Q9

#18 Find all the poles of ~~each~~ the fcn and give the order of each pole of $\dfrac{1}{z^2+\cancel{2}z+1}$.

By the quad. formula, we have zeros for $\dfrac{1}{z^2+2z+1}$ at $z = \dfrac{-1 \pm \sqrt{1-4}}{2} = \dfrac{-1 \pm i\sqrt{3}}{2}$.

Thus $\dfrac{1}{z^2+z+1} = \dfrac{1}{\left(z - \left(\frac{-1+i\sqrt{3}}{2}\right)\right)\left(z - \left(\frac{-1-i\sqrt{3}}{2}\right)\right)}$

and there are poles of order 1 @ $z = \dfrac{-1 \pm i\sqrt{3}}{2}$.

Q10

#4 p. 356-7

(a) State the location and (b) order of each pole and (c) find the corresponding residue, for $\dfrac{1}{z} - \dfrac{e^z}{z(z+1)} + \dfrac{1}{(z-1)^4}$.

(a) Poles @ $z = \pm 1$

<u>Not @ z=0</u>: $\lim\limits_{z \to 0} z f(z) = \lim\limits_{z \to 0}\left(1 - \dfrac{e^z}{(z+1)} + \dfrac{z}{(z-1)^4}\right)$

$= 1 - 1 + 0 = 0$

So the L. exp @ 0 has the form $f(z) = \sum\limits_{n=0}^{\infty} c_n z^n$.

(b) + (c)

$\underline{z = -1}$ Pole of order 1

$$\mathrm{Res}\,[f(z), -1] = \lim_{z \to -1} (z+1)\, f(z)$$

$$= \frac{-e^{-1}}{-1} = e^{-1}$$

$\underline{z = 1}$ Pole of order 4

Using Rule III p. 353 gives

$$\mathrm{Res}\,[f(z), 1] = 0$$

since we can factor $(z-1)$ out of $\frac{d^3}{dz^3}\left[(z-1)^4 f(z)\right]$.

Q11

$\underline{\#10}$ Same (a), (b), (c) as Q10, but for

$$\frac{1}{\left(\mathrm{Log}(z/e) - 1\right)^2}$$

(a) We have a pole when $\mathrm{Log}\left(\frac{z}{e}\right) = 1$ as the denom then vanishes. ie.

$$1 = \mathrm{Log}\, z - \mathrm{Log}\, e = \mathrm{Log}\, z - 1 \iff \boxed{z = e^2}$$

(b) + (c) Note that since $\mathrm{Log}\left(\frac{z}{e}\right) - 1 = 0$ at $z = e^2$

~~By taking derivatives an~~

it has a power series expansion $\sum_{n=1}^{\infty} c_n (z - e^2)^n$

By taking derivatives and evaluating at $z=e^2$, we get $c_1 = \frac{1}{e^2}$, $2c_2 = \frac{-1}{(e^2)^2} \iff c_2 = \frac{-1}{2e^4}$

Thus

$$\left(\operatorname{Log}\frac{z}{e} - 1\right)^2 = \left(e^{-2}(z-e^2) - \frac{1}{2}e^{-4}(z-e^2)^2 + \cdots\right)^2$$

$$= e^{-4}(z-e^2)^2 - \frac{1}{2}e^{-6}(z-e^2)^3 + \cdots$$

By long division, we recover that

$$\frac{1}{\left(\operatorname{Log}\left(\frac{z}{e}\right)-1\right)^2} = e^4(z-e^2)^{-2} + e^2(z-e^2)^{-1} + \text{high order terms.}$$

Thus the pole has order 2 (@ $z=e^2$) and the residue is e^2.

Q12

#14 Same (a) (b) (c) as above with

$$\frac{1}{e^{2z}+e^z+1}.$$

(a) Look for solutions to $e^{2z}+e^z+1=0$.

By quadratic formula:

$$e^z = \frac{-1 \pm i\sqrt{3}}{2} = e^{\pm 2\pi i/3}$$

Taking logs: $z = 2\pi i\left[k \pm \frac{2}{3}\right]$, $k \in \mathbb{Z}$

Thus we have simple poles at

(b) $$z = 2\pi i\left[k \pm \frac{1}{3}\right] \qquad k \in \mathbb{Z}$$

since the zeros are order 1 in the denom.

(c) By rule IV, for any pole z_0, its residue is given by

$$\frac{1}{2e^{2z_0}+e^{z_0}} = \frac{1}{2(-1-e^{z_0})+e^{z_0}} = \frac{-1}{2+e^{z_0}}$$

since $e^{2z_0}+e^{z_0}+1=0$

$$= \frac{-1}{2+\left(\frac{-1\pm i\sqrt{3}}{2}\right)} = \boxed{\frac{-2}{3\pm i\sqrt{3}}}$$

Q 13

#2 p. 364-5

Establish

(a) $$\int_0^{2\pi} \frac{d\theta}{k-\sin\theta} = \frac{2\pi}{\sqrt{k^2-1}} \quad \text{for } k>1.$$

(b) Does your result hold for $k<-1$? Explain.

Using the Δ of vars $z=e^{i\theta}$, $dz = e^{i\theta} i\, d\theta$

we get $$I = \int_0^{2\pi} \frac{d\theta}{k-\sin\theta} = \oint_{|z|=1} \frac{1}{k-\left(\frac{z-z^{-1}}{2i}\right)}\frac{dz}{iz}$$

$$= \oint_{|z|=1} \frac{-2\, dz}{z^2 - 2ikz - 1}$$

This has simple poles at $z = \dfrac{2ik \pm \sqrt{(2ik)^2 + 4}}{2}$

$$= ik \pm i\sqrt{k^2-1}$$

$$= i\left[k \pm \sqrt{k^2-1}\right]$$

At the poles, $\overset{+}{z_0}, \overset{-}{z_1}$, $|z_i| = 2k^2 - 1 \pm 2\sqrt{k^2-1}$

and since $|z_0 z_1| = 1$, since $|z_1| < |z_0|$,

$z_1 = i\left(k - \sqrt{k^2-1}\right)$ must be in the unit circle for $k > 1$.

Thus, for $k > 1$,

$$I = 2\pi i \,\mathrm{Res}\left[f(z), z_1\right]$$

$$= 2\pi i \left(\frac{-2}{2z_1 - 2ik}\right)$$

$$= \frac{-2\pi}{\sqrt{k^2-1}}$$

For $k < -1$, $|z_0| < 1$ and $|z_1| > 1$.

Thus $I = 2\pi i \,\mathrm{Res}\left[f(z), z_0\right]$

$$= \frac{-2\pi}{\sqrt{k^2-1}}$$

So it also holds for $k < -1$.

Q14

#6

Establish $\displaystyle\int_0^{2\pi} \cos^m\theta\, d\theta = \frac{2\pi}{2^m}\frac{m!}{\left[\left(\frac{m}{2}\right)!\right]^2}$ for $m \geq 0$ even.

Using $z = e^{i\theta}$,

$$I = \int_0^{2\pi} \cos^m\theta\, d\theta = \oint_{|z|=1} \left(\frac{z+z^{-1}}{2}\right)^m \frac{dz}{iz}$$

$$= \oint_{|z|=1} \frac{1}{i\,2^m} \frac{(z^2+1)^m}{z^{m+1}}\, dz$$

$$= \frac{2\pi}{2^m} \operatorname{Res}\left[\left\{\frac{(z^2+1)^m}{z^{m+1}}\right\}, 0\right]$$

Since $\displaystyle\frac{(z^2+1)^m}{z^{m+1}} = \left(\sum_{k=0}^{m} \frac{m!}{(m-k)!\,k!} (z^2)^{m-k}\right)\frac{1}{z^{m+1}}$

when $m \geq 0$ even, we get the -1^{th} ~~term~~ term when $2m-2k-m-1 = -1$, so when $k = \frac{m}{2}$.

$\Rightarrow m = 2k$

So $\displaystyle I = \frac{2\pi}{2^m}\operatorname{Res}\left[\cdots, 0\right] = \frac{2\pi}{2^m}\frac{m!}{\left[\left(\frac{m}{2}\right)!\right]^2}$