yan [5] in dimensions $n \geq 5$ show that the energy assumption in Druet's result is necessary at least in the case of the standard sphere.

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In case $f \equiv \frac{n(n-2)}{4}$ and $(M,g) = (\mathbb{S}^n, g_0)$, the positive solutions of (1.2) have been classified by Obata [12] (see also Caffarelli, Gidas, and Spruck [4]). In this case, the solutions are not bounded in $L^\infty(\mathbb{S}^n)$ but they all have the same energy. We refer to Brendle [2], Brendle and Marques [3], Khuri, Marques, and Schoen [10] and the references therein for results on the set of solutions of (1.2) in case $f \equiv \frac{n-2}{4(n-1)} \operatorname{Scal}_g$ and $(M,g) \neq (\mathbb{S}^n, g_0)$. On the other hand, in case $f < \frac{n-2}{4(n-1)} \operatorname{Scal}_g$ on a general closed manifold, Druet [7] obtained pointwise a priori bounds on the set of positive solutions of (1.2). Remark that if moreover $0 < f < \frac{n-2}{4(n-1)} \operatorname{Scal}_g$ is constant, then Bidaut-Véron and Véron [1] obtained that $u \equiv f^{(n-2)/4}$ is the unique positive solution of (1.2). We refer to the books of Druet, Hebey, and Robert [8] and Hebey [9] for more results on equations of type (1.1) on a closed manifold.

As in the paper of Chen, Wei, and Yan [5], we obtain Theorem 1.1 by proving a more general result in case $(M,g) = (\mathbb{R}^4, \delta_0)$ where δ_0 is the Euclidean metric on $\mathbb{R}^4$. We let $D^{1,2}(\mathbb{R}^4)$ be the completion of the set of smooth functions with compact support in $\mathbb{R}^4$ with respect to the norm $\|u\|_{D^{1,2}(\mathbb{R}^4)} = \|\nabla u\|_{L^2(\mathbb{R}^4)}$. For simplicity, we will denote $\Delta := \Delta_{\delta_0}$, $\langle \cdot, \cdot \rangle := \langle \cdot, \cdot \rangle_{\delta_0}$, and $|\cdot| := |\cdot|_{\delta_0}$. We say that the operator $\Delta + f$ is coercive in $D^{1,2}(\mathbb{R}^4)$ if

$$\int_{\mathbb{R}^4} \left(|\nabla u|^2 + f u^2 \right) dx \geq C \|u\|^2_{D^{1,2}(\mathbb{R}^4)} \qquad \forall u \in D^{1,2}\left(\mathbb{R}^4\right)$$

for some constant $C > 0$. We obtain the following result:

Theorem 1.2. *Assume that $(M,g) = (\mathbb{R}^4, \delta_0)$ and $f \in C^{0,\alpha}(\mathbb{R}^4) \cap L^2(\mathbb{R}^4)$ is radially symmetric about the point 0. Assume moreover that the operator $\Delta + f$ is coercive in $D^{1,2}(\mathbb{R}^4)$ and the function $r \mapsto r^2 f(r)$ has a strict local maximum point $r_0 > 0$ such that $f(r_0) > 0$. Then there exists a family of positive solutions $(u_\varepsilon)_{\varepsilon > 0}$ in $C^{2,\alpha}(\mathbb{R}^4) \cap D^{1,2}(\mathbb{R}^4)$ of (1.1) such that $\|\nabla u_\varepsilon\|_{L^2(\mathbb{R}^4)} \to \infty$ as $\varepsilon \to 0$.*

The proof of Theorem 1.2 relies on a Lyapunov–Schmidt-type method as in the paper of Chen, Wei, and Yan [5]. This method for constructing solutions with infinitely many peaks was invented and successfully used in previous works by Wang, Wei and Yan [13, 14] and Wei and Yan [15–18]. A specificity in our case is that the number of peaks in the construction behaves as a logarithm of the peak's height while it behaves as a power of the peak's height in the higher dimensional case (see the paper of Chen, Wei, and Yan [5]). Due to this logarithm behavior, we need to introduce some suitable changes of variables in order to find the critical points of the reduced energy in this case (see the proof of Theorem 1.2 at the end of Section 2).

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2. Proof of Theorems 1.1 and 1.2

This section is devoted to the proof of Theorems 1.1 and 1.2. For any integer $k \geq 1$, we let H_k be the set of all functions $u \in D^{1,2}(\mathbb{R}^4)$ such that u is even in x_2, x_3, x_4 and

$$u\left(r\cos(\theta), r\sin(\theta), x_3, x_4\right) = u\left(r\cos(\theta + 2\pi/k), r\sin(\theta + 2\pi/k), x_3, x_4\right)$$

for all $r > 0$ and $\theta, x_3, x_4 \in \mathbb{R}$. Assuming that the operator $\Delta + f$ is coercive in $D^{1,2}(\mathbb{R}^4)$, we can equip H_k with the inner product

$$\langle u, v\rangle_{H_k} := \int_{\mathbb{R}^4} \left(\langle \nabla u, \nabla v\rangle + fuv\right) dx \qquad \forall u, v \in H_k$$

and the norm

$$\|u\|_{H_k} := \sqrt{\langle u, u\rangle_{H_k}} \qquad \forall u \in H_k.$$

For any $k \geq 1$ and $r, \mu > 0$, we define

$$W_{k,r,\mu} := \sum_{i=1}^{k} U_{i,k,r,\mu}$$

where

$$U_{i,k,r,\mu}(x) := \frac{2\sqrt{2}\mu}{1 + \mu^2 \left|x - x_{i,k,r}\right|^2} \qquad \forall x \in \mathbb{R}^4$$

and

$$x_{i,k,r} := \left(r\cos\left(2(i-1)\pi/k\right), r\sin\left(2(i-1)\pi/k\right), 0, 0\right).$$

Moreover, we define

$$P_{k,r,\mu} := \left\{\phi \in H_k : \sum_{i=1}^{k} \langle \phi, Z_{i,j,k,r,\mu}\rangle_{H_k} = 0 \quad \forall j \in \{1,2\}\right\}$$

where

$$Z_{i,1,k,r,\mu} := \frac{1}{\mu}\frac{d}{dr}\left[U_{i,k,r,\mu}\right] \qquad \text{and} \qquad Z_{i,2,k,r,\mu} := \mu\frac{d}{d\mu}\left[U_{i,k,r,\mu}\right].$$

First, in Proposition 2.1 below, we solve the equation

$$Q_{k,r,\mu}\left(W_{k,r,\mu} + \phi - (\Delta + f)^{-1}\left((W_{k,r,\mu} + \phi)_+^3\right)\right) = 0 \tag{2.1}$$

where $\phi \in P_{k,r,\mu}$ is the unknown function, $Q_{k,r,\mu}$ is the orthogonal projection of H_k onto $P_{k,r,\mu}$, and $u_+ := \max(u, 0)$ for all $u : \mathbb{R}^4 \to \mathbb{R}$.

We will prove the following result in Section 3:

Proposition 2.1. *Let $f \in C^{0,\alpha}(\mathbb{R}^4) \cap L^2(\mathbb{R}^4)$ be a radially symmetric function about the point 0 and such that the operator $\Delta + f$ is coercive in $D^{1,2}(\mathbb{R}^4)$. Then for any $a, b, c, d > 0$ such that $a < b$ and $c < d$, there exist constants $k_0 > 0$ and $C_0 > 0$ such that for any $k \geq k_0$, $r \in [a, b]$, and $\mu \in \left[e^{ck^2}, e^{dk^2}\right]$, there exists a unique solution $\phi_{k,r,\mu} \in P_{k,r,\mu}$ of* (2.1) *such that*

$$\|\phi_{k,r,\mu}\|_{H_k} \leq C_0 k/\mu. \tag{2.2}$$

Moreover, the map $(r, \mu) \mapsto \phi_{k,r,\mu}$ is continuously differentiable and if there exists a critical point $(r_k, \mu_k) \in [a, b] \times \left[e^{ck^2}, e^{dk^2}\right]$ of the function

$$(r, \mu) \longmapsto \mathcal{I}_k(r, \mu) := I(W_{k,r,\mu} + \phi_{k,r,\mu})$$

where

$$I(u) := \frac{1}{2}\int_{\mathbb{R}^4}\left(|\nabla u| + fu^2\right)dx - \frac{1}{4}\int_{\mathbb{R}^4} u_+^4 dx,$$

then the function $W_{k,r_k,\mu_k} + \phi_{k,r_k,\mu_k}$ is a positive solution in $C^{2,\alpha}(\mathbb{R}^4) \cap H_k$ of the equation

$$\Delta u + fu = u^3 \qquad in\ \mathbb{R}^4. \tag{2.3}$$

Then we will prove the following result in Section 4:

Proposition 2.2. *Let $f \in C^{0,\alpha}(\mathbb{R}^4) \cap L^2(\mathbb{R}^4)$ be a radially symmetric function about the point 0 and such that the operator $\Delta + f$ is coercive in $D^{1,2}(\mathbb{R}^4)$. Then there exist constants $c_0, c_1, c_2 > 0$ such that for any $a, b, c, d > 0$ such that $a < b$ and $c < d$,*

$$I(W_{k,r,\mu} + \phi_{k,r,\mu}) = c_0 k + c_1 f(r)\frac{k \ln \mu}{\mu^2} - \frac{c_2 k^3}{r^2 \mu^2} + \mathrm{o}\left(\frac{k^3}{\mu^2}\right) \tag{2.4}$$

as $k \to \infty$ uniformly in $r \in [a, b]$ and $\mu \in \left[e^{ck^2}, e^{dk^2}\right]$ where $\phi_{k,r,\mu}$ is as in Proposition 2.1.

Now, we prove Theorem 1.2 by using Propositions 2.1 and 2.2.

Proof of Theorem 1.2. Since $f(r_0) > 0$ and r_0 is a strict local maximum point of the function $r \mapsto r^2 f(r)$, we obtain that there exists $\delta_0 > 0$ such that

$$0 < r^2 f(r) < r_0^2 f(r_0) \qquad \forall r \in [r_0 - \delta_0, r_0 + \delta_0]. \tag{2.5}$$

For any $k \geq 1$ and $s > 0$, we define $\mu_k(s) := e^{sk^2}$. By applying Proposition 2.2, we obtain

$$\mathcal{I}_k(r, \mu_k(s)) = c_0 k + k^3 e^{-2sk^2}\left(c_1 f(r)\, s - \frac{c_2}{r^2} + \mathrm{o}(1)\right) \tag{2.6}$$

as $k \to \infty$ uniformly in (r, s) in compact subsets of $(0, \infty)^2$. Remark that the function

$$s \longmapsto e^{-2sk^2}\left(c_1 f(r)\, s - \frac{c_2}{r^2}\right)$$

attains its maximal value at the point

$$s_k(r) := \frac{c_2}{c_1 f(r) r^2} + \frac{1}{2k^2}$$

for all $k \geq 1$ and $r \in [r_0 - \delta_0, r_0 + \delta_0]$. We define

$$\mathcal{J}_k(r,t) := \mathcal{I}_k(r, \mu_k(s_k(r) + t)).$$

By using (2.5), we obtain that there exists $t_0 > 0$ such that

$$t_0 < \min\left(\frac{s_k(r_0)}{2}, \frac{2}{3}(s_k(r_0+\delta_0) - s_k(r_0)), \frac{2}{3}(s_k(r_0-\delta_0) - s_k(r_0))\right) \tag{2.7}$$

for all $k \geq 1$. Since $t_0 < s_k(r_0)/2$, it follows from (2.6) that

$$\mathcal{J}_k(r,t) = c_0 k + k^3 e^{-2(s_k(r)+t)k^2}(c_1 f(r) t + \mathrm{o}(1)) \tag{2.8}$$

as $k \to \infty$ uniformly in $(r,t) \in [r_0 - \delta_0, r_0 + \delta_0] \times [-t_0, t_0]$. Since $s_k(r) > s_k(r_0)$ and $f(r) > 0$, it follows from (2.8) that

$$\mathcal{J}_k(r, t_0) < \mathcal{J}_k(r_0, t_0/2) \tag{2.9}$$

and

$$\mathcal{J}_k(r, -t_0) < \mathcal{J}_k(r_0, t_0/2) \tag{2.10}$$

as $k \to \infty$ uniformly in $r \in [r_0 - \delta_0, r_0 + \delta_0]$. Moreover, by using (2.7) and (2.8), we obtain

$$\mathcal{J}_k(r_0 \pm \delta_0, t) < \mathcal{J}_k(r_0, t_0/2) \tag{2.11}$$

as $k \to \infty$ uniformly in $t \in [-t_0, t_0]$. It follows from (2.9)–(2.11) that the function $\mathcal{J}_k$ has a local maximum point $(r_k, t_k) \in [r_0 - \delta_0, r_0 + \delta_0] \times [-t_0, t_0]$ for large k. We then obtain $\nabla \mathcal{I}_k(r_k, \mu_k(s_k(r_k) + t_k)) = 0$ and so by applying the second part of Proposition 2.1, we obtain that the function $W_{k,r_k,\mu_k(s_k(r_k)+t_k)} + \phi_{k,r_k,\mu_k(s_k(r_k)+t_k)}$ is a positive solution of the equation (2.3). Moreover, by using (2.2) together with the definition of $W_{k,r_k,\mu_k(s_k(r_k)+t_k)}$, we easily obtain

$$\left\|\nabla\left(W_{k,r_k,\mu_k(s_k(r_k)+t_k)} + \phi_{k,r_k,\mu_k(s_k(r_k)+t_k)}\right)\right\|_{L^2} \to \infty$$

as $k \to \infty$. This ends the proof of Theorem 1.2. □

Finally, we prove Theorem 1.1 by using Theorem 1.2.

Proof of Theorem 1.1. By using a stereographic projection, we can see that the equation (1.1) on $(M,g) = (\mathbb{S}^4, g_0)$ is equivalent to the problem

$$\begin{cases} \Delta u + \dfrac{4(f-2)}{\left(1+|y|^2\right)^2} u = u^3 & \text{in } \mathbb{R}^4 \\ u \in D^{1,2}\left(\mathbb{R}^4\right). \end{cases} \tag{2.12}$$

It is easy to check that if $f > 2$ is a constant, then the potential function in (2.12) satisfies the assumptions of Theorem 1.2. With this remark, Theorem 1.1 becomes a direct corollary of Theorem 1.2. □

3. Proof of Proposition 2.1

We prove Proposition 2.1 in this section. Throughout this section, we assume that $f \in C^{0,\alpha}(\mathbb{R}^4) \cap L^2(\mathbb{R}^4)$ is radially symmetric about the point 0 and the operator $\Delta + f$ is coercive in $D^{1,2}(\mathbb{R}^4)$.

We rewrite (2.1) as

$$L_{k,r,\mu}(\phi) = Q_{k,r,\mu}\left(N_{k,r,\mu}(\phi) + R_{k,r,\mu}\right)$$

where

$$\begin{aligned} &L_{k,r,\mu}(\phi) := Q_{k,r,\mu}\left(\phi - (\Delta + f)^{-1}\left(3W_{k,r,\mu}^2 \phi\right)\right),\\ &N_{k,r,\mu}(\phi) := (\Delta + f)^{-1}\left((W_{k,r,\mu} + \phi)_+^3 - W_{k,r,\mu}^3 - 3W_{k,r,\mu}^2 \phi\right),\\ &R_{k,r,\mu} := (\Delta + f)^{-1}\left(W_{k,r,\mu}^3\right) - W_{k,r,\mu}. \end{aligned}$$

First, we obtain the following result:

Lemma 3.1. *For any $a, b, c, d > 0$ such that $a < b$ and $c < d$, there exist constants $k_1 > 0$ and $C_1 > 0$ such that for any $k \geq k_1$, $r \in [a, b]$, and $\mu \in \left[e^{ck^2}, e^{dk^2}\right]$, $L_{k,r,\mu}$ is an isomorphism from $P_{k,r,\mu}$ to itself and*

$$\left\|L_{k,r,\mu}(\phi)\right\|_{H_k} \geq C_2 \left\|\phi\right\|_{H_k} \qquad \forall \phi \in P_{k,r,\mu}.$$

Proof. The proof of this result follows the same lines as in the paper of Chen, Wei, and Yan [5]. □

We then estimate the error term $R_{k,r,\mu}$. We obtain the following result:

Lemma 3.2. *For any $a, b, c, d > 0$ such that $a < b$ and $c < d$, there exist constants $k_2 > 0$ and $C_2 > 0$ such that*

$$\left\|R_{k,r,\mu}\right\|_{H_k} \leq C_2 k / \mu. \tag{3.1}$$

for all $k \geq k_2$, $r \in [a, b]$, and $\mu \in \left[e^{ck^2}, e^{dk^2}\right]$.

Proof. For any $\phi \in H_k$, by integrating by parts, we obtain

$$\begin{aligned} \left\langle R_{k,r,\mu}, \phi \right\rangle_{H_k} &= \int_{\mathbb{R}^4} \left(W_{k,r,\mu}^3 - \Delta W_{k,r,\mu} - f W_{k,r,\mu}\right) \phi dx\\ &= \int_{\mathbb{R}^4} \left(W_{k,r,\mu}^3 - \sum_{i=1}^k U_{i,k,r,\mu}^3 - f W_{k,r,\mu}\right) \phi dx\\ &= \mathrm{O}\left(\sum_{i=1}^k \int_{\mathbb{R}^4} \left(\sum_{j \neq i} \sum_{l=1}^k U_{j,k,r,\mu} U_{l,k,r,\mu} + |f|\right) U_{i,k,r,\mu} \left|\phi\right| dx\right). \end{aligned} \tag{3.2}$$

By using Hölder's inequality and Sobolev's inequality, it follows from (3.2) that

$$\left\|R_{k,r,\mu}\right\|_{H_k} = \sum_{i=1}^k \mathrm{O}\left(k \sum_{j \neq i} \left\|U_{i,k,r,\mu}^2 U_{j,k,r,\mu}\right\|_{L^{4/3}} + \left\|f U_{i,k,r,\mu}\right\|_{L^{4/3}}\right). \tag{3.3}$$

We start with estimating the first term in (3.3). For any $\alpha \in \{1, \dots, k\}$, we define

$$\Omega_{\alpha,k,r} := \left\{ (y_1, y_2, y_3, y_4) \in \mathbb{R}^4 : \left\langle (y_1, y_2, 0, 0), x_{\alpha,k,r} \right\rangle \geq \cos(\pi/k) \right\}.$$

We then write

$$\int_{\mathbb{R}^4} U_{i,k,r,\mu}^{8/3} U_{j,k,r,\mu}^{4/3} dx = \sum_{\alpha=1}^{k} \int_{\Omega_{\alpha,k,r}} U_{i,k,r,\mu}^{8/3} U_{j,k,r,\mu}^{4/3} dx. \tag{3.4}$$

We observe that if $\alpha \neq j$, then

$$|x - x_{j,k,r}| \geq |x - x_{\alpha,k,r}| \quad \text{and} \quad |x - x_{j,k,r}| \geq \frac{1}{2} |x_{\alpha,k,r} - x_{j,k,r}| \tag{3.5}$$

for all $x \in \Omega_{\alpha,k,r}$. For any $i, j, \alpha \in \{1, \dots, k\}$ such that $i \neq j$, by using (3.4), we obtain

$$U_{i,k,r,\mu}(x)^{8/3} U_{j,k,r,\mu}(x)^{4/3} \leq \begin{cases} \dfrac{2^{8/3} \left(2\sqrt{2}\right)^4 \mu^{4/3}}{\left(1 + \mu^2 |x - x_{i,k,r}|^2\right)^{8/3} |x_{i,k,r} - x_{j,k,r}|^{8/3}} & \text{if } \alpha = i \\ \dfrac{2^{8/3} \left(2\sqrt{2}\right)^4 \mu^{4/3}}{\left(1 + \mu^2 |x - x_{\alpha,k,r}|^2\right)^{8/3} |x_{i,k,r} - x_{\alpha,k,r}|^{8/3}} & \text{if } \alpha \neq i \end{cases} \tag{3.6}$$

for all $x \in \Omega_{\alpha,k,r} \setminus \{x_{\alpha,k,r}\}$. By using (3.4) and (3.6) and straightforward estimates, we obtain

$$\begin{aligned} \int_{\mathbb{R}^4} U_{i,k,r,\mu}^{8/3} U_{j,k,r,\mu}^{4/3} dx &= \mathrm{O}\left(\frac{\mu^{-8/3}}{|x_{i,k,r} - x_{j,k,r}|^{8/3}} + \sum_{\alpha \neq i} \frac{\mu^{-8/3}}{|x_{i,k,r} - x_{\alpha,k,r}|^{8/3}} \right) \\ &= \mathrm{O}\left(\frac{\mu^{-8/3}}{|x_{i,k,r} - x_{j,k,r}|^{8/3}} + \frac{k^{8/3}}{\mu^{8/3}} \right). \end{aligned} \tag{3.7}$$

It follows from (3.7) that

$$\sum_{j \neq i} \left\| U_{i,k,r,\mu}^2 U_{j,k,r,\mu} \right\|_{L^{4/3}} = \mathrm{O}\left(k (k/\mu)^2 \right) \tag{3.8}$$

Now, we estimate the second term in (3.4). Since $f \in L^\infty(\mathbb{R}^4) \cap L^2(\mathbb{R}^4)$, by applying Hölder's inequality and straightforward estimates, we obtain

$$\begin{aligned} \int_{\mathbb{R}^4 \setminus B(x_{i,k,r},1)} |f U_{i,k,r,\mu}|^{4/3} dx &= \mathrm{O}\left(\left(\int_{\mathbb{R}^4 \setminus B(x_{i,k,r},1)} |U_{i,k,r,\mu}|^4 dx \right)^{1/3} \right) \\ &= \mathrm{O}\left(\mu^{-4/3} \right) \end{aligned} \tag{3.9}$$

and

$$\int_{B(x_{i,k,r},1)} |fU_{i,k,r,\mu}|^{4/3}\, dx = \mathrm{O}\left(\int_{B(x_{i,k,r},1)} |U_{i,k,r,\mu}|^{4/3}\, dx\right) = \mathrm{O}\left(\mu^{-4/3}\right). \tag{3.10}$$

It follows from (3.9) and (3.10) that

$$\|fU_{i,k,r,\mu}\|_{L^{4/3}} = \mathrm{O}\left(1/\mu\right). \tag{3.11}$$

Finally, (3.1) follows from (3.8) and (3.11). □

We can now prove Proposition 2.1 by using Lemmas 3.1 and 3.2.

Proof of Proposition 2.1. We define

$$T_{k,r,\mu}(\phi) := L^{-1}_{k,r,\mu}\left(Q_{k,r,\mu}\left(N_{k,r,\mu}(\phi) + R_{k,r,\mu}\right)\right) \qquad \forall \phi \in P_{k,r,\mu}$$

and

$$V_{k,r,\mu} := \left\{\phi \in P_{k,r,\mu} :\ \|\phi\|_{H_k} \le C_0 k/\mu\right\}$$

where $C_0 > 0$ is a constant to be fixed later on. It follows from Lemmas 3.1 and 3.2 that

$$\|T_{k,r,\mu}(\phi)\|_{H_k} \le C_1\big(\|N_{k,r,\mu}(\phi)\|_{H_k} + C_2 k/\mu\big) \tag{3.12}$$

for all $k \ge k_2$, $r \in [a,b]$, and $\mu \in \left[e^{ck^2}, e^{dk^2}\right]$. By integrating by parts and using Hölder's inequality, Sobolev's inequality, and straightforward estimates, we obtain

$$\langle N_{k,r,\mu}(\phi), \psi\rangle_{H_k} = \int_{\mathbb{R}^4} \left((W_{k,r,\mu} + \phi)^3_+ - W^3_{k,r,\mu} - 3W^2_{k,r,\mu}\phi\right)\psi dx = \mathrm{O}\left(\left(\|W_{k,r,\mu}\|_{L^4} \|\phi\|^2_{H_k} + \|\phi\|^3_{H_k}\right)\|\psi\|_{H_k}\right) \tag{3.13}$$

for all $\psi \in H_k$. Proceeding as in (3.4)–(3.8), we obtain

$$\int_{\mathbb{R}^4} W^4_{k,r,\mu} dx = \mathrm{O}\left(\sum_{i=1}^{k} \int_{\mathbb{R}^4} \Big(U^4_{i,k,r,\mu} + \sum_{j\ne i} U^2_{i,k,r,\mu} U^2_{j,k,r,\mu}\Big) dx\right) = \mathrm{O}\left(k + k\left(k/\mu\right)^4 \ln \mu\right). \tag{3.14}$$

It follows from (3.13) and (3.14) that

$$\|N_{k,r,\mu}(\phi)\|_{H_k} = \mathrm{O}\left(k^{1/4}\|\phi\|^2_{H_k} + \|\phi\|^3_{H_k}\right). \tag{3.15}$$

Letting C_0 be large enough so that $C_0 > C_1 C_2$, it follows from (3.12) and (3.15) that there exists a constant $k_3 > 0$ such that

$$T_{k,r,\mu}(V_{k,r,\mu}) \subset V_{k,r,\mu} \tag{3.16}$$

for all $k \ge k_3$, $r \in [a,b]$, and $\mu \in \left[e^{ck^2}, e^{dk^2}\right]$. Now, we prove that if k is large enough, then $T_{k,r,\mu}$ is a contraction map from $V_{k,r,\mu}$ to itself, i.e.

$$\|T_{k,r,\mu}(\phi_1) - T_{k,r,\mu}(\phi_2)\|_{H_k} \le C\|\phi_1 - \phi_2\|_{H_k} \quad \forall \phi_1, \phi_2 \in V_{k,r,\mu}. \tag{3.17}$$

for some constant $C \in (0,1)$. It follows from Lemma 3.1 that

$$\left\| T_{k,r,\mu}\left(\phi_1\right) - T_{k,r,\mu}\left(\phi_2\right) \right\|_{H_k} \le C_1 \left\| N_{k,r,\mu}\left(\phi_1\right) - N_{k,r,\mu}\left(\phi_2\right) \right\|_{H_k} \quad (3.18)$$

By integrating by parts and using Hölder's inequality, Sobolev's inequality, and (3.14), we obtain

$$\begin{aligned}
&\left\langle N_{k,r,\mu}\left(\phi_1\right) - N_{k,r,\mu}\left(\phi_2\right), \psi \right\rangle_{H_k} \\
&\quad = \int_{\mathbb{R}^4} \left(\left(W_{k,r,\mu} + \phi_1\right)_+^3 - \left(W_{k,r,\mu} + \phi_2\right)_+^3 - 3W_{k,r,\mu}^2 \left(\phi_1 - \phi_2\right) \right) \psi dx \\
&\quad = \mathrm{O}\Big(\left(\left\| W_{k,r,\mu} \right\|_{L^4} + \left\| \phi_1 \right\|_{H_k} + \left\| \phi_2 \right\|_{H_k} \right) \\
&\qquad \times \left(\left\| \phi_1 \right\|_{H_k} + \left\| \phi_2 \right\|_{H_k} \right) \left\| \phi_1 - \phi_2 \right\|_{H_k} \left\| \psi \right\|_{H_k} \Big) \\
&\quad = \mathrm{O}\Big(\left(k^{1/4} + \left\| \phi_1 \right\|_{H_k} + \left\| \phi_2 \right\|_{H_k} \right) \\
&\qquad \times \left(\left\| \phi_1 \right\|_{H_k} + \left\| \phi_2 \right\|_{H_k} \right) \left\| \phi_1 - \phi_2 \right\|_{H_k} \left\| \psi \right\|_{H_k} \Big) \qquad (3.19)
\end{aligned}$$

It follows from (3.19) that

$$\left\| N_{k,r,\mu}\left(\phi_1\right) - N_{k,r,\mu}\left(\phi_2\right) \right\|_{H_k} = \mathrm{o}\left(\left\| \phi_1 - \phi_2 \right\|_{H_k} \right) \qquad (3.20)$$

as $k \to \infty$ uniformly in $r \in [a,b]$, $\mu \in \left[e^{ck^2}, e^{dk^2}\right]$, and $\phi_1, \phi_2 \in V_{k,r,\mu}$. We then obtain (3.17) by putting together (3.18) and (3.20). It follows from (3.16) and (3.17) that there exists a constant $k_4 \ge k_3$ such that for any $k \ge k_4$, $r \in [a,b]$, and $\mu \in \left[e^{ck^2}, e^{dk^2}\right]$, there exists a unique solution $\phi_{k,r,\mu} \in V_{k,r,\mu}$ of (2.1). The continuous differentiability of $(r,\mu) \mapsto \phi_{k,r,\mu}$ is standard.

Now, we prove the last part of Proposition 2.1. We let $(r_k, \mu_k) \in [a,b] \times \left[e^{ck^2}, e^{dk^2}\right]$ be a critical point of $\mathcal{I}_k$. Since $\phi_{k,r,\mu}$ is a solution of (2.1), we obtain that there exist $c_{1,k}$ and $c_{2,k}$ such that

$$DI\left(W_{k,r_k,\mu_k} + \phi_{k,r_k,\mu_k}\right) = \sum_{j=1}^{2} c_{j,k} \sum_{i=1}^{k} \left\langle Z_{i,j,k,r_k,\mu_k}, \cdot \right\rangle_{H_k}. \qquad (3.21)$$

It follows from (3.21) that

$$\begin{aligned}
0 &= \frac{\partial \mathcal{I}_k}{\partial r}\left(r_k, \mu_k\right) \\
&= \sum_{j=1}^{2} c_{j,k} \sum_{i=1}^{k} \left\langle Z_{i,j,k,r_k,\mu_k}, \frac{d}{dr}\left[W_{k,r,\mu_k} + \phi_{k,r,\mu_k}\right]_{r=r_k} \right\rangle_{H_k} \\
&= \sum_{j=1}^{2} c_{j,k} \sum_{i=1}^{k} \Bigg(\mu_k \sum_{\alpha=1}^{k} \left\langle Z_{i,j,k,r_k,\mu_k}, Z_{\alpha,1,k,r_k,\mu_k} \right\rangle_{H_k} \\
&\qquad + \left\langle Z_{i,j,k,r_k,\mu_k}, \frac{d}{dr}\left[\phi_{k,r,\mu_k}\right]_{r=r_k} \right\rangle_{H_k} \Bigg) \qquad (3.22)
\end{aligned}$$

and

$$\begin{aligned} 0 &= \frac{\partial \mathcal{I}_k}{\partial \mu}(r_k, \mu_k) \\ &= \sum_{j=1}^{2} c_{j,k} \sum_{i=1}^{k} \left\langle Z_{i,j,k,r_k,\mu_k}, \frac{d}{d\mu}\left[W_{k,r_k,\mu} + \phi_{k,r_k,\mu}\right]_{\mu=\mu_k} \right\rangle_{H_k} \\ &= \sum_{j=1}^{2} c_{j,k} \sum_{i=1}^{k} \left(\frac{1}{\mu_k} \sum_{\alpha=1}^{k} \left\langle Z_{i,j,k,r_k,\mu_k}, Z_{\alpha,2,k,r_k,\mu_k} \right\rangle_{H_k} \right. \\ &\quad \left. + \left\langle Z_{i,j,k,r_k,\mu_k}, \frac{d}{d\mu}\left[\phi_{k,r_k,\mu}\right]_{\mu=\mu_k} \right\rangle_{H_k} \right). \end{aligned} \tag{3.23}$$

For any $i,\alpha \in \{1,\dots,k\}$ and $j,\beta \in \{1,2\}$, direct calculations yield

$$\left\langle Z_{i,j,k,r_k,\mu_k}, Z_{\alpha,\beta,k,r_k,\mu_k} \right\rangle_{H_k} = \Lambda_j \delta_{i\alpha}\delta_{j\beta} + \mathrm{o}(1) \tag{3.24}$$

as $k \to \infty$ where $\Lambda_j > 0$ is a constant and $\delta_{i\alpha} := 1$ if $\alpha = i$ and $\delta_{i\alpha} := 0$ if $\alpha \neq i$. Moreover, since $\phi_{k,r,\mu} \in P_{k,r,\mu}$, we obtain

$$\begin{aligned} &\sum_{i=1}^{k} \left\langle Z_{i,j,k,r_k,\mu_k}, \frac{d}{dr}\left[\phi_{k,r,\mu_k}\right]_{r=r_k} \right\rangle_{H_k} \\ &\qquad = -\sum_{i=1}^{k} \left\langle \frac{d}{dr}\left[Z_{i,j,k,r,\mu_k}\right]_{r=r_k}, \phi_{k,r_k,\mu_k} \right\rangle_{H_k} \end{aligned}$$

and therefore by using Cauchy–Schwartz inequality and (2.2), we obtain

$$\begin{aligned} &\left| \sum_{i=1}^{k} \left\langle Z_{i,j,k,r_k,\mu_k}, \frac{d}{dr}\left[\phi_{k,r,\mu_k}\right]_{r=r_k} \right\rangle_{H_k} \right| \\ &\qquad \leq \left\| \sum_{i=1}^{k} \frac{d}{dr}\left[Z_{i,j,k,r,\mu_k}\right]_{r=r_k} \right\|_{H_k} \left\| \phi_{k,r_k,\mu_k} \right\|_{H_k} = \mathrm{o}(k\mu_k). \end{aligned} \tag{3.25}$$

Similarly, we obtain

$$\begin{aligned} &\left| \sum_{i=1}^{k} \left\langle Z_{i,j,k,r_k,\mu_k}, \frac{d}{d\mu}\left[\phi_{k,r_k,\mu}\right]_{\mu=\mu_k} \right\rangle_{H_k} \right| \\ &\qquad \leq \left\| \sum_{i=1}^{k} \frac{d}{d\mu}\left[Z_{i,j,k,r_k,\mu}\right]_{\mu=\mu_k} \right\|_{H_k} \left\| \phi_{k,r_k,\mu_k} \right\|_{H_k} = \mathrm{o}\left(k\mu_k^{-1}\right). \end{aligned} \tag{3.26}$$

It follows from (3.22)–(3.26) that if k is large enough, then $c_{1,k} = c_{2,k} = 0$, i.e. the function $W_{k,r_k,\mu_k} + \phi_{k,r_k,\mu_k}$ is a weak solution of the equation

$$\Delta u + fu = u_+^3 \qquad \text{in } \mathbb{R}^4.$$

By using the coercivity of the operator $\Delta + f$ in $D^{1,2}(\mathbb{R}^4)$, we obtain that $u \geq 0$ a.e. in $\mathbb{R}^4$. It then follows from standard elliptic regularity theory and the strong maximum principle that $W_{k,r_k,\mu_k}+\phi_{k,r_k,\mu_k}$ is a strong positive solution in $C^{2,\alpha}\left(\mathbb{R}^4\right)$ of (2.3). $\square$

4. Proof of Proposition 2.2

We prove Proposition 2.2 in this section. Throughout this section, we assume that $f\in C^{0,\alpha}\left(\mathbb{R}^4\right)\cap L^2\left(\mathbb{R}^4\right)$ is radially symmetric about the point 0 and the operator $\Delta+f$ is coercive in $D^{1,2}\left(\mathbb{R}^4\right)$. First, we obtain the following result:

Lemma 4.1. *There exist constants $c_0,c_1,c_2>0$ such that for any $a,b,c,d>0$ such that $a<b$ and $c<d$,*

$$I\left(W_{k,r,\mu}\right)=c_0k+c_1f\left(r\right)\frac{k\ln\mu}{\mu^2}-\frac{c_2k^3}{r^2\mu^2}+\mathrm{o}\left(\frac{k^3}{\mu^2}\right) \tag{4.1}$$

as $k\to\infty$ uniformly in $r\in[a,b]$ and $\mu\in\left[e^{ck^2},e^{dk^2}\right]$.

Proof. By integrating by parts, we obtain

$$\begin{aligned}
I\left(W_{k,r,\mu}\right)&=\frac{1}{2}\int_{\mathbb{R}^4}\left(\Delta W_{k,r,\mu}+fW_{k,r,\mu}\right)W_{k,r,\mu}dx-\frac{1}{4}\int_{\mathbb{R}^4}W_{k,r,\mu}^4dx\\
&=\frac{1}{2}\int_{\mathbb{R}^4}\left(\sum_{i,j=1}^{k}U_{i,k,r,\mu}^3U_{j,k,r,\mu}+fW_{k,r,\mu}^2-\frac{1}{2}W_{k,r,\mu}^4\right)dx\\
&=\frac{1}{2}\sum_{i=1}^{k}\int_{\mathbb{R}^4}\Bigg(fU_{i,k,r,\mu}^2+\frac{1}{2}U_{i,k,r,\mu}^4-\sum_{j\neq i}U_{i,k,r,\mu}^3U_{j,k,r,\mu}\\
&\quad+f\sum_{j\neq i}U_{i,k,r,\mu}U_{j,k,r,\mu}\Bigg)dx\\
&\quad+\mathrm{O}\left(\sum_{i,l=1}^{k}\sum_{j\neq i}\sum_{m\neq l}\int_{\mathbb{R}^4}U_{i,k,r,\mu}U_{j,k,r,\mu}U_{l,k,r,\mu}U_{m,k,r,\mu}dx\right)\\
&=\frac{1}{2}\sum_{i=1}^{k}\int_{\mathbb{R}^4}\Bigg(fU_{i,k,r,\mu}^2+\frac{1}{2}U_{i,k,r,\mu}^4-\sum_{j\neq i}U_{i,k,r,\mu}^3U_{j,k,r,\mu}\\
&\quad+f\sum_{j\neq i}U_{i,k,r,\mu}U_{j,k,r,\mu}\Bigg)dx+\mathrm{O}\left(k^2\sum_{i=1}^{k}\sum_{j\neq i}\int_{\mathbb{R}^4}U_{i,k,r,\mu}^2U_{j,k,r,\mu}^2dx\right)
\end{aligned} \tag{4.2}$$

Direct calculations yield

$$\int_{\mathbb{R}^4}U_{i,k,r,\mu}^4dx=\left(2\sqrt{2}\right)^4\int_{\mathbb{R}^4}\frac{dx}{\left(1+|x|^2\right)^4}. \tag{4.3}$$

and

$$\int_{\mathbb{R}^4}fU_{i,k,r,\mu}^2dx=16\pi^2f\left(r\right)\frac{\ln\mu}{\mu^2}+\mathrm{o}\left(\frac{\ln\mu}{\mu^2}\right) \tag{4.4}$$

as $k\to\infty$ uniformly in $r\in[a,b]$ and $\mu\in\left[e^{ck^2},e^{dk^2}\right]$. By splitting the integral as in (3.4) and estimating each term, we obtain

$$\begin{aligned}\sum_{j\neq i}\int_{\mathbb{R}^4}U_{i,k,r,\mu}^3U_{j,k,r,\mu}dx&=\sum_{j\neq i}\int_{\Omega_{i,k,r}}\frac{64\mu^2}{\left(1+\mu^2\left|x-x_{i,k,r}\right|^2\right)^3}\\&\quad\times\left(\frac{1+\mathrm{O}\left(\mathbf{1}_{\Omega_{i,k,r}\backslash B\left(x_i,\left|x_{1,k,r}-x_{2,k,r}\right|/2\right)}\right)}{\left|x_{i,k,r}-x_{j,k,r}\right|^2}\right.\\&\qquad\left.+\mathrm{O}\left(\frac{\mu^{-2}+\left|x-x_{i,k,r}\right|\left|x_{i,k,r}-x_{j,k,r}\right|}{\left|x_{i,k,r}-x_{j,k,r}\right|^4}\mathbf{1}_{B\left(x_i,\left|x_{1,k,r}-x_{2,k,r}\right|/2\right)}\right)\right)dx\\&\qquad+\mathrm{O}\left(\sum_{\alpha\neq i}\frac{k\mu}{\left|x_{i,k,r}-x_{\alpha,k,r}\right|^3}\int_{\Omega_{\alpha,k,r}}\frac{dx}{\left(1+\mu^2\left|x-x_{\alpha,k,r}\right|^2\right)^{5/2}}\right)\\&=\sum_{j\neq i}\left(\frac{64\mu^{-2}}{\left|x_{i,k,r}-x_{j,k,r}\right|^2}\int_{\mathbb{R}^4}\frac{dx}{\left(1+|x|^2\right)^3}+\mathrm{O}\left(\frac{k\mu^{-3}}{\left|x_{i,k,r}-x_{j,k,r}\right|^3}\right)\right)\\&=\frac{32k^2}{\pi^2r^2\mu^2}\int_{\mathbb{R}^4}\frac{dx}{\left(1+|x|^2\right)^3}\sum_{j=1}^\infty\frac{1}{j^2}+\mathrm{o}\left(\frac{k^2}{\mu^2}\right)\end{aligned}\tag{4.5}$$

as $k\to\infty$ uniformly in $r\in[a,b]$ and $\mu\in\left[e^{ck^2},e^{dk^2}\right]$. Moreover, straightforward estimates give

$$\begin{aligned}&\sum_{j\neq i}\int_{\mathbb{R}^4\backslash(B(x_{i,k,r,\mu},|x_{i,k,r,\mu}-x_{j,k,r,\mu}|/2)\cup B(x_{j,k,r,\mu},|x_{i,k,r,\mu}-x_{j,k,r,\mu}|/2))}U_{i,k,r,\mu}^2U_{j,k,r,\mu}^2dx\\&=\mathrm{O}\left(\mu^{-4}\sum_{j\neq i}\int_{\mathbb{R}^4\backslash(B(x_{i,k,r,\mu},|x_{i,k,r,\mu}-x_{j,k,r,\mu}|/2)\cup B(x_{j,k,r,\mu},|x_{i,k,r,\mu}-x_{j,k,r,\mu}|/2))}\left|x-x_{i,k,r,\mu}\right|^{-4}\left|x-x_{j,k,r,\mu}\right|^{-4}dx\right)\\&=\mathrm{O}\left(\sum_{j\neq i}\frac{\mu^{-4}}{\left|x_{i,k,r,\mu}-x_{j,k,r,\mu}\right|^4}\right)=\mathrm{O}\left((k/\mu)^4\right),\end{aligned}\tag{4.6}$$

$$\begin{aligned}&\sum_{j\neq i}\int_{B(x_{i,k,r,\mu},|x_{i,k,r,\mu}-x_{j,k,r,\mu}|/2)\cup B(x_{j,k,r,\mu},|x_{i,k,r,\mu}-x_{j,k,r,\mu}|/2)}U_{i,k,r,\mu}^2U_{j,k,r,\mu}^2dx\\&\qquad=\mathrm{O}\left(\sum_{j\neq i}\frac{\mu^{-4}}{|x_{i,k,r,\mu}-x_{j,k,r,\mu}|^4}\int_{B(0,\mu|x_{i,k,r,\mu}-x_{j,k,r,\mu}|/2)}\frac{dx}{\left(1+|x|^2\right)^2}\right)\\&\qquad=\mathrm{O}\left(\sum_{j\neq i}\frac{\mu^{-4}\ln\mu}{|x_{i,k,r,\mu}-x_{j,k,r,\mu}|^4}\right)=\mathrm{O}\left((k/\mu)^4\ln\mu\right),\end{aligned}\tag{4.7}$$

$$\begin{aligned}&\sum_{j\neq i}\int_{\mathbb{R}^4\backslash(B(x_{i,k,r,\mu},1)\cup B(x_{j,k,r,\mu},1))}fU_{i,k,r,\mu}U_{j,k,r,\mu}dx\\&\qquad=\mathrm{O}\left(\sum_{j\neq i}\left(\int_{\mathbb{R}^4\backslash B(x_{j,k,r,\mu},1)}U_{j,k,r,\mu}^4dx\right)^{1/2}\right)\end{aligned}$$

$$= \mathrm{O}\left(k\left(\int_{\mathbb{R}^4\setminus B(0,\mu)}\frac{dx}{\left(1+|x|^2\right)^4}\right)^{1/2}\right) = \mathrm{O}\left(\frac{k}{\mu^2}\right), \tag{4.8}$$

and

$$\begin{aligned}
&\sum_{j\neq i}\int_{B(x_{i,k,r,\mu},1)\cup B(x_{j,k,r,\mu},1)} fU_{i,k,r,\mu}U_{j,k,r,\mu}dx\\
&\quad = \mathrm{O}\left(\sum_{j\neq i}\int_{B(x_{i,k,r,\mu},1)} U_{i,k,r,\mu}U_{j,k,r,\mu}dx\right)\\
&\quad = \mathrm{O}\left(\sum_{j\neq i}\int_{B(x_{i,k,r,\mu},1)} \frac{\mu^{-2}dx}{|x-x_{i,k,r,\mu}|^2\,|x-x_{j,k,r,\mu}|^2}\right)\\
&\quad = \mathrm{O}\left(\mu^{-2}\sum_{j\neq i}\ln\frac{1}{|x_{i,k,r,\mu}-x_{j,k,r,\mu}|}\right) = \mathrm{O}\left(\frac{k\ln k}{\mu^2}\right)
\end{aligned} \tag{4.9}$$

as $k\to\infty$ uniformly in $r\in[a,b]$ and $\mu\in\left[e^{ck^2},e^{dk^2}\right]$. Finally, (4.1) follows from (4.2)–(4.9). □

We can now prove Proposition 2.2 by using Lemma 4.1.

Proof of Proposition 2.2. By integrating by parts, we obtain

$$\begin{aligned}
I\left(W_{k,r,\mu}+\phi_{k,r,\mu}\right) &= I\left(W_{k,r,\mu}\right) - \left\langle R_{k,r,\mu},\phi_{k,r,\mu}\right\rangle_{H_k} + \frac{1}{2}\left\|\phi_{k,r,\mu}\right\|_{H_k}^2\\
&\quad -\frac{1}{4}\int_{\mathbb{R}^4}\left(\left(W_{k,r,\mu}+\phi_{k,r,\mu}\right)_+^4 - W_{k,r,\mu}^4 - 4W_{k,r,\mu}^3\phi_{k,r,\mu}\right)dx.
\end{aligned} \tag{4.10}$$

By using Cauchy–Schwartz inequality, Lemma 3.1, and Proposition 2.1, we obtain

$$-\left\langle R_{k,r,\mu},\phi_{k,r,\mu}\right\rangle_{H_k} + \frac{1}{2}\left\|\phi_{k,r,\mu}\right\|_{H_k}^2 = \mathrm{O}\left((k/\mu)^2\right). \tag{4.11}$$

Moreover, by using Hölder's inequality, Sobolev's inequality, (3.14), and Lemma 3.1, we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^4}\left(\left(W_{k,r,\mu}+\phi_{k,r,\mu}\right)_+^4 - W_{k,r,\mu}^4 - 4W_{k,r,\mu}^3\phi_{k,r,\mu}\right)dx\\
&\quad = \mathrm{O}\left(\int_{\mathbb{R}^4}\left(W_{k,r,\mu}^2+\phi_{k,r,\mu}^2\right)\phi_{k,r,\mu}^2dx\right)\\
&\quad = \mathrm{O}\left(\left\|W_{k,r,\mu}\right\|_{L^4}^2\left\|\phi_{k,r,\mu}\right\|_{H_k}^2 + \left\|\phi_{k,r,\mu}\right\|_{H_k}^4\right)\\
&\quad = \mathrm{O}\left(\sqrt{k}\,(k/\mu)^2 + (k/\mu)^4\right).
\end{aligned} \tag{4.12}$$

Finally, (2.4) follows from (4.10)–(4.12). □

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