Q15

#10

Establish $\displaystyle\int_0^{2\pi} \frac{\cos\theta}{1-2a\cos\theta+a^2}\,d\theta = I = \frac{2\pi a}{1-a^2}$ for $|a|<1$.

Setting $z=e^{i\theta}$ we get

$$I = \oint_{|z|=1} \frac{\left(\frac{z+z^{-1}}{2}\right)}{1-\cancel{2}a(z+z^{-1})+a^2}\,\frac{dz}{iz}$$

$$= \frac{1}{2i}\oint_{|z|=1} \frac{(z^2+1)}{z\left(-\cancel{2}az^2+(a^2+1)z-a\right)}\,dz$$

$$= \frac{1}{2i}\oint_{|z|=1} \frac{(z+i)(z-i)}{-az\left[z^2-\left(\frac{a^2+1}{a}\right)z+1\right]}$$

So there are poles at $z=0,\ a,\ \frac{1}{a}$.

Since $|a|<1$,

$$I = 2\pi i\left[\text{Res}(a) + \text{Res}(0)\right]$$

$$\text{Res}(0) = \frac{1}{2i}\,\frac{1}{-a(1)+0(\cdots)} = \frac{-1}{2ai}$$

$$\text{Res}(a) = \frac{1}{2i}\,\frac{a^2+1}{(-a^2)\left(2a-\left(\frac{a^2+1}{a}\right)\right)} = \frac{-1}{2ia}\left(\frac{1+a^2}{-1+a^2}\right)$$

So

$$I = \frac{-2\pi i}{2ai}\left[1 + \frac{a^2+1}{a^2-1}\right] = \frac{-\pi}{a}\left(\frac{2a^2}{a^2-1}\right) = \frac{2\pi}{1-a^2} \quad \boxtimes$$

Q16

#20 p.370-1

Evaluate $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{(x^2+x+1)(x^2+1)} = I$

By thm 4,

$$I = 2\pi i \sum_{\substack{z_0 \text{ poles} \\ \text{in } \mathcal{H}}} \operatorname{Res}[f(z), z_0]$$

where $f(z) = \dfrac{1}{(z^2+z+1)(z^2+1)}$

Poles @ $z = \pm i, \dfrac{-1 \pm i\sqrt{3}}{2}$

So the poles in the upper-half plane are $\overset{z_1}{\overset{\shortparallel}{i}}, \overset{z_2}{\overset{\shortparallel}{\dfrac{-1+i\sqrt{3}}{2}}}$

Res @ z_1: $\dfrac{1}{(z_1^2+z_1+1)(2z_1)} = \dfrac{1}{2z_1^2} = -\dfrac{1}{2}$

Res @ z_2: $\dfrac{1}{(2z_2+1)(z_2^2+1)} = \dfrac{-1}{2z_2^2+z_2}$

$$= \frac{1}{(i\sqrt{3})\left(\frac{1-i\sqrt{3}}{2}\right)} = \frac{1}{2}\left(\frac{1+i\sqrt{3}}{i\sqrt{3}}\right) = \frac{1}{2} + \frac{1}{2i\sqrt{3}}$$

Thus,

$$I = 2\pi i \left(\frac{1}{2i\sqrt{3}}\right) = \boxed{\frac{\pi}{\sqrt{3}}}$$

Q17

#24

Evaluate $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{(x+a)^2+b^2} = I \quad a, b \in \mathbb{R},\ b > 0$

By Thm 4,

$$I = 2\pi i \sum_{\substack{z_0 \text{ is a} \\ \text{pole in the} \\ \text{U.H.P.}}} \operatorname{Res}[f(z), z_0]$$

where $f(z) = \dfrac{1}{(z+a)^2+b^2}$

Poles @ $(z+a)^2 = -b^2$

$$z+a = \pm ib$$
$$z = -a \pm ib$$

Since $b > 0$, $z = -a+ib$ is the only pole in the U.H.P.

$$\operatorname{Res}[f(z), -a+ib] = \frac{1}{2((-a+ib)+a)} = \frac{1}{2ib}$$

Thus, $I = 2\pi i \left(\dfrac{1}{2ib}\right) = \dfrac{\pi}{b}$ ☒

Q18

#2 p.381-382

Evaluate $\displaystyle\int_{-\infty}^{\infty} \frac{x\sin(2x)}{x^2+3}\,dx = I$

By Eqn 6.6 - 12(b),

$$I = \int_{-\infty}^{\infty} \sin(2x)\,\frac{x}{x^2+3} = \mathrm{Im}\left[2\pi i \sum_{\substack{z_0 \in \mathcal{H} \\ \text{pole}}} \mathrm{Res}\left(\frac{z\, e^{i2z}}{z^2+3}, z_0\right)\right]$$

$\frac{ze^{i2z}}{z^2+3}$ has simple poles at $\pm i\sqrt{3}$, so we need the residue at $z_0 = i\sqrt{3} \in \mathcal{H}$, which gives

$\cancel{2\pi i \frac{z_0 e^{i2z_0}}{2z_0} = \frac{i e^{-2\sqrt{3}}}{2}}$

~~Thus,~~ $\cancel{I = \mathrm{Im}\left[2\pi i\left(i e^{-2\sqrt{3}}\right)\right]}$

$$\frac{\cancel{z_0}\, e^{i2z_0}}{2\cancel{z_0}}$$

Thus,

$$I = \mathrm{Im}\left[\frac{2\pi i}{2}\left(e^{i2(i\sqrt{3})}\right)\right] = \pi e^{-2\sqrt{3}}$$

Q19

#5

Evaluate $\displaystyle\int_{-\infty}^{\infty} \frac{x\sin(2x)}{x^2+x+1}\,dx$

Again, by Eqn 6.6 − 12(b)

$$I = \int_{-\infty}^{\infty} \sin(2x)\, \frac{x}{(x-\omega)(x-\omega^2)} = \mathrm{Im}\left[2\pi i \,\mathrm{Res}\left(\frac{z e^{i2z}}{(z-\omega)(z-\omega^2)}, \omega\right)\right]$$

where $\omega = \dfrac{-1+i\sqrt{3}}{2}$

$$= \mathrm{Im}\left[2\pi i \, \frac{\omega e^{i2\omega}}{2\omega+1}\right]$$

$$= \mathrm{Im}\left[2\pi \cancel{i}\, \frac{e^{2\pi i/3}\, e^{-i-\sqrt{3}}}{\cancel{i}\sqrt{3}}\right]$$

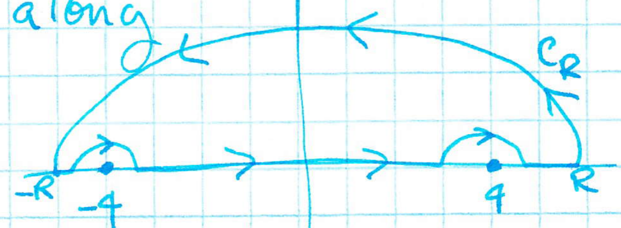
$$= \frac{2\pi}{\sqrt{3}}\, e^{-\sqrt{3}} \sin\left(\frac{2\pi}{3} - 1\right)$$

Q 20

#5 p. 393−4

$$\int_{-\infty}^{\infty} \frac{\cos 2x}{x^2-16}\, dx = I$$

To avoid poles at $-4, 4$ we consider the contour integral along



Thus, writing $f(x) = \dfrac{e^{2ix}}{x^2-16}$, $f(z) = \dfrac{e^{2iz}}{z^2-16}$ we get

$$\int_{-R}^{-4-\varepsilon} f(x)\,dx + \int_{|z+4|=\varepsilon} f(z)\,dz + \int_{-4+\varepsilon}^{4-\varepsilon} f(x)\,dx + \int_{|z+4|=\varepsilon} f(z)\,dz + \int_{4+\varepsilon}^{R} f(x)\,dx + \int_{C_R} f(z)\,dz$$

Sums to zero since there are no poles on the interior of the boundary. Taking $\lim_{\substack{R\to\infty\\ \varepsilon\to 0}}$ we get

$$\int_{-\infty}^{\infty}\frac{e^{2ix}}{x^2-16}\,dx-\pi i\left[\mathrm{Res}\left[f(z),4\right]\right)-\pi i\left(\mathrm{Res}\left[f(z),-4\right]\right)=0$$

Thus we find

~~$I = Re$~~

$$\mathrm{Res}\left[f(z),4\right]=\frac{e^{8i}}{8}\qquad \mathrm{Res}\left[f(z),-4\right]=\frac{e^{-8i}}{-8}$$

and

$$I=\mathrm{Re}\left(\pi i\left(\frac{e^{i8}}{8}+\frac{e^{-i8}}{-8}\right)\right)$$

$$=\mathrm{Re}\left(\frac{-\pi}{4}\,\frac{e^{i8}-e^{-i8}}{2i}\right)$$

$$=\mathrm{Re}\left(-\frac{\pi}{4}\sin(8)\right)$$

$$=-\frac{\pi}{4}\sin(8)$$

Q21

#9

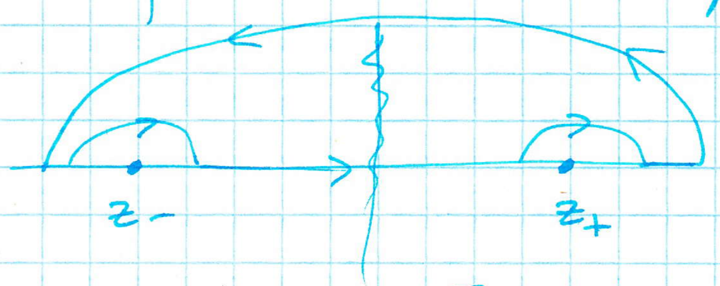
Prove $I=\displaystyle\int_{-\infty}^{\infty}\frac{\cos mx}{ax^2+bx+c}=\frac{-2\pi\cos\left(\frac{mb}{2a}\right)\sin\left(m\frac{\sqrt{b^2-4ac}}{2a}\right)}{\sqrt{b^2-4ac}}$

where $m\geq 0$, $a,b,c\in\mathbb{R}$, $b^2>4ac$, $a\neq 0$.

<u>Note</u>: $\dfrac{e^{imz}}{az^2+bz+c} = f(z)$ has poles at

$$z = \frac{-b \pm \sqrt{b^2-4ac}}{2a} \quad \text{call the roots, } z_+, z_-$$

Since $b^2 > 4ac$, $a \neq 0$, $z_+, z_- \in \mathbb{R}$ and hence, taking the contour integral of $f(z)$ along



gives

$$\int_{-\infty}^{\infty} f(x)\,dx = \pi i \left[\operatorname{Res}[f(z), z_+] + \operatorname{Res}[f(z), z_-] \right]$$

So

$$I = \operatorname{Re}\left(\pi i \left(\frac{e^{imz_+}}{2az_+ + b} + \frac{e^{imz_-}}{2az_- + b} \right) \right)$$

$$= \operatorname{Re}\left[\pi i \, \frac{e^{imz_+} - e^{imz_-}}{\sqrt{b^2-4ac}} \right] \quad \text{using } \sin z = \frac{e^{iz}-e^{-iz}}{2i}$$

$$= \operatorname{Re}\left[\frac{\pi i}{\sqrt{b^2-4ac}} \cdot 2i \, e^{-imb/2a} \sin\left(\frac{m\sqrt{b^2-4ac}}{2a} \right) \right]$$

$$= \frac{-2\pi}{\sqrt{b^2-4ac}} \cos\left(\frac{mb}{2a} \right) \sin\left(\frac{m\sqrt{b^2-4ac}}{2a} \right)$$

Q22

#2 p. 469–970

Find the ~~Lap~~ inverse Laplace transform for $t \geq 0$ where $a, b \neq 0 \in \mathbb{R}$, $a \neq b$.

$$\frac{s}{(s-a)(s+b)}$$

$$f(t) = \operatorname{Res}\left[\frac{se^{st}}{(s-a)(s+b)}, a\right] + \operatorname{Res}\left[\frac{se^{st}}{(s-a)(s+b)}, -b\right]$$

$$= \frac{ae^{at}}{a+b} + \frac{(-b)\,e^{-bt}}{-b-a} = \frac{1}{a+b}\left(ae^{at} + be^{-bt}\right)$$

Q23

#3

$$\frac{1}{(s+a)^2}$$

$$f(t) = \operatorname{Res}\left[\frac{e^{st}}{(s+a)^2}, -a\right]$$

$$= \lim_{s \to -a} \frac{d}{ds} e^{st}$$

$$= \lim_{s \to -a} te^{st} = te^{-at}$$

Q24

#7

$$\frac{1}{(s^2+a^2)^2}$$

Poles at $s = \pm ai$ ~~so~~ of order 2, so

$$f(t) = \mathrm{Res}\left[\frac{e^{st}}{(s^2+a^2)^2}, ai\right] + \mathrm{Res}\left[\frac{e^{st}}{(s^2+a^2)^2}, -ai\right]$$

$$= \lim_{s \to ai} \frac{d}{ds}\left(\frac{e^{st}}{(s+ai)^2}\right) + \lim_{s \to -ai} \frac{d}{ds}\left(\frac{e^{st}}{(s-ai)^2}\right)$$

$$= \left.\frac{te^{st}(s+ai)^2 - 2e^{st}(s+ai)}{(s+ai)^4}\right|_{ai} + \frac{te^{st}(s-ai)^2 - 2e^{st}(s-ai)}{(s-ai)^4}$$

$$= \frac{-4a^2te^{ait} - 4ai\,e^{ait}}{16a^3} + \frac{-4a^2te^{-ait} + 4ai\,e^{-ait}}{16a^3}$$

$$= \frac{1}{16a^3}\left(-(4i)(2i)\sin(at)\right) - \frac{4at}{16a^3}\left(2\cos(at)\right)$$

$$= \frac{1}{2a^3}\sin(at) - \frac{t}{2a^2}\cos(at)$$