ESI Summer School on Quantum Chaos

L^2 Restriction bounds for eigenfunctions

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Background

- (M^n, g) a compact, closed Riemannian manifold with Laplacian $\Delta_g : C^\infty(M) \to C^\infty(M)$ and eigenfunctions $\phi_{\lambda_j} \in C^\infty$:

$$-\Delta_g \phi_{\lambda_j} = \lambda_j^2 \phi_{\lambda_j}; \ \|\phi_{\lambda_j}\|_{L^2} = 1.$$

- $H \subset M^n$ an orientable smooth hypersurface. In some cases, H can be a higher-codimension submanifold.

- **Problem:** Estimate the L^2 restrictions

$$\int_H |\phi_\lambda(s)|^2 \, d\sigma(s). \qquad (1)$$

- **Rationale:** 1) Want to understand the large$-\lambda$ behaviour of the ϕ_λ's. Pointwise

L^∞ results are very hard; difficult to improve on the bound

$$\|\phi_\lambda\|_{L^\infty(M)} = \mathcal{O}(\lambda^{\frac{n-1}{2}}).$$

The problem in (1) is easier but still very non-trivial.

2) Quantum ergodicity: Recent results (Zelditch-T, Dyatlov-Zworski) on Quantum Ergodic Restriction:

$$\lim_{\lambda\to\infty} \langle Op_H(a)\phi_\lambda|_H, \phi_\lambda|_H\rangle_{L^2(H)}$$

$$= \int_{B^*H} a(s,\sigma)\gamma(s,\sigma)\,ds\,d\sigma.$$

3) Restriction bounds naturally arise in study of eigenfunction nodal sets, etc...

- Most results extend to semiclassical Schrödinger operators $P(h) = -h^2\Delta + V(x)$ with eigenfunctions ϕ_h satisfying $P(h)\phi_h = E(h)\phi_h$, $|E(h)-E| = o(1)$, E a regular energy level.

General Results

- For general Laplace eigenfunctions with $\|\phi_\lambda\|_{L^2(M)} = 1$, Burq-Gérard-Tzvetkov [BGT] prove that

$$\int_H |\phi_\lambda|^2 \, d\sigma(s) = \mathcal{O}(\lambda^{\frac{1}{2}}), \ (n = 2). \tag{2}$$

- The universal bound (2) is achieved on S^2 with $H = \{(x, y, z) \in S^2; z = 0\}$ the equator and $\phi_n(x, y, z) = c_0 n^{\frac{1}{4}}(x + iy)^n; n = 1, 2, 3, ...$, the highest-weight harmonics.

- In the case where H has positive geodesic curvature, the bound (2) improves to

$$\int_H |\phi_\lambda|^2 \, d\sigma(s) = \mathcal{O}(\lambda^{\frac{1}{3}}); \ (n = 2).$$

BGT also obtain sharp general L^p bounds for $p \neq 2$ in any dimension and Hu generalized the positively-curved results to any dimension. Hassell-Tacy have extended these L^p bounds to the semiclassical case where $P(h) = -h^2\Delta + V(x)$.

- For flat tori with dim $= 2, 3$, Bourgain-Rudnick have proved sharp upper and lower L^2-restriction bounds when H is curved.

Quantum Completely Integrable (QCI) Case

- Here, we assume (M^2, g) compact surface, $P_1(h) = -h^2\Delta + V(x)$ and assume there is $P_2(h) \in Op_h(S^*)$ with

$$[P_1(h), P_2(h)] = 0.$$

Let $p_1, p_2 \in C^\infty(T^*M)$ be the corresponding principal symbols; in particular, $p_1(x, \xi) = |\xi|_g^2 + V(x)$.

- Let $(E, F) \in R^2$ be joint energy-levels for (p_1, p_2) with $dp_1|_{p_1^{-1}(E)} \neq 0$ and assume that $H \subset M$ is a smooth hypersurface (ie. a curve).

- Examples include compact spheres and tori of revolution, Liouville surfaces, ellipsoids, C. Neumann oscillators, ...

- **Admissibility** Consider the $2n-2$ dimensional submanifold of T^*M given by $N = p_1^{-1}(E) \cap T^*_H M$. The integral $p_2 \in C^\infty(T^*M)$ is **admissible** provided $p_2|_N$ is Morse.

- In the homogeneous case, $p_1(x, \xi) = |\xi|_g^2$ and $E = 1$ so admissibility requirement on p_2 is that $p_2|_{S^*_H M}$ is Morse.

- **Theorem [T (CMP)]** Let ϕ_h be L^2-normalized joint eigenfunctions of $(P_1(h), P_2(h))$ with joint eigenvalues $(E_1(h), E_2(h))$ and $E_1(h) = E_1 + O(h)$. Assuming H is admissible, for $h \in (0, h_0]$ with $h_0 > 0$ sufficiently small,

$$\int_H |\phi_h(s)|^2 \, d\sigma(s) = \mathcal{O}(|\log h|).$$

- **Example:** (convex surface of revolution) In geodesic polar coordinates $(t, \phi) \in (0,1) \times [0, 2\pi]$, $a(t) \geq 0, a(0) = a(1) = 0$ with single non-degenerate maximum at $t = t_0$.

$$p_1(t, \phi, \xi_t, \xi_\phi) = \xi_t^2 + a^{-1}(t)\xi_\phi^2,$$

$$p_2(t, \phi, \xi_t, \xi_\phi) = \xi_\phi^2.$$

Consider the equator $H = \{(t, \phi); t = t_0\}$. Then,

$$p_2|_{S^*_H M}(\phi, \xi_t) = a(t_0)(1 - \xi_t^2)$$

and this fails to be Morse. Along the equator $t = t_0$ we already know that there are ϕ_h's such that $\int_H |\phi_h|^2 d\sigma(s) \sim h^{-\frac{1}{2}}$.

- When H is a graph over the meridian of the form $H = \{(t, \phi(t))\}$, it is admissible. Similarily, when H is a graph over the equator of the form $H = \{(f(\phi), \phi)\}$, H is admissible as long as $f'(\phi) \neq 0$.

Quantum Ergodic Restriction (QER)

- Consider the opposite case where (M^n, g) compact, Riemannian manifold with **ergodic** geodesic flow

$$G^t : S^*M \to S^*M$$

with respect to Liouville measure $d\mu$ on S^*M.

- The set $S^*_H M$ of unit co-vectors to M with footpoints on H forms a cross-section to the flow in the sense that almost every trajectory of the geodesic flow intersects $S^*_H M$ transversally. In particular, almost every trajectory from $S^*_H M$ returns to $S^*_H M$.

Cauchy data along H

- Consider the eigenvalue problem on M

$$-\Delta_g \phi_j = \lambda_j^2 \phi_j, \quad \langle \phi_j, \phi_k \rangle = \delta_{jk}$$

$$B\phi_j = 0 \text{ on } \partial M,$$

where $\langle f, g \rangle = \int_M f\bar{g} dV$ (dV is the volume form of the metric) and where B is the boundary operator, e.g. $B\phi = \phi|_{\partial M}$ in the Dirichlet case or $B\phi = \partial_\nu \phi|_{\partial M}$ in the Neumann case. We also allow $\partial M = \emptyset$.

- Let $h_j = \lambda_j^{-1}$ and ϕ_{h_j} be a corresponding orthonormal basis of eigenfunctions with eigenvalue h_j^{-2}, so that the eigenvalue problem takes the semi-classical form,

$$(-h^2 \Delta_g - 1)\phi_h = 0,$$

$$B\phi_h = 0 \text{ on } \partial M$$

where $B = I$ or $B = hD_\nu$ in the Dirichlet or Neumann cases respectively.

- Consider the semiclassical Cauchy data along H:

$$CD(\phi_h) := \{(\phi_h|_H,\, hD_\nu \phi_h|_H)\}.$$

- **Theorem 1 [Christianson-Zelditch-T]** Suppose $H \subset M$ is a smooth, codimension 1 embedded orientable separating hypersurface and assume $H \cap \partial M = \emptyset$ if $\partial M \neq \emptyset$. Assume that $\{\phi_h\}$ is an interior quantum ergodic sequence. Then the appropriately renormalized Cauchy data $d\Phi_h^{CD}$ of ϕ_h is quantum ergodic in the sense that for any $a^w \in \Psi^0(H)$, there exists a sub-sequence of eigenvalues of density one so that as $h_j \to 0^+$,

$$\langle a^w hD_\nu \phi_h|_H, hD_\nu \phi_h|_H \rangle_{L^2(H)}$$

$$+\langle a^w (1 + h^2 \Delta_H) \phi_h|_H, \phi_h|_H \rangle_{L^2(H)}$$

$$\rightarrow_{h \rightarrow 0^+} \frac{4}{\mu(S^*M)} \int_{B^*H} a_0(x', \xi')(1 - |\xi'|^2)^{1/2} d\sigma,$$

where $a_0(x', \xi')$ is the principal symbol of a^w, $-h^2 \Delta_H$ is the induced tangential (semi-classical) Laplacian with principal symbol $|\xi'|^2$, μ is the Liouville measure on S^*M, and $d\sigma$ is the standard symplectic volume form on B^*H.

- Result holds for *all* interior hypersurfaces and generalizes results of Hassell-Zelditch and Burq in the boundary case (ie. $H = \partial\Omega$.)

Dirichlet data along H

- The *first return time* $T(s, \xi)$ on $S_H^* M$ defined to be

  $$T(s, \xi) = \inf\{t > 0 : G^t(s, \xi) \in S_H^* M, (s, \xi) \in S_H^* M)\}$$

  By definition $T(s, \xi) = +\infty$ if the trajectory through (s, ξ) fails to return to H. The domain of T (where it is finite) is denoted by $\mathcal{L}$ (loopset).

- Define the first return map on the same domain by

  $$\Phi : \mathcal{L} \to S_H^* M, \quad \Phi(s, \xi) = G^{T(s,\xi)}(s, \xi) \tag{3}$$

  When G^t is ergodic, Φ is defined almost everywhere and is also ergodic with respect to Liouville measure $\mu_{L,H}$ on $S_H^* M$. The jth return time $T^{(j)}(s, \xi)$ to $S_H^* M$ and the jth return map Φ^j are defined inductively when the return times are finite.

- **Definition:** Let $r_H : T^*_H M \to T^*_H M$ be reflection through T^*H. H is asymmetric with respect to geodesic flow if

$$\mu_{L,H}\,(\bigcup_{j\neq 0}^{\infty}\{(s,\xi)\in S^*_H M :$$

$$r_H G^{T^{(j)}(s,\xi)}(s,\xi) = G^{T^{(j)}(s,\xi)} r_H(s,\xi)\}\,) = 0. \tag{4}$$

- **Theorem 2 (QER) [Zelditch-T (GAFA)]** Let (M,g) be a compact manifold with ergodic geodesic flow, and let $H \subset M$ be a hypersurface that is asymmetric with respect to geodesic flow. Then, there exists a density-one subset S of N such that for $a \in S^{0,0}(T^*H \times [0,h_0))$,

$$\lim_{h_j \to 0^+; j \in S} \langle Op_{h_j}(a)\gamma_H \phi_{h_j}, \gamma_H \phi_{h_j}\rangle_{L^2(H)} = \omega(a),$$

where

$$\omega(a) = \frac{1}{vol(S^*M)} \int_{S^*_H M} a_0(s,\sigma)\, d\mu_{L,H}.$$

- Result applies to geodesic circles, closed horocycles and generic closed geodesics on a hyperbolic surface.

- The analogue of QER for piecewise smooth bounded domains in R^n was proved in [Zelditch-T] (AHP).

- Results have subsequently been generalized by Dyatlov-Zworski to semiclassical Schrödinger operators $P(h) = -h^2\Delta + V$ and arbitrary manifolds with boundary.

- **Sketch of Proof of Theorem 2:** Assume $a \in S^0$ is homogeneous (semiclassical case follows similarily). Let $U(t) = \exp(it\sqrt{\Delta}) : C^\infty(M) \to C^\infty(M)$ and $\gamma_H : C^0(M) \to C^0(H)$ be restriction to H. We study matrix elements

$$\langle Op_H(a)\phi_j|_H, \phi_j|_H\rangle_{L^2(H)}.$$

$$\langle Op_H(a)\gamma_H\phi_j, \gamma_H\phi_j\rangle_{L^2(H)}$$

$$= \langle \gamma_H^* Op_H(a)\gamma_H U(t)\phi_j, U(t)\phi_j\rangle_{L^2(M)}$$

$$= \langle V(t;a)\phi_j, \phi_j\rangle_{L^2(M)} = \langle \bar{V}_T(a)\phi_j, \phi_j\rangle_{L^2(M)} \quad (*)$$

where,

$$V(t;a) := U(-t)\gamma_H^* Op_H(a)\gamma_H U(t),$$

$$\bar{V}_T(a) := \frac{1}{2T}\int_{-\infty}^{\infty} \chi(T^{-1}t)\, V(t;a)\, dt.$$

- Here, $\chi \in C_0^\infty(R)$ with $\int_{-\infty}^{\infty} \chi(t)dt = 1$.

- Composition of wave fronts gives

$$WF'(\bar{V}_T(a)) := \{(x,\xi,x',\xi') \in T^*M \times T^*M :$$

$$\exists t \in (-T,T), \exp_x t\xi = \exp_{x'} t\xi' = s \in H,$$

$$G^t(x,\xi)|_{T_sH} = G^t(x',\xi')|_{T_sH}, \;\; |\xi| = |\xi'|\}.$$

- Modulo some technical issues regarding tangential and normal directions to H, one decompose $\bar{V}_T(a)$ into a pseudo-differential and a Fourier integral part according to the dichotomy that points $(x, \xi, x', \xi') \in WF'(\bar{V}_T(a))$ satisfy either

$$\begin{aligned} &(i)\ G^t(x,\xi) = G^t(x',\xi'), \text{ or} \\ &(ii)\ G^t(x',\xi') = r_H G^t(x,\xi), \end{aligned} \tag{5}$$

  where r_H is the reflection map of T^*H.

- One has the following decomposition: $\bar{V}_T(a)$ is a Fourier integral operator with local canonical graph, and possesses the decomposition

$$\bar{V}_T(a) = P_T(a) + F_T(a) + R_T(a).$$

(i) $P_T(a) \in Op_{cl}(S^0(T^*M))$ is a pseudo-differential operator of order zero with principal symbol

$$\sigma(P_T(a))(x,\xi) = \frac{1}{T}\sum_{j\in Z}(\gamma^{-1}a_H)(G^{t_j(x,\xi)}(x,\xi))$$

$$\times \chi(T^{-1}t_j(x,\xi))$$

where, $t_j(x,\xi) \in C^\infty(T^*M)$ are the impact times of the geodesic $\exp_x(t\xi)$ with H, $a_H(s,\xi) = a(s,\xi|_H) \in S^0(T^*_H M)$ and $\gamma \in S^0(T^*_H M)$

(ii) $F_T(a)$ is a Fourier integral operator of order zero with canonical relation Γ_T.

$$F_T(a) = \sum_{j=1}^{N_T} F_T^{(j)}(a), \tag{6}$$

where the $F_T^{(j)}(a)$; $j = 1, ..., N_{T,\epsilon}$ are zeroth-order homogeneous Fourier integral operators.

Here, $WF'(F_T^{(j)}(a))$ is in the reflection piece of (5); explicity

$$WF'(F_T^{(j)}(a)) = \{(x,\xi;\mathcal{R}_j(x,\xi))\},$$

$$\mathcal{R}_j(x,\xi) = G^{t_j(x,\xi)} r_H G^{-t_j(x,\xi)}(x,\xi).$$

Symbol is given by

$$\sigma(F_T^{(j)})(x,\xi) = \frac{1}{T}(\gamma^{-1}a_H)(G^{t_j(x,\xi)}(x,\xi))$$

$$\times\,\chi(T^{-1}t_j(x,\xi))\ |dxd\xi|^{\frac{1}{2}}.$$

(iii) $R_T(a)$ is a smoothing operator.

- It suffices to show that

$$\limsup_{\lambda\to\infty} \frac{1}{N(\lambda)} \sum_{j:\lambda_j\leq\lambda} \left|\langle \bar{V}_T(a)\phi_j,\phi_j\rangle_{L^2(M)} - \omega(a)\right|^2$$

$$= o(1)\ \ (\text{as } T\to\infty).$$

- Use the L^2 ergodic theorem to show that

$$\limsup_{\lambda\to\infty} \frac{1}{N(\lambda)} \sum_{j:\lambda_j\le\lambda} \left| \langle P_T(a)\phi_j, \phi_j\rangle_{L^2(M)} - \omega(a) \right|^2 = 0,$$

- Reduced to showing that

$$\limsup_{\lambda\to\infty} \frac{1}{N(\lambda)} \sum_{j:\lambda_j\le\lambda} \left| \langle F_T(a)\phi_j, \phi_j\rangle_{L^2(M)} \right|^2 = o(1)$$

as $T \to \infty$. Here, we need the microlocal asymmetry condition on H.

- First use the Schwarz inequality

$$\frac{1}{N(\lambda)} \sum_{j:\lambda_j\le\lambda} \left| \langle F_T(a)\phi_j, \phi_j\rangle_{L^2(M)} \right|^2$$

$$\le \frac{1}{N(\lambda)} \sum_{j:\lambda_j\le\lambda} \langle F_T(a)^* F_T(a)\phi_j, \phi_j\rangle_{L^2(M)} \; (**)$$

to bound the variance sum by a trace and use the local Weyl law for homogeneous Fourier integral operators $F : C^\infty(M) \to C^\infty(M)$ [Z] to prove that the right side of $(**)$ tends to zero under geodesic asymmetry condition.

- In the case of local canonical graphs, the local Weyl law states that

$$\frac{1}{N(\lambda)} \sum_{j:\lambda_j \leq \lambda} \langle F\phi_{\lambda_j}, \phi_{\lambda_j}\rangle \to \int_{S\Gamma_F \cap \Delta_{T^*M}} \sigma_\Delta(F) d\mu_L, \tag{7}$$

where Γ_F is the canonical relation of F, $S\Gamma_F$ is the set of vectors of norm one, and $S\Gamma_F \cap \Delta_{T^*M}$ is its intersection with the diagonal of $T^*M \times T^*M$. Also, $\sigma_\Delta(F)$ is the (scalar) symbol in this set and $d\mu_L$ is Liouville measure. Thus, if Γ_F is a local canonical graph, the right side is zero unless the intersection has dimension $m = \dim M$, i.e.

the trace sifts out the 'pseudo-differential part' of F.

- Application of (7) to $F = F_T(a)^* F_T(a)$ gives:

$$\lim_{\lambda\to\infty} \frac{1}{N(\lambda)} \sum_{\lambda_j \le \lambda} \sum_{k,\ell=1}^{N_T} \langle F_T^{(\ell)}(a)^* F_T^{(k)}(a) \phi_{\lambda_j}, \phi_{\lambda_j} \rangle$$

$$= \frac{1}{T^2} \int_{S^*M} \sum_{j=1}^{N_T} \left| \chi(\frac{t_j(x,\xi)}{T}) \gamma^{-1} a_H(G^{t_j(x,\xi)}(x,\xi)) \right|^2 d\mu$$

$$+ \frac{1}{T^2} \int_{S\{\mathcal{R}_j = \mathcal{R}_k\}} \sum_{j \neq k}^{N_T} \chi(\frac{t_j(x,\xi)}{T}) \, \gamma^{-1} a_H(G^{t_j(x,\xi)}(x,\xi))$$

$$\times \chi(\frac{t_k(x,\xi)}{T}) \, \gamma^{-1} a_H(G^{t_k(x,\xi)}(x,\xi)) \, d\mu_L. \ (\#)$$

- Since $N_T = \mathcal{O}(T)$ and $|\chi| \le 1$, the first term on the right side (#)

$$O\left(\frac{1}{T} \|a_H\|^2_{C^0(S^* M_H)} \right).$$

- The second term on the RHS in (#) vanishes as $T \to \infty$ from the geodesic asymmetry condition on H which can be written in the form

$$\mu_L \left(S\{\mathcal{R}_j = \mathcal{R}_k\} \right) = 0$$

for $j \neq k$.

Eigenfunction Nodal Sets

- $\Omega \subset R^2$ a piecewise analytic bounded domain and consider Neumann (or Dirichlet) problem:

$$-\Delta_\Omega \phi_\lambda = \lambda^2 \phi_\lambda,$$

$$\partial_\nu \phi_\lambda|_{\partial\Omega} = 0.$$

- Say that an interior C^ω curve $H \subset \Omega$ is *good* if for some constant $C > 0$

$$\int_H |\phi_\lambda|^2 \, d\sigma \geq e^{-C\lambda}.$$

- Define the nodal intersection counting function

$$n_D(\lambda, H) = \#\{N_{\phi_\lambda} \cap H\},$$

where, $N_{\phi_\lambda} = \{x \in \Omega; \phi_\lambda(x) = 0\}$.

- **Theorem [Zelditch-T (JDG)]** Assume that $H \subset int(\Omega)$ is good. Then,

$$n_D(\lambda, H) = O_H(\lambda), \text{ as } \lambda \to \infty.$$

- Not all curves H are good: Consider the stadium with bisector $H = \{(x,y) \in \Omega; x = 0\}$.

- Let H_ϵ be a complex tube of width $\epsilon > 0$ containing H as the totally-real part. Let $(\phi_\lambda|_H)^{\mathbf{C}}(z)$ be the holomorphic continuation of $\phi_\lambda|_H$ to H_ϵ.

- Goodness condition in above result can be weakened to

$$\sup_{z \in H_\epsilon} |(\phi_\lambda|_H)^{\mathbf{C}}(z)| \geq e^{-C\lambda} \ (*)$$

for some $C > 0$, which is easier to verify.

- **Theorem [El-Hajj-T]** Let $H \subset \Omega$ be any interior C^ω curve. Then, under the weakened condition $(*)$,

$$n_D(\lambda, H) = \mathcal{O}_H(\lambda).$$